DWORK–TYPE CONGRUENCES AND p-ADIC *KZ* CONNECTION

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Abstract. We show that the p-adic *KZ* connection associated with the family of curves $y^q = (t-z_1)\dots(t-z_{qg+1})$ has an invariant subbundle of rank g, while the corresponding complex *KZ* connection has no nontrivial proper subbundles due to the irreducibility of its monodromy representation. The construction of the invariant subbundle is based on new Dwork–type congruences for associated Hasse–Witt matrices.

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1. Introduction

The Knizhnik–Zamolodchikov (KZ) differential equations are objects of conformal field theory, representation theory, enumerative geometry, see for example [KZ, Dr, EFK, MO, V2]. The solutions of the KZ equations have the form of multidimensional hypergeometric functions, see [SV1]. In this paper we discuss the analog of hypergeometric solutions of the KZ equations considered over a p-adic field instead of the field of complex numbers.

More precisely, we consider the KZ equations in the special case, in which the complex hypergeometric solutions are given by the integrals of the form

$$I(z_1,\dots,z_{qg+1}) = \int_C \frac{R(t,z_1,\dots,z_{qg+1})\,dt}{\sqrt[1/q]{(t-z_1)\dots(t-z_{qg+1})}} \tag{1.1}$$

where q, g are positive integer parameters, and $R(t,z)$ are suitable rational functions.

In this case the space of solutions of the KZ equations is a qg-dimensional complex vector space. We also consider the p-adic version of the same differential equations. We assume that q is a prime number (that is a technical assumption) and show that the qg-dimensional space of local solutions of these p-adic KZ equations has a remarkable g-dimensional subspace of solutions which can be p-adic analytically continued as a subspace to a large domain $\mathfrak{D}^{(m),o}_{\mathrm{KZ}}$ in the space where the KZ equations are defined, see Theorems 6.10 and 6.12 for precise statements. This g-dimensional global subspace of solutions is defined as the uniform p-adic limit of a g-dimensional space of polynomial solutions of these KZ equations modulo p^s as $s\to\infty$. For $q=2$ and $g=1$ this construction was deduced in [V5] from the classical B. Dwork's paper [Dw], see also [VZ1]. For $q=2$ and any g the corresponding construction was developed in [VZ2].

In [SV2] general KZ equations were considered over the field $\mathbb{F}_p$ and their polynomial solutions were constructed as p-approximations of hypergeometric integrals. In the current paper that construction is modified to obtain polynomial solutions modulo p^s of the KZ equations related to the integrals in formula (1.1). The polynomial solutions are vectors of polynomials with integer coefficients. We call them the p^s-hypergeometric solutions. While the complex analytic integrals in (1.1) give the whole qg-dimensional space of all solutions of the complex KZ equations, the p^s-hypergeometric solutions span only a g-dimensional subspace. Then the p-adic limit of that subspace as $s\to\infty$ gives the desired globally defined subspace of solutions.

On other p-approximations of hypergeometric periods see [SV2, RV1, RV2, VZ1, VZ2].

In order to prove Theorems 6.10 and 6.12 we develop new matrix Dwork–type congruences in Section 2. In Section 3 we show how our Dwork–type congruences imply the uniform p-adic convergence of certain sequences of matrices on suitable domains of the space of their parameters. In Section 4 we define our KZ equations and construct their complex holomorphic solutions. In Section 5 we describe the p^s-hypergeometric solutions of the same equations. In Section 6 we formulate and prove the main Theorems 6.10 and 6.12.

This paper may be viewed as a continuation of the paper [VZ2] where the case $q = 2$ is developed.

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2. Dwork–type congruences

The Dwork–type congruences were originated by B. Dwork in the classical paper [Dw]. On Dwork–type congruences see for example [Dw, Me, MV, Vl, VZ1, VZ2].

In this paper p is an odd prime. We denote by $\mathbb{Z}_p[w^{\pm 1}]$ the ring of Laurent polynomials in variables w with coefficients in $\mathbb{Z}_p$. A congruence $F(w) \equiv G(w) \pmod{p^s}$ for two Laurent polynomials from the ring is understood as the divisibility by p^s of all coefficients of $F(w) - G(w)$.

For a Laurent polynomial $G(w)$ we define $\sigma(G(w)) = G(w^p)$.

We denote $x = (t, z)$, where $t = (t_1, \dots, t_r)$ and $z = (z_1, \dots, z_n)$ are two groups of variables.

2.1. **Definition of ghosts.** Let $\mathbf{e} = (e_1, \dots, e_l)$ be a tuple of positive integers and $\Lambda = (\Lambda_0(x), \Lambda_1(x), \dots, \Lambda_l(x))$ a tuple of Laurent polynomials in $\mathbb{Z}_p[x^{\pm 1}]$.

Define $V_0(x) = \Lambda_0(x)$. For $s = 1, \dots, l$, define $V_s(x)$ by the recursive formula

$$\begin{aligned} \Lambda_0(x)\Lambda_1(x)^{p^{e_1}} \dots \Lambda_s(x)^{p^{e_1+\dots+e_s}} &= V_s(x) + V_{s-1}(x)\Lambda_s(x^{p^{e_1+\dots+e_s}}) + \\ &+ V_{s-2}(x)\Lambda_{s-1}(x^{p^{e_1+\dots+e_{s-1}}})\Lambda_s(x^{p^{e_1+\dots+e_{s-1}}})^{p^{e_s}} + \dots + \\ &+ V_0(x)\Lambda_1(x^{p^{e_1}})\Lambda_2(x^{p^{e_1}})^{p^{e_2}} \cdots \Lambda_s(x^{p^{e_1}})^{p^{e_2+\dots+e_s}}, \end{aligned} \tag{2.1}$$

The Laurent polynomials $V_0(x), \dots, V_l(x) \in \mathbb{Z}_p[x^{\pm 1}]$ are called the *ghosts* associated with the tuples $\mathbf{e}$ and Λ.

