

A parametrized quasi-normal mode framework for non-Schwarzschild metrics

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In this work we comment in more detail on what happens to the parametrized framework first presented by Cardoso *et al.* when there are departures from the Schwarzschild background metric, as well as possible deviations in the “dynamics”. We treat possible deviations in the background metric with additional coefficients with respect to the original works. The advantages of this reformulation are clear when applied to a parameter estimation problem, since the coefficients are always real, and many of them do not depend on the overtone number and angular momentum of the frequency, thus eventually reducing the total amount of parameters to be inferred.

I. INTRODUCTION

Ongoing and future efforts of gravitational wave detectors provide unprecedented access to test general relativity (GR) in the strong field regime [1–4]. Among the several exciting possibilities, one is to perform black hole spectroscopy in order to verify the validity of the requirements of the no-hair theorem and also general relativity itself [5–7]. Our current understanding of binary black hole mergers predicts that the final stage of such events should be described by a superposition of linear perturbations of the background, the so-called quasi-normal modes [8–10], see also Refs. [11–14] for reviews.

The spectrum of quasi-normal modes is sensitive to the background perturbed metric, but also to the dynamical behaviour of the theory in which the perturbations are excited. The purpose of this work is to provide a framework which allows to study quasi-normal modes in spherical symmetry for metrics which are different from the Schwarzschild metric when the modifications are small. Our framework is following a prescription similar to the one presented by Cardoso *et al.* [15] and McManus *et al.* [16]. In these works, a modified (system) of wave equations has been introduced and the corresponding quasi-normal mode spectrum has been computed up to quadratic order for a set of small deviation parameters. We explicitly show how possible deviations in the background metric could affect the quasi-normal frequencies, in a different fashion from [15, 16]. While the general idea is similar, our approach allows for a better description of non-Schwarzschild backgrounds, purely real deviation parameters in the framework, as well as a differentiation between the parameters that depend on the angular momentum of the frequencies, and those which do not.

This work is structured as follows. In Sec. II we introduce the metric dependent parametrized framework. We apply it to two different cases in Sec. III. Conclusions are discussed in Sec. IV. Throughout this work we use units in which $G = c = 1$.

II. THEORETICAL FRAMEWORK

Let us start by qualitatively reviewing how gravitational perturbations in gravity theories can be studied. In general, one assumes a linear perturbation of the metric as

$$g_{ab} \simeq \bar{g}_{ab} + h_{ab}, \quad (1)$$

for which the background metric \bar{g}_{ab} and the perturbation metric h_{ab} satisfy the 0th order and 1st order equations of motion, respectively. They are obtained from the linear expansion of the full system of equations

$$G_{ab} \simeq \bar{G}_{ab} + H_{ab} = 0, \quad (2)$$

where G_{ab} are the equations of motion of a given gravity theory. Let us assume that the background metric is given by some non-GR metric. We remark that we do not know *a priori* which equations of motion this metric is satisfying. However, as long as it is perturbatively close, with respect to some parameter δ , to the Schwarzschild/Kerr metric, it must satisfy Einstein’s equations of GR up to some error linear in δ

$$\bar{G}_{ab}^{\text{GR}}[\bar{g}_{cd}] = 0 + \mathcal{O}(\delta). \quad (3)$$

The presence of the $\mathcal{O}(\delta)$ term can be interpreted as the fact that the equations of motion of a new theory whose static solution is approximated by the metric \bar{g}_{ab} must have different dynamics, not captured with the GR equations by a linear factor.

We can conjecture that the first order equations share the same property. For GR, one would have

$$H_{ab}^{\text{GR}}[h_{cd}; \bar{g}_{cd}^{\text{SCH}}] = 0, \quad (4)$$

but when $\bar{g}_{cd}^{\text{SCH}}$ is substituted by \bar{g}_{cd} , the equations become

$$H_{ab}^{\text{GR}}[h_{cd}; \bar{g}_{cd}] = 0 + \mathcal{O}(\delta). \quad (5)$$

One must bear in mind that this notation generically includes the 0th and 1st order metric as well as their derivatives.

