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ON LINKED MODULES OVER THE SUPER-YANGIAN OF THE SUPERALGEBRA $Q(1)$

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Abstract. Let $Q(n)$ be the queer Lie superalgebra. We determine conditions under which two 1-dimensional modules over the super-Yangian of $Q(1)$ can be extended nontrivially, and thus belong to the same block of the subcategory of finite-dimensional $YQ(1)$-modules admitting generalized central character $\chi = 0$. We use these results to determine conditions under which two 1-dimensional modules over the finite W-algebra for $Q(n)$ can be extended nontrivially. We describe blocks in the category of finite-dimensional modules over the finite W-algebra for $Q(2)$. In certain cases we determine conditions under which two simple finite-dimensional $YQ(1)$-modules admitting central character $\chi \neq 0$ can be extended nontrivially and propose a conjecture in the general case.

1. Introduction

The *queer* Lie superalgebra $Q(n)$ is a fixed point subalgebra of the general linear Lie superalgebra $\mathfrak{gl}(n|n)$ relative to certain involutive automorphism.

We started to study the representation theory of the finite W-algebra W^n for $Q(n)$ associated with the principal nilpotent coadjoint orbits in [8]. We have shown that all irreducible representations of W^n are finite-dimensional. In [10] we classified irreducible representations of W^n (Theorem 4.7). We used these results to classify irreducible finite-dimensional representations of the super-Yangian $YQ(1)$ of $Q(1)$ (Theorem 5.13). A natural problem is to describe blocks in the subcategory of finite-dimensional $YQ(1)$-modules and in the subcategory of finite-dimensional W^n-modules admitting a given generalized central character χ. We initiated the study of blocks in these subcategories in [11, 12]. If $\chi = 0$, then the simple modules in these subcategories are 1-dimensional. In this paper we determine when two 1-dimensional $YQ(1)$-modules can be extended nontrivially, and thus belong to the same block (Theorem 11.1). We use these results and results of [10] to determine when two 1-dimensional W^n-modules can be extended nontrivially (Theorem 14.1). Using Theorem 14.1, we describe blocks in the category of finite-dimensional modules over W^2 (Theorem 15.2).

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Every simple finite-dimensional module over $YQ(1)$ is isomorphic (up to change of parity) to $V(\mathbf{s}) \otimes \Gamma_f$, where $V(\mathbf{s})$ is a simple $YQ(1)$-module parameterized by an n-tuple of nonzero complex numbers $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ such that $s_i + s_j \neq 0$ for all $i < j$, and Γ_f is a 1-dimensional $YQ(1)$-module, which is defined by certain generating function $f(u) \in YQ(1)[[u^{-2}]]$. If $n \geq 1$, then $V(\mathbf{s}) \otimes \Gamma_f$ admits a nontrivial central character. In the case when $\mathbf{s} = (s_1)$, we determine conditions under which two simple modules of type $V(s_1) \otimes \Gamma_f$ can be extended nontrivially (Proposition 12.1). We propose a conjecture in the general case when $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ (Conjecture 12.3).

2. The Lie superalgebra $Q(n)$

Consider the general linear Lie superalgebra $\mathfrak{gl}(n|n)$ with the standard basis E_{ij}, where $i, j = \pm 1, \ldots, \pm n$. Define the parity of i by

$$p(i) = 0 \text{ if } i > 0 \text{ and } p(i) = 1 \text{ if } i < 0.$$

Let η be an involutive automorphism of $\mathfrak{gl}(n|n)$ defined by

$$\eta(E_{ij}) = E_{-i,-j}.$$

The *queer* Lie superalgebra $Q(n)$ is the fixed point subalgebra in $\mathfrak{gl}(n|n)$ relative to η. Recall that $Q(n)$ can also be defined as follows (see [3]). Equip $\mathbb{C}^{n|n}$ with the odd operator ζ such that $\zeta^2 = -\operatorname{Id}$. Then $Q(n)$ is the centralizer of ζ in the Lie superalgebra $\mathfrak{gl}(n|n)$. Let $\zeta = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. It is easy to see that $Q(n)$ consists of matrices of the form

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix},$$

where A, B are $n \times n$-matrices. Let

$$\{e_{i,j}, f_{i,j} \,|\, i, j = 1, \ldots, n\}$$

denote the basis in $Q(n)$ consisting of elementary even and odd matrices. Set

$$\xi_i := (-1)^{i+1} f_{i,i}, \;\; x_i := \xi_i^2 = e_{i,i}. \tag{2.1}$$

3. The finite W-algebra for $Q(n)$

Let W^n be the *finite W-algebra* associated with a principal even nilpotent element φ in the coadjoint representation of $\mathfrak{g} = Q(n)$. Let us recall its definition (see [13]). We fix the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ to be the set of matrices with diagonal A and B. By $\mathfrak{n}^+$ (respectively, $\mathfrak{n}^-$) we denote the nilpotent subalgebras consisting of matrices with strictly upper triangular (respectively, low triangular) A and B.

The Lie superalgebra $\mathfrak{g}$ has the triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, and we set $\mathfrak{b} = \mathfrak{n}^+ \oplus \mathfrak{h}$. Choose $\varphi \in \mathfrak{g}^*$ such that

$$\varphi(f_{i,j}) = 0, \quad \varphi(e_{i,j}) = \delta_{i,j+1}.$$

Let I_φ be the left ideal in $U(\mathfrak{g})$ generated by $x-\varphi(x)$ for all $x\in\mathfrak{n}^-$. Let $\pi : U(\mathfrak{g})\to U(\mathfrak{g})/I_\varphi$ be the natural projection. Then

$$W^n = \{\pi(y)\in U(\mathfrak{g})/I_\varphi \mid \mathrm{ad}(x)y\in I_\varphi \text{ for all } x\in\mathfrak{n}^-\}.$$

Using the identification of $U(\mathfrak{g})/I_\varphi$ with the Whittaker module $U(\mathfrak{g})\otimes_{U(\mathfrak{n}^-)}\mathbb{C}_\varphi\simeq U(\mathfrak{b})\otimes\mathbb{C}$, we can consider W^n as a subalgebra of $U(\mathfrak{b})$. The natural projection $\vartheta : U(\mathfrak{b})\to U(\mathfrak{h})$ with the kernel $\mathfrak{n}^+U(\mathfrak{b})$ is called the *Harish-Chandra homomorphism.* It is proven in [8] that the restriction of ϑ to W^n is injective. We will identify W^n with $\vartheta(W^n)\subset U(\mathfrak{h})$.

Example 3.1. $n=2$, $\mathfrak{h}=\mathrm{span}\{x_1,x_2\mid\xi_1,\xi_2\}$. Then W^2 realized as a subalgebra of $U(\mathfrak{h})$ has the following generators:

$$z_0 = x_1+x_2,\ z_1 = x_1x_2-\xi_1\xi_2 \text{ (even)},$$

$$\phi_0=\xi_1+\xi_2,\ \phi_1 = x_2\xi_1 - x_1\xi_2 \text{ (odd)}.$$

4. The super Yangian of $Q(1)$

The Yangians $YQ(n)$ associated with the Lie superalgebras $Q(n)$ were defined by M. L. Nazarov ([5, 6]). Recall that $YQ(1)$ is the associative unital superalgebra over $\mathbb{C}$ with the countable set of generators $T^{(m)}_{i,j}$, where $m=1,2,\dots$ and $i,j=\pm1$. The $\mathbb{Z}_2$-grading of $YQ(1)$ is defined as follows:

$$p(T^{(m)}_{i,j}) = p(i)+p(j),\ \text{where } p(1)=0 \text{ and } p(-1)=1.$$

To write the defining relations for these generators, we employ the formal series in $YQ(1)[[u^{-1}]]$:

$$T_{i,j}(u) = \delta_{ij}\cdot 1 + T^{(1)}_{i,j}u^{-1} + T^{(2)}_{i,j}u^{-2}+\dots.$$

