Mode Stability for Gravitational Instantons of Type D

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Abstract

We study Ricci-flat perturbations of gravitational instantons of Petrov type D. Analogously to the Lorentzian case, the Weyl curvature scalars of extreme spin-weight satisfy a Riemannian version of the separable Teukolsky equation. As a step towards rigidity of the type D Kerr and Taub-bolt families of instantons, we prove mode stability, i.e. that the Teukolsky equation admits no solutions compatible with regularity and asymptotic (local) flatness.

1 Introduction

A gravitational instanton is a complete and non-compact Ricci-flat Riemannian four-manifold with quadratic curvature decay. There are a number of families of known examples, such as the Riemannian Kerr instanton, and the Taub–NUT and Taub-bolt instantons. There are some known results about gravitational instantons, a large part of which hold under various symmetry assumptions, such as the existence of a $U(1)$ or $U(1) \times U(1)$ isometry group, see e.g. [1, 2, 7, 11]. In the compact case, we have the Besse conjecture [6], stating that all compact Ricci-flat manifolds have special holonomy. This is a wide-open conjecture; there are no known examples of compact Ricci-flat four-manifolds with generic holonomy, i.e. holonomy group $SO(4)$. This is in contrast to the the non-compact case, since there are examples of gravitational instantons with generic holonomy. However, all known examples still satisfy the weaker requirement of Hermiticity, and it is therefore natural to conjecture that all gravitational instantons are Hermitian. A first step towards such a result is given by proving *rigidity*, i.e. for various known examples of gravitational instantons, showing that there are no other Ricci-flat metrics close to that metric.

It was shown in [16], that perturbations of the Lorentzian Kerr metric whose frequency lies in the upper half plane, and satisfying certain boundary conditions, the perturbations of the Weyl scalars of extreme spin weight vanish identically. This result is known as *mode stability*. Furthermore, mode stability for frequencies on the real axis was shown in [5]. As was shown in [15], see also [4], perturbations of the Lorentzian Kerr metric whose Weyl scalars of extreme spin weight vanish identically must be perturbations within the Kerr family, modulo gauge.

The following two main theorems of this paper show that the Riemannian analog of mode stability holds in the ALF type D case.[1]

Theorem 1. *For Ricci-flat AF perturbations of the Riemannian Kerr metric, the perturbed Weyl scalars* $\dot{\Psi}_0, \dot{\tilde{\Psi}}_0$ *vanish identically.*

[1]By type D, we mean Petrov type D^+D^-, cf. [1, 7]. The only ALF instantons of type D are the Riemannian Kerr and the Taub-bolt metrics.

Theorem 2. *For Ricci-flat ALF perturbations of the Taub-bolt metric, the perturbed Weyl scalars* $\dot{\Psi}_0, \dot{\tilde{\Psi}}_0$ *vanish identically.*

As a consequence, one might conjecture that rigidity in the above mentioned sense holds for these two instantons.

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2 The Newman–Penrose Formalism in Riemannian Signature

The Newman–Penrose formalism [10], commonly used in general relativity, can be adapted to a Riemannian signature (cf. [3, 9]). Let $(l, \bar{l}, m, \overline{m})$ be a tetrad of vector fields with complex coefficients, in which the metric has the form

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{1}$$

When viewed as first order differential operators, we denote the vector fields $l, \bar{l}, m, \overline{m}$ by $D, \Delta, \delta, -\tilde{\delta}$, respectively. With respect to the tetrad $(l, \bar{l}, m, \overline{m})$, the Levi-Civita connection is represented by 24 *spin coefficients*, denoted by Greek letters and defined to be the coefficients in the right hand sides of the equations

$$\frac{1}{2}(\bar{l}^a \nabla_b l_a - m^a \nabla_b \overline{m}_a) = \gamma l_b + \epsilon \bar{l}_b - \alpha m_b + \beta \overline{m}_b, \tag{2}$$
$$\bar{l}^a \nabla_b \overline{m}_a = -\nu l_b - \pi \bar{l}_b + \lambda m_b - \mu \overline{m}_b, \tag{3}$$
$$m^a \nabla_b l_a = \tau l_b + \kappa \bar{l}_b - \rho m_b + \sigma \overline{m}_b, \tag{4}$$
$$\frac{1}{2}(\bar{l}^a \nabla_b l_a + \overline{m}^a \nabla_b m_a) = \tilde{\gamma} l_b + \tilde{\epsilon} \bar{l}_b - \tilde{\beta} m_b + \tilde{\alpha} \overline{m}_b, \tag{5}$$
$$\bar{l}^a \nabla_b m_a = \tilde{\nu} l_b + \tilde{\pi} \bar{l}_b - \tilde{\mu} m_b + \tilde{\lambda} \overline{m}_b, \tag{6}$$
$$\overline{m}^a \nabla_b l_a = -\tilde{\tau} l_b - \tilde{\kappa} \bar{l}_b + \tilde{\sigma} m_b - \tilde{\rho} \overline{m}_b. \tag{7}$$

We also have the Weyl scalars:

$$\begin{aligned} &\Psi_0 = -W(l, m, l, m), \quad \Psi_1 = -W(l, \bar{l}, l, m), \quad \Psi_2 = W(l, m, \bar{l}, \overline{m}), \\ &\Psi_3 = W(l, \bar{l}, \bar{l}, \overline{m}), \quad \Psi_4 = -W(\bar{l}, \overline{m}, \bar{l}, \overline{m}), \end{aligned} \tag{8}$$

$$\begin{aligned} &\tilde{\Psi}_0 = -W(l, \overline{m}, l, \overline{m}), \quad \tilde{\Psi}_1 = -W(l, \bar{l}, l, \overline{m}), \quad \tilde{\Psi}_2 = W(l, \overline{m}, \bar{l}, m), \\ &\tilde{\Psi}_3 = W(l, \bar{l}, \bar{l}, m), \quad \tilde{\Psi}_4 = -W(\bar{l}, m, \bar{l}, m), \end{aligned} \tag{9}$$

where W denotes the Weyl curvature tensor.

From the definitions, one sees immediately that $\overline{\Psi}_k = \Psi_{4-k}$ and $\overline{\tilde{\Psi}}_k = \tilde{\Psi}_{4-k}$, so that the Weyl tensor is determined by the six scalars $\Psi_0, \Psi_1, \Psi_2, \tilde{\Psi}_0, \tilde{\Psi}_1, \tilde{\Psi}_2$, and that Ψ_2 and $\tilde{\Psi}_2$ are real. The Weyl scalars of *extreme spin weight* are defined to be $\Psi_0, \Psi_4, \tilde{\Psi}_0$ and $\tilde{\Psi}_4$.

A tetrad $(l, \bar{l}, m, \overline{m})$ is said to be *principal* if $\Psi_0 = \Psi_1 = \tilde{\Psi}_0 = \tilde{\Psi}_1 = 0$. It can be shown that a Ricci-flat four-manifold admits a principal tetrad if and only if it has type D. A proof of this fact in the Lorentzian case can be found in [12, Chapter 7].

2.1 The Perturbation Equations

When referring to a *perturbation* $\dot{g}$ of a metric g, we are referring to a linear perturbation of g, i.e. a symmetric two-tensor $\dot{g}$. When g is Ricci-flat, we say that $\dot{g}$ is a *Ricci-flat perturbation* if it is Ricci-flat to first order, i.e. if $\dot{g} \in \ker((D\,\mathrm{Ric})_g)$. In general, for a quantity depending on the metric g, we let a dot above the quantity denote its derivative in the direction $\dot{g}$. Then $\dot{g}$ is a Ricci-flat perturbation if and only if $\dot{\mathrm{Ric}} = 0$.

Ricci-flat perturbations of the Lorentzian Kerr metric have been studied extensively, and in [14], Teukolsky derived a well-known equation for the perturbation of the Weyl scalars of extreme spin weight, for such perturbations of the metric. The following theorem gives a Riemannian analog of that perturbation equation.

