Twistor quadrics and black holes

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A simple procedure is given to construct curved, non-self-dual (complexified) Kähler metrics on space-time in terms of deformations of holomorphic quadric surfaces in flat twistor space, as a variation of Penrose's original twistor construction. Imposing Lorentzian reality conditions, astrophysically relevant space-times such as the standard (non-self-dual) Schwarzschild, Kerr, and Plebański-Demiański metrics, among others, are derived as examples of the construction, and an interpretation of the associated space-time singularities is provided.

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I. INTRODUCTION

Penrose's twistor theory is a radical approach to fundamental physics deeply based on complex geometry [1–4]. Originally a proposal for quantum gravity, and motivated by the fact that the celestial sphere of any event in Minkowski space has a natural complex structure and is thus a Riemann sphere $\mathbb{CP}^1$ (cf. [5]), the space-time manifold emerges as the moduli space of $\mathbb{CP}^1$'s ("twistor lines") in twistor space, in a highly nonlocal correspondence. These Riemann spheres are holomorphic, which implies that our intuition about lines and their intersection in $\mathbb{R}^3$ continues to hold in twistor space, and leads to a remarkable construction of the conformal structure of space-time in terms of intersection of lines. Gravitation, in the twistor view, should correspond to deformations of the flat twistor structure. The nonlinear graviton construction [2] shows that this holds true for self-dual (or half-flat) curved space-times.

Self-dual curvature, however, is a strong restriction for general relativity (GR), as it implies (conformally) flat space-times, so the connection between twistors and GR has historically been obstructed by this condition. Nevertheless, black hole space-times, while non-self-dual, are known to posses the Lorentzian analogue of complex, conformally Kähler structures [6,7]. The profound insights provided by the twistor program regarding the nature of space-time structure, together with the richness of Kähler geometry, the astrophysical relevance of black holes, and the recent interest in applications of methods from high-energy physics (e.g., twistor-strings, spinor helicity, double-copy, scattering amplitudes) to gravitational wave physics in the strong-field regime [8,9], motivate the question of whether the twistor construction can be adapted to describe fully nonlinear black holes. A hint towards addressing this issue is given by a result in Riemannian geometry: Pontecorvo [10] showed that a Kähler metric in the conformal structure of a conformally half-flat four-manifold corresponds to a holomorphic section of (the square root of) the anticanonical bundle of the twistor space, with two zeros on each twistor line. Equivalently, this can be understood as a surface that intersects each twistor line at two points.

In this note, inspired by Pontecorvo's construction and by the nonlinear graviton, we describe a variation of the twistor construction that produces non-self-dual (complexified) Kähler metrics on space-time as deformations of holomorphic quadric surfaces in flat twistor space. Part of the basic intuition is elementary: two points define a line, and this carries over to twistor space since everything is holomorphic (the two points in question being the intersection points of twistor lines with the quadric). Imposing Lorentzian reality conditions, we shall show that the Plebański-Demiański class of space-times [11,12], which includes the classical black hole metrics of GR such as Schwarzschild, Kerr, etc., is recovered by this construction. In addition, the quadric has the appealing physical interpretation of representing the momentum-angular momentum structure of a linearized approximation to a black hole, supplemented by a choice of real slice. Its deformations then represent the full Kähler and nonlinear structure of the black hole. The approach also provides a novel description of space-time singularities: they correspond to the intersection of two divisors in twistor space. This is intimately connected to the Lorentzian character of space-time.

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The basic object that underlies our method (a twistor quadric) also appears, from quite different angles, in several *a priori* unrelated constructions, including scattering amplitudes [13] and the double-copy [14], Chern-Simons theory [15], space-time foam [16], celestial holography [17,18], and perturbation theory [19].

