Twistor quadrics and black holes

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Abstract

A simple procedure is given to construct curved, non-self-dual (complexified) Kähler metrics on space-time in terms of deformations of holomorphic quadric surfaces in flat twistor space. Imposing Lorentzian reality conditions, the Schwarzschild, Kerr, and Plebański-Demiański space-times (among others) are derived as examples of the construction.

1 Introduction

Twistor theory constitutes a remarkable approach to the description of the complex structure of space-time [1, 2, 3, 4]. Motivated by the fact that the celestial sphere of any event in Minkowski space has a natural complex structure and is thus a Riemann sphere $\mathbb{CP}^1$ (cf. [5]), the space-time manifold emerges as the moduli space of $\mathbb{CP}^1$'s ('twistor lines') in twistor space. These lines are holomorphic, which implies that our intuition about lines and their intersection in $\mathbb{R}^3$ continues to hold in twistor space, and allows to define the conformal structure of space-time by intersection of lines. Gravitation, in the twistor view, should correspond to deformations of the flat twistor structure. Penrose's non-linear graviton construction [2] shows that this holds true for self-dual (or half-flat) curved space-times.

If a geometry admits a twistor space and, in addition, it has some further special structure, this extra structure is often holomorphically encoded in the twistor space. For example, in Riemannian geometry, Pontecorvo showed [6] that a Kähler metric in the conformal structure of a conformally half-flat 4-manifold corresponds to a holomorphic section of (the square root of) the anti-canonical bundle of the twistor space, with two zeros on each twistor line. This can also be understood in terms of a holomorphic surface that intersects each twistor line at two points.

Self-dual curvature is a strong restriction for general relativity, as it implies (conformally) flat space-time. In this note, inspired by Pontecorvo's construction and by the non-linear graviton, we describe a variation of the twistor construction that produces non-self-dual (complexified) Kähler metrics on space-time as deformations of holomorphic quadric surfaces in flat twistor space. Part of the basic intuition is that two points define a line; this carries over to twistor space since everything is holomorphic (the two points in question being the intersection points of twistor lines with the quadric; see below). Imposing Lorentzian reality conditions, we show that the general Plebański-Demiański class of space-times [7, 8], which includes the standard black hole metrics of general relativity such as Schwarzschild, Kerr, etc., is recovered by this construction.

2 Twistor quadrics and Kähler metrics

2.1 Preliminaries

We start off by introducing some basic definitions to set our conventions. Let (M, g) be a 4-dimensional, orientable Riemannian or Lorentzian manifold. We also allow complexified geometries. We say that a

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$(1,1)$ tensor J is an almost-complex structure if it satisfies $J^2 = -\mathbb{I}$ and its $(\pm \mathrm{i})$-eigenspaces $T^\pm$ have the same rank. We say that J is, in addition, compatible with g if it holds $g(J\cdot, J\cdot) = g(\cdot,\cdot)$; in this case we refer to (g, J) as an almost-Hermitian structure. The $(\pm \mathrm{i})$-eigenspaces $T^\pm$ of J split the tangent bundle as $T^+ \oplus T^-$. We say that J is integrable, and is thus a complex structure, if $T^\pm$ are both involutive under the Lie bracket, i.e. $[T^\pm, T^\pm] \subset T^\pm$ (for both signs). If this is satisfied and J is also compatible with g, then (g, J) is a Hermitian structure.

Given an almost-Hermitian structure, the fundamental 2-form is defined by $\kappa(\cdot,\cdot) = g(J\cdot,\cdot)$. The 2-form κ and J are compatible in the sense that $\kappa(J\cdot, J\cdot) = \kappa(\cdot,\cdot)$. Note that any two of g, J, κ determines the third; in particular, given compatible κ, J, the metric g is determined by $g(\cdot,\cdot) = \kappa(\cdot, J\cdot)$. Finally, we say that the geometry is (complexified) Kähler if (g, J) is Hermitian and $\mathrm{d}\kappa = 0$. In this case, κ is also referred to as the symplectic form.

