A Tight Universal Relation between the Shape Eccentricity and the Moment of Inertia for Rotating Neutron Stars

Yong Gao[1,2] , Lijing Shao[2,3] , and Jan Steinhoff[4]

[1] Department of Astronomy, School of Physics, Peking University, Beijing 100871, People's Republic of China
[2] Kavli Institute for Astronomy and Astrophysics, Peking University, Beijing 100871, People's Republic of China; lshao@pku.edu.cn
[3] National Astronomical Observatories, Chinese Academy of Sciences, Beijing 100012, People's Republic of China
[4] Max Planck Institute for Gravitational Physics (Albert Einstein Institute), Am Mühlenberg 1, Potsdam D-14476, Germany

Abstract

Universal relations that are insensitive to the equation of state are useful in reducing the parameter space when measuring global quantities of neutron stars (NSs). In this paper, we reveal a new universal relation that connects the eccentricity to the radius and moment of inertia of rotating NSs. We demonstrate that the universality of this relation holds for both conventional NSs and bare quark stars (QSs) in the slow-rotation approximation, albeit with different relations. The maximum relative deviation is approximately 1% for conventional NSs and 0.1% for QSs. Additionally, we show that the universality still exists for fast-rotating NSs if we use the dimensionless spin to characterize their rotation. The new universal relation will be a valuable tool to reduce the number of parameters used to describe the shape and multipoles of rotating NSs, and it may also be used to infer the eccentricity or moment of inertia of NSs in future X-ray observations.

1. Introduction

Neutron stars (NSs) are the densest stars in the Universe, offering a unique laboratory to study supranuclear matter and gravity in the strong-field regime. Currently, the equation of state (EOS) for the cores of NSs is still poorly understood. Many EOS models with varying compositions and states of dense matter have been developed, leading to significantly different predictions of global properties for NSs (Lattimer & Prakash 2001). Therefore, observed global properties of NSs, such as the mass and the radius, can be used to constrain EOS models.

Despite the fact that the global properties of NSs depend sensitively on the EOS models, there exist EOS-insensitive relations that connect various quantities of NSs. These relations are said to be universal because they are insensitive to EOS models to a high degree of accuracy. For instance, a universal relation connecting the frequency and damping time of the quadrupolar f mode to the mass and moment of inertia of NSs was discovered by Lau et al. (2010). Yagi & Yunes (2013a) found the famous I–Love–Q relation for slowly rotating NSs, which links the mass, the moment of inertia, the tidal Love number, and the spin-induced quadrupole moment. Universal relations for NSs are of great significance in both astrophysics and fundamental physics. By providing EOS-insensitive connections between different quantities, these relations allow us to extract global properties of NSs with higher accuracy, and help us study the inverse problem of determining the EOS. The universal relations are also a probe for nonperfect fluid inside NSs. For instance, it has been shown that anisotropic pressure (Yagi & Yunes 2015), strong magnetic fields (Haskell et al. 2014), and ultrahigh elasticity (Lau et al. 2017, 2019) can affect the global structure of NSs and potentially break the I–

Love–Q universal relation. Moreover, universal relations can break the degeneracy between gravity theories and the uncertainties in EOS, making NSs the ideal laboratories for testing gravity (Shao & Yagi 2022). We refer readers to Yagi & Yunes (2013b), Doneva & Pappas (2018), and references therein for reviews.

The exploration of the universal relations for rotating NSs has garnered lots of attention since the discovery of the I–Love–Q relation. The calculations of the I–Q relation were quickly extended to fast rotation by Doneva et al. (2013), and it was shown that the universality of the relation is lost and becomes increasingly EOS-dependent as the spin frequency increases. However, Pappas & Apostolatos (2014) and Chakrabarti et al. (2014) demonstrated that the I–Q relation remains universal if dimensionless quantities are used to characterize the spin amplitude instead of the spin frequency f. Pappas & Apostolatos (2014) and Yagi et al. (2014) discovered that the first four multipole moments of rotating NSs are universal to some extent. This relation allows for a more accurate description of the spacetime geometry around an NS with fewer parameters. Additionally, Luk & Lin (2018) found another universal relation connecting the radius and orbital frequency of the innermost stable circular orbit (ISCO) to the mass and spin frequency of rotating NSs.