For every $0 \leqslant j \leqslant s \leqslant l$, denote

$$\begin{aligned} W_s(x) &:= \Lambda_0(x)\Lambda_1(x)^{p^{e_1}} \cdots \Lambda_s(x)^{p^{e_1+\dots+e_s}}, \\ W_s^{(j)}(x) &:= \Lambda_j(x)\Lambda_{j+1}(x)^{p^{e_{j+1}}} \cdots \Lambda_s(x)^{p^{e_{j+1}+\dots+e_s}}. \end{aligned}$$

Then (2.1) can be formulates as

$$W_s(x) = V_s(x) + \sum_{j=1}^{s} V_{j-1}(x) W_s^{(j)}(x^{p^{e_1+\dots+e_j}}), \tag{2.2}$$

or as

$$W_s(x) = V_s(x) + \sum_{j=1}^{s} V_{j-1}(x) \sigma^{e_1+\dots+e_j}(W_s^{(j)}(x)). \tag{2.3}$$

Lemma 2.1. *For $s = 0, 1, \dots, l$, we have $V_s(x) \equiv 0 \pmod{p^s}$.*

Proof. In the proof we use the congruence $F(x^p)^{p^{i-1}} \equiv F(x)^{p^i} \pmod{p^i}$ valid for $i > 0$.

For $s = 0$ we have $V_0(x) = \Lambda_0(x)$ and no requirements on divisibility. For $s = 1$, we have

$$V_1(x) = \Lambda_0(x)\Lambda_1(x)^{p^{e_1}} - V_0(x)\Lambda_1(x^{p^{e_1}}) = \Lambda_0(x)(\Lambda_1(x)^{p^{e_1}} - \Lambda_1(x^{p^{e_1}})),$$

and

(2.4)

$$\Lambda_1(x^{p^{e_1}}) \overset{(\mathrm{mod}\ p)}{\equiv} \Lambda_1(x^{p^{e_1-1}})^p \overset{(\mathrm{mod}\ p^2)}{\equiv} \Lambda_1(x^{p^{e_1-2}})^{p^2} \overset{(\mathrm{mod}\ p^3)}{\equiv} \dots \overset{(\mathrm{mod}\ p^{e_1})}{\equiv} \Lambda_1(x)^{p^{e_1}}.$$

This proves the lemma for $s=1$.

For $s>1$ the proof is by induction on s. Assume that the lemma is proved for all $j<s$. Then similarly to (2.4) we obtain $\Lambda_s(x^{p^{e_1+\dots+e_j}})^{p^{e_{j+1}+\dots+e_s}} \equiv \Lambda_s(x)^{p^{e_1+\dots+e_s}} \pmod{p^{1+e_{j+1}+\dots+e_s}}$ and hence

$$\begin{aligned} V_{j-1}(x)\Lambda_s(x^{p^{e_1+\dots+e_j}})^{p^{e_{j+1}+\dots+e_s}} &\equiv V_{j-1}(x)\Lambda_s(x)^{p^{e_1+\dots+e_s}} \pmod{p^{j+e_{j+1}+\dots+e_s}} \\ &\equiv V_{j-1}(x)\Lambda_s(x)^{p^{e_1+\dots+e_s}} \pmod{p^s} \end{aligned}$$

since $e_i \geqslant 1$ for all i. Then we deduce modulo p^s:

$$\begin{aligned} V_s(x) = {} & W_{s-1}(x)\Lambda_s(x)^{p^{e_1+\dots+e_s}} - \sum_{j=1}^{s-1} V_{j-1}(x) W_{s-1}^{(j)}(x^{p^{e_1+\dots+e_j}})\Lambda_s(x^{p^{e_1+\dots+e_j}})^{p^{e_{j+1}+\dots+e_s}} \\ & - V_{s-1}(x)\Lambda_s(x^{p^{e_1+\dots+e_s}}) \equiv \\ & \equiv \Big(W_{s-1}(x) - \sum_{j=1}^{s-1} V_{j-1}(x) W_{s-1}^{(j)}(x^{p^{e_1+\dots+e_j}}) - V_{s-1}(x)\Big)\Lambda_s(x)^{p^{e_1+\dots+e_s}} = 0, \end{aligned}$$

obtaining the required statement. $\square$

For a Laurent polynomial $F(t,z)$ in t,z, let $N(F)\subset\mathbb{R}^r$ be the Newton polytope of $F(t,z)$ with respect to the t *variables only*.

Lemma 2.2. *For $s=0,1,\dots,l$, we have*

$$N(V_s) \subset N(\Lambda_0) + p^{e_1} N(\Lambda_1) + \dots + p^{e_1+\dots+e_s} N(\Lambda_s)\,.$$

Proof. This follows from (2.2) by induction on s. $\square$

2.2. **Convex polytopes.** Let $\Delta = (\Delta_0,\dots,\Delta_l)$ be a tuple of nonempty finite subsets of $\mathbb{Z}^r$ of the same size $\#\Delta_j = g$ for some positive integer g.

Definition 2.3. *A tuple $(N_0,N_1,\dots,N_l)$ of convex polytopes in $\mathbb{R}^r$ is called $(\Delta,\mathbf{e})$*-admissible *if for any $0\leqslant i\leqslant j<l$ we have*

(2.5) $$\big(\Delta_i + N_i + p^{e_{i+1}}N_{i+1} + \dots + p^{e_{i+1}+\dots+e_j}N_j\big) \cap p^{e_{i+1}+\dots+e_{j+1}}\mathbb{Z}^r \subset p^{e_{i+1}+\dots+e_{j+1}}\Delta_{j+1}\,.$$

Notice that any sub-tuple $(N_i,N_{i+1},\dots,N_j)$ of a $(\Delta,\mathbf{e})$-admissible tuple $(N_0,N_1,\dots,N_l)$ is $(\Delta',\mathbf{e}')$-admissible where $\Delta'=(\Delta_i,\dots,\Delta_j)$ and $\mathbf{e}'=(e_{i+1},\dots,e_j)$.

Definition 2.4. *A tuple $(\Lambda_0(t,z),\Lambda_1(t,z),\dots,\Lambda_l(t,z))$ of Laurent polynomials is called $(\Delta,\mathbf{e})$*-admissible *if the tuple $\big(N(\Lambda_0),N(\Lambda_1),\dots,N(\Lambda_l)\big)$ is $(\Delta,\mathbf{e})$-admissible.*

Example. Let $r=1$, $n=13$, $\mathbf{e}=(2,2,\dots,2)$, $\Gamma=\{1,2,3,4\}\subset\mathbb{Z}$, $\Delta=(\Gamma,\Gamma,\dots,\Gamma)$, $N=[0,13(p^2-1)/3]\subset\mathbb{R}$, $F(t_1,z)=\prod_{i=1}^{13}(t_1-z_i)^{(p^2-1)/3}$. Then the tuple $(N,N,\dots,N)$ of intervals in $\mathbb{R}$ and the tuple of polynomials $(F(t_1,z),F(t_1,z),\dots,F(t_1,z))$ are $(\Delta,\mathbf{e})$-admissible.