Concerning reality conditions, one can show that given an almost-Hermitian structure, the fundamental 2-form is an eigenform of the Hodge star operator, so it is self-dual (SD) or anti-self-dual (ASD). In Riemann signature, (A)SD 2-forms are real, whereas in Lorentz signature they are complex. Therefore, the tensor J is real-valued in Riemann signature, and complex-valued in Lorentz signature. Lorentzian Kähler geometry was thoroughly investigated by Flaherty [9].

2.2 Twistor space

Let $\mathbb{CM}$ denote complexified Minkowski space, with complexified inertial coordinates $t_\mathrm{c}, x_\mathrm{c}, y_\mathrm{c}, z_\mathrm{c}$ and flat holomorphic metric $\eta = \mathrm{d}t_\mathrm{c}^2 - \mathrm{d}x_\mathrm{c}^2 - \mathrm{d}y_\mathrm{c}^2 - \mathrm{d}z_\mathrm{c}^2$. Introduce double null coordinates $u_\mathrm{c}, v_\mathrm{c}, w_\mathrm{c}, \tilde{w}_\mathrm{c}$ by $u_\mathrm{c} = \frac{1}{\sqrt{2}}(t_\mathrm{c} + z_\mathrm{c})$, $v_\mathrm{c} = \frac{1}{\sqrt{2}}(t_\mathrm{c} - z_\mathrm{c})$, $w_\mathrm{c} = \frac{1}{\sqrt{2}}(x_\mathrm{c} + \mathrm{i}y_\mathrm{c})$, $\tilde{w}_\mathrm{c} = \frac{1}{\sqrt{2}}(x_\mathrm{c} - \mathrm{i}y_\mathrm{c})$. Twistor space is the manifold $\mathbb{PT} = \mathbb{CP}^3 \backslash \mathbb{CP}^1$, and it is related to space-time via the incidence relation

$$\begin{pmatrix} Z^0 \\ Z^1 \end{pmatrix} = \mathrm{i} \begin{pmatrix} u_\mathrm{c} & w_\mathrm{c} \\ \tilde{w}_\mathrm{c} & v_\mathrm{c} \end{pmatrix} \begin{pmatrix} Z^2 \\ Z^3 \end{pmatrix} \tag{1}$$

where we use homogeneous coordinates $Z^\alpha = (Z^0, Z^1, Z^2, Z^3)$ on $\mathbb{CP}^3$ (the $\mathbb{CP}^1$ removed corresponding to $Z^2 = Z^3 = 0$). The twistor correspondence (1) is non-local. Fixing $Z^\alpha \in \mathbb{PT}$, the set of space-time points satisfying (1) is a totally null 2-surface in $\mathbb{CM}$, called 'α-surface'. Fixing $(u_\mathrm{c}, v_\mathrm{c}, w_\mathrm{c}, \tilde{w}_\mathrm{c}) \in \mathbb{CM}$, the set of Z^α satisfying (1) is a (holomorphic, linearly embedded) Riemann sphere $L_x \cong \mathbb{CP}^1$, which is called 'twistor line'. Space-time is the moduli space of twistor lines in $\mathbb{PT}$, and twistor space is the moduli space of α-surfaces in $\mathbb{CM}$. Inhomogeneous local coordinates for twistor space are given by the equations that define α-surfaces, that is (putting $\zeta = Z^3/Z^2$, in a region with $Z^2 \neq 0$)

$$\omega^0 = u_\mathrm{c} + \zeta w_\mathrm{c}, \qquad \omega^1 = \tilde{w}_\mathrm{c} + \zeta v_\mathrm{c}, \qquad \zeta. \tag{2}$$

For fixed $(u_\mathrm{c}, v_\mathrm{c}, w_\mathrm{c}, \tilde{w}_\mathrm{c})$ and variable ζ, these three quantities describe a twistor line in $\mathbb{PT}$. For fixed $(\omega^0, \omega^1, \zeta)$ and variable $(u_\mathrm{c}, v_\mathrm{c}, w_\mathrm{c}, \tilde{w}_\mathrm{c})$, (2) describe an α-surface in $\mathbb{CM}$.

