Prepared for submission to JHEP

Holographic Euclidean thermal correlator

Song He[a,b] **and Yi Li**[a]

[a] *Center for Theoretical Physics and College of Physics, Jilin University, Changchun 130012, People's Republic of China*

[b] *Max Planck Institute for Gravitational Physics (Albert Einstein Institute), Am Mühlenberg 1, 14476 Golm, Germany*

E-mail: hesong@jlu.edu.cn, liyi@fudan.edu.cn

Abstract: In this paper, we compute holographic Euclidean thermal correlators of the stress tensor and $U(1)$ current from the AdS planar black hole. To this end, we set up perturbative boundary value problems for Einstein's gravity and Maxwell theory in the spirit of Gubser-Klebanov-Polyakov-Witten, with appropriate gauge fixing and regularity boundary conditions at the horizon of the black hole. The linearized Einstein equation and Maxwell equation in the black hole background are related to the Heun equation of degenerate local monodromy. Leveraging the connection relation of local solutions of the Heun equation, we partly solve the boundary value problem and obtain exact two-point thermal correlators for $U(1)$ current and stress tensor in the scalar and shear channel.

Contents

1 Introduction

As an embodiment of the holographic principle [1, 2], the Anti-de Sitter gravity/conformal field theory (AdS/CFT) correspondence [3–5] establishes a connection between a quantum gravity theory in AdS space and a boundary conformal field theory. This equivalence is encapsulated in the concise Gubser-Klebanov-Polyakov-Witten (GKPW) relation, where the partition function of the boundary conformal field theory with operator sources equals the gravity partition function with prescribed boundary conditions

$$\langle e^{\int \phi_0 O} \rangle_{CFT} = Z_{\mathrm{G}}[\phi_0] \tag{1.1}$$

In the most helpful limit to exploit this correspondence, the classical gravity on-shell action becomes the generating functional of connected correlators of the strongly-coupled CFT

$$I_{CFT}[\phi_0] = I_{\mathrm{G,on-shell}}[\phi_0] \tag{1.2}$$

Correlators are computed by functional differentiation of the generating functional, which amounts to solving the perturbative boundary value problem for the bulk fields' equation of motion. This involves varying the boundary value of the bulk fields and solving for the corresponding variation of the on-shell configuration in the bulk. The near-boundary behavior is well-established, allowing the extraction of holographic correlators [6–9]. However, solving the global boundary value problem is generally intricate, exemplified in cases like pure gravity [10].

Although the prescription is clear, explicit computation of Euclidean holographic correlators in the GKPW approach has been limited to pure AdS space and its quotient spaces, such as thermal AdS where the method of images can be applied (e.g., [11] for thermal bootstrap emphasis). In our prior work [12], we computed holographic torus correlators of stress tensor. This study focuses on Euclidean thermal two-point correlators of the stress tensor and $U(1)$ current in four-dimensional CFTs. Beyond the Hawking-Page transition [13], the thermal state corresponds holographically to a five-dimensional Euclidean AdS planar black hole [14]. Correlators are derived by solving perturbative boundary value problems in Einstein's gravity and Maxwell theory for the $U(1)$ gauge field in the black hole background. Two important steps are involved. The first is to appropriately fix the gauge and impose regularity boundary conditions at the horizon, ensuring a unique solution. The second step identifies specific equations of motion as the Heun equation [15], and solves the boundary value problem with the connection relation of local solutions. The general connection relation was established in [16], applied to exact thermal correlators in Minkowski signature [17, 18], and employed in various black hole perturbation problems [19, 20]. In our case, the Heun equations feature degenerate local monodromy, with characteristic exponents differing by an integer. We compute the connection relation by taking a limit of the generic case. Ultimately, we obtain exact two-point correlators for the $U(1)$ current and stress tensor in the scalar and shear channels (as defined in [21]).

Thermal two-point correlators, also known as thermal spectral functions, have many important applications and have been studied in [21] using gauge invariants in each channel. In the final discussion section, we comment on our approach to holographic computation and relevant applications.

2 Holographic setup

We start by reviewing the basics of holographic computation independent of the bulk background geometry. For Einstein's gravity, it's customary to work in the Fefferman-Graham gauge [6, 22]

$$ds^2 = \frac{dr^2}{r^2} + \frac{1}{r^2}\mathbf{g}_{ij}(r,x)dx^i dx^j \tag{2.1}$$

and in dimension four, we have the series expansion

$$\mathbf{g}_{ij} = \mathbf{g}^{(0)}_{ij} + r^2\mathbf{g}^{(2)}_{ij} + r^4\mathbf{g}^{(4)}_{ij} + r^4\log r\,\mathbf{h}^{(4)}_{ij} + \dots \tag{2.2}$$

The background metric of the holographic field theory γ_{ij} corresponds to $\mathbf{g}^{(0)}_{ij}$, and the one-point correlator of the stress tensor, with appropriate renormalization, is given by [7]

$$\langle T_{ij}\rangle = \frac{4}{16\pi G}\left[\mathbf{g}^{(4)}_{ij} - \frac{1}{8}\mathbf{g}^{(0)}_{ij}(\mathbf{P}^{(0)2} - \mathbf{P}^{(0)}_{ij}\mathbf{P}^{(0)ij}) - \frac{1}{2}\mathbf{P}^{(0)}_{ik}\mathbf{P}^{(0)k}_{j} + \frac{1}{4}\mathbf{P}^{(0)}\mathbf{P}^{(0)}_{ij}\right] \tag{2.3}$$

where $\mathbf{P}^{(0)}_{ij}$ is the Schouten tensor of $\mathbf{g}^{(0)}_{ij}$. In our case, the holographic field theory lives on a flat background, so the terms of the Schouten tensor do not contribute. We have

$$\langle T_{ij}\rangle = \frac{1}{4\pi G}\mathbf{g}^{(4)}_{ij} \tag{2.4}$$

The Einstein equation near the conformal boundary determines the series (2.2) in terms of $\mathbf{g}^{(0)}_{ij}$ and $\mathbf{g}^{(4)}_{ij}$ or equivalently the one point correlator $\langle T_{ij}\rangle$, and imposes holographic Ward identities of conservation and Weyl anomaly on $\langle T_{ij}\rangle$. Near boundary solutions of the Einstein equation are in one-to-one correspondence to the pair $(\gamma_{ij}, \langle T_{ij}\rangle)$. The global geometry of the bulk spacetime fully determines the one-point correlator as a function of the boundary metric, from which we can compute multi-point correlators by taking functional derivatives.

Similarly, for the $U(1)$ gauge field, we can put it in the radial gauge near the conformal boundary using the Fefferman-Graham coordinates of the bulk metric

$$A = \mathbf{A}_i(r, x)dx^i \tag{2.5}$$

with the series expansion

$$\mathbf{A}_i = \mathbf{A}_i^{(0)} + r^2\mathbf{A}_i^{(2)} + r^2 \log r\mathbf{B}_i^{(2)} + \ldots \tag{2.6}$$

The one-point correlator with appropriate renormalization is given by

$$\langle J_i\rangle = -2\mathbf{A}_i^{(2)} \tag{2.7}$$

The Maxwell equation near the conformal boundary determines the series (2.6) in terms of $\mathbf{A}_i^{(0)}$ and $\mathbf{A}_i^{(2)}$ or equivalently the one point correlator $\langle J_i\rangle$, and imposes holographic Ward identities of conservation on $\langle J_i\rangle$. The global geometry of the bulk spacetime fully determines $\langle J_i\rangle$.

Now we specialize in the holographic correlators from the five-dimensional AdS planar black hole. The black hole is a solid cylinder $\mathbb{B}^2 \times \mathbb{R}^3$ with the metric

$$ds^2 = \frac{1}{\rho^2}[(1 - \frac{\rho^4}{\rho_0^4})^{-1}d\rho^2 + (1 - \frac{\rho^4}{\rho_0^4})dt^2 + d\vec{x}^2] \tag{2.8}$$

The period of Euclidean time t, namely the inverse temperature, is $\beta = \pi\rho_0$. The conformal boundary is at $\rho = 0$, and the horizon is at $\rho = \rho_0$, being the $\mathbb{R}^3$ axis of the cylinder. The standard Fefferman-Graham radial coordinate r is related to ρ by

$$\rho = \frac{r}{\sqrt{1 + \frac{r^4}{4\rho_0^4}}} \tag{2.9}$$

For simplicity, we set $\rho_0 = 1$ in the metric, effectively working in the unit of ρ_0, and we will recover ρ_0 dependence when final results are obtained. As a convention, we label bulk spacetime coordinate indices by Greek alphabets $\mu, \nu, \rho, \ldots$, the boundary spacetime coordinate indices by Roman alphabets $i, j, k, \ldots$ and the boundary space indices by $a, b, c, \ldots$.