Apart from multipoles and the ISCO, the eccentricity is another important parameter to describe rotating NSs. The oblateness induced by rotation has a large impact on the X-ray emissions from the surface of X-ray pulsars. To study this effect, Morsink et al. (2007) parameterized the oblate shape with the compactness of NSs and discovered that the geometric effect induced by the oblateness can rival the Doppler effect in certain configurations. To reduce the parameter space of X-ray modeling, Bauböck et al. (2013) derived a universal relation between the eccentricity and the compactness of slowly rotating NSs. AlGendy & Morsink (2014) also found an EOS-insensitive fit of the eccentricity, using a slightly different parameterization for the surface other than the one used in the

Hartle–Thorne formalism. Apart from multipoles and the ISCO, the oblate shape is also important. In the canonical pulse-profile modeling of X-ray pulsars, photons are emitted from an oblate surface and assumed to propagate in a Schwarzschild background, which is the so-called "Oblate + Schwarzschild" (O + S) approximation. To give an analytical formula to describe the rotation-induced oblateness, Morsink et al. (2007) and AlGendy & Morsink (2014) parameterized the oblate shape with the spin frequency and compactness of NSs. Recently, Silva et al. (2021b) developed a more accurate fitting formula compared to Morsink et al. (2007) and AlGendy & Morsink (2014), which better describes the large deformation of the surface for very rapid rotation. These fitting formulas can capture the shape of NSs at a wide range of spin frequencies, compactnesses, and EOSs (i.e., universal to some extent). In slow rotation, Bauböck et al. (2013) also explored the universal relation of rotating NSs. They showed that both the moment of inertia and the surface eccentricity can be approximately represented by a single parameter, the compactness. Frieben & Rezzolla (2012) uncovered quasi-universal relations relating surface distortion to spin frequency and magnetic field, which can be used to calculate surface distortion up to significant levels of rotation and magnetization.

In this paper, we discover a new universal relation between the surface eccentricity and the moment of inertia for rotating NSs. The paper is structured as follows. In Section 2, we provide a definition of multipoles and eccentricity in the slow-rotation approximation and present the universal relation for both conventional NSs and quark stars (QSs). In Section 3, we investigate the universal relation for fast-rotating NSs. A discussion of possible applications and connections of the new universal relation to early work is shown in Section 4. Throughout the paper, we use geometric units with $G = c = 1$.

2. A New Universal Relation in the Slow-rotation Approximation

2.1. Multipole Moments and Shape Parameters

To study the universal relation, we first give an overview of the structures and shape parameters of slowly rotating NSs. Following Hartle (1967) and Hartle & Thorne (1968), we construct these stars by solving the Einstein equations perturbatively in a slow-rotation expansion to quadratic order in the spin. At the zeroth order in spin, we obtain the mass M and the radius R of the nonrotating background. At the first order in spin, we extract the angular momentum J, from which we can define the moment of inertia I and the dimensionless spin χ as $I \equiv J/\Omega$ and $\chi \equiv J/M^2$, where Ω is the angular frequency of the rotating star. Universal relations usually connect dimensionless quantities. The dimensionless moment of inertia $\bar{I}$ is usually defined as $\bar{I} \equiv I/M^3$. At the second order in spin, the star is deformed into an oblate shape, and we get the spin-induced quadrupole moment $Q \equiv -J^2/M - 8KM^3/5$. The parameter K depends on the EOS of NSs and equals to zero for Kerr black holes according to the no-hair theorem. The dimensionless quadrupole moment is defined as $\bar{Q} \equiv -Q/M^3\chi^2$. The I–Q relation connects the dimensionless quantities $\bar{I}$ and $\bar{Q}$. The exterior spacetime of a slowly rotating NS can be fully described up to the quadratic order in spin by the mass M, the angular momentum J, and the quadrupole moment Q (Hartle & Thorne 1968).

