arXiv:2309.05617v1 [gr-qc] 11 Sep 2023

Hidden symmetries of generalised gravitational instantons

Bernardo Araneda*
Max-Planck-Institut für Gravitationsphysik
(Albert-Einstein-Institut), Am Mühlenberg 1,
D-14476 Potsdam, Germany

September 12, 2023

Abstract

For conformally Kähler Riemannian four-manifolds with a Killing field, we develop a framework to solve the field equations for generalised gravitational instantons corresponding to conformal self-duality and to cosmological Einstein-Maxwell. We obtain generic identities for the curvature of such manifolds without assuming field equations. After applying the framework to recover standard solutions, we find conformally self-dual generalisations of the Page-Pope, Plebański-Demiański, and Chen-Teo solutions, which are neither hyper-Kähler nor quaternionic-Kähler, giving new self-dual gravitational instantons in conformal gravity.

1 Introduction

Gravitational instantons are four-dimensional, complete, Ricci-flat Riemannian manifolds with sufficiently fast curvature decay, typically ALE, ALF, or AF (cf. [1] for precise definitions). They are expected to give the dominant contributions to the path integral for Euclidean quantum gravity. Particular cases are metrics with self-dual Riemann tensor (i.e. hyper-Kähler manifolds), while more general cases correspond to generalisations of the Ricci-flat condition. These generalisations include the addition of a cosmological constant, solutions to Einstein-Maxwell theory, conformally self-dual geometries (i.e. metrics with self-dual Weyl tensor), Bach-flat metrics, etc. Such solutions are interesting in high-energy physics, as Einstein-Maxwell theory coincides with the bosonic sector of $N = 2$ supergravity in four dimensions, and conformally self-dual and Bach-flat geometries are solutions to conformal gravity. Examples of cosmological Einstein-Maxwell instantons have been studied in [2, 3, 4, 5], while instantons in conformal gravity were considered in [6, 7] and more recently in [8, 9, 10, 11, 12]. Generalised instantons are also interesting in Riemannian geometry, concerning open problems about the classification of these spaces. Examples of classifications in the Ricci-flat case include ALE hyper-Kähler [13], and ALF toric-Hermitian [14]. In the non-Ricci-flat case, there are classifications of *compact* complex surfaces, including compact Einstein-Hermitian [15] (i.e. with non-trivial cosmological constant) and compact Bach-flat Kähler [16].

A curious property about Ricci-flat gravitational instantons (also common to the more general classifications mentioned above) is that all known examples are Hermitian, cf. [17, Question 1.4], which implies (using Bianchi identities) that they are conformally Kähler, and have at least one Killing field (as long as they are not Kähler themselves). Motivated by this, in this work we study

*Email: bernardo.araneda@aei.mpg.de

generalised gravitational instantons corresponding to the conformally self-dual and cosmological Einstein-Maxwell equations, under the assumption of a geometry which is conformally Kähler with a Killing field. We will show that in both cases the field equations reduce to a single scalar equation: the $SU(\infty)$ (continuous) Toda equation. (Or the modified Toda equation in the case of Einstein-Maxwell with non-zero cosmological constant.) We derive a number of useful identities for conformally Kähler metrics, cf. in particular our main Theorem 2.9 for the Ricci form. Our results provide a generalisation of Tod's work [18], which is for the (non-conformally-self-dual) Ricci-flat case. In the conformally-self-dual (non-Ricci-flat) case, the reduction was already known from LeBrun's work [19].

We apply the construction to the study of a large number of metric ansätze, including the spherically symmetric, Kerr-Newman, Page-Pope [20], Plebański-Demiański [21], and Chen-Teo [22, 23] classes. In particular, we show that the Page-Pope class of metrics on bundles over Riemann surfaces is generically ambi-Kähler without assuming any field equations, and we classify all conformally self-dual solutions. We also construct a Plebański-Demiański self-dual gravitational instanton in conformal gravity, which depends on 5 parameters and is not Einstein, so it is different from the standard self-dual limit of Plebański-Demiański. More generally, part of our motivation comes from open questions concerning the Chen-Teo instanton [22], which is a 2-parameter, Ricci-flat AF metric that gives a counterexample to the classical Euclidean Black Hole Uniqueness Conjecture [1]. This instanton was generalised in [23] to a 5-parameter, Ricci-flat (singular) family, which includes both the Plebański-Demiański and the (triple-collinearly-centred) Gibbons-Hawking spaces. The construction of the cosmological Einstein-Maxwell Chen-Teo solution is a challenging open problem [23], and in future works we will apply the framework developed in this work to obtain that solution. In the current paper, we give a family of conformally self-dual generalisations.

Our work also provides an explicit Toda formulation of all the examples mentioned above. In particular, we give a simple trick to solve the Toda equation (with an extra symmetry) for complicated metric ansätze.

Concerning conformal gravity, the field equations are the vanishing of the Bach tensor, which is a conformally invariant condition. Any conformally (anti-)self-dual space satisfies these equations. Einstein metrics are also Bach-flat, so if one has an Einstein space then any conformal transformation of it will be a solution to conformal gravity, but this will simply be coming from a solution to ordinary Einstein gravity. Bach-flat metrics which are *not* conformally Einstein are thus more intriguing from the conformal gravity point of view. Now, in this work we are interested in conformally Kähler metrics, and Derdziński showed [24, Proposition 4] that a Kähler metric with non-self-dual Weyl tensor is Bach-flat *if and only if* it is (locally) conformally Einstein[1]. Thus, since we restrict to conformally Kähler geometry, we will not worry about the Bach-flat equations. In particular, since we show that the Page-Pope class [20] is always ambi-Kähler, this implies that Bach-flat instantons such as generalised Eguchi-Hanson and generalised Taub-NUT (considered recently in [12] in the conformal gravity context) are conformally Einstein.

A natural question is then whether there are non-self-dual Bach-flat instantons which are not conformally Kähler: such solutions would be Bach-flat but not conformally Einstein, so more interesting for conformal gravity. In fact, at least in Lorentz signature such solutions exist: see [25, 26]. (We mention however that many of these solutions are Petrov type N and thus do not have Euclidean sections, so the situation for instantons is less clear.)

Overview. The core of our framework is developed in section 2, where we obtain a number of identities for conformally Kähler metrics whose Ricci tensor is invariant under the complex

[1] That is: if $\hat{g}$ is Kähler, with Ricci scalar $\hat{R} \neq 0$, then the Bach tensor vanishes iff $\hat{R}^{-2}\hat{g}$ is Einstein [24].

structure: we give a reduction of the conformally self-dual and of the cosmological Einstein-Maxwell equations (Prop. 2.1 and Prop. 2.2 resp.), and obtain generic expressions for the metric (Prop. 2.4), Ricci scalar (Prop. 2.8) and Ricci form (Theorem 2.9). We also comment on the special case of ambi-Kähler structures (section 2.5), and we give some basic examples (section 2.6). In particular, the Kerr-Newman example in section 2.6 allows us to illustrate in a simple case the trick to solve the Toda equation mentioned above; this will be used in more complicated cases in later sections of the paper. In section 3 we study the Page-Pope class [20], solving the conformally self-dual and cosmological Einstein-Maxwell equations, and in section 4 we do the same for the Plebański-Demiański class [21]. In section 5 we analyse the Chen-Teo class [22, 23], giving a Toda formulation and finding conformally self-dual generalisations. We present our conclusions in section 6. We include appendices A, B with some basic background, definitions, and identities. We also mention that our construction is purely local[2].

2 Conformally Kähler geometry

2.1 Preliminaries

For general definitions and background, we refer to appendix A. Let (M, g_{ab}) be a conformally Kähler 4-manifold, with complex structure $J^a{}_b$ and fundamental 2-form $\kappa_{ab} = g_{bc}J^c{}_a$. Recall that κ_{ab} is necessarily self-dual (SD) or anti-self-dual (ASD) w.r.t. to the Hodge star; we choose κ_{ab} ASD for concreteness. Then it can be written in 2-spinor language as $\kappa_{ab} = j_{AB}\epsilon_{A'B'}$, where j_{AB} is symmetric (and satisfies $j^A{}_C j^C{}_B = -\delta^A_B$). The conformally rescaled 2-form is $\hat{\kappa}_{ab} = \Omega j_{AB}\hat{\epsilon}_{A'B'}$, where $\hat{\epsilon}_{A'B'} = \Omega\epsilon_{A'B'}$. The conformal Kähler property is $\hat{\nabla}_a\hat{\kappa}_{bc} = 0$, where $\hat{\nabla}_a$ is the Levi-Civita connection of $\hat{g}_{ab}$. In spinors, this translates into $\hat{\nabla}_{AA'}(\Omega j_{BC}) = 0$. Using the relation between $\hat{\nabla}_{AA'}$ and $\nabla_{AA'}$ (see [27, 28, 29]), one deduces that (M, g_{ab}) possesses a valence-2 Killing spinor:

$$\nabla_{A'(A}K_{BC)} = 0, \qquad K_{AB} := \Omega^{-1}j_{AB}. \tag{2.1}$$

Define now $Z_{ab} := K_{AB}\epsilon_{A'B'}$. A calculation using the Killing spinor equation (see [28, Eq. (6.4.6)]) shows that

$$\nabla_a Z_{bc} = \nabla_{[a}Z_{bc]} - 2g_{a[b}\xi_{c]}, \qquad \xi_a := \tfrac{1}{3}\nabla^b Z_{ab}. \tag{2.2}$$

The first equation is the conformal Killing-Yano (CKY) equation. In terms of the fundamental 2-form, the CKY tensor is $Z_{ab} = \Omega^{-1}\kappa_{ab}$.

Notice that ξ_b has always zero divergence, $\nabla^a\xi_a = 0$ (this follows from $\nabla^a\nabla^b Z_{ab} = 0$ since Z_{ab} is a 2-form). In addition, a calculation shows that ξ_b can be expressed as

$$\xi_b = J^a{}_b\partial_a\Omega^{-1}. \tag{2.3}$$

From this expression, we can deduce that the vector field $\xi = \xi^a\partial_a$ preserves both the conformal factor Ω and the fundamental 2-form κ_{ab}. For the first, we notice from (2.3) that $\pounds_\xi\Omega = \xi^a\partial_a\Omega = 0$. For the second, recall Cartan's formula for a generic vector field v and 2-form ω: $\pounds_v\omega = \mathrm{d}(v\lrcorner\omega) + v\lrcorner\mathrm{d}\omega$. Then $\pounds_\xi\hat{\kappa} = \mathrm{d}(\xi\lrcorner\hat{\kappa})$. Now, $(\xi\lrcorner\hat{\kappa})_b = \xi^a\hat{\kappa}_{ab} = -\Omega^2\xi_a J^a{}_b = -\partial_b\Omega$ (we use g_{ab} to lower indices). Thus $\pounds_\xi\hat{\kappa} = 0$, and since $\pounds_\xi\Omega = 0$, it follows that also $\pounds_\xi\kappa = 0$. So the conformal factor Ω is a Hamiltonian for ξ^a w.r.t. the symplectic structure $\hat{\kappa}_{ab}$.

Let us now show that the vector field ξ^a is a Killing vector of g_{ab} if and only if the Ricci tensor is invariant under the complex structure, meaning that $R_{ab} = R_{cd}J^c{}_a J^d{}_b$. (Notice that R_{ab}

[2] In particular, some of our examples include the 4-sphere S^4, which does not admit a global complex structure.

can be replaced by its trace-free part in this equation.) First, we apply an additional covariant derivative to the Killing spinor equation (2.1), $0 = \nabla_{AA'}\nabla^{(A}_{B'}K^{BC)}$. Symmetrizing over $A'B'$, this leads to

$$\Phi_{A'B'C}{}^{(A}K^{B)C} = -\nabla^{(A}_{(A'}\xi^{B)}_{B')}, \tag{2.4}$$

where $\Phi_{A'B'AB}$ represents the trace-free Ricci tensor, $\Phi_{ab} = -\frac{1}{2}(R_{ab} - \frac{R}{4}g_{ab})$. The right hand side of (2.4) is the conformal Killing operator applied to ξ_b, since $\nabla_{(A|(A'}\xi_{B')|B)} = \nabla_{(a}\xi_{b)} - \frac{1}{4}g_{ab}\nabla_c\xi^c$. But we noticed that $\nabla_c\xi^c = 0$, so it reduces to the ordinary Killing operator. The left hand side of (2.4) can be written (lowering the indices AB) as $-\Omega^{-1}\Phi_{c(a}J^c{}_{b)}$, where we used that $Z_a{}^b = -\Omega^{-1}J^b{}_a$. Multiplying by $J^a{}_d$ and renaming indices, (2.4) is equivalent to

$$R_{ab} - R_{cd}J^c{}_aJ^d{}_b = 4J^c{}_a\nabla_{(c}\xi_{b)}, \tag{2.5}$$

which proves our assertion about the Killing property of ξ^a.

The conformal Kähler condition also imposes restrictions on the (ASD) Weyl tensor. Again this can be seen from the Killing spinor equation: the condition $0 = \nabla_{A'(A}\nabla^{A'}_{B}K_{CD)}$ leads to $\Psi_{(ABC}{}^EK_{D)E} = 0$ (where Ψ_{ABCD} is the ASD Weyl curvature spinor), which implies that Ψ_{ABCD} is type D in the Petrov classification (the full Weyl tensor is generically type $D \otimes I$). A simple way to show this is to use $K_{AB} = \Omega^{-1}j_{AB}$ and decompose j_{AB} into principal spinors as in the first identity in (A.5): $j_{AB} = 2\mathrm{i}o_{(A}o^\dagger_{B)}$ (where $o_Ao^{\dagger A} = 1$). Then the condition $\Psi_{(ABC}{}^Ej_{D)E} = 0$ implies $\Psi_{ABCD} = 6\Psi_2 o_{(A}o_Bo^\dagger_Co^\dagger_{D)}$, where

$$\Psi_2 := \Psi_{ABCD}o^Ao^Bo^{\dagger C}o^{\dagger D} = C_{abcd}\ell^am^b\tilde{m}^cn^d = -\tfrac{1}{8}C_{abcd}J^{ac}J^{bd} \tag{2.6}$$

and $\ell^a, n^a, m^a, \tilde{m}^a$ is a (complex) null tetrad associated to o^A (cf. equation (A.6)).

2.2 Field equations

We will focus on the conformally self-dual and cosmological Einstein-Maxwell equations. The former are automatically solutions to conformal gravity. In view of Derdziński's result [24], cf. the introduction 1, we will not be interested in the Bach-flat equations per se.

2.2.1 Conformal self-duality

We say that a 4-dimensional, orientable Riemannian manifold is conformally (A)SD (or conformally half-flat) if the Weyl tensor satisfies

$$C_{abcd} = \pm{}^*C_{abcd} = \pm\tfrac{1}{2}\varepsilon_{ab}{}^{mn}C_{mncd} \tag{2.7}$$

where ε_{abcd} is the volume form, and where SD corresponds to the + sign and ASD to the − sign. In spinors, the SD equation is equivalent to $\Psi_{ABCD} \equiv 0$, and the ASD equation is equivalent to $\tilde{\Psi}_{A'B'C'D'} \equiv 0$.

For a conformally Kähler manifold (M, g_{ab}, κ_{ab}), we saw in (2.6) that the only non-trivial component of the ASD Weyl spinor is Ψ_2, so conformal self-duality reduces simply to the scalar equation $\Psi_2 = 0$. A convenient form for this equation can be obtained from the following:

Proposition 2.1. *Let (M, g_{ab}, κ_{ab}) be conformally Kähler. Let $\hat{g}_{ab} = \Omega^2 g_{ab}$ be the corresponding Kähler metric, and let $\hat{R}$ be its Ricci scalar. Then*

$$\Psi_2 = \Omega^2\frac{\hat{R}}{12}. \tag{2.8}$$

Proof. If $J^a{}_b = \kappa_{bc} g^{ca}$ is the complex structure and $\hat{\nabla}_a$ is the Levi-Civita connection of $\hat{g}_{ab}$, then $\hat{\nabla}_a J^b{}_c = 0$. From the integrability condition $[\hat{\nabla}_a, \hat{\nabla}_b] J^c{}_d = 0$, we get $\hat{R}_{abcd} = \hat{R}_{abef} J^e{}_c J^f{}_d$, where $\hat{R}_{abcd}$ is the Riemann tensor of $\hat{g}_{ab}$. Contracting with $\hat{g}^{ac}\hat{g}^{bd}$ and defining $\hat{J}^{ac} = \hat{g}^{ec} J^a{}_e$, we find the Ricci scalar $\hat{R} = \hat{R}_{abcd}\hat{J}^{ac}\hat{J}^{bd}$. Writing this in terms of the Weyl tensor (cf. [30, Eq. (3.2.28)]), one gets $\hat{R} = -\frac{3}{2}\hat{C}_{abcd}\hat{J}^{ac}\hat{J}^{bd}$. The conformal transformation of (2.6) is $\hat{\Psi}_2 = -\frac{1}{8}\hat{C}_{abcd}\hat{J}^{ac}\hat{J}^{bd}$. Since the conformal weights of $\hat{C}_{abcd}$ and $\hat{J}^{ac}$ are $+2$ and -2 respectively, we get $\hat{\Psi}_2 = \Omega^{-2}\Psi_2$. Putting everything together, (2.8) follows. □

2.2.2 Cosmological Einstein-Maxwell

Given a 4-manifold (M, g_{ab}) and a 2-form F_{ab}, the Einstein-Maxwell equations with cosmological constant λ (or cosmological Einstein-Maxwell, or Einstein-Maxwell-λ for short) are

$$
\begin{gathered}
R_{ab} - \frac{R}{2} g_{ab} + \lambda g_{ab} = 2F_{ac}F_b{}^c - \frac{1}{2} g_{ab} F_{cd}F^{cd}, \\
\nabla^a F_{ab} = 0 = \nabla_{[a}F_{bc]}.
\end{gathered}
\tag{2.9}
$$

If $\lambda < 0$, the system (2.9) is the bosonic part of the field equations of gauged $N = 2$ supergravity in four dimensions.