II. TWISTOR QUADRICS AND KÄHLER METRICS

Let $\mathbb{CM}$ be complexified Minkowski space, with complexified inertial coordinates t_c, x_c, y_c, z_c and flat holomorphic metric $\eta = \mathrm{d}t_c^2 - \mathrm{d}x_c^2 - \mathrm{d}y_c^2 - \mathrm{d}z_c^2$. Introduce double null coordinates $u_c = \frac{1}{\sqrt{2}}(t_c + z_c)$, $v_c = \frac{1}{\sqrt{2}}(t_c - z_c)$, $w_c = \frac{1}{\sqrt{2}}(x_c + \mathrm{i}y_c)$, $\tilde{w}_c = \frac{1}{\sqrt{2}}(x_c - \mathrm{i}y_c)$. The twistor space of $\mathbb{CM}$ is $\mathbb{PT} = \mathbb{CP}^3 \backslash \mathbb{CP}^1$, and it is related to space-time via the incidence relation,

$$\begin{pmatrix} Z^0 \\ Z^1 \end{pmatrix} = \mathrm{i} \begin{pmatrix} u_c & w_c \\ \tilde{w}_c & v_c \end{pmatrix} \begin{pmatrix} Z^2 \\ Z^3 \end{pmatrix}, \tag{1}$$

where we use homogeneous coordinates $Z^\alpha = (Z^0, Z^1, Z^2, Z^3)$ on $\mathbb{CP}^3$ (the $\mathbb{CP}^1$ removed corresponding to $Z^2 = Z^3 = 0$). The twistor correspondence (1) is nonlocal: fixing $Z^\alpha \in \mathbb{PT}$, the set of space-time points satisfying (1) is a totally null two-surface in $\mathbb{CM}$ called "α surface", and fixing $(u_c, v_c, w_c, \tilde{w}_c) \in \mathbb{CM}$, the set of Z^α satisfying (1) is a (holomorphic, linearly embedded) Riemann sphere $L_x \cong \mathbb{CP}^1$ called "twistor line". Space-time is the moduli space of twistor lines in $\mathbb{PT}$, and twistor space is the moduli space of α surfaces in $\mathbb{CM}$. Inhomogeneous local coordinates for $\mathbb{PT}$ are

$$\omega^0 = u_c + \zeta w_c, \qquad \omega^1 = \tilde{w}_c + \zeta v_c, \qquad \zeta, \tag{2}$$

where $\zeta = Z^3/Z^2$ (in a region with $Z^2 \neq 0$). For fixed $(u_c, v_c, w_c, \tilde{w}_c)$ and variable ζ, (2) describes a twistor line in $\mathbb{PT}$. For fixed $(\omega^0, \omega^1, \zeta)$ and variable $(u_c, v_c, w_c, \tilde{w}_c)$, (2) describes an α surface in $\mathbb{CM}$.

Twistor space is fibered over $\mathbb{CP}^1$, being the total space of the bundle $\mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathbb{CP}^1$. Here, $\mathcal{O}(-1)$ is the tautological line bundle over $\mathbb{CP}^1$, and $\mathcal{O}(k) = \mathcal{O}(-1)^{*\otimes k}$ $(k > 0)$. The base of the fibration has homogeneous coordinates Z^2, Z^3, or inhomogeneous coordinate ζ, and the fibers have coordinates Z^0, Z^1. Each fiber has an $\mathcal{O}(2)$-valued symplectic structure $\mu = \mathrm{d}Z^0 \wedge \mathrm{d}Z^1$. All of this is valid for flat space-times. One of the main ideas in twistor theory is that gravitation, namely curved space-times, should correspond to deformations of twistor structures. Penrose showed [2] that this is true for half-flat space-times: a complex anti-self-dual (ASD) vacuum space-time corresponds to a three-dimensional complex manifold $\mathcal{PT}$ obtained as a deformation of $\mathbb{PT}$ that preserves the fibration $\mathcal{PT} \to \mathbb{CP}^1$ and the fiberwise symplectic structure μ.