Observationally, some observation of a rotating NS depends on the geometry of its surface. We use the eccentricity e_s to describe the oblate shape of an NS,

$$e_s \equiv \sqrt{\left(\frac{R_{eq}}{R_p}\right)^2 - 1}, \tag{1}$$

where R_{eq} and R_p are the equatorial and polar radii in a specific coordinate. In the Hartle–Thorne coordinate, the isodensity surface at radial coordinate r in the nonrotating star is displaced to

$$r \rightarrow r + \xi_0(r) + \xi_2(r)P_2(\cos\theta) \tag{2}$$

in the rotating configuration, where ξ_0 and ξ_2 are spherical and quadrupole displacements, respectively, and $P_2(\cos\theta)$ is the Legendre polynomial. Combining Equations (1)–(2), we get the surface eccentricity in the Hartle–Thorne coordinate as

$$e_{HT} = [-3(\xi_2(R)/R)]^{1/2}. \tag{3}$$

Equation (2) describes the isodensity surface in a particular coordinate system. By embedding the isodensity surface into a three-dimensional flat space (denoted by r^*, θ^*, ϕ^*), Hartle & Thorne (1968) found an invariant parameterization of the oblate surface. To the second order of the spin, the desired surface is a spheroid with

$$\begin{aligned} r^*(\theta^*) = r + \xi_0(r) \\ + \{\xi_2(r) + r[v_2(r) - h_2(r)]\}P_2(\cos\theta^*). \end{aligned} \tag{4}$$

Here v_2 and h_2 are metric functions at the second order in spin. The eccentricity of the stellar surface embedded in flat space is then given by

$$e_* = \{-3[v_2(R) - h_2(R) + \xi_2(R)/R]\}^{1/2}, \tag{5}$$

where the superscript "*" denotes the eccentricity observed in the flat space.

2.2. A Universal Relation for the Eccentricity of NSs

The universal relation that we discovered connects the quantity $e_s/R\Omega$ and the dimensionless moment of inertia $\bar{I}$. Similar to the I–Q relation (Yagi & Yunes 2013a) and the three-hair relation for the multipole moments (Yagi et al. 2014), the normalization factors M and R are quantities of the nonrotating background in the slow-rotation approximation. For convenience, we define a dimensionless radius, $\hat{R} \equiv R\Omega$. We have verified that the universal relation exists for both the eccentricity in the Hartle–Thorne coordinate, e_{HT}, and the eccentricity of the embedding surface, e_*. In the following, we use e_* to illustrate the results.

We first study the universal relation for conventional NSs. Our selection of realistic EOSs includes BSK21 (Goriely et al. 2010), AU (Wiringa et al. 1988), HLPS (Hebeler et al. 2013), PAL1 (Prakash et al. 1988), APR (Akmal et al. 1998), SLy4 (Douchin & Haensel 2001), MS0 (Mueller & Serot 1996), and ENG (Engvik et al. 1994). As shown in Figure 1, these models cover a wide range in the mass–radius diagram of static NSs, and all of them have a maximal NS mass larger than $2\,M_\odot$. Although the very stiff EOSs MS0 and PAL1 have been ruled out by the tidal deformability from GW170817 (Abbott et al. 2017), we include them to demonstrate that the universality exists for a large family of EOSs.
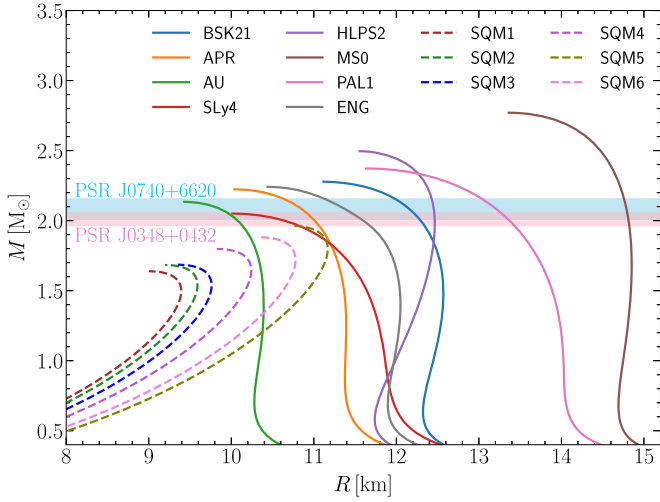
Figure 1. The mass–radius relation for selected EOS models of NSs (solid) and QSs (dashed). The 1σ regions of the mass measurements of PSR J0348+0432 (Antoniadis et al. 2013) and PSR J0740+6620 (Fonseca et al. 2021) are illustrated.