Let us now take into account a spherically symmetric background metric defined as follows

$$\bar{g}_{ab}dx^a dx^b = -f(r)A(r)dt^2 + \frac{B^2(r)}{f(r)A(r)}dr^2 + r^2d\Omega, \quad (6)$$

with $f(r) = 1 - r_H/r$, being r_H the location of the event horizon. Then, we can expand the functions A and B with a set of *real* coefficients $a_M^{(k)}, b_M^{(k)}$ in a post-Newtonian fashion [17]

$$A(r) = 1 + \delta \sum_{k=1}^K a_M^{(k)} \left(\frac{r_H}{r}\right)^k, \quad (7)$$

$$B(r) = 1 + \delta \sum_{k=1}^K b_M^{(k)} \left(\frac{r_H}{r}\right)^k. \quad (8)$$

We make the series terminate at some finite number K starting from $k = 1$ to ensure asymptotic flatness.

From the prescription suggested by passing from Eq. (4) to Eq. (5) one could guess the following form for the master perturbation equation

$$fZ \frac{d}{dr} \left(fZ \frac{d\Phi}{dr} \right) + \left[\omega^2 W - fZ \bar{V}_\ell \right] \Phi = 0, \quad (9)$$

where one has enough generality to define the perturbation variable Φ to choose $Z = A/B$ such that the tortoise coordinate r_* arising from this equation is $dr_* fA/B = dr$. Finally, we expect small deviations from GR in the other parameters as well: $W = 1 + \delta w(r)$, $\bar{V}_\ell = \bar{V}_\ell^{\text{GR}}(r) + \delta \bar{V}_\ell(r)$. Note that in this definition we made explicit that radial functions modifying the potential depend on the angular momentum ℓ of the quasi-normal frequencies.

In the following we try to provide a heuristic explanation of why we expect this form for the master equation. The first step is to write down the master equation for a perturbation in GR

$$F_{\text{GR}} \frac{d}{dr} \left(F_{\text{GR}} \frac{d\Phi}{dr} \right) + \left[\omega^2 - \bar{V}^{\text{GR}} \right] \Phi = 0, \quad (10)$$

where $F_{\text{GR}} \equiv \sqrt{\bar{g}_{tt}^{\text{Sch}} \bar{g}_{rr}^{\text{Sch}}} = f(r)$ is the ‘‘natural’’ tortoise function for the Schwarzschild solution, and

$$\bar{V}^{\text{GR}} = \bar{g}_{tt}^{\text{Sch}} \frac{\Lambda}{r^2} + G(r) \frac{F_{\text{GR}}}{r} \frac{dF_{\text{GR}}}{dr}, \quad (11)$$

where $\Lambda = \ell(\ell + 1)$, $G(r) = 1$ denotes scalar perturbations, $G(r) = -3$ for axial perturbations and

$$G(r) = -3 \frac{1 + 3F_{\text{GR}}^2(r) - \Lambda^2}{[1 - 3F_{\text{GR}}(r) + \Lambda]^2} \quad (12)$$

for polar perturbations. We intentionally wrote the eikonal term $\bar{g}_{tt}^{\text{Sch}} \Lambda/r^2$ to make its correspondence with geodesic motion evident.

If we want to assume that the background metric is the metric of Eq. (6), we can substitute all the $\bar{g}_{ab}^{\text{Sch}}$ terms with \bar{g}_{ab} everywhere in equation (10). For example, one should transform F_{GR} into fA/B . Moreover, to take into account that the theory can be different, one should add a correction δZ to the tortoise coordinate and one $\delta \bar{V}$ to the potential. These two corrections take into account the fact that the derivation of the perturbation equation might be done in a theory which is not GR [cfr. Eq. (5)]. The resulting equation is

$$f(Z + \delta Z) \frac{d}{dr} \left[f(Z + \delta Z) \frac{d\Phi}{dr} \right] + \left[\omega^2 - fZ \left(V_\ell^{\text{GR}} + \delta \bar{V}_\ell \right) \right] \Phi = 0. \quad (13)$$

One can always re-define the perturbation function Φ and re-scale the equation to remove the δZ term from the derivative term. This would ensure that the second order derivative is acted upon the tortoise coordinate of the metric. However, by doing so, one would introduce some terms multiplying the frequency, obtaining Eq. (9). The term multiplying the frequency can be mapped into the modification of the tortoise coordinate $\delta w(r) = -2\delta Z(r)$, and we expand it as

$$\delta w(r) = \sum_{k=1}^K w^{(k)} \left(\frac{r_H}{r}\right)^k. \quad (14)$$

It is worth noting that this term appears just because the background metric (and thus the tortoise coordinate) is different from the Schwarzschild metric.