Theorem 3. *Consider a Ricci-flat perturbation $\dot{g}$ of a Ricci-flat type D metric g. Relative to a principal tetrad, the perturbation $\dot{\Psi}_0$ satisfies the equation*

$$((D - 3\epsilon + \tilde{\epsilon} - \tilde{\rho} - 4\rho)(\Delta - 4\gamma + \mu) - (\delta - \tilde{\alpha} - 3\beta + \tilde{\pi} - 4\tau)(\tilde{\delta} - 4\alpha + \pi) - 3\Psi_2)\dot{\Psi}_0 = 0, \tag{10}$$

and the perturbation $\dot{\tilde{\Psi}}_0$ satisfies the equation

$$((D - 3\tilde{\epsilon} + \epsilon - \rho - 4\tilde{\rho})(\Delta - 4\tilde{\gamma} + \tilde{\mu}) - (\tilde{\delta} - \alpha - 3\tilde{\beta} + \pi - 4\tilde{\tau})(\delta - 4\tilde{\alpha} + \tilde{\pi}) - 3\tilde{\Psi}_2)\dot{\tilde{\Psi}}_0 = 0. \tag{11}$$

Proof. Since we have a principal tetrad, $\Psi_0 = \tilde{\Psi}_0 = \Psi_1 = \tilde{\Psi}_1 = \kappa = \tilde{\kappa} = \sigma = \tilde{\sigma} = 0$. The perturbed versions of (116) and (121) become

$$(\Delta - 4\gamma + \mu)\dot{\Psi}_0 = (\delta - 4\tau - 2\beta)\dot{\Psi}_1 + 3\dot{\sigma}\Psi_2 \tag{12}$$

and

$$(\tilde{\delta} - 4\alpha + \pi)\dot{\Psi}_0 = (D - 4\rho - 2\epsilon)\dot{\Psi}_1 + 3\dot{\kappa}\Psi_2 \tag{13}$$

respectively. Operating on (12) with D and on (13) with δ, subtracting the resulting equations and using the commutator relation (74), we get

$$(D(\Delta - 4\gamma + \mu) - \delta(\tilde{\delta} - 4\alpha + \pi))\dot{\Psi}_0 = ([D, \delta] - 4D\tau - 2D\beta + 4\delta\rho + 2\delta\epsilon)\dot{\Psi}_1 + (3D\dot{\sigma} - 3\delta\dot{\kappa})\Psi_2 \tag{14}$$

$$\begin{aligned} &= (-(\tilde{\alpha} + 3\beta - \tilde{\pi} + 4\tau)D + (3\epsilon - \tilde{\epsilon} + \tilde{\rho} + 4\rho)\delta \\ &\quad - (D(4\tau + 2\beta)) + (\delta(4\rho + 2\epsilon)))\dot{\Psi}_1 + (3D\dot{\sigma} - 3\delta\dot{\kappa})\Psi_2. \end{aligned} \tag{15}$$

We eliminate the first two terms in the first bracket on the right:

$$((D-3\epsilon+\tilde{\epsilon}-\tilde{\rho}-4\rho)(\Delta-4\gamma+\mu)-(\delta-\tilde{\alpha}-3\beta+\tilde{\pi}-4\tau)(\tilde{\delta}-4\alpha+\pi))\dot{\Psi}_0=A_1+A_2, \tag{16}$$

where

$$\begin{aligned}A_1=((-3\epsilon+\tilde{\epsilon}-\tilde{\rho}-4\rho)(-4\tau-2\beta)-(-\tilde{\alpha}-3\beta+\tilde{\pi}-4\tau)(-4\rho-2\epsilon)\\ -(D(4\tau+2\beta))+(\delta(4\rho+2\epsilon)))\end{aligned} \tag{17}$$

and

$$A_2=(3(D-3\epsilon+\tilde{\epsilon}-\tilde{\rho}-4\rho)\dot{\sigma}-3(\delta-\tilde{\alpha}-3\beta+\tilde{\pi}-4\tau)\dot{\kappa})\Psi_2. \tag{18}$$

By using (81), (87) and (112), we see that $A_1=0$. Also, from the fact that our metric has type D, together with (120) and a suitable linear combination of (117) and (122), we have

$$D\Psi_2=3\rho\Psi_2, \qquad \delta\Psi_2=3\tau\Psi_2. \tag{19}$$

Therefore, by the Leibniz rule,

$$\begin{aligned}A_2&=3D\Psi_2-3\delta\Psi_2+(3((D-3\epsilon+\tilde{\epsilon}-\tilde{\rho}-4\rho)\dot{\sigma})-3((\delta-\tilde{\alpha}-3\beta+\tilde{\pi}-4\tau)\dot{\kappa}))\Psi_2 &(20)\\ &=3(((D-3\epsilon+\tilde{\epsilon}-\tilde{\rho}-\rho)\dot{\sigma})-((\delta-\tilde{\alpha}-3\beta+\tilde{\pi}-\tau)\dot{\kappa}))\Psi_2 &(21)\\ &=3\dot{\Psi}_0\Psi_2, &(22)\end{aligned}$$

where we used the linearization of (89) in the last step, showing that (10) holds. The proof of (11) is similar, referring to the tilded equivalents of the NP equations instead. □

3 The Riemannian Kerr Instanton

In Boyer-Lindquist coordinates (t,r,θ,ϕ), the Riemannian Kerr family of metrics is given by the expression

$$g=\frac{\Sigma}{\Delta}\,dr^2+\Sigma\,d\theta^2+\frac{\Delta}{\Sigma}(dt-a\sin^2\theta\,d\phi)^2+\frac{\sin^2\theta}{\Sigma}((r^2-a^2)\,d\phi+a\,dt)^2. \tag{23}$$

Here, $M>0$ and $a\in\mathbb{R}$ are the parameters of the family, $\Delta=\Delta(r)=r^2-2Mr-a^2$ and $\Sigma=r^2-a^2\cos^2\theta$, and the coordinates have the ranges $r>r_+$, $0<\theta<\pi$, where $r_\pm=M\pm\sqrt{M^2+a^2}$ are the roots of Δ. Like its Lorentzian counterpart, this metric is Ricci-flat. Note that Δ has a different meaning than in Section 2, and it will retain this new meaning throughout this section.

Now define new coordinates $(\tilde{t},\tilde{r},\theta,\tilde{\phi})$ by

$$\begin{cases} r &= M+\sqrt{M^2+a^2}\cosh\tilde{r},\\ t &= \frac{1}{\kappa}\tilde{t},\\ \phi &= \tilde{\phi}-\frac{\Omega}{\kappa}\tilde{t},\end{cases} \tag{24}$$

where $\kappa=\frac{\sqrt{M^2+a^2}}{2Mr_+}$ and $\Omega=\frac{a}{2Mr_+}$. Then r is a smooth function of $\tilde{r}^2$, and (23) gives

$$g=\Sigma(d\tilde{r}^2+d\theta^2+(\tilde{r}^2+O(\tilde{r}^4))\,d\tilde{t}^2+(\sin^2\theta+O(\sin^4\theta))\,d\tilde{\phi}^2). \tag{25}$$

Letting $(\tilde{r},\tilde{t})$ be polar coordinates on $\mathbb{R}^2$ and letting $(\theta,\tilde{\phi})$ be spherical coordinates on S^2, it follows that g extends to a complete metric on $\mathbb{R}^2\times S^2$, provided that we identify $\tilde{t}$ and $\tilde{\phi}$ with period 2π independently. Note that this is equivalent to performing the identifications $(t,\phi)\sim(t+\frac{2\pi}{\kappa},\phi-\frac{2\pi\Omega}{\kappa})\sim(t,\phi+2\pi)$.

3.1 The Separated Perturbation Equations in Coordinates

We shall be interested in a particular choice of complex null tetrad $(l, \bar{l}, m, \overline{m})$, called the *Carter tetrad*, defined by

$$l = \frac{1}{\sqrt{2\Delta\Sigma}}\left((r^2 - a^2)\frac{\partial}{\partial t} - a\frac{\partial}{\partial\phi}\right) + i\sqrt{\frac{\Delta}{2\Sigma}}\frac{\partial}{\partial r}, \tag{26}$$

$$m = \frac{1}{\sqrt{2\Sigma}}\frac{\partial}{\partial\theta} - \frac{i}{\sqrt{2\Sigma}}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} + a\sin\theta\frac{\partial}{\partial t}\right). \tag{27}$$

Note that $|l|_g = |m|_g = 1$. The spin coefficients for the Carter tetrad are given explicitly in Section A.1. For this tetrad we have

$$\Psi_2 = \frac{M}{(r - a\cos\theta)^3}, \qquad \tilde{\Psi}_2 = \frac{M}{(r + a\cos\theta)^3}, \tag{28}$$

and all other Weyl scalars vanish. In particular, this is a principal tetrad.

We shall now analyze the perturbation equations in the Carter tetrad. The relevant properties of the equations are given in the following four lemmas.