Our universal relation is described with great accuracy by

$$\frac{e_*}{\hat{R}} = \sum_{k=0}^{3} a_k (\ln \bar{I})^k, \qquad (6)$$

where a_k's are fitting coefficients with $a_0 = -0.855572$, $a_1 = 2.185502$, $a_2 = -0.428061$, and $a_3 = 0.051177$. Note that we use the dimensionless moment of inertia $\bar{I} \leqslant 100$, which corresponds to $M \gtrsim 0.5\,M_\odot$ for selected models. We define the relative deviation to Equation (6) as

$$\Delta = \frac{e_*/\hat{R} - (e_*/\hat{R})_{\text{fit}}}{(e_*/\hat{R})_{\text{fit}}}. \qquad (7)$$

As shown in Figure 2, the relative deviation to the universal relation is smaller than 1% for selected models of EOSs. NSs with $M \gtrsim 1\,M_\odot$ are more relevant for astrophysics, and, in this case, the dimensionless moment of inertia $\bar{I} \lesssim 30$ for selected EOS models, and the universal relation takes a simpler form,

$$\frac{e_*}{\hat{R}} = 0.11418 + 1.04115 \ln \bar{I}. \qquad (8)$$

The relative deviation to this relation is less than $\sim$1%.

For QSs, we use the phenomenological MIT bag model to describe the quark matter. This model assumes a nearly equal number of u, d, and s quarks and a small fraction of electrons confined within a bag of vacuum energy density B (Farhi & Jaffe 1984; Witten 1984). We account for the mass of the s quark, m_s, and include the quark–gluon interaction to the lowest order in $\alpha_c = g^2/4\pi$. To investigate the universal relation for QSs, we employ six different EOSs with varying combinations of m_s, α_c, and B in Table 1. The resulting mass–radius relation is displayed in Figure 1.

The relation between $e_*/\hat{R}$ and $\bar{I}$ is different from that of NSs, but it is still universal and can be well fitted by

$$\frac{e_*}{\hat{R}} = \sum_{k=0}^{4} b_k (\ln \bar{I})^k, \qquad (9)$$

with coefficients $b_0 = -1.499749$, $b_1 = 2.911859$, $b_2 = -0.749237$, $b_3 = 0.137057$, and $b_4 = -0.007801$.
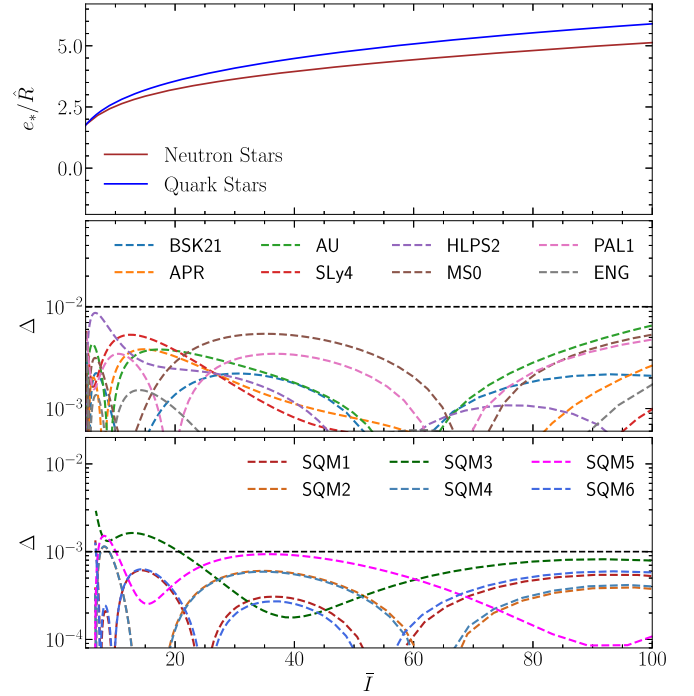
Figure 2. The $e_*/\hat{R}$–$\bar{I}$ universal relation for slowly rotating models. The upper panel shows the fitted universal relations for both NSs and QSs. The middle (lower) panel presents the relative deviation for NSs (QSs).