3 $U(1)$ current

Now, we work on the boundary value problem of the $U(1)$ gauge field, beginning with gauge-fixing. We put it in the radial gauge $A_\rho = 0$ in the region $0 \le \rho < 1$ (excluding the horizon) by a $U(1)$ gauge transformation. For a global solution, its restriction to the region $0 \le \rho < 1$ must have a regular limit going to the horizon $\rho = 1$. Therefore, we formulate the boundary value problem in the radial gauge, with the boundary condition that the solution has a regular limit as $\rho \to 1$ after a gauge transformation. To work out the explicit form of the boundary condition, we introduce the "cylindrical radial coordinate" $\mathfrak{s}$

$$\cosh 2\mathfrak{s} = \frac{1}{\rho^2} \tag{3.1}$$

Near the horizon, the metric takes the form of Euclidean metric in "cylindrical coordinates"

$$ds^2 \sim d\mathfrak{s}^2 + \mathfrak{s}^2 d(2t)^2 + d\vec{x}^2 \tag{3.2}$$

and the horizon is properly covered by the "Cartesian coordinates"

$$\begin{aligned} X &= \mathfrak{s} \cos 2t \\ Y &= \mathfrak{s} \sin 2t \\ \vec{x} &= \vec{x} \end{aligned} \tag{3.3}$$

The gauge field is regular at the horizon if and only if its components in a coordinate chart that properly covers the horizon, for example, the "Cartesian coordinates" are regular. That is, we have

$$A = \mathbf{A}_i dx^i \tag{3.4}$$

and there exists a $U(1)$ gauge transformation Λ, such that

$$\lim_{\mathfrak{s}\to 0} A + d\Lambda = A_X^*(\vec{x})dX + A_Y^*(\vec{x})dY + A_a^*(\vec{x})dx^a \tag{3.5}$$

The components on the right-hand side can only depend on $\vec{x}$ because the t-circle shrinks to a point as $\mathfrak{s} \to 0$. We find

$$\lim_{\mathfrak{s}\to 0} \partial_{\mathfrak{s}}\Lambda = A_X^*(\vec{x}) \cos 2t + A_Y^*(\vec{x}) \sin 2t \tag{3.6}$$

$$\lim_{\mathfrak{s}\to 0} \frac{\mathbf{A}_t + \partial_t \Lambda}{\mathfrak{s}} = -2A_X^*(\vec{x}) \sin 2t + 2A_Y^*(\vec{x}) \cos 2t \tag{3.7}$$

$$\lim_{\mathfrak{s}\to 0} \mathbf{A}_a + \partial_a \Lambda = A_a^*(\vec{x}) \tag{3.8}$$

From (3.6) we see Λ can be approximated as a linear function of $\mathfrak{s}$ as $\mathfrak{s} \to 0$ (or $\rho \to 1$), then from (3.8) we know

$$\mathbf{A}_a \text{ regular as } \rho \to 1 \tag{3.9}$$

In addition, by integrating (3.7) over t we find

$$\int_0^\pi dt \mathbf{A}_t|_{\rho=1} = 0 \tag{3.10}$$

This gauge fixing and regularity boundary conditions at the horizon, together with the boundary value

$$\mathbf{A}_i|_{\rho=0} = \mathcal{A}_i \tag{3.11}$$

as a turned-on source on the CFT side, determine a unique solution to the Maxwell equation as we will see.

We utilize the translational symmetry in $t, \vec{x}$ direction and work with Fourier modes $\tilde{\mathbf{A}}_i$ with Matsubara frequency $\omega = 2m, m \in \mathbb{Z}$ and spatial momentum $\vec{p}$. For simplicity, we also rotate the spatial momentum to the x^1 direction. The Maxwell equation

$$d * F = 0 \tag{3.12}$$

then decouples to the transverse channel for $\tilde{\mathbf{A}}_2, \tilde{\mathbf{A}}_3$ and the longitudinal channel for $\tilde{\mathbf{A}}_t, \tilde{\mathbf{A}}_1$. For the transverse component $\tilde{\mathbf{A}}_2$ (and the same for $\tilde{\mathbf{A}}_3$) we have

$$(\partial_z^2 - \frac{2z}{1-z^2}\partial_z - \frac{\omega^2 + p^2(1-z^2)}{4z(1-z^2)^2})\tilde{\mathbf{A}}_2 = 0 \tag{3.13}$$

where we used the convenient coordinate $z = \rho^2$. This is an ordinary differential equation with four regular singularities $z = 0, 1, -1, \infty$. By the substitution $\tilde{\mathbf{A}}_2(z) = (1-z^2)^{-\frac{1}{2}}w(z)$, we get a Heun equation in the normal form for $w(z)$

$$(\partial_z^2 + \frac{\frac{1}{4} - (\frac{1}{2})^2}{z^2} + \frac{\frac{1}{4} - (\frac{m}{2})^2}{(z-1)^2} + \frac{\frac{1}{4} - (\frac{m}{2}i)^2}{(z+1)^2} + \frac{p^2 + 4m^2 - 2}{8z(z-1)} - \frac{p^2 + 4m^2 + 2}{8z(z+1)})w(z) = 0 \tag{3.14}$$

with

$$t = -1, a_0 = \frac{1}{2}, a_1 = \frac{|m|}{2}, a_t = \frac{m}{2}i, a_\infty = \frac{1}{2}, u = -\frac{p^2 + 4m^2 + 2}{8} \tag{3.15}$$

We refer the readers to Appendix A for a brief review of Fuchsian differential equations, the Heun equation, its connection problem, and notational conventions. By the boundary condition (3.9), $\tilde{\mathbf{A}}_2$ is regular at $z = 1$, so it must be proportional to the solution of exponent $\frac{|m|}{2}$ at $z = 1$. The constant of proportionality is determined by the boundary condition $\tilde{\mathbf{A}}_2|_{z=0} = \tilde{\mathcal{A}}_2$ and the connection relation (A.12). We find

$$\begin{aligned}\tilde{\mathbf{A}}_2(\omega = 2m, p, z) = \tilde{\mathcal{A}}_2(\omega, p)(1-z^2)^{-\frac{1}{2}}\big[w_-^{(0)} + \frac{p^2 + 4m^2}{4}(-2\psi(1) - 1 \\ + \frac{1}{2}\sum_{\theta,\sigma=\pm}\psi(\theta\frac{m}{2} + \sigma a) - \frac{1}{2}\partial_{a_0}^2 F|_{a_0=\frac{1}{2}} - \frac{2}{p^2 + 4m^2}(1 + 2\partial_t\partial_{a_0}F|_{a_0=\frac{1}{2}, t=-1}))w_+^{(0)}\big]\end{aligned} \tag{3.16}$$

For the longitudinal components $\tilde{\mathbf{A}}_t, \tilde{\mathbf{A}}_1$, we have

$$\partial_z^2\tilde{\mathbf{A}}_t - \frac{p^2}{4z(1-z^2)}\tilde{\mathbf{A}}_t + \frac{2mp}{4z(1-z^2)}\tilde{\mathbf{A}}_1 = 0 \tag{3.17}$$

$$\partial_z^2\tilde{\mathbf{A}}_1 - \frac{2z}{1-z^2}\partial_z\tilde{\mathbf{A}}_1 - \frac{4m^2}{4z(1-z^2)^2}\tilde{\mathbf{A}}_1 + \frac{2mp}{4z(1-z^2)^2}\tilde{\mathbf{A}}_t = 0 \tag{3.18}$$

$$\frac{2m}{1-z^2}\partial_z\tilde{\mathbf{A}}_t + p\partial_z\tilde{\mathbf{A}}_1 = 0 \tag{3.19}$$

Plugging (3.19) into $\partial_z\big(z(1-z^2)(3.17)\big)$, we obtain

$$(\partial_z^2 + \frac{1-3z^2}{z(1-z^2)}\partial_z - \frac{p^2(1-z^2) + 4m^2}{4z(1-z^2)^2})\partial_z\tilde{\mathbf{A}}_t = 0 \tag{3.20}$$

When $m \neq 0$, the solution to this third-order differential equation is determined by the three boundary conditions

$$\begin{aligned}&\tilde{\mathbf{A}}_1|_{z=1} \text{ regular}\\&\tilde{\mathbf{A}}_1|_{z=0} = \tilde{\mathcal{A}}_1,\ \tilde{\mathbf{A}}_t|_{z=0} = \tilde{\mathcal{A}}_t\end{aligned} \tag{3.21}$$