2.3. **Hasse–Witt matrices.** For $v \in \mathbb{Z}^r$ denote by $\operatorname{Coeff}_v F(t,z)$ the coefficient of t^v in the Laurent polynomial $F(t,z)$. This is a Laurent polynomial in z.

Given $m \geqslant 1$ and finite subsets $\Delta', \Delta'' \subset \mathbb{Z}^r$, define the *Hasse–Witt matrix* of the Laurent polynomial $F(t,z)$ by the formula

$$A(m, \Delta', \Delta'', F(t,z)) := \big(\operatorname{Coeff}_{p^m v - u} F(t,z)\big)_{u\in\Delta', v\in\Delta''}. \tag{2.6}$$

Lemma 2.5. *Let Λ be a $(\Delta, \mathbf{e})$-admissible tuple of Laurent polynomials in $\mathbb{Z}_p[x^{\pm 1}] = \mathbb{Z}_p[t^{\pm 1}, z^{\pm 1}]$. Then for $0 \leqslant s \leqslant l$ we have*

(i) $A(e_1 + \cdots + e_{s+1}, \Delta_0, \Delta_{s+1}, V_s) \equiv 0 \pmod{p^s}$;

(ii) $A\big(e_1 + \cdots + e_{s+1}, \Delta_0, \Delta_{s+1}, W_s\big) =$
$$= A\big(e_1, \Delta_0, \Delta_1, V_0\big) \cdot \sigma^{e_1}\big(A\big(e_2 + \cdots + e_{s+1}, \Delta_1, \Delta_{s+1}, W_s^{(1)}\big)\big) +$$
$$+ A\big(e_1 + e_2, \Delta_0, \Delta_2, V_1\big) \cdot \sigma^{e_1+e_2}\big(A\big(e_3 + \cdots + e_{s+1}, \Delta_2, \Delta_{s+1}, W_s^{(2)}\big)\big) + \cdots +$$
$$+ A\big(e_1 + \cdots + e_s, \Delta_0, \Delta_s, V_{s-1}\big) \cdot \sigma^{e_1+\cdots+e_s}\big(A\big(e_{s+1}, \Delta_s, \Delta_{s+1}, W_s^{(s)}\big)\big) +$$
$$+ A\big(e_1 + \cdots + e_{s+1}, \Delta_0, \Delta_{s+1}, V_s\big).$$

Notice that all these matrices are $g \times g$-matrices.

Proof. Part (i) follows from Lemma 2.1. To prove (ii) consider the identity

$$\Lambda_0(t,z)\Lambda_1(t,z)^{p^{e_1}} \ldots \Lambda_s(t,z)^{p^{e_1+\cdots+e_s}} = \sum_{j=1}^{s} V_{j-1}(t,z)\Lambda_j(t^{p^{e_1+\cdots+e_j}}, z^{p^{e_1+\cdots+e_j}}) \times$$
$$\times \Lambda_{j+1}(t^{p^{e_1+\cdots+e_j}}, z^{p^{e_1+\cdots+e_j}})^{p^{e_{j+1}}} \ldots \Lambda_s(t^{p^{e_1+\cdots+e_j}}, z^{p^{e_1+\cdots+e_j}})^{p^{e_{j+1}+\cdots+e_s}} + V_s(t,z), \tag{2.7}$$

which is nothing else but (2.1). Let $u \in \Delta_0, v \in \Delta_{s+1}$. In order to calculate the coefficient of $t^{p^{e_1+\cdots+e_{s+1}}v-u}$ in the j-th summand on the right-hand side of (2.7), we look for all pairs of vectors $w \in N(V_{j-1})$ and $y \in N(\Lambda_j(t,z)\ldots\Lambda_s(t,z)^{p^{e_{j+1}+\cdots+e_{s+1}}})$ such that

$$w + p^{e_1+\cdots+e_j} y = p^{e_1+\cdots+e_{s+1}} v - u.$$

Hence $u + w \in p^{e_1+\cdots+e_j}\mathbb{Z}^r$. On the other hand, it follows from Lemma 2.2 that $w \in N(\Lambda_0) + p^{e_1}N(\Lambda_1) + \cdots + p^{e_1+\cdots+e_{j-1}}N(\Lambda_{j-1})$, so that

$$u + w \in \Delta_0 + N(\Lambda_0) + pN(\Lambda_1) + \cdots + p^{e_1+\cdots+e_{j-1}}N(\Lambda_{j-1}).$$

From the $(\Delta, \mathbf{e})$-admissibility we deduce that $u + w = p^{e_1+\cdots+e_j}\delta$ for some $\delta \in \Delta_j$, thus $w = p^{e_1+\cdots+e_j}\delta - u,\ \ y = p^{e_{j+1}+\cdots+e_{s+1}}v - \delta$ and

$$\operatorname{Coeff}_{p^{e_1+\cdots+e_{s+1}}v-u}\big(V_{j-1}(t,z)\Lambda_j(t^{p^{e_1+\cdots+e_j}}, z^{p^{e_1+\cdots+e_j}})\ldots\Lambda_s(t^{p^{e_1+\cdots+e_j}}, z^{p^{e_1+\cdots+e_j}})^{p^{e_{j+1}+\cdots+e_s}}\big) =$$
$$= \sum_{\delta\in\Delta_j} \operatorname{Coeff}_{p^{e_1+\cdots+e_j}\delta-u}(V_{j-1}(t,z)) \cdot$$
$$\cdot\ \sigma^{e_1+\cdots+e_j}\big(\operatorname{Coeff}_{p^{e_{j+1}+\cdots+e_{s+1}}v-\delta}\big(\Lambda_j(t,z)\Lambda_{j+1}(t,z)^{p^{e_{j+1}}}\ldots\Lambda_s(t,z)^{p^{e_{j+1}+\cdots+e_s}}\big)\big).$$

This proves (ii). $\square$

2.4. **Congruences.** The next results discuss congruences of the type $F_1(z)F_2(z)^{-1} \equiv G_1(z)G_2(z)^{-1} \pmod{p^s}$, where F_1, F_2, G_1, G_2 are $g\times g$ matrices whose entries are Laurent polynomials in z. We consider such congruences when the determinants $\det F_2(z)$ and $\det G_2(z)$ are Laurent polynomials both nonzero modulo p. Using Cramer's rule we write the entries of the inverse matrix $F_2(z)^{-1}$ in the form $f_{ij}(z)/\det F_2(z)$ for $f_{ij}(z) \in \mathbb{Z}_p[z^{\pm 1}]$ and do a similar computation for $G_2(z)$. This presents the congruence $F_1(z)F_2(z)^{-1} \equiv G_1(z)G_2(z)^{-1} \pmod{p^s}$ in the form

$$\frac{1}{\det F_2(z)} \cdot F(z) \;\equiv\; \frac{1}{\det G_2(z)} \cdot G(z) \pmod{p^s} \tag{2.8}$$

for some $g \times g$ matrices $F(z), G(z)$ with entries in $\mathbb{Z}_p[z^{\pm1}]$, while (2.8) is nothing else but the congruence $F(z)\cdot \det G_2(z) \equiv G(z) \cdot \det F_2(z) \pmod{p^s}$.