A description of Kähler geometry in twistor terms is available in Euclidean signature, where twistor space can also be defined as the space of almost-complex structures compatible with a four-dimensional Riemannian conformal structure [20]. This space coincides with the projective spin bundle $\mathbb{PS}$ (so there is a fibration $\mathbb{PS} \to M$), which is a complex three-manifold (the twistor space $\mathcal{PT}$) if and only if the conformal structure is ASD. In this context, Pontecorvo showed [10] that a (necessarily scalar-flat) Kähler metric corresponds to a global holomorphic section of $K_{\mathcal{PT}}^{-1/2}$ which vanishes at two points in each twistor line, where $K_{\mathcal{PT}}^{-1/2}$ is the square-root of the anticanonical line bundle of $\mathcal{PT}$. To see this, notice first that the bundle $K_{\mathcal{PT}}^{-1/2}$ restricted to a twistor line is $\mathcal{O}(2)$. Hitchin showed in [21] that the space $H^0(\mathcal{PT}, \mathcal{O}(k))$ of global, holomorphic sections of $\mathcal{O}(k)$ can be identified with the kernel of the valence-k twistor operator. This implies that a section χ of $K_{\mathcal{PT}}^{-1/2}$ corresponds to a valence-2 twistor spinor, or Killing spinor. The requirement that χ vanishes at two points in each twistor line means that the Killing spinor is nondegenerate. Using [22], this gives a conformal Kähler structure on (Euclidean) space-time.

In complexified flat space-time, we can also describe Pontecorvo's construction in terms of holomorphic quadric surfaces in flat twistor space $\mathbb{PT}$; this is the perspective we need in this note. The space $H^0[\mathbb{CP}^1, \mathcal{O}(k)]$ consists of degree k homogeneous polynomials in $\mathbb{C}^2$. Thus, a section χ of $K_{\mathbb{PT}}^{-1/2}$, when restricted to a twistor line, is of the form,

$$\chi = A\zeta^2 + 2B\zeta + C, \tag{3}$$

for some A, B, C, where we use an inhomogeneous coordinate ζ on $\mathbb{CP}^1$. We can then think of χ as a holomorphic quadratic function $\chi(Z^\alpha) = Q_{\alpha\beta} Z^\alpha Z^\beta$ for some symmetric $Q_{\alpha\beta}$. Equation (3) follows after using the incidence relation (1), which also shows that $A = A(v_c, w_c)$, $B = B(u_c, v_c, w_c, \tilde{w}_c)$, and $C = C(u_c, \tilde{w}_c)$. The zero set of χ is a holomorphic quadric, $\mathbb{Q} = \{Z^\alpha \in \mathbb{PT} | \chi(Z^\alpha) = 0\}$. From Kerr's theorem [1], the surface $\mathbb{Q}$ corresponds to a shear-free, null geodesic congruence in $\mathbb{CM}$. The condition that χ vanishes at two points in each twistor line L_x is the same as saying that L_x intersects $\mathbb{Q}$ at two points, corresponding to the two roots $\zeta_\pm$ of the quadratic polynomial (3), that is $\chi = A(\zeta - \zeta_+)(\zeta - \zeta_-)$, where $\zeta_\pm = (-B \pm \sqrt{B^2 - AC})/A$. We allow, however, the possibility of twistor lines where the roots coincide; these correspond to caustics in the ray congruence on space-time (and will later correspond to curvature singularities). The quadric is divided into two regions (divisors),

$$\mathbb{Q} = \mathbb{A}_+ \cup \mathbb{A}_-, \tag{4}$$

where $\mathbb{A}_\pm$ can be described in local coordinates by any two of [see (2)]

$$\omega^0_\pm = u_c + \zeta_\pm w_c, \qquad \omega^1_\pm = \tilde{w}_c + \zeta_\pm v_c, \qquad \zeta_\pm, \qquad (5)$$

or by any function of them. For fixed + or −, the three coordinates in (5) are functionally dependent as a consequence of the quadric equation $\chi(\omega^0_\pm, \omega^1_\pm, \zeta_\pm) = 0$. A simple example to illustrate this (and to have in mind in general) is a product of planes, that is, $Q_{\alpha\beta} = A^+_{(\alpha}A^-_{\beta)}$ for some fixed $A^\pm_\alpha$. The two regions in (4) are in this case two planes $\mathbb{A}_\pm = \{Z^\alpha | A^\pm_\alpha Z^\alpha = 0\}$, and the roots coincide in the twistor line corresponding to the intersection of the planes; see, e.g., [[23], Figs. 6–11].