By the substitution $\partial_z \tilde{\mathbf{A}}_t = z^{-\frac{1}{2}}(1-z^2)^{-\frac{1}{2}} w(z)$, (3.20) can be transformed to the normal Heun equation

$$\Big(\partial_z^2 + \frac{\frac{1}{4}-0^2}{z^2} + \frac{\frac{1}{4}-(\frac{m}{2})^2}{(z-1)^2} + \frac{\frac{1}{4}-(\frac{m}{2}i)^2}{(z+1)^2} + \frac{p^2+4m^2-6}{8z(z-1)} - \frac{p^2+4m^2+6}{8z(z+1)}\Big)w(z) = 0 \tag{3.22}$$

with

$$t=-1, a_0=0, a_1=\frac{|m|}{2}, a_t=\frac{m}{2}i, a_\infty=1, u=-\frac{p^2+4m^2+6}{8} \tag{3.23}$$

By (3.19) the solution must be proportional to $w_+^{(1)}$ for $\tilde{\mathbf{A}}_1$ to be regular at $z=1$. The constant of proportionality can be further determined by using the connection relation (A.12) and evaluating (3.17) at $z=0$. We find

$$\begin{aligned}z^{\frac{1}{2}}\sqrt{1-z^2}\,\partial_z\tilde{\mathbf{A}}_t &= \frac{2mp\tilde{\mathcal{A}}_1 - p^2\tilde{\mathcal{A}}_t}{4}\Big[-w_-^{(0)}(z)\\&+\big(2\psi(1) - \frac{1}{2}\sum_{\theta,\sigma=\pm}\psi(\frac{1}{2}+\theta\frac{m}{2}+\sigma a) + \frac{1}{2}\partial_{a_0}^2 F\big)w_+^{(0)}\Big]\end{aligned} \tag{3.24}$$

Then we integrate to obtain $\tilde{\mathbf{A}}_t$ with the constant of integration given by the boundary value $\tilde{\mathcal{A}}_t$

$$\begin{aligned}\tilde{\mathbf{A}}_t &= \tilde{\mathcal{A}}_t + \frac{2mp\tilde{\mathcal{A}}_1 - p^2\tilde{\mathcal{A}}_t}{4}\Big[-(z\log z+\ldots)\\&+\big(2\psi(1)+1-\frac{1}{2}\sum_{\theta,\sigma=\pm}\psi(\frac{1}{2}+\theta\frac{m}{2}+\sigma a)+\frac{1}{2}\partial_{a_0}^2F\big)(z+\ldots)\Big]\end{aligned} \tag{3.25}$$

We get $\tilde{\mathbf{A}}_1$ by plugging $\tilde{\mathbf{A}}_t$ back to (3.19)

$$\begin{aligned}\tilde{\mathbf{A}}_1 &= \tilde{\mathcal{A}}_1(1+\ldots)\\&+\frac{2m(p\tilde{\mathcal{A}}_t - 2m\tilde{\mathcal{A}}_1)}{4}\big(2\psi(1)+1-\frac{1}{2}\sum_{\theta,\sigma=\pm}\psi(\frac{1}{2}+\theta\frac{m}{2}+\sigma a)+\frac{1}{2}\partial_{a_0}^2F\big)(z+\ldots)\end{aligned} \tag{3.26}$$

When $m=0$, we get $\tilde{\mathbf{A}}_1 = \tilde{\mathcal{A}}_1$ from (3.19). We still solve for $z^{\frac{1}{2}}\sqrt{1-z^2}\,\partial_z\tilde{\mathbf{A}}_t$ from (3.20), which is a linear combination of $w_+^{(1)} = \sqrt{1-z}(1+\ldots)$ and $w_-^{(1)} = \sqrt{1-z}(\log(1-z)+\ldots)$. Then we plug it into (3.17) and evaluate at $z=1$. We have $\tilde{\mathbf{A}}_t(m=0)|_{z=1}=0$ from the boundary condition (3.10), and we find $z^{\frac{1}{2}}\sqrt{1-z^2}\,\partial_z\tilde{\mathbf{A}}_t$ must be proportional to $w_+^{(1)}$, the same as the previous case when $m\neq 0$. So, we can carry over the results for $m \neq 0$ and set $m=0$ in the expression.

To obtain the holographic correlators, we recover the dependence on ρ_0 or the inverse temperature $\beta = \pi\rho_0$, and read off $\mathbf{A}_i^{(2)}$ from the bulk gauge field $\mathbf{A}_i$ (in our case the coefficient of z^1)

$$\begin{aligned}
\tilde{\mathbf{A}}_2^{(2)}(\omega = \frac{2m}{\rho_0}, p) &= -\frac{p^2+\omega^2}{4}\tilde{\mathcal{A}}_2(\omega,p)\mathcal{C}_1(\omega = \frac{2m}{\rho_0}, p) \\
\tilde{\mathbf{A}}_t^{(2)}(\omega = \frac{2m}{\rho_0}, p) &= \frac{\omega p\tilde{\mathcal{A}}_1(\omega,p) - p^2\tilde{\mathcal{A}}_t(\omega,p)}{4}\mathcal{C}_2(\omega = \frac{2m}{\rho_0}, p) \\
\tilde{\mathbf{A}}_1^{(2)}(\omega = \frac{2m}{\rho_0}, p) &= \frac{\omega p\tilde{\mathcal{A}}_t(\omega,p) - \omega^2\tilde{\mathcal{A}}_1(\omega,p)}{4}\mathcal{C}_2(\omega = \frac{2m}{\rho_0}, p)
\end{aligned} \tag{3.27}$$

where

$$\begin{aligned}
&\mathcal{C}_1(\omega = \frac{2m}{\rho_0}, p) = (2\psi(1) + 1 - \frac{1}{2}\sum_{\theta,\sigma=\pm}\psi(\theta\frac{m}{2} + \sigma a) \\
&+ \frac{1}{2}\partial_{a_0}^2 F + \frac{2}{\rho_0^2p^2 + 4m^2}(1 + \partial_t\partial_{a_0}F))\Big|_{t=-1, a_0=\frac{1}{2}, a_1=\frac{|m|}{2}, a_t=\frac{m}{2}i, a_\infty=\frac{1}{2}, u=-\frac{\rho_0^2p^2+4m^2+2}{8}} \\
&\mathcal{C}_2(\omega = \frac{2m}{\rho_0}, p) = (2\psi(1) + 1 - \frac{1}{2}\sum_{\theta,\sigma=\pm}\psi(\frac{1}{2} + \theta\frac{m}{2} + \sigma a) \\
&+ \frac{1}{2}\partial_{a_0}^2 F)\Big|_{t=-1, a_0=0, a_1=\frac{|m|}{2}, a_t=\frac{m}{2}i, a_\infty=1, u=-\frac{\rho_0^2p^2+4m^2+6}{8}}
\end{aligned} \tag{3.28}$$

We compute two-point correlators by the formula for renormalized one-point correlators (2.7). Rotating the spatial momentum to a general direction, we find

$$\begin{aligned}
\langle \tilde{J}_t(\omega,p)\tilde{J}_t(-\omega,-p)\rangle &= \frac{p^2}{2}\mathcal{C}_2(\omega,p) \\
\langle \tilde{J}_t(\omega,p)\tilde{J}_b(-\omega,-p)\rangle &= -\frac{\omega}{2}\mathcal{C}_2(\omega,p)p_b \\
\langle \tilde{J}_a(\omega,p)\tilde{J}_b(-\omega,-p)\rangle &= \frac{p^2+\omega^2}{2}\mathcal{C}_1(\omega,p)(\delta_{ab} - \frac{p_ap_b}{p^2}) + \frac{\omega^2}{2}\mathcal{C}_2(\omega,p)\frac{p_ap_b}{p^2}
\end{aligned} \tag{3.29}$$