Theorem 2.6. *Let $(\Lambda_0(t,z), \Lambda_1(t,z), \dots, \Lambda_l(t,z))$ be a $(\Delta, \mathbf{e})$-admissible tuple of Laurent polynomials in $\mathbb{Z}_p[x^{\pm1}] = \mathbb{Z}_p[t^{\pm1}, z^{\pm1}]$.*

(i) *For $0 \leqslant s \leqslant l$ we have*

$$A\big(e_1+\dots+e_{s+1}, \Delta_0, \Delta_{s+1}, \Lambda_0(x)\Lambda_1(x)^{p^{e_1}}\cdots\Lambda_s(x)^{p^{e_1+\dots+e_s}}\big) \equiv$$
$$\equiv A\big(e_1, \Delta_0, \Delta_1, \Lambda_0(x)\big)\cdot \sigma^{e_1}\big(A\big(e_2, \Delta_1, \Delta_2, \Lambda_1(x)\big)\big)\cdots \sigma^{e_1+\dots+e_s}\big(A\big(e_{s+1}, \Delta_s, \Delta_{s+1}, \Lambda_s(x)\big)\big)$$

modulo p.

(ii) *Assume that the determinants of the matrices $A\big(e_{i+1}, \Delta_i, \Delta_{i+1}, \Lambda_i(t,z)\big)$, $i = 0, 1, \dots, l$, are Laurent polynomials all nonzero modulo p. Then for $1 \leqslant s \leqslant l$ the determinant of the matrix $A\big(e_2+\dots+e_{s+1}, \Delta_1, \Delta_{s+1}, \Lambda_1(x)\Lambda_2(x)^{p^{e_2}}\cdots\Lambda_s(x)^{p^{e_2+\dots+e_s}}\big)$ is a Laurent polynomial nonzero modulo p and we have modulo p^s:*

$$\begin{aligned} &A\big(e_1+\dots+e_{s+1}, \Delta_0, \Delta_{s+1}, \Lambda_0(x)\Lambda_1(x)^{p^{e_1}}\cdots\Lambda_s(x)^{p^{e_1+\dots+e_s}}\big)\cdot \\ &\cdot \sigma^{e_1}\big(A\big(e_2+\dots+e_{s+1}, \Delta_1, \Delta_{s+1}, \Lambda_1(x)\Lambda_2(x)^{p^{e_2}}\cdots\Lambda_s(x)^{p^{e_2+\dots+e_s}}\big)\big)^{-1} \equiv \\ &\equiv A\big(e_1+\dots+e_s, \Delta_0, \Delta_s, \Lambda_0(x)\Lambda_1(x)^{p^{e_1}}\cdots\Lambda_{s-1}(x)^{p^{e_1+\dots+e_{s-1}}}\big)\cdot \\ &\cdot \sigma^{e_1}\big(A\big(e_2+\dots+e_s, \Delta_1, \Delta_s, \Lambda_1(x)\Lambda_2(x)^{p^{e_2}}\cdots\Lambda_{s-1}(x)^{p^{e_2+\dots+e_{s-1}}}\big)\big)^{-1}, \end{aligned} \tag{2.9}$$

where in this congruence for $s=1$ we understand the second factor on the right-hand side as the $g\times g$ identity matrix, see formula (2.10) below.

Proof. By Lemma 2.5 we have

$$A\big(e_1+\dots+e_{s+1}, \Delta_0, \Delta_{s+1}, \Lambda_0(x)\Lambda_1(x)^{p^{e_1}}\dots\Lambda_s(x)^{p^{e_1+\dots+e_s}}\big) \equiv$$
$$\equiv A\big(e_1, \Delta_0, \Delta_1, \Lambda_0(x)\big)\cdot\sigma^{e_1}\big(A\big(e_2+\dots+e_{s+1}, \Delta_1, \Delta_{s+1}, \Lambda_1(x)\Lambda_2(x)^{p^{e_2}}\dots\Lambda_s(x)^{p^{e_2+\dots+e_s}}\big)\big)$$

modulo p. Iteration gives part (i) of the theorem.

If the determinants of the matrices $A\big(e_{i+1}, \Delta_i, \Delta_{i+1}, \Lambda_i(t,z)\big)$, $i=0,1,\dots,l$, are Laurent polynomials all nonzero modulo p, then part (i) implies that the determinant

$$\det A\big(e_2+\dots+e_{s+1}, \Delta_1, \Delta_{s+1}, \Lambda_1(x)\Lambda_2(x)^{p^{e_2}}\cdots\Lambda_s(x)^{p^{e_2+\dots+e_s}}\big) \equiv$$
$$\equiv \textstyle\prod_{j=1}^s \det \sigma^{e_2+\dots+e_j}\big(A\big(e_{j+1}, \Delta_j, \Delta_{j+1}, \Lambda_j(t,z)\big)\big) \pmod{p},$$

is a Laurent polynomial nonzero modulo p. This proves the first statement of part (ii) of the theorem and allows us to consider the inverse matrices in the congruence of part (ii).

We prove part (ii) by induction on s. For $s=1$, congruence (2.9) takes the form

$$A\big(e_1+e_2,\Delta_0,\Delta_2,\Lambda_0(x)\Lambda_1(x)^{p^{e_1}}\big)\cdot\sigma^{e_1}\big(A\big(e_2,\Delta_1,\Delta_2,\Lambda_1(x)\big)\big)^{-1}\equiv A\big(e_1,\Delta_2,\Delta_1,\Lambda_0(x)\big) \tag{2.10}$$

modulo p. This congruence follows from part (i).

