

Conservative (failed)-tail effects at the fifth post-Newtonian order

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This work deals with the tail and “failed” tail sectors of the conservative dynamics for compact binary systems at the 5PN order. We employ the Fokker Lagrangian method with dimensional regularization, and our results for the tail sector are perfectly consistent with the previous EFT computations. As for the “failed” tail sector, we have good hopes that this new computation will help solving the current discrepancy in the literature.

I. INTRODUCTION

The post-Newtonian (PN) approximation scheme is a very efficient framework that relies on weak-field and slow-velocity approximations to perturbatively solve Einstein’s equation. It has been notably implemented through a large class of methods to resolve the dynamics of bound compact binaries system, see *e.g.* [1–3] for reviews. In the conservative sector,¹ the current accuracy is the fourth PN order (*i.e.* the $(v/c)^8$ correction to the Newtonian energy and angular momentum), that was obtained by means of the canonical Hamiltonian formalism [8–10], the Fokker method [11–15], and effective field theory (EFT) approach [16–22]. Starting at this 4PN precision, the conservative dynamics can be split between an “instantaneous” sector and a “hereditary” one. The latter takes into account the back-reaction of emitted radiation onto the dynamics of the binary, which induces non-local in time effects (thus the name). The computation of the instantaneous dynamics has been completed at 5PN by a large variety of methods [23–26], and pushed up to the 6PN precision [27]. As for the hereditary sector, due to the very subtle nature of the computations, only partial results exist. For instance, the tail sector (due to the scattering of waves onto the static curvature induced by the ADM mass) has been computed by means of EFT [28]. As for the “failed” tail² (due to the scattering of waves onto the static curvature induced by the ADM angular momentum), it was also derived within the EFT framework. However, there is a discrepancy between previous results [29, 30] and the recent work of [31]. Note also that the logarithmic dependencies in the binding energy (due to this hereditary sector) are known up to the 7PN order [32].

The aim of this work is to derive the (failed) tail effects by means of the Fokker method using dimensional regularization. We thus work with $d = 3 + \varepsilon$ space-like dimensions and the d -dimensional gravitational strength, G , is linked to the usual Newton constant G_N by a new length scale ℓ_0 as $G = \ell_0^\varepsilon G_N$ (this regularization constant is directly related to the scale μ used in EFT framework [28, 31] as $\ell_0 = \mu^{-1}$). The tail interactions entering at 5PN involve the constant ADM mass M , the mass octupole M_{ijk} and the current quadrupole S_{ijk} . As for the failed tail, it describes the interaction between the constant angular momentum S_{ij} and the mass quadrupole M_{ij} . Note that, as we work in d dimensions, we use the notations of [33] for current moments, notably $S_{ij} \equiv \varepsilon_{ijk} L^k$, where L^i is the usual angular momentum. Our result reads

$$\begin{aligned} \mathcal{S}_{\text{tail}} = & -\frac{G^2 M}{c^{10}} \int dt \int_0^\infty d\tau \left\{ \frac{1}{189} \left(K_\varepsilon(\tau) - \frac{82}{35} \right) M_{ijk}^{(4)}(t) M_{ijk}^{(5)}(t - \tau) + \frac{16}{45} \left(K_\varepsilon(\tau) - \frac{49}{20} \right) S_{k|ji}^{(3)}(t) S_{k|ji}^{(4)}(t - \tau) \right\} \\ & + \frac{G^2}{30 c^{10}} \int dt S_{ij} M_{ik}^{(3)}(t) M_{jk}^{(4)}(t), \end{aligned} \quad (1.1)$$

where parenthesis denote time derivations and we have dressed the pole as

$$K_\varepsilon(\tau) \equiv \frac{1}{\varepsilon} - 2 \ln \left(\frac{c\sqrt{q}\tau}{2\ell_0} \right), \quad \text{with} \quad \bar{q} \equiv 4\pi e^{\gamma_E}, \quad (1.2)$$

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¹ This paper focuses on the conservative sector, *i.e.* the study of the (conserved) dynamics of the system. Nevertheless, those PN frameworks are also in use to solve the dissipative sector, *i.e.* to derive the waveform. Notably, using matched post-Newtonian and multipolar-post-Minkowskian methods [4], the gravitational flux at 4PN and phase at 4.5PN were recently obtained [5, 6]. Note also that the 2PN sector of the gravitational flux has been confirmed by EFT means [7].