Lemma 1. *For the Carter tetrad, the perturbation equation* (10) *is equivalent to the equation*[2] $\mathbf{L}\Phi = 0$*, where* $\Phi = \Psi_2^{-2/3}\dot{\Psi}_0$ *and*

$$\mathbf{L} = \frac{\partial}{\partial r}\Delta\frac{\partial}{\partial r} + \frac{1}{\Delta}\left((r^2 - a^2)\frac{\partial}{\partial t} - a\frac{\partial}{\partial\phi} + 2i(r - M)\right)^2 + 8i(r + a\cos\theta)\frac{\partial}{\partial t}$$
$$+ \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\left(a\sin^2\theta\frac{\partial}{\partial t} + \frac{\partial}{\partial\phi} - 2i\cos\theta\right)^2. \tag{29}$$

Furthermore, if Φ is a solution to this equation coming from a perturbation of the metric, then we can write

$$\Phi(t, r, \theta, \phi) = \sum_{m,\omega,\Lambda} e^{i(m\phi - \omega t)} R_{m,\omega,\Lambda}(r) S_{m,\omega,\Lambda}(\theta), \tag{30}$$

where m runs over $\mathbb{Z}$, ω runs over $\Omega + \kappa\mathbb{Z}$, and for each choice of m, ω, Λ, the function $R = R_{m,\omega,\Lambda}$ solves the equation $\mathbf{R}R = 0$. The function $S = S_{m,\omega,\Lambda}$ is the unique solution to the boundary value problem $\mathbf{S}S = 0$, $S'(0) = S'(\pi) = 0$, where

$$\mathbf{R} = \frac{d}{dr}\Delta\frac{d}{dr} + U(r), \tag{31}$$

$$U(r) = -\frac{((r^2 - a^2)\omega + am + 2(r - M))^2}{\Delta} + 8r\omega - \Lambda \tag{32}$$

and

$$\mathbf{S} = \frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta} + V(\cos\theta), \tag{33}$$

$$V(x) = 8a\omega x - \frac{1}{1 - x^2}\left(a\omega(1 - x^2) - m + 2x\right)^2 + \Lambda. \tag{34}$$

[2] Note that the equation $\mathbf{L}\Phi = 0$ is the same equation as that occuring in the Lorentzian case (see [5, 16]), but with t replaced with it, a replaced with $-ia$, and with $s = -2$.

Here, S is normalized with respect to the L^2 product with measure $\sin\theta\, d\theta$, and the separation constant Λ runs over the (countable set of) values for which such an S exists.

The same statement holds for the perturbation equation (11)*, if Φ is replaced by $\tilde{\Phi}$, where $\tilde{\Phi} = \tilde{\Psi}_2^{-2/3}\dot{\tilde{\Psi}}_0$.*

Proof. The fact that (10) is equivalent to $\mathbf{L}\Phi = 0$ follows from a direct computation, using the expressions for the spin coefficients in Section A.1.

Now note that the boundary value problem $\mathbf{S}S = 0$, $S'(0) = S'(\pi) = 0$ is a Sturm-Liouville problem. Thus, there exists an orthonormal L^2 basis of functions $\{S_{m,\omega,\Lambda}\}_\Lambda$ solving it, and furthermore, we can perform a Fourier series decompositions in the coordinates (t, ϕ). From these considerations, we can write (30), where

$$R_{m,\omega,\Lambda}(r) = \frac{\kappa}{4\pi^2} \int_0^{2\pi/\kappa} \int_0^{2\pi} \int_0^{\pi} e^{-i(m\phi - \omega t)} \Phi(t, r, \theta, \phi) S_{m,\omega,\Lambda}(\theta) \sin\theta \, d\theta \, d\phi \, dt. \tag{35}$$

The fact that $R = R_{m,\omega,\Lambda}$ satisfies $\mathbf{R}R = 0$ now follows directly from (35), along with the fact that $\mathbf{L}\Phi = 0$.

For the statement involving $\tilde{\Phi}$, the proof is entirely analogous. □

Lemma 2. *The equation $\mathbf{R}R = 0$ is an ordinary differential equation in a complex variable r, which has regular singular points at $r = r_\pm$. The point $r = \infty$ is an irregular singular point of rank 1, except when $\omega = 0$, in which case it is a regular singular point. Thus, the equation $\mathbf{R}R = 0$ is a confluent Heun equation (see [13, Section 3]) when $\omega \neq 0$, and a hypergeometric equation (see [13, Section 2]) when $\omega = 0$. The characteristic exponents at $r = r_+$ are*

$$\pm\left(1 + \frac{2Mr_+ + am}{r_+ - r_-}\right), \tag{36}$$

and those at $r = r_-$ are

$$\pm\left(-1 + \frac{2Mr_- + am}{r_+ - r_-}\right). \tag{37}$$

When $\omega = 0$, we have $\Lambda \geq 0$, and the characteristic exponents at $r = \infty$ are

$$-\frac{3}{2} \pm i\sqrt{\frac{7}{2} + \Lambda}. \tag{38}$$

When $\omega \neq 0$, the equation $\mathbf{R}R = 0$ admits normal solutions (see [8, Section 3.2]), near $r = \infty$, of the asymptotic form

$$R \sim e^{\pm r\omega} r^{-1 \pm 2(M\omega - 1)}. \tag{39}$$

Proof. The fact that $r = r_\pm$ are regular singular points follows directly from the fact that $\Delta = (r - r_+)(r - r_-)$, the statement about the type and rank of the singular point at $r = \infty$ follows directly from the discussion in [8, Section 3.1], and the expressions for the characteristic exponents can be seen from the discussion in [13, Section 1.1.3]. Letting $R = y/\sqrt{\Delta}$, the equation $\mathbf{R}R = 0$ is transformed into

$$\frac{d^2y}{dr^2} + qy = 0, \tag{40}$$

where

$$q(r) = \frac{U(r)}{\Delta} + \left(\frac{r_+ - r_-}{2\Delta}\right)^2 = -\omega^2 - \frac{4\omega(M\omega - 1)}{r} + O(r^{-2}). \tag{41}$$

Following [8, Section 3.2], the equation $\mathbf{R}R = 0$ therefore has normal solutions of the asymptotic form

$$R \sim e^{\pm r\omega} r^{-1\pm 2(M\omega-1)}. \tag{42}$$

□

Lemma 3. *For a solution to the equation $\mathbf{R}R = 0$ coming from a (globally smooth) perturbation of the Kerr metric, the corresponding characteristic exponent at $r = r_+$ is*

$$\left|1 + \frac{2Mr_+ + am}{r_+ - r_-}\right|. \tag{43}$$

Proof. Since the set corresponding to $r = r_+$ is compact, and by assumption, the perturbation $\dot{W}$ of the Weyl tensor is continuous, $\dot{W}$ has bounded norm in a neighborhood of this set. Consequently, since l and m have norm 1, it follows that $\dot{\Psi}_0 = -\dot{W}(l, m, l, m)$ is bounded near $r = r_+$. Since $\Psi_2^{-2/3} = O(r^2)$, it follows that Φ, and therefore R, is bounded near $r = r_+$. The statement now follows immediately. □

Lemma 4. *Let R be a solution to the equation $\mathbf{R}R = 0$ coming from an asymptotically flat perturbation of the Kerr metric. When $\omega = 0$, none of the characteristic exponents at $r = \infty$ are compatible with the asymptotic flatness assumption. When $\omega \neq 0$, exactly one of the asymptotic normal solutions is compatible with this assumption, namely*

$$R \sim e^{-r|\omega|} r^{-1-2(M\omega-1)\,\mathrm{sgn}(\omega)}. \tag{44}$$

Proof. The assumption of asymptotic flatness means that $\dot{g} = O(r^{-1})$ as $r \to \infty$, with corresponding decay on derivatives. In particular, $W = O(r^{-3})$, which means that Φ, and therefore R, decays as r^{-1} as $r \to \infty$. In particular, we must have $\lim_{r\to\infty} R(r) = 0$, and the result now follows immediately. □

3.2 Mode Stability

Equipped with the lemmas of the previous subsection, we are now in a position to prove Theorem 1.

Proof of Theorem 1. For $r > r_+$ and $-1 < x < 1$, note that

$$\begin{aligned} U(r) + V(x) = &-\frac{16M(r+ax)}{(r-ax)^2} \\ &- \frac{(a^2x(mx-2) + 2a(x^2-1)(M(r\omega-1)+r) + r(m-2x)(2M-r))^2}{(1-x^2)(\Delta + (1-x^2)a^2)\Delta} \\ &- \frac{(2M(ax+3r) + (r-ax)(r+ax)(-ax\omega + r\omega - 2))^2}{(r-ax)^2(\Delta + (1-x^2)a^2)} \\ < 0. \end{aligned} \tag{45}$$

Here, the strict negativity follows from that of the first term, which holds because $r_+ > |a|$. By an integration of parts, we have

$$U(r) \leq U(r) + \int_0^\pi \left(\frac{dS}{d\theta}\right)^2 \sin\theta\, d\theta = U(r) + \int_0^\pi \left(\mathbf{S}S - \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dS}{d\theta}\right)\right) S\sin\theta\, d\theta \tag{46}$$

$$= U(r) + \int_0^\pi V(\cos\theta) S^2 \sin\theta\, d\theta = \int_0^\pi (U(r) + V(\cos\theta))S^2 \sin\theta\, d\theta < 0, \tag{47}$$

where the last equality follows from (45) and the normalization of S. Multiplying (31) by $\overline{R}$ and integrating, the first term being integrated by parts, we get

$$0 = \left[\Delta \frac{dR}{dr}\overline{R}\right]_{r=r_+}^{r=\infty} - \int_{r_+}^\infty \left(\Delta\left|\frac{dR}{dr}\right|^2 - U|R|^2\right) dr. \tag{48}$$

We claim that the first term vanishes; to see this, we consider the endpoints separately. Near $r = \infty$, we have $\Delta \sim r^2$, while R and its derivative decays exponentially. Thus, the term in square brackets decays exponentially, and in particular it goes to zero as $r \to \infty$. Near $r = r_+$, we know that $\overline{R}$ is bounded. The characteristic exponent corresponding to R is either positive, in which case $\frac{dR}{dr} = o(r^{-1})$, or R is analytic in a neighborhood of $r = r_+$, in which case $\frac{dR}{dr}$ is bounded. In either case, the product $\Delta\frac{dR}{dr}$ goes to zero as $r \to r_+$, and from this it immediately follows that the term in square brackets goes to zero.