4 Stress tensor

Gauge fixing and regularity boundary conditions at the horizon for Einstein's gravity follow the same line as the Maxwell theory. We can make the solid cylinder coordinates $\rho, t, \vec{x}$ the Fefferman-Graham coordinates of the perturbed bulk metric in the region $0 \leq \rho < 1$ by a diffeomorphism. Then, the boundary value problem is formulated in this gauge with the boundary condition that the metric has a regular limit as $\rho \to 1$ after a diffeomorphism. For a first-order perturbation of the bulk metric, we have

$$\delta ds^2 = \delta\mathbf{g}_{ij}dx^idx^j \tag{4.1}$$

And to the first order, the diffeomorphism is characterized by a vector V, then the regularity boundary condition at the horizon is the variation of the bulk metric

$$\mathcal{L}_V(ds^2) + \delta ds^2 \tag{4.2}$$

has a regular limit as $\rho \to 1$ (or $\mathfrak{s} \to 0$), that is, its components in the "Cartesian coordinates" (3.3) are regular. We find

$$\lim_{\mathfrak{s}\to 0} 2\partial_{\mathfrak{s}} V^{\mathfrak{s}} = \cos^2 2t \delta g^*_{XX} + 2\cos 2t \sin 2t \delta g^*_{XY} + \sin^2 2t \delta g^*_{YY} \tag{4.3}$$

$$\lim_{\mathfrak{s}\to 0} \frac{\partial_t V^{\mathfrak{s}} + \frac{\sinh^2 2\mathfrak{s}}{\cosh 2\mathfrak{s}} \partial_{\mathfrak{s}} V^t}{2\mathfrak{s}}$$

$$= -\cos 2t \sin 2t \delta g^*_{XX} + (\cos^2 2t - \sin^2 2t)\delta g^*_{XY} + \cos 2t \sin 2t \delta g^*_{YY} \tag{4.4}$$

$$\lim_{\mathfrak{s}\to 0} \partial_a V^{\mathfrak{s}} + \cosh 2\mathfrak{s} \partial_{\mathfrak{s}} V^x = \cos 2t \delta g^*_{Xa} + \sin 2t \delta g^*_{Ya} \tag{4.5}$$

$$\lim_{\mathfrak{s}\to 0} \frac{\delta \mathbf{g}_{tt} + \frac{\sinh 2\mathfrak{s}}{\cosh^2 2\mathfrak{s}}((3 + 4\cosh 4\mathfrak{s})V^{\mathfrak{s}} + \sinh 4\mathfrak{s} \partial_t V^t)}{4\mathfrak{s}^2}$$

$$= \sin^2 2t \delta g^*_{XX} - 2\cos 2t \sin 2t \delta g^*_{XY} + \cos^2 2t \delta g^*_{YY} \tag{4.6}$$

$$\lim_{\mathfrak{s}\to 0} \delta \mathbf{g}_{ta} + \frac{\sinh^2 2\mathfrak{s}}{\cosh 2\mathfrak{s}} \partial_a V^t + \cosh 2\mathfrak{s} \partial_t V^a = -2\mathfrak{s} \sin 2t \delta g^*_{Xa} + 2\mathfrak{s} \cos 2t \delta g^*_{Ya} \tag{4.7}$$

$$\lim_{\mathfrak{s}\to 0} \delta \mathbf{g}_{ab} + \cosh 2\mathfrak{s}(\partial_a V^b + \partial_b V^a) + 2\sinh 2\mathfrak{s} V^{\mathfrak{s}} \delta_{ab} = \delta g^*_{ab} \tag{4.8}$$

(4.3) and (4.5) show that $V^{\mathfrak{s}}$ and V^a can be approximated by linear function in $\mathfrak{s}$ as $\mathfrak{s} \to 0$. Plugging into (4.8), we see

$$\delta \mathbf{g}_{ab} \text{ regular as } \rho \to 1 \tag{4.9}$$

By (4.4) we know

$$V^t = O(\frac{1}{\mathfrak{s}}) \tag{4.10}$$

as $\mathfrak{s} \to 0$. Then by integrating (4.7) over t we find

$$\int_0^\pi dt \delta \mathbf{g}_{ta}|_{\rho=1} = 0 \tag{4.11}$$

Similar to the case of $U(1)$ gauge field, we work in Fourier modes and rotate the spatial momentum to the x^1 direction. And for simplicity, we use the variable $\mathbf{h}_{ij} = \rho^2 \delta \mathbf{g}_{ij}$ which on the conformal boundary equals the variation of the CFT background metric

$$\mathbf{h}_{ij}|_{\rho=0} = \delta\gamma_{ij} \tag{4.12}$$

The linearized Einstein equation

$$\frac{1}{2}(\nabla^\lambda \nabla_\mu \delta g_{\lambda\nu} + \nabla^\lambda \nabla_\nu \delta g_{\lambda\mu} - \nabla^\lambda \nabla_\lambda \delta g_{\mu\nu} - \nabla_\mu \nabla_\nu \delta g^\lambda_\lambda) + 4\delta g_{\mu\nu} = 0 \tag{4.13}$$

decouples to the scalar channel of $\tilde{\mathbf{h}}_{23}$ and $\tilde{\mathbf{h}}_{22} - \tilde{\mathbf{h}}_{33}$, the shear channel of $\tilde{\mathbf{h}}_{t2}, \tilde{\mathbf{h}}_{12}$ and $\tilde{\mathbf{h}}_{t3}, \tilde{\mathbf{h}}_{13}$, and the sound channel of $\tilde{\mathbf{h}}_{tt}, \tilde{\mathbf{h}}_{11}, \tilde{\mathbf{h}}_{22} + \tilde{\mathbf{h}}_{33}, \tilde{\mathbf{h}}_{t1}$. In the scalar channel, we have

$$\partial_z^2 \tilde{\mathbf{h}}_{23} - \frac{1 + z^2}{z(1 - z^2)} \partial_z \tilde{\mathbf{h}}_{23} - \frac{p^2(1 - z^2) + \omega^2}{4z(1 - z^2)^2} \tilde{\mathbf{h}}_{23} = 0 \tag{4.14}$$

and in the shear channel, we have

$$\partial_z^2\tilde{\mathbf{h}}_{t2} - \frac{1}{z}\partial_z\tilde{\mathbf{h}}_{t2} - \frac{p^2}{4z(1-z^2)}\tilde{\mathbf{h}}_{t2} + \frac{2mp}{4z(1-z^2)}\tilde{\mathbf{h}}_{12} = 0 \tag{4.15}$$

$$\partial_z^2\tilde{\mathbf{h}}_{12} - \frac{1+z^2}{z(1-z^2)}\partial_z\tilde{\mathbf{h}}_{12} - \frac{4m^2}{4z(1-z^2)^2}\tilde{\mathbf{h}}_{12} + \frac{2mp}{4z(1-z^2)^2}\tilde{\mathbf{h}}_{t2} = 0 \tag{4.16}$$

$$\frac{2m}{1-z^2}\partial_z\tilde{\mathbf{h}}_{t2} + p\partial_z\tilde{\mathbf{h}}_{12} = 0 \tag{4.17}$$

The computation is very similar to that of the transverse channel and longitudinal channel of the $U(1)$ gauge field in the previous section, so we present the results of correlators without showing the detailed computation

$$\begin{aligned}
\langle\tilde{T}_{23}(\omega=\frac{2m}{\rho_0},p)\tilde{T}_{23}(-\omega,-p)\rangle &= \frac{1}{4\pi G}\frac{(p^2+\omega^2)^2}{32}\mathcal{C}_3(\omega=\frac{2m}{\rho_0},p)\\
\langle\tilde{T}_{t2}(\omega=\frac{2m}{\rho_0},p)\tilde{T}_{t2}(-\omega,-p)\rangle &= \frac{1}{4\pi G}\frac{p^2+\omega^2}{32}p^2\mathcal{C}_4(\omega=\frac{2m}{\rho_0},p)\\
\langle\tilde{T}_{t2}(\omega=\frac{2m}{\rho_0},p)\tilde{T}_{12}(-\omega,-p)\rangle &= -\frac{1}{4\pi G}\frac{p^2+\omega^2}{32}\omega p\mathcal{C}_4(\omega=\frac{2m}{\rho_0},p)\\
\langle\tilde{T}_{12}(\omega=\frac{2m}{\rho_0},p)\tilde{T}_{12}(-\omega,-p)\rangle &= \frac{1}{4\pi G}\frac{p^2+\omega^2}{32}\omega^2\mathcal{C}_4(\omega=\frac{2m}{\rho_0},p)
\end{aligned} \tag{4.18}$$

with

$$\begin{aligned}
&\mathcal{C}_3(\omega=\frac{2m}{\rho_0},p) = \big[2\psi(1)+\frac{5}{2}-\frac{1}{2}\sum_{\theta,\sigma=\pm}\psi(-\frac{1}{2}+\theta\frac{m}{2}+\sigma a)\\
&+\frac{1}{2}\partial_{a_0}^2F - \frac{16}{(\rho_0^2p^2+4m^2)^2}\big(4a^2-2a^2m^2+\frac{1}{4}m^4+4(\partial_tF)^2+(-8a^2+2m^2)\partial_tF\\
&-4\partial_tF\partial_t\partial_{a_0}F+(-2+4a^2-m^2)\partial_t\partial_{a_0}F\big)\big]\Big|_{t=-1,a_0=1,a_1=\frac{|m|}{2},a_t=\frac{m}{2}i,a_\infty=0,u=-\frac{\rho_0^2p^2+4m^2-2}{8}}\\
&\mathcal{C}_4(\omega=\frac{2m}{\rho_0},p) = \big(2\psi(1)+1-\frac{1}{2}\sum_{\theta,\sigma=\pm}\psi(\theta\frac{m}{2}+\sigma a)\\
&+\frac{1}{2}\partial_{a_0}^2F+\frac{2}{\rho_0^2p^2+4m^2}(1+2\partial_t\partial_{a_0}F)\big)\Big|_{t=-1,a_0=\frac{1}{2},a_1=\frac{|m|}{2},a_t=\frac{m}{2}i,a_\infty=\frac{3}{2},u=-\frac{\rho_0^2p^2+4m^2+10}{8}}
\end{aligned} \tag{4.19}$$