Proposition 2.2. *Let (M, g_{ab}, κ_{ab}) be a conformally Kähler Riemannian 4-manifold, whose Ricci tensor is invariant under the complex structure (equivalently, (2.3) is a Killing vector). Then the cosmological Einstein-Maxwell equations (2.9) are equivalent to the constancy of the scalar curvature: $R = 4\lambda$. The corresponding Maxwell field is $F_{ab} = F^-_{ab} + F^+_{ab}$, where*

$$
F^-_{ab} = \Omega^2 \kappa_{ab}, \qquad F^+_{ab} = \tfrac{1}{4}\Omega^{-2}(\rho_{ab} - \lambda\kappa_{ab}), \tag{2.10}
$$

and $\rho_{ab} = R_{bc}J^c{}_a$ is the Ricci form.

Remark 2.3. *Proposition 2.2 is a generalisation of Flaherty's result [31], who showed that scalar-flat Kähler metrics are automatically solutions to the Einstein-Maxwell system. The extension to the conformally Kähler case has been a subject of interest in the mathematical literature of recent years, see e.g. [32, 33, 34].*

Proof of Proposition 2.2. Let $\kappa_{ab} = j_{AB}\epsilon_{A'B'}$ be the fundamental 2-form. The symplectic form is $\hat{\kappa}_{ab} = \Omega^2\kappa_{ab}$. Since $\hat{\kappa}_{ab}$ is ASD and closed, it satisfies Maxwell equations $\nabla_{[a}\hat{\kappa}_{bc]} = 0 = \nabla^a\hat{\kappa}_{ab}$. In spinors, we have $\hat{\kappa}_{ab} = \varphi_{AB}\epsilon_{A'B'}$, with $\varphi_{AB} = \Omega^2 j_{AB}$ and

$$
\nabla^{AA'}\varphi_{AB} = 0. \tag{2.11}
$$

The Ricci tensor is constrained by eq. (2.4) (or equiv. (2.5)). We see from (2.4)-(2.5) that ξ^a is a Killing vector if and only if the Ricci tensor R_{ab}, or equivalently its trace-free part Φ_{ab}, is invariant under $J^a{}_b$. That is, iff $\Phi_{A'B'C(A}j^C{}_{B)} = 0$. Assuming this to be the case, we get

$$
\Phi_{ABA'B'} = 2\varphi_{AB}\phi_{A'B'}, \tag{2.12}
$$

where $\phi_{A'B'} \equiv \frac{1}{4}\Omega^{-4}\varphi^{AB}\Phi_{ABA'B'}$. Now, the contracted Bianchi identities in spinor form are $\nabla^{AA'}\Phi_{ABA'B'} + \frac{1}{8}\nabla_{BB'}R = 0$, see [27, Eq. (4.10.8)]. In view of (2.11), a short calculation gives

$$
\nabla^{AA'}\phi_{A'B'} = -\tfrac{\Omega^{-4}}{16}\varphi^{AB}\nabla_{BB'}R.
$$

Thus, we see that $\phi_{A'B'}$ also satisfies Maxwell equations $\nabla^{AA'}\phi_{A'B'} = 0$ if and only if R is a constant, say $R \equiv 4\lambda$. But (2.12) together with $R = 4\lambda$ are precisely the Einstein-Maxwell equations [27, Eq. (5.2.6)] (adapted to Euclidean signature, and setting Newton's gravitational constant equal to one). □

2.3 An expression for the metric

Proposition 2.4. *Let (M, g_{ab}, κ_{ab}) be a conformally Kähler Riemannian 4-manifold, whose Ricci tensor is invariant under the complex structure. Then there are local coordinates (ψ, x, y, z), real functions $W(x, y, z), u(x, y, z)$, and a 1-form $A(x, y, z)$ (with $\partial_\psi \lrcorner A = 0$) such that the metric and the fundamental 2-form can be written respectively as*

$$g = W^{-1}(\mathrm{d}\psi + A)^2 + W[\mathrm{d}z^2 + e^u(\mathrm{d}x^2 + \mathrm{d}y^2)], \tag{2.13}$$

$$\kappa = (\mathrm{d}\psi + A) \wedge \mathrm{d}z + We^u \mathrm{d}x \wedge \mathrm{d}y. \tag{2.14}$$

Remark 2.5. *The expression* (2.13) *appears in many constructions related to Kähler geometry in four dimensions, under different assumptions. LeBrun [19] deduced* (2.13) *for scalar-flat Kähler metrics with symmetry, and Tod deduced* (2.13) *for one-sided-type-D Ricci-flat metrics [18]. In the current work, we only assume the conformally Kähler condition with symmetry.*

Proof of Proposition 2.4. We start by choosing an orthonormal coframe $(\beta^0, \beta^1, \beta^2, \beta^3)$, and we define the almost-complex structure $J^a{}_b = \kappa_{bc} g^{ca}$, where $\kappa = \beta^0 \wedge \beta^1 + \beta^2 \wedge \beta^3$ as in (A.2). We assume that we chose the coframe such that J is the integrable complex structure of the hypothesis, and that the fundamental 2-form κ satisfies $\mathrm{d}(\Omega^2\kappa) = 0$ for some non-constant scalar field Ω. The hypothesis of J-invariance of the Ricci tensor of g_{ab} implies that the covector field ξ_a given by (2.3) is Killing, $\nabla_{(a}\xi_{b)} = 0$. We now construct a new orthonormal coframe $(\theta^0, \theta^1, \theta^2, \theta^3)$ as described in appendix A, using an almost-hyper-Hermitian structure (J_1, J_2, J_3) with $J_1 \equiv J$. First, introduce a coordinate ψ parametrizing the orbits of ξ^a, that is $\xi^a\partial_a = \partial_\psi$. Defining

$$W^{-1} := g_{ab}\xi^a\xi^b \tag{2.15}$$

and lowering an index, we have $\xi_a \mathrm{d}x^a = W^{-1}(\mathrm{d}\psi + A)$ for some 1-form A. We normalize as $e_0 := W^{1/2}\xi$, and we define $\theta^0 := g(e_0, \cdot) = W^{-1/2}(\mathrm{d}\psi + A)$. We also put $e_1 := Je_0$ and $\theta^1 := g(e_1, \cdot)$. From (2.3) we see that $\xi_a J^a{}_b = -\partial_b \Omega^{-1}$, so it follows that $\theta^1 = W^{1/2}\xi \lrcorner \kappa = W^{1/2}\mathrm{d}z$, where

$$z := \Omega^{-1}. \tag{2.16}$$

The remaining two elements θ^2, θ^3 of the new coframe are obtained by first defining $e_2 := J_2 e_0$, $e_3 := J_3 e_0$, and then $\theta^2 := g(e_2, \cdot) = W^{1/2}\xi \lrcorner \kappa_2$, $\theta^3 := g(e_3, \cdot) = W^{1/2}\xi \lrcorner \kappa_3$. We see that

$$\theta^2 + \mathrm{i}\theta^3 = W^{1/2}\, \xi \lrcorner (\kappa_2 + \mathrm{i}\kappa_3).$$

Now, we see from (A.3) that $\kappa_2 + \mathrm{i}\kappa_3 = 2\ell \wedge m$, where $\ell = \frac{1}{\sqrt{2}}(\beta^0 + \mathrm{i}\beta^1)$, $m = \frac{1}{\sqrt{2}}(\beta^2 + \mathrm{i}\beta^3)$ are type-$(1,0)$ forms of J_1. Integrability of J_1 implies the existence of holomorphic coordinates z^0, z^1 such that $\mathrm{d}z^0, \mathrm{d}z^1$ span type-$(1,0)$ forms. So ℓ and m can be expressed as linear combinations of $\mathrm{d}z^0, \mathrm{d}z^1$. In particular, this implies that $\mathrm{d}z^0 \wedge \mathrm{d}z^1 = \chi \ell \wedge m$ for some real scalar field χ. On the other hand, using Cartan's formula for the Lie derivative, we have $\pounds_\xi(\mathrm{d}z^0 \wedge \mathrm{d}z^1) = \mathrm{d}[\xi \lrcorner (\mathrm{d}z^0 \wedge \mathrm{d}z^1)]$. Since ξ preserves the complex structure, we can choose holomorphic coordinates such that $\pounds_\xi(\mathrm{d}z^0 \wedge \mathrm{d}z^1) = 0$, thus, there is a complex scalar ζ such that $\xi \lrcorner (\mathrm{d}z^0 \wedge \mathrm{d}z^1) = \mathrm{d}\zeta$. So:

$$\xi \lrcorner (\kappa_2 + \mathrm{i}\kappa_3) = 2\, \xi \lrcorner (\ell \wedge m) = 2\chi^{-1}\xi \lrcorner (\mathrm{d}z^0 \wedge \mathrm{d}z^1) = 2\chi^{-1}\mathrm{d}\zeta. \tag{2.17}$$

Separating ζ into real and imaginary parts as $\zeta \equiv \frac{1}{\sqrt{2}}(x + \mathrm{i}y)$, we thus get

$$\theta^2 + \mathrm{i}\theta^3 = \sqrt{2}W^{1/2}\chi^{-1}(\mathrm{d}x + \mathrm{i}\mathrm{d}y).$$

Finally, defining a real function u by $e^u := 2\chi^{-2}$, and putting everything together, we get (2.13). The expression (2.14) follows form $\kappa = \beta^0 \wedge \beta^1 + \beta^2 \wedge \beta^3 = \theta^0 \wedge \theta^1 + \theta^2 \wedge \theta^3$. □

To summarize, the key variables are defined by:

$$z = \Omega^{-1}, \qquad W^{-1} = g_{ab}\xi^a\xi^b, \qquad e^{u/2}(\mathrm{d}x + \mathrm{i}\mathrm{d}y) = 2\xi \lrcorner (\ell \wedge m). \tag{2.18}$$

2.3.1 The monopole equation

We now derive a few identities that will be useful for the proof of some results below. The Hermitian expression of the metric is $g = 2g_{\alpha\bar{\beta}}\mathrm{d}z^\alpha \mathrm{d}\bar{z}^\beta$, where $g_{\alpha\bar{\beta}} = g(\partial_\alpha, \partial_{\bar{\beta}})$ (with $\partial_\alpha = \partial/\partial z^\alpha$, $\partial_{\bar{\alpha}} = \partial/\partial\bar{z}^\alpha$), and $z^\alpha = (z^0, z^1)$ are complex holomorphic coordinates. These coordinates can be obtained from the fact that $\mathrm{d}z^\alpha$ must be linear combinations of type-$(1,0)$ forms, which are spanned e.g. by $\theta^0 + \mathrm{i}\theta^1$ and $\theta^2 + \mathrm{i}\theta^3$. Recalling that ∂_ψ is Killing, we have

$$\mathrm{d}z^0 = \tfrac{1}{\sqrt{2}}\left[\mathrm{d}\psi + A + iW\mathrm{d}z + (f+ih)\mathrm{d}x + (-h+\mathrm{i}f)\mathrm{d}y\right], \tag{2.19a}$$

$$\mathrm{d}z^1 = \tfrac{1}{\sqrt{2}}(\mathrm{d}x + \mathrm{i}\mathrm{d}y), \tag{2.19b}$$

for some real functions f, h (which must exist due to integrability of J). Using that (by definition) $J(\mathrm{d}z^1) = \mathrm{i}\mathrm{d}z^1$, and recalling (2.3) and (2.16), we also note the identities

$$J(\mathrm{d}x) = -\mathrm{d}y, \qquad J(\mathrm{d}y) = \mathrm{d}x, \qquad J(\mathrm{d}z) = \xi. \tag{2.20}$$

The vector fields ∂_α can be computed using $\mathrm{d}z^\alpha(\partial_\beta) = \delta^\alpha_\beta$, $\mathrm{d}\bar{z}^\alpha(\partial_\beta) = 0$. Tedious calculations then give

$$\partial_{z^0} = \tfrac{1}{\sqrt{2}}\left[(1 + \tfrac{\mathrm{i}A_z}{W})\partial_\psi - \tfrac{\mathrm{i}}{W}\partial_z\right], \tag{2.21a}$$

$$\partial_{z^1} = \tfrac{1}{\sqrt{2}}\left[-\left[(1+\tfrac{\mathrm{i}A_z}{W})(f+\mathrm{i}h) + (A_x - \mathrm{i}A_y)\right]\partial_\psi + \mathrm{i}\tfrac{(f+\mathrm{i}h)}{W}\partial_z + \partial_x - \mathrm{i}\partial_y\right], \tag{2.21b}$$

where we decomposed $A \equiv A_x\mathrm{d}x + A_y\mathrm{d}y + A_z\mathrm{d}z$. We then find the metric coefficients $g_{\alpha\bar{\beta}}$ to be

$$g_{0\bar{0}} = \frac{1}{W}, \qquad g_{0\bar{1}} = -\frac{(f-\mathrm{i}h)}{W}, \qquad g_{1\bar{0}} = -\frac{(f+\mathrm{i}h)}{W}, \qquad g_{1\bar{1}} = \frac{f^2+h^2}{W} + We^u. \tag{2.22}$$

We deduce from here that

$$e^u = g_{0\bar{0}}g_{1\bar{1}} - g_{0\bar{1}}g_{1\bar{0}} = \det(g_{\alpha\bar{\beta}}). \tag{2.23}$$

The condition $\mathrm{d}^2 z^\alpha = 0$ gives

$$f_x = h_y, \tag{2.24a}$$

$$f_z = W_y = \partial_x A_z - \partial_z A_x, \tag{2.24b}$$

$$h_z = W_x = -\partial_y A_z + \partial_z A_y, \tag{2.24c}$$

$$h_x + f_y = \partial_x A_y - \partial_y A_x = -z^2\partial_z(\tfrac{We^u}{z^2}), \tag{2.24d}$$

where the last equality in (2.24d) follows from the conformal Kähler condition $\mathrm{d}(z^{-2}\kappa) = 0$ (using that κ is given by (2.14)). Noting that the last three equations give an expression for $\mathrm{d}A$ in terms of derivatives of W, the integrability condition $\mathrm{d}^2 A = 0$ then leads to

$$W_{xx} + W_{yy} + \partial_z\left[z^2\partial_z(\tfrac{We^u}{z^2})\right] = 0. \tag{2.25}$$

All of the above identities are valid for a generic Hermitian metric g of the form (2.13). In particular this also applies to the Kähler metric $\hat{g} = z^{-2}g$ and the (closed) Kähler form $\hat{\kappa} = z^{-2}\kappa$, which can be written as

$$\hat{g} = \hat{W}^{-1}(\mathrm{d}\psi + A)^2 + \hat{W}(\mathrm{d}\hat{z}^2 + e^{\hat{u}}(\mathrm{d}x^2 + \mathrm{d}y^2)), \tag{2.26}$$

$$\hat{\kappa} = (\mathrm{d}\psi + A)\wedge\mathrm{d}\hat{z} + \hat{W}e^{\hat{u}}\mathrm{d}x\wedge\mathrm{d}y, \tag{2.27}$$

where we defined

$$\hat{W} = z^2 W, \qquad \hat{z} = -\frac{1}{z}, \qquad e^{\hat{u}} = \frac{e^u}{z^4}. \tag{2.28}$$

The expression for $\mathrm{d}A$ obtained before can now be written as

$$\mathrm{d}A = (\hat{W}e^{\hat{u}})_{\hat{z}} \mathrm{d}x \wedge \mathrm{d}y + \hat{W}_x \mathrm{d}y \wedge \mathrm{d}\hat{z} + \hat{W}_y \mathrm{d}\hat{z} \wedge \mathrm{d}x, \tag{2.29}$$

whereas eq. (2.25) becomes a monopole equation:

$$\hat{W}_{xx} + \hat{W}_{yy} + (\hat{W}e^{\hat{u}})_{\hat{z}\hat{z}} = 0. \tag{2.30}$$

Remark 2.6. *The above identities relate the three unknowns u, W, A. If we know u, then we solve* (2.30) *to find W, and then we find A by integrating* (2.29) *(or* (2.24b),(2.24c),(2.24d)*). To find an equation for u, we must impose field equations, since so far the only assumption is the conformal Kähler condition with symmetry.*

2.4 Curvature

2.4.1 The Ricci scalar and the $SU(\infty)$ Toda equation

Proposition 2.7. *Consider a Kähler 4-manifold $(M, \hat{g}_{ab}, \hat{\kappa}_{ab})$, where the metric and Kähler form can be written as in* (2.26)-(2.27)*, and ∂_ψ is a Killing field. Then the Ricci scalar of $\hat{g}_{ab}$ is*

$$\hat{R} = -\frac{1}{\hat{W}e^{\hat{u}}}\left[\hat{u}_{xx} + \hat{u}_{yy} + (e^{\hat{u}})_{\hat{z}\hat{z}}\right]. \tag{2.31}$$

Proof. We use a well-known formula for the Ricci scalar of any Kähler metric:

$$\hat{R} = -2\hat{g}^{\alpha\bar{\beta}}\partial_\alpha \partial_{\bar{\beta}} \log \hat{\Delta}, \tag{2.32}$$

where $\hat{\Delta} := \det \hat{g}_{\alpha\bar{\beta}}$, and $\hat{g}^{0\bar{0}} = \hat{\Delta}^{-1}\hat{g}_{1\bar{1}}$, $\hat{g}^{0\bar{1}} = -\hat{\Delta}^{-1}\hat{g}_{1\bar{0}}$, $\hat{g}^{1\bar{0}} = -\hat{\Delta}^{-1}\hat{g}_{0\bar{1}}$, $\hat{g}^{1\bar{1}} = \hat{\Delta}^{-1}\hat{g}_{0\bar{0}}$. From the hatted version of (2.23) we see that $\hat{\Delta} = e^{\hat{u}}$, and using also (2.22) we find

$$\hat{R} = -\frac{2}{e^{\hat{u}}}\left[(f^2 + h^2 + \hat{W}^2 e^{\hat{u}})\partial_0\partial_{\bar{0}} + f(\partial_0\partial_{\bar{1}} + \partial_1\partial_{\bar{0}}) + \mathrm{i}h(\partial_0\partial_{\bar{1}} - \partial_1\partial_{\bar{0}}) + \partial_1\partial_{\bar{1}}\right]\hat{u}.$$