We have thus shown that

$$\int_{r_+}^\infty \left(\Delta\left|\frac{dR}{dr}\right|^2 - U|R|^2\right) dr = 0. \tag{49}$$

Since $U < 0$, the terms in the integrand are both non-negative, and must therefore vanish. We conclude that R vanishes identically. $\square$

4 The Taub-Bolt Instanton

The general *Taub–NUT* family of Ricci-flat metrics, depending on two parameters $M, N > 0$, is given in coordinates (t, r, θ, ϕ) by

$$g = \frac{\Sigma}{\Delta} dr^2 + 4N^2 \frac{\Delta}{\Sigma}(dt + \cos\theta\, d\phi)^2 + \Sigma(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{50}$$

where $\Delta = r^2 - 2Mr + N^2$ and $\Sigma = r^2 - N^2$. Setting $M = N$ yields the self-dual Taub–NUT metric, a complete metric on $\mathbb{R}^4$.

Another metric of interest, the *Taub-bolt metric*, arises by letting $M = \frac{5}{4}N$, which we shall do from now. We will now give a brief account of the regularity of this metric. Introducing the coordinate system $(\tilde{t}, \tilde{r}, \tilde{\theta}, \phi)$ by

$$\begin{cases} r &= \frac{N}{4}(5 + 3\cosh\tilde{r}), \\ t &= 2\tilde{t} - \phi, \\ \theta &= 2\arctan(\frac{\tilde{\theta}}{2}), \end{cases} \tag{51}$$

we see that r is smooth as a function of $\tilde{r}^2$, and that

$$g = \Sigma(d\tilde{r}^2 + (\tilde{r}^2 + O(\tilde{r}^4))\, d\tilde{t}^2 + (1 + O(\tilde{\theta}^2))\, d\tilde{\theta}^2 + (\tilde{\theta}^2 + O(\tilde{\theta}^4))\, d\phi^2 + O(\tilde{r}^2\tilde{\theta}^2)\, d\tilde{t}d\phi). \tag{52}$$

Viewing $(\tilde{\theta}, \phi)$ as polar coordinates on $\mathbb{R}^2 \times \{y\} \subseteq \mathbb{R}^4$, and viewing $(\tilde{r}, \tilde{t})$ as polar coordinates on $\{x\} \times \mathbb{R}^2 \subseteq \mathbb{R}^4$, it follows that g extends to a smooth metric on $\mathbb{R}^4 \cong \mathbb{C}^2$, provided that we identify $\tilde{t}$ and ϕ with period 2π independently. This is equivalent to making the identifications $(t, \phi) \sim (t + 4\pi, \phi) \sim (t + 2\pi, \phi + 2\pi)$.

We can also introduce another coordinate system $(\hat{t}, \tilde{r}, \hat{\theta}, \phi)$ by

$$\begin{cases} t &= 2\hat{t} + \phi, \\ \theta &= 2\,\mathrm{arccot}(\frac{\hat{\theta}}{2}), \end{cases} \tag{53}$$

so that

$$g = \Sigma(d\tilde{r}^2 + (\tilde{r}^2 + O(\tilde{r}^4))\, d\hat{t}^2 + (1 + O(\hat{\theta}^2))\, d\hat{\theta}^2 + (\hat{\theta}^2 + O(\hat{\theta}^4))\, d\phi^2 + O(\tilde{r}^2\hat{\theta}^2)\, d\hat{t}d\phi). \tag{54}$$

In the same way as for the previous coordinate system, this shows that g extends to a smooth metric on another copy of $\mathbb{C}^2$. Note that the identifications made to ensure regularity in the coordinate system $(\tilde{t}, \tilde{r}, \tilde{\theta}, \phi)$ also ensure regularity in the coordinate system $(\hat{t}, \tilde{r}, \hat{\theta}, \phi)$. When defined on the union of these copies of $\mathbb{C}^2$, this metric is complete.

Computing the transition map between the two coordinate systems, we see that they are related by $(\hat{t}, \tilde{r}, \hat{\theta}, \phi) = (\tilde{t} - \phi, \tilde{r}, \frac{4}{\tilde{\theta}}, \phi)$. In other words, the two copies of $\mathbb{C}^2$ are glued together according to the map

$$\begin{aligned} (\mathbb{C} \setminus \{0\}) \times \mathbb{C} &\to (\mathbb{C} \setminus \{0\}) \times \mathbb{C}, \\ (z_1, z_2) &\mapsto \left(\frac{4}{z_1}, z_2 \cdot \frac{|z_1|}{z_1}\right). \end{aligned}$$

Topologically, this is the same thing as gluing two such copies along the map $(z_1, z_2) \mapsto (\frac{1}{z_1}, z_2 \cdot \frac{|z_1|}{z_1})$, or equivalently, gluing together two copies of $\overline{D}^2 \times \mathbb{C}$ along the map

$$\begin{aligned} S^1 \times \mathbb{C} &\to S^1 \times \mathbb{C}, \\ (z_1, z_2) &\mapsto \left(\frac{1}{z_1}, \frac{z_2}{z_1}\right). \end{aligned} \tag{55}$$

We now claim that the manifold is diffeomorphic to $\mathbb{C}P^2$ minus a point. To see this, consider two of the projective coordinate charts for $\mathbb{C}P^2$, (U_0, ϕ_0) and (U_1, ϕ_1), where

$$U_i = \{[Z_0 : Z_1 : Z_2] \in \mathbb{C}P^2 \mid Z_i \neq 0\}, \tag{56}$$

and

$$\begin{aligned} \phi_0 : U_0 &\to \mathbb{C}^2, \\ [Z_0 : Z_1 : Z_2] &\mapsto \left(\frac{Z_1}{Z_0}, \frac{Z_2}{Z_0}\right), \\ \phi_1 : U_1 &\to \mathbb{C}^2, \\ [Z_0 : Z_1 : Z_2] &\mapsto \left(\frac{Z_0}{Z_1}, \frac{Z_2}{Z_1}\right). \end{aligned}$$

Since $U_0 \cup U_1 = \mathbb{C}P^2 \setminus \{[0:0:1]\}$, it follows that the latter is topologically equivalent to two copies of $\mathbb{C}^2$, glued together along the transition map

$$(\mathbb{C} \setminus \{0\}) \times \mathbb{C} \to (\mathbb{C} \setminus \{0\}) \times \mathbb{C},$$
$$(z_1, z_2) \mapsto \left(\frac{1}{z_1}, \frac{z_2}{z_1}\right).$$

Again, topologically this is the same thing as gluing together two copies of $\overline{D}^2 \times \mathbb{C}$ along the map (55). This shows that the manifold is *homeomorphic* to $\mathbb{C}P^2$ minus a point. To show that these are *diffeomorphic*, we can replace the closed disk by an open disk of radius slightly larger than 1, gluing the two spaces together along a thin open strip around S^1. The gluing map will then be isotopic to the corresponding transition map in $\mathbb{C}P^2$.

4.1 The Separated Perturbation Equations in Coordinates

As for Kerr, we are interested in a particular choice of complex null tetrad $(l, \bar{l}, m, \overline{m})$, in this case given by

$$l = \frac{1}{\sqrt{2\Sigma}}\left(\frac{1}{\sin\theta}\left(\cos\theta\frac{\partial}{\partial t} - \frac{\partial}{\partial\phi}\right) + i\frac{\partial}{\partial\theta}\right), \tag{57}$$

$$m = \sqrt{\frac{\Delta}{2\Sigma}}\frac{\partial}{\partial r} + \frac{i\sqrt{\Sigma/2\Delta}}{2N}\frac{\partial}{\partial t}, \tag{58}$$

satisfying $|l|_g = |m|_g = 1$.