Then for all possible indices i,j,k,l we have the relations

$$\begin{aligned}&(u^2-v^2)[T_{i,j}(u),T_{k,l}(v)]\cdot(-1)^{p(i)p(k)+p(i)p(l)+p(k)p(l)}\\ &= (u+v)(T_{k,j}(u)T_{i,l}(v) - T_{k,j}(v)T_{i,l}(u))\\ &-(u-v)(T_{-k,j}(u)T_{-i,l}(v)-T_{k,-j}(v)T_{i,-l}(u))\cdot(-1)^{p(k)+p(l)}.\end{aligned}\tag{4.1}$$

Here v is a formal parameter independent of u, so that (4.1) is an equality in the algebra of formal Laurent series in u^{-1},v^{-1} with coefficients in $YQ(1)$. For all indices i,j we also have the relations

$$T_{i,j}(-u) = T_{-i,-j}(u).\tag{4.2}$$

The relations (4.1) and (4.2) are equivalent to the following defining relations:

$$([T^{(m+1)}_{i,j}, T^{(r-1)}_{k,l}] - [T^{(m-1)}_{i,j}, T^{(r+1)}_{k,l}]) \cdot (-1)^{p(i)p(k)+p(i)p(l)+p(k)p(l)} =$$

$$T^{(m)}_{k,j} T^{(r-1)}_{i,l} + T^{(m-1)}_{k,j} T^{(r)}_{i,l} - T^{(r-1)}_{k,j} T^{(m)}_{i,l} - T^{(r)}_{k,j} T^{(m-1)}_{i,l} \tag{4.3}$$

$$+(-1)^{p(k)+p(l)}(-T^{(m)}_{-k,j} T^{(r-1)}_{-i,l} + T^{(m-1)}_{-k,j} T^{(r)}_{-i,l} + T^{(r-1)}_{k,-j} T^{(m)}_{i,-l} - T^{(r)}_{k,-j} T^{(m-1)}_{i,-l}),$$

$$T^{(m)}_{-i,-j} = (-1)^m T^{(m)}_{i,j}, \tag{4.4}$$

where $m, r = 1, \dots$ and $T^{(0)}_{i,j} = \delta_{ij}$. Recall that $YQ(1)$ is a *Hopf superalgebra* (see [6]) with comultiplication given by the formula

$$\Delta(T^{(r)}_{i,j}) = \sum_{s=0}^{r} \sum_{k} (-1)^{(p(i)+p(k))(p(j)+p(k))} T^{(s)}_{i,k} \otimes T^{(r-s)}_{k,j}.$$

The *evaluation homomorphism* $ev : YQ(1) \to U(Q(1))$ is defined as follows:

$$T^{(1)}_{1,1} \mapsto -e_{1,1}, \quad T^{(1)}_{1,-1} \mapsto f_{1,1}, \quad T^{(0)}_{i,j} \mapsto \delta_{i,j}, \quad T^{(r)}_{i,j} \mapsto 0 \text{ for } r > 1, i, j = \pm 1.$$

5. W^n IS A QUOTIENT OF $YQ(1)$

Definition 5.1. (a) Define $\Delta_l : YQ(1) \longrightarrow YQ(1)^{\otimes l}$ by

$$\Delta_l := \Delta_{l-1,l} \circ \cdots \circ \Delta_{2,3} \circ \Delta.$$

(b) Let $\varphi_n : YQ(1) \to U(Q(1))^{\otimes n} \simeq U(\mathfrak{h})$ be $\varphi_n := ev^{\otimes n} \circ \Delta_n$.

Note that $\varphi_n(T^{(r)}_{1,1}) = \varphi_n(T^{(r)}_{1,-1}) = 0$ if $r > n$.

Proposition 5.2. *([9], Corollary 5.16) The map φ_n is a surjective homomorphism from $YQ(1)$ onto W^n, realized as a subalgebra of $U(\mathfrak{h})$:*

$$\varphi_n(YQ(1)) = \vartheta(W^n) \simeq W^n.$$

Note that W^{m+n} is a subalgebra of $W^m \otimes W^n$ ([10], Lemma 3.3). The following diagram commutes:

$$\begin{array}{ccc} YQ(1) & \xrightarrow{\Delta} & YQ(1) \otimes YQ(1) \\ {\scriptstyle \varphi_{m+n}}\downarrow & & \downarrow{\scriptstyle \varphi_m \otimes \varphi_n} \\ W^{m+n} & \longrightarrow & W^m \otimes W^n \end{array} \tag{5.1}$$

6. Simple modules over associative superalgebras

We work in the category of vector superspaces over $\mathbb{C}$. We denote the *parity* of a homogeneous vector v of a superspace by $p(v) \in \mathbb{Z}_2$. All tensor products are over $\mathbb{C}$.

Let $\mathcal{A}$ be a superalgebra. By an $\mathcal{A}$-module M we mean a $\mathbb{Z}_2$-graded left $\mathcal{A}$-module. A submodule of M is a $\mathbb{Z}_2$-graded submodule. By Π we denote the parity functor $\Pi(M) = M \otimes \mathbb{C}^{0|1}$. For a module M over an associate superalgebra $\mathcal{A}$, $\Pi(M)$ has the same underlying vector space but with the opposite $\mathbb{Z}$-grading. The new action of $a \in \mathcal{A}$ on $m \in \Pi(M)$ is given in terms of the old action by $a \cdot m := (-1)^{p(a)} am$.

Recall that if M is a simple finite-dimensional $\mathcal{A}$-module over some associative superalgebra $\mathcal{A}$, then by Schur's Lemma $\mathrm{End}_{\mathcal{A}}(M)$ is either one-dimensional, or two-dimensional and has basis $\{\mathrm{Id}_M, \epsilon_M\}$, where ϵ_M is a (unique up to a sign) odd involution on M: $\epsilon_M^2 = \mathrm{Id}_M$. Note that ϵ_M provides an $\mathcal{A}$ isomorphism $M \longrightarrow \Pi(M)$. We say that M is an *irreducible of* M-*type* in the former case and an *irreducible of* Q-*type* in the latter (see [4, 1]).

Let $\mathcal{A}$ and $\mathcal{B}$ be two superalgebras. The tensor product $\mathcal{A}\otimes\mathcal{B}$ is again a superalgebra, where multiplication is given by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{p(b_1)p(a_2)} a_1 a_2 \otimes b_1 b_2$$

for $a_i \in \mathcal{A}, b_i \in \mathcal{B}$. Let M and N be two modules over associative superalgebras $\mathcal{A}$ and $\mathcal{B}$. Then $M \otimes N$ is naturally a module over $\mathcal{A} \otimes \mathcal{B}$ where

$$(a \otimes b)(m \otimes n) = (-1)^{p(b)p(m)} am \otimes bn,$$

where $a \in \mathcal{A}, b \in \mathcal{B}$ and $m \in M, n \in N$. If M and N are two simple finite-dimensional modules over associative superalgebras $\mathcal{A}$ and $\mathcal{B}$, then the module $M \otimes N$ might be not simple. In fact, if M and N are both of M-type, then $M \otimes N$ is simple of M-type. If one of these modules is of M-type, and the other is of Q-type, then $M \otimes N$ is simple of Q-type. However, if M and N are both of Q-type, then $M \otimes N$ is not simple. Let ϵ_M and ϵ_N be odd involutions of M and N, respectively. Then the map $\epsilon_M \otimes \epsilon_N$ defined by

$$(\epsilon_M \otimes \epsilon_N)(m \otimes n) = (-1)^{p(m)} \epsilon_M(m) \otimes \epsilon_N(n)$$

is an even $\mathcal{A} \otimes \mathcal{B}$-automorphism of $M \otimes N$, and its square is $-\mathrm{Id}_{M\otimes N}$. In this case $M \otimes N$ decomposes into a direct sum of two $\mathcal{A} \otimes \mathcal{B}$-submodules, which are formed by the $\pm\mathbf{i}$-eigenspaces of $\epsilon_M \otimes \epsilon_N$. We can choose either submodule and denote it by $M \boxtimes N$. Then

$$M \otimes N \simeq M \boxtimes N \oplus \Pi(M \boxtimes N).$$

Both submodules are simple and of M-type.