Twistor space is fibered over $\mathbb{CP}^1$, being the total space of the fiber bundle $\mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathbb{CP}^1$. Here, $\mathcal{O}(-1)$ is the tautological line bundle over $\mathbb{CP}^1$, and $\mathcal{O}(k) = \mathcal{O}(-1)^{*\otimes k}$ $(k > 0)$. The base of the fibration has homogeneous coordinates Z^2, Z^3, or inhomogeneous coordinate ζ, and the fibers have coordinates Z^0, Z^1. Each fiber has an $\mathcal{O}(2)$-valued symplectic structure $\mu = \mathrm{d}Z^0 \wedge \mathrm{d}Z^1$. All of this is valid for flat space-times. One of the main ideas in twistor theory is that gravitation, namely curved space-times, should correspond to deformations of twistor structures. Penrose showed [2] that this is true for half-flat space-times: he proved that an ASD, Ricci-flat, complex space-time corresponds to a 3-dimensional complex manifold $\mathcal{PT}$ obtained as a deformation of $\mathbb{PT}$ that preserves the fibration $\mathcal{PT} \to \mathbb{CP}^1$ and the fiberwise symplectic structure μ.

2.3 Quadrics

In a Riemannian setting, twistor space can also be defined as the space of almost-complex structures compatible with a 4-dimensional Riemannian conformal structure. This space coincides with the (6-real-dimensional) projective spin bundle, which can be shown to be a complex 3-manifold (the twistor space $\mathcal{PT}$) if and only if the conformal structure is ASD.

In this context, one may ask what a Kähler metric in the conformal structure corresponds to in twistor space. This was studied by Pontecorvo [6], who showed that a (necessarily scalar-flat) Kähler metric corresponds to a preferred (global, holomorphic) section of $K_{\mathcal{PT}}^{-1/2}$ which vanishes at two points in each twistor line, where $K_{\mathcal{PT}}^{-1/2}$ is the square-root of the anti-canonical line bundle of $\mathcal{PT}$. We can see this by first noticing that the bundle $K_{\mathcal{PT}}^{-1/2}$ restricted to a twistor line is $\mathcal{O}(2)$. Hitchin showed in [10, Section 2] that the space $H^0(\mathcal{PT},\mathcal{O}(k))$ of global, holomorphic sections of $\mathcal{O}(k)$ can be identified with $\ker(T_k)$, where T_k is the valence-k twistor operator. This implies that a section χ of $K_{\mathcal{PT}}^{-1/2}$ corresponds to a valence-2 twistor-spinor, or Killing spinor. The requirement that χ vanishes at two points in each twistor line means that the Killing spinor is non-degenerate. Using then [11, Lemma 2.1], this corresponds to a conformal Kähler structure on space-time.

In complexified flat space-time, we can also describe Pontecorvo's construction in terms of holomorphic quadric surfaces in flat twistor space $\mathbb{PT}$. The space $H^0(\mathbb{CP}^1,\mathcal{O}(k))$ consists of degree k homogeneous polynomials in $\mathbb{C}^2$. Thus, a section χ of $K_{\mathbb{PT}}^{-1/2}$, when restricted to a twistor line, is of the form