It is worth mentioning that, while the description of $\mathbb{Q}$ we are giving is adapted to the physical applications we need below, a somewhat more conceptual description can be given in Euclidean signature [24], where twistor space $\mathcal{PT} = \mathbb{PS}$ is equipped with an antiholomorphic involution σ (acting as the antipodal map $\zeta \to -1/\bar{\zeta}$ on each twistor line $\mathbb{CP}^1$). In this case, the quadric $\mathbb{Q}$ is the union of the images of two sections of the fibration $\mathbb{PS} \to M$ that are interchanged by σ. That is, $\mathbb{A}_+$ is defined by a projective spinor $\alpha_{A'}$ and $\mathbb{A}_- = \sigma(\mathbb{A}_+)$ by its conjugate $\alpha^\dagger_{A'}$. Then, unlike in the Lorentzian and complex cases, the intersection $\mathbb{A}_+ \cap \mathbb{A}_-$ is a single twistor line $L_q = \mathbb{CP}^1$, corresponding to a point q on (Euclidean) space-time. This point can be thought of as conformal infinity, and removing it from the manifold leaves us with an asymptotically flat (noncompact) space [24]. However, this description is not valid in Lorentz signature, since in that case $\mathbb{A}_+$ and $\mathbb{A}_-$ are associated to two independent spinor fields. The intersection region $\mathbb{A}_+ \cap \mathbb{A}_-$ is thus more complicated in the Lorentzian case, and, as we shall see, it describes space-time singularities.

Away from $\mathbb{A}_+ \cap \mathbb{A}_-$, a twistor line L_x intersects $\mathbb{Q}$ at two points, and since two points define a line, they can also be used to characterize L_x. In other words, the space-time manifold is recovered from the quadric $\mathbb{Q}$. More concretely, varying the line L_x, the intersection points serve as a coordinate system on space-time. Given local holomorphic coordinates on $\mathbb{A}_\pm$ [obtained, e.g., from (5)], say $z^A_\pm$ with $A = 0, 1$, the pair (z^A_+, z^A_-) is the desired coordinate system on $\mathbb{CM}$. The complex structure J induced on space-time from $\mathbb{Q}$ is then

$$J = \mathrm{i}(\partial_{z^A_+} \otimes \mathrm{d}z^A_+ - \partial_{z^A_-} \otimes \mathrm{d}z^A_-), \qquad (6)$$

where the Einstein summation convention is assumed. The tensor (6) is compatible with the Minkowski metric, in the sense that $\eta(J\cdot, J\cdot) = \eta(\cdot,\cdot)$. In particular, the vectors $\partial_{z^A_\pm}$ are null. In fact, the construction so far is conformally invariant. Conformal invariance is broken by choosing a symplectic form: the spin 1 Penrose transform for the twistor function $f = \chi^{-2}$ computes the (closed) Kähler form to be

$$\kappa = \frac{\mathrm{i}}{(B^2 - AC)^{3/2}}[A\mathrm{d}u_c \wedge \mathrm{d}\tilde{w}_c - B(\mathrm{d}u_c \wedge \mathrm{d}v_c + \mathrm{d}w_c \wedge \mathrm{d}\tilde{w}_c) + C\mathrm{d}w_c \wedge \mathrm{d}v_c] \qquad (7)$$

[A, B, C are defined in (3)]. In terms of quadric coordinates $z^A_\pm$, (7) is

$$\kappa = \kappa_{A\tilde{B}}\mathrm{d}z^A_+ \wedge \mathrm{d}z^B_-, \qquad \kappa_{A\tilde{B}} = \kappa(\partial_{z^A_+}, \partial_{z^B_-}). \qquad (8)$$

Indices $A, \tilde{B}, \ldots$ are numerical and take values 0, 1 (and again Einstein summation is used). The distinction between an index 'B' and an index '$\tilde{B}$' is only intended to remind that they are associated to the two different halves of the quadric, and in equations like (8), they are summed over as usual.