² We borrow this nomenclature to [29], where it has been dubbed “failed” as, although it comes as an hereditary effect, this interaction fails to induce a non-local-in-time sector.

where γ_E is the Euler constant. The first line of (1.1), corresponding to the tail interactions, is in perfect agreement with Eqs. (5) and (9) of [28]. As for the failed tail (the second line), we fully agree with the recent result of [31], obtained by an independent method.

The plan of this paper is as follows. Sec. II describes the method employed to derive the (failed) tail effects, namely the Fokker method with dimensional regularization. This method is then applied to each interaction separately in Sec. III. Finally, Sec. IV concludes our work.

II. GENERAL METHOD

In order to perform the computation of the conservative (failed) tail sector at 5PN, we follow the method that was used for the lowest-order tail $M \times M_{ij} \times M_{ij}$, entering at 4PN [13, 14]. The following section briefly recalls and discuss its main steps.³

A. Tail effects in the action

The starting point of the method is naturally an action composed of two sectors: the gravitational kinetic term and the matter description. For the first one, we work with the usual Landau-Lifschitz Lagrangian, together with a gauge-fixing term (see *e.g.* [11])

$$\mathcal{S}_g = \frac{c^4}{16\pi G} \int dt d^d x \sqrt{-g} \left[g^{\mu\nu} \left(\Gamma_{\mu\rho}^\lambda \Gamma_{\nu\lambda}^\rho - \Gamma_{\mu\nu}^\lambda \Gamma_{\rho\lambda}^\rho \right) - \frac{1}{2} g_{\mu\nu} \Gamma^\mu \Gamma^\nu \right], \quad (2.1)$$

where $\Gamma_{\nu\rho}^\mu$ are the Christoffel symbols and the last term enforces the gauge $\Gamma^\mu \equiv g^{\alpha\beta} \Gamma_{\alpha\beta}^\mu = 0$. In terms of the so-called ‘‘gothic metric’’ $\mathfrak{g}^{\mu\nu} \equiv \sqrt{-g} g^{\mu\nu}$, this action becomes

$$\mathcal{S}_g = \frac{c^4}{32\pi G} \int dt d^d x \left[\mathfrak{g}_{\alpha\beta} \left(\partial_\mu \mathfrak{g}^{\alpha\nu} \partial_\nu \mathfrak{g}^{\beta\mu} - \partial_\mu \mathfrak{g}^{\alpha\mu} \partial_\nu \mathfrak{g}^{\beta\nu} \right) - \frac{1}{2} \mathfrak{g}^{\alpha\beta} \mathfrak{g}_{\mu\nu} \mathfrak{g}_{\sigma\tau} \left(\partial_\alpha \mathfrak{g}^{\mu\sigma} \partial_\beta \mathfrak{g}^{\nu\tau} - \frac{1}{d-1} \partial_\alpha \mathfrak{g}^{\mu\nu} \partial_\beta \mathfrak{g}^{\sigma\tau} \right) \right]. \quad (2.2)$$

As for the matter sector, we consider structureless, non-spinning point-particles, thus described by the action

$$\mathcal{S}_{\text{pp}} = -c \sum_A m_A \int d\tau_A = -c^2 \sum_A m_A \int dt d^d x \frac{\delta_A}{u_A^0}, \quad (2.3)$$

where m_A is the mass of the particle A , τ_A its proper time, $u_A^0 \equiv [-(g_{\mu\nu})_A v_A^\mu v_A^\nu / c^2]^{-1/2}$ is the associated Lorentz factor, $v_A^\mu = (c, v_A^i)$ (with v_A^i the usual velocity), and the d -dimensional Dirac distribution $\delta_A \equiv \delta[\mathbf{x} - \mathbf{y}_A(t)]$ locates the Lagrangian on the world-line of the particles. As we are interested by the dynamics of binary systems, we will run A only over two values.