A lengthy and tedious computation of $\partial_0\partial_{\bar{0}}\hat{u}$, etc. (using (2.21) and recalling $\partial_\psi \hat{u} = 0$) gives

$$\hat{R} = -\frac{1}{\hat{W}e^{\hat{u}}}\left[\hat{u}_{xx} + \hat{u}_{yy} + e^{\hat{u}}\hat{u}_{\hat{z}\hat{z}} - \frac{1}{\hat{W}}(e^{\hat{u}}\hat{W}_{\hat{z}} + h_x + f_y)\hat{u}_{\hat{z}}\right].$$

Using the hatted version of (2.24d), we see that $h_x + f_y = -\partial_{\hat{z}}(\hat{W}e^{\hat{u}})$, thus (2.31) follows. □

Proposition 2.8. *Let (M, g_{ab}, κ_{ab}) be a conformally Kähler Riemannian 4-manifold, where the metric and fundamental 2-form can be written as in* (2.13)-(2.14)*, and where the conformal factor is $\Omega = z^{-1}$ and ∂_ψ is a Killing field. (In particular, the covector* (2.3) *is not assumed to be Killing.) Then the Ricci scalar of g_{ab} is*

$$R = -\frac{1}{We^u}\left[u_{xx} + u_{yy} + (e^u)_{zz}\right]. \tag{2.33}$$

Proof. For a general metric g_{ab}, if $\hat{g}_{ab} \equiv \Omega^2 g_{ab}$, the Ricci scalars of g_{ab} and $\hat{g}_{ab}$ are related by a standard formula [30, Eq. (D.9)], which can be written as

$$R = \Omega^2(\hat{R} + 6\Omega^{-3}\Box\Omega). \tag{2.34}$$

In our case, we assume $\hat{g}_{ab}$ to be Kähler, and $\Omega = z^{-1}$. $\hat{R}$ was computed in (2.31), so we see that we must compute $z^3\Box z^{-1}$. To do this, we use the general formula $\Box\Phi = \frac{1}{\sqrt{\mathrm{g}}}\partial_a(\sqrt{\mathrm{g}}\, g^{ab}\partial_b\Phi)$ valid for an arbitrary function Φ, with $\sqrt{\mathrm{g}} = \sqrt{\det(g_{ab})}$. In terms of $g_{\alpha\bar{\beta}}$, we have $\det(g_{ab}) = [\det(g_{\alpha\bar{\beta}})]^2$, so $\sqrt{\mathrm{g}} = e^u$, and

$$\begin{aligned}\Box\Phi = e^{-u}[&\partial_0(g_{1\bar{1}}\partial_{\bar{0}} - g_{1\bar{0}}\partial_{\bar{1}})\Phi + \partial_1(g_{0\bar{0}}\partial_{\bar{1}} - g_{0\bar{1}}\partial_{\bar{0}})\Phi \\ &+ \partial_{\bar{0}}(g_{1\bar{1}}\partial_0 - g_{0\bar{1}}\partial_1)\Phi + \partial_{\bar{1}}(g_{0\bar{0}}\partial_1 - g_{1\bar{0}}\partial_0)\Phi].\end{aligned}$$

Using formulas (2.21) and (2.22), and assuming that Φ depends only on z, $\Phi = \Phi(z)$, we get $\Box\Phi = \frac{1}{We^u}\partial_z(e^u\partial_z\Phi)$. Replacing now $\Phi = 1/z$,

$$\Box z^{-1} = \frac{2}{Wz^3}\left(1 - \frac{zu_z}{2}\right). \tag{2.35}$$

Using then (2.34) and (2.31),

$$R = \frac{1}{z^2}\left[-\frac{z^2}{We^u}\left[u_{xx} + u_{yy} + z^2\partial_z\left(z^2\partial_z\left(\frac{e^u}{z^4}\right)\right)\right] - \frac{12}{W}\left(\frac{zu_z}{2} - 1\right)\right]$$

which then gives (2.33), after using the identity $z^2\partial_z(z^2\partial_z(F/z^4)) = F_{zz} - 6F_z/z + 12F/z^2$ valid for any function F. $\Box$

The $SU(\infty)$ Toda equation for a function v is $v_{xx} + v_{yy} + (e^v)_{zz} = 0$. In view of Propositions 2.1, 2.2, 2.7 and 2.8, we see that both the conformal self-duality equations and the Einstein-Maxwell-λ equations with $\lambda = 0$ reduce to the Toda equation, for $\hat{u}$ and u respectively.

2.4.2 The Ricci form

Theorem 2.9. *Let (M, g_{ab}, κ_{ab}) be a conformally Kähler Riemannian 4-manifold whose Ricci tensor is invariant under the complex structure, so that ξ_a given by* (2.3) *is a Killing field and g_{ab} and κ_{ab} have the expressions* (2.13)-(2.14)*. Then the Ricci form $\rho_{ab} = R_{bc}J^c{}_a$ is*

$$\rho = \frac{1}{2}We^uR\,\mathrm{d}x\wedge\mathrm{d}y - \frac{W}{z^2}\left[\tilde{*}\mathrm{d} - \xi\wedge\mathrm{d}\right]\left(\frac{W_0}{W}\right), \tag{2.36}$$

where R is the Ricci scalar (2.33)*, we defined*

$$W_0 := z\left(1 - \frac{zu_z}{2}\right) \tag{2.37}$$

and, for an arbitrary function ϕ, the operator $\tilde{*}\mathrm{d}$ is

$$\tilde{*}\mathrm{d}\phi := \phi_x\mathrm{d}y\wedge\mathrm{d}z + \phi_y\mathrm{d}z\wedge\mathrm{d}x + e^u\phi_z\mathrm{d}x\wedge\mathrm{d}y. \tag{2.38}$$

In addition, the trace-free Ricci form can be expressed as

$$\begin{aligned}\rho - \tfrac{R}{4}\kappa = {}& \left[\tfrac{R}{4} - \tfrac{1}{z^2}\partial_z(\tfrac{W_0}{W})\right](-(\mathrm{d}\psi + A)\wedge\mathrm{d}z + We^u\mathrm{d}x\wedge\mathrm{d}y) \\ &+ \frac{W}{z^2}\left[\xi\wedge(\mathrm{d}x\partial_x + \mathrm{d}y\partial_y) - \mathrm{d}z\wedge(\mathrm{d}x\partial_y - \mathrm{d}y\partial_x)\right](\tfrac{W_0}{W}).\end{aligned} \tag{2.39}$$

Remark 2.10. *1. From* (2.36) *we see that Ricci-flatness $\rho_{ab} = 0$ reduces to $\frac{W_0}{W} = \gamma = \text{const.}$ together with the $SU(\infty)$ Toda equation $u_{xx} + u_{yy} + (e^u)_{zz} = 0$, so we recover Tod's result [18]. Comparison to the Schwarzschild case (cf. section 2.6 below) suggests to use the notation $\gamma \equiv -M$, where M is Schwarzschild's mass.*

2. *From* (2.39)*, the Einstein condition $\rho - \frac{R}{4}\kappa = 0$ (i.e. $R_{ab} = \lambda g_{ab}$) is satisfied if and only if u satisfies the modified Toda equation*

$$u_{xx} + u_{yy} + (e^u)_{zz} = -4\lambda W e^u \tag{2.40}$$

and $\frac{W_0}{W}$ is a function of only z satisfying $\frac{1}{z^2}\frac{\mathrm{d}}{\mathrm{d}z}(\frac{W_0}{W}) = \lambda$, whose solution is

$$W \equiv W_\lambda := \frac{W_0}{\frac{\lambda}{3}z^3 + \gamma} = \frac{z\left(1 - \frac{zu_z}{2}\right)}{\frac{\lambda}{3}z^3 + \gamma} \tag{2.41}$$

where γ is an integration constant. This was also obtained by Tod in [18].

3. *In the Einstein-Maxwell-λ case $R = 4\lambda$, in view of* (2.10)*, formula* (2.39) *gives us an explicit expression for the SD part of the Maxwell field.*

Proof of Theorem 2.9. We start by recalling that for any two metrics g_{ab} and $\hat{g}_{ab} = \Omega^2 g_{ab}$, whose Ricci tensors are R_{ab} and $\hat{R}_{ab}$ respectively, the relation between R_{ab} and $\hat{R}_{ab}$ is given by a standard conformal transformation formula (our reference is [30, Eq. (D.8)]), that in four dimensions can be written as

$$R_{ab} = \hat{R}_{ab} - 2\Omega\hat{\nabla}_a\hat{\nabla}_b\Omega^{-1} + 4\Omega^2(\hat{\nabla}_a\Omega^{-1})(\hat{\nabla}_b\Omega^{-1}) - \Sigma\hat{g}_{ab}, \tag{2.42}$$

with

$$\Sigma := \Omega\hat{g}^{ab}\hat{\nabla}_a\hat{\nabla}_b\Omega^{-1} + \Omega^2\hat{g}^{ab}(\hat{\nabla}_a\Omega^{-1})(\hat{\nabla}_b\Omega^{-1}). \tag{2.43}$$

Assuming now that $\hat{g}_{ab}$ is Kähler, we recall from the previous sections that $\hat{\nabla}_a J^b{}_c = 0$, $\hat{\nabla}_a\Omega^{-1} = -\xi_b J^b{}_a$, where ξ_a is defined in (2.2). At this point we are not assuming that ξ_a is Killing. Contracting (2.42) with $J^b{}_c$, we get

$$R_{ab}J^b{}_c = -\hat{\rho}_{ac} - 2\Omega\hat{\nabla}_a\xi_c - 4\Omega^2 J^b{}_a\xi_b\xi_c + \Sigma\hat{\kappa}_{ac}, \tag{2.44}$$

where $\hat{\rho}_{ac} \equiv \hat{R}_{cb}J^b{}_b$ is the Ricci form of $\hat{g}_{ab}$. From (2.5), we know that the Ricci tensor R_{ab} is J-invariant if and only if ξ_a is Killing, $\nabla_{(a}\xi_{b)} = 0$. Assuming this to be the case, $R_{ab}J^b{}_c$ is anti-symmetric and so we can define the Ricci form of g_{ab}, $\rho_{ac} := R_{cb}J^b{}_a = -\rho_{ca}$. Using that the two terms with ξ_a in (2.44) are anti-symmetric in ac (not separately but together), after some manipulations we get the formula

$$\rho = \hat{\rho} - \Sigma\hat{\kappa} + \Omega^{-1}\mathrm{d}(\Omega^2\xi). \tag{2.45}$$

Now, we want to express (2.45) in terms of u, W. For the scalar Σ, defined in (2.43), note that it can be written as $\Sigma = \Omega\hat{\Box}\Omega^{-1} + W^{-1}$, where $\hat{\Box} = \hat{g}^{ab}\hat{\nabla}_a\hat{\nabla}_b$. Alternatively, we have $\Omega\hat{\Box}\Omega^{-1} = -\Omega^{-3}\Box\Omega = -z^3\Box z^{-1}$. We already computed $\Box z^{-1}$ in (2.35), so

$$\Sigma = \frac{zu_z - 1}{W}. \tag{2.46}$$

For the Ricci form $\hat{\rho}$ of the Kähler metric $\hat{g}_{ab}$, we use the well-known formula $\hat{\rho} = -\mathrm{i}\partial\bar{\partial}\log\det(\hat{g}_{\alpha\bar{\beta}})$, where $\partial = \mathrm{d}z^\alpha \wedge \partial_\alpha$ and $\bar{\partial} = \mathrm{d}\bar{z}^\alpha \wedge \partial_{\bar{\alpha}}$ are Dolbeault operators defined by the complex structure.

Recalling from (the hatted version of) formula (2.23) that $\log\det(\hat{g}_{\alpha\bar{\beta}}) = \hat{u}$, we have $\hat{\rho} = -i\partial\bar{\partial}\hat{u}$. In addition, for any smooth function f, we have the identity $-i\partial\bar{\partial}f = \frac{1}{2}\mathrm{d}(J\mathrm{d}f)$. Putting then $f = \hat{u}$ and using identities (2.20), we get

$$\hat{\rho} = \tfrac{1}{2}\left[-(\hat{u}_{xx} + \hat{u}_{yy})\mathrm{d}x \wedge \mathrm{d}y - \hat{u}_{xz}\mathrm{d}z \wedge \mathrm{d}y + \hat{u}_{yz}\mathrm{d}z \wedge \mathrm{d}x + \mathrm{d}\hat{u}_z \wedge \xi + \hat{u}_z \mathrm{d}\xi\right].$$

Replacing this expression, together with (2.46) and (2.27), in equation (2.45):

$$\begin{aligned}\rho = &\left[-\tfrac{1}{2}(\hat{u}_{xx} + \hat{u}_{yy}) - \tfrac{(zu_z-1)}{z^2}e^u\right]\mathrm{d}x \wedge \mathrm{d}y - \tfrac{1}{2}\hat{u}_{xz}\mathrm{d}z \wedge \mathrm{d}y + \tfrac{1}{2}\hat{u}_{yz}\mathrm{d}z \wedge \mathrm{d}x \\ &+ \left[\tfrac{1}{2}\mathrm{d}\hat{u}_z + \tfrac{(zu_z-1)}{z^2}\mathrm{d}z - \tfrac{2}{z^2}\mathrm{d}z\right] \wedge \xi + \left(\tfrac{1}{2}\hat{u}_z + \tfrac{1}{z}\right)\mathrm{d}\xi.\end{aligned}$$

Using the relation (2.28) between $\hat{u}$ and u, after some tedious computations we arrive at the unenlightening expression

$$\begin{aligned}\rho = &-\tfrac{1}{2}\left[u_{xx} + u_{yy} + \tfrac{2(zu_z-1)}{z^2}e^u + \tfrac{2z}{W}(\tfrac{zu_z}{2} - 1)\partial_z(\tfrac{We^u}{z^2})\right]\mathrm{d}x \wedge \mathrm{d}y \\ &+ \tfrac{1}{2}\left[u_{yz} - \tfrac{2}{zW}(\tfrac{zu_z}{2} - 1)W_y\right](\mathrm{d}z \wedge \mathrm{d}x + \mathrm{d}y \wedge \mathrm{d}\xi) \\ &+ \tfrac{1}{2}\left[u_{xz} - \tfrac{2}{zW}(\tfrac{zu_z}{2} - 1)W_x\right](-\mathrm{d}z \wedge \mathrm{d}y + \mathrm{d}x \wedge \mathrm{d}\xi) \\ &+ \tfrac{1}{2}\left[\tfrac{1}{z^2}(z^2u_{zz} + 2zu_z - 2) - \tfrac{2}{zW}(\tfrac{zu_z}{2} - 1)W_z\right]\mathrm{d}z \wedge \xi.\end{aligned}$$

Now, defining W_0 as in (2.37), we have the identities

$$\begin{aligned}u_{yz} - \tfrac{2}{zW}\left(\tfrac{zu_z}{2} - 1\right)W_y &= -\tfrac{2W}{z^2}\partial_y\left(\tfrac{W_0}{W}\right), \\ u_{xz} - \tfrac{2}{zW}\left(\tfrac{zu_z}{2} - 1\right)W_x &= -\tfrac{2W}{z^2}\partial_x\left(\tfrac{W_0}{W}\right), \\ \tfrac{1}{z^2}(z^2u_{zz} + 2zu_z - 2) - \tfrac{2}{zW}(\tfrac{zu_z}{2} - 1)W_z &= -\tfrac{2W}{z^2}\partial_z\left(\tfrac{W_0}{W}\right),\end{aligned}$$

which lead to

$$\begin{aligned}\rho = &-\tfrac{1}{2}\left[u_{xx} + u_{yy} + \tfrac{2e^u}{z^2}(1 - \tfrac{W_0}{W}W_z - W_0u_z)\right]\mathrm{d}x \wedge \mathrm{d}y \\ &- \tfrac{W}{z^2}\mathrm{d}z \wedge (\mathrm{d}x\,\partial_y - \mathrm{d}y\,\partial_x)(\tfrac{W_0}{W}) - \tfrac{W}{z^2}\mathrm{d}(\tfrac{W_0}{W}) \wedge \xi.\end{aligned}$$

Defining the operator $\tilde{*}\mathrm{d}$ as in (2.38), we have

$$\mathrm{d}z \wedge (\mathrm{d}x\,\partial_y - \mathrm{d}y\,\partial_x)(\tfrac{W_0}{W}) = \tilde{*}\mathrm{d}(\tfrac{W_0}{W}) - e^u\partial_z(\tfrac{W_0}{W})\mathrm{d}x \wedge \mathrm{d}y,$$

which then leads to our final formula (2.36). Having shown this, the proof of (2.39) requires only a few more tedious but straightforward computations, so we will omit them. □

2.4.3 The ASD Weyl tensor

From eq. (2.6) we know that the only non-trivial component of the ASD Weyl tensor is Ψ_2. In addition, from eq. (2.8) we also know that Ψ_2 is essentially given by the Ricci scalar $\hat{R}$ of the Kähler metric, which in turn is given by (2.31) in terms of u, W. An alternative expression that can be useful in practice can be given in terms of the function W_0 defined in (2.37): a short calculation gives

$$\Psi_2 = -\frac{1}{z^3}\frac{W_0}{W} + \frac{R}{12}. \tag{2.47}$$

In particular, notice that in the Ricci-flat and Einstein cases, we recover a well-known relation between Ψ_2 and the conformal factor: $\Psi_2 \propto \Omega^3$ (recall $\Omega = z^{-1}$).