III. DEFORMED QUADRICS

Consider a holomorphic quadric $\mathbb{Q}$ in twistor space, which is arbitrary except for the assumption that, generically, twistor lines intersect $\mathbb{Q}$ at two points, so that the quadric is divided into two regions $\mathbb{A}_\pm$ as in (4). Choose holomorphic coordinates $z^A_\pm$ on $\mathbb{A}_\pm$. We now introduce a deformed quadric as

$$\mathcal{Q} = \mathcal{A}_+ \cup \mathcal{A}_-, \qquad (9)$$

where $\mathcal{A}_+$ and $\mathcal{A}_-$ are the level sets of the four functions $\dot{z}^A_+$ and $\dot{z}^A_-$ defined by

$$\dot{z}^A_+ = z^A_+, \qquad \dot{z}^A_- = z^A_- + f^A(z^B_+, z^B_-), \qquad (10)$$

for some functions f^A, such that $\mathrm{d}\dot{z}^0_+ \wedge \mathrm{d}\dot{z}^1_+ \wedge \mathrm{d}\dot{z}^0_- \wedge \mathrm{d}\dot{z}^1_- \neq 0$. Although one half of the quadric remains undeformed, $\mathbb{A}_+ = \mathcal{A}_+$, the other half $\mathbb{A}_-$ is deformed to $\mathcal{A}_-$ and $\mathcal{Q}$ is, in general, *not* inside twistor space (a point in $\mathcal{A}_-$ is not in twistor space, since it does not correspond to an α-surface in $\mathbb{CM}$). The functions f^A must depend on both z^A_+ and z^A_-. Although our construction is inspired by the nonlinear graviton (cf. [2,25]), the sense in which (9) is a deformed quadric does not seem to be the same as the complex-structure-deformations of twistor theory. More formally, (10) can be understood as an integrable deformation of only *one* of the eigenspaces of (6). This notion escapes the standard Euclidean framework as in that case the eigenspaces are complex conjugates [equivalently, $\mathbb{A}_- = \sigma(\mathbb{A}_+)$ as we saw].

Recalling that the complex structure on space-time induced by the original quadric is given by (6), we associate the deformed quadric to a new complex structure,

$$\dot{J} = \mathrm{i}(\partial_{\dot{z}^A_+} \otimes \mathrm{d}\dot{z}^A_+ - \partial_{\dot{z}^A_-} \otimes \mathrm{d}\dot{z}^A_-). \qquad (11)$$

This is an integrable almost-complex structure on the (complexified) space-time manifold, but it is *not* compatible with the Minkowski metric: $\eta(\dot{J}\cdot,\dot{J}\cdot) \neq \eta(\cdot,\cdot)$. In particular, unlike the undeformed quadric, the new vectors $\partial_{\dot{z}^A_+}$ are not null (they are linear combinations of $\partial_{z^A_+}$ and $\partial_{z^A_-}$). We then interpret the deformation (10) of the quadric as a deformation of the conformal structure on space-time: new conformal structures are introduced by requiring that their light cones contain $\partial_{\dot{z}^A_\pm}$.

This requirement alone, however, does not fix a metric. In order to do this, we must ask additional conditions on the deformations (10). To this end, we choose to restrict to quadric deformations that preserve the symplectic structure induced on space-time, κ. We can take inspiration for this restriction from the nonlinear graviton, where the twistor deformations preserve the fiberwise symplectic structure (which allows to reconstruct the space-time metric); however, this is not the same since we are here dealing with a symplectic structure on space-time (not on twistor space). Regardless, the symplectic-form-preserving condition allows to fix a metric,

$$g(X,Y) \coloneqq \kappa(X,\dot{J}Y), \tag{12}$$

for all vectors X, Y, where the symmetry property of this map follows from requiring κ and $\dot{J}$ to be compatible, which in turn is the same as requiring $\partial_{\dot{z}^A_\pm}$ to be null. The metric (12) is then

$$g = 2g_{A\tilde{B}}\mathrm{d}\dot{z}^A_+ \odot \mathrm{d}\dot{z}^B_-, \tag{13}$$

where $g_{A\tilde{B}} = g(\partial_{\dot{z}^A_+},\partial_{\dot{z}^B_-})$. A calculation shows that the deformations (10) preserve the symplectic structure κ if and only if the functions $g_{A\tilde{B}}$ and f^A satisfy

$$g_{A\tilde{C}}\left(\delta^C_B + \frac{\partial f^C}{\partial z^B_-}\right) = \mathrm{i}\kappa_{A\tilde{B}}, \tag{14a}$$

$$\epsilon^{AC} g_{A\tilde{B}}\frac{\partial f^B}{\partial z^C_+} = 0, \tag{14b}$$

where the four functions $\kappa_{A\tilde{B}}$ are defined in (8), and $\epsilon^{AC} = -\epsilon^{CA}$, $\epsilon^{01} = 1$. The functions $g_{A\tilde{B}}$ in (13) can then be computed from Eq. (14a), and the deformation functions f^A are not completely arbitrary but are restricted by the condition (14b).