$$\chi = A\zeta^2 + 2B\zeta + C \tag{3}$$

for some A, B, C, where we are using an inhomogeneous coordinate ζ on $\mathbb{CP}^1$. We can then think of χ as a holomorphic quadratic function $\chi(Z^\alpha) = Q_{\alpha\beta}Z^\alpha Z^\beta$ for some symmetric $Q_{\alpha\beta}$. The expression (3) follows after using the incidence relation (1), which also shows that $A = A(v_{\rm c}, w_{\rm c})$, $B = B(u_{\rm c}, v_{\rm c}, w_{\rm c}, \tilde{w}_{\rm c})$ and $C = C(u_{\rm c}, \tilde{w}_{\rm c})$. The zero set of χ is a holomorphic quadric, $\mathbb{Q} = \{Z^\alpha \in \mathbb{PT}\,|\,\chi(Z^\alpha) = 0\}$. From Kerr's theorem [1], the surface $\mathbb{Q}$ corresponds to a shear-free, null geodesic congruence in $\mathbb{CM}$. The condition that χ vanishes at two points in each twistor line L_x is the same as saying that L_x intersects $\mathbb{Q}$ at two points, corresponding to the two roots $\zeta_\pm$ of the quadratic polynomial (3), that is $\chi = A(\zeta-\zeta_+)(\zeta-\zeta_-)$, where $\zeta_\pm = (-B \pm \sqrt{B^2 - AC})/A$. We allow, however, the possibility of twistor lines where the roots coincide; these correspond to caustics in the ray congruence on space-time (and will later correspond to curvature singularities for the curved, non-self-dual metric we will construct). Importantly, the quadric is divided into two regions in twistor space:

$$\mathbb{Q} = \mathbb{A}_+ \cup \mathbb{A}_-, \tag{4}$$

where $\mathbb{A}_\pm$ can be described in local coordinates by any two of (see (2))

$$\omega^0_\pm = u_{\rm c} + \zeta_\pm w_{\rm c}, \qquad \omega^1_\pm = \tilde{w}_{\rm c} + \zeta_\pm v_{\rm c}, \qquad \zeta_\pm, \tag{5}$$

or by any function of them. For fixed $+$ or $-$, the three coordinates in (5) are functionally dependent as a consequence of the quadric equation $\chi(\omega^0_\pm, \omega^1_\pm, \zeta_\pm) = 0$. A simple example to illustrate this (and to have in mind in general) is a product of planes, that is, a quadric given by $Q_{\alpha\beta} = A^+_{(\alpha}A^-_{\beta)}$ for some fixed $A^\pm_\alpha$. The two regions in (4) are in this case two planes $\mathbb{A}_\pm = \{Z^\alpha | A^\pm_\alpha Z^\alpha = 0\}$, and the roots coincide in the twistor line corresponding to the intersection of the planes, see e.g. [12, Fig. 6-11].

Since, generically, a twistor line L_x intersects $\mathbb{Q}$ at two points, and since two points define a unique line through them, the two intersection points can also be used to characterise the twistor line L_x. In other words, varying the line L_x, the intersection points serve as a coordinate system on space-time. Given local holomorphic coordinates on $\mathbb{A}_\pm$ (obtained e.g. from (5)), say $z^A_\pm$ with $A = 0, 1$, the pair (z^A_+, z^A_-) is the desired coordinate system on $\mathbb{CM}$. The complex structure J induced on space-time from the quadric $\mathbb{Q}$ can then be shown to be

$$J = {\rm i}(\partial_{z^A_+} \otimes {\rm d}z^A_+ - \partial_{z^A_-} \otimes {\rm d}z^A_-) \tag{6}$$

where the Einstein summation convention is assumed. The tensor (6) is compatible with the Minkowski metric. In particular, the vectors $\partial_{z^A_\pm}$ are null. In fact, the construction so far is conformally invariant. The Kähler structure is obtained from the symplectic form, which can be shown to be

$$\kappa = \frac{\rm i}{(B^2 - AC)^{3/2}}[A{\rm d}u_{\rm c} \wedge {\rm d}\tilde{w}_{\rm c} - B({\rm d}u_{\rm c} \wedge {\rm d}v_{\rm c} + {\rm d}w_{\rm c} \wedge {\rm d}\tilde{w}_{\rm c}) + C{\rm d}w_{\rm c} \wedge {\rm d}v_{\rm c}] \tag{7}$$

(A, B, C are defined in (3)). One can show this using, for example, the Penrose transform for spin 1, with the twistor function $f(Z^\alpha) = [\chi(Z^\alpha)]^{-2}$. In terms of quadric coordinates $z^A_\pm$, the symplectic form (7) is

$$\kappa = \kappa_{A\tilde{B}}{\rm d}z^A_+ \wedge {\rm d}z^B_-, \qquad \kappa_{A\tilde{B}} = \kappa(\partial_{z^A_+}, \partial_{z^B_-}). \tag{8}$$

Indices $A, \tilde{B}, \ldots$ are numerical and take values $0, 1$ (and again Einstein summation is used). The distinction between an index 'B' and an index '$\tilde{B}$' is only intended to remind that they are associated to the two different halves of the quadric, and in equations like (8) they are summed over as usual.