From the gothic metric, we define the exact perturbation

$$h^{\mu\nu} \equiv \mathfrak{g}^{\mu\nu} - \eta^{\mu\nu}, \quad (2.4)$$

for which the gauge condition $\Gamma^\mu = 0$ translates into the usual harmonic gauge $\partial_\nu h^{\mu\nu} = 0$. This perturbation obeys a wave equation source by the Landau-Lifschitz pseudo-tensor $\tau^{\mu\nu}$

$$\square h^{\mu\nu} = \tau^{\mu\nu} = \frac{16\pi G}{c^4} |g| T^{\mu\nu} + \Lambda^{\mu\nu}, \quad (2.5)$$

where $T^{\mu\nu}$ is the stress-energy tensor of the matter distribution and $\Lambda^{\mu\nu}$ encrypts the non-linearities intrinsic to GR (its expression is given *e.g.* in Eq. (175) of [1]).

³ The conventions employed throughout this work are as follows: we work with a mostly plus signature; greek letters denote spacetime indices and latin ones, purely spatial indices; bold font denotes d -dimensional vectors, *e.g.* $\mathbf{y}_A = y_A^i$; we use the multi-index notations of [33] (coming from Young tableaux), *i.e.* $M_L = M_{i_1 i_2 \dots i_\ell}$ and $S_{i|L} = S_{i|i_\ell \dots i_2 i_1}$; hats and angular brackets denote a symmetric and trace-free operator, $\hat{x}_L = x_{\langle L} = \text{STF}[x_L]$; the d’Alembertian operator is defined with respect to the flat Minkowski metric $\square \equiv \eta^{\mu\nu} \partial_{\mu\nu} = \Delta - c^{-2} \partial_t^2$; (anti-)symmetrizations are weighted, *e.g.* $A_{(ij)} = (A_{ij} + A_{ji})/2$; the Lagrangian and Lagrangian density are denoted as $S = \int dt \mathcal{L} = \int dt d^d x L$, and we will refer to ‘‘Lagrangian’’ for ‘‘Lagrangian density’’ henceforth; finally, and as usual, we dubb ‘‘ n PN’’ a quantity of order $\mathcal{O}(c^{-2n})$.

We are interested here in the near-zone behavior of the metric, *i.e.* we aim at solving $\square h_{\text{NZ}}^{\mu\nu} = \bar{\tau}^{\mu\nu}$, where $\bar{\tau}^{\mu\nu}$ is the PN expansion of $\tau^{\mu\nu}$. The solution to such wave equation can be split in two sectors, as

$$h_{\text{NZ}}^{\mu\nu} = \bar{h}^{\mu\nu} + \mathcal{H}^{\mu\nu}. \quad (2.6)$$

The first sector, $\bar{h}^{\mu\nu}$, is a particular solution of the wave equations, corresponding to the *potential* modes of the EFT framework. It is computed by applying the PN-expanded, regularized Green function on $\bar{\tau}^{\mu\nu}$, see *e.g.* Eq. (2.5) of [13], and its expression is known up to 4PN [34]. Due to PN expansions and regularization of the Green function, the metric $\bar{h}^{\mu\nu}$ is not the correct prescription, and we have to add an homogeneous solution, $\mathcal{H}^{\mu\nu}$. This solution is a consequence of the matching equation linking the near- and far-zone behaviors of the metric [1, 4], and its construction is the purpose of the next section. As will be clear there, it corresponds to the conservative sector of the waves radiated by the source, and thus one can assimilate it to the *radiative* modes of the EFT framework.

Following the spirit of the Fokker method, we inject the near-zone metric (2.6) into the conservative action (2.2)–(2.3), in order to obtain a resulting Lagrangian depending only on the matter variables (which accounts to integrating out the gravitational modes). This yields an action mixing potential and radiative modes. The sector free from any $\mathcal{H}^{\mu\nu}$ is the usual, instantaneous action, computed at 5PN by EFT means in *e.g.* [25, 26], and we let its re-computation within the Fokker framework for future studies. What interests us here is the linear-in- $\mathcal{H}^{\mu\nu}$ sector of the action, encompassing the leading order (failed) tail effects.⁴ This linear sector can be interpreted as the backreaction of the scattered wave, $\mathcal{H}^{\mu\nu}$, onto the dynamics of the binary, thus describing indeed a tail effect. This point of view corresponds to the closure of radiative Feynmann diagrams, performed in [31].