For $1<s<l$ we substitute the expressions for $A\big(e_1+\cdots+e_{s+1},\Delta_0,\Delta_{s+1},\Lambda_0(x)\Lambda_1(x)^{p^{e_1}}\cdots\cdots\Lambda_s(x)^{p^{e_1+\cdots+e_s}}\big)$ and $A\big(e_1+\cdots+e_s,\Delta_0,\Delta_s,\Lambda_0(x)\Lambda_1(x)^{p^{e_1}}\cdots\Lambda_{s-1}(x)^{p^{e_1+\cdots+e_{s-1}}}\big)$ from part (ii) of Lemma 2.5 into the two sides of the desired congruence:

$$\begin{aligned}
&A\big(\textstyle\sum_{a=1}^{s+1}e_a,\Delta_0,\Delta_{s+1},\Lambda_0(x)\Lambda_1(x)^{p^{e_1}}\cdots\Lambda_s(x)^{p^{e_1+\cdots+e_s}}\big)\cdot\\
&\cdot\sigma^{e_1}\big(A\big(\textstyle\sum_{a=2}^{s+1}e_a,\Delta_1,\Delta_{s+1},\Lambda_1(x)\Lambda_2(x)^{p^{e_2}}\cdots\Lambda_s(x)^{p^{e_2+\cdots+e_s}}\big)\big)^{-1}=A\big(e_1,\Delta_0,\Delta_1,V_0\big)+\\
&+\textstyle\sum_{j=2}^{s}A\big(\sum_{a=1}^{j}e_a,\Delta_0,\Delta_{j+1},V_{j-1}\big)\cdot\sigma^{\sum_{a=1}^{j}e_a}\big(A\big(\sum_{a=j+1}^{s+1}e_a,\Delta_j,\Delta_{s+1},W_s^{(j)}\big)\big)\cdot\\
&\cdot\sigma^{e_1}\big(A\big(\textstyle\sum_{a=2}^{s+1}e_a,\Delta_1,\Delta_{s+1},W_s^{(1)}\big)\big)^{-1}+\\
&+A\big(\textstyle\sum_{a=1}^{s+1}e_a,\Delta_0,\Delta_{s+1},V_s\big)\cdot\sigma^{e_1}\big(A\big(\sum_{a=2}^{s+1}e_a,\Delta_1,\Delta_{s+1},W_s^{(1)}\big)\big)^{-1}
\end{aligned}\tag{2.11}$$

and

$$\begin{aligned}
&A\big(\textstyle\sum_{a=1}^{s}e_a,\Delta_0,\Delta_s,\Lambda_0(x)\Lambda_1(x)^{p^{e_1}}\cdots\Lambda_{s-1}(x)^{p^{e_1+\cdots+e_{s-1}}}\big)\cdot\\
&\cdot\sigma^{e_1}\big(A\big(\textstyle\sum_{a=2}^{s}e_a,\Delta_1,\Delta_s,\Lambda_1(x)\Lambda_2(x)^{p^{e_2}}\cdots\Lambda_{s-1}(x)^{p^{e_2+\cdots+e_{s-1}}}\big)\big)^{-1}=A\big(e_1,\Delta_0,\Delta_1,V_0\big)+\\
&+\textstyle\sum_{j=2}^{s}A\big(\sum_{a=1}^{j}e_a,\Delta_0,\Delta_{j+1},V_{j-1}\big)\cdot\sigma^{\sum_{a=1}^{j}e_a}\big(A\big(\sum_{a=j+1}^{s}e_a,\Delta_j,\Delta_s,W_{s-1}^{(j)}\big)\big)\cdot\\
&\cdot\sigma^{e_1}\big(A\big(\textstyle\sum_{a=2}^{s}e_a,\Delta_1,\Delta_s,W_{s-1}^{(1)}\big)\big)^{-1}.
\end{aligned}\tag{2.12}$$

Since we want to compare these two expressions modulo p^s, the last term in (2.11) containing $V_s\equiv 0\pmod{p^s}$ can be ignored.

Given $j=2,\dots,s$, we use the inductive hypothesis as follows:

$$\begin{aligned}
&A\big(\textstyle\sum_{a=i+1}^{s+1}e_a,\Delta_i,\Delta_{s+1},W_s^{(i)}\big)\cdot\sigma^{e_{i+1}}\big(A\big(\sum_{a=i+2}^{s+1}e_a,\Delta_{i+1},\Delta_{s+1},W_s^{(i+1)}\big)\big)^{-1}\equiv\\
&\equiv A\big(\textstyle\sum_{a=i+1}^{s}e_a,\Delta_i,\Delta_s,W_{s-1}^{(i)}\big)\cdot\sigma^{e_{i+1}}\big(A\big(\sum_{a=i+2}^{s}e_a,\Delta_{i+1},\Delta_s,W_{s-1}^{(i+1)}\big)\big)^{-1}\pmod{p^{s-i}}
\end{aligned}$$

for $i=1,\dots,j-1$. Applying $\sigma^{\sum_{a=1}^{i}e_a}$ to the i-th congruence and multiplying them out lead to telescoping products on both sides:

$$\begin{aligned}
&\sigma^{e_1}\big(A\big(\textstyle\sum_{a=2}^{s+1}e_a,\Delta_1,\Delta_{s+1},W_s^{(1)}\big)\big)\cdot\sigma^{\sum_{a=1}^{j}e_a}\big(A\big(\sum_{a=j+1}^{s+1}e_a,\Delta_j,\Delta_{s+1},W_s^{(j)}\big)\big)^{-1}\equiv\\
&\equiv\sigma^{e_1}\big(A\big(\textstyle\sum_{a=2}^{s}e_a,\Delta_1,\Delta_s,W_{s-1}^{(1)}\big)\big)\cdot\sigma^{\sum_{a=1}^{j}e_a}\big(A\big(\sum_{a=j+1}^{s}e_a,\Delta_j,\Delta_s,W_{s-1}^{(j)}\big)\big)^{-1}
\end{aligned}$$

modulo p^{s-j+1}. By our assumptions these four matrices are invertible. Therefore, we can invert them to obtain the congruence

$$\begin{aligned}
&\sigma^{\sum_{a=1}^{j}e_a}\big(A\big(\textstyle\sum_{a=j+1}^{s+1}e_a,\Delta_j,\Delta_{s+1},W_s^{(j)}\big)\big)\cdot\sigma^{e_1}\big(A\big(\sum_{a=2}^{s+1}e_a,\Delta_1,\Delta_{s+1},W_s^{(1)}\big)\big)^{-1}\equiv\\
&\equiv\sigma^{\sum_{a=1}^{j}e_a}\big(A\big(\textstyle\sum_{a=j+1}^{s}e_a,\Delta_j,\Delta_s,W_{s-1}^{(j)}\big)\big)\cdot\sigma^{e_1}\big(A\big(\sum_{a=2}^{s}e_a,\Delta_1,\Delta_s,W_{s-1}^{(1)}\big)\big)^{-1}
\end{aligned}\tag{2.13}$$