3 Deformed quadrics and non-self-dual Kähler metrics

Consider a holomorphic quadric $\mathbb{Q}$ in twistor space, which is arbitrary except for the assumption that, generically, twistor lines intersect $\mathbb{Q}$ at two points, so that the quadric is divided into two regions $\mathbb{A}_\pm$ as in (4). Choose holomorphic coordinates $z^A_\pm$ on $\mathbb{A}_\pm$. We now introduce a "deformed" quadric as

$$\mathcal{Q} = \mathcal{A}_+ \cup \mathcal{A}_-, \tag{9}$$

where $\mathcal{A}_+$ and $\mathcal{A}_-$ are the level sets of the four functions $\dot{z}^A_+$ and $\dot{z}^A_-$ defined by

$$\dot{z}^A_+ = z^A_+, \qquad \dot{z}^A_- = z^A_- + f^A(z^B_+, z^B_-) \tag{10}$$

for some functions f^A, such that $\mathrm{d}\dot{z}^0_+ \wedge \mathrm{d}\dot{z}^1_+ \wedge \mathrm{d}\dot{z}^0_- \wedge \mathrm{d}\dot{z}^1_- \neq 0$. Although one half of the quadric remains "undeformed", $\mathbb{A}_+ = \mathcal{A}_+$, the other half $\mathbb{A}_-$ is deformed to $\mathcal{A}_-$ and $\mathcal{Q}$ is, in general, *not* inside twistor space (a point in $\mathcal{A}_-$ is not in twistor space, since it does not correspond to an α-surface in $\mathbb{CM}$). We also note that in order to get a non-trivial construction, the functions f^A must depend on both z^A_+ and z^A_-, otherwise (10) would just be a diffeomorphism on the quadric. Although our construction is inspired by the non-linear graviton, the sense in which (9) is a deformed quadric does not seem to be the same as the complex-structure-deformations of twistor theory.

Recalling that the complex structure on space-time induced by the original quadric is given by (6), we associate the deformed quadric to a new complex structure:

$$\dot{J} = \mathrm{i}(\partial_{\dot{z}^A_+} \otimes \mathrm{d}\dot{z}^A_+ - \partial_{\dot{z}^A_-} \otimes \mathrm{d}\dot{z}^A_-). \tag{11}$$

This is an integrable almost-complex structure on the (complexified) space-time manifold, but it is *not* Hermitian: it is not compatible with the Minkowski metric. In particular, unlike the undeformed quadric, the new vectors $\partial_{\dot{z}^A_+}$ are not null (they are linear combinations of $\partial_{z^A_+}$ and $\partial_{z^A_-}$). We then interpret the deformation (10) of the quadric as a deformation of the conformal structure on space-time: new conformal structures are introduced by requiring that their null cones contain $\partial_{\dot{z}^A_\pm}$.

This requirement alone, however, does not fix a metric. In order to do this, we must ask additional conditions on the deformations (10). To this end, we choose to restrict to quadric deformations that preserve the symplectic structure induced on space-time, κ. In some sense, we can take inspiration for this restriction from the non-linear graviton, where the twistor deformations preserve the fiberwise symplectic structure (which allows to reconstruct the space-time metric); however, this is not the same since we are here dealing with a symplectic structure on space-time (not on twistor space). Regardless, the symplectic-form-preserving condition allows to fix a metric:

$$g(X, Y) := \kappa(X, \dot{J}Y), \tag{12}$$

for all vectors X, Y, where the symmetry property of this map follows from requiring κ and $\dot{J}$ to be compatible, which in turn is the same as requiring $\partial_{\dot{z}^A_\pm}$ to be null. The metric (12) is then