On the potential side, from the construction, we can split the contribution to the potential coming from the metric and from the ‘‘dynamics’’. We have that

$$\bar{V}_\ell^{\text{GR}} + \delta \bar{V}_\ell = \bar{V}_\ell^{\text{MOD}} + \delta \bar{V}_\ell^{\text{D}}, \quad (15)$$

where we made the following identifications

$$\bar{V}_\ell^{\text{MOD}} = B \frac{\ell(\ell + 1)}{r^2} + \frac{G(r)}{r} \frac{d}{dr} \left(f \frac{A}{B} \right), \quad (16)$$

$$\delta \bar{V}_\ell^{\text{D}} = \frac{\delta}{r_H^2} \sum_{k=0}^K \beta_{\text{D},\ell}^{(k)} \left(\frac{r_H}{r}\right)^k. \quad (17)$$

This last term, linear in δ , expresses in a $1/r$ basis our ignorance on the correct equations of motion for the dynamics of the theories under scrutiny.

Collectively we label all the *real* coefficients, from metric and from dynamics as

$$\beta^{(i)} = \left(\beta_{\text{D},\ell}^{(0)}, z^{(1)}, b_M^{(1)}, w^{(1)}, \beta_{\text{D},\ell}^{(1)}, z^{(2)}, b_M^{(2)}, w^{(2)}, \beta_{\text{D},\ell}^{(2)}, \dots \right), \quad (18)$$

where we identified $z^{(k)} = a_M^{(k)} - b_M^{(k)}$ from the definition of Z . In the rest of the paper we will refer to this set of parameters as *mixed* parametrization or *metric-potential* parametrization to distinguish it from the *potential-only*

parametrization introduced in [15]. A quadratic expansion of ω in the parameters $\beta^{(i)}$ is simply given by

$$\omega \simeq \omega^0 + c_{(i)}\beta^{(i)} + \frac{1}{2}v_{(ij)}\beta^{(i)}\beta^{(j)}. \quad (19)$$

We can infer the coefficients $c_{(i)}$ and $v_{(ij)}$ from the coefficients $d_{(k)}$ and $e_{(ks)}$ obtained in [15, 16]¹ by comparing the respective master equations. In order to do so, we must manipulate Eq. (9). By performing the change of variables $\phi = \sqrt{Z}\Phi$, the master equation takes the form

$$f \frac{d}{dr} \left[f \frac{d\phi}{dr} \right] + \left[\omega^2 \frac{W_H}{Z_H^2} - fV \right] \phi = 0, \quad (20)$$

$$V = \frac{\bar{V}}{Z} - \frac{f(Z')^2 - 2Z(fZ)'}{4Z^2} - \frac{\omega^2}{f} \left(\frac{W}{Z^2} - \frac{W_H}{Z_H^2} \right), \quad (21)$$

with $Z_H = Z(r_H)$ and $W_H = W(r_H)$. Also the last term of (21) can be put in the $1/r$ form with the following manipulation

$$\begin{aligned} \frac{\omega^2}{f} \left(\frac{W}{Z^2} - \frac{W_H}{Z_H^2} \right) &= \omega^2 \sum_k \frac{2z^{(k)} - w^{(k)}}{r^{k-1}} \frac{r^k - r_H^k}{r - r_H} \\ &= \omega^2 \sum_k \left(2z^{(k)} - w^{(k)} \right) \sum_{p=0}^{k-1} \left(\frac{r_H}{r} \right)^p. \end{aligned} \quad (22)$$

In this way, we can define a first order expansion of the potential in δ as

$$\delta V \equiv V - V_{\text{GR}} = \frac{\delta}{r_H^2} \sum_{k=0}^{\infty} \alpha^{(k)} \left(\frac{r_H}{r} \right)^k. \quad (23)$$