7. Simple W^n-modules

We classified simple W^n-modules in [10] (Theorem 4.7). Here we recall their construction.

7.1. **W^n-modules $V(\mathbf{s})$.** Let $\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{C}^n$. We call $\mathbf{s}$ *regular* if $s_i \neq 0$ for all $i \leq n$ and *typical* if $s_i + s_j \neq 0$ for all $1 \leq i < j \leq n$. Note that we have the natural embedding of the Lie superalgebras

$$Q(1) \oplus Q(1) \oplus \cdots \oplus Q(1) \hookrightarrow Q(n). \tag{7.1}$$

Let $\mathfrak{h}_1$ denote the Cartan subalgebra of $Q(1)$. Then $\mathfrak{h}_1 = \mathrm{span}\{x_1 \mid \xi_1\}$ with $x_1 = \xi_1^2$, and $U(\mathfrak{h}_1) \simeq \mathbb{C}([\xi_1])$. Let $V(\mathbf{s}_i)$ be a $(1|1)$-dimensional $U(\mathfrak{h}_1)$-module, where the action is given by

$$\xi \mapsto \begin{pmatrix} 0 & \sqrt{s_i} \\ \sqrt{s_i} & 0 \end{pmatrix}, \quad x \mapsto \begin{pmatrix} s_i & 0 \\ 0 & s_i \end{pmatrix} \qquad \text{for } i = 1, 2.$$

The embedding (7.1) induces the isomorphism

$$U(\mathfrak{h}) \simeq U(\mathfrak{h}_1) \otimes U(\mathfrak{h}_1) \otimes \cdots \otimes U(\mathfrak{h}_1).$$

Then $V(\mathbf{s}) := V(s_1) \boxtimes V(s_2) \boxtimes \cdots \boxtimes V(s_n)$ is a simple $U(\mathfrak{h})$-module. Consider the restriction of $U(\mathfrak{h})$ to W^n. Let $\mathbf{s} = (s_1, \dots, s_n)$ be regular typical. Then $V(\mathbf{s})$ is a simple W^n-module, and if $\mathbf{s}' = \sigma(\mathbf{s})$ for some permutation of coordinates, then $V(\mathbf{s})$ is isomorphic to $V(\mathbf{s}')$ as a W^n-module, see [10].

7.2. **Construction of simple W^n-modules.** Let Γ_t be the simple W^2-module of dimension $(1|0)$ on which ϕ_0, ϕ_1 and z_0 act by zero and z_1 acts by the scalar t.

Let $r, p, q \in \mathbb{N}$ and $r + 2p + q = n$, $\mathbf{t} = (t_1, \dots, t_p) \in \mathbb{C}^p$, and $\lambda = (\lambda_1, \dots, \lambda_q) \in \mathbb{C}^q$, where $\mathbf{t}$ is regular and λ is regular typical. Recall that there is an embedding $W^n \hookrightarrow W^r \otimes (W^2)^{\otimes p} \otimes W^q$ ([10], Corollary 3.4). Set

$$S(\mathbf{t}, \lambda) := \mathbb{C} \boxtimes \Gamma_{t_1} \boxtimes \cdots \boxtimes \Gamma_{t_p} \boxtimes V(\lambda),$$

where the first term $\mathbb{C}$ in the tensor product denotes the trivial W^r-module. For $q = 0$ we use the notation $S(\mathbf{t})$ and set $V(\lambda) = \mathbb{C}$.

Proposition 7.1. *(see* [10], *Theorem 4.7) (a) Every simple W^n-module is isomorphic to $S(\mathbf{t}, \lambda)$ up to change of parity.*
(b) Two simple W^n-modules $S(\mathbf{t}, \lambda)$ and $S(\mathbf{t}', \lambda')$ are isomorphic if and only if $p' = p$, $q' = q$, $\mathbf{t}' = \sigma(\mathbf{t})$ and $\lambda' = \tau(\lambda)$ for some $\sigma \in S_p$ and $\tau \in S_q$.

8. Central characters

The center of $U(\mathfrak{g})$ for $\mathfrak{g} = Q(n)$ is described in [7]. The center of $U(\mathfrak{h})$ coincides with $\mathbb{C}[x_1, \dots, x_n]$ and the image of the center of $U(\mathfrak{g})$ under the Harish-Chandra homomorphism ϑ is generated by the polynomials $p_k = x_1^{2k+1} + \cdots + x_n^{2k+1}$ for all $k \in \mathbb{N}$. These polynomials are called Q-symmetric polynomials.

In [8] we proved that the center Z^n of W^n coincides with $W^n \bigcap \mathbb{C}[x_1, \dots, x_n] = \vartheta(Z(U(\mathfrak{g})))$ and hence can be also identified with the ring of Q-symmetric polynomials.

Every $\mathbf{s}$ defines the central character $\chi_{\mathbf{s}} : Z^n \to \mathbb{C}$. Furthermore, it follows from the description of simple W^n-modules in [10] (Theorem 4.6) that every simple W^n-module admits central character $\chi_{\mathbf{s}}$ for some $\mathbf{s}$. For every $\mathbf{s} = (s_1, \ldots, s_n)$ we define the *core* $c(\mathbf{s}) = (s_{i_1}, \ldots, s_{i_m})$ as a subsequence obtained from $\mathbf{s}$ by removing all $s_j = 0$ and all pairs (s_i, s_j) such that $s_i + s_j = 0$. Up to a permutation this result does not depend on the order of removing. Thus, the core is well defined up to permutation. We call m the length of the core.

Example 8.1. Let $\mathbf{s} = (1, 0, 3, -1, -1)$, then $c(\mathbf{s}) = (3, -1)$.

The following is a reformulation of the central character description in [7].

Lemma 8.2. *Let* $\mathbf{s}, \mathbf{s}' \in \mathbb{C}^n$. *Then* $\chi_{\mathbf{s}} = \chi_{\mathbf{s}'}$ *if and only if* $\mathbf{s}$ *and* $\mathbf{s}'$ *have the same core (up to permutation).*

It follows from Lemma 8.2 that the core depends only on the central character $\chi_{\mathbf{s}}$, we denote it $c(\chi)$.

9. Simple finite-dimensional $YQ(1)$-modules

We classified simple finite-dimensional $YQ(1)$-modules in [10]. First we recall the description of 1-dimensional $YQ(1)$-modules.

Remark 9.1. Note that $[T_{1,1}^{(k)}, T_{1,1}^{(m)}] = 0$ if $k + m$ is even (see [8], Proposition 6.4).

Definition 9.2. Let $\mathbf{A}$ be the commutative subalgebra in $YQ(1)$ generated by $T_{1,1}^{(2k)}$ for $k \geq 0$. Let

$$f(u) = 1 + \sum_{k>0} f_{2k} u^{-2k}.$$

Let Γ_f be the corresponding 1-dimensional $\mathbf{A}$-module, where the action of

$$T_{1,1}(u^{-2}) = \sum_{k \geq 0} T_{1,1}^{(2k)} u^{-2k}$$

is given by the generating function $f(u)$.

Recall that for any Hopf superalgebra R, the ideal (R_1) generated by all odd elements is a Hopf ideal and the quotient $R/(R_1)$ is a Hopf algebra.

Proposition 9.3. *([10], Lemma 5.11) The quotient* $YQ(1)/(YQ(1)_1)$ *is isomorphic to* $\mathbf{A} \simeq \mathbb{C}[T_{1,1}^{(2k)}]_{k>0}$, *with comultiplication*

$$\Delta T_{1,1}(u^{-2}) = T_{1,1}(u^{-2}) \otimes T_{1,1}(u^{-2}).$$

Thus we can lift an $\mathbf{A}$-module Γ_f to a $YQ(1)$-module.