$$g = 2 g_{A\tilde{B}} \mathrm{d}\dot{z}^A_+ \odot \mathrm{d}\dot{z}^B_-, \tag{13}$$

where $g_{A\tilde{B}} = g(\partial_{\dot{z}^A_+}, \partial_{\dot{z}^B_-})$. As before, all indices here are numerical, see below eq. (8) for our conventions. A calculation shows that the deformations (10) preserve the symplectic structure κ if and only if the functions $g_{A\tilde{B}}$ and f^A satisfy

$$g_{A\tilde{C}} \left(\delta^C_B + \frac{\partial f^C}{\partial z^B_-} \right) = \mathrm{i}\kappa_{A\tilde{B}}, \tag{14a}$$

$$\epsilon^{AC} g_{A\tilde{B}} \frac{\partial f^B}{\partial z^C_+} = 0, \tag{14b}$$

where the four functions $\kappa_{A\tilde{B}}$ are defined in (8). The functions $g_{A\tilde{B}}$ in (13) can then be computed from eq. (14a) (by inverting the matrix inside the brackets on the left), and the deformation functions f^A are not completely arbitrary but are restricted by the condition (14b).

In summary: the result of this construction is a new metric (13)-(14) on space-time that is generically curved, non-(A)SD, and automatically (complexified) Kähler.

4 Black holes

Consider the holomorphic quadric $\mathbb{Q} \subset \mathbb{PT}$ given as the zero set of the following quadratic function:

$$\chi(Z^\alpha) = Z^0 Z^3 - Z^1 Z^2. \tag{15}$$

On twistor lines, this adopts the form (3) with $A = w_c$, $B = \frac{1}{2}(u_c - v_c)$, $C = -\tilde{w}_c$ (we are omitting an irrelevant overall factor of i coming from (1)). The roots are then easily computed to be

$$\zeta_\pm = \frac{-z_c \pm r_c}{x_c + \mathrm{i}y_c}, \qquad r_c := \sqrt{x_c^2 + y_c^2 + z_c^2}. \tag{16}$$

In some sense, we could say that ζ_+ and ζ_- are "related" by a complexified antipodal map: if $\tilde{\zeta}_\pm = (-z_c \pm r_c)/(x_c - \mathrm{i}y_c)$, then $\zeta_+ = -1/\tilde{\zeta}_-$. Twistor lines with $r_c = 0$ intersect $\mathbb{Q}$ only once; the Kähler structure on $\mathbb{CM}$ is not well-defined at these points. The symplectic form (7) is

$$\kappa = \frac{\mathrm{i}}{r_c^3}\left[x_c(\mathrm{d}t_c \wedge \mathrm{d}x_c + \mathrm{i}\mathrm{d}y_c \wedge \mathrm{d}z_c) + y_c(\mathrm{d}t_c \wedge \mathrm{d}y_c + \mathrm{i}\mathrm{d}z_c \wedge \mathrm{d}x_c) + z_c(\mathrm{d}t_c \wedge \mathrm{d}z_c + \mathrm{i}\mathrm{d}x_c \wedge \mathrm{d}y_c)\right]. \tag{17}$$

Recalling (4) and (5), we choose the following quadric coordinates $z^0_\pm, z^1_\pm$ on $\mathbb{A}_\pm$:

$$z^0_\pm = \omega^0_\pm, \qquad z^1_\pm = \tfrac{\mathrm{i}}{\sqrt{2}}\log(\pm\zeta_\pm). \tag{18}$$

We now impose reality conditions: we take the real Lorentzian slice in $\mathbb{CM}$ defined by

$$t_c = t, \quad x_c = x, \quad y_c = y, \quad z_c = z - \mathrm{i}a \tag{19}$$

where t, x, y, z are all real, and a is a real parameter. The function r_c in (16) is complex: we denote by r its real part, so that (from the definition of r_c) we must have $r_c = r - \mathrm{i}az/r$. Let us introduce a real coordinate system (r, p, φ) related to Cartesian coordinates (x, y, z) by

$$x + \mathrm{i}y = \sqrt{(r^2 + a^2)(1 - p^2)}e^{\mathrm{i}\varphi}, \qquad z = rp \tag{20}$$