Table 1
Parameters for QSs in the MIT Bag Model

Model	$B(\text{MeV fm}^{-3})$	$m_s(\text{MeV})$	α_c
SQM1	80	100	0
SQM2	80	50	0.1
SQM3	70	150	0
SQM4	70	50	0.3
SQM5	60	0	0
SQM6	60	100	0.4

Interestingly, the deviation from the universal relation for QSs is much smaller than that for NSs, with a relative deviation of less than $\sim$0.1%, as shown in the lower panel of Figure 2. For QSs in our study, the condition for $M \gtrsim 1\,M_\odot$ corresponds to $\bar{I} \lesssim 20$. Within this range, the universal relation can be approximated by a simpler fitting formula,

$$\frac{e_*}{\hat{R}} = -0.043372 + 1.210546 \ln(\bar{I} - 0.470579). \qquad (10)$$

The relative deviation from this fitting formula is less than $\sim$0.3%.

Compared to the parameterization in Baubӧck et al. (2013), the new universal relation that we propose incorporates an extra parameter, namely the moment of inertia, in addition to the parameters R, M, and Ω. But the new universal relation is much tighter than that of Baubӧck et al. (2013).

3. Universal Relation for Fast-rotating NSs

Fast rotation is relevant for submillisecond pulsars, nascent NSs after supernovae, and NSs formed in binary NS mergers. Rapid rotation causes NSs to develop a more obvious oblate shape. In this section, we explore the universal relation for

rapidly rotating NSs using the RNS code developed by Stergioulas & Friedman (1995).

The RNS code uses a quasi-isotropic coordinate system to represent the line element of the stationary axisymmetric spacetime,

$$ds^2 = -e^{2\nu}dt^2 + B^2r^2\sin^2\theta e^{-2\nu}(d\phi - \omega dt)^2 + e^{2(\xi-\nu)}(dr^2 + r^2d\theta^2), \quad (11)$$

where ν, B, ω, and ξ are metric functions that depend on r and θ. Assuming a perfect fluid and uniform rotation, we obtain the stellar structure and spacetime metric. The conserved angular momentum J can be computed from a volume integration over the matter field. The moment of inertia I and the dimensionless spin have the same definition as before. The quadrupole moment Q can be obtained from the asymptotic expansion of the metric functions. The surface eccentricity is formally given by Equation (1), with the eccentricity, and equatorial and polar radii defined in the quasi-isotropic coordinate. We use the notation e_i to denote the eccentricity in the quasi-isotropic coordinate. The surface eccentricity is formally given by Equation (1), with the eccentricity e_i, equatorial radius R_{eq}^i, and polar radius R_q^i defined in the quasi-isotropic coordinate. Note that, unlike in the slow-rotation approximation, the normalization factor M is the mass for the rotating configuration, and $\hat{R} \equiv R_{eq}^i \Omega$.

To study universal relations for rapidly rotating NSs, it is necessary to use a suitable parameter to characterize their spin amplitude. As demonstrated by Doneva et al. (2013), if one uses the spin frequency f as the parameter, the I–Q relation for fast-rotating NSs is lost. Similarly, the universal relation that we discovered also breaks down for fixed spin frequencies. However, Pappas & Apostolatos (2014) and Chakrabarti et al. (2014) found that the I–Q relation is still universal for fast-rotating NSs if one chooses dimensionless spin parameters such as χ, Mf, and Rf, instead of the dimensionful f. Inspired by their work, we use χ to characterize the spin amplitude and find that the $e_i/\hat{R}$–$\bar{I}$ relation for both conventional NSs and strange QSs is still universal.