Proposition 9.4. *([10], Lemma 5.12) The isomorphism classes of* 1*-dimensional* $YQ(1)$*-modules are in bijection with the set* $\{\Gamma_f\}$. *Furthermore, we have the identity* $\Gamma_f \otimes \Gamma_g \simeq \Gamma_{fg}$.

Let $\mathbf{s} \in \mathbb{C}^n$ be regular typical. Then we can lift the W^n-module $V(\mathbf{s})$ to a simple $YQ(1)$-module. Note that $T_{1,1}^{(r)}$ and $T_{1,-1}^{(r)}$ act on $V(\mathbf{s})$ by zero if $r > n$.

Proposition 9.5. *([10], Theorem 5.13) Any simple finite-dimensional $YQ(1)$-module is isomorphic to $V(\mathbf{s}) \otimes \Gamma_f$ or $\Pi V(\mathbf{s}) \otimes \Gamma_f$ for some regular typical $\mathbf{s}$ and $f(u) = 1 + \sum_{k>0} f_{2k} u^{-2k}$. Furthermore, $V(\mathbf{s}) \otimes \Gamma_f$ and $V(\mathbf{s}') \otimes \Gamma_g$ are isomorphic up to change of parity if and only if $\mathbf{s}'$ is obtained from $\mathbf{s}$ by permutation of coordinates and $f(u) = g(u)$.*

Proposition 9.6. *([10], Proposition 5.19) The simple $YQ(1)$-module $V(\mathbf{s}) \otimes \Gamma_f$ is lifted from some W^{m+n}-module if and only if $f \in \mathbb{C}[u^{-2}]$. Moreover, the smallest m is equal to the degree of the polynomial f.*

Remark 9.7. Note that $m = 2p$ is even. $S(t_1, \ldots, t_p, \lambda) \simeq V(\lambda) \otimes \Gamma_f$ where

$$f = \prod_{i=1}^{p} (1 + t_i u^{-2}).$$

10. The category $YQ(1)$–mod

We described the center Z of $YQ(1)$ in [10]. Let

$$\eta_i = (-\frac{1}{2})^i \operatorname{ad}^i T_{1,1}^{(2)} (T_{1,-1}^{(1)}), \quad Z_{2i} = \frac{1}{2}[\eta_0, \eta_{2i}], \tag{10.1}$$

where $\operatorname{ad}^i T_{1,1}^{(2)}$ is the i-power of the adjoint endomorphism $\operatorname{ad} T_{1,1}^{(2)}$. The elements $\{Z_{2i} \mid i \in \mathbb{N}\}$ are algebraically independent generators of Z.

Let $YQ(1)$–mod be the category of finite-dimensional $YQ(1)$-modules. A $YQ(1)$-module M admits *generalized central character* χ if for any $z \in Z$ and $m \in M$, there exists $n \in \mathbb{Z}_{\geq 0}$ such that $(z - \chi(z))^n \cdot m = 0$. Let $(YQ(1))^\chi$–mod be the full subcategory of modules admitting generalized central character χ. The category $YQ(1)$–mod is the direct sum of the subcategories $(YQ(1))^\chi$–mod, as χ ranges over the central characters for which $(YQ(1))^\chi$–mod is nonempty.

Lemma 10.1. *Every simple $YQ(1)$-module in the subcategory $(YQ(1))^\chi$–mod is isomorphic up to change of parity to $V(\mathbf{s}) \otimes \Gamma_f$, where $\mathbf{s} = (s_1, \ldots, s_n)$ is regular typical, which is unique up to permutation.*

Proof. Let $\mathbf{C} \subset YQ(1)$ be the unital subalgebra generated by $\{\eta_i \mid i \in \mathbb{N}\}$. Then $V(\mathbf{s})$ and $V(\mathbf{s}) \otimes \Gamma_f$ are isomorphic $\mathbf{C}$-modules. Indeed, $\eta_0 = T_{1,-1}^{(1)}$ and by (10.1)

$$\eta_{i+1} = (-\frac{1}{2})[T_{1,1}^{(2)}, \eta_i]. \tag{10.2}$$

Note that

$$\Delta(T_{1,-1}^{(1)}) = T_{1,-1}^{(1)} \otimes 1 + 1 \otimes T_{1,-1}^{(1)}.$$

Hence η_0 acts on $V(\mathbf{s}) \otimes \Gamma_f$ as $\eta_0 \otimes 1$. Then it follows by induction from (10.2) that η_i acts on $V(\mathbf{s}) \otimes \Gamma_f$ as $\eta_i \otimes 1$ for all i. Then every $\zeta \in \mathbf{C}$ acts as $\zeta \otimes 1$. Hence $V(\mathbf{s})$

and $V(\mathbf{s})\otimes\Gamma_f$ are isomorphic $\mathbf{C}$-modules, and they admit the same central character χ.

On the other hand, $YQ(1)$-modules $V(\mathbf{s})$ and $V(\mathbf{s}')$, where $\mathbf{s} = (s_1, \dots, s_n)$ and $\mathbf{s}' = (s'_1, \dots, s'_m)$ are regular typical, have the same central character χ if and only if $n = m$ and $\mathbf{s}'$ is a permutation of $\mathbf{s}$.

Indeed, a $YQ(1)$-module $V(\mathbf{s})$ admits a central character χ. It can be presented using the generating function

$$\chi(u) = \sum_{i=0}^{\infty} \chi_{2i} u^{-2i-1},$$

where $\chi_{2i} = \chi(Z_{2i})$. Let σ_k denote the k-th elementary symmetric polynomial. We proved in [10] that

$$\chi(u) = \frac{\sum_{i=0}^{\infty} \sigma_{2i+1}(\mathbf{s}) u^{-2i-1}}{1 + \sum_{i=1}^{\infty} \sigma_{2i}(\mathbf{s}) u^{-2i}}.$$

Note that $V(s_1, \dots, s_n)$ and $V(s_1, \dots, s_n, 0)$ have the same central character. Suppose that $V(\mathbf{s})$ and $V(\mathbf{s}')$ have the same central character χ and $\mathbf{s}$, $\mathbf{s}'$ are regular typical. Assume that $n \geq m$. Extend $\mathbf{s}'$ to the n-tuple $\mathbf{s}'' = (s'_1, \dots, s'_m, 0, 0, \dots, 0)$. Then $V(\mathbf{s}'')$ and $V(\mathbf{s})$ have the same central character χ. Note that $\varphi_n(Z) = Z^n$ (see [10]). Thus $\chi = \chi_{\mathbf{s}} \circ \varphi_n = \chi_{\mathbf{s}''} \circ \varphi_n$. Then $\chi_{\mathbf{s}} = \chi_{\mathbf{s}''}$. Hence by Lemma 8.2, $\mathbf{s}$ and $\mathbf{s}''$ have the same core (up to permutation). Hence $m = n$ and $\mathbf{s}'$ is a permutation of $\mathbf{s}$. Clearly, if $\mathbf{s}'$ is a permutation of $\mathbf{s}$, then $\chi_{\mathbf{s}} = \chi_{\mathbf{s}'}$, and hence $V(\mathbf{s})$ and $V(\mathbf{s}')$ have the same central character χ. □

Recall that simple modules are partitioned into *blocks*. If two simple modules M_1 and M_2 can be extended nontrivially, i.e., if there is a non-split short exact sequence $0 \longrightarrow M_i \longrightarrow M \longrightarrow M_j \longrightarrow 0$ with $\{i, j\} = \{1, 2\}$, then M_1 and M_2 belong to the same block, and we will say that they are *linked*. Here we agree that M_i is linked to itself. More generally, if there is a finite sequence of simple modules $M = M_1, M_2, \dots, M_n = N$ such that adjacent pairs belong to the same block, then modules M and N belong to this block. A module M belongs to a block if all its composition factors do. Each block lies in a single $(YQ(1))^{\chi}$–mod. However, different blocks can belong to the same $(YQ(1))^{\chi}$–mod: see [2].