The spin coefficients for this tetrad are given explicitly in Section A.2. For this tetrad, we have

$$\Psi_2 = \frac{N}{4(r-N)^3}, \qquad \tilde{\Psi}_2 = \frac{9N}{4(r+N)^3}, \tag{59}$$

and the rest of the Weyl scalars vanish. Thus, this is a principal tetrad, and we see that the Taub-bolt metric is of type D.

The following four lemmas give the relevant properties of the perturbation equations (10) (11) for our analysis. The proofs are entirely analogous to those in Section 3.1, and are therefore omitted.

Lemma 5. *For the tetrad given in* (57) *and* (58)*, the perturbation equation* (10) *is equivalent to the equation* $\mathbf{L}\Phi = 0$*, where* $\Phi = \Psi_2^{-2/3}\dot{\Psi}_0$ *and*

$$\mathbf{L} = \frac{\partial}{\partial r}\Delta\frac{\partial}{\partial r} - \frac{4N(r+N)}{(r-N)^2} + \frac{\Sigma^2}{4N^2\Delta}\left(\frac{\partial}{\partial t} - i\frac{N(4r^2 - 11Nr + 3N^2)}{\Sigma(r-N)}\right)^2$$
$$+ \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\left(\cos\theta\frac{\partial}{\partial t} - \frac{\partial}{\partial\phi} - 2i\cos\theta\right)^2. \tag{60}$$

Furthermore, if Φ *is a solution to this equation coming from a perturbation of the metric, then we can write*

$$\Phi(t, r, \theta, \phi) = \sum_{m,\omega,\Lambda} e^{i(m\phi - \omega t)} R_{m,\omega,\Lambda}(r) S_{m,\omega,\Lambda}(\theta), \tag{61}$$

where m runs over $\frac{1}{2}\mathbb{Z}$, and ω runs over $m+\mathbb{Z}$, and for each choice of m, ω, Λ, the function $R = R_{m,\omega,\Lambda}$ solves the equation $\mathbf{R}R = 0$, and the function $S = S_{m,\omega,\Lambda}$ solves the boundary value problem $\mathbf{S}S = 0$, $S'(0) = S'(\pi) = 0$, where

$$\mathbf{R} = \frac{d}{dr}\Delta\frac{d}{dr} + U(r), \tag{62}$$

$$U(r) = -\frac{4N(r+N)}{(r-N)^2} - \frac{\Sigma^2}{4N^2\Delta}\left(\omega + \frac{N(4r^2 - 11Nr + 3N^2)}{\Sigma(r-N)}\right)^2 - \Lambda \tag{63}$$

and

$$\mathbf{S} = \frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta} + V(\cos\theta), \tag{64}$$

$$V(x) = -\frac{((\omega+2)x + m)^2}{1-x^2} + \Lambda. \tag{65}$$

The separation constant Λ runs over the (countable set of) values for which such an S exists, all of which are non-negative.

The same statement holds if Φ is replaced by $\tilde{\Phi} = \tilde{\Psi}_2^{-2/3}\dot{\tilde{\Psi}}_0$, the operator $\mathbf{L}$ is replaced by $\tilde{\mathbf{L}}$, and the operator $\mathbf{R}$ replaced by $\tilde{\mathbf{R}}$, defined in the same way but using a potential $\tilde{U}$ in place of U. Here,

$$\tilde{\mathbf{L}} = \frac{\partial}{\partial r}\Delta\frac{\partial}{\partial r} - \frac{36N(r-N)}{(r+N)^2} + \frac{\Sigma^2}{4N^2\Delta}\left(\frac{\partial}{\partial t} + i\frac{N(4r^2 - 19Nr + 13N^2)}{\Sigma(r+N)}\right)^2 \\ + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\left(\cos\theta\frac{\partial}{\partial t} - \frac{\partial}{\partial\phi} - 2i\cos\theta\right)^2 \tag{66}$$

and

$$\tilde{U}(r) = -\frac{36N(r-N)}{(r+N)^2} - \frac{\Sigma^2}{4N^2\Delta}\left(\omega - \frac{N(4r^2 - 19Nr + 13N^2)}{\Sigma(r+N)}\right)^2 - \Lambda. \tag{67}$$

Lemma 6. *The equation $\mathbf{R}R = 0$ is an ordinary differential equation in a complex variable r, which has regular singular points at $r = 2N$ and $r = N/2$. The point $r = \infty$ is an irregular singular point of rank 1, except when $\omega = 0$, in which case it is a regular singular point. Thus, the equation $\mathbf{R}R = 0$ is a confluent Heun equation (see [13, Section 3]) when $\omega \neq 0$, and a hypergeometric equation (see [13, Section 2]) when $\omega = 0$. The characteristic exponents at $r = 2N$ are*

$$\pm(\omega - 1), \tag{68}$$

and those at $r = N/2$ are

$$\pm\left(\frac{\omega}{4} - 1\right). \tag{69}$$

When $\omega = 0$, the characteristic exponents at $r = \infty$ are

$$-\frac{3}{2} \pm i\sqrt{\frac{7}{2} + \Lambda}. \tag{70}$$

When $\omega \neq 0$, the equation $\mathbf{R}R = 0$ admits normal solutions (see [8, Section 3.2]), near $r = \infty$, of the asymptotic form

$$R \sim e^{\pm r\omega/2N} r^{-1\pm(5\omega/4-2)}. \tag{71}$$

Lemma 7. *For a solution to the equation* $\mathbf{R}R = 0$ *coming from a (globally smooth) perturbation of the Taub-bolt metric, the corresponding characteristic exponent at* $r = 2N$ *is* $|\omega - 1|$.

Lemma 8. *Let* R *be a solution to the equation* $\mathbf{R}R = 0$ *coming from an asymptotically locally flat perturbation of the Taub-bolt metric. When* $\omega = 0$*, none of the characteristic exponents at* $r = \infty$ *are compatible with the assumption of asymptotic local flatness. When* $\omega \neq 0$*, exactly one of the asymptotic normal solutions is compatible with this assumption, namely*

$$R \sim e^{-r|\omega|/2N} r^{-1-(5\omega/4-2)\,\mathrm{sgn}(\omega)}. \tag{72}$$

Corresponding lemmas regarding the asymptotics of the equation $\tilde{\mathbf{R}}R = 0$ also hold. We omit them, since they are entirely analogous.

4.2 Mode Stability

Proof of Theorem 2. In this case, we see directly that $U(r) < 0$, and integrating the equation $\mathbf{R}R = 0$ by parts like in the proof of Theorem 1, we see that R vanishes identically. The case involving the equation $\tilde{\mathbf{R}}R = 0$ is entirely analogous. $\square$

A Newman–Penrose Equations

We have the Newman–Penrose commutation relations:

$$[\Delta, D]\eta = (\gamma + \tilde{\gamma})D\eta + (\epsilon + \tilde{\epsilon})\Delta\eta - (\pi + \tilde{\tau})\delta\eta - (\tilde{\pi} + \tau)\tilde{\delta}\eta, \tag{73}$$
$$[D, \delta]\eta = -(\tilde{\alpha} + \beta - \tilde{\pi})D\eta - \kappa\Delta\eta + (\epsilon - \tilde{\epsilon} + \tilde{\rho})\delta\eta + \sigma\tilde{\delta}\eta, \tag{74}$$
$$[\delta, \Delta]\eta = -\tilde{\nu}D\eta - (\tilde{\alpha} + \beta - \tau)\Delta\eta + (-\gamma + \tilde{\gamma} + \mu)\delta\eta + \tilde{\lambda}\tilde{\delta}\eta, \tag{75}$$
$$[\tilde{\delta}, D]\eta = (\alpha + \tilde{\beta} - \pi)D\eta + \tilde{\kappa}\Delta\eta - \tilde{\sigma}\delta\eta + (\epsilon - \tilde{\epsilon} - \rho)\tilde{\delta}\eta, \tag{76}$$
$$[\tilde{\delta}, \Delta]\eta = -\nu D\eta - (\alpha + \tilde{\beta} - \tilde{\tau})\Delta\eta + \lambda\delta\eta + (\gamma - \tilde{\gamma} + \tilde{\mu})\tilde{\delta}\eta, \tag{77}$$
$$[\tilde{\delta}, \delta]\eta = (-\mu + \tilde{\mu})D\eta + (-\rho + \tilde{\rho})\Delta\eta + (\alpha - \tilde{\beta})\delta\eta + (-\tilde{\alpha} + \beta)\tilde{\delta}\eta. \tag{78}$$