As will be explicit hereafter, the different components of the radiative metric at a given PN order will follow $\mathcal{H}^{\mu\nu} = \mathcal{O}(c^{-2n-2}, c^{-2n-1}, c^{-2n})$ with $\mathcal{H}^{kk} = \mathcal{O}(c^{-2n-2})$ (in particular, $n = 5$ for this work). The leading PN order of the linear-in- $\mathcal{H}^{\mu\nu}$ sector of the action reads

$$\mathcal{S}_{\text{LO}}^{\text{tails}} = - \int dt d^d x \left\{ \frac{m_1 c^2}{8} \left[\mathcal{H}^{00ii} - \frac{4v_1^i}{c} \mathcal{H}^{0i} + \frac{2v_1^{ij}}{c^2} \mathcal{H}^{ij} \right] \delta_1 + \frac{(d-1)\mathcal{H}^{ij}}{64(d-2)\pi G} \partial_i V \partial_j V \right\} + (1 \leftrightarrow 2), \quad (2.7)$$

where we have shortened $\mathcal{H}^{00ii} \equiv \frac{2}{d-1}[(d-2)\mathcal{H}^{00} + \mathcal{H}^{kk}]$. From the compact terms (proportional to the Dirac distribution), one will be able to reconstruct the Newtonian value of the moments.⁵ As for the non-compact term (the last piece), it is treated by using the generalized Riesz formulae displayed in App. A of [35].

B. Computation of the radiative metric

The radiative metric $\mathcal{H}^{\mu\nu}$ corresponds to an homogeneous solution of the wave equations (2.5), regular in the source (when $r \rightarrow 0$). This means that it has the structure

$$\mathcal{H}^{\mu\nu} = \sum_{j, \ell \geq 0} \Delta^{-j} \hat{x}_L \left(\frac{d}{cdt} \right)^{2j} f_L^{\mu\nu}(t), \quad \text{with} \quad \Delta^{-j} \hat{x}_L = \frac{\Gamma(\frac{d}{2} + \ell)}{\Gamma(\frac{d}{2} + \ell + j)} \frac{r^{2j} \hat{x}_L}{2^{2j} j!}, \quad (2.8)$$

where the functions $f_L^{\mu\nu}(t)$ are determined by the matching equation [1, 4], *i.e.* by imposing that the near- and far-zone expansions of the metric agree in some overlapping region. Therefore it is clear that $\mathcal{H}^{\mu\nu}$ encodes the fact that the dynamics of the system is sensitive to the gravitational waves radiated at spatial infinity.

This fact is even more evident when looking at the practical computation of $\mathcal{H}^{\mu\nu}$. As derived in [14], the matching equation imposes that $\mathcal{H}^{\mu\nu}$ is nothing but an homogeneous solution of the far-zone expansion of the wave equations (2.5). It is thus sourced by the expansion of $\Lambda^{\mu\nu}$ at spatial infinity⁶ and can be computed by means of the d -dimensional usual multipolar-post-Minkowskian algorithm.

This algorithm starts with the d -dimensional generalization of Thorne's linearized metric [36], namely [33]

$$h_1^{00} = -\frac{4}{c^2} \sum_{\ell \geq 0} \frac{(-)^\ell}{\ell!} \hat{\partial}_L \tilde{\mathcal{M}}_L, \quad (2.9a)$$

⁴ As the constant (ADM) masses and angular momentum do not radiate, the quadratic-in- $\mathcal{H}^{\mu\nu}$ sector of the action cannot contribute to tail effects at leading order. Note however that, at 5PN, this quadratic sector can contribute to the memory effect and to the 1PN corrections to the $M \times M_{ij} \times M_{ij}$ tail effect. The study of both those effects are let for future works.

⁵ For example, if $\mathcal{H}^{00ii} = \hat{x}^{ijk} F_{ijk}(t)$, where F_{ijk} only depends on time, then $\int d^d x m_1 \mathcal{H}^{00ii} \delta_1 + (1 \leftrightarrow 2) = O_{ijk} F_{ijk}$, where $O_{ijk} = m_1 \hat{y}_1^{ijk} + m_2 \hat{y}_2^{ijk}$ is the Newtonian mass octupole moment.