11. The subcategory $(YQ(1))^{\chi=0}$–mod

It follows from Proposition 9.5 that simple modules in the subcategory $(YQ(1))^{\chi=0}$–mod are exactly the 1-dimensional modules Γ_f up to change of parity. Let Γ_f and Γ_g be two $YQ(1)$-modules, where

$$f(u) = \sum_{k\geq 0} a_{2k} u^{-2k}, \quad g(u) = \sum_{k\geq 0} b_{2k} u^{-2k}, \quad a_0 = b_0 = 1. \tag{11.1}$$

Recall that Γ_f is linked to itself. If $f \neq g$, then one can easily check that the short exact sequence

$$0 \longrightarrow \Gamma_f \longrightarrow M \longrightarrow \Gamma_g \longrightarrow 0$$

splits. Indeed, we have the following relations in $YQ(1)$:

$$[T_{1,1}^{(2k)}, T_{1,-1}^{(1)}] = 2T_{1,-1}^{(2k)}. \tag{11.2}$$

$$[T_{1,1}^{(2)}, T_{1,-1}^{(2k)}] = 2T_{1,-1}^{(2k+1)} + 2T_{1,-1}^{(2k)} - 2T_{1,1}^{(2k)}T_{1,-1}^{(1)}. \tag{11.3}$$

$$[T_{1,-1}^{(1)}, T_{1,-1}^{(2k+1)}] = -2T_{1,1}^{(2k+1)}. \tag{11.4}$$

All odd generators $T_{1,-1}^{(r)}$ act on M by zero, since M is a purely even module. Then $T_{1,1}^{(2k+1)}$ also acts on M by zero by (11.4). Note that $T_{1,1}^{(2k)}$ acts on M as $\begin{pmatrix} a_{2k} & c_{2k} \\ 0 & b_{2k} \end{pmatrix}$, and there exists m such that $a_{2m} \neq b_{2m}$, since $f \neq g$. We can choose a basis in M so that $c_{2m} = 0$. Then $c_{2k} = 0$ for all k, since $T_{1,1}^{(2k)}$ commute. Hence $M \simeq \Gamma_f \oplus \Gamma_g$.

We will determine when Γ_f is linked with $\Pi(\Gamma_g)$. Let $x_k = \frac{1}{2}(a_{2k} - b_{2k})$.

Theorem 11.1. *$Ext^1(\Pi(\Gamma_g), \Gamma_f) \neq 0$ if and only if x_1 is an arbitrary complex number and x_k for $k > 1$ satisfies the recurrence relation*

$$x_{k+1} = (x_1 x_k - x_k + a_{2k})x_1. \tag{11.5}$$

Proof. Note that the short exact sequence

$$0 \longrightarrow \Gamma_f \longrightarrow M \longrightarrow \Pi(\Gamma_g) \longrightarrow 0$$

is non-split if and only if $T_{1,-1}^{(1)}$ does not act by zero. Indeed, if $T_{1,-1}^{(1)}$ acts by zero, then $T_{1,-1}^{(2k)}$ and $T_{1,-1}^{(2k+1)}$ also act by zero for all k by (11.2) and (11.3), but then $M \simeq \Gamma_f \oplus \Pi(\Gamma_g)$. Clearly, if $M \simeq \Gamma_f \oplus \Pi(\Gamma_g)$, then all odd generators act by zero.

Hence $\mathrm{Ext}^1(\Pi(\Gamma_g), \Gamma_f) \neq 0$ if and only if one can define a representation $\rho : YQ(1) \longrightarrow \mathrm{End}(\mathbb{C}^{1|1})$ such that (up to equivalence)

$$\rho(T_{1,1}^{(2k)}) = \begin{pmatrix} a_{2k} & 0 \\ 0 & b_{2k} \end{pmatrix}, \quad \rho(T_{1,-1}^{(1)}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{11.6}$$

Then

$$\rho(T_{1,-1}^{(2k)}) = \begin{pmatrix} 0 & \frac{1}{2}(a_{2k} - b_{2k}) \\ 0 & 0 \end{pmatrix}, \tag{11.7}$$

$$\rho(T_{1,-1}^{(2k+1)}) = \begin{pmatrix} 0 & \frac{1}{4}(a_2 - b_2)(a_{2k} - b_{2k}) + \frac{1}{2}(a_{2k} + b_{2k}) \\ 0 & 0 \end{pmatrix}, \tag{11.8}$$

$$\rho(T_{1,1}^{(2k+1)}) = 0. \tag{11.9}$$

Here (11.7) follows from (11.6) and the relation (11.2), (11.8) follows from (11.6), (11.7), and (11.3), and (11.9) follows from (11.8) and (11.4).

Let $x_k = \frac{1}{2}(a_{2k} - b_{2k})$. Then from (11.7)

$$\rho(T_{1,-1}^{(2k)}) = \begin{pmatrix} 0 & x_k \\ 0 & 0 \end{pmatrix}, \tag{11.10}$$

and from (11.8)

$$\rho(T_{1,-1}^{(2k+1)}) = \begin{pmatrix} 0 & x_1 x_k - x_k + a_{2k} \\ 0 & 0 \end{pmatrix}, \tag{11.11}$$

The recurrence relation (4.3) with $m = 2k-1$ and $r = 2p+2$ gives the relation

$$([T_{1,1}^{(2k)}, T_{1,-1}^{(2p+1)}] - [T_{1,1}^{(2k-2)}, T_{1,-1}^{(2p+3)}]) =$$

$$T_{1,1}^{(2k-1)} T_{1,-1}^{(2p+1)} + T_{1,1}^{(2k-2)} T_{1,-1}^{(2p+2)} - T_{1,1}^{(2p+1)} T_{1,-1}^{(2k-1)} - T_{1,1}^{(2p+2)} T_{1,-1}^{(2k-2)} \tag{11.12}$$

$$+T_{-1,1}^{(2k-1)} T_{-1,-1}^{(2p+1)} - T_{-1,1}^{(2k-2)} T_{-1,-1}^{(2p+2)} - T_{1,-1}^{(2p+1)} T_{1,1}^{(2k-1)} + T_{1,-1}^{(2p+2)} T_{1,1}^{(2k-2)}.$$

From (11.12) and (11.6), (11.10), (11.11), (11.9) we obtain the relation

$$x_1 x_p x_k + (a_{2p} - x_p) x_k - x_1 x_{p+1} x_{k-1} = x_{p+1}(a_{2k-2} - x_{k-1}).$$

If $p = 0$ (and $a_0 = 1, x_0 = 0$) we have

$$x_k - x_1^2 x_{k-1} = x_1(a_{2k-2} - x_{k-1}),$$

which is equivalent to (11.5). On the other hand, one can check that ρ defined by (11.6), (11.9), (11.10) and (11.11), with x_k satisfying (11.5), preserves the relations (4.3). □

Corollary 11.2. *Let $x_k = \frac{1}{2}(b_{2k} - a_{2k})$. Then $Ext^1(\Gamma_f, \Pi(\Gamma_g)) \neq 0$ if and only if x_1 is an arbitrary complex number and x_k for $k > 1$ satisfies the recurrence relation*

$$x_{k+1} = (x_1 x_k + x_k + a_{2k}) x_1. \tag{11.13}$$

12. Towards the general case: the subcategory $(YQ(1))^{\chi}$–mod

Making use of Lemma 10.1, we would like to determine conditions under which two $YQ(1)$-modules $V(\mathbf{s}) \otimes \Gamma_f$ and $V(\mathbf{s}) \otimes \Gamma_g$ can be extended nontrivially.

We will consider the case, when $\mathbf{s} = (s)$. We denote $V(\mathbf{s})$ by $V(s)$.