The coefficients $\alpha^{(k)}$ can be obtained up to second-order by combinations of the coefficients $\beta^{(i)}$. For generic index k and tensor axial perturbation, we have that linear contributions are

$$\begin{aligned} \alpha^{(k)} &= \left[(k-2)z^{(k-2)} - (k-3)z^{(k-3)} \right] \frac{k-7}{2} \\ &+ \Lambda \left(b_M^{(k-2)} - z^{(k-2)} \right) + \beta_{D,\ell}^{(k)} \\ &+ \omega^2 r_H^2 \sum_{p=k+1}^K \left(w^{(p)} - 2z^{(p)} \right), \end{aligned} \quad (24)$$

while we do not display the quadratic ones since the expression is lengthy and uninformative. The formula for scalar perturbations has an analogue form, while for the tensor polar case, one should first decompose the term (12) such that it yields $1/r$ terms. We suggest that the best-motivated way to proceed is to perform

an eikonal expansion of the term (*i.e.*, around $\ell \rightarrow \infty$) and include eikonal terms until the deviation in the coefficients is smaller than the numerical error of the coefficients themselves.

The manipulation shown with the previous calculations has brought the equation (20) in the same form shown of that in Ref. [15]. Now, one can apply the quadratic expansion of the quasi-normal frequencies from Refs. [15, 16] to Eq. (20), yielding

$$\begin{aligned} \omega \simeq \frac{Z_H}{W_H^{1/2}} \left(\omega^0 + d_{(k)}\alpha^{(k)} \right. \\ \left. + d_{(k)}d_{(s)}\alpha^{(k)}\partial_\omega\alpha^{(s)} + \frac{1}{2}e_{(ks)}\alpha^{(k)}\alpha^{(s)} \right). \end{aligned} \quad (25)$$

Due to the relation between the coefficients $\alpha^{(k)}$ and $\beta^{(i)}$, we can always do the mapping

$$\begin{aligned} c_{(i)} &\equiv \frac{\partial\omega}{\partial\beta^{(i)}} = \omega^0 \frac{\partial}{\partial\beta^{(i)}} \frac{Z_H}{W_H^{1/2}} + d_{(k)} \frac{\partial\alpha^{(k)}}{\partial\beta^{(i)}}, \quad (26) \\ v_{(ij)} &\equiv \frac{\partial^2\omega}{\partial\beta^{(i)}\partial\beta^{(j)}} = \omega^0 \frac{\partial^2}{\partial\beta^{(i)}\partial\beta^{(j)}} \frac{Z_H}{W_H^{1/2}} \\ &+ d_{(k)} \left(\frac{\partial}{\partial\beta^{(i)}} \frac{Z_H}{W_H^{1/2}} \frac{\partial\alpha^{(k)}}{\partial\beta^{(j)}} + \frac{\partial}{\partial\beta^{(j)}} \frac{Z_H}{W_H^{1/2}} \frac{\partial\alpha^{(k)}}{\partial\beta^{(i)}} \right) \\ &+ d_{(k)}d_{(s)} \left(\frac{\partial\alpha^{(k)}}{\partial\beta^{(i)}} \frac{\partial^2\alpha^{(s)}}{\partial\beta^{(j)}\partial\omega} + \frac{\partial\alpha^{(k)}}{\partial\beta^{(j)}} \frac{\partial^2\alpha^{(s)}}{\partial\beta^{(i)}\partial\omega} \right) \\ &+ d_{(k)} \frac{\partial^2\alpha^{(k)}}{\partial\beta^{(i)}\partial\beta^{(j)}} + e_{(ks)} \frac{\partial\alpha^{(k)}}{\partial\beta^{(i)}} \frac{\partial\alpha^{(s)}}{\partial\beta^{(j)}}. \end{aligned} \quad (27)$$

It is worth noting that since all the coefficients $\beta^{(i)}$ are real, any term that would make the coefficients $\alpha^{(k)}$ complex comes from the fact that the background metric is not Schwarzschild. Indeed, we remind that the coefficients $w^{(k)}$ denote the deviation of the “natural” tortoise coordinate of the master equation from the tortoise coordinate of the metric, nonetheless defined by the coefficients $z^{(k)}$.²