According to Lo & Lin (2011), the maximum value of the dimensionless spin parameter χ for NSs rotating at the Keplerian frequency is about 0.7 for various EOS models. This limit is nearly independent of the mass of the NS if the mass is larger than $1\,M_\odot$. However, for QSs in the MIT bag model, the spin parameter can be larger than unity and does not have a universal upper limit. Its value also depends strongly on the bag constant and the mass of the star. Therefore, in Figure 3, we display three representative cases with $\chi = 0.4$ and 0.6 for conventional NSs. The relative deviation from the universal relation is less than $\sim 1\%$. The cases for QSs with $\chi = 0.6$ and $\chi = 0.8$ are shown in Figure 4, and the relative deviation is of the order of 0.1%, which is again much tighter than conventional NSs. Therefore, we generate data points in the regime $0.2 \leqslant \chi \leqslant 0.6$ for NSs and $0.2 \leqslant \chi \leqslant 0.8$ for QSs.

In Figure 3, we show the universal relation for both NSs and QSs with $\chi = 0.6$. The relative errors are of the order of 0.3%. More explicitly, the $e_i/\hat{R}$–$\bar{I}$ relation with a dependence on χ can be fitted by

$$\frac{e_i}{\hat{R}} = \sum_{i,j} \mathcal{A}_{ij}\chi^i\log^j\bar{I}, \quad (12)$$
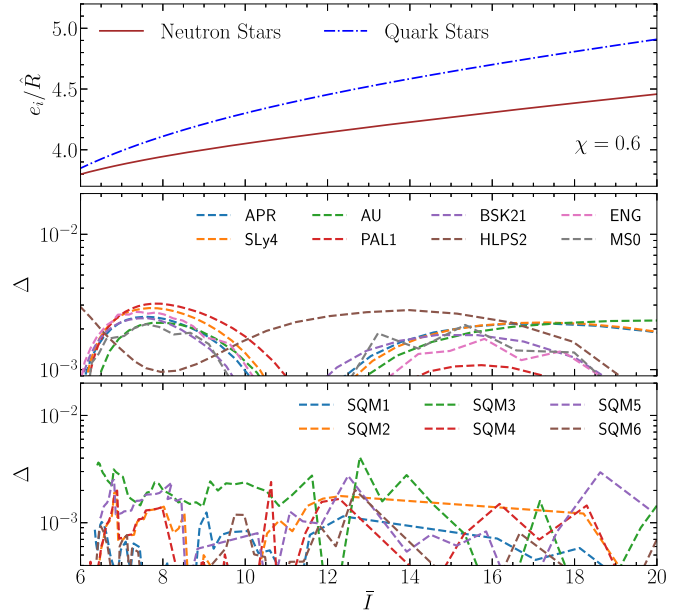
Figure 3. The $e_i/\hat{R}$–$\bar{I}$ universal relation for fast-rotating models with the dimensionless spin $\chi = 0.6$. The upper panel shows the fitted universal relations for both NSs and QSs. The middle (lower) panel presents the relative deviation for NSs (QSs).
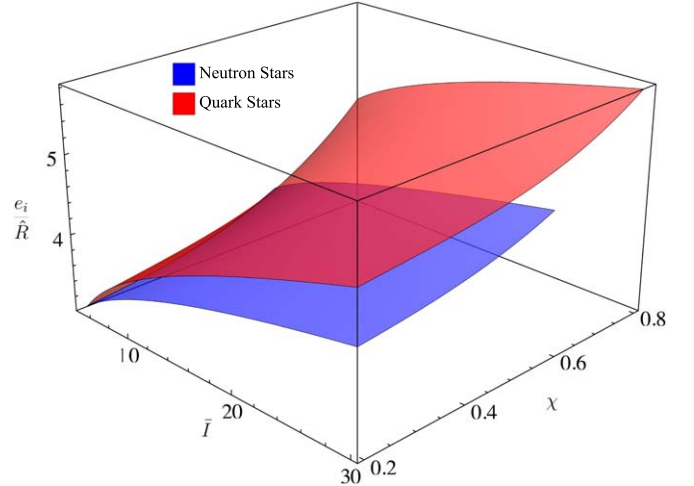
Figure 4. The surface of the fitting formula Equation (12) for NSs (blue) and QSs (red). Models of NSs and QSs fall on two well-defined surfaces.

where the numerical coefficients $\mathcal{A}_{ij}$ for NSs and QSs are given in Table 2. The maximum relative errors of the fitting formula are of the order of 1% for NSs and 0.3% for QSs. In Figure 4, we present the surfaces described by the fitting formula Equation (12). For a given χ and $\bar{I}$, the value of $e_i/\hat{R}$ for QSs is always larger than that for NSs. At the maximum mass limit, the two surfaces become closest.