2.5 Ambi-Kähler structures

It may happen that a geometry (M, g_{ab}) is conformally Kähler w.r.t. *both* ASD and SD orientations: this is called an ambi-Kähler structure [35]. In this case we have two integrable complex structures $(J_\pm)^a{}_b$ and two Kähler metrics $\hat{g}^\pm_{ab} = \Omega^2_\pm g_{ab}$. As in (2.3), we now have

$$\xi^\pm_b = (J_\pm)^a{}_b \partial_a \Omega^{-1}_\pm. \tag{2.48}$$

If at least one of $\xi^a_\pm$ is a Killing vector, then all of the results of the previous sections apply w.r.t. the corresponding orientation $\pm$. If both $\xi^a_\pm$ are Killing, then we will have two Toda formulations, one for each orientation: the corresponding Toda variables $(u_\pm, W_\pm, z_\pm, x_\pm, y_\pm)$ are the analogue of (2.18),

$$z_\pm = \Omega^{-1}_\pm, \qquad W^{-1}_\pm = g_{ab}\xi^a_\pm \xi^b_\pm, \qquad e^{u_\pm/2}(\mathrm{d}x_\pm + \mathrm{i}\mathrm{d}y_\pm) = 2\xi_\pm \lrcorner (\ell \wedge m^\pm), \tag{2.49}$$

where in the last equality we defined $m^+ \equiv m$, $m^- \equiv \bar{m}$. Analogously to (2.37), we also put $W^\pm_0 = z_\pm(1 - \frac{1}{2} z_\pm \partial_{z_\pm} u_\pm)$.

The Weyl tensor of an ambi-Kähler structure is of Petrov type $D \otimes D$: this means that both Weyl curvature spinors Ψ_{ABCD} and $\tilde{\Psi}_{A'B'C'D'}$ are type D. The only non-trivial components are $\Psi^-_2 \equiv \Psi_2$ and $\Psi^+_2 \equiv \tilde{\Psi}_2$, which can be computed as

$$\Psi^\pm_2 = \Omega^2_\pm \frac{\hat{R}_\pm}{12} = -\frac{1}{z^3_\pm}\frac{W^\pm_0}{W_\pm} + \frac{R}{12}, \tag{2.50}$$

where $\hat{R}_\pm$ are the Ricci scalars of the Kähler metrics $\hat{g}^\pm_{ab}$. (Recall that (2.50) is valid regardless of whether (2.48) are Killing or not.)

2.6 Examples

Flat space. Consider the function $u = u(x, z)$ given by

$$e^u = \frac{z^2}{\cosh^2 x}. \tag{2.51}$$

Replacing in (2.33) and (2.37), we get $R = 0 = W_0$. Using then formulas (2.36) and (2.47), we see that $\rho_{ab} = 0 = \Psi_2$, which means $R_{ab} = 0 = \Psi_{ABCD}$, thus the solution is hyper-Kähler (as it is self-dual and Ricci-flat). The remaining function W is determined by solving (2.25), which then determines the 1-form A by integrating (2.24b)-(2.24c)-(2.24d). Different choices of solutions to (2.25) will give different hyper-Kähler metrics.

The simple case $W = z^{-1}$, $A = \tanh(x)\mathrm{d}y$ gives (locally) flat space. This can be seen by making the coordinate transformation $x = \log\tan(\theta/2)$, $y = -\varphi$, $z = \varrho^2/4$, which brings the metric to the form

$$g = \mathrm{d}\varrho^2 + \frac{\varrho^2}{4}[(\mathrm{d}\psi + \cos\theta \mathrm{d}\varphi)^2 + (\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2)], \tag{2.52}$$

which is Euclidean 4-space expressed in terms of Euler angles (ψ, θ, φ).

Conformally hyper-Kähler. Consider $u = u(z)$ given by

$$e^u = z^4. \tag{2.53}$$

Using (2.28), this gives $e^{\hat{u}} = 1$, so $\hat{u} = 0$. The Kähler metric $\hat{g}$ thus satisfies $\hat{\rho} = 0$ (i.e. $\hat{R}_{ab} = 0$), so it is Ricci-flat and therefore hyper-Kähler. Thus, (2.53) corresponds to the case in which

g is conformally hyper-Kähler (which, in particular, implies that g is self-dual). Alternatively, replacing $\hat{u} = 0$ in (2.26) we see that the Kähler metric adopts a Gibbons-Hawking form, and eq. (2.29) becomes $\mathrm{d}A = *_3\mathrm{d}\hat{W}$ (where $*_3$ is the Hodge star in $\mathbb{R}^3$), which implies that $\hat{g}$ is hyper-Kähler, see [36, Chapter 9].

Spherical symmetry. We now start from a metric Ansatz: we consider a manifold with local real coordinates (τ, r, θ, ϕ) and a Riemannian metric

$$g = f(r)\mathrm{d}\tau^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2), \tag{2.54}$$

where f is an arbitrary smooth function of r. Choose the coframe $\beta^0 = f^{1/2}\mathrm{d}\tau$, $\beta^1 = f^{-1/2}\mathrm{d}r$, $\beta^2 = r\mathrm{d}\theta$, $\beta^3 = r\sin\theta\mathrm{d}\varphi$, define the fundamental 2-forms $\kappa^{\pm} = \beta^0\wedge\beta^1 \mp \beta^2\wedge\beta^3$ (κ^+ is SD and κ^- is ASD), and the associated almost-complex structures $(J_{\pm})^a{}_b = \kappa^{\pm}_{bc}g^{ca}$. The type-$(1,0)$ eigenspaces of $J_{\pm}$ are generated by $\ell = \frac{1}{\sqrt{2}}(\beta^0 + \mathrm{i}\beta^1)$, $m^{\pm} = \frac{1}{\sqrt{2}}(\beta^2 \mp \mathrm{i}\beta^3)$. Putting $a^0_{\pm} = f^{-1/2}$, $b^0_{\pm} = 0$, and $a^1_{\pm} = 0$, $b^1_{\pm} = (r\sin\theta)^{-1}$, we see that the type $(1,0)$-forms $a^{\alpha}_{\pm}\ell + b^{\alpha}_{\pm}m^{\pm}$ ($\alpha = 0,1$) are closed, so both J_+ and J_- are integrable. Furthermore, if $\Omega_{\pm} \equiv r^{-1}$ then it is straightforward to see that $\mathrm{d}[\Omega^2_{\pm}\kappa^{\pm}] = 0$, so the geometry (2.54) is ambi-Kähler. Finally, computing the vector fields (2.48), we get $\xi^a_{\pm}\partial_a = \partial_{\tau}$, which is Killing. Thus, regardless of the form of the arbitrary function $f(r)$, the geometry is conformally Kähler (in fact ambi-Kähler) with a J-invariant Ricci tensor. We then compute the variables (2.49):

$$z_{\pm} = r, \qquad W^{-1}_{\pm} = f, \qquad e^{u_{\pm}} = fr^2\sin^2\theta, \qquad \mathrm{d}x_{\pm} = \frac{\mathrm{d}\theta}{\sin\theta}, \qquad \mathrm{d}y_{\pm} = \mp\mathrm{d}\varphi. \tag{2.55}$$

Using the formulas of previous sections, a short calculation gives

$$R = \frac{2 - (r^2 f)''}{r^2}, \qquad \Psi^{\pm}_2 = \frac{\{2 - r^2[r^2(f/r^2)']'\}}{12r^2}, \qquad \frac{W^{\pm}_0}{W_{\pm}} = -\frac{r^2 f'}{2}, \tag{2.56}$$

where a prime $'$ represents a derivative w.r.t. r.

The Einstein-Maxwell-λ equations are $R = 4\lambda$, which gives

$$f(r) = 1 + \frac{a_1}{r} + \frac{a_2}{r^2} - \frac{\lambda}{3}r^2 \tag{2.57}$$

for arbitrary constants a_1, a_2. The Weyl scalars $\Psi^{\pm}_2$ and the two pieces $F^{\pm}$ of the Maxwell field are:

$$\Psi^{\pm}_2 = -\frac{a_1}{2r^3} - \frac{a_2}{r^4}, \qquad F^{\pm} = (-a_2/4)^{\frac{1\pm 1}{2}}\left(\frac{1}{r^2}\mathrm{d}\tau\wedge\mathrm{d}r \mp \sin\theta\mathrm{d}\theta\wedge\mathrm{d}\varphi\right). \tag{2.58}$$

Setting $a_1 \equiv -2M$, $a_2 \equiv Q^2$, we recognise the Euclidean Reissner-Nördstrom-(A)dS solution.

We can alternatively look for $f(r)$ such that the ansatz (2.54) is conformally self-dual, eq. (2.7). Recall that this is equivalent to $\Psi^-_2 \equiv \Psi_2 = 0$. Since $\Psi^-_2 = \Psi^+_2$, we see that $\Psi_2 = 0$ gives $C_{abcd} \equiv 0$, so the self-dual solution to the ansatz (2.54) is conformally flat. The condition $\Psi^{\pm}_2 = 0$ gives $f(r) = 1 + b_1 r + b_2 r^2$ for arbitrary constants b_1, b_2. The Ricci scalar and trace-free Ricci form are

$$R = -6\left(\frac{b_1}{r} + 2b_2\right), \qquad \rho^{\pm} - \frac{R}{4}\kappa^{\pm} = \frac{2b_1}{r}\left(\mathrm{d}\tau\wedge\mathrm{d}r \pm r^2\sin\theta\mathrm{d}\theta\wedge\mathrm{d}\varphi\right). \tag{2.59}$$

We see that the geometry is Einstein iff $b_1 = 0$, in which case it reduces to Euclidean (anti-)de Sitter space with cosmological constant $-3b_2$ (which is S^4 if $b_2 < 0$).

The Kerr-Newman ansatz. Our final example will allow us to illustrate a trick to solve the Toda equation, that is also useful for more complicated metric ansätze. Consider the metric

$$g = \frac{\Delta}{\Sigma}(\mathrm{d}\tau - a\sin^2\theta\mathrm{d}\varphi)^2 + \frac{\sin^2\theta}{\Sigma}[a\mathrm{d}\tau + (r^2-a^2)\mathrm{d}\varphi]^2 + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma\mathrm{d}\theta^2, \tag{2.60}$$

where a is a real constant, $\Sigma = r^2 - a^2\cos^2\theta$, and $\Delta = \Delta(r)$ is an arbitrary function of r. As in the previous example, we start by choosing a coframe: $\beta^0 = (\frac{\Delta}{\Sigma})^{1/2}(\mathrm{d}\tau - a\sin^2\theta\mathrm{d}\varphi)$, $\beta^1 = (\frac{\Sigma}{\Delta})^{1/2}\mathrm{d}r$, $\beta^2 = \sqrt{\Sigma}\mathrm{d}\theta$, $\beta^3 = \frac{\sin\theta}{\sqrt{\Sigma}}[a\mathrm{d}\tau + (r^2-a^2)\mathrm{d}\varphi]$; we define $\kappa^\pm = \beta^0\wedge\beta^1 \mp \beta^2\wedge\beta^3$ and the almost-complex structures $(J_\pm)^a{}_b = \kappa^\pm_{bc}g^{ca}$. Type-$(1,0)$ forms for $J_\pm$ are spanned by $\ell = \frac{1}{\sqrt{2}}(\beta^0 + \mathrm{i}\beta^1)$ and $m^\pm = \frac{1}{\sqrt{2}}(\beta^2 \mp \mathrm{i}\beta^3)$. Putting $a^0_\pm = \frac{(r^2-a^2)}{\sqrt{\Delta\Sigma}}$, $b^0_\pm = \frac{\pm\mathrm{i}a\sin\theta}{\sqrt{\Sigma}}$, and $a^1_\pm = \frac{\pm\mathrm{i}a}{\sqrt{\Delta\Sigma}}$, $b^1_\pm = \frac{1}{\sin\theta\sqrt{\Sigma}}$, we see that $a^\alpha_\pm\ell + b^\alpha_\pm m^\pm$ $(\alpha = 0,1)$ are closed, so J_+ and J_- are integrable. Defining $\Omega_\pm = (r\mp a\cos\theta)^{-1}$, a short calculation shows that $\mathrm{d}[\Omega^2_\pm\kappa^\pm] = 0$, so the metric (2.60) is ambi-Kähler. The vector fields (2.48) can be computed to be $\xi^a_\pm\partial_a = \partial_\tau$, so they are Killing. Therefore, independently of the form of $\Delta(r)$, the metric (2.60) is conformally Kähler (ambi-Kähler) with a J-invariant Ricci tensor. We can then compute the variables (2.49):

$$z_\pm = r\mp a\cos\theta, \quad W_\pm = \frac{\Sigma}{\Delta + a^2\sin^2\theta}, \quad e^{u_\pm} = \Delta\sin^2\theta, \quad \mathrm{d}x_\pm = \frac{\mp a\mathrm{d}r}{\Delta} + \frac{\mathrm{d}\theta}{\sin\theta}, \quad \mathrm{d}y_\pm = \mp\mathrm{d}\varphi. \tag{2.61}$$

For concreteness let us focus on z_-, x_-, y_-, etc., and let us omit the subscript $-$. The Einstein-Maxwell equations with $\lambda = 0$ reduce to the $SU(\infty)$ Toda equation with symmetry: $u_{xx} + (e^u)_{zz} = 0$. From (2.61) we find the vector fields ∂_x, ∂_z:

$$\partial_x = \tfrac{\sin\theta\Delta}{\Delta + a^2\sin^2\theta}\left(a\sin\theta\partial_r + \partial_\theta\right), \qquad \partial_z = \tfrac{1}{\Delta + a^2\sin^2\theta}\left(\Delta\partial_r - a\sin\theta\partial_\theta\right). \tag{2.62}$$

One can check that $u_{xx} + (e^u)_{zz} = 0$ then becomes a quite complicated differential equation: we do not seem to gain anything in passing from (x,z) to (r,θ). However, we can do the following trick: introduce an auxiliary variable σ by

$$u_x = \sigma_z, \qquad (e^u)_z = -\sigma_x. \tag{2.63}$$

Using (2.62), these equations lead respectively to:

$$a\sin^2\theta\dot{\Delta} + 2\cos\theta\Delta = \Delta\,\partial_r\sigma - a\sin\theta\,\partial_\theta\sigma, \tag{2.64a}$$

$$-\sin\theta\dot{\Delta} + 2a\sin\theta\cos\theta = a\sin\theta\,\partial_r\sigma + \partial_\theta\sigma, \tag{2.64b}$$

where $\dot{\Delta} = \frac{\mathrm{d}\Delta}{\mathrm{d}r}$. Now, from (2.64b) we get an expression for $\partial_r\sigma$, and we replace this in (2.64a). The resulting equation relates $\dot{\Delta}$ and $\partial_\theta\sigma$. We then replace this new expression for $\partial_\theta\sigma$ back in (2.64b). After these manipulations, we get the following system:

$$\partial_\theta\sigma = -\sin\theta\dot{\Delta}, \qquad \partial_r\sigma = 2\cos\theta. \tag{2.65}$$

Using now the identity $\partial_r\partial_\theta\sigma = \partial_\theta\partial_r\sigma$, we immediately get:

$$\ddot{\Delta} = 2 \qquad \Rightarrow \qquad \Delta = r^2 + c_1 r + c_2, \tag{2.66}$$

for some constants c_1, c_2. The metric is automatically Einstein-Maxwell, but to interpret c_1, c_2, we compute the rest of the curvature (recall from (2.10) that the Ricci form is $\rho = 4\Omega^2 F^+$):

$$\Psi^\pm_2 = -\frac{c_1/2}{(r\mp a\cos\theta)^3} - \frac{(a^2+c_2)}{(r\mp a\cos\theta)^3(r\pm a\cos\theta)}, \tag{2.67}$$

$$F^\pm = \frac{[-\frac14(a^2+c_2)]^{\frac{1\pm1}{2}}}{(r\mp a\cos\theta)^2}\left[\mathrm{d}\tau\wedge\mathrm{d}(r\mp a\cos\theta) - \sin\theta\mathrm{d}\varphi\wedge(a\sin\theta\mathrm{d}r \mp (r^2-a^2)\mathrm{d}\theta)\right]. \tag{2.68}$$

Setting $c_1 \equiv -2M$, $a^2 + c_2 \equiv Q^2$, we recognise the Euclidean Kerr-Newman solution.

If, instead of solving the Einstein-Maxwell equations, we focus on the conformal SD equation (2.7), or equivalently $\hat{R} = 0$, one in principle expects that $\hat{R} = 0$ may give a different form for Δ. The condition $\hat{R} = 0$ is again equivalent to the Toda equation $\hat{u}_{xx} + (e^{\hat{u}})_{\hat{z}\hat{z}} = 0$ (where $\hat{u}, \hat{z}$ are defined in (2.28)), so we can solve it by doing the same trick as in (2.63). We now get

$$(r + a\cos\theta)^2\ddot{\Delta} - 6(r + a\cos\theta)\dot{\Delta} + 12\Delta = 2(r + a\cos\theta)^2 - 12a\cos\theta(r + a\cos\theta) - 12a^2\sin\theta. \tag{2.69}$$

Assuming $a \neq 0$, and applying $\partial_\theta^2\partial_r^2$ to the above equation, we are led to $\ddot{\Delta} = 2$, so again we find $\Delta = r^2 + c_3 r + c_4$ for some constants c_3, c_4. Replacing back in (2.69), we find $c_3 = 0$, $c_4 = -a^2$, so $\Delta = r^2 - a^2$. We already computed the curvature of the metric when Δ is a quadratic polynomial: the Ricci scalar vanishes, and $\Psi_2^\pm$ and the Ricci form are (2.67), (2.68) with c_1, c_2 replaced by c_3, c_4 respectively. Since $c_3 = 0$, $c_4 = -a^2$, we see that the rest of the curvature vanishes. Therefore, the self-dual solution of the Kerr-Newman ansatz (2.60) is simply flat space.