The advantage of this formulation over the potential-only parametrization is evident when reconstructing the modified potential after the detection of more than one quasi-normal mode [19, 21, 22]. In fact, even if from a first look it seems that the metric-potential formalism has more parameters than the potential-only one, one must be aware that the coefficients of the latter are complex (because they can depend on the frequency) and they are different for each mode measured, as they are sensible to n and ℓ . This means twice $K+1$ coefficients for each

¹ See also [18, 19] for alternative ways to compute these coefficients

² An additional frequency-dependent contribution not shown here could come from a non-spherically symmetric background metric, that reduces to Schwarzschild in the zero coupling limit (*e.g.*, the slow-rotation expansion [20]).

mode measured. On the other hand, the mixed formalism has $K + 1$ coefficients $\beta_{\text{D},\ell}^{(k)}$ which change with the angular momentum of the mode, and $K - 1$ coefficients $z^{(k)}$, $b_{\text{M}}^{(k)}$ and $w^{(k)}$. None of them depends on the overtone number. By simple counting, one can find that the total number of coefficients for the observation of n_ℓ different angular momentum modes and n_n additional (*i.e.* on top of the fundamental one) overtones for the potential-only case is $2(n_n + n_\ell)(K + 1)$, while for the metric-potential case one has $n_\ell(K + 1) + 3(K - 1)$.

III. APPLICATIONS

A. Reissner-Nordström Metric

As first non-trivial application of the framework we consider the Reissner-Nordström (RN) metric [23–26]. With the current notation it is given by the metric Eq. (6), with

$$A = 1 - \frac{r_-}{r}, \quad B = 1, \quad (28)$$

where r_- can be related to the mass M and a (small) electric charge Q through

$$r_- = M - \sqrt{M^2 - Q^2} \approx \frac{Q^2}{2M}, \quad (29)$$

and the location of the horizon is given by

$$r_{\text{H}} = M + \sqrt{M^2 - Q^2}. \quad (30)$$

The perturbation equation in the odd-parity sector is close to the Regge-Wheeler equation [13]. From its expression, one can either infer the axial deviation parameters for potential-only formalism (*cfr.* Ref. [15]) or the parameters for the mixed framework

$$z^{(1)} = -\frac{r_-}{r_{\text{H}}}, \quad (31)$$

$$\beta_{\text{D},\ell}^{(3)} = -\frac{4r_-}{3r_{\text{H}}}(\Lambda - 2), \quad \beta_{\text{D},\ell}^{(4)} = -2\frac{r_-}{r_{\text{H}}}. \quad (32)$$

In Fig. 1 we compare the application of the original parametrized framework with the new metric dependent one for the RN $n = 0, 1, 2$, $\ell = 2$ quasi-normal modes. In the two panels we show the relative error $\delta\omega = |\omega - \omega^{\text{RN}}|/\omega^{\text{RN}}$ between the parametrized quasi-normal modes ω (potential-only parametrization is represented by solid lines while potential-metric parametrization by dashed lines) and the numerically computed one ω^{RN} . One can clearly see that the mixed parametrization gives an even more accurate prediction of the frequencies for each case considered. We also checked different angular momentum ℓ which gives analog features.