Proposition 12.1. *Let $V(s) \otimes \Gamma_f$ and $V(s) \otimes \Gamma_g$ be $YQ(1)$-modules, where $s \neq 0$, let $f(u)$ and $g(u)$ be given by (11.1), and let $x_k = \frac{1}{2}(a_{2k} - b_{2k})$. Then $Ext^1(V(s) \otimes \Pi(\Gamma_g), V(s) \otimes \Gamma_f) \neq 0$ if and only if x_k satisfies the recurrence relation (11.5).*

Proof. First, show that if

$$0 \longrightarrow \Gamma_f \longrightarrow \mathbb{C}^{1|1} \longrightarrow \Pi(\Gamma_g) \longrightarrow 0 \tag{12.1}$$

is a non-split short exact sequence of $YQ(1)$-modules, then the short exact sequence of $YQ(1)$-modules

$$0 \longrightarrow V(s)\otimes\Gamma_f \longrightarrow V(s)\otimes\mathbb{C}^{1|1} \longrightarrow V(s)\otimes\Pi(\Gamma_g) \longrightarrow 0 \tag{12.2}$$

is non-split if and only if $x_1 \neq s$.

Let $\mathbb{C}^{1|1} =< \imath \mid \bar{\imath} >$ and $V(s) =< v \mid w >$, where $\imath$ and v are even and $\bar{\imath}$ and w are odd. Note that $T_{1,1}^{(1)}$ and $T_{1,-1}^{(1)}$ act on $V(s)$ as $\begin{pmatrix} -s & 0 \\ 0 & -s \end{pmatrix}$ and $\begin{pmatrix} 0 & \sqrt{s} \\ \sqrt{s} & 0 \end{pmatrix}$, respectively, and $T_{1,1}^{(r)}$ and $T_{1,-1}^{(r)}$ act by zero if $r \geq 2$. Let $\Gamma_f =< \imath >$. Suppose that $V(s)\otimes\mathbb{C}^{1|1} = V(s)\otimes\Gamma_f \oplus M$ is a direct sum of $YQ(1)$-modules. Then $M =< X \mid Y >$, where

$$\begin{aligned} X &= a(v\otimes\imath) + (w\otimes\bar{\imath}), \quad a\in\mathbb{C}, \\ Y &:= T_{1,-1}^{(1)}(X) = (a\sqrt{s}-1)(w\otimes 1) + \sqrt{s}(v\otimes\bar{\imath}). \end{aligned} \tag{12.3}$$

Obviously, $T_{1,1}^{(n)}(X) = \lambda_n X$ for some $\lambda_n \in \mathbb{C}$. If n is even this implies that $2x_1 a = \sqrt{s}$, and if n is odd, then $2a\sqrt{s} = 1$. Hence $x_1 = s$. One can easily check that if $x_1 = s$, then M defined by (12.3), where $a = \frac{1}{2\sqrt{s}}$, is a $YQ(1)$-submodule of $V(s)\otimes\mathbb{C}^{1|1}$.

Next, suppose that there is a non-split short exact sequence of $YQ(1)$-modules

$$0 \longrightarrow V(s)\otimes\Gamma_f \longrightarrow M \longrightarrow V(s)\otimes\Pi(\Gamma_g) \longrightarrow 0. \tag{12.4}$$

Describe the action of $YQ(1)$ on $V(s)\otimes\Gamma_f$. Recall that $YQ(1)$ is generated by $T_{1,1}^{(2k)}$ and $T_{1,-1}^{(1)}$ (see [10], Lemma 5.1). Let $\{v, w\}$ be a basis of $V(s)$, and $\Gamma_f =< \imath >$. Then $V(s)\otimes\Gamma_f =< v_1 \mid w_1 >$, where $v_1 = v\otimes\imath$ and $w_1 = w\otimes\imath$. The action of $T_{1,1}^{(2k)}$ and $T_{1,-1}^{(1)}$ with respect to this basis is given by the matrices $\begin{pmatrix} a_{2k} & 0 \\ 0 & a_{2k} \end{pmatrix}$ and $\begin{pmatrix} 0 & \sqrt{s} \\ \sqrt{s} & 0 \end{pmatrix}$, respectively.

Let f and g be given by (11.1), and let $x_k = \frac{1}{2}(a_{2k} - b_{2k})$. Assume that $f \neq g$. Then there exists m such that $a_{2m} \neq b_{2m}$. We can choose a basis in M: $\{v_1, w_1, v_2, w_2\}$, where v_1 and v_2 are even and w_1 and w_2 are odd, with respect to which the action of $T_{1,1}^{(2m)}$ is given by a diagonal matrix, and since all $T_{1,1}^{(2k)}$ commute, they also act by diagonal matrices:

$$\begin{pmatrix} a_{2k} & 0 & 0 & 0 \\ 0 & a_{2k} & 0 & 0 \\ 0 & 0 & b_{2k} & 0 \\ 0 & 0 & 0 & b_{2k} \end{pmatrix}. \tag{12.5}$$

We choose a basis so that in addition $T^{(1)}_{1,-1}$ acts by

$$\begin{pmatrix} 0 & \sqrt{s} & 0 & 1 \\ \sqrt{s} & 0 & -1 & 0 \\ 0 & 0 & 0 & \sqrt{s} \\ 0 & 0 & \sqrt{s} & 0 \end{pmatrix}. \tag{12.6}$$

Using (11.2)-(11.4) we obtain that the action of $T^{(2k)}_{1,-1}$ and $T^{(2k+1)}_{1,-1}$ on M is given by the following matrices, respectively:

(12.7)
$$\begin{pmatrix} 0 & 0 & 0 & x_k \\ 0 & 0 & -x_k & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a_{2k}\sqrt{s} & 0 & x_1x_k - x_k + a_{2k} \\ a_{2k}\sqrt{s} & 0 & -(x_1x_k - x_k + a_{2k}) & 0 \\ 0 & 0 & 0 & b_{2k}\sqrt{s} \\ 0 & 0 & b_{2k}\sqrt{s} & 0 \end{pmatrix},$$

and $T^{(2k+1)}_{1,1}$ acts by

$$\begin{pmatrix} -a_{2k}s & 0 & 0 & x_k\sqrt{s} \\ 0 & -a_{2k}s & -x_k\sqrt{s} & 0 \\ 0 & 0 & -b_{2k}s & 0 \\ 0 & 0 & 0 & -b_{2k}s \end{pmatrix}. \tag{12.8}$$

Then using the relation (11.12), we can show that x_k are determined exactly by the recurrence relation (11.5). If $f = g$, then all x_k are zero. Clearly, they satisfy (11.5). On the other hand, one can define the action of $YQ(1)$ on M by (12.5), (12.7) and (12.8), where x_k satisfy the recurrence relation (11.5), and show that this actions respects (4.3). □

Remark 12.2. Note that if $x_1 \neq 0, s$, then M is isomorphic to the $YQ(1)$-module $V(s) \otimes \mathbb{C}^{1|1}$ defined by (12.2). Indeed, let $V(s) =< v \mid w >$ and $\mathbb{C}^{1|1} =< \imath \mid \bar{\imath} >$. Then $V(s) \otimes \mathbb{C}^{1|1} =< v\otimes \imath, w \otimes \imath \mid w \otimes \bar{\imath}, v \otimes \bar{\imath} >$. Note that in this basis $T^{(2)}_{1,1}$ acts by

$$\begin{pmatrix} a_2 & 0 & -\sqrt{s} & 0 \\ 0 & a_2 & 0 & \sqrt{s} \\ 0 & 0 & b_2 & 0 \\ 0 & 0 & 0 & b_2 \end{pmatrix}. \tag{12.9}$$

If $x_1 \neq 0$, one can choose a basis so that the matrix of $T^{(2)}_{1,1}$ is diagonal, and correspondingly, the matrices for all $T^{(2k)}_{1,1}$ are given by (12.5). Also, if $x_1 \neq s$, then multiplying $v \otimes \imath$ and $w \otimes \imath$ by $1 - \frac{s}{x_1}$, we obtain that $T^{(1)}_{1,-1}$ acts in this basis by the matrix (12.6). Then $M \simeq V(s) \otimes \mathbb{C}^{1|1}$, since $YQ(1)$ is generated by $T^{(2k)}_{1,1}$ and $T^{(1)}_{1,-1}$.