3 The Page-Pope class

3.1 Preliminaries

Consider a Riemann surface Σ with a Riemannian metric $g_\Sigma = 2h\,\mathrm{d}\zeta\mathrm{d}\bar{\zeta}$, where $\zeta = \frac{1}{\sqrt{2}}(x + \mathrm{i}y)$ is a holomorphic coordinate and h is a real positive function. Let $\kappa_\Sigma = \mathrm{i}h\,\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}$ be the Kähler form. Since $\mathrm{d}\kappa_\Sigma = 0$, there is, locally, a 1-form A such that $\kappa_\Sigma = \mathrm{d}A$ and a Kähler potential K_Σ with $h = \partial_\zeta\partial_{\bar{\zeta}}K_\Sigma$. We now define a manifold M as the total space of a fibre bundle over Σ with 2-dimensional fibers parametrized by r, ψ, and we introduce a Riemannian metric g on M by

$$g = F(r)\mathrm{d}r^2 + G(r)(\mathrm{d}\psi + A)^2 + H(r)g_\Sigma \tag{3.1}$$

where F, G, H are arbitrary (non-negative) functions of r. Note that, by redefining the coordinate r, the three functions F, G, H can be reduced to two. For the moment we will focus on the form (3.1), but we will later make use of this freedom.

The class of metrics (3.1) includes geometries such as Fubini-Study, Eguchi-Hanson, Taub-NUT, Kähler surfaces of Calabi type (cf. [37, 38]), particular cases of the Bianchi IX class, etc. It is the restriction to four dimensions of the geometries considered by Page and Pope in [20]. In [20], the conditions on the functions F, G, H so that the metric (3.1) is Einstein are determined, and they find that, under the Einstein assumption, the metric is conformal to two different Kähler metrics. We will first show that this ambi-Kähler structure is actually independent of the form of F, G, H, and so it is independent of field equations; then we will use this result to study generalised instantons.

Proposition 3.1. *For any functions F, G, H, the class of metrics* (3.1) *is (locally) ambi-Kähler.*

Proof. Write the metric on the Riemann surface as $g_\Sigma = h(\mathrm{d}x^2 + \mathrm{d}y^2)$, and choose the coframe $\beta^0 = \sqrt{G}(\mathrm{d}\psi + A)$, $\beta^1 = \sqrt{F}\mathrm{d}r$, $\beta^2 = \sqrt{Hh}\,\mathrm{d}x$, $\beta^3 = \sqrt{Hh}\,\mathrm{d}y$. We define the fundamental 2-forms (with opposite orientation)

$$\kappa^\pm = \beta^0 \wedge \beta^1 \mp \beta^2 \wedge \beta^3 = \sqrt{FG}(\mathrm{d}\psi + A) \wedge \mathrm{d}r \mp H\kappa_\Sigma, \tag{3.2}$$

and the associated almost-complex structures $(J_\pm)^a{}_b = \kappa^\pm_{bc}g^{ca}$. Type-(1,0) forms for $J_\pm$ are $\ell = \frac{1}{\sqrt{2}}(\beta^0 + \mathrm{i}\beta^1)$, $m^\pm = \frac{1}{\sqrt{2}}(\beta^2 \mp \mathrm{i}\beta^3)$. Let $a^0_\pm = \frac{1}{\sqrt{G}}$, $b^0_\pm = \frac{\mp\mathrm{i}}{\sqrt{2Hh}}\partial_\zeta K_\Sigma$, $a^1_\pm = 0$, $b^1_\pm = \frac{1}{\sqrt{Hh}}$.

Then a short calculation shows that the type-$(1,0)$ forms $a^\alpha_\pm \ell + b^\alpha_\pm m^\pm$ $(\alpha = 0,1)$ are closed, so J_+ and J_- are both integrable. Furthermore, using (3.2) we find that regardless of the form of F, G, H we always have

$$\mathrm{d}[\Omega^2_\pm \kappa^\pm] = 0, \qquad \Omega^2_\pm \equiv \frac{c_\pm}{H} \exp\left[\pm \int \frac{\sqrt{FG}}{H} \mathrm{d}r\right], \tag{3.3}$$

where $c_\pm$ are arbitrary constants. Thus (3.1) is (locally) ambi-Kähler, independently of the form of F, G, H. □

The vector fields (2.48) are

$$\xi^a_\pm \partial_a = \frac{1}{\sqrt{FG}} \frac{\mathrm{d}(\Omega^{-1}_\pm)}{\mathrm{d}r} \partial_\psi. \tag{3.4}$$

Since ∂_ψ is a Killing vector of (3.1), we see that (3.4) are in general not Killing. Requiring (3.4) to be Killing imposes restrictions on $\Omega_\pm$, which means restrictions on F, G, H.

3.2 Conformally self-dual solutions

Let $\hat{R}_\pm$ be the Ricci scalars of the Kähler metrics $\hat{g}^\pm_{ab}$. Recall that the SD equation $*C = +C$ is equivalent to $\hat{R}_- = 0$, and the ASD equation $*C = -C$ is equivalent to $\hat{R}_+ = 0$. To solve the equation $\hat{R}_\pm = 0$, we use Proposition 2.7.

Defining

$$\hat{F}_\pm := \Omega^2_\pm F, \qquad \hat{G}_\pm := \Omega^2_\pm G, \qquad \hat{H}_\pm := \Omega^2_\pm H \tag{3.5}$$

and using (3.3), we see that $\mathrm{d}\hat{H}_\pm = \pm\sqrt{\hat{F}_\pm \hat{G}_\pm}\mathrm{d}r$. Thus, if we further define

$$\hat{z}_\pm := -\hat{H}_\pm, \qquad \hat{W}_\pm := \hat{G}^{-1}_\pm, \qquad e^{\hat{u}_\pm} := \hat{G}_\pm \hat{H}_\pm h, \tag{3.6}$$

then the Kähler metrics $\hat{g}^\pm = \Omega^2_\pm g$ and Kähler forms $\hat{\kappa}^\pm = \Omega^2_\pm \kappa$ become

$$\hat{g}^\pm = \hat{W}^{-1}_\pm (\mathrm{d}\psi + A)^2 + \hat{W}_\pm [\mathrm{d}\hat{z}^2_\pm + e^{\hat{u}_\pm}(\mathrm{d}x^2 + \mathrm{d}y^2)], \tag{3.7}$$

$$\hat{\kappa}^\pm = \mp [(\mathrm{d}\psi + A) \wedge \mathrm{d}\hat{z}_\pm + \hat{W}_\pm e^{\hat{u}_\pm} \mathrm{d}x \wedge \mathrm{d}y]. \tag{3.8}$$

A straightforward calculation using (2.31) gives

$$\hat{R}_\pm = \frac{1}{\hat{H}_\pm}\left[R_\Sigma + \frac{\mathrm{d}^2}{\mathrm{d}\hat{z}^2_\pm}(\hat{G}_\pm \hat{z}_\pm)\right], \qquad R_\Sigma := -h^{-1}(\partial^2_x + \partial^2_y)\log h. \tag{3.9}$$

The (A)SD equations then reduce to

$$\frac{\mathrm{d}^2}{\mathrm{d}\hat{z}^2_\pm}(\hat{G}_\pm \hat{z}_\pm) = -R_\Sigma. \tag{3.10}$$

The left side is a function of $\hat{z}_\pm$ only, while the right side is a function of (x, y) only. Thus, (3.10) demands R_Σ to be constant. Assuming Σ to be simply connected, this implies that g_Σ is isometric to the standard metric of either the 2-sphere ($R_\Sigma > 0$), the Euclidean 2-plane ($R_\Sigma = 0$), or the hyperbolic plane ($R_\Sigma < 0$). The solution to (3.10) is $\hat{G}_\pm \hat{z}_\pm = -\frac{R_\Sigma}{2}\hat{z}^2_\pm + b_\pm \hat{z}_\pm - a_\pm$, so

$$\hat{G}_\pm = -\frac{R_\Sigma}{2}\hat{z}_\pm + b_\pm - \frac{a_\pm}{\hat{z}_\pm}, \tag{3.11}$$

where $a_\pm, b_\pm$ are arbitrary constants. Recalling $\hat{z}_\pm = -\hat{H}_\pm$, expressing the above equation in terms of $G, H, \Omega_\pm$, and summarising:

Theorem 3.2. *The metric* (3.1) *is a solution to the conformally (A)SD equations* $*C = \mp C$ *(i.e.* $\hat{R}_\pm = 0$*) iff* g_Σ *has constant curvature* R_Σ *and the functions* F, G, H *satisfy*

$$G = \frac{R_\Sigma}{2} H + \frac{b_\pm}{\Omega_\pm^2} + \frac{a_\pm}{\Omega_\pm^4 H}. \tag{3.12}$$

where $a_\pm, b_\pm$ *are arbitrary constants and* $\Omega_\pm$ *are defined in* (3.3).

Remark 3.3 (Classification)**.** *It is worth mentioning a different perspective on the above solutions. From* (3.6)*, the function* $\hat{u}_\pm$ *is "separable" in the sense that* $\hat{u}_\pm = f(x,y) + g(z)$*, where* $f(x,y) = \log h$ *and* $g(z) = \log(\hat{G}_\pm \hat{H}_\pm)$*. Thus, we solved the Toda equation (here* $\hat{R}_\pm = 0$*) when the Toda variable is separable. All separable solutions to the Toda equation were classified by Tod in [39]: the classification is in terms of three constants* k, a, b*, which in our notation are* $k \equiv -\frac{R_\Sigma}{2}$*,* $a \equiv b_\pm$*,* $b \equiv -a_\pm$*.*

Let us see some examples. In all three examples that follow, we take $\Sigma = \mathbb{CP}^1 \cong S^2$ with the round 2-metric, which has $R_\Sigma = 2$.

Fubini-Study. Taking the functions $F = \frac{1}{(1+r^2)^2}$, $G = \frac{r^2}{4(1+r^2)^2}$, $H = \frac{r^2}{4(1+r^2)}$, the ambi-Kähler class (3.1) becomes the Fubini-Study metric on $M = \mathbb{CP}^2$. Putting $c_+ = \frac{1}{4}$, $c_- = 4$ in (3.3), we find $\Omega_+^2 = 1$ and $\Omega_-^2 = H^{-2}$, so the metric is actually Kähler w.r.t. J_+ (and of course conformally Kähler w.r.t. J_-). We have $\hat{G}_+ = G$, $\hat{H}_+ = H$, $\hat{G}_- = \frac{4}{r^2}$, $\hat{H}_- = \frac{4(1+r^2)}{r^2}$. This gives $\hat{z}_\pm = H^{\pm 1}$. Replacing in (3.9), we find $\hat{R}_+ = 24$, $\hat{R}_- = 0$. This is of course consistent with the fact that Fubini-Study is Einstein with cosmological constant equal to 6 and has a self-dual Weyl tensor. Using (3.4), we also note that ξ_+^a vanishes and $\xi_-^a \partial_a \equiv \partial_\psi$ is Killing.

Generalised Eguchi-Hanson. Let $F = \frac{1}{f(r)}$, $G = \frac{r^2 f(r)}{4}$, $H = \frac{r^2}{4}$, where $f(r)$ is an arbitrary function of r. The metric (3.1) is then a "generalised Eguchin-Hanson" space. Setting $c_+ = \frac{1}{4}$, $c_- = 4$ in (3.3), we find $\Omega_+^2 = 1$ and $\Omega_-^2 = H^{-2}$ (so the metric is Kähler w.r.t. J_+). The form of $f(r)$ that makes the space conformally (A)SD can now be easily found by solving the algebraic equation (3.12):

$$*C = \mp C \qquad \Longleftrightarrow \qquad f(r) = 1 + b_\pm \left(\frac{2}{r}\right)^{\pm 2} + a_\pm \left(\frac{2}{r}\right)^{\pm 4}. \tag{3.13}$$

We also note that ξ_+^a vanishes and $\xi_-^a \partial_a \equiv \partial_\psi$ is Killing, so the rest of the curvature can be computed using the results of section 2. The ordinary Eguchi-Hanson instanton corresponds to (3.13) with $*C = -C$, $b_+ = 0$ and $a_+ = -a/16$ (the Ricci tensor then vanishes and the space is hyper-Kähler). The case (3.13) with $*C = -C$ and $b_+ \neq 0$ was studied in [12] in the context of conformal gravity, where the term $4b_+/r^2$ is referred to as the b-mode.

Generalised Taub-NUT. Letting $F = \frac{1}{f(r)}$, $G = 4n^2 f(r)$, $H = r^2 - n^2$, where $f(r)$ is an arbitrary function of r and n is a constant, the metric (3.1) is a "generalised Taub-NUT" space. Putting $c_\pm = 1$ in (3.3), we find $\Omega_\pm^2 = (r \pm n)^{-2}$. The algebraic equation (3.12) now gives:

$$*C = \mp C \qquad \Longleftrightarrow \qquad f(r) = \frac{1}{4n^2}\left[r^2 - n^2 + b_\pm (r \pm n)^2 + a_\pm \frac{(r \pm n)^3}{(r \mp n)}\right]. \tag{3.14}$$

Using (3.4), we get $\xi_\pm^a \partial_a = \frac{1}{2n}\partial_\psi$, so $\xi_+^a = \xi_-^a$ is Killing and we can compute the rest of the curvature using the results of section 2.

3.3 Cosmological Einstein-Maxwell solutions

For concreteness, let us focus on the ASD side κ^-. Introducing new variables (z, W, u) by

$$\mathrm{d}z = \sqrt{FG}\mathrm{d}r, \qquad W = G^{-1}, \qquad e^u = GHh, \tag{3.15}$$

the metric (3.1) and fundamental 2-form κ^- (3.2) adopt the form (2.13)-(2.14). From (3.4), we get $\xi^a_-\partial_a = \frac{\mathrm{d}(\Omega_-^{-1})}{\mathrm{d}z}\partial_\psi$. To apply the construction of section 2.2.2, we need to restrict to the case in which ξ^a_- is Killing. This is true iff $\mathrm{d}(\Omega_-^{-1})/\mathrm{d}z$ is a constant. Given that z and Ω_- are defined up to addition and multiplication by a constant respectively, we can then simply set $\Omega_-^{-1} \equiv z$. From the conformally Kähler condition $\Omega_-^2 + \frac{\mathrm{d}}{\mathrm{d}z}(\Omega_-^2 H) = 0$ it follows that $\frac{\mathrm{d}}{\mathrm{d}z}(\frac{H}{z^2}) = -\frac{1}{z^2}$, so we deduce

$$H = z + kz^2, \tag{3.16}$$

where k is an arbitrary constant. Now, from (3.3) we have $\Omega_+^2 = \frac{c_+c_-}{(H\Omega_-)^2}$. Setting $c_+c_- = k$ for later convenience, we deduce $\Omega_+^{-1} = \frac{(1+kz)}{k}$. This implies $\mathrm{d}(\Omega_+^{-1})/\mathrm{d}z = 1$, so $\xi^a_+ \equiv \xi^a_-$.

The Einstein-Maxwell-λ equations are $R = 4\lambda$, where R is given by (2.33). Using that formula and the definitions (3.15), we find

$$R = \frac{1}{H}\left[R_\Sigma - \frac{\mathrm{d}^2(GH)}{\mathrm{d}z^2}\right] \tag{3.17}$$

where R_Σ was defined in (3.9). It is convenient to have a formulation that is more symmetric in the SD and ASD sides. To this end, introduce ϱ by $z = \frac{1}{\sqrt{k}}(\varrho - n)$, where $n := \frac{1}{2\sqrt{k}}$. Then $H = \varrho^2 - n^2$ and $\Omega_\pm^{-1} = \frac{1}{\sqrt{k}}(\varrho \pm n)$. The equation $R = 4\lambda$ can be easily solved: from (3.17), we find that R_Σ must be constant and

$$kG = \frac{-\frac{\lambda}{3}\varrho^4 + (\frac{R_\Sigma}{2} + 2\lambda n^2)\varrho^2 + \alpha\varrho + \beta}{\varrho^2 - n^2}, \tag{3.18}$$

where α, β are arbitrary constants of integration. The solution then depends on 5 parameters: k (or n), $R_\Sigma, \lambda, \alpha, \beta$. To interpret them, we compute the rest of the curvature.

Recalling formulas (2.50) and (3.9), and using $\hat{G}_\pm \hat{z}_\pm = -\Omega_\pm^4 GH$, we have

$$\Psi_2^\pm = \frac{1}{12(\varrho^2 - n^2)}\left[R_\Sigma - (\varrho \pm n)^2 \tfrac{\mathrm{d}}{\mathrm{d}\varrho}\left((\varrho \pm n)^2 \tfrac{\mathrm{d}}{\mathrm{d}\varrho}\left(\tfrac{kGH}{(\varrho \pm n)^4}\right)\right)\right].$$

Using (3.18), we find

$$\Psi_2^\pm = -\frac{\frac{1}{2}(\alpha \mp (R_\Sigma n + \frac{8}{3}\lambda n^3))}{(\varrho \pm n)^3} - \frac{(\beta - \frac{R_\Sigma}{2}n^2 - \lambda n^4)}{(\varrho \mp n)(\varrho \pm n)^3}. \tag{3.19}$$

The SD piece of the Maxwell field is $F^+ = \frac{z^2}{4}(\rho - \lambda\kappa)$. Recalling (2.39), we get

$$F^+ = -\frac{1}{4k}\frac{(\beta - \frac{R_\Sigma}{2}n^2 - \lambda n^4)}{(\varrho + n)^2}\left[(\mathrm{d}\psi + A) \wedge \tfrac{\mathrm{d}\varrho}{\sqrt{k}} - (\varrho^2 - n^2)h\mathrm{d}x \wedge \mathrm{d}y\right]. \tag{3.20}$$

Formulas (3.19)-(3.20) suggest to define

$$Q := \beta - \tfrac{R_\Sigma}{2}n^2 - \lambda n^4, \qquad \mu := -\tfrac{1}{2}\alpha, \qquad \nu := \tfrac{1}{2}(R_\Sigma n + \tfrac{8}{3}\lambda n^3), \tag{3.21}$$

and to identify Q with "electromagnetic charge", μ with "mass", and ν with a sort of "NUT charge". The geometry is Einstein ($R_{ab} = \lambda g_{ab}$) iff $Q = 0$, and it is self-dual ($\Psi_{ABCD} = 0$) iff $\mu = \nu$ and $Q = 0$. In the latter case, the space is actually quaternionic-Kähler (that is, $R_{ab} = \lambda g_{ab}$ and $\Psi_{ABCD} = 0$). The hyper-Kähler case ($R_{ab} = 0 = \Psi_{ABCD}$) corresponds to $Q = \mu - \nu = \lambda = 0$, and, assuming $\Sigma = \mathbb{CP}^1$ (so $R_\Sigma = 2$), it reduces to the Taub-NUT instanton with NUT charge $\nu = n$.