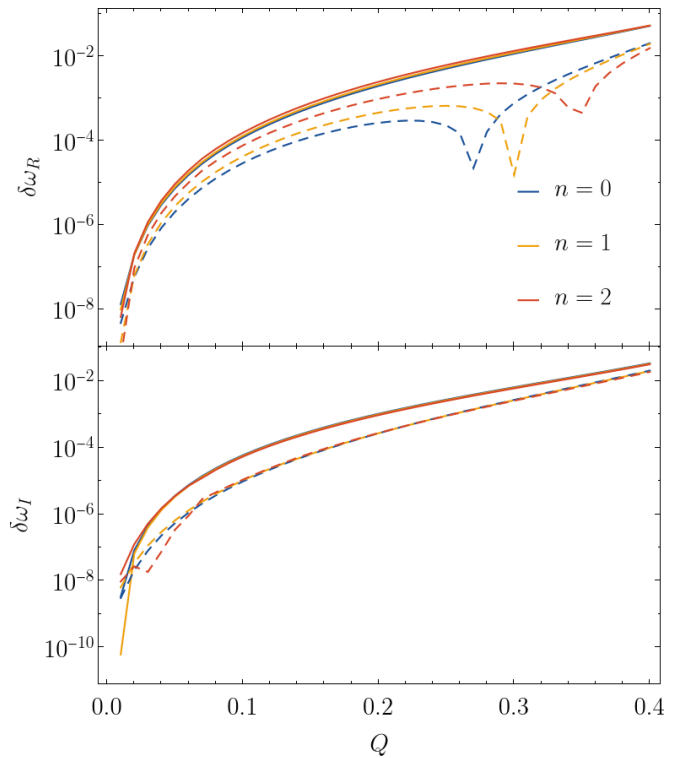


FIG. 1. Here we show the relative errors $\delta\omega = |\omega - \omega^{\text{RN}}|/\omega^{\text{RN}}$ for real (upper panel) and imaginary (lower panel) part of the axial $\ell = 2$ RN quasi-normal modes for $n = 0, 1, 2$ using the potential parametrized framework (solid lines) and the new metric dependent one (dashed lines).

B. Einstein-dilaton-Gauss-Bonnet gravity

Einstein-dilaton-Gauss-Bonnet (EdGB) gravity is a well-studied alternative theory of gravity, where a dynamical scalar field is coupled to the Gauss-Bonnet invariant through a coupling α_{GB} (more details can be found in [27] and references therein). In the small-coupling limit it provides a background metric and the perturbation equations of the form we analysed here. The master equation for axial perturbation can be recast in the form of Eq. (9) retaining up to ζ^2 terms, where we introduced the dimensionless coupling $\zeta = 4\alpha_{\text{GB}}/r_{\text{H}}^2$. The relevant functions are (according to the notation of [28])

$$B = 1 - \zeta^2 \left(\frac{r_{\text{H}}^2}{8r^2} + \frac{r_{\text{H}}^3}{6r^3} + \frac{7r_{\text{H}}^4}{16r^4} + \frac{2r_{\text{H}}^5}{5r^5} + \frac{3r_{\text{H}}^6}{8r^6} \right), \quad (33)$$

$$Z = 1 + \zeta^2 \left(-\frac{49r_{\text{H}}}{80r} - \frac{39r_{\text{H}}^2}{80r^2} - \frac{97r_{\text{H}}^3}{240r^3} + \frac{49r_{\text{H}}^4}{120r^4} + \frac{61r_{\text{H}}^5}{120r^5} + \frac{7r_{\text{H}}^6}{12r^6} \right), \quad (34)$$

$$W = 1 + \zeta^2 \left(-\frac{2r_{\text{H}}^3}{r^3} - \frac{r_{\text{H}}^4}{r^4} - \frac{r_{\text{H}}^5}{r^5} + \frac{4r_{\text{H}}^6}{r^6} \right), \quad (35)$$