In our study, we have defined the eccentricity in the quasi-isotropic coordinate system, where the radial coordinate r corresponds to the isotropic Schwarzschild coordinate in the limit of zero spin. However, Morsink et al. (2007), AlGendy & Morsink (2014), and Silva et al. (2021b) define the eccentricity differently. We know that circles centered about the symmetric axis have circumference $2\pi\bar{r}$, where $\bar{r}$ is related to r and θ by

$$\bar{r} = e^{-\nu(r,\theta)}B(r,\theta)r\sin\theta = r_c(r,\theta)\sin\theta. \quad (13)$$

Table 2
Numerical Coefficients for the Two-parameter Fitting Formula Equation (12)

$i =$	0	1	2	3
	Coefficients for neutron stars			
$\mathcal{A}_{i0}$	2.149039	−0.146205	−1.867329	5.65915
$\mathcal{A}_{i1}$	1.23962	2.21818	0.942935	−2.87271
$\mathcal{A}_{i2}$	−0.165024	−2.63504	2.28603	−2.53576
$\mathcal{A}_{i3}$	0.158013	1.04597	−1.80829	2.23285
	Coefficients for quark stars			
$\mathcal{A}_{i0}$	0.98273	−5.06930	12.9114	5.65915
$\mathcal{A}_{i1}$	4.29319	16.3498	−48.8296	30.4711
$\mathcal{A}_{i2}$	−3.12737	−12.9528	50.1446	−37.8440
$\mathcal{A}_{i3}$	1.30675	3.27539	−16.5473	13.8336

Here r_c corresponds to the Schwarzschild coordinate in the limit of zero spin. By using r_c, the equatorial and polar radii can be defined as

$$R_{\mathrm{eq}}^c = r_c\left(R_{\mathrm{eq}}^i, \theta = \frac{\pi}{2}\right), \quad R_{\mathrm{p}}^c = r_c(R_{\mathrm{p}}^i, \theta = 0), \quad (14)$$

where R_{eq}^c is the circumferential radius of the star in the equatorial plane. Then the surface eccentricity e_c can be obtained from Equation (1).

We have found that the universality still holds for the eccentricity e_c, but its relation differs from that of e_i. Specifically, for given EOS families and a fixed value of $\bar{I}$, the value of $e_i/\hat{R}$ is larger than $e_c/\hat{R}$. To compare the universal relations for these two eccentricities, we have included an example in Figure 5. The figure shows the $e_i/\hat{R}$–$\bar{I}$ and $e_c/\hat{R}$–$\bar{I}$ relations for NSs and QSs at $\chi = 0.5$, which highlights the differences of the universal relations for these two eccentricities.

4. Discussion

The relation between $e_s/\hat{R}$ and $\bar{I}$ adds a new tight universal relation to the known ones. Here, e_s is the surface eccentricity formally defined in Equation (1) without assigning a specific coordinate system. In our paper, it includes e_s^{HT}, e_s^*, and $\tilde{e}_s$, which are eccentricities defined in different coordinate systems. The effect of gauge choice on the eccentricity is very small. All of these eccentricities satisfy the universal relation very well. The universal relation between $e_s/\hat{R}$ and $\bar{I}$ adds a new tight universal relation to the known ones. Here, e_s is the surface eccentricity formally defined in Equation (1), including e_{HT}, e_*, e_i, and e_c in our work. It is important to note that these different definitions of eccentricity lead to different universal relations. Since the I–Q relation (Yagi & Yunes 2013a; Chakrabarti et al. 2014) connects $\bar{I}$ to $\bar{Q}$, and the three-hair relation (Pappas & Apostolatos 2014; Yagi et al. 2014) connects $\bar{Q}$ to two other higher-order multipoles, the relation between $e_s/\hat{R}$ and these normalized multipole moments is also universal. Combined with the I–Love relation (Yagi & Yunes 2013a), we have a universal relation between $e_s/\hat{R}$ and the dimensionless tidal Love number. Moreover, the universal relation between the f-mode oscillation and $\bar{I}$ (Lau et al. 2010) helps us connect $e_s/\hat{R}$ to the frequency and damping time of the quadrupolar f mode.