4 The Plebański-Demiański class

4.1 Preliminaries

The Plebański-Demiański ansatz [21] is the 4-dimensional family of Riemannian metrics given in local real coordinates (τ, ϕ, p, q) by

$$g = \frac{1}{(p-q)^2}\left[-Q\frac{(\mathrm{d}\tau - p^2\mathrm{d}\phi)^2}{(1-p^2q^2)} + P\frac{(\mathrm{d}\phi - q^2\mathrm{d}\tau)^2}{(1-p^2q^2)} + (1-p^2q^2)\left(\frac{\mathrm{d}p^2}{P} - \frac{\mathrm{d}q^2}{Q}\right)\right], \tag{4.1}$$

where P and Q are arbitrary functions of p and q respectively, and we assume $P > 0$, $Q < 0$. The vector fields $\partial_\tau, \partial_\phi$ are Killing. We will first show that regardless of the form of P, Q, the geometry is ambi-Kähler.

Consider the following orthonormal coframe:

$$\begin{aligned} \beta^0 &= \tfrac{1}{(p-q)}\sqrt{\tfrac{-Q}{1-p^2q^2}}(\mathrm{d}\tau - p^2\mathrm{d}\phi), & \beta^1 &= \tfrac{1}{(p-q)}\sqrt{\tfrac{1-p^2q^2}{-Q}}\mathrm{d}q, \\ \beta^2 &= \tfrac{1}{(p-q)}\sqrt{\tfrac{1-p^2q^2}{P}}\mathrm{d}p, & \beta^3 &= \tfrac{1}{(p-q)}\sqrt{\tfrac{P}{1-p^2q^2}}(\mathrm{d}\phi - q^2\mathrm{d}\tau). \end{aligned} \tag{4.2}$$

Defining the 2-forms

$$\kappa^{\pm} = \beta^0 \wedge \beta^1 \mp \beta^2 \wedge \beta^3 = \frac{(\mathrm{d}\tau - p^2\mathrm{d}\phi) \wedge \mathrm{d}q \mp \mathrm{d}p \wedge (\mathrm{d}\phi - q^2\mathrm{d}\tau)}{(p-q)^2} \tag{4.3}$$

(κ^+ is SD and κ^- is SD), the tensor fields $(J_\pm)^a{}_b = \kappa^{\pm}_{bc}g^{ca}$ are almost-complex structures with opposite orientation. The type-$(1,0)$ eigenspace of $J_\pm$ is spanned by $\ell = \frac{1}{\sqrt{2}}(\beta^0 + \mathrm{i}\beta^1)$, $m^\pm = \frac{1}{\sqrt{2}}(\beta^2 \mp \mathrm{i}\beta^3)$. Setting

$$a^0_\pm := \tfrac{(p-q)}{\sqrt{-Q(1-p^2q^2)}}, \quad b^0_\pm := \pm\mathrm{i}\tfrac{(p-q)p^2}{\sqrt{P(1-p^2q^2)}}, \quad a^1_\pm := \mathrm{i}\tfrac{(p-q)q^2}{\sqrt{-Q(1-p^2q^2)}}, \quad b^1_\pm := \mp\tfrac{(p-q)}{\sqrt{P(1-p^2q^2)}}, \tag{4.4}$$

a straightforward calculation shows that the 1-forms $\omega^\alpha_\pm := a^\alpha_\pm\ell + b^\alpha_\pm m^\pm$ (with $\alpha = 0,1$) are

$$\omega^0_\pm = \frac{1}{\sqrt{2}}\left(\mathrm{d}\tau - \mathrm{i}\frac{\mathrm{d}q}{Q} \pm \mathrm{i}\frac{p^2\mathrm{d}p}{P}\right), \qquad \omega^1_\pm = \frac{1}{\sqrt{2}}\left(\mathrm{i}\mathrm{d}\phi + \frac{q^2\mathrm{d}q}{Q} \mp \frac{\mathrm{d}p}{P}\right), \tag{4.5}$$

so $\mathrm{d}\omega^\alpha_\pm = 0$. Since $\omega^\alpha_\pm$ are type-(1,0) forms for $J_\pm$, we see that both almost-complex structures $J_\pm$ are integrable[3]. Additionally, we find

$$\mathrm{d}[\Omega^2_\pm\kappa^\pm] = 0, \qquad \Omega_\pm = \frac{p-q}{1 \pm pq}, \tag{4.6}$$

thus, the class of metrics (4.1) is ambi-Kähler for any functions $P(p), Q(q)$. A computation shows that the vector fields defined by (2.48) are

$$\xi^a_\pm\partial_a = \partial_\tau \mp \partial_\phi, \tag{4.7}$$

so both $\xi^a_\pm$ are Killing. We can then compute the Toda variables (2.49):

$$\begin{aligned} z_\pm &= \frac{1 \pm pq}{p-q}, \qquad W^{-1}_\pm = \frac{(1 \pm q^2)^2P - (1 \pm p^2)^2Q}{(p-q)^2(1-p^2q^2)}, \qquad e^{u_\pm} = \frac{-PQ}{(p-q)^4}, \\ \mathrm{d}x_\pm &= \frac{(1 \pm p^2)}{P}\mathrm{d}p - \frac{(1 \pm q^2)}{Q}\mathrm{d}q, \qquad y_\pm = -\tau \pm \phi. \end{aligned} \tag{4.8}$$

Note that $\partial_{y_\pm}$ are Killing fields.

[3]Note that from (4.5) we can also read off holomorphic coordinates for $J_\pm$, since $\omega^\alpha_\pm \equiv \mathrm{d}z^\alpha_\pm$.

4.2 Conformally self-dual solutions

Theorem 4.1. *The metric* (4.1) *satisfies the conformally self-dual equation* $*C = C$ *if and only if the functions* P *and* Q *are given by*

$$\begin{aligned} P &= a_0 + a_1 p + a_2 p^2 + a_3 p^3 + a_4 p^4, \\ Q &= a_4 + a_3 q + a_2 q^2 + a_1 q^3 + a_0 q^4, \end{aligned} \tag{4.9}$$

where $a_0, ..., a_4$ *are arbitrary constants. The solution is conformally half-flat but generically non-Einstein. Furthermore, the space is:*

(i) Quaternionic-Kähler (i.e. Einstein) iff $a_1 = a_3$*,*

(ii) Hyper-Kähler (i.e. Ricci-flat) iff $a_1 = a_3$ *and* $a_0 = a_4$*,*

(iii) Flat iff $a_1 = a_3 = 0$ *and* $a_0 = a_4$*.*

Remark 4.2. *We stress that the conformally self-dual solution* (4.9) *is different from the self-dual limit of the standard Plebański-Demiański solution, which is a quaternionic-Kähler space corresponding to case* (*i*) *above (see the next subsection). The solution* (4.9) *can be regarded (locally) as a generalisation of the standard Plebański-Demiański space to a self-dual gravitational instanton in conformal gravity.*

Proof of Theorem 4.1. Recall that the SD equation $*C = C$ is equivalent to $\hat{R}_- = 0$, where $\hat{R}_-$ is given by (2.31). For notational convenience, let us denote $x \equiv x_-$, $y \equiv y_-$, $z \equiv z_-$, $u \equiv u_-$. Since ∂_y is Killing, we see that the metric (4.1) is SD if and only if

$$\hat{u}_{xx} + (e^{\hat{u}})_{\hat{z}\hat{z}} = 0, \tag{4.10}$$

where $\hat{u} = u - 4 \log z$ and $\hat{z} = -\frac{1}{z}$. If one writes the Toda equation (4.10) in terms of the variables p, q, P, Q (using (4.8) for the $-$ sign) and tries to solve for P, Q by brute force, the equation becomes too complicated and we were not able to solve it in this way. Instead, in order to solve (4.10) we recall the trick (2.63) that we used to solve the Toda equation in the Kerr-Newman case (section 2.6): we introduce an auxiliary variable $\hat{\sigma}$ by

$$\hat{u}_x = \hat{\sigma}_{\hat{z}}, \qquad (e^{\hat{u}})_{\hat{z}} = -\hat{\sigma}_x. \tag{4.11}$$

The vector fields ∂_x, ∂_z can be computed from (4.8): we find

$$\partial_x = \frac{PQ}{F}\left[(1-p^2)\partial_p + (1-q^2)\partial_q\right], \qquad \partial_z = \frac{(p-q)^2}{F}\left[(1-q^2)P\partial_p + (1-p^2)Q\partial_q\right], \tag{4.12}$$

where $F \equiv (1-p^2)^2 Q - (1-q^2)^2 P$. Noticing that $\partial_{\hat{z}} = z^2 \partial_z$, eqs. (4.11) lead, respectively, to

$$\frac{(1-p^2)Q\dot{P}}{(1-pq)^2} + \frac{(1-q^2)P\dot{Q}}{(1-pq)^2} + \frac{4(p+q)PQ}{(1-pq)^2} = (1-q^2)P\frac{\partial\hat{\sigma}}{\partial p} + (1-p^2)Q\frac{\partial\hat{\sigma}}{\partial q}, \tag{4.13a}$$

$$\frac{(1-q^2)\dot{P}}{(1-pq)^2} + \frac{(1-p^2)\dot{Q}}{(1-pq)^2} + \frac{4q(1-q^2)P}{(1-pq)^3} + \frac{4p(1-p^2)Q}{(1-pq)^3} = (1-p^2)\frac{\partial\hat{\sigma}}{\partial p} + (1-q^2)\frac{\partial\hat{\sigma}}{\partial q}, \tag{4.13b}$$

where $\dot{P} = \frac{dP}{dp}$, $\dot{Q} = \frac{dQ}{dq}$. Now, from (4.13b) we find an expression for $\partial_p \hat{\sigma}$, and we then replace this in (4.13a). When we do this, $\dot{Q}$ disappears from the resulting equation, leaving us with an equation for $\partial_q\hat{\sigma}$ and $\dot{P}$ only. We then replace this new expression for $\partial_q\hat{\sigma}$ in (4.13b), and we end up with an equation for $\partial_p\hat{\sigma}$ and $\dot{Q}$ only. Explicitly, we find:

$$\frac{\partial\hat{\sigma}}{\partial q} = \frac{\dot{P}}{(1-pq)^2} + \frac{4qP}{(1-pq)^3}, \qquad \frac{\partial\hat{\sigma}}{\partial p} = \frac{\dot{Q}}{(1-pq)^2} + \frac{4pQ}{(1-pq)^3}.$$

Using these equations and the identity $\partial_p\partial_q\hat{\sigma} = \partial_q\partial_p\hat{\sigma}$, a short calculation leads to

$$(1-pq)^2\ddot{Q} + 6p(1-pq)\dot{Q} + 12p^2Q = (1-pq)^2\ddot{P} + 6q(1-pq)\dot{P} + 12q^2P. \tag{4.14}$$

Applying ∂_q^2 to this equation, and then ∂_p^2 to the resulting expression, we get

$$q^2\,\dddot{Q} - 2q\,\ddot{Q} + 2\ddot{Q} = p^2\,\dddot{P} - 2p\,\ddot{P} + 2\ddot{P},$$

which can be rewritten as

$$q^3\frac{\mathrm{d}^2}{\mathrm{d}q^2}\left(\frac{\ddot{Q}}{q}\right) = p^3\frac{\mathrm{d}^2}{\mathrm{d}p^2}\left(\frac{\ddot{P}}{p}\right).$$

Since the left side is a function of q only, and the right side is a function of p only, the equation is easy to solve: we find that P, Q must be fourth order polynomials in p and q, respectively. In addition, the fact that P, Q must satisfy (4.14) imposes relations between the coefficients of the polynomials: this then leads to the form (4.9).

It remains to prove the assertion concerning the special limits $(i), (ii), (iii)$. This can be done using formula (B.2) with $b_0 = a_4$, $b_1 = a_3$, $b_2 = a_2$, $b_3 = a_1$, $b_4 = a_0$. We find

$$\frac{W_0^-}{W_-} = z_-^3\left[(a_4 - a_0) + \frac{(a_3 - a_1)}{2}\frac{(p+q)}{(1+pq)}\right]. \tag{4.15}$$

Now we use Theorem 2.9, from where we see that the solution will be Einstein iff $\frac{1}{z_-^2}\partial_{z_-}(\frac{W_0^-}{W_-}) = \frac{R}{4} = \lambda$. Since it is conformally self-dual, the Einstein condition will imply that it is quaternionic-Kähler. From (4.15) we see that this is true iff $a_1 = a_3$. The cosmological constant is $\lambda = 3(a_4 - a_0)$, and the only non-vanishing component of the SD Weyl spinor is

$$\Psi_2^+ = -\frac{a_1}{z_+^3}. \tag{4.16}$$

In addition, the solution will be hyper-Kähler iff $R_{ab} = 0$, which from the above reduces to $a_1 = a_3$ and $a_0 = a_4$. The only non-trivial part of the curvature is now (4.16). Finally, from these considerations and eq. (4.16) we see that the solution will be flat iff $a_1 = a_3 = 0$ and $a_0 = a_4$. □

Note that in the flat limit there are still two parameters left (a_0 and a_2), so we actually get a 2-parameter family of flat metrics, as is expected from the analysis in [21].

4.3 Cosmological Einstein-Maxwell solutions

Although the Plebański-Demiański solution to the system (2.9) is well-known [21], here we re-derive the result as an application of the framework developed in section 2. This illustrates that one actually does not need to solve the full Einstein equations as in [21], but just $R = 4\lambda$. This example also allows us to give a trick to solve the modified Toda equation.

Proposition 4.3. *The metric* (4.1) *satisfies the cosmological Einstein-Maxwell equations* (2.9) *if and only if the functions P and Q are given by*

$$\begin{aligned} P &= a_0 + a_1 p + a_2 p^2 + a_3 p^3 + a_4 p^4, \\ Q &= (a_0 + \tfrac{1}{3}\lambda) + a_1 q + a_2 q^2 + a_3 q^3 + (a_4 - \tfrac{1}{3}\lambda) q^4, \end{aligned} \tag{4.17}$$

where $a_0, ..., a_4$ are arbitrary constants.

Proof. For concreteness, we choose to work with the ASD side, and we denote $x \equiv x_-$, $y \equiv y_-$, $z \equiv z_-$, $u \equiv u_-$, $W \equiv W_-$. Since the metric (4.1) is conformally Kähler with symmetry, from Propositions 2.2 and 2.8 we know that the Einstein-Maxwell-λ equation reduces to

$$u_{xx} + (e^u)_{zz} = -4\lambda W e^u \tag{4.18}$$

(as ∂_y is Kiling). To solve the modified Toda equation (4.18), we use a slight variation of the trick used in (4.11): we introduce two variables σ, T by

$$u_x = \sigma_z + T, \qquad (e^u)_z = -\sigma_x. \tag{4.19}$$

Equation (4.18) becomes $T_x = -4\lambda W e^u$, and, using (4.12), this gives

$$(1-p^2)\partial_p T + (1-q^2)\partial_q T = -4\lambda \frac{(1-p^2q^2)}{(p-q)^2}. \tag{4.20}$$

Equations (4.19) lead to

$$\begin{aligned} \tfrac{(1-p^2)}{(p-q)^2} Q\dot{P} + \tfrac{(1-q^2)}{(p-q)^2} P\dot{Q} + \tfrac{4(p+q)}{(p-q)^2} PQ &= (1-q^2)P\partial_p\sigma + (1-p^2)Q\partial_q\sigma + \tfrac{F}{(p-q)^2} T, \\ \tfrac{(1-q^2)}{(p-q)^2}\dot{P} + \tfrac{(1-p^2)}{(p-q)^2}\dot{Q} - \tfrac{4(1-q^2)}{(p-q)^3} P + \tfrac{4(1-p^2)}{(p-q)^3} Q &= (1-p^2)\partial_p\sigma + (1-q^2)\partial_q\sigma, \end{aligned}$$

where $F = (1-p^2)^2 Q - (1-q^2)^2 P$. Proceeding as in the proof of Theorem 4.1, we now arrive at the system

$$\frac{\partial\sigma}{\partial q} = \frac{\dot{P}}{(p-q)^2} - \frac{4P}{(p-q)^3} - \frac{(1-p^2)}{(p-q)^2} T, \qquad \frac{\partial\sigma}{\partial p} = \frac{\dot{Q}}{(p-q)^2} + \frac{4Q}{(p-q)^3} + \frac{(1-q^2)}{(p-q)^2} T.$$

Using the identity $\partial_p\partial_q\sigma = \partial_q\partial_p\sigma$ and eq. (4.20), we get

$$(p-q)^2\ddot{Q} + 6(p-q)\dot{Q} + 12Q = (p-q)^2\ddot{P} - 6(p-q)\dot{P} + 12P + 4\lambda(1-p^2q^2). \tag{4.21}$$

Applying ∂_q^2 and then ∂_p^2 we are led to

$$\ddddot{Q} - \ddddot{P} = -8\lambda. \tag{4.22}$$

Taking additional derivatives ∂_p and ∂_q, and using that P and Q depend only on p and q respectively, we see that P and Q must be fourth order polynomials, $P = \sum_{i=0}^4 a_i p^i$, $Q = \sum_i^4 b_i q^i$. Replacing back in (4.21), we get $b_0 = a_0 + \frac{1}{3}\lambda$, $b_1 = a_1$, $b_2 = a_2$, $b_3 = a_3$, $b_4 = a_4 - \frac{1}{3}\lambda$, so the result (4.17) follows. $\square$

Using formulas (B.2) and (2.50), we find:

$$\frac{W_0^{\pm}}{W_{\pm}} = \frac{(a_3 \pm a_1)}{2} - (a_0 - a_4 + \frac{1}{3}\lambda)\left(\frac{p+q}{1 \mp pq}\right) + \frac{\lambda}{3}\left(\frac{1 \pm pq}{p-q}\right)^3, \tag{4.23}$$

$$\Psi_2^{\pm} = -\frac{(a_3 \pm a_1)}{2}\left(\frac{p-q}{1 \pm pq}\right)^3 + (a_0 - a_4 + \frac{1}{3}\lambda)\left(\frac{p+q}{1 \mp pq}\right)\left(\frac{p-q}{1 \pm pq}\right)^3. \tag{4.24}$$

From the above formulas we see that the conformally SD limit $\Psi_2^- = 0$ corresponds to $a_3 = a_1$ and $a_0 - a_4 + \frac{\lambda}{3} = 0$, which implies $\frac{W_0^{\pm}}{W_{\pm}} = \frac{\lambda}{3} z_{\pm}^3$. Using then Theorem 2.9, in this limit we get $\rho = \lambda\kappa$, so the space is Einstein. Thus, the conformally SD limit of the standard Plebański-Demiański solution (4.17) is indeed different from the generalisation found in Theorem 4.1.