modulo p^{s-j+1}. Since $V_{j-1}\equiv 0 \pmod{p^{j-1}}$, we obtain the congruence

$$\begin{gathered}
A\big(\sum_{a=1}^{j} e_a, \Delta_0, \Delta_{j+1}, V_{j-1}\big)\cdot\\
\cdot\, \sigma^{\sum_{a=1}^{j} e_a}\big(A\big(\sum_{a=j+1}^{s+1} e_a, \Delta_j, \Delta_{s+1}, W_s^{(j)}\big)\big)\cdot \sigma^{e_1}\big(A\big(\sum_{a=2}^{s+1} e_a, \Delta_1, \Delta_{s+1}, W_s^{(1)}\big)\big)^{-1}\equiv\\
\equiv A\big(\sum_{a=1}^{j} e_a, \Delta_0, \Delta_{j+1}, V_{j-1}\big)\cdot\\
\cdot\, \sigma^{\sum_{a=1}^{j} e_a}\big(A\big(\sum_{a=j+1}^{s} e_a, \Delta_j, \Delta_{s}, W_{s-1}^{(j)}\big)\big)\cdot \sigma^{e_1}\big(A\big(\sum_{a=2}^{s} e_a, \Delta_1, \Delta_{s}, W_{s-1}^{(1)}\big)\big)^{-1}
\end{gathered}$$

modulo p^s. This shows that the j-th summands in (2.11) and (2.12) are congruent modulo p^s. The theorem is proved. ☐

Corollary 2.7. *Under the assumptions of part* (ii) *of Theorem* 2.6 *for* $1\leqslant s\leqslant l$ *we have:*

$$\begin{aligned}
&\det A\big(e_1+\dots+e_{s+1}, \Delta_0, \Delta_{s+1}, \Lambda_0(x)\Lambda_1(x)^{p^{e_1}}\cdots\Lambda_s(x)^{p^{e_1+\dots+e_s}}\big)\cdot\\
&\cdot \det\sigma^{e_1}\big(A\big(e_2+\dots+e_s, \Delta_1, \Delta_s, \Lambda_1(x)\Lambda_2(x)^{p^{e_2}}\cdots\Lambda_{s-1}(x)^{p^{e_2+\dots+e_{s-1}}}\big)\big)\equiv\\
&\equiv \det A\big(e_1+\dots+e_s, \Delta_0, \Delta_s, \Lambda_0(x)\Lambda_1(x)^{p^{e_1}}\cdots\Lambda_{s-1}(x)^{p^{e_1+\dots+e_{s-1}}}\big)\cdot\\
&\cdot \det\sigma^{e_1}\big(A\big(e_2+\dots+e_{s+1}, \Delta_1, \Delta_{s+1}, \Lambda_1(x)\Lambda_2(x)^{p^{e_2}}\cdots\Lambda_{s}(x)^{p^{e_2+\dots+e_{s}}}\big)\big)
\end{aligned}$$

modulo p^s.

2.5. **Derivations.** Recall that $z=(z_1,\dots,z_n)$. Denote

$$D_v=\frac{\partial}{\partial z_v},\qquad v=1,\dots,n.$$

Let $F_1(z), F_2(z), G_1(z), G_2(z)\in\mathbb{Z}_p[z^{\pm1}]$ and $\ell\geqslant 1$. If

$$D_v(F_1(z))\cdot F_2(z)\equiv D_v(G_1(z))\cdot G_2(z)\pmod{p^s},$$

then

$$\begin{aligned}
(2.14)\qquad & D_v(\sigma^\ell(F_1(z)))\cdot\sigma^\ell(F_2(z)) - D_v(\sigma^\ell(G_1(z)))\cdot\sigma^\ell(G_2(z)) =\\
&= D_v(F_1(z^{p^\ell}))\cdot F_2(z^{p^\ell}) - D_v(G_1(z^{p^\ell}))\cdot G_2(z^{p^\ell}) =\\
&= p^\ell z_v^{p^\ell-1}\big(D_v(F_1(z))\cdot F_2(z) - D_v(G_1(z))\cdot G_2(z)\big)\big|_{z\to z^{p^\ell}}\equiv\\
&\equiv 0\pmod{p^{s+\ell}}.
\end{aligned}$$

Theorem 2.8. *Let* $(\Lambda_0(t,z),\Lambda_1(t,z),\dots,\Lambda_l(t,z))$ *be a* $(\Delta,\mathbf{e})$*-admissible tuple of Laurent polynomials in* $\mathbb{Z}_p[x^{\pm1}]=\mathbb{Z}_p[t^{\pm1},z^{\pm1}]$*. Let* $D=D_v$ *for some* $v=1,\dots,n$*. Then under the assumptions of part* (ii) *of Theorem* 2.6 *we have*

$$\begin{aligned}
(2.15)\qquad & D\big(\sigma^\ell\big(A\big(\sum_{a=1}^{s+1} e_a, \Delta_0, \Delta_{s+1}, W_s\big)\big)\big)\cdot\sigma^\ell\big(A\big(\sum_{a=1}^{s+1} e_a, \Delta_0, \Delta_{s+1}, W_s\big)\big)^{-1}\equiv\\
&\equiv D\big(\sigma^\ell\big(A\big(\sum_{a=1}^{s} e_a, \Delta_0, \Delta_{s}, W_{s-1}\big)\big)\big)\cdot\sigma^\ell\big(A\big(\sum_{a=1}^{s} e_a, \Delta_0, \Delta_{s}, W_{s-1}\big)\big)^{-1}\pmod{p^{s+\ell}}
\end{aligned}$$

for $1\leqslant s\leqslant l$ *and* $0\leqslant\ell$.

Proof. Notice that it is sufficient to establish the congruences (2.15) for $\ell=0$, as the general ℓ case follows from (2.14). So, we assume that $\ell=0$ and proceed by induction on $s\geqslant 0$. For $s=0$ the statement is trivially true.