⁶ We consider compact binaries, and so a compact-supported matter stress-energy tensor: at spatial infinity, $\tau^{\mu\nu}$ reduces to $\Lambda^{\mu\nu}$.

$$h_1^{0i} = \frac{4}{c^3} \sum_{\ell \geq 1} \frac{(-)^\ell}{\ell!} \left[\hat{\partial}_{L-1} \widetilde{\mathcal{M}}_{iL-1}^{(1)} + \frac{\ell}{\ell+1} \hat{\partial}_L \widetilde{\mathcal{S}}_{i|L} \right], \quad (2.9b)$$

$$h_1^{ij} = -\frac{4}{c^4} \sum_{\ell \geq 2} \frac{(-)^\ell}{\ell!} \left[\hat{\partial}_{L-2} \widetilde{\mathcal{M}}_{ijL-2}^{(2)} + \frac{2\ell}{\ell+1} \hat{\partial}_{L-1} \widetilde{\mathcal{S}}_{(i|\underline{L-1}j)}^{(1)} + \frac{\ell-1}{\ell+1} \hat{\partial}_L \widetilde{\mathcal{K}}_{ij|L} \right]. \quad (2.9c)$$

Underlined indices are excluded from symmetrization, and we have introduced the notation

$$\widetilde{\mathcal{M}}_L(r, t) \equiv \frac{\tilde{k}}{r^{d-2}} \int_1^{+\infty} dy \gamma_{\frac{1-d}{2}}(y) M_L \left(t - \frac{yr}{c} \right), \quad (2.10)$$

where

$$\tilde{k} \equiv \frac{\Gamma\left(\frac{d-2}{2}\right)}{\pi^{\frac{d-2}{2}}} = 1 - \frac{\varepsilon}{2} \ln \bar{q} + \mathcal{O}(\varepsilon^2) \quad \text{and} \quad \gamma_k(z) \equiv \frac{2\sqrt{\pi}}{\Gamma(1+k)\Gamma(-\frac{1}{2}-k)} (z^2 - 1)^k \quad (2.11)$$

is such that $\lim_{d \rightarrow 3} \widetilde{\mathcal{M}}_L(r, t) = M_L(t - r/c)/r$. Note the presence of the additional set of moments $K_{ij|L}$, which are a pure artifact of working in $d \neq 3$ dimensions.

For our practical purpose, we will only consider interactions between a static moment (either the ADM mass M or angular momentum $S_{i|j}$) and a propagating one. Therefore, injecting in $\Lambda^{\mu\nu}$ the sectors of the linear metric (2.9) that are of interest for us, the quadratic sources are of the form

$$N(\mathbf{x}, t) = \hat{n}_L \frac{\ell_0^{q\varepsilon}}{r^{p+q\varepsilon}} \int_1^{+\infty} dz \gamma_{\frac{1-d}{2}}(z) z^k F \left(t - \frac{zr}{c} \right), \quad (2.12)$$

where $F(t)$ represents a product of $\{M, S_{i|j}\}$ with (temporal derivatives of) one of the moments $\{M_{ij}, M_{ijk}, S_{i|jk}\}$. Following the computation performed in [14], we then define the homogeneous solution $\mathcal{U}^{\mu\nu}$ corresponding to the source (2.12) as

$$\mathcal{U} = \frac{(-)^{p+\ell}}{d+2\ell-2} \text{PF}_{B=0} \frac{\Gamma(q\varepsilon - B)}{\Gamma(p+\ell-1+q\varepsilon - B)} C_\ell^{k,p,q} \sum_{j \in \mathbb{N}} \Delta^{-j} \hat{x}_L \int_0^\infty d\tau \frac{\tau^{B-q\varepsilon}}{r_0^B} \frac{F^{(2j+\ell+p-1)}(t-\tau)}{c^{2j+\ell+p+q\varepsilon-B}}, \quad (2.13)$$

where the PF operator corresponds to the *finite part* operation when $B \rightarrow 0$ [37], and $C_\ell^{k,p,q}$ reads

$$C_\ell^{k,p,q} \equiv \int_1^{+\infty} dy \gamma_{\frac{1-d}{2}-\ell}(y) \int_1^{+\infty} dz \gamma_{\frac{1-d}{2}}(z) z^k (y+z)^{\ell-2+p+q\varepsilon-B}. \quad (2.14)$$

These coefficients are generalizations for $q \in \mathbb{Z}$ of the ones introduced in [14], and can be computed following the lines of the App. D of that work.⁷