IV. BLACK HOLES

Consider the holomorphic quadric $\mathbb{Q} \subset \mathbb{PT}$ given as the zero set of the following quadratic function:

$$\chi(Z^\alpha) = Z^0 Z^3 - Z^1 Z^2. \tag{15}$$

On twistor lines, this adopts the form (3) with $A = w_c$, $B = \frac{1}{2}(u_c - v_c)$, $C = -\tilde{w}_c$ [we are omitting an irrelevant overall factor of i coming from (1)]. The roots are then easily computed to be

$$\zeta_\pm = \frac{-z_c \pm r_c}{x_c + \mathrm{i}y_c}, \qquad r_c \coloneqq \sqrt{x_c^2 + y_c^2 + z_c^2}, \tag{16}$$

and they are "related" by a complexified antipodal map: if $\tilde{\zeta}_\pm = (-z_c \pm r_c)/(x_c - \mathrm{i}y_c)$, then $\zeta_+ = -1/\tilde{\zeta}_-$ (recall the involution σ in the Euclidean case mentioned before). Twistor lines with $r_c = 0$ intersect $\mathbb{Q}$ only once; the Kähler structure on $\mathbb{CM}$ is not well-defined at these points. The symplectic form (7) is

$$\begin{aligned}\kappa = \mathrm{i}r_c^{-3}[&x_c(\mathrm{d}t_c \wedge \mathrm{d}x_c + \mathrm{i}\mathrm{d}y_c \wedge \mathrm{d}z_c) \\ &+ y_c(\mathrm{d}t_c \wedge \mathrm{d}y_c + \mathrm{i}\mathrm{d}z_c \wedge \mathrm{d}x_c) \\ &+ z_c(\mathrm{d}t_c \wedge \mathrm{d}z_c + \mathrm{i}\mathrm{d}x_c \wedge \mathrm{d}y_c)].\end{aligned} \tag{17}$$

Recalling (4) and (5), we choose the following quadric coordinates $z^0_\pm$, $z^1_\pm$ on $\mathbb{A}_\pm$:

$$z^0_\pm = \omega^0_\pm, \qquad z^1_\pm = \frac{\mathrm{i}}{\sqrt{2}}\log(\pm\zeta_\pm). \tag{18}$$

We now impose reality conditions: we take the real Lorentzian slice in $\mathbb{CM}$ defined by

$$t_c = t, \qquad x_c = x, \qquad y_c = y, \qquad z_c = z - \mathrm{i}a, \tag{19}$$

where t, x, y, z are all real, and a is a real parameter. The function r_c in (16) is complex: we denote by r its real part, so that (from the definition of r_c) we must have $r_c = r - \mathrm{i}az/r$. Let us introduce a real coordinate system (r,p,φ) related to Cartesian coordinates (x,y,z) by

$$x + \mathrm{i}y = \sqrt{(r^2+a^2)(1-p^2)}e^{\mathrm{i}\varphi}, \qquad z = rp \tag{20}$$

(where we assume $p^2 < 1$). The symplectic form (17) becomes

$$\begin{aligned}\kappa = \frac{\mathrm{i}}{(r-\mathrm{i}ap)^2}&[\mathrm{d}t \wedge \mathrm{d}(r - \mathrm{i}ap) \\ &- \mathrm{d}\varphi \wedge (a(1-p^2)\mathrm{d}r - \mathrm{i}(r^2+a^2)\mathrm{d}p)],\end{aligned} \tag{21}$$

and the quadric coordinates (18) are

$$\begin{aligned}z^0_\pm &= \frac{1}{\sqrt{2}}[t \pm (r - \mathrm{i}ap)], \\ z^1_\pm &= \frac{1}{\sqrt{2}}\left[\varphi \pm \left(-\arctan(a/r) - \frac{\mathrm{i}}{2}\log\left(\frac{1+p}{1-p}\right)\right)\right].\end{aligned} \tag{22}$$

After some calculations, we find the components $\kappa_{A\tilde{B}}$ in (8) to be $\kappa_{0\tilde{0}} = \mathrm{i}r_c^{-2}$, $\kappa_{0\tilde{1}} = 0 = \kappa_{1\tilde{0}}$, $\kappa_{1\tilde{1}} = -\mathrm{i}r_c^{-2}(r^2+a^2)\times(1-p^2)$, where $r_c^2 = (r - \mathrm{i}ap)^2$.