Using part (ii) of Lemma 2.5 we can write

$$
\begin{aligned}
(2.16)\qquad & D\big(A\big(\textstyle\sum_{a=1}^{s+1} e_a, \Delta_0, \Delta_{s+1}, W_s\big)\big) \cdot A\big(\sum_{a=1}^{s+1} e_a, \Delta_0, \Delta_{s+1}, W_s\big)^{-1} = \\
&= \textstyle\sum_{j=1}^{s+1} D\big(A\big(\sum_{a=1}^{j} e_a, \Delta_0, \Delta_{j+1}, V_{j-1}\big)\big) \cdot \sigma^{\sum_{a=1}^{j} e_a}\big(A\big(\sum_{a=j+1}^{s+1} e_a, \Delta_j, \Delta_{s+1}, W_s^{(j)}\big)\big) \cdot \\
&\qquad \cdot A\big(\textstyle\sum_{a=1}^{s+1} e_a, \Delta_0, \Delta_{s+1}, W_s\big)^{-1} + \\
&+ \textstyle\sum_{j=1}^{s+1} A\big(\sum_{a=1}^{j} e_a, \Delta_0, \Delta_{j+1}, V_{j-1}\big) \cdot D\big(\sigma^{\sum_{a=1}^{j} e_a}\big(A\big(\sum_{a=j+1}^{s+1} e_a, \Delta_j, \Delta_{s+1}, W_s^{(j)}\big)\big)\big) \cdot \\
&\qquad \cdot A\big(\textstyle\sum_{a=1}^{s+1} e_a, \Delta_0, \Delta_{s+1}, W_s\big)^{-1}
\end{aligned}
$$

and

$$
\begin{aligned}
(2.17)\qquad & D\big(A\big(\textstyle\sum_{a=1}^{s} e_a, \Delta_0, \Delta_{s}, W_{s-1}\big)\big) \cdot A\big(\sum_{a=1}^{s} e_a, \Delta_0, \Delta_{s}, W_{s-1}\big)^{-1} = \\
&= \textstyle\sum_{j=1}^{s} D\big(A\big(\sum_{a=1}^{j} e_a, \Delta_0, \Delta_{j+1}, V_{j-1}\big)\big) \cdot \sigma^{\sum_{a=1}^{j} e_a}\big(A\big(\sum_{a=j+1}^{s} e_a, \Delta_j, \Delta_{s}, W_{s-1}^{(j)}\big)\big) \cdot \\
&\qquad \cdot A\big(\textstyle\sum_{a=1}^{s} e_a, \Delta_0, \Delta_{s}, W_{s-1}\big)^{-1} + \\
&+ \textstyle\sum_{j=1}^{s} A\big(\sum_{a=1}^{j} e_a, \Delta_0, \Delta_{j+1}, V_{j-1}\big) \cdot D\big(\sigma^{\sum_{a=1}^{j} e_a}\big(A\big(\sum_{a=j+1}^{s} e_a, \Delta_j, \Delta_{s}, W_{s-1}^{(j)}\big)\big)\big) \cdot \\
&\qquad \cdot A\big(\textstyle\sum_{a=1}^{s} e_a, \Delta_0, \Delta_{s}, W_{s-1}\big)^{-1}.
\end{aligned}
$$

The summands corresponding to $j = s+1$ in (2.16) vanish modulo p^s and can be ignored since $V_s \equiv 0 \pmod{p^s}$.

For the same reason

$$
(2.18)\qquad D\big(A\big(\textstyle\sum_{a=1}^{j} e_a, \Delta_0, \Delta_{j+1}, V_{j-1}\big)\big) \equiv 0 \pmod{p^{j-1}}.
$$

We also have

$$
\begin{aligned}
(2.19)\qquad & \sigma^{\sum_{a=1}^{j} e_a}\big(A\big(\textstyle\sum_{a=j+1}^{s+1} e_a, \Delta_j, \Delta_{s+1}, W_s^{(j)}\big)\big) \cdot A\big(\sum_{a=1}^{s+1} e_a, \Delta_0, \Delta_{s+1}, W_s\big)^{-1} \equiv \\
&\equiv \sigma^{\sum_{a=1}^{j} e_a}\big(A\big(\textstyle\sum_{a=j+1}^{s} e_a, \Delta_j, \Delta_{s}, W_{s-1}^{(j)}\big)\big) \cdot A\big(\sum_{a=1}^{s} e_a, \Delta_0, \Delta_{s}, W_{s-1}\big)^{-1} \pmod{p^{s-j+1}}.
\end{aligned}
$$

This follows from (2.13), in which we take $j+1$ and $s+1$ for j and s and use W_s instead of $W_{s+1}^{(1)}$.

Multiplying congruences (2.18) and (2.19) we get

$$
\begin{aligned}
(2.20)\qquad & D\big(A\big(\textstyle\sum_{a=1}^{j} e_a, \Delta_0, \Delta_{j+1}, V_{j-1}\big)\big) \cdot \sigma^{\sum_{a=1}^{j} e_a}\big(A\big(\sum_{a=j+1}^{s+1} e_a, \Delta_j, \Delta_{s+1}, W_s^{(j)}\big)\big) \cdot \\
&\qquad \cdot A\big(\textstyle\sum_{a=1}^{s+1} e_a, \Delta_0, \Delta_{s+1}, W_s\big)^{-1} \equiv \\
&\equiv D\big(A\big(\textstyle\sum_{a=1}^{j} e_a, \Delta_0, \Delta_{j+1}, V_{j-1}\big)\big) \cdot \sigma^{\sum_{a=1}^{j} e_a}\big(A\big(\sum_{a=j+1}^{s} e_a, \Delta_j, \Delta_{s}, W_{s-1}^{(j)}\big)\big) \cdot \\
&\qquad \cdot A\big(\textstyle\sum_{a=1}^{s} e_a, \Delta_0, \Delta_{s}, W_{s-1}\big)^{-1} \pmod{p^s}.
\end{aligned}
$$

Congruence (2.20) implies that the first sum in (2.16) is congruent to the first sum in (2.17) modulo p^s.