In the sound channel, we have

$$\partial_z^2\tilde{\mathbf{h}}_{tt} - \frac{3-5z^2}{2z(1-z^2)}\partial_z\tilde{\mathbf{h}}_{tt} - \frac{1+z^2}{2z}\partial_z(\tilde{\mathbf{h}}_{11}+\tilde{\mathbf{h}}_{22}+\tilde{\mathbf{h}}_{33}) + \frac{-4z+12z^3-p^2(1-z^2)}{4z(1-z^2)^2}\tilde{\mathbf{h}}_{tt}$$
$$-\frac{4m^2}{4z(1-z^2)}(\tilde{\mathbf{h}}_{11}+\tilde{\mathbf{h}}_{22}+\tilde{\mathbf{h}}_{33}) + \frac{2mp}{2z(1-z^2)}\tilde{\mathbf{h}}_{t1} = 0 \tag{4.20}$$

$$\partial_z^2\tilde{\mathbf{h}}_{11} - \frac{3+z^2}{2z(1-z^2)}\partial_z\tilde{\mathbf{h}}_{11} - \frac{1}{2z(1-z^2)}\partial_z\tilde{\mathbf{h}}_{tt} - \frac{1}{2z}\partial_z(\tilde{\mathbf{h}}_{22}+\tilde{\mathbf{h}}_{33})$$
$$-\frac{4m^2}{4z(1-z^2)^2}\tilde{\mathbf{h}}_{11} - \frac{p^2+4z}{4z(1-z^2)^2}\tilde{\mathbf{h}}_{tt} - \frac{p^2}{4z(1-z^2)}(\tilde{\mathbf{h}}_{22}+\tilde{\mathbf{h}}_{33}) + \frac{2mp}{2z(1-z^2)^2}\tilde{\mathbf{h}}_{t1} = 0 \tag{4.21}$$

$$\partial_z^2(\tilde{\mathbf{h}}_{22}+\tilde{\mathbf{h}}_{33}) - \frac{2}{z(1-z^2)}\partial_z(\tilde{\mathbf{h}}_{22}+\tilde{\mathbf{h}}_{33}) - \frac{1}{z(1-z^2)}\partial_z\tilde{\mathbf{h}}_{tt} - \frac{1}{z}\partial_z\tilde{\mathbf{h}}_{11}$$
$$-\frac{4m^2+p^2(1-z^2)}{4z(1-z^2)^2}(\tilde{\mathbf{h}}_{22}+\tilde{\mathbf{h}}_{33}) - \frac{2}{(1-z^2)^2}\tilde{\mathbf{h}}_{tt} = 0 \tag{4.22}$$

$$\partial_z^2\tilde{\mathbf{h}}_{t1} - \frac{1}{z}\partial_z\tilde{\mathbf{h}}_{t1} - \frac{2mp}{4z(1-z^2)}(\tilde{\mathbf{h}}_{22}+\tilde{\mathbf{h}}_{33}) = 0 \tag{4.23}$$

$$\partial_z^2(\tilde{\mathbf{h}}_{11}+\tilde{\mathbf{h}}_{22}+\tilde{\mathbf{h}}_{33}) + \frac{1}{1-z^2}\partial_z^2\tilde{\mathbf{h}}_{tt} - \frac{z}{1-z^2}\partial_z(\tilde{\mathbf{h}}_{11}+\tilde{\mathbf{h}}_{22}+\tilde{\mathbf{h}}_{33})$$
$$+\frac{z}{(1-z^2)^2}\partial_z\tilde{\mathbf{h}}_{tt} + \frac{2}{(1-z^2)^3}\tilde{\mathbf{h}}_{tt} = 0 \tag{4.24}$$

$$2m\partial_z(\tilde{\mathbf{h}}_{11}+\tilde{\mathbf{h}}_{22}+\tilde{\mathbf{h}}_{33}) + \frac{2mz}{1-z^2}\partial_z(\tilde{\mathbf{h}}_{11}+\tilde{\mathbf{h}}_{22}+\tilde{\mathbf{h}}_{33}) - p\partial_z\tilde{\mathbf{h}}_{t1} - \frac{2pz}{1-z^2}\tilde{\mathbf{h}}_{t1} = 0 \tag{4.25}$$

$$p\partial_z(\tilde{\mathbf{h}}_{22}+\tilde{\mathbf{h}}_{33}) + \frac{p}{1-z^2}\partial_z\tilde{\mathbf{h}}_{tt} - \frac{2m}{1-z^2}\partial_z\tilde{\mathbf{h}}_{t1} + \frac{pz}{(1-z^2)^2}\tilde{\mathbf{h}}_{tt} = 0 \tag{4.26}$$

We don't know how to analytically solve the boundary value problem in the sound channel. For future reference, we can reduce the sound channel to a five-dimensional first order equation of variables $\tilde{\mathbf{h}}_{tt}, \tilde{\mathbf{h}}_{11}, \frac{\tilde{\mathbf{h}}_{22}+\tilde{\mathbf{h}}_{33}}{2}, \tilde{\mathbf{h}}_{t1}, \partial_z\tilde{\mathbf{h}}_{t1}$ (a similar equation can be found in [21]), and by the substitution

$$\begin{pmatrix} \tilde{\mathbf{h}}_{tt} \\ \tilde{\mathbf{h}}_{11} \\ \frac{\tilde{\mathbf{h}}_{22}+\tilde{\mathbf{h}}_{33}}{2} \\ \tilde{\mathbf{h}}_{t1} \\ \partial_z\tilde{\mathbf{h}}_{t1} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{3}(1-z^2)^2 & \frac{2}{3}z(1-z^2) & 0 & 0 \\ -z^2 & 1-z^2 & \frac{2}{3}z & 0 & 0 \\ \frac{1}{2}z^2 & 0 & -\frac{1}{3}z & 0 & 0 \\ 0 & 0 & 0 & 1-z^2 & 0 \\ 0 & 0 & 0 & 0 & z \end{pmatrix} H \tag{4.27}$$

we can transform the equation into a Fuchsian system of normal form

$$\partial_z H = \left(\frac{M_0}{z} + \frac{M_1}{z-1} + \frac{M_{-1}}{z+1}\right)H \tag{4.28}$$

with

$$M_0 = \begin{pmatrix} -2 & -\frac{2}{3} & 0 & \frac{p}{3m} & \frac{12m^2+p^2}{6mp} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -m^2 + \frac{p^2}{12} & -1 & mp & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{mp}{3} & 0 & 2 \end{pmatrix}$$

$$M_1 = \begin{pmatrix} 0 & 0 & -\frac{1}{3} & 0 & -\frac{m}{p} \\ 0 & -\frac{1}{2} & 0 & -\frac{p}{2m} & -\frac{p}{4m} \\ \frac{p^2}{8} & \frac{-1+m^2}{2} & 2 & \frac{p(1-m^2)}{2m} & \frac{p}{4m} \\ 0 & 0 & 0 & -1 & -\frac{1}{2} \\ -\frac{mp}{4} & 0 & \frac{mp}{6} & 0 & 0 \end{pmatrix}$$

$$M_{-1} = \begin{pmatrix} 0 & 0 & \frac{1}{3} & 0 & -\frac{m}{p} \\ 0 & -\frac{1}{2} & 0 & -\frac{p}{2m} & -\frac{p}{4m} \\ \frac{p^2}{8} & \frac{1+m^2}{2} & 2 & -\frac{p(1+m^2)}{2m} & -\frac{p}{4m} \\ 0 & 0 & 0 & -1 & -\frac{1}{2} \\ \frac{mp}{4} & 0 & \frac{mp}{6} & 0 & 0 \end{pmatrix} \tag{4.29}$$

5 Summary and discussion

In our study, we calculated holographic Euclidean thermal correlators of the $U(1)$ current and stress tensor for four-dimensional CFTs using the AdS_5 planar black hole, following the approach of GKPW. By utilizing the connection relation of local solutions of the Heun equation, we obtained exact correlators for the $U(1)$ current and stress tensor in both the scalar and shear channels.