If $\mathcal{U}^{\mu\nu}$ is of the form (2.8), namely an homogeneous solution regular in the source, it is not yet the homogeneous solution $\mathcal{H}^{\mu\nu}$ that we seek. At this stage, $\mathcal{U}^{\mu\nu}$ has no reason to be divergenceless and thus does not verify in general the harmonic condition. So to construct the correct solution, we add to $\mathcal{U}^{\mu\nu}$ a suited homogeneous solution, $\mathcal{V}^{\mu\nu}$, which cancels its divergence, following the standard procedure described *e.g.* in [1, 4]

$$\mathcal{U}^{\mu\nu} \longrightarrow \mathcal{V}^{\mu\nu} = H(\partial_\mu \mathcal{U}^{\mu\nu}) \longrightarrow \mathcal{H}^{\mu\nu} \equiv \mathcal{U}^{\mu\nu} + \mathcal{V}^{\mu\nu}. \quad (2.15)$$

In this method, $\mathcal{V}^{\mu\nu}$ is uniquely determined *via* the harmonicity algorithm given by Eqs. (47)–(48) in [1] and dubbed H here. Note that a similar removal of the divergence was employed in the EFT computation of [31], and was in fact a crucial step to obtain the failed tail.

Once the divergenceless $\mathcal{H}^{\mu\nu}$ is known, one can inject it in the action (2.7) and compute the integrals to obtain the desired effects. In order to simplify the procedure, one can also implement a gauge transformation to “push” the ij components of the metric to higher PN orders, and thus only have compact integrals to perform. The gothic metric transforms under the gauge transformation with vector ξ^μ as

$$\mathcal{H}^{\mu\nu} \rightarrow \mathcal{H}'^{\mu\nu} = \mathcal{H}^{\mu\nu} + \partial^\mu \xi^\nu + \partial^\nu \xi^\mu - \partial_\rho \xi^\rho \eta^{\mu\nu} + \mathcal{O}(\xi^2). \quad (2.16)$$

⁷ In the case of the memory interaction, the two moments under consideration are propagating, and thus sources are of the form $N \propto \int dy \gamma_{\frac{1-d}{2}}(y) y^k F(t - yr/c) \int dz \gamma_{\frac{1-d}{2}}(z) z^m G(t - zr/c)$. In such cases, we were not able to write the homogeneous solution in a form as simple as (2.13), notably because there are no factorization similar to (2.14).

So, by choosing ξ^μ adequately, one can cancel the leading order of \mathcal{H}^{ij} . In the next section, both raw and shifted metrics are displayed for each interaction, and we have naturally verified that they give the same result.

Note that we have also performed another consistency check on the results presented in the next section. By employing the method developed in [38], we have first derived the metric $\mathcal{H}^{\mu\nu}$ by using a three-dimensional, purely Hadamard regularization procedure. Then, we have computed the contribution induced by the difference between the d -dimensional regularization scheme and the Hadamard one, using techniques elaborated in [39]. By summing those two contributions, we recover the metric computed directly in d dimensions.

III. RESULTS AT 5PN

Let us implement the method described in the previous section (with extensive use of the *xAct* library from the *Mathematica* software [40]) in the cases of the tails appearing at 5PN in the conservative action, composed of the $M \times M_{ijk} \times M_{ijk}$, $M \times S_{i|jk} \times S_{i|jk}$ and $S_{i|j} \times M_{ij} \times M_{ij}$ interactions. We recall that pole is dressed as in Eq. (1.2). By summing the separate results, we obtain our main result (1.1).

A. Mass octupole tail

The divergenceless metric for the $M \times M_{ijk}$ interaction reads at the leading order

$$\mathcal{H}_{M \times M_{ijk}}^{00ii} = \frac{4G^2M \hat{x}^{ijk}}{315 c^{12}} \int_0^\infty d\tau \left(K_\varepsilon(\tau) - \frac{199}{70} \right) M_{ijk}^{(9)}(t - \tau) + \mathcal{O}(c^{-14}), \quad (3.1a)$$

$$\mathcal{H}_{M \times M_{ijk}}^{0i} = -\frac{4G^2M \hat{x}^{jk}}{45 c^{11}} \int_0^\infty d\tau \left(K_\varepsilon(\tau) - \frac{1189}{420} \right) M_{ijk}^{(8)}(t - \tau) + \mathcal{O}(c^{-13}), \quad (3.1b)$$

$$\mathcal{H}_{M \times M_{ijk}}^{ij} = \frac{4G^2M \hat{x}^k}{9 c^{10}} \int_0^\infty d\tau \left(K_\varepsilon(\tau) - \frac{113}{42} \right) M_{ijk}^{(7)}(t - \tau) + \mathcal{O}(c^{-12}). \quad (3.1c)$$