(where we assume $p^2 < 1$). The symplectic form (17) and the quadric coordinates (18) become

$$\kappa = \frac{\mathrm{i}}{(r - \mathrm{i}ap)^2}\left[\mathrm{d}t \wedge \mathrm{d}(r - \mathrm{i}ap) - \mathrm{d}\varphi \wedge (a(1 - p^2)\mathrm{d}r - \mathrm{i}(r^2 + a^2)\mathrm{d}p)\right], \tag{21}$$

$$z^0_\pm = \tfrac{1}{\sqrt{2}}[t \pm (r - \mathrm{i}ap)], \qquad z^1_\pm = \tfrac{1}{\sqrt{2}}\left[\varphi \pm \left(-\arctan(a/r) - \tfrac{\mathrm{i}}{2}\log\left(\tfrac{1+p}{1-p}\right)\right)\right]. \tag{22}$$

After some calculations, we find the components $\kappa_{A\tilde{B}}$ in (8) to be $\kappa_{0\tilde{0}} = \mathrm{i}r_c^{-2}$, $\kappa_{0\tilde{1}} = 0 = \kappa_{1\tilde{0}}$, $\kappa_{1\tilde{1}} = -\mathrm{i}r_c^{-2}(r^2 + a^2)(1 - p^2)$, where $r_c^2 = (r - \mathrm{i}ap)^2$.

Following the prescription (10), we now deform the quadric given by (15) to a new quadric $\mathcal{Q} = \mathcal{A}_+ \cup \mathcal{A}_-$ according to

$$\dot{z}^A_+ = z^A_+, \qquad \dot{z}^A_- = z^A_- + R^A(r) + P^A(p), \tag{23}$$

for arbitrary functions $R^A(r), P^A(p)$, where $r = r(z^B_+, z^B_-)$ and $p = p(z^B_+, z^B_-)$ are given by inverting the relations (22). A calculation shows that the symplectic-form-preserving requirement (14b) reduces to

$$(r^2 + a^2)\frac{\partial R^1}{\partial r} - a\frac{\partial R^0}{\partial r} = 0, \qquad a(1 - p^2)\frac{\partial P^1}{\partial p} - \frac{\partial P^0}{\partial p} = 0 \tag{24}$$

so the functions R^0, R^1 and P^0, P^1 in (23) are not independent but are related by this condition. The new metric on space-time is given by (13), (14a), and, as mentioned, it is curved, non-(A)SD, and (complexified) Kähler. Furthermore, it turns out that this simple prescription already identifies the Plebański-Demiański class [7, 8]: to see this, we define four functions $\mathrm{T}(t, r, p), \Phi(\varphi, r, p), \Delta_r(r), \Delta_p(p)$ by

$$\mathrm{T} := t + \tfrac{1}{\sqrt{2}}(R^0(r) + P^0(p)), \qquad \Phi := \varphi + \tfrac{1}{\sqrt{2}}(R^1(r) + P^1(p)), \tag{25}$$

$$\Delta_r := (r^2 + a^2)\left(1 - \tfrac{1}{\sqrt{2}}\tfrac{\partial R^0}{\partial r}\right)^{-1}, \qquad \Delta_p := \mathrm{i}a(1 - p^2)\left(\mathrm{i}a + \tfrac{1}{\sqrt{2}}\tfrac{\partial P^0}{\partial p}\right)^{-1}. \tag{26}$$

After some lengthy calculations, the new metric (13), (14a) is

$$
\begin{aligned}
g = \frac{1}{r_{\mathrm{c}}^2}\Bigg[&\frac{(\Delta_r - a^2\Delta_p)}{\Sigma}\mathrm{d}\mathrm{T}^2 + \frac{2a[(r^2+a^2)\Delta_p - (1-p^2)\Delta_r)}{\Sigma}\mathrm{d}\mathrm{T}\mathrm{d}\Phi \\
&+ \frac{[a^2(1-p^2)^2\Delta_r - (r^2+a^2)^2\Delta_p]}{\Sigma}\mathrm{d}\Phi^2 - \frac{\Sigma}{\Delta_r}\mathrm{d}r^2 - \frac{\Sigma}{\Delta_p}\mathrm{d}p^2\Bigg],
\end{aligned} \tag{27}
$$

where $\Sigma := r^2 + a^2p^2$. By choosing a specific form for Δ_r, Δ_p, this is the Kähler metric associated to the Plebański-Demiański space-time [13].