Conjecture 12.3. Let S be a simple finite-dimensional $YQ(1)$-module. Let $n \geq 2$, $\mathbf{s} = (s_1, s_2, \dots, s_n)$ be regular typical, and let $f(u)$ and $g(u)$ be given by (11.1). Let $x_k = \frac{1}{2}(a_{2k} - b_{2k})$. Then

$$Ext^1(S, V(\mathbf{s}) \otimes \Gamma_f) \neq 0,$$

if and only if $S \simeq V(\mathbf{s}) \otimes \Pi(\Gamma_g)$, where x_1 is an arbitrary complex number and x_k for $k > 1$ satisfies the recurrence relation (11.5). The short exact sequence (12.2) is non-split.

13. The category W^n–mod

Let W^n–mod be the category of finite-dimensional W^n-modules. Let $(W^n)^\chi$–mod be the full subcategory of modules admitting generalized central character χ. The category W^n–mod is the direct sum of subcategories $(W^n)^\chi$–mod, as χ ranges over the central characters χ for which $(W^n)^\chi$–mod is nonempty. We proved in [10] (Lemma 4.12) that a simple W^n-module S belongs to $(W^n)^\chi -$ mod if and only if it is isomorphic (up to change of parity) to $S(\mathbf{t}, \lambda)$ with $\lambda = c(\chi)$.

14. The subcategory $(W^n)^{\chi=0}$–mod

Note that simple modules in the subcategory $(W^n)^{\chi=0}$–mod are exactly the 1-dimensional modules $S(\mathbf{t})$ up to change of parity (see [10]).

Theorem 14.1. *Fix $\mathbf{t} = (t_1, \dots, t_p)$ and $\mathbf{t}' = (t'_1, \dots, t'_q)$, where p and q are less than or equal to $\frac{n}{2}$. Consider the W^n-modules $S(\mathbf{t})$ and $S(\mathbf{t}')$. Define $a_{2k} = \sigma_k(t_1, \dots, t_p)$ for $k = 1, \dots, p$, $a_{2k} = 0$ for $k > p$. Similarly, define $b_{2k} = \sigma_k(t'_1, \dots, t'_q)$ for $k = 1, \dots, q$, $b_{2k} = 0$ for $k > q$. Let $x_k = \frac{1}{2}(a_{2k} - b_{2k})$.*
(a) If $S(\mathbf{t})$ is a nontrivial W^n-module, then $Ext^1(\Pi(S(\mathbf{t}')), S(\mathbf{t})) \neq 0$ if and only if $x_1 \neq 0$ and x_k for $k > 1$ satisfies the recurrence relation (11.5) or $S(\mathbf{t}')$ is isomorphic to $S(\mathbf{t})$ and $n > 2p$.
(b) If $S(\mathbf{t}) = \mathbb{C}^{1|0}$ is the trivial W^n-module, then $Ext^1(\Pi(S(\mathbf{t}')), S(\mathbf{t})) \neq 0$ if and only if $S(\mathbf{t}') = \mathbb{C}^{1|0}$ or $\mathbf{t}' = (t'_1)$ with $t'_1 = -2$.

Proof. Suppose that $\mathrm{Ext}^1(\Pi(S(\mathbf{t}')), S(\mathbf{t})) \neq 0$. Lift $S(\mathbf{t})$ and $S(\mathbf{t}')$ to $YQ(1)$-modules Γ_f and Γ_g, respectively, where f and g are given by (11.1). Then $\mathrm{Ext}^1(\Pi(\Gamma_g), \Gamma_f)) \neq 0$. Hence by Theorem 11.1, x_k satisfy (11.5). Note that if $x_1 = 0$, then all $x_k = 0$ and hence $S(\mathbf{t}')$ is isomorphic to $S(\mathbf{t})$. One can show that if $S(\mathbf{t})$ is a nontrivial W^n-module, which is linked with $\Pi(S(\mathbf{t}))$, then $n > 2p$. Indeed, suppose that $n = 2p$. Then there exists a non-split short exact sequence

$$0 \longrightarrow S(\mathbf{t}) \longrightarrow M \longrightarrow \Pi(S(\mathbf{t})) \longrightarrow 0. \qquad (14.1)$$

We lift the W^n-module M to a $YQ(1)$-module. Then the action of $YQ(1)$ on M is given by (11.6)-(11.9) (up to equivalence), where $a_{2k} = b_{2k}$ for all k. Then

$$\rho(T_{1,-1}^{(2p+1)}) = \begin{pmatrix} 0 & a_{2p} \\ 0 & 0 \end{pmatrix}. \tag{14.2}$$

Note that $\rho(T_{1,-1}^{(2p+1)}) = 0$, since $2p+1 > n$. Hence $a_{2p} = 0$, but

$$a_{2p} = \sigma_p(t_1, \dots, t_p) = t_1 \cdot \ldots \cdot t_p.$$

Hence $t_i = 0$ for some i. A contradiction, since all t_i are nonzero.

Conversely, show that if x_k satisfy (11.5), then the lifted modules Γ_f and $\Pi(\Gamma_g)$ are linked (see (12.1)). Assume that $x_1 \neq 0$. Then (11.5) implies that

$$x_1 x_k - x_k + a_{2k} = 0$$

for $2k \geq n$, if n is even, and for $2k \geq n-1$, if n is odd. Hence $\rho(T_{1,-1}^{(r)}) = 0$ if $r > n$ by (11.10) and (11.11), and $\rho(T_{1,1}^{(r)}) = 0$ if $r > n$ by (11.6) and (11.9). Recall that the kernel of the surjective homomorphism $\varphi_n : YQ(1) \longrightarrow W^n$ is generated by $T_{1,1}^{(r)}$ and $T_{1,-1}^{(r)}$, where $r > n$. This allows one to define a representation $\mu : W^n \longrightarrow \mathrm{End}(\mathbb{C}^{1|1})$ such that $\rho = \mu \circ \varphi_n$. Thus $S(\mathbf{t})$ is linked with $\Pi(S(\mathbf{t}'))$.

If $S(\mathbf{t}) = \mathbb{C}^{1|0}$, then $a_{2k} = 0$ for $k \geq 1$. From (11.5), $x_1 = 0$ or $x_1 = 1$. In the first case $x_k = 0$ and $b_{2k} = 0$ for $k \geq 1$. Hence $S(\mathbf{t}')$ is the trivial module. In the second case, $x_1 = 1$ and $x_k = 0$ for $k \geq 2$, $b_2 = -2$, and $b_{2k} = 0$ for $k \geq 2$. Hence $\mathbf{t}' = (t_1')$ with $t_1' = -2$.

Finally, assume that $S(\mathbf{t})$ is a nontrivial W^n-module and $n > 2p$. Let $r = n - 2p$. Recall that there is an embedding $W^n \hookrightarrow W^r \otimes (W^2)^{\otimes p}$, and

$$S(\mathbf{t}) = \mathbb{C} \boxtimes \Gamma_{t_1} \boxtimes \cdots \boxtimes \Gamma_{t_p},$$

where the first term $\mathbb{C}$ in the tensor product denotes the trivial W^r-module. By (b), there exists a non-split short exact sequence of W^r-modules

$$0 \longrightarrow \mathbb{C}^{1|0} \longrightarrow \mathbb{C}^{1|1} \longrightarrow \mathbb{C}^{0|1} \longrightarrow 0. \tag{14.3}$$