Extensive research has focused on thermal two-point correlators (thermal spectral functions). Notably, [21] demonstrated the presence of gauge invariants in each channel that diagonalize coupled differential equations. These invariants and their derivatives render the on-shell action quadratic. Thermal two-point correlators have been computed using this formalism numerically or analytically by approximations [23, 24]. For example in the longitudinal channel the gauge invariant is $E_L = p\tilde{\mathbf{A}}_t - \omega\tilde{\mathbf{A}}_1$ and we have

$$\partial_z^2 E_L - \frac{2\omega^2 z}{(1-z^2)(\omega^2 + p^2(1-z^2))}\partial_z E_L - \frac{\omega^2 + p^2(1-z^2)}{4z(1-z^2)^2} E_L = 0 \tag{5.1}$$

This is a Fuchsian differential equation with six singularities. The two singularities $z = \pm\sqrt{1 + \frac{\omega^2}{p^2}}$ are apparent singularities since they don't appear in the equation of the fields. One can verify that these apparent singularities cannot be transformed away by a substitution $E_L(z) = P(z)f(z)$ where $P(z)$ is a meromorphic function that does not introduce new singularities. In essence, these apparent singularities remain inherent to the equation. We don't know how to relate this equation to the Heun equation and obtain the exact holographic correlators. From the technical standpoint, we want to work with equations of fields, and in the Euclidean signature, the boundary conditions of fields with gauge/diffeomorphism symmetry are clearly specified. This is the technical reason for our approach of holographic computation, in addition to giving an illustrative example of Euclidean boundary value problems.

Thermal two-point correlators find diverse applications. They characterize the linear response to perturbations in thermal equilibrium. They can be used to compute transport coefficients such as shear viscosity, thermal conductivity and electric conductivity [25, 26], and higher order coefficients (such as the coefficient of the response of electric current to the third derivative of electric potential). In addition, we can probe the chaotic dynamics by studying the pole-skipping of the correlators [27–29]. These correlators also encode the information of operator product expansion (OPE) of holographic CFTs. For instance, [30–32] computed holographic correlators in the OPE limit via near-boundary analysis, extracting OPE coefficients for multi-stress tensors. For integer operator dimension with operator mixing, exact two-point correlators are necessary for complete OPE coefficient extraction.

Acknowledgments

We would like to thank Cristoforo Iossa and Yunda Zhang for their helpful discussion. S.H. would appreciate the financial support from the Fundamental Research Funds for the Central Universities and Max Planck Partner Group and the Natural Science Foundation of China (NSFC) Grants No. 12075101 and No. 12235016.

A Fuchsian ODE, Heun equation and connection problem

In this appendix, we briefly review Fuchsian differential equations, the Heun equation, and its connection relation we used in the computation in the main text.

An ordinary differential equation (ODE) is called Fuchsian if the coefficients are rational functions and all singularities are regular. Eigenvectors of the local monodromy constitute a natural basis of local solutions around singularities. When eigenvalues of the local monodromy are all distinct, eigenvectors span the space of local solutions, and they take the form of a series

$$w_k^{(a)} = (z-a)^{\rho_k} \sum_{i=0}^{\infty} c_i (z-a)^i \tag{A.1}$$

where a is the singularity, k labels the local solution and the prefactor $(z-a)^{\rho_k}$ captures the local monodromy. The characteristic exponents ρ_k are computed as the roots of the indicial equation. When we have repeated eigenvalues of the local monodromy, that is some characteristic exponents differ by integers, we may need generalized eigenvectors to span the space of local solutions, and they are expressed as series with logarithms. For a second order ODE, we label the two characteristic exponents as ρ^+, ρ^-, with $\mathrm{Re}\rho_+ \geq \mathrm{Re}\rho_-$. There is always a series solution (with no logarithm) $w_+^{(a)}$ with the exponent ρ^+. If two exponents differ by an integer, the other solution $w_-^{(a)}$ may contain a logarithm. There is also no canonical choice of $w_-^{(a)}$ to form a basis since we can add any constant multiple of

$w_+^{(a)}$ to $w_-^{(a)}$. For computational convenience, we choose the convention that the coefficient of the power $(z-a)^{\rho_+}$ is zero in $w_-^{(a)}$.

The Heun equation is the second-order Fuchsian ODE with four regular singularities. In its normal form, it is

$$(\partial_z^2 + \frac{\frac{1}{4}-a_0^2}{z^2} + \frac{\frac{1}{4}-a_1^2}{(z-1)^2} + \frac{\frac{1}{4}-a_t^2}{(z-t)^2} - \frac{\frac{1}{2}-a_1^2-a_t^2-a_0^2+a_\infty^2+u}{z(z-1)} + \frac{u}{z(z-t)})w(z) = 0 \tag{A.2}$$

The four singularities with exponents at these points are

$$\begin{aligned} z=0, \rho &= \frac{1}{2} \pm a_0 \\ z=1, \rho &= \frac{1}{2} \pm a_1 \\ z=t, \rho &= \frac{1}{2} \pm a_t \\ z=\infty, \rho &= -\frac{1}{2} \pm a_\infty \end{aligned} \tag{A.3}$$

The connection relation of the local solutions in the generic case (that is, characteristic exponents do not differ by an integer) was studied in [16] by relating the Heun equation to the Belavin-Polyakov-Zamolodchikov (BPZ) equation [33] satisfied by conformal blocks with degenerate insertion in the Liouville field theory in the semiclassical limit. By the Alday-Gaiotto-Tachikawa (AGT) correspondence, the Liouville correlators can be exactly computed by localization in supersymmetric gauge theories [34–38]. Without losing generality, let $z=0$ and $z=1$ be two adjacent singularities, the connection relation between local solutions around these two points is

$$w_\theta^{(1)}(z) = \sum_{\theta'=\pm} \mathcal{M}_{\theta\theta'}(a_1, a_0; a) e^{(\frac{\theta}{2}\partial_{a_1} - \frac{\theta'}{2}\partial_{a_0})F(\begin{smallmatrix} a_t & a_1 \\ a_\infty & a & a_0 \end{smallmatrix}; \frac{1}{t})} w_{\theta'}^{(0)}(z) \tag{A.4}$$

where

$$\mathcal{M}_{\theta\theta'}(a_1, a_0; a) = \frac{\Gamma(-2\theta' a_0)\Gamma(1+2\theta a_1)}{\Gamma(\frac{1}{2}+\theta a_1 - \theta' a_0 + a)\Gamma(\frac{1}{2}+\theta a_1 - \theta' a_0 - a)} \tag{A.5}$$

and F is the Nekrasov-Shatashvili function, defined as power series in $\frac{1}{t}$ with rational functions in other parameters as coefficients. We refer the reader to Appendix C in [17] (or Appendix C in [16]) for the detailed definition [1]. The exchange momentum a is to be implicitly determined from the relation

$$u = -\frac{1}{4} - a^2 + a_t^2 + a_0^2 + t\partial_t F \tag{A.6}$$

[1]In our application, a_t and a_1 are swapped and t is replaced by $\frac{1}{t}$ in the arguments of the function compared to the expression in [17]. That's because we consider the connection relation between $z=0$ and $z=1$, a different pair of points than that in [17].