In terms of the spin coefficients, the vacuum Einstein equations become

$$-D\gamma + \Delta\epsilon = \Psi_2 - \tilde{\gamma}\epsilon - \gamma(2\epsilon + \tilde{\epsilon}) - \kappa\nu + \beta\pi + \alpha\tilde{\pi} + \alpha\tau + \pi\tau + \beta\tilde{\tau}, \tag{79}$$
$$-D\tilde{\gamma} + \Delta\tilde{\epsilon} = \tilde{\Psi}_2 - \gamma\tilde{\epsilon} - \tilde{\gamma}(\epsilon + 2\tilde{\epsilon}) - \tilde{\kappa}\tilde{\nu} + \tilde{\alpha}\pi + \tilde{\beta}\tilde{\pi} + \tilde{\beta}\tau + \tilde{\alpha}\tilde{\tau} + \tilde{\pi}\tilde{\tau}, \tag{80}$$
$$-D\tau + \Delta\kappa = \Psi_1 - 3\gamma\kappa - \tilde{\gamma}\kappa + \tilde{\pi}\rho + \pi\sigma + \epsilon\tau - \tilde{\epsilon}\tau + \rho\tau + \sigma\tilde{\tau}, \tag{81}$$
$$-D\tilde{\tau} + \Delta\tilde{\kappa} = \tilde{\Psi}_1 - \gamma\tilde{\kappa} - 3\tilde{\gamma}\tilde{\kappa} + \pi\tilde{\rho} + \tilde{\pi}\tilde{\sigma} + \tilde{\sigma}\tau - \epsilon\tilde{\tau} + \tilde{\epsilon}\tilde{\tau} + \tilde{\rho}\tilde{\tau}, \tag{82}$$
$$-D\nu + \Delta\pi = \Psi_3 - 3\epsilon\nu - \tilde{\epsilon}\nu + \gamma\pi - \tilde{\gamma}\pi + \mu\pi + \lambda\tilde{\pi} + \lambda\tau + \mu\tilde{\tau}, \tag{83}$$
$$-D\tilde{\nu} + \Delta\tilde{\pi} = \tilde{\Psi}_3 - \epsilon\tilde{\nu} - 3\tilde{\epsilon}\tilde{\nu} + \tilde{\lambda}\pi - \gamma\tilde{\pi} + \tilde{\gamma}\tilde{\pi} + \tilde{\mu}\tilde{\pi} + \tilde{\mu}\tau + \tilde{\lambda}\tilde{\tau}, \tag{84}$$
$$-\Delta\beta + \delta\gamma = \tilde{\alpha}\gamma + 2\beta\gamma - \alpha\tilde{\lambda} - \beta(\tilde{\gamma} + \mu) + \epsilon\tilde{\nu} + \nu\sigma - \gamma\tau - \mu\tau, \tag{85}$$
$$\Delta\tilde{\alpha} - \delta\tilde{\gamma} = \tilde{\Psi}_3 - \beta\tilde{\gamma} + \tilde{\beta}\tilde{\lambda} + \tilde{\alpha}(-\gamma + \mu) - \tilde{\epsilon}\tilde{\nu} - \tilde{\nu}\tilde{\rho} + \tilde{\gamma}\tau + \tilde{\lambda}\tilde{\tau}, \tag{86}$$
$$-D\beta + \delta\epsilon = \Psi_1 - \tilde{\alpha}\epsilon - \beta\tilde{\epsilon} - \gamma\kappa - \kappa\mu + \epsilon\tilde{\pi} + \beta\tilde{\rho} + \alpha\sigma + \pi\sigma, \tag{87}$$
$$-D\tilde{\alpha} + \delta\tilde{\epsilon} = -\beta\tilde{\epsilon} - \tilde{\gamma}\kappa - \tilde{\kappa}\tilde{\lambda} + \tilde{\epsilon}\tilde{\pi} + \tilde{\pi}\tilde{\rho} + \tilde{\alpha}(\epsilon - 2\tilde{\epsilon} + \tilde{\rho}) + \tilde{\beta}\sigma, \tag{88}$$

$$-D\sigma + \delta\kappa = \Psi_0 - \tilde{\alpha}\kappa - 3\beta\kappa + \kappa\tilde{\pi} + 3\epsilon\sigma - \tilde{\epsilon}\sigma + \rho\sigma + \tilde{\rho}\sigma - \kappa\tau, \tag{89}$$

$$D\tilde{\rho} - \delta\tilde{\kappa} = 3\tilde{\alpha}\tilde{\kappa} + \beta\tilde{\kappa} - \tilde{\kappa}\tilde{\pi} - \epsilon\tilde{\rho} - \tilde{\epsilon}\tilde{\rho} - \tilde{\rho}^2 - \sigma\tilde{\sigma} + \kappa\tilde{\tau}, \tag{90}$$

$$\Delta\mu - \delta\nu = \lambda\tilde{\lambda} + \gamma\mu + \tilde{\gamma}\mu + \mu^2 - \tilde{\alpha}\nu - 3\beta\nu - \tilde{\nu}\pi + \nu\tau, \tag{91}$$

$$-\Delta\tilde{\lambda} + \delta\tilde{\nu} = -\tilde{\Psi}_4 + \gamma\tilde{\lambda} - 3\tilde{\gamma}\tilde{\lambda} - \tilde{\lambda}\mu - \tilde{\lambda}\tilde{\mu} + \tilde{\nu}(3\tilde{\alpha} + \beta + \tilde{\pi}) - \tilde{\nu}\tau, \tag{92}$$

$$-D\mu + \delta\pi = \Psi_2 - \epsilon\mu - \tilde{\epsilon}\mu - \kappa\nu - \tilde{\alpha}\pi + \beta\pi + \pi\tilde{\pi} + \mu\tilde{\rho} + \lambda\sigma, \tag{93}$$

$$-D\tilde{\lambda} + \delta\tilde{\pi} = \epsilon\tilde{\lambda} - 3\tilde{\epsilon}\tilde{\lambda} - \kappa\tilde{\nu} + \tilde{\alpha}\tilde{\pi} - \beta\tilde{\pi} + \tilde{\pi}^2 + \tilde{\lambda}\tilde{\rho} + \tilde{\mu}\sigma, \tag{94}$$

$$-\Delta\sigma + \delta\tau = \kappa\tilde{\nu} - \tilde{\lambda}\rho + 3\gamma\sigma - \tilde{\gamma}\sigma - \mu\sigma + \tilde{\alpha}\tau - \beta\tau - \tau^2, \tag{95}$$

$$\Delta\tilde{\rho} - \delta\tilde{\tau} = \tilde{\Psi}_2 - \tilde{\kappa}\tilde{\nu} - \gamma\tilde{\rho} - \tilde{\gamma}\tilde{\rho} + \mu\tilde{\rho} + \tilde{\lambda}\tilde{\sigma} + \tilde{\alpha}\tilde{\tau} - \beta\tilde{\tau} + \tau\tilde{\tau}, \tag{96}$$

$$-\delta\tilde{\beta} + \tilde{\delta}\tilde{\alpha} = \tilde{\Psi}_2 - \alpha\tilde{\alpha} + 2\tilde{\alpha}\tilde{\beta} - \beta\tilde{\beta} + \tilde{\epsilon}\mu - \tilde{\epsilon}\tilde{\mu} + \tilde{\gamma}\rho - \tilde{\gamma}\tilde{\rho} - \tilde{\mu}\tilde{\rho} + \tilde{\lambda}\tilde{\sigma}, \tag{97}$$

$$\delta\alpha - \tilde{\delta}\beta = \Psi_2 - \alpha\tilde{\alpha} + 2\alpha\beta - \beta\tilde{\beta} - \epsilon\mu + \epsilon\tilde{\mu} - \gamma\rho - \mu\rho + \gamma\tilde{\rho} + \lambda\sigma, \tag{98}$$

$$\Delta\alpha - \tilde{\delta}\gamma = \Psi_3 - \tilde{\beta}\gamma - \alpha\tilde{\gamma} + \beta\lambda + \alpha\tilde{\mu} - \epsilon\nu - \nu\rho + \lambda\tau + \gamma\tilde{\tau}, \tag{99}$$

$$-\Delta\tilde{\beta} + \tilde{\delta}\tilde{\gamma} = \alpha\tilde{\gamma} - \tilde{\alpha}\lambda - \tilde{\beta}(\gamma - 2\tilde{\gamma} + \tilde{\mu}) + \tilde{\epsilon}\nu + \tilde{\nu}\tilde{\sigma} - \tilde{\gamma}\tilde{\tau} - \tilde{\mu}\tilde{\tau}, \tag{100}$$

$$D\alpha - \tilde{\delta}\epsilon = 2\alpha\epsilon + \tilde{\beta}\epsilon + \gamma\tilde{\kappa} + \kappa\lambda - \epsilon\pi - \pi\rho - \alpha(\tilde{\epsilon} + \rho) - \beta\tilde{\sigma}, \tag{101}$$