By applying the following shift

$$\xi_{M \times M_{ijk}}^0 = -\frac{G^2M \hat{x}^{ijk}}{189 c^{11}} \int_0^\infty d\tau \left(K_\varepsilon(\tau) - \frac{149}{70} \right) M_{ijk}^{(8)}(t - \tau), \quad (3.2a)$$

$$\xi_{M \times M_{ijk}}^i = -\frac{G^2M \hat{x}^{ijk}}{9 c^{10}} \int_0^\infty d\tau \left(K_\varepsilon(\tau) - \frac{113}{42} \right) M_{ijk}^{(7)}(t - \tau), \quad (3.2b)$$

the metric becomes of order $\mathcal{H}'_{M \times M_{ijk}}{}^{\mu\nu} = \mathcal{O}(c^{-12}, c^{-13}, c^{-12})$ and reads

$$\mathcal{H}'_{M \times M_{ijk}}{}^{00ii} = \frac{8G^2M \hat{x}^{ijk}}{189 c^{12}} \int_0^\infty d\tau \left(K_\varepsilon(\tau) - \frac{82}{35} \right) M_{ijk}^{(9)}(t - \tau) + \mathcal{O}(c^{-14}), \quad (3.3)$$

which, injected in the action (2.7), yields (upon integrations by parts)

$$\mathcal{S}_{M \times M_{ijk}} = -\frac{G^2M}{189 c^{10}} \int dt \int_0^\infty d\tau \left(K_\varepsilon(\tau) - \frac{82}{35} \right) M_{ijk}^{(4)}(t) M_{ijk}^{(5)}(t - \tau) + \mathcal{O}(c^{-12}). \quad (3.4)$$

B. Current quadrupole tail

The divergenceless metric for the $M \times S_{i|jk}$ interaction reads at the leading order

$$\mathcal{H}_{M \times S_{i|jk}}^{00ii} = \mathcal{O}(c^{-14}), \quad (3.5a)$$

$$\mathcal{H}_{M \times S_{i|jk}}^{0i} = \frac{8G^2M \hat{x}^{jk}}{45 c^{11}} \int_0^\infty d\tau \left(K_\varepsilon(\tau) - \frac{71}{30} \right) S_{i|jk}^{(7)}(t - \tau) + \mathcal{O}(c^{-13}), \quad (3.5b)$$

$$\mathcal{H}_{M \times S_{i|jk}}^{ij} = -\frac{16G^2M \hat{x}^k}{9 c^{10}} \int_0^\infty d\tau \left(K_\varepsilon(\tau) - \frac{73}{30} \right) S_{(i|\underline{k}j)}^{(6)}(t - \tau) + \mathcal{O}(c^{-12}). \quad (3.5c)$$

By applying the following shift

$$\xi_{\mathcal{M} \times \mathcal{S}_{i|jk}}^0 = 0, \quad \xi_{\mathcal{M} \times \mathcal{S}_{i|jk}}^i = \frac{8G^2 M \hat{x}^{jk}}{9c^{10}} \int_0^\infty d\tau \left(K_\varepsilon(\tau) - \frac{73}{30} \right) S_{i|jk}^{(6)}(t - \tau), \quad (3.6)$$

the metric becomes of order $\mathcal{H}'_{\mathcal{M} \times \mathcal{S}_{i|jk}}{}^{\mu\nu} = \mathcal{O}(c^{-14}, c^{-11}, c^{-12})$, and reads

$$\mathcal{H}'_{\mathcal{M} \times \mathcal{S}_{i|jk}}{}^{0i} = -\frac{32G^2 M \hat{x}^{jk}}{45c^{11}} \int_0^\infty d\tau \left(K_\varepsilon(\tau) - \frac{49}{20} \right) S_{i|jk}^{(7)}(t - \tau) + \mathcal{O}(c^{-13}), \quad (3.7)$$

which, injected in the action (2.7), yields (upon integrations by parts)

$$\mathcal{S}_{\mathcal{M} \times \mathcal{S}_{i|jk}} = -\frac{16G^2 M}{45c^{10}} \int dt \int_0^\infty d\tau \left(K_\varepsilon(\tau) - \frac{49}{20} \right) S_{i|jk}^{(3)}(t) S_{i|jk}^{(4)}(t - \tau) + \mathcal{O}(c^{-12}). \quad (3.8)$$