Following the prescription (10), we now deform the quadric given by (15) to a new quadric $\mathcal{Q} = \mathcal{A}_+ \cup \mathcal{A}_-$ according to

$$\dot{z}_+^A = z_+^A, \qquad \dot{z}_-^A = z_-^A + R^A(r) + P^A(p), \tag{23}$$

for arbitrary functions $R^A(r)$, $P^A(p)$, where $r = r(z_+^B, z_-^B)$ and $p = p(z_+^B, z_-^B)$ are given by inverting the relations (22). A calculation shows that the symplectic-form-preserving requirement (14b) reduces to

$$(r^2 + a^2)\frac{\mathrm{d}R^1}{\mathrm{d}r} - a\frac{\mathrm{d}R^0}{\mathrm{d}r} = 0 = a(1-p^2)\frac{\mathrm{d}P^1}{\mathrm{d}p} - \frac{\mathrm{d}P^0}{\mathrm{d}p}, \tag{24}$$

so the functions R^0, R^1 and P^0, P^1 in (23) are not independent but are related by this condition. The new metric on space-time is given by (13), (14a): it is curved, non-(A)SD, and (complexified) Kähler. Furthermore, it turns out that this simple prescription already identifies the Plebański-Demiański class [11,12]: to see this, we define four functions $\mathrm{T}(t,r,p)$, $\Phi(\varphi,r,p)$, $\Delta_r(r)$, $\Delta_p(p)$ by

$$\begin{aligned}
\mathrm{T} &:= t + \frac{1}{\sqrt{2}}(R^0(r) + P^0(p)),\\
\Phi &:= \varphi + \frac{1}{\sqrt{2}}(R^1(r) + P^1(p)),\\
\Delta_r &:= (r^2+a^2)\left(1 - \frac{1}{\sqrt{2}}\frac{\mathrm{d}R^0}{\mathrm{d}r}\right)^{-1},\\
\Delta_p &:= \mathrm{i}a(1-p^2)\left(\mathrm{i}a + \frac{1}{\sqrt{2}}\frac{\mathrm{d}P^0}{\mathrm{d}p}\right)^{-1}.
\end{aligned} \tag{25}$$

After some lengthy calculations, the new metric (13), (14a) is

$$\begin{aligned}
g = \frac{1}{r_\mathrm{c}^2}\Bigg[&\frac{(\Delta_r - a^2\Delta_p)}{\Sigma}\mathrm{dT}^2 - \frac{\Sigma}{\Delta_r}\mathrm{d}r^2 - \frac{\Sigma}{\Delta_p}\mathrm{d}p^2\\
&+ \frac{2a[(r^2+a^2)\Delta_p - (1-p^2)\Delta_r)}{\Sigma}\mathrm{dTd\Phi}\\
&+ \frac{[a^2(1-p^2)^2\Delta_r - (r^2+a^2)^2\Delta_p]}{\Sigma}\mathrm{d}\Phi^2\Bigg],
\end{aligned} \tag{26}$$

where $\Sigma := r^2 + a^2p^2$. Choosing a specific form for Δ_r, Δ_p, this is the Kähler metric associated to the Plebański-Demiański space-time [7].

We emphasize that the definitions (25) are introduced only to recover the familiar form (26): all necessary information about the metric (26) is already contained in the deformed quadric (23).