In our computation, the masslessness of the bulk fields leads to an integer difference in exponents at $z = 0$ (the conformal boundary), specifically, a_0 becomes a half-integer. This degenerate scenario can be derived as a limit of the generic case, as a specific solution to the Heun equation continuously depends on the parameters. The emergence of logarithm and the discontinuity of the local monodromy basis reflect a qualitative change in the local monodromy, rather than a specific solution. The solution $w_+^{(1)}$ remains well-defined and continuously depends on parameters including a_0, even when a_1 approaches half-integers (assuming $\mathrm{Re}a_1 > 0$) [2]. We proceed to take the limit $a_0 \to \frac{N}{2}, N \in \mathbb{N}$ while keeping other parameters, such as t, a_1, a_t, a_∞, a, fixed [3]. For $a_0 = 0$ we have

$$
\begin{aligned}
w_+^{(1)} = \lim_{a_0 \to 0} \frac{1}{2a_0} \Big[& \frac{\Gamma(1+2a_1)\Gamma(1+2a_0)}{\Gamma(\frac{1}{2}+a_1+a_0+a)\Gamma(\frac{1}{2}+a_1+a_0-a)} e^{(\frac{1}{2}\partial_{a_1}+\frac{1}{2}\partial_{a_0})F} z^{\frac{1}{2}-a_0}(1+\dots) \\
& - \frac{\Gamma(1+2a_1)\Gamma(1-2a_0)}{\Gamma(\frac{1}{2}+a_1-a_0+a)\Gamma(\frac{1}{2}+a_1-a_0-a)} e^{(\frac{1}{2}\partial_{a_1}-\frac{1}{2}\partial_{a_0})F} z^{\frac{1}{2}+a_0}(1+\dots) \Big]
\end{aligned}
\tag{A.7}
$$

The quantity in the square bracket must vanish when $a_0 = 0$ for the limit to exist. It indeed vanish because $\partial_{a_0} F|_{a_0=0} = 0$ with F being an even function of a_0. Then the limit becomes the derivative with respect to a_0, and we get

$$
\begin{aligned}
w_+^{(1)} =& \frac{\Gamma(1+2a_1)}{\Gamma(\frac{1}{2}+a_1+a)\Gamma(\frac{1}{2}+a_1-a)} e^{\frac{1}{2}\partial_{a_1}F} z^{\frac{1}{2}} \\
& (2\psi(1) - \psi(\frac{1}{2}+a_1+a) - \psi(\frac{1}{2}+a_1-a) + \frac{1}{2}\partial_{a_0}^2 F - \log z + \dots) \\
=& \frac{\Gamma(1+2a_1)}{\Gamma(\frac{1}{2}+a_1+a)\Gamma(\frac{1}{2}+a_1-a)} e^{\frac{1}{2}\partial_{a_1}F} \\
& \big[- w_-^{(0)} + \big(2\psi(1) - \psi(\frac{1}{2}+a_1+a) - \psi(\frac{1}{2}+a_1-a) + \frac{1}{2}\partial_{a_0}^2 F\big) w_+^{(0)} \big]
\end{aligned}
\tag{A.8}
$$

[2] $w_-^{(1)}$ is not continuous when a_1 approaches half-integers. When both a_0 and a_1 are half-integers, the complete connection relation is computed by solving two linear equations obtained from the limits of $w_+^{(0)}$ and $w_+^{(1)}$.

[3] Another curve in the parameter space can also be chosen to approach the limit, such as fixing u, an explicit parameter in the Heun equation, instead of a. However, as the connection coefficients explicitly depend on a, fixing a yields a relatively simple form for obtaining the limit.

where ψ denotes the digamma function. For $a_0 = \frac{1}{2}$ we find

$$
\begin{aligned}
w_+^{(1)} = &\lim_{a_0\to\frac{1}{2}} \Big[\frac{\Gamma(1+2a_1)\Gamma(2a_0)}{\Gamma(\frac{1}{2}+a_1+a_0+a)\Gamma(\frac{1}{2}+a_1+a_0-a)} e^{(\frac{1}{2}\partial_{a_1}+\frac{1}{2}\partial_{a_0})F} \\
&\times z^{\frac{1}{2}-a_0}(1+\frac{-\frac{t}{2}+t(a_0^2+a_1^2+a_t^2-a_\infty^2)+(1-t)u}{(1-2a_0)t}z+\ldots) \\
&+\frac{\Gamma(1+2a_1)\Gamma(-2a_0)}{\Gamma(\frac{1}{2}+a_1-a_0+a)\Gamma(\frac{1}{2}+a_1-a_0-a)} e^{(\frac{1}{2}\partial_{a_1}-\frac{1}{2}\partial_{a_0})F} z^{\frac{1}{2}+a_0}(1+\ldots)\Big] \\
=&\frac{\Gamma(1+2a_1)}{\Gamma(1+a_1+a)\Gamma(1+a_1-a)} e^{(\frac{1}{2}\partial_{a_1}+\frac{1}{2}\partial_{a_0})F} + \Gamma(1+2a_1) e^{\frac{1}{2}\partial_{a_1}F} z \\
&\times \lim_{a_0\to\frac{1}{2}} \frac{1}{1-2a_0}\Big[\frac{\Gamma(2a_0)}{\Gamma(\frac{1}{2}+a_1+a_0+a)\Gamma(\frac{1}{2}+a_1+a_0-a)} e^{\frac{1}{2}\partial_{a_0}F} \\
&\times \frac{-\frac{t}{2}+t(a_0^2+a_1^2+a_t^2-a_\infty^2)+(1-t)u}{t} z^{\frac{1}{2}-a_0} \\
&-\frac{\Gamma(2-2a_0)}{2a_0\Gamma(\frac{1}{2}+a_1-a_0+a)\Gamma(\frac{1}{2}+a_1-a_0-a)} e^{-\frac{1}{2}\partial_{a_0}F} z^{-\frac{1}{2}+a_0}+\ldots\Big]
\end{aligned}
\tag{A.9}
$$

The quantity in the square bracket must vanish when $a_0 = \frac{1}{2}$ for the limit to exist, that is, we must have

$$
\begin{aligned}
&e^{\partial_{a_0}F} \frac{-\frac{t}{2}+t(a_0^2+a_1^2+a_t^2-a_\infty^2)+(1-t)u}{t}\Big|_{a_0=\frac{1}{2}} \\
&= e^{\partial_{a_0}F} \frac{-\frac{1+t}{4}+ta_0^2+ta_1^2+(1-t)a^2-a_\infty^2-(1-t)t\partial_t F}{t}\Big|_{a_0=\frac{1}{2}} \\
&= a_1^2 - a^2
\end{aligned}
\tag{A.10}
$$

By the expansion of F

$$
F = \frac{(\frac{1}{4}-a^2-a_t^2+a_\infty^2)(\frac{1}{4}-a^2-a_1^2+a_0^2)}{\frac{1}{2}-2a^2}\frac{1}{t} + O(\frac{1}{t^2}) \tag{A.11}
$$

one can verify (A.10) holds to the order of expansion. Again, the limit becomes the derivative with respect to a_0 and we find

$$
\begin{aligned}
w_+^{(1)} =& \frac{\Gamma(1+2a_1)}{\Gamma(1+a_1+a)\Gamma(1+a_1-a)} e^{(\frac{1}{2}\partial_{a_1}+\frac{1}{2}\partial_{a_0})F} \\
&+ \frac{\Gamma(1+2a_1)}{\Gamma(a_1+a)\Gamma(a_1-a)} e^{(\frac{1}{2}\partial_{a_1}-\frac{1}{2}\partial_{a_0})F} z\big[-2\psi(1)-1+\frac{1}{2}\psi(1+a_1+a)+\frac{1}{2}\psi(1+a_1-a) \\
&+ \frac{1}{2}\psi(a_1+a)+\frac{1}{2}\psi(a_1-a)-\frac{1}{2}\partial_{a_0}^2 F+\log z-\frac{t+t(1-t)\partial_t\partial_{a_0}F}{2(-\frac{t}{2}+t(a_0^2+a_1^2+a_t^2-a_\infty^2)+(1-t)u)}\big]+\ldots \\
=& \frac{\Gamma(1+2a_1)}{\Gamma(1+a_1+a)\Gamma(1+a_1-a)} e^{(\frac{1}{2}\partial_{a_1}+\frac{1}{2}\partial_{a_0})F} w_-^{(0)} \\
&+ \frac{\Gamma(1+2a_1)}{\Gamma(a_1+a)\Gamma(a_1-a)} e^{(\frac{1}{2}\partial_{a_1}-\frac{1}{2}\partial_{a_0})F}\big[-2\psi(1)-1+\frac{1}{2}\psi(1+a_1+a)+\frac{1}{2}\psi(1+a_1-a) \\
&+ \frac{1}{2}\psi(a_1+a)+\frac{1}{2}\psi(a_1-a)-\frac{1}{2}\partial_{a_0}^2 F-\frac{t+t(1-t)\partial_t\partial_{a_0}F}{2(-\frac{t}{2}+t(a_0^2+a_1^2+a_t^2-a_\infty^2)+(1-t)u)}\big] w_+^{(0)} \\
=& \frac{\Gamma(1+2a_1)}{\Gamma(a_1+a)\Gamma(a_1-a)} e^{(\frac{1}{2}\partial_{a_1}-\frac{1}{2}\partial_{a_0})F}\big[\frac{t}{-\frac{t}{2}+t(a_0^2+a_1^2+a_t^2-a_\infty^2)+(1-t)u} w_-^{(0)} \\
&+ \big(-2\psi(1)-1+\frac{1}{2}\psi(1+a_1+a)+\frac{1}{2}\psi(1+a_1-a)+\frac{1}{2}\psi(a_1+a)+\frac{1}{2}\psi(a_1-a) \\
&- \frac{1}{2}\partial_{a_0}^2 F-\frac{t+t(1-t)\partial_t\partial_{a_0}F}{2(-\frac{t}{2}+t(a_0^2+a_1^2+a_t^2-a_\infty^2)+(1-t)u)}\big) w_+^{(0)}\big] \qquad \text{(A.12)}
\end{aligned}
$$