We emphasise that the definitions (25)-(26) are introduced only to recover the familiar form (27): all necessary information about the metric (27) is already contained in the deformed quadric (23).

As an example, put first $R^1 = P^0 = P^1 = 0$ and then $a = 0$ (so that Δ_p reduces to $1 - p^2$), define $\cos\theta := p$ and $f(r) := \Delta_r/r^2$; then (27) multiplied by r^2 is the (real, ordinary) Schwarzschild metric if one sets $\Delta_r = r^2 - 2Mr$. Similarly, the Reissner-Nördstrom metric, and cosmological versions, etc., are obtained by choosing different functions Δ_r. Space-time points with $r = 0$, corresponding to twistor lines intersecting the undeformed quadric $\mathbb{Q} \subset \mathbb{PT}$ only once, are curvature singularities.

As another example, put $P^0 = P^1 = 0$. Defining $\cos\theta := p$, and setting $\Delta_r = r^2 - 2Mr + a^2$, the metric (27) multiplied by r_{c}^2 is the (real) Kerr metric. The Kerr-Newman metric corresponds to $\Delta_r = r^2 - 2Mr + a^2 + Q^2$, and to obtain the cosmological versions one must include non-trivial P^0, P^1. Twistor lines intersecting $\mathbb{Q}$ only once are those with $r_{\mathrm{c}} = 0$, which is the same as $r = 0 = \cos\theta$ and correspond to ring singularities.

5 Final remarks

The particular quadric (15) used in the derivation of the Plebański-Demiański space-time can be given some sort of physical interpretation, by writing it as $Q_{\alpha\beta}Z^\alpha Z^\beta$ and noticing that $Q_{\alpha\beta}$ is the angular-momentum twistor corresponding to a static, spin-less particle at rest in a complex space-time. In fact, the Penrose transform can be used here to show that the associated twistor functions produce the spin 2 field of linearized black holes [3]. This idea has been revived in recent interesting work on scattering amplitudes, see e.g. [14]. Our construction shows, however, that the exact non-linear solutions are associated to a deformation of the quadric (which is not inside twistor space), in line with the general twistor philosophy that curved space-times should correspond to deformed twistor structures.

The approach in this work has been to take a complexified space-time as a starting point, and then recover real slices by the imposition of (Lorentzian) reality conditions. This is why we needed to consider only one quadric (15) to recover different space-times. If, on the contrary, we assume from the beginning that $t_{\mathrm{c}}, x_{\mathrm{c}}, y_{\mathrm{c}}, z_{\mathrm{c}}$ in (1) are real, then the quadrics for (say) Schwarzschild and Kerr are different. Also, we have chosen to work with the form of the Plebański-Demiański space-time given in [8], as this allows to recover standard black holes in a straightforward manner. If we wish to work with the original Plebański-Demiański coordinates [7], one possibility is to start from a twistor quadric different from (15). The corresponding quadric is $\chi = Z^0Z^1 + cZ^2Z^3$, as was found by Haslehurst and Penrose [15].

There are many open questions concerning our construction that we believe deserve further investigation. Can any (Riemannian or Lorentzian) Kähler metric be obtained by this procedure? In particular, the Chen-Teo instanton [16]? Also, the Einstein equations in the non-linear graviton are automatically encoded in the deformed twistor space; how are field equations encoded in the deformations considered in this work? It would also be desirable to obtain a more intrinsic (i.e. not coordinate dependent) approach to the deformations. Finally, "non-integrable" deformations of the quadric might be related to black hole perturbation theory, since metric perturbations constructed from the Teukolsky equations still possess one family (but not two) of α-surfaces [17].

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