C. Angular momentum failed tail

Finally, the divergenceless metric for the $\mathcal{S}_{i|j} \times \mathcal{M}_{ij}$ interaction reads at the leading order

$$\mathcal{H}_{\mathcal{S}_{i|j} \times \mathcal{M}_{ij}}{}^{00ii} = \frac{4G^2}{45c^{12}} \hat{x}^{jk} S_{i|k} M_{ij}^{(7)}(t) + \mathcal{O}(c^{-14}), \quad (3.9a)$$

$$\mathcal{H}_{\mathcal{S}_{i|j} \times \mathcal{M}_{ij}}{}^{0i} = \frac{4G^2}{9c^{11}} \hat{x}^k S_{j|k} M_{ij}^{(6)}(t) - \frac{G^2}{9c^{11}} \hat{x}^k S_{i|j} M_{jk}^{(6)}(t) + \mathcal{O}(c^{-13}), \quad (3.9b)$$

$$\mathcal{H}_{\mathcal{S}_{i|j} \times \mathcal{M}_{ij}}{}^{ij} = -\frac{22G^2}{15c^{10}} S_{k|(i} M_{j)k}^{(5)}(t) + \mathcal{O}(c^{-12}). \quad (3.9c)$$

By applying the shift

$$\xi_{\mathcal{S}_{i|j} \times \mathcal{M}_{ij}}^0 = \frac{4G^2}{45c^{11}} \hat{x}^{jk} S_{i|k} M_{ij}^{(6)}(t), \quad (3.10a)$$

$$\xi_{\mathcal{S}_{i|j} \times \mathcal{M}_{ij}}^i = \frac{8G^2}{15c^{10}} \hat{x}^k S_{j|k} M_{ij}^{(5)}(t) - \frac{3G^2}{15c^{10}} \hat{x}^k S_{i|j} M_{jk}^{(5)}(t). \quad (3.10b)$$

the metric becomes of order $\mathcal{H}'_{\mathcal{S}_{i|j} \times \mathcal{M}_{ij}}{}^{\mu\nu} = \mathcal{O}(c^{-12}, c^{-13}, c^{-12})$, and reads

$$\mathcal{H}'_{\mathcal{S}_{i|j} \times \mathcal{M}_{ij}}{}^{00ii} = -\frac{4G^2}{15c^{12}} \hat{x}^{jk} S_{i|k} M_{ij}^{(7)}(t) + \mathcal{O}(c^{-14}), \quad (3.11)$$

which, injected in the action (2.7), yields (upon integrations by parts)

$$\mathcal{S}_{\mathcal{S}_{i|j} \times \mathcal{M}_{ij}} = \frac{G^2}{30c^{10}} S_{i|j} \int dt M_{ik}^{(3)}(t) M_{jk}^{(4)}(t) + \mathcal{O}(c^{-12}). \quad (3.12)$$

IV. SUMMARY AND CONCLUSION

In this work, we have derived the leading order tail and ‘‘failed’’ tail sectors appearing at the 5PN order in the conservative dynamics of compact binaries, by employing the Fokker Lagrangian framework. Making use of dimensional regularization, we have computed the homogeneous solution of the near-zone metric, and have integrated it out in the action. Our result, given in Eq. (1.1), is consistent with previous works performed within the EFT framework: the tail sector agrees with [28], and the failed tail one, with [31]. With this new computation at hand, we hope that the current discrepancy in EFT results for the failed tail sector will be fully understood and resolved.

The last step towards completion of the 5PN conservative dynamics is the *memory* effect, *i.e.* the interaction of three mass quadrupoles. In order to compute it within the Fokker Lagrangian framework, the method presented in this work has to be enhanced, as briefly discussed in the footnote 7. This subtle computation is thus left for future work.

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