5 The Chen-Teo class

In this section we show how to apply the framework of section 2 to the Chen-Teo class [22, 23], but we leave the detailed construction of the generalised solutions for future works. Unlike all examples considered so far, the Chen-Teo class is generically *not* ambi-Kähler, but at most conformally Kähler w.r.t. *only one* orientation. (Correspondingly, in general it does not have Lorentzian sections.)

5.1 Toda formulation

Consider the 4-dimensional family of metrics given in local coordinates (τ, ϕ, x_1, x_2) by

$$g = \frac{(F\mathrm{d}\tau + G\mathrm{d}\phi)^2}{(x_1 - x_2)HF} + \frac{kH}{(x_1 - x_2)^3}\left(\frac{\mathrm{d}x_1^2}{X_1} - \frac{\mathrm{d}x_2^2}{X_2} - \frac{X_1X_2}{kF}\mathrm{d}\phi^2\right), \tag{5.1}$$

where k is a constant, $G(x_1, x_2), H(x_1, x_2), X_1(x_1), X_2(x_2)$ are arbitrary functions of their arguments, and

$$F = x_2^2 X_1 - x_1^2 X_2. \tag{5.2}$$

The vector fields $\partial_\tau, \partial_\phi$ are Killing. For a specific choice of the functions G, H, X_1, X_2, the metric (5.1) is the Ricci-flat Chen-Teo geometry, see [23, Eq. (2.1)] [4].

Let c be an arbitrary constant, and define new variables

$$\begin{gathered} W := \frac{k}{c^2}\frac{(x_1 - x_2)H}{F}, \qquad \psi := \frac{\sqrt{k}}{c}\tau, \qquad y := \frac{c}{\sqrt{k}}\phi, \qquad \tilde{G} := \frac{k}{c^2}\frac{G}{F}, \qquad A := \tilde{G}\mathrm{d}y \\ \mathrm{d}x := c\left[\frac{x_1}{X_1}\mathrm{d}x_1 - \frac{x_2}{X_2}\mathrm{d}x_2\right], \qquad \mathrm{d}z := c\frac{(x_2\mathrm{d}x_1 - x_1\mathrm{d}x_2)}{(x_1 - x_2)^2}, \qquad e^u := \frac{-X_1X_2}{(x_1 - x_2)^4}. \end{gathered} \tag{5.3}$$

Then a calculation shows that (5.1) adopts the form (2.13):

$$g = W^{-1}(\mathrm{d}\psi + A)^2 + W[\mathrm{d}z^2 + e^u(\mathrm{d}x^2 + \mathrm{d}y^2)]. \tag{5.4}$$

The Killing fields are now $\partial_\psi, \partial_y$.

Remark 5.1 (The Chen-Teo parameter ν)**.** *From the expression for* $\mathrm{d}z$ *in* (5.3) *we can find* z *by integration: the solution is* $z = \frac{cx_2}{x_2 - x_1} + \nu$*, where* ν *is an arbitrary constant. We are free to choose any relation between* c *and* ν *we want; in particular, setting*

$$c \equiv -(1 + \nu), \tag{5.5}$$

we get $z = \frac{\nu x_1 + x_2}{x_1 - x_2}$*, which, in the Ricci-flat Chen-Teo case, is the (inverse of the) conformal factor that makes the metric Kähler. The parameter* ν *is particularly important in the Ricci-flat case [23]: the Chen-Teo solution is a one-parameter* ($-1 \leq \nu \leq 1$) *family of metrics interpolating between the Plebański-Demiański* ($\nu = 1$) *and Gibbons-Hawking* ($\nu = -1$) *spaces.*

The fact that the metric (5.1) can be written as (5.4) does not imply, of course, that the geometry (5.1) is necessarily conformally Kähler. To investigate this, we choose the coframe $\beta^0 = W^{-1/2}(\mathrm{d}\psi + A)$, $\beta^1 = \sqrt{W}\mathrm{d}z$, $\beta^2 = \sqrt{W}e^{u/2}\mathrm{d}x$, $\beta^3 = \sqrt{W}e^{u/2}\mathrm{d}y$ for (5.4). The 2-form $\kappa = \beta^0 \wedge \beta^1 + \beta^2 \wedge \beta^3$ is equal to (2.14), and defines the almost-complex structure $J^a{}_b = \kappa_{bc}g^{ca}$.

[4] To compare our notation to that of [23], set $X_1 \equiv X$, $X_2 \equiv Y$, $x_1 \equiv x$, $x_2 \equiv y$.

The type-$(1,0)$ eigenspace is spanned by $\ell = \frac{1}{\sqrt{2}}(\beta^0 + \mathrm{i}\beta^1)$, $m = \frac{1}{\sqrt{2}}(\beta^2 + \mathrm{i}\beta^3)$. In particular, the following are type-(1,0) forms:

$$\omega^0 = \mathrm{d}\psi + A + \mathrm{i}W\mathrm{d}z + B(\mathrm{d}x + \mathrm{i}\mathrm{d}y), \qquad \omega^1 = \mathrm{d}x + \mathrm{i}\mathrm{d}y \tag{5.6}$$

where B is an arbitrary complex function. Since $\mathrm{d}\omega^1 = 0$, we see that J will be integrable if $\mathrm{d}\omega^0 = 0$. This gives the Hermitian condition, and, assuming that it holds, the conformally Kähler condition is $\mathrm{d}(z^{-2}\kappa) = 0$. These two conditions lead respectively to:

$$Z_1 := \tilde{G}_z - W_x = 0, \tag{5.7a}$$

$$Z_2 := \tilde{G}_x + z^2\partial_z\left(\tfrac{We^u}{z^2}\right) = 0 \tag{5.7b}$$

(recall (2.24c), (2.24d)). The geometry (5.1) will be conformally Kähler (for the given choice of almost-complex structure (5.6)) iff the conditions (5.7) are satisfied. The vector field (2.3) is the Killing vector ∂_ψ.

Remark 5.2. *To have some intuition about* (5.7)*, we express Z_1 in terms of the original variables:*

$$Z_1 = \frac{k(x_1 - x_2)X_1X_2}{c^2F}\left[\partial_{x_2}\left(\frac{x_1G}{X_1F} + \frac{x_2H}{(x_1-x_2)F}\right) + \partial_{x_1}\left(\frac{x_2G}{X_2F} + \frac{x_1H}{(x_1-x_2)F}\right)\right]. \tag{5.8}$$

Then, for the original Chen-Teo Ricci-flat metric [23], using [17, Eqs. (3.7a)-(3.7b)] we see that indeed $Z_1 = 0$, which justifies our choice of almost-complex structure (5.6) *for the general class* (5.1)*. (In the Ricci-flat case, $Z_2 = 0$ follows form $Z_1 = 0$.)*

Having identified the Toda variables for (5.1), the Ricci-flat Chen-Teo metric [23] can now be obtained as an application of the framework of section 2; we briefly sketch the procedure in what follows. The metric (5.1) will be Ricci-flat iff u satisfies the Toda equation and $W = \gamma W_0$, where γ is a constant and W_0 is given by (2.37). The Toda equation is $u_{xx} + (e^u)_{zz} = 0$. The trick to solve it is the same that we used in previous cases, see e.g. the proof of Theorem 4.1: we introduce an auxiliary variable σ by $u_x = \sigma_z$, $(e^u)_z = -\sigma_x$. Using

$$\partial_x = -\frac{X_1X_2}{cF}(x_1\partial_{x_1} + x_2\partial_{x_2}), \qquad \partial_z = \frac{(x_1-x_2)^2}{cF}(x_2X_1\partial_{x_1} + x_1X_2\partial_{x_2}), \tag{5.9}$$

we deduce

$$\frac{x_1X_2\dot{X}_1}{(x_1-x_2)^2} + \frac{x_2X_1\dot{X}_2}{(x_1-x_2)^2} - \frac{4X_1X_2}{(x_1-x_2)^2} = -x_2X_1\frac{\partial\sigma}{\partial x_1} - x_1X_2\frac{\partial\sigma}{\partial x_2},$$

$$\frac{x_2\dot{X}_1}{(x_1-x_2)^2} + \frac{x_1\dot{X}_2}{(x_1-x_2)^2} - \frac{4x_2X_1}{(x_1-x_2)^3} + \frac{4x_1X_2}{(x_1-x_2)^3} = -x_1\frac{\partial\sigma}{\partial x_1} - x_2\frac{\partial\sigma}{\partial x_2}$$

where $\dot{X}_1 \equiv \frac{\mathrm{d}X_1}{\mathrm{d}x_1}$, $\dot{X}_2 \equiv \frac{\mathrm{d}X_2}{\mathrm{d}x_2}$. This leads to

$$\frac{\partial\sigma}{\partial x_2} = -\frac{\dot{X}_1}{(x_1-x_2)^2} + \frac{4X_1}{(x_1-x_2)^3}, \qquad \frac{\partial\sigma}{\partial x_1} = -\frac{\dot{X}_2}{(x_1-x_2)^2} - \frac{4X_2}{(x_1-x_2)^3}.$$

Using then $\partial_{x_1}\partial_{x_2}\sigma = \partial_{x_2}\partial_{x_1}\sigma$, after some calculations we arrive at

$$(x_1-x_2)^2\ddot{X}_2 + 6(x_1-x_2)\dot{X}_2 + 12X_2 = (x_1-x_2)^2\ddot{X}_1 - 6(x_1-x_2)\dot{X}_2 + 12X_1. \tag{5.10}$$

Applying $\partial^2_{x_1}\partial^2_{x_2}$, we get $\ddddot{X}_1 = \ddddot{X}_2$, which implies that X_1, X_2 are fourth order polynomials, and replacing in (5.10) we see that they must have the same coefficients,

$$X_1 = a_0 + a_1x_1 + a_2x_1^2 + a_3x_1^3 + a_4x_1^4, \qquad X_2 = a_0 + a_1x_2 + a_2x_2^2 + a_3x_2^3 + a_4x_2^4. \tag{5.11}$$

This determines u, which in turn determines W via $W = \gamma z(1 - \frac{z}{2}u_z)$. Using (5.3) we find $H = \frac{c^2}{k}\frac{FW}{(x_1-x_2)}$. Finally, G is found via equations (5.7). This way we recover [23, Eq. (2.1)].

5.2 A conformally self-dual family

Theorem 5.3. *Assume the family of metrics* (5.1) *to be conformally Kähler w.r.t the almost-complex structure* (5.6) *(that is, conditions* (5.7) *are satisfied). Choose the relation* (5.5) *between the parameters c and ν. Then* (5.1) *is conformally self-dual $*C = C$ if and only if the functions X_1, X_2 are given by*

$$\begin{aligned} X_1 &= a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3 + a_4 x_1^4, \\ X_2 &= a_0 \nu^2 - a_1 \nu x_2 + a_2 x_2^2 - \tfrac{a_3}{\nu} x_2^3 + \tfrac{a_4}{\nu^2} x_2^4. \end{aligned} \tag{5.12}$$

Proof. The proof is very similar to the proof of Theorem 4.1. We use that the SD equation $*C = C$ is equivalent to $\hat{R} = 0$, where $\hat{R}$ is given by (2.31): $\hat{u}_{xx} + (e^{\hat{u}})_{\hat{z}\hat{z}} = 0$, with $\hat{u} = u - 4\log z$, $\hat{z} = -1/z$ (we used that ∂_y is Killing). Introducing $\hat{\sigma}$ by $\hat{u}_x = \hat{\sigma}_{\hat{z}}$, $(e^{\hat{u}})_{\hat{z}} = -\hat{\sigma}_x$, and using that the vector fields ∂_x, ∂_z are given by (5.9), we are led to

$$\begin{aligned} \frac{x_1 X_2 \dot{X}_1}{(\nu x_1 + x_2)^2} + \frac{x_2 X_1 \dot{X}_2}{(\nu x_1 + x_2)^2} - \frac{4 X_1 X_2}{(\nu x_1 + x_2)^2} &= -x_2 X_1 \frac{\partial \hat{\sigma}}{\partial x_1} - x_1 X_2 \frac{\partial \hat{\sigma}}{\partial x_2}, \\ \frac{x_2 \dot{X}_1}{(\nu x_1 + x_2)^2} + \frac{x_1 \dot{X}_2}{(\nu x_1 + x_2)^2} - \frac{4\nu x_2 X_1}{(\nu x_1 + x_2)^3} - \frac{4 x_1 X_2}{(\nu x_1 + x_2)^3} &= -x_1 \frac{\partial \hat{\sigma}}{\partial x_1} - x_2 \frac{\partial \hat{\sigma}}{\partial x_2}, \end{aligned}$$

from where we deduce

$$\frac{\partial \hat{\sigma}}{\partial x_2} = -\frac{\dot{X}_1}{(\nu x_1 + x_2)^2} + \frac{4\nu X_1}{(\nu x_1 + x_2)^3}, \qquad \frac{\partial \hat{\sigma}}{\partial x_1} = -\frac{\dot{X}_2}{(\nu x_1 + x_2)^2} + \frac{4 X_1}{(\nu x_1 + x_2)^3}.$$

Using $\partial_{x_1}\partial_{x_2}\hat{\sigma} = \partial_{x_2}\partial_{x_1}\hat{\sigma}$, we find

$$(\nu x_1 + x_2)^2 \ddot{X}_1 - 6(\nu x_1 + x_2)\dot{X}_1 + 12\nu^2 X_1 = (\nu x_1 + x_2)^2 \ddot{X}_2 - 6(\nu x_1 + x_2)\dot{X}_2 + 12 X_2. \tag{5.13}$$

Applying $\partial^2_{x_2}\partial^2_{x_1}$, we get

$$\ddddot{X}_1 = \nu^2 \ddddot{X}_2, \tag{5.14}$$

which implies that X_1 and X_2 are fourth order polynomials in x_1 and x_2 respectively. Replacing in (5.13), we find relations between the coefficients and we get (5.12). □

Remark 5.4. *Similarly to Theorem 4.1, the solution* (5.12) *can be regarded (locally) as a generalisation of the Ricci-flat Chen-Teo metric to a self-dual gravitational instanton in conformal gravity. However, unlike* (4.9)*, the conformally self-dual equation now determines X_1, X_2 in* (5.1) *to be given by* (5.12)*, but it does not determine the other arbitrary functions G, H in* (5.1)*. These are constrained by the conformally Kähler condition* (5.7)*, but this restriction does not determine G, H uniquely. A detailed analysis of this issue is left for future work.*

6 Final comments

We studied generalised gravitational instantons corresponding to conformally Kähler 4-manifolds whose Ricci tensor is invariant under the complex structure. (The latter condition is equivalent to the existence of a Killing vector.) We obtained generic identities for the metric, Ricci scalar and Ricci form, and we used this to show that a class of field equations reduce to the scalar $SU(\infty)$ Toda equation. More precisely, we showed this for the conformally self-dual and cosmological Einstein-Maxwell field equations. (In the latter case, the scalar equation is the modified Toda equation if the cosmological constant is non-zero.) We applied the construction to a large number of examples, and we gave a trick to solve the Toda equation (with an extra symmetry) in the most complicated cases among these.

The reduction in the conformally SD case was already known from the work of LeBrun [19] on scalar-flat Kähler geometry, so the novelty in this sense is the application to the construction of conformally self-dual generalisations of the Page-Pope, Plebański-Demiański, and Chen-Teo metrics, which give new self-dual (generically non-Einstein) gravitational instantons in conformal gravity. In the Page-Pope case (3.1) the solutions can be classified, cf. Theorem 3.2 and Remark 3.3. For the Plebański-Demiański ansatz (4.1), the solution can be found in closed form, cf. Theorem 4.1: it is a 5-parameter non-Einstein metric (and thus different from the self-dual limit of the standard Plebański-Demiański space). For the Chen-Teo class (5.1), we found a family of conformally SD solutions, but the metric cannot be given in closed form: the functions G, H in the Chen-Teo ansatz (although restricted by the conformally Kähler condition (5.7)) remain undetermined. The analysis of this issue is left for future work, together with thermodynamical aspects of the new solutions.