For $a_0 = 1$ we have

$$
\begin{aligned}
w_+^{(1)} =& \frac{\Gamma(1+2a_1)}{2\Gamma(-\frac{1}{2}+a_1+a)\Gamma(-\frac{1}{2}+a_1-a)} e^{(\frac{1}{2}\partial_{a_1}-\frac{1}{2}\partial_{a_0})F}\big[-\frac{1}{(2-2a_0)c_2|_{a_0=1}} w_{\tilde{-}}^{(0)} \\
&+ (2\psi(1)+\frac{5}{2}-\frac{1}{2}\psi(-\frac{1}{2}+a_1+a)-\frac{1}{2}\psi(-\frac{1}{2}+a_1-a)-\frac{1}{2}\psi(\frac{3}{2}+a_1+a)-\frac{1}{2}\psi(\frac{3}{2}+a_1-a) \\
&+ \frac{1}{2}\partial_{a_0}^2 F+\frac{\partial_{a_0}((2-2a_0)c_2)}{2(2-2a_0)c_2}|_{a_0=1}) w_{\tilde{+}}^{(0)}\big] \qquad \text{(A.13)}
\end{aligned}
$$

In general, the coefficient c_N in the series solution $z^{\frac{1}{2}-a_0}\sum_{k=0}^{\infty} c_k z^k$ and the coefficient for $w_+^{(0)}$ simultaneously take $a_0 = \frac{N}{2}$ as a pole, so the limit $a_0 \to \frac{N}{2}$ always becomes a differentiation with respect to a_0. We use Mathematica to compute the connection relation in the degenerate case for higher values of N.

References

[1] G. 't Hooft, *Dimensional reduction in quantum gravity*, *Conf. Proc. C* **930308** (1993) 284 [gr-qc/9310026].

[2] L. Susskind, *The World as a hologram*, *J. Math. Phys.* **36** (1995) 6377 [hep-th/9409089].

[3] J.M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[4] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys. Lett. B* **428** (1998) 105 [hep-th/9802109].

[5] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[6] C. Fefferman and C.R. Graham, *Conformal invariants*, *Astérisque* **S131** (1985) 95.

[7] S. de Haro, S.N. Solodukhin and K. Skenderis, *Holographic reconstruction of space-time and renormalization in the AdS / CFT correspondence*, *Commun. Math. Phys.* **217** (2001) 595 [hep-th/0002230].

[8] M. Bianchi, D.Z. Freedman and K. Skenderis, *Holographic renormalization*, *Nucl. Phys. B* **631** (2002) 159 [hep-th/0112119].

[9] C. Fefferman and C.R. Graham, *The ambient metric*, *Ann. Math. Stud.* **178** (2011) 1 [0710.0919].

[10] M.T. Anderson, *Geometric aspects of the AdS / CFT correspondence*, *IRMA Lect. Math. Theor. Phys.* **8** (2005) 1 [hep-th/0403087].

[11] L.F. Alday, M. Kologlu and A. Zhiboedov, *Holographic correlators at finite temperature*, *JHEP* **06** (2021) 082 [2009.10062].

[12] S. He, Y. Li, Y.-Z. Li and Y. Zhang, *Holographic torus correlators of stress tensor in AdS_3/CFT_2*, *JHEP* **06** (2023) 116 [2303.13280].

[13] S.W. Hawking and D.N. Page, *Thermodynamics of Black Holes in anti-De Sitter Space*, *Commun. Math. Phys.* **87** (1983) 577.

[14] E. Witten, *Anti-de Sitter space, thermal phase transition, and confinement in gauge theories*, *Adv. Theor. Math. Phys.* **2** (1998) 505 [hep-th/9803131].

[15] H. K., *Zur Theorie der Riemann'schen Functionen zweiter Ordnung mit vier Verzweigungspunkten*, *Math. Ann.* **33** (1888) 161.

[16] G. Bonelli, C. Iossa, D. Panea Lichtig and A. Tanzini, *Irregular Liouville Correlators and Connection Formulae for Heun Functions*, *Commun. Math. Phys.* **397** (2023) 635 [2201.04491].

[17] M. Dodelson, A. Grassi, C. Iossa, D. Panea Lichtig and A. Zhiboedov, *Holographic thermal correlators from supersymmetric instantons*, *SciPost Phys.* **14** (2023) 116 [2206.07720].

[18] A. Bhatta and T. Mandal, *Exact thermal correlators of holographic CFTs*, *JHEP* **02** (2023) 222 [2211.02449].

[19] G. Bonelli, C. Iossa, D.P. Lichtig and A. Tanzini, *Exact solution of Kerr black hole perturbations via CFT2 and instanton counting: Greybody factor, quasinormal modes, and Love numbers*, *Phys. Rev. D* **105** (2022) 044047 [2105.04483].

[20] G. Aminov, P. Arnaudo, G. Bonelli, A. Grassi and A. Tanzini, *Black hole perturbation theory and multiple polylogarithms*, 2307.10141.

[21] P.K. Kovtun and A.O. Starinets, *Quasinormal modes and holography*, *Phys. Rev. D* **72** (2005) 086009 [hep-th/0506184].

[22] C.R. Graham and J.M. Lee, *Einstein metrics with prescribed conformal infinity on the ball*, *Adv. Math.* **87** (1991) 186.

[23] P. Kovtun and A. Starinets, *Thermal spectral functions of strongly coupled N=4 supersymmetric Yang-Mills theory*, *Phys. Rev. Lett.* **96** (2006) 131601 [hep-th/0602059].

[24] D. Teaney, *Finite temperature spectral densities of momentum and R-charge correlators in N=4 Yang Mills theory*, *Phys. Rev. D* **74** (2006) 045025 [hep-ph/0602044].

[25] G. Policastro, D.T. Son and A.O. Starinets, *The Shear viscosity of strongly coupled N=4 supersymmetric Yang-Mills plasma*, *Phys. Rev. Lett.* **87** (2001) 081601 [hep-th/0104066].

[26] S.A. Hartnoll, A. Lucas and S. Sachdev, *Holographic quantum matter*, 1612.07324.

[27] S. Grozdanov, K. Schalm and V. Scopelliti, *Black hole scrambling from hydrodynamics*, *Phys. Rev. Lett.* **120** (2018) 231601 [1710.00921].

[28] M. Blake, H. Lee and H. Liu, *A quantum hydrodynamical description for scrambling and many-body chaos*, *JHEP* **10** (2018) 127 [1801.00010].

[29] M. Blake, R.A. Davison, S. Grozdanov and H. Liu, *Many-body chaos and energy dynamics in holography*, *JHEP* **10** (2018) 035 [1809.01169].

[30] A.L. Fitzpatrick and K.-W. Huang, *Universal Lowest-Twist in CFTs from Holography*, *JHEP* **08** (2019) 138 [1903.05306].

[31] Y.-Z. Li, Z.-F. Mai and H. Lü, *Holographic OPE Coefficients from AdS Black Holes with Matters*, *JHEP* **09** (2019) 001 [1905.09302].

[32] R. Karlsson, A. Parnachev, V. Prilepina and S. Valach, *Thermal stress tensor correlators, OPE and holography*, *JHEP* **09** (2022) 234 [2206.05544].

[33] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, *Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory*, *Nucl. Phys. B* **241** (1984) 333.

[34] L.F. Alday, D. Gaiotto and Y. Tachikawa, *Liouville Correlation Functions from Four-dimensional Gauge Theories*, *Lett. Math. Phys.* **91** (2010) 167 [0906.3219].

[35] N.A. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, *Adv. Theor. Math. Phys.* **7** (2003) 831 [hep-th/0206161].

[36] N. Nekrasov and A. Okounkov, *Seiberg-Witten theory and random partitions*, *Prog. Math.* **244** (2006) 525 [hep-th/0306238].

[37] L.F. Alday, D. Gaiotto, S. Gukov, Y. Tachikawa and H. Verlinde, *Loop and surface operators in N=2 gauge theory and Liouville modular geometry*, *JHEP* **01** (2010) 113 [0909.0945].

[38] N.A. Nekrasov and S.L. Shatashvili, *Quantization of Integrable Systems and Four Dimensional Gauge Theories*, in *16th International Congress on Mathematical Physics*, pp. 265–289, 8, 2009, DOI [0908.4052].