Consider the W^n-module $M = \mathbb{C}^{1|1} \boxtimes \Gamma_{t_1} \boxtimes \cdots \boxtimes \Gamma_{t_p}$. Then we obtain an exact sequence (14.1), which is non-split. Indeed, let ξ_i for $i = 1, \dots, r$ be odd elementary matrices in $Q(r)$, and ξ_j^1, ξ_j^2 be odd elementary matrices in $Q(2)$ for $j = 1, \dots, p$, see (2.1). Recall that there is a surjective homomorphism $\varphi_n : YQ(1) \longrightarrow W^n$ for every n, see Proposition 5.2. Note that $\varphi_r(T_{1,-1}^{(1)}) = \xi_1 + \ldots + \xi_r$ by (2.6) in [10]. Because the exact sequence (14.3) is non-split, the action of $\varphi_r(T_{1,-1}^{(1)})$ on $\mathbb{C}^{1|1}$ is nonzero. Also,

$$\varphi_{r+2p}(T_{1,-1}^{(1)}) = \sum_{i=1}^{r} \xi_i \otimes 1^{\otimes p} + \sum_{j=1}^{p} 1 \otimes 1^{\otimes (j-1)} \otimes (\xi_j^1 + \xi_j^2) \otimes 1^{\otimes (p-j)}$$

by (2.6) and (3.10) in [10]. Note that ξ_j^1 and ξ_j^2 act on Γ_{t_j} by zero for $j = 1, \dots, p$. Hence $\varphi_{r+2p}(T^{(1)}_{1,-1})$ acts on M as $(\xi_1 + \ldots + \xi_r) \otimes 1^{\otimes p}$, and this action is nonzero. Thus $\mathrm{Ext}^1(\Pi(S(\mathbf{t})), S(\mathbf{t})) \neq 0$. □

Remark 14.2. Suppose that Γ_f is lifted from a nontrivial module $S(\mathbf{t})$, and assume that $\mathrm{Ext}^1(\Pi(\Gamma_g), \Gamma_f) \neq 0$. Note that Γ_g is lifted from some W^n-module $S(\mathbf{t}')$ which is not isomorphic to $S(\mathbf{t})$, if and only if $x_{\frac{n+2}{2}} = 0$ if n is even and $x_{\frac{n+1}{2}} = 0$ if n is odd, see (11.5). This means that x_1 is a (nonzero) root of the polynomial of degree n (respectively, $n-1$) defined by the recurrence relation (11.5) if n is even (respectively, odd). Then we set $b_{2k} = a_{2k} - 2x_k$ for all k and find $\mathbf{t}' = (t'_1, \dots, t'_q)$ such that $b_{2k} = \sigma_k(\mathbf{t}')$. Here $\mathbf{t}'$ is defined up to permutation of $t'_1, \dots, t'_q$, and we delete all zero entries. Then $\mathrm{Ext}^1(\Pi(S(\mathbf{t}')), S(\mathbf{t})) \neq 0$. Also, if $n > 2p$, then $\mathrm{Ext}^1(\Pi(S(\mathbf{t})), S(\mathbf{t})) \neq 0$. Moreover, all modules $S(\mathbf{t}')$ satisfying the above formula are obtained in this way.

Corollary 14.3. *(a) If $S(\mathbf{t})$ is a nontrivial W^n-module, then $\mathrm{Ext}^1(S(\mathbf{t}), \Pi(S(\mathbf{t}'))) \neq 0$ if and only if $x_k := \frac{1}{2}(b_{2k} - a_{2k})$ satisfies the recurrence relation (11.13) for $k > 0$ and $x_1 \neq 0$ or $S(\mathbf{t}')$ is isomorphic to $S(\mathbf{t})$ and $n > 2p$.*
(b) If $S(\mathbf{t})$ is a trivial W^n-module, then $\mathrm{Ext}^1(S(\mathbf{t}), \Pi(S(\mathbf{t}'))) \neq 0$ if and only if $S(\mathbf{t}') = \mathbb{C}^{1|0}$ or $\mathbf{t}' = (t'_1)$ with $t'_1 = -2$.

15. Blocks in the category W^2–mod

Lemma 15.1. *Let $n = 2$. A simple W^2-module S belongs to $(W^2)^\chi - \mathrm{mod}$ if and only if one of the following three cases takes place:*
(1) $S \simeq V(s_1, s_2)$ for $s_1 \neq -s_2, s_1, s_2 \neq 0$ and $c(\chi) = (s_1, s_2)$,
(2) $S \simeq V(s, 0)$ for $s \neq 0$ and $c(\chi) = (s)$,
(3) $S \simeq \Gamma_t$ or $\Pi(\Gamma_t)$ and $\chi = 0$.

Proof. Follows from Lemma 4.12 in [10]. □

Theorem 15.2. *(1) Each simple W^2-module $V(s_1, s_2)$ for $s_1 \neq -s_2, s_1, s_2 \neq 0$ forms a block in $(W^2)^\chi$-mod, where $c(\chi) = (s_1, s_2)$.*
(2) Each simple W^2-module $V(s, 0)$ for $s \neq 0$ forms a block in $(W^2)^\chi$-mod, where $c(\chi) = (s)$.
(3) The blocks in the subcategory $(W^2)^{\chi=0}$-mod are described as follows. Let $a \in \mathbb{C}$. Define

$$a_n = a - n^2 + n\sqrt{1 - 4a} \text{ for } n = 0, \pm 1, \pm 2, \dots \tag{15.1}$$

Then Γ_a lies in the block formed by Γ_{a_n} if n is even and $\Pi\Gamma_{a_n}$, if n is odd. $\Pi\Gamma_a$ lies in the block formed by $\Pi\Gamma_{a_n}$ if n is even and Γ_{a_n}, if n is odd.

Proof. Statements (1) and (2) follow from Lemma 8.2 and Lemma 15.1. To prove (3), first we will show that Γ_a is linked with $\Pi\Gamma_b$ if and only if

$$b = a - 1 \pm \sqrt{1 - 4a}. \tag{15.2}$$

Let $S(\mathbf{t}) = \Gamma_a$ and $S(\mathbf{t}') = \Gamma_b$. Set $a_0 = b_0 = 1$, $a_2 = a$, $b_2 = b$ and $a_{2k} = b_{2k} = 0$ for $k > 1$, $x_k = \frac{1}{2}(a_{2k} - b_{2k})$ for $k \geq 0$. Suppose $a \neq 0$, then by Theorem 14.1 (a)

$$x_2 = (x_1^2 - x_1 + a_2)x_1,$$

and x_1 must satisfy $x_1^2 - x_1 + a_2 = 0$. Hence $x_1 = \frac{1}{2}(1 \pm \sqrt{1-4a})$. Thus $b = a - 1 \pm \sqrt{1-4a}$. Note that Corollary 14.3 (a) gives the same result.

If $a = 0$, then by Theorem 14.1 (b) we have that $b = 0$ or $b = -2$. Hence (15.2) holds.

Note that b is a root of the equation

$$b^2 + (2 - 2a)b + (a^2 + 2a) = 0. \tag{15.3}$$

The sum of the roots of equation (15.3) is $2a - 2$. This gives the relation

$$a_{n-1} + a_{n+1} = 2a_n - 2 \quad (a_n = a). \tag{15.4}$$

Then (15.2) and (15.4) imply (15.1). □

Example 15.3. (1) $a = 0$, then $a_n = n(1-n)$ and Γ_0 lies in the block

$$\ldots, \Gamma_{-30}, \Pi\Gamma_{-20}, \Gamma_{-12}, \Pi\Gamma_{-6}, \Gamma_{-2}, , \Pi\Gamma_0, \Gamma_0, \Pi\Gamma_{-2}, \Gamma_{-6}, \Pi\Gamma_{-12}, \Gamma_{-20}, \Pi\Gamma_{-30}, \ldots$$

(2) $a = \frac{1}{4}$, then $a_n = \frac{1}{4} - n^2$ and $\Gamma_{\frac{1}{4}}$ lies in the block

$$\Gamma_{\frac{1}{4}}, \Pi\Gamma_{-\frac{3}{4}}, \Gamma_{-\frac{15}{4}}, \ldots$$

(3) $a = 1$, then $a_n = 1 - n^2 + n\sqrt{-3}$ and Γ_1 lies in the block

$$\ldots, \Pi\Gamma_{-3\sqrt{-3}-8}, \Gamma_{-2\sqrt{-3}-3}, \Pi\Gamma_{-\sqrt{-3}}, \Gamma_1, \Pi\Gamma_{\sqrt{-3}}, \Gamma_{2\sqrt{-3}-3}, \Pi\Gamma_{3\sqrt{-3}-8}, \ldots$$

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