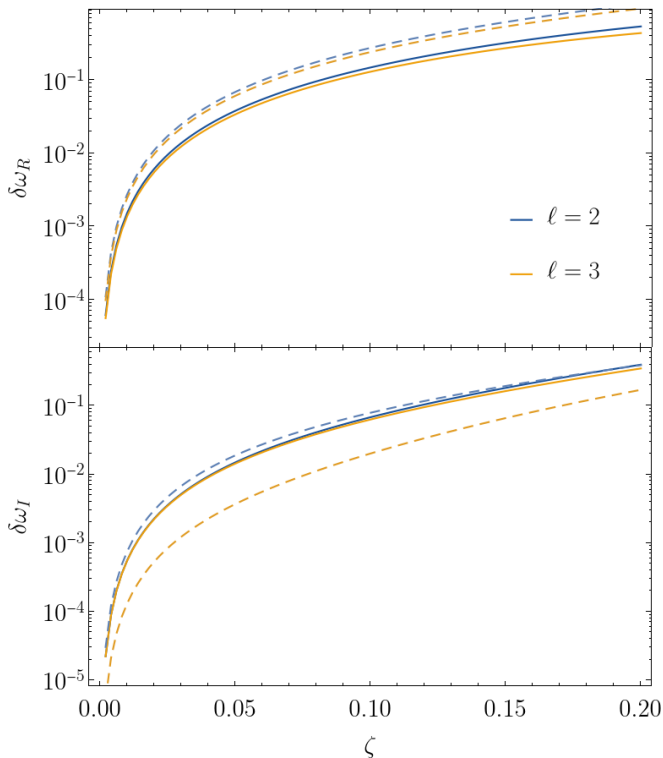


FIG. 2. Relative errors $\delta\omega = |\omega - \omega^{\text{GB}}|/\omega^{\text{GB}}$ for real and imaginary part of the axial $n = 0$ GB quasi-normal modes for $\ell = 2, 3$ using the potential-only parametrized framework (solid lines) and the metric-potential dependent one (dashed lines).

$$\begin{aligned} \bar{V}_\ell = & V_\ell^{\text{GR}} + \zeta^2 \frac{r_{\text{H}}}{r^3} \left[-\frac{147}{80} + \frac{(6 - \Lambda)r_{\text{H}}}{8r} \right. \\ & + \left(\frac{99}{4} - \frac{19\Lambda}{6} \right) \frac{r_{\text{H}}^2}{r^2} + \left(-\frac{15\Lambda}{16} - \frac{29}{4} \right) \frac{r_{\text{H}}^3}{r^3} \\ & \left. - \frac{9(\Lambda - 5)r_{\text{H}}^4}{10r^4} + \left(\frac{41\Lambda}{8} - \frac{712}{5} \right) \frac{r_{\text{H}}^5}{r^5} + \frac{519r_{\text{H}}^6}{4r^6} \right]. \quad (36) \end{aligned}$$

As shown in section II, one can transform the equation such that it is in the form suitable to express the modifications from GR as coefficients of the potential formalism. Hence, in Fig. 2, we compare the $\ell = 2, 3$ fundamental modes obtained with the two different formalism against the fits of the numerical quasi-normal mode frequencies ω^{GB} computed in [27]. In this case, the metric-potential parametrization under-performs the potential-only one in most cases, even though the error with respect to the numerical fits are still comparable.

IV. CONCLUSIONS

In this work we have introduced a framework to compute quasi-normal modes from non-GR metrics where

modifications are small. Our ansatz is very similar and can be mapped to the one proposed by Cardoso *et al.* [15]. This allows for a quick computation of QNMs of a given metric. We have found comparable precision among the two methods in approximating the numerically computed frequencies both for RN and EdGB axial perturbations. In the paper we showed how to perform the mapping between the two formalisms for scalar and tensor axial perturbations only, but we also provided an explanation on how to do it for the tensor polar case.

The main advantage of the mixed method is evident when one has in mind the so-called inverse problem, which refers to the reconstruction of the perturbation equation in a theory-agnostic approach [19, 21, 22]. Essential condition for the inverse problem is the detection of more angular modes/overtones simultaneously, even if the capability of doing so with current observations raised an ongoing discussion in the literature [29–34]. The advantage in the inverse problem comes from a reduction of the number of parameters, all real, and only with the $\beta_{\text{D},\ell}^{(k)}$ depending on the angular momentum of the perturbation. This is not the case for the potential-only parametrization, for which all coefficients $\alpha^{(k)}$ are in principle complex, and depending on angular momentum and overtone number as well. Moreover, it can help to give insights to discern whether a modification comes from the background or from the dynamics.

The possibility to explicitly separate the two contributions could also be useful when complementary constraints on the metric can be provided, but not on the dynamics. Such a case may be possible with activities of the Event Horizon Telescope Collaboration [35–37], which can in principle be used to study possible deviations from the Schwarzschild/Kerr metric of supermassive black holes, see e.g. Refs. [38–45] for some recent works. Other complementary constraints can also be obtained using x-ray spectroscopy, see e.g. Refs. [46–48]. Although it seems unlikely that observations of the same objects can be done using gravitational waves and one of these techniques in the foreseeable future, one can still combine constraints on individual metrics under certain assumptions, *e.g.*, assuming the way how the metric is modified happens in a similar way for all sources.

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