Universal relations are a powerful tool to reduce modeling uncertainties and infer NS parameters. As we discussed before,
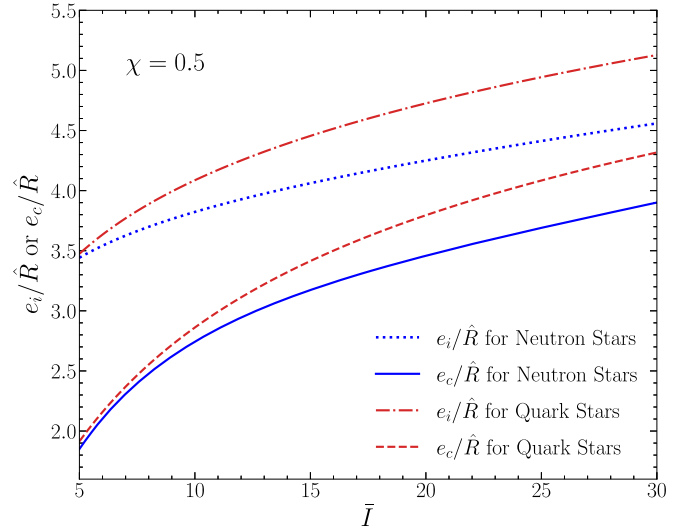
Figure 5. The $e_i/\hat{R}$–$\bar{I}$ and $e_c/\hat{R}$–$\bar{I}$ relations for NSs (blue) and QSs (red). Here we take the dimensionless spin $\chi = 0.5$.

the eccentricity of rotating NSs is observable and is an important input for X-ray modeling. The oblateness induced by rotation at frequencies above 300 Hz produces a geometric effect that has imprints in the pulse profile of X-ray pulsars (Morsink et al. 2007). For some emission configurations, the oblateness effect can rival the Doppler effect. As a result, the effects of oblateness need to be taken into account when measuring the radii of NSs from rotationally broadened atomic lines (Baubock et al. 2013).

On one hand, our new universal relation reduces the number of parameters used to describe the shape and multipoles of rotating NSs. On the other hand, the new universal relation can be used to infer the eccentricity of NSs if the mass, radius, spin frequency, and moment of inertia are inferred in future X-ray observations. The new universal relation we obtained can be used to help infer NS properties. For example, if future X-ray observations can measure the eccentricity and radius of pulsars, one can use the universal relation to forecast the moment of inertia of NSs with similar masses and use the I–Love relation to test gravity (Silva et al. 2021a). Conversely, if the moment of inertia is obtained through observations of, say, the double pulsar system (Lattimer & Schutz 2005; Hu et al. 2020; Kramer et al. 2021) or gravitational waves from a binary NS inspiral (Lau et al. 2010), our universal relation can be employed to improve the inference of the radii of NSs through X-ray observations.

For very rapid rotation, deviations from a simple ellipse become potentially important. Currently, telescopes such as NICER observe only slowly rotating NSs, for which approximating the shape as an ellipse is accurate enough. However, if NICER or similar telescopes were to observe highly rapidly rotating pulsars, the full shape function for the surface would be required. Previous studies, such as Morsink et al. (2007), AlGendy & Morsink (2014), and Silva et al. (2021b), have parameterized the oblate shape of NSs using the spin frequency and compactness as parameters. In comparison to these works, our newly discovered universal relation incorporates an additional parameter, the moment of inertia, and only focuses on the eccentricity of the star. In the future, it is necessary to extend our framework to accurately fit the shape of the star at

all latitudes, especially when interpreting the observations of rapidly rotating NSs.

ORCID iDs

Yong Gao ⦿ https://orcid.org/0000-0003-1390-5477
Lijing Shao ⦿ https://orcid.org/0000-0002-1334-8853
Jan Steinhoff ⦿ https://orcid.org/0000-0002-1614-0214

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