For the cosmological Einstein-Maxwell equations, we showed that the solution for the Page-Pope ansatz is a generalised Taub-NUT geometry that depends on 5 parameters, which can be identified with mass, electromagnetic charge, NUT charge, cosmological constant, and curvature of the Riemann surface over which it is fibered. In the Plebański-Demiański case, we recovered the standard Euclidean version of the cosmological electro-vacuum solution [21]. For the Chen-Teo class, the construction of the corresponding cosmological Einstein-Maxwell solution remains an open problem, but in future works we will apply the framework developed in this paper to achieve this goal. The purely Einstein and purely Einstein-Maxwell cases are independently interesting, and the difficulties in their construction are different. In particular, the purely Einstein case is likely to be relevant for a possible generalisation of the (Ricci-flat) instanton classification of [14] to Einstein metrics and its relation to the compact case [15].

Acknowledgements. I am very grateful to Maciej Dunajski and Paul Tod for very helpful conversations about the topics of this work, and to the Institut Mittag-Leffler in Djursholm, Sweden for hospitality during the conference "Einstein Spaces and Special Geometry" in July 2023. I would also like to thank the Alexander von Humboldt Foundation and the Max Planck Society for financial support.

A Some background

Basic definitions. Let (M, g_{ab}) be a 4-dimensional, orientable Riemannian manifold (signature $(++++)$), and let ∇_a be the Levi-Civita connection of g_{ab}. We say that (M, g_{ab}) is almost-Hermitian if there is an almost-complex structure $J^a{}_b$ which is compatible with g_{ab} (i.e. $J^a{}_c J^c{}_b = -\delta^a_b$ and $g_{cd} J^c{}_a J^d{}_b = g_{ab}$). The tensor field $J^a{}_b$ produces a decomposition $TM = T^+ \oplus T^-$, where $T^\pm$ is the eigenspace with eigenvalue $\pm$i. Similarly, the cotangent bundle splits as $T^*M = T^{*+} \oplus T^{*-}$. Elements of T^{*+} are referred to as type-$(1,0)$ forms, and elements of T^{*-} are type-$(0,1)$ forms. We say that the manifold J is Hermitian if J is integrable, that is, if T^+ is involutive under the Lie bracket. Equivalently, J is integrable iff the differential ideal generated by type-$(1,0)$ forms is closed under exterior differentiation. If the integrability condition is satisfied, there exist complex scalars $z^\alpha = (z^0, z^1)$ (called holomorphic coordinates) such that T^{*+} is spanned by $\mathrm{d}z^0, \mathrm{d}z^1$.

The Hermitian condition is common to the conformal class of g_{ab}. We say that $(M, g_{ab}, J^a{}_b)$ is Kähler if it is Hermitian and the fundamental 2-form $\kappa_{ab} \equiv g_{bc} J^c{}_a$ is closed, $\mathrm{d}\kappa = 0$. Alternatively, the Kähler condition is equivalent to $\nabla_a J^b{}_c = 0$. We say that $(M, g_{ab}, J^a{}_b)$ is conformally Kähler if there is a positive scalar field Ω such that $(M, \hat{g}_{ab}, J^a{}_b)$ is Kähler, where $\hat{g}_{ab} = \Omega^2 g_{ab}$.

The fundamental 2-form κ is always an eigenform of the Hodge star $*$, i.e., it is self-dual (SD) or anti-self-dual (ASD), $*\kappa = \pm\kappa$, and we say that the SD and ASD cases have opposite orientation. We say that (M, g_{ab}) is ambi-Hermitian if there are two integrable almost-complex structures $J^{\pm}$, with opposite orientation, which are compatible with g_{ab}. We say that (M, g_{ab}) is ambi-Kähler if it is ambi-Hermitian and there are two positive scalar fields $\Omega_{\pm}$ such that the metrics $g^{\pm}_{ab} = \Omega^2_{\pm} g_{ab}$ are Kähler.

Frames. An orthonormal coframe is a set of four 1-forms $(\beta^0, \beta^1, \beta^2, \beta^3)$ such that

$$g = \beta^0 \otimes \beta^0 + \beta^1 \otimes \beta^1 + \beta^2 \otimes \beta^2 + \beta^3 \otimes \beta^3. \tag{A.1}$$

A null coframe $(\ell, n, m, \tilde{m})$ can be constructed as $\ell := \frac{1}{\sqrt{2}}(\beta^0 + \mathrm{i}\beta^1)$, $m := \frac{1}{\sqrt{2}}(e^2 + \mathrm{i}e^3)$, $n = \bar{\ell}$, $\tilde{m} = -\bar{m}$, so that the metric is $g = 2(\ell \odot n - m \odot \tilde{m})$. The volume form is $\varepsilon = -\beta^0 \wedge \beta^1 \wedge \beta^2 \wedge \beta^3$ (we follow the conventions of [27, 28]). With this convention, a basis of ASD 2-forms is given by

$$\kappa_1 = \beta^0 \wedge \beta^1 + \beta^2 \wedge \beta^3, \qquad \kappa_2 = \beta^0 \wedge \beta^2 - \beta^1 \wedge \beta^3, \qquad \kappa_3 = \beta^0 \wedge \beta^3 + \beta^1 \wedge \beta^2. \tag{A.2}$$

Raising an index with the inverse metric g^{ab}, we get three almost-Hermitian structures $(J_i)^a{}_b := (\kappa_i)_{bc} g^{ca}$, satisfying the quaternion algebra $J_i J_j = -\delta_{ij} + \epsilon_{ijk} J_k$. The triple (J_1, J_2, J_3) is called an almost-hyper-Hermitian structure. The type-$(1,0)$ forms of J_1 are spanned by ℓ, m. We also note that

$$\kappa_2 + \mathrm{i}\kappa_3 = 2\ell \wedge m. \tag{A.3}$$

Given an arbitrary vector ξ, the triple (J_1, J_2, J_3) can be used to construct a new (non-normalized) orthogonal frame: $(\xi, J_1\xi, J_2\xi, J_3\xi)$. Defining $W^{-1} := g(\xi, \xi)$ and $e_0 := W^{1/2}\xi$, $e_i := J_i e_0$, the set $\theta^{\mathbf{a}} := g(e_{\mathbf{a}}, \cdot)$ $(\mathbf{a} = 0, ..., 3)$ is a new orthonormal coframe.

Spinors. The spin group in four dimensions and Riemannian signature is $SU(2)_L \times SU(2)_R$. Spinors transforming under $SU(2)_L$ (resp. $SU(2)_R$) have unprimed (resp. primed) indices. The spin spaces are equipped with symplectic structures $\epsilon_{AB}, \epsilon_{A'B'}$ (with inverses $\epsilon^{AB}, \epsilon^{A'B'}$), and with an anti-holomorphic involution denoted by $\dagger$, so that the complex conjugates of $o^A, \alpha^{A'}$ are $o^{\dagger A}, \alpha^{\dagger A'}$ respectively. If $\mathbb{S}, \mathbb{S}'$ are the spin bundles, and $\Lambda^2_{\pm}$ the bundles of (anti-)self-dual 2-forms, we have the isomorphisms

$$TM \otimes \mathbb{C} \cong \mathbb{S} \otimes \mathbb{S}', \qquad \Lambda^2_+ \cong \mathbb{S}'^* \odot \mathbb{S}'^*, \qquad \Lambda^2_- \cong \mathbb{S}^* \odot \mathbb{S}^*. \tag{A.4}$$

Locally, the space of almost-Hermitian structures (with a given orientation) is the projective spin bundle, whose fibers are $\mathbb{CP}^1$s. This means that an almost-Hermitian structure is locally represented by a projective spinor field. With our conventions, ASD orientation corresponds to unprimed spinors. The triple $(\kappa_1, \kappa_2, \kappa_3)$ can be defined using a single spinor, say o_A, together with its complex conjugate $o^{\dagger}_A$. Explicitly, choosing the normalization $\epsilon^{AB} o_A o^{\dagger}_B = 1$, we have (recall (A.4))

$$(\kappa_1)_{ab} = 2\mathrm{i} o_{(A} o^{\dagger}_{B)} \epsilon_{A'B'}, \quad (\kappa_2)_{ab} = (o_A o_B + o^{\dagger}_A o^{\dagger}_B)\epsilon_{A'B'}, \quad (\kappa_3)_{ab} = \mathrm{i}(o^{\dagger}_A o^{\dagger}_B - o_A o_B)\epsilon_{A'B'}. \tag{A.5}$$

Choosing also an arbitrary primed spinor $\alpha^{A'}$, with complex conjugate $\alpha^{\dagger A'}$, a null frame can be constructed as

$$\ell^a = o^A \alpha^{A'}, \qquad n^a = o^{\dagger A} \alpha^{\dagger A'}, \qquad m^a = o^A \alpha^{\dagger A'}, \qquad \tilde{m}^a = o^{\dagger A} \alpha^{A'}. \tag{A.6}$$

B Additional details for the Plebański-Demiański ansatz

Consider the variables (4.8), and take P, Q to be given by

$$P = a_0 + a_1 p + a_2 p^2 + a_3 p^3 + a_4 p^4, \qquad Q = b_0 + b_1 q + b_2 q^2 + b_3 q^3 + b_4 q^4, \tag{B.1}$$

where a_i, b_i are arbitrary constants. Then we find:

$$\begin{aligned}\frac{W_0^\pm}{W_\pm} = \frac{(-1)}{2(1 \mp pq)(p-q)^3} &\{2(a_0 - b_0) + (a_1 - b_1)(p+q) + (b_3 \pm a_1)(q^3 \mp pq^4) \\ &+ 2(b_4 - a_0)q^4 + (b_1 \pm a_3)(\mp p^3 + p^4 q) + 2(b_0 - a_4)p^4 + (a_3 - b_3)(p^4 q^3 + p^3 q^4) \\ &+ 2(a_4 - b_4)p^4 q^4 + 2[a_2 - b_2 \pm 2(a_0 - b_0)]pq \pm 2[a_2 - b_2 \mp 2(b_4 - a_0)]pq^3 \\ &\mp 2[b_2 - a_2 \mp 2(a_4 - b_0)]p^3 q + 2[a_2 - b_2 \pm 2(a_4 - b_4)]p^3 q^3 \\ &\mp 3(b_3 \pm b_1)(p^3 q^2 \pm pq^2) + 3(a_3 \pm a_1)(p^2 q \pm p^2 q^3)\}\,.\end{aligned} \tag{B.2}$$

References

[1] G. W. Gibbons, *Gravitational instantons: A survey*, in: Osterwalder, K. (eds) Mathematical Problems in Theoretical Physics. Lecture Notes in Physics, vol 116. Springer, Berlin, Heidelberg (1980)

[2] M. Dunajski and S. A. Hartnoll, *Einstein-Maxwell gravitational instantons and five dimensional solitonic strings*, Class. Quant. Grav. **24** (2007), 1841-1862 [arXiv:hep-th/0610261 [hep-th]].

[3] M. Dunajski, J. Gutowski, W. Sabra and P. Tod, *Cosmological Einstein-Maxwell Instantons and Euclidean Supersymmetry: Anti-Self-Dual Solutions*, Class. Quant. Grav. **28** (2011), 025007 [arXiv:1006.5149 [hep-th]].

[4] J. B. Gutowski and W. A. Sabra, *Gravitational Instantons and Euclidean Supersymmetry*, Phys. Lett. B **693** (2010), 498-502 [arXiv:1007.2421 [hep-th]].

[5] M. Dunajski, J. B. Gutowski, W. A. Sabra and P. Tod, *Cosmological Einstein-Maxwell Instantons and Euclidean Supersymmetry: Beyond Self-Duality*, JHEP **03** (2011), 131 [arXiv:1012.1326 [hep-th]].

[6] A. Strominger, G. T. Horowitz and M. J. Perry, *Instantons in Conformal Gravity*, Nucl. Phys. B **238** (1984), 653-664

[7] A. V. Smilga, *Vacuum structure in quantum gravity*, Nucl. Phys. B **234** (1984), 402-412

[8] M. Dunajski and P. Tod, *Self-Dual Conformal Gravity*, Commun. Math. Phys. **331** (2014), 351-373 [arXiv:1304.7772 [hep-th]].

[9] J. Li, H. S. Liu, H. Lu and Z. L. Wang, *Fermi Surfaces and Analytic Green's Functions from Conformal Gravity*, JHEP **02** (2013), 109 [arXiv:1210.5000 [hep-th]].

[10] H. Lu, Y. Pang, C. N. Pope and J. F. Vazquez-Poritz, *AdS and Lifshitz Black Holes in Conformal and Einstein-Weyl Gravities*, Phys. Rev. D **86** (2012), 044011 [arXiv:1204.1062 [hep-th]].

[11] H. S. Liu and H. Lu, *Charged Rotating AdS Black Hole and Its Thermodynamics in Conformal Gravity*, JHEP **02** (2013), 139 [arXiv:1212.6264 [hep-th]].

[12] C. Corral, G. Giribet and R. Olea, *Self-dual gravitational instantons in conformal gravity: Conserved charges and thermodynamics*, Phys. Rev. D **104** (2021) no.6, 064026 [arXiv:2105.10574 [hep-th]].

[13] P. B. Kronheimer, *The construction of ALE spaces as hyper-Kähler quotients*, J. Diff. Geom. 29, no.3, 665-683 (1989)

[14] O. Biquard and P. Gauduchon, *On Toric Hermitian ALF Gravitational Instantons*, Commun. Math. Phys. **399** (2023) no.1, 389-422 [arXiv:2112.12711 [math.DG]].

[15] C. LeBrun, *On Einstein, Hermitian 4-manifolds*, Journal of Differential Geometry, 90(2), 277-302 (2012).

[16] C. LeBrun, *Bach-Flat Kähler Surfaces*, J. Geom. Analysis 30 (2020) 2491–2514.

[17] S. Aksteiner and L. Andersson, *Gravitational Instantons and special geometry*, [arXiv:2112.11863 [gr-qc]].

[18] P. Tod, *One-sided type-D Ricci-flat metrics*, [arXiv:2003.03234 [gr-qc]].

[19] C. LeBrun, *Explicit self-dual metrics on* $\mathbb{CP}^2\#...\#\mathbb{CP}^2$, J. Differential Geom. 34(1), 223-253, (1991)

[20] D. N. Page and C. N. Pope, *Inhomogeneous Einstein metrics on complex line bundles*, Class. Quant. Grav. **4** (1987), 213-225

[21] J. F. Plebański and M. Demiański, *Rotating, charged, and uniformly accelerating mass in general relativity*, Annals of Physics, 98(1), 98-127 (1976)

[22] Y. Chen and E. Teo, *A New AF gravitational instanton*, Phys. Lett. B **703** (2011), 359-362 [arXiv:1107.0763 [gr-qc]].

[23] Y. Chen and E. Teo, *Five-parameter class of solutions to the vacuum Einstein equations*, Phys. Rev. D **91** (2015) no.12, 124005 [arXiv:1504.01235 [gr-qc]].

[24] A. Derdziński, *Self-dual Kähler manifolds and Einstein manifolds of dimension four*, Compositio Math., 49 (1983), pp. 405-433.

[25] P. Nurowski and J. F. Plebanski, *Nonvacuum twisting type N metrics*, Class. Quant. Grav. **18** (2001), 341-351 [arXiv:gr-qc/0007017 [gr-qc]].

[26] H. S. Liu, H. Lü, C. N. Pope and J. F. Vázquez-Poritz, *Not Conformally-Einstein Metrics in Conformal Gravity*, Class. Quant. Grav. **30** (2013), 165015 [arXiv:1303.5781 [hep-th]].

[27] R. Penrose and W. Rindler, *Spinors and space-time: Volume 1, Two-spinor calculus and relativistic fields*, volume 1. Cambridge University Press, 1984.

[28] R. Penrose and W. Rindler, *Spinors and space-time: Volume 2, Spinor and twistor methods in space-time geometry*, Cambridge University Press, 1986.

[29] M. Dunajski and P. Tod, *Four–Dimensional Metrics Conformal to Kahler*, Math. Proc. Cambridge Phil. Soc. **148** (2010), 485 [arXiv:0901.2261 [math.DG]].

[30] R. M. Wald, *General Relativity*, Chicago, Usa: Univ. Pr. (1984) 491p

[31] E. J. Flaherty, *The nonlinear graviton in interaction with a photon*, Gen. Relat. Gravit. **9**, 961-978 (1978)

[32] C. LeBrun, *The Einstein-Maxwell Equations and Conformally Kähler Geometry*, Commun. Math. Phys. 344, 621–653 (2016)

[33] V. Apostolov and G. Maschler, *Conformally Kähler, Einstein-Maxwell geometry*, J. Eur. Math. Soc. 21 (2019), no. 5, pp. 1319-1360

[34] C. Koca, C. W. Tønnesen-Friedman, *Strongly Hermitian Einstein-Maxwell solutions on ruled surfaces*, Ann. Glob. Anal. Geom. **50**, 29-46 (2016)

[35] V. Apostolov, D. M. J. Calderbank and P. Gauduchon, *Ambitoric geometry I: Einstein metrics and extremal ambikähler structures*, J. Reine Angew. Math. **2016** (2016) no.721, 109-147 [arXiv:1302.6975 [math.DG]].

[36] M. Dunajski, *Solitons, instantons, and twistors* (Oxford graduate texts in mathematics. 19)

[37] V. Apostolov, D. Calderbank and P. Gauduchon, *The Geometry of Weakly Self-dual Kähler Surfaces*, Compositio Mathematica, 135(3), 279-322 (2003).

[38] J. Lucietti, P. Ntokos and S. G. Ovchinnikov, *On the uniqueness of supersymmetric AdS(5) black holes with toric symmetry*, Class. Quant. Grav. **39** (2022) no.24, 245006 [arXiv:2208.00896 [hep-th]].

[39] K. P. Tod, *Scalar-flat Kähler and hyper-Kähler metrics from Painlevé-III*, Classical and Quantum Gravity 12.6 (1995): 1535.