$$-D\tilde{\beta} + \tilde{\delta}\tilde{\epsilon} = \tilde{\Psi}_1 - \alpha\tilde{\epsilon} - \tilde{\gamma}\tilde{\kappa} - \tilde{\kappa}\tilde{\mu} + \tilde{\epsilon}\pi + \tilde{\beta}(-\epsilon + \rho) + \tilde{\alpha}\tilde{\sigma} + \tilde{\pi}\tilde{\sigma}, \tag{102}$$

$$D\rho - \tilde{\delta}\kappa = 3\alpha\kappa + \tilde{\beta}\kappa - \kappa\pi - \epsilon\rho - \tilde{\epsilon}\rho - \rho^2 - \sigma\tilde{\sigma} + \tilde{\kappa}\tau, \tag{103}$$

$$-D\tilde{\sigma} + \tilde{\delta}\tilde{\kappa} = \tilde{\Psi}_0 - \alpha\tilde{\kappa} - 3\tilde{\beta}\tilde{\kappa} + \tilde{\kappa}\pi - \epsilon\tilde{\sigma} + 3\tilde{\epsilon}\tilde{\sigma} + \rho\tilde{\sigma} + \tilde{\rho}\tilde{\sigma} - \tilde{\kappa}\tilde{\tau}, \tag{104}$$

$$-\delta\tilde{\mu} + \tilde{\delta}\tilde{\lambda} = \tilde{\Psi}_3 - \alpha\tilde{\lambda} + 3\tilde{\beta}\tilde{\lambda} - \tilde{\alpha}\tilde{\mu} - \beta\tilde{\mu} + \mu\tilde{\pi} - \tilde{\mu}\tilde{\pi} + \tilde{\nu}\rho - \tilde{\nu}\tilde{\rho}, \tag{105}$$

$$\delta\lambda - \tilde{\delta}\mu = \Psi_3 - \tilde{\alpha}\lambda + 3\beta\lambda - \alpha\mu - \tilde{\beta}\mu - \mu\pi + \tilde{\mu}\pi - \nu\rho + \nu\tilde{\rho}, \tag{106}$$

$$-\Delta\lambda + \tilde{\delta}\nu = -\Psi_4 - 3\gamma\lambda + \tilde{\gamma}\lambda - \lambda\mu - \lambda\tilde{\mu} + \nu(3\alpha + \tilde{\beta} + \pi) - \nu\tilde{\tau}, \tag{107}$$

$$\Delta\tilde{\mu} - \tilde{\delta}\tilde{\nu} = \lambda\tilde{\lambda} + \gamma\tilde{\mu} + \tilde{\gamma}\tilde{\mu} + \tilde{\mu}^2 - \alpha\tilde{\nu} - 3\tilde{\beta}\tilde{\nu} - \nu\tilde{\pi} + \tilde{\nu}\tilde{\tau}, \tag{108}$$

$$D\lambda - \tilde{\delta}\pi = 3\epsilon\lambda - \tilde{\epsilon}\lambda + \tilde{\kappa}\nu - \alpha\pi + \tilde{\beta}\pi - \pi^2 - \lambda\rho - \mu\tilde{\sigma}, \tag{109}$$

$$-D\tilde{\mu} + \tilde{\delta}\tilde{\pi} = \tilde{\Psi}_2 - \epsilon\tilde{\mu} - \tilde{\epsilon}\tilde{\mu} - \tilde{\kappa}\tilde{\nu} - \alpha\tilde{\pi} + \tilde{\beta}\tilde{\pi} + \pi\tilde{\pi} + \tilde{\mu}\rho + \tilde{\lambda}\tilde{\sigma}, \tag{110}$$

$$-\delta\tilde{\sigma} + \tilde{\delta}\tilde{\rho} = \tilde{\Psi}_1 + \tilde{\kappa}(\mu - \tilde{\mu}) - \alpha\tilde{\rho} - \tilde{\beta}\tilde{\rho} + 3\tilde{\alpha}\tilde{\sigma} - \beta\tilde{\sigma} + \rho\tilde{\tau} - \tilde{\rho}\tilde{\tau}, \tag{111}$$

$$\delta\rho - \tilde{\delta}\sigma = \Psi_1 + \kappa(-\mu + \tilde{\mu}) - \tilde{\alpha}\rho - \beta\rho + 3\alpha\sigma - \tilde{\beta}\sigma - \rho\tau + \tilde{\rho}\tau, \tag{112}$$

$$\Delta\rho - \tilde{\delta}\tau = \Psi_2 - \kappa\nu - \gamma\rho - \tilde{\gamma}\rho + \tilde{\mu}\rho + \lambda\sigma + \alpha\tau - \tilde{\beta}\tau + \tau\tilde{\tau}, \tag{113}$$

$$-\Delta\tilde{\sigma} + \tilde{\delta}\tilde{\tau} = \tilde{\kappa}\nu - \lambda\tilde{\rho} - \gamma\tilde{\sigma} + 3\tilde{\gamma}\tilde{\sigma} - \tilde{\mu}\tilde{\sigma} + \alpha\tilde{\tau} - \tilde{\beta}\tilde{\tau} - \tilde{\tau}^2. \tag{114}$$

Finally, we have the Bianchi identities:

$$\begin{aligned} D\tilde{\Psi}_3 - \Delta\Psi_1 + \delta\Psi_2 - \delta\tilde{\Psi}_2 = {} & \tilde{\Psi}_4\tilde{\kappa} + 2\tilde{\Psi}_1\tilde{\lambda} + 2\Psi_1(\gamma - \mu) + \Psi_0\nu - 3\tilde{\Psi}_2\tilde{\pi} \\ & + 2\tilde{\Psi}_3(\tilde{\epsilon} - \tilde{\rho}) + 2\Psi_3\sigma - 3\Psi_2\tau, \end{aligned} \tag{115}$$

$$-\Delta\Psi_0 + \delta\Psi_1 = 4\Psi_0\gamma - \Psi_0\mu + 3\Psi_2\sigma - 2\Psi_1(\beta + 2\tau), \tag{116}$$

$$D\tilde{\Psi}_2 - \delta\tilde{\Psi}_1 = 2\tilde{\Psi}_3\tilde{\kappa} + \tilde{\Psi}_0\tilde{\lambda} + 2\tilde{\Psi}_1(\tilde{\alpha} - \tilde{\pi}) - 3\tilde{\Psi}_2\tilde{\rho}, \tag{117}$$

$$D\tilde{\Psi}_4 - \delta\tilde{\Psi}_3 = 4\tilde{\Psi}_4\tilde{\epsilon} + 3\tilde{\Psi}_2\tilde{\lambda} - 2\tilde{\Psi}_3(\tilde{\alpha} + 2\tilde{\pi}) - \tilde{\Psi}_4\tilde{\rho}, \tag{118}$$

$$-\Delta\Psi_2 + \delta\Psi_3 = -3\Psi_2\mu + 2\Psi_1\nu + \Psi_4\sigma + 2\Psi_3(\beta - \tau), \tag{119}$$

$$-\Delta\Psi_1 + \delta\Psi_2 = 2\Psi_1(\gamma - \mu) + \Psi_0\nu + 2\Psi_3\sigma - 3\Psi_2\tau, \tag{120}$$

$$D\Psi_1 - \tilde{\delta}\Psi_0 = 3\Psi_2\kappa + \Psi_0(4\alpha - \pi) - 2\Psi_1(\epsilon + 2\rho), \tag{121}$$

$$\begin{aligned} D\Psi_2 + D\tilde{\Psi}_2 - \delta\tilde{\Psi}_1 - \tilde{\delta}\Psi_1 = {} & 2\Psi_3\kappa + 2\tilde{\Psi}_3\tilde{\kappa} + \Psi_0\lambda + \tilde{\Psi}_0\tilde{\lambda} + 2\Psi_1(\alpha - \pi) \\ & + 2\tilde{\Psi}_1(\tilde{\alpha} - \tilde{\pi}) - 3\Psi_2\rho - 3\tilde{\Psi}_2\tilde{\rho}, \end{aligned} \tag{122}$$

$$\begin{aligned} D\Psi_3 - \Delta\tilde{\Psi}_1 - \tilde{\delta}\Psi_2 + \tilde{\delta}\tilde{\Psi}_2 = {} & \Psi_4\kappa + 2\Psi_1\lambda + 2\tilde{\Psi}_1(\tilde{\gamma} - \tilde{\mu}) + \tilde{\Psi}_0\tilde{\nu} - 3\Psi_2\pi \\ & + 2\Psi_3(\epsilon - \rho) + 2\tilde{\Psi}_3\tilde{\sigma} - 3\tilde{\Psi}_2\tilde{\tau}, \end{aligned} \tag{123}$$