As an example, put first $R^1 = P^0 = P^1 = 0$ and then $a = 0$ (so that Δ_p reduces to $1 - p^2$), define $\cos\theta := p$ and $f(r) := \Delta_r/r^2$; then (26) multiplied by r^2 is the (real, ordinary) Schwarzschild metric if one sets $\Delta_r = r^2 - 2Mr$. Similarly, the Reissner-Nördstrom metric, and cosmological versions, etc., are obtained by choosing different functions Δ_r. Space-time points with $r = 0$, corresponding to twistor lines intersecting the undeformed quadric $\mathbb{Q} \subset \mathbb{PT}$ only once, are curvature singularities.

As another example, put $P^0 = P^1 = 0$. Defining $\cos\theta := p$, and setting $\Delta_r = r^2 - 2Mr + a^2$, the metric [11] multiplied by r_c^2 is the (real, ordinary) Kerr metric. The Kerr-Newman metric corresponds to $\Delta_r = r^2 - 2Mr + a^2 + Q^2$, and to obtain the cosmological versions, one must include nontrivial P^0, P^1. Twistor lines intersecting $\mathbb{Q}$ only once are those with $r_\mathrm{c} = 0$, which is the same as $r = 0 = \cos\theta$ and correspond to ring singularities.

V. DISCUSSION

The quadric (15) used in the derivation of the Plebański-Demiański space-time can be given a physical interpretation by writing it as $Q_{\alpha\beta}Z^\alpha Z^\beta$ and noticing that $Q_{\alpha\beta}$ is the angular-momentum twistor corresponding to a static, spinless particle at rest in a complex space-time. In fact, the Penrose transform can be used here to show that the associated twistor functions produce the spin 2 field of linearized black holes [3]. This idea has been revived in recent interesting work on scattering amplitudes; see, e.g., [13]. Our construction shows, however, that the exact nonlinear solutions are associated to a deformation of the quadric (which is not inside twistor space), in line with the general twistor philosophy that curved space-times should correspond to deformed twistor structures.

The space-time manifold is recovered from the quadric, the only problematic region being the intersection of the two divisors. When the quadric structure is deformed, this intersection is preserved, and, in space-time, this translates into curvature singularities of the (Lorentzian) black hole metrics. By contrast, Euclidean reality conditions force the intersection to be a single twistor line, which is then identified with conformal infinity in compactified Euclidean space.

The approach in this work has been to take a complexified space-time as a starting point and then recover real slices by the imposition of (Lorentzian) reality conditions. This is why we needed to consider only one quadric (15) to recover different space-times. If, on the contrary, we assume from the beginning that t_c, x_c, y_c, z_c in (1) are real, then the quadrics for (say) Schwarzschild and Kerr are different. Also, we have chosen to work with the form of the Plebański-Demiański space-time given in [12], as this allows us to recover standard black holes in a straightforward manner. If we wish to work with the original Plebański-Demiański coordinates [11], one possibility is to start from a twistor quadric different from (15). The corresponding quadric is $\chi = Z^0Z^1 + cZ^2Z^3$, as was found by Haslehurst and Penrose [26]. A simple framework for relating different solutions via complex coordinate transformations has been recently given in [27].

The construction in this paper suggests several research directions and connections with modern developments in geometry, gravitational, and high-energy physics, that we believe are worth pursuing further. In a geometric context: can any Kähler metric be obtained as a deformed twistor quadric? This might be interesting in relation to recent advances in the classification of non-self-dual gravitational instantons [28–30]. It would also be desirable to have a mathematically rigorous treatment of the deformations (and possibly of cohomological issues), perhaps following [24]. Related to this and to the original twistor program: the Einstein equations in the nonlinear graviton are automatically encoded in the deformed twistor space, so it would be very interesting to understand how field equations are encoded in the deformations considered in this work. In the context of gravitational wave physics and black hole stability, nonintegrable deformations of the quadric may be related to perturbation theory, since metric perturbations constructed from the Teukolsky equations still possess one family (but not two) of α surfaces [19]. Relations between (scalar-flat) Kähler geometry and (self-dual) gravity, especially concerning Pontecorvo's construction and the quadrics studied in this paper, have been proposed in the description of space-time foam in twistor string theory [16], where the quadric is interpreted as giving rise to a natural meromorphic three-form in twistor space (in turn associated to D1-brane charge). This also plays a basic role in recent constructions concerning Chern-Simons theory [15] and celestial holography [17,18].

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