$$\begin{aligned} -\Delta\Psi_2 - \Delta\tilde{\Psi}_2 + \delta\Psi_3 + \tilde{\delta}\tilde{\Psi}_3 = {} & -3\Psi_2\mu - 3\tilde{\Psi}_2\tilde{\mu} + 2\Psi_1\nu + 2\tilde{\Psi}_1\tilde{\nu} + \Psi_4\sigma \\ & + \tilde{\Psi}_4\tilde{\sigma} + 2\Psi_3(\beta - \tau) + 2\tilde{\Psi}_3(\tilde{\beta} - \tilde{\tau}), \end{aligned} \tag{124}$$

$$-\Delta\tilde{\Psi}_3 + \tilde{\delta}\tilde{\Psi}_4 = -2\tilde{\Psi}_3(\tilde{\gamma} + 2\tilde{\mu}) + 3\tilde{\Psi}_2\tilde{\nu} + \tilde{\Psi}_4(4\tilde{\beta} - \tilde{\tau}), \tag{125}$$

$$D\tilde{\Psi}_1 - \delta\tilde{\Psi}_0 = 3\tilde{\Psi}_2\tilde{\kappa} + \tilde{\Psi}_0(4\tilde{\alpha} - \tilde{\pi}) - 2\tilde{\Psi}_1(\tilde{\epsilon} + 2\tilde{\rho}), \tag{126}$$

$$D\tilde{\Psi}_2 - \delta\tilde{\Psi}_1 = 2\tilde{\Psi}_3\tilde{\kappa} + \tilde{\Psi}_0\tilde{\lambda} + 2\tilde{\Psi}_1(\tilde{\alpha} - \tilde{\pi}) - 3\tilde{\Psi}_2\tilde{\rho}, \tag{127}$$

$$-\Delta\tilde{\Psi}_0 + \tilde{\delta}\tilde{\Psi}_1 = 4\tilde{\Psi}_0\tilde{\gamma} - \tilde{\Psi}_0\tilde{\mu} + 3\tilde{\Psi}_2\tilde{\sigma} - 2\tilde{\Psi}_1(\tilde{\beta} + 2\tilde{\tau}), \tag{128}$$

$$-\Delta\Psi_3 + \delta\Psi_4 = -2\Psi_3(\gamma + 2\mu) + 3\Psi_2\nu + \Psi_4(4\beta - \tau), \tag{129}$$

$$-\Delta\Psi_2 + \delta\Psi_3 = -3\Psi_2\mu + 2\Psi_1\nu + \Psi_4\sigma + 2\Psi_3(\beta - \tau), \tag{130}$$

$$D\Psi_4 - \tilde{\delta}\Psi_3 = 4\Psi_4\epsilon + 3\Psi_2\lambda - 2\Psi_3(\alpha + 2\pi) - \Psi_4\rho, \tag{131}$$

$$-\Delta\Psi_1 + \delta\Psi_2 = 2\Psi_1(\gamma - \mu) + \Psi_0\nu + 2\Psi_3\sigma - 3\Psi_2\tau, \tag{132}$$

$$D\Psi_3 - \tilde{\delta}\Psi_2 = \Psi_4\kappa + 2\Psi_1\lambda - 3\Psi_2\pi + 2\Psi_3(\epsilon - \rho), \tag{133}$$

$$D\Psi_3 - \tilde{\delta}\Psi_2 = \Psi_4\kappa + 2\Psi_1\lambda - 3\Psi_2\pi + 2\Psi_3(\epsilon - \rho). \tag{134}$$

A.1 Spin Coefficients for Kerr

$$\alpha = \frac{r\cos\theta - a}{(r - a\cos\theta)2\sqrt{2\Sigma}\sin\theta}, \tag{135}$$

$$\beta = \frac{r\cos\theta - a}{(r - a\cos\theta)2\sqrt{2\Sigma}\sin\theta}, \tag{136}$$

$$\gamma = \frac{i(-\Delta/(r - a\cos\theta) + r - M)}{2\sqrt{2\Delta\Sigma}}, \tag{137}$$

$$\epsilon = \frac{i(-\Delta/(r - a\cos\theta) + r - M)}{2\sqrt{2\Delta\Sigma}}, \tag{138}$$

$$\kappa = 0, \tag{139}$$

$$\lambda = 0, \tag{140}$$

$$\mu = -\frac{i\sqrt{\Delta/2\Sigma}}{r - a\cos\theta}, \tag{141}$$

$$\nu = 0, \tag{142}$$

$$\pi = -\frac{a\sin\theta}{(r - a\cos\theta)\sqrt{2\Sigma}}, \tag{143}$$

$$\rho = -\frac{i\sqrt{\Delta/2\Sigma}}{r - a\cos\theta}, \tag{144}$$

$$\sigma = 0, \tag{145}$$

$$\tau = -\frac{a\sin\theta}{(r-a\cos\theta)\sqrt{2\Sigma}}, \tag{146}$$

$$\tilde{\alpha} = -\frac{r\cos\theta + a}{(r+a\cos\theta)2\sqrt{2\Sigma}\sin\theta}, \tag{147}$$

$$\tilde{\beta} = -\frac{r\cos\theta + a}{(r+a\cos\theta)2\sqrt{2\Sigma}\sin\theta}, \tag{148}$$

$$\tilde{\gamma} = \frac{i(-\Delta/(r+a\cos\theta)+r-M)}{2\sqrt{2\Delta\Sigma}}, \tag{149}$$

$$\tilde{\epsilon} = \frac{i(-\Delta/(r-a\cos\theta)+r-M)}{2\sqrt{2\Delta\Sigma}}, \tag{150}$$

$$\tilde{\kappa} = 0, \tag{151}$$

$$\tilde{\lambda} = 0, \tag{152}$$

$$\tilde{\mu} = -\frac{i\sqrt{\Delta/2\Sigma}}{r+a\cos\theta}, \tag{153}$$

$$\tilde{\nu} = 0, \tag{154}$$

$$\tilde{\pi} = -\frac{a\sin\theta}{(r+a\cos\theta)\sqrt{2\Sigma}}, \tag{155}$$

$$\tilde{\rho} = -\frac{i\sqrt{\Delta/2\Sigma}}{r+a\cos\theta}, \tag{156}$$

$$\tilde{\sigma} = 0, \tag{157}$$

$$\tilde{\tau} = -\frac{a\sin\theta}{(r+a\cos\theta)\sqrt{2\Sigma}}. \tag{158}$$

A.2 Spin Coefficients for Taub-Bolt

$$\alpha = \frac{N(r+N)^2}{8\Sigma\sqrt{2\Delta\Sigma}}, \tag{159}$$

$$\beta = \frac{N(r+N)^2}{8\Sigma\sqrt{2\Delta\Sigma}}, \tag{160}$$

$$\gamma = \frac{i\cot\theta}{2\sqrt{2\Sigma}}, \tag{161}$$

$$\epsilon = \frac{i\cot\theta}{2\sqrt{2\Sigma}}, \tag{162}$$

$$\kappa = 0, \tag{163}$$

$$\lambda = 0, \tag{164}$$

$$\mu = 0, \tag{165}$$

$$\nu = 0, \tag{166}$$

$$\pi = -\frac{(r+N)\sqrt{\Delta/2\Sigma}}{\Sigma}, \tag{167}$$
$$\rho = 0, \tag{168}$$
$$\sigma = 0, \tag{169}$$
$$\tau = -\frac{(r+N)\sqrt{\Delta/2\Sigma}}{\Sigma}, \tag{170}$$
$$\tilde{\alpha} = -\frac{9N(r-N)}{8(r+N)\sqrt{\Delta\Sigma}}, \tag{171}$$
$$\tilde{\beta} = -\frac{9N(r-N)}{8(r+N)\sqrt{\Delta\Sigma}}, \tag{172}$$
$$\tilde{\gamma} = \frac{i\cot\theta}{2\sqrt{2\Sigma}}, \tag{173}$$
$$\tilde{\epsilon} = \frac{i\cot\theta}{2\sqrt{2\Sigma}}, \tag{174}$$
$$\tilde{\kappa} = 0, \tag{175}$$
$$\tilde{\lambda} = 0, \tag{176}$$
$$\tilde{\mu} = 0, \tag{177}$$
$$\tilde{\nu} = 0, \tag{178}$$
$$\tilde{\pi} = \frac{\sqrt{\Delta/2\Sigma}}{r+N}, \tag{179}$$
$$\tilde{\rho} = 0, \tag{180}$$
$$\tilde{\sigma} = 0, \tag{181}$$
$$\tilde{\tau} = \frac{\sqrt{\Delta/2\Sigma}}{r+N}. \tag{182}$$

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