To match the second sums we recall the inductive hypothesis in the form

(2.21)
$$D\big(\sigma^{\sum_{a=1}^j e_a}\big(A\big(\sum_{a=j+1}^{s+1} e_a, \Delta_j, \Delta_{s+1}, W_s^{(j)}\big)\big)\big)\cdot$$
$$\cdot\, \sigma^{\sum_{a=1}^j e_a}\big(A\big(\sum_{a=j+1}^{s+1} e_a, \Delta_j, \Delta_{s+1}, W_s^{(j)}\big)\big)^{-1} \equiv$$
$$\equiv\, D\big(\sigma^{\sum_{a=1}^j e_a}\big(A\big(\sum_{a=j+1}^{s} e_a, \Delta_j, \Delta_{s}, W_{s-1}^{(j)}\big)\big)\big)\cdot$$
$$\cdot\, \sigma^{\sum_{a=1}^j e_a}\big(A\big(\sum_{a=j+1}^{s} e_a, \Delta_j, \Delta_{s}, W_{s-1}^{(j)}\big)\big)^{-1} \pmod{p^s},$$

and notice that both sides in (2.21) are congruent to zero modulo $\sigma^{\sum_{a=1}^j e_a}$ by formula (2.14). Therefore, multiplying congruences (2.21) and (2.19) we obtain

$$D\big(\sigma^{\sum_{a=1}^j e_a}\big(A\big(\sum_{a=j+1}^{s+1} e_a, \Delta_j, \Delta_{s+1}, W_s^{(j)}\big)\big)\big)\cdot A\big(\sum_{a=1}^{s+1} e_a, \Delta_0, \Delta_{s+1}, W_s\big)^{-1} \equiv$$
$$\equiv D\big(\sigma^{\sum_{a=1}^j e_a}\big(A\big(\sum_{a=j+1}^{s} e_a, \Delta_j, \Delta_{s}, W_{s-1}^{(j)}\big)\big)\big)\cdot A\big(\sum_{a=1}^{s} e_a, \Delta_0, \Delta_{s}, W_{s-1}\big)^{-1} \pmod{p^s}.$$

Multiplying both sides of this congruence by $A\big(\sum_{a=1}^j e_a, \Delta_0, \Delta_{j+1}, V_{j-1}\big)$ we conclude that the second sum in (2.16) is congruent to the second sum in (2.17) modulo p^s. The theorem is proved. □

There are similar congruences for higher order derivatives of the matrices $A\big(\sum_{a=1}^{s+1} e_a, \Delta_0, \Delta_{s+1}, W_s\big)$. We restrict ourselves with the second order derivatives.

Theorem 2.9. *Let $(\Lambda_0(t,z), \Lambda_1(t,z), \dots, \Lambda_l(t,z))$ be a $(\Delta, \mathbf{e})$-admissible tuple of Laurent polynomials in $\mathbb{Z}_p[x^{\pm1}] = \mathbb{Z}_p[t^{\pm1}, z^{\pm1}]$. Then under the assumptions of part* (ii) *of Theorem 2.6 we have*

$$D_u\big(D_v\big(A\big(\sum_{a=1}^{s+1} e_a, \Delta_0, \Delta_{s+1}, W_s\big)\big)\big)\,\cdot\, A\big(\sum_{a=1}^{s+1} e_a, \Delta_0, \Delta_{s+1}, W_s\big)^{-1} \equiv \tag{2.22}$$
$$\equiv D_u\big(D_v\big(A\big(\sum_{a=1}^{s} e_a, \Delta_0, \Delta_{s}, W_{s-1}\big)\big)\big)\cdot A\big(\sum_{a=1}^{s} e_a, \Delta_0, \Delta_{s}, W_{s-1}\big)^{-1} \pmod{p^s}$$

for all $1 \leqslant u, v \leqslant n$ and $0 \leqslant s \leqslant l$.

Proof. Notice that, for an invertible matrix $F(z)$ and a derivation D, we have $D(F^{-1}) = -F^{-1}\,D(F)\,F^{-1}$.

We apply the derivation D_u to congruence (2.15) with $D = D_v$:

$$D_u\big(D_v\big(A\big(\sum_{a=1}^{s+1} e_a, \Delta_0, \Delta_{s+1}, W_s\big)\big)\big)\cdot A\big(\sum_{a=1}^{s+1} e_a, \Delta_0, \Delta_{s+1}, W_s\big)^{-1} +$$
$$+\, D_v\big(A\big(\sum_{a=1}^{s+1} e_a, \Delta_0, \Delta_{s+1}, W_s\big)\big)\cdot A\big(\sum_{a=1}^{s+1} e_a, \Delta_0, \Delta_{s+1}, W_s\big)^{-1}\cdot$$
$$\cdot\, D_u\big(A\big(\sum_{a=1}^{s+1} e_a, \Delta_0, \Delta_{s+1}, W_s\big)\big)\cdot A\big(\sum_{a=1}^{s+1} e_a, \Delta_0, \Delta_{s+1}, W_s\big)^{-1} \equiv$$
$$\equiv D_u\big(D_v\big(A\big(\sum_{a=1}^{s} e_a, \Delta_0, \Delta_{s}, W_{s-1}\big)\big)\big)\cdot A\big(\sum_{a=1}^{s} e_a, \Delta_0, \Delta_{s}, W_{s-1}\big)^{-1} +$$
$$+\, D_v\big(A\big(\sum_{a=1}^{s} e_a, \Delta_0, \Delta_{s}, W_{s-1}\big)\big)\cdot A\big(\sum_{a=1}^{s} e_a, \Delta_0, \Delta_{s}, W_{s-1}\big)^{-1}\cdot$$
$$\cdot\, D_u\big(A\big(\sum_{a=1}^{s} e_a, \Delta_0, \Delta_{s}, W_{s-1}\big)\big)\cdot A\big(\sum_{a=1}^{s} e_a, \Delta_0, \Delta_{s}, W_{s-1}\big)^{-1}$$

modulo p^s. It remains to apply (2.15) with $D = D_u$ and $D = D_v$ and $\ell = 0$ to see that the second terms on both sides agree modulo p^s. After their cancellation we are left with the required congruences in (2.22). □

Remark. The results of Section 2 in the case $\mathbf{e} = (e_1, \dots, e_l) = (1, \dots, 1)$ and $\Delta = (\Delta_0, \dots, \Delta_l)$ such that $\Delta_0 = \dots = \Delta_l$ were obtained in [VZ2].

3. Convergence

3.1. **Unramified extensions of $\mathbb{Q}_p$.** We fix an algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$. For every m, there is a unique unramified extension of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}_p}$ of degree m, denoted by $\mathbb{Q}_p^{(m)}$. This can be obtained by attaching to $\mathbb{Q}_p$ a primitive root of 1 of order $p^m - 1$. The norm $|\cdot|_p$ on $\mathbb{Q}_p$ extends to a norm $|\cdot|_p$ on $\mathbb{Q}_p^{(m)}$. Let

$$\mathbb{Z}_p^{(m)} = \{a \in \mathbb{Q}_p^{(m)} \mid |a|_p \leqslant 1\}$$

denote the ring of integers in $\mathbb{Q}_p^{(m)}$. The ring $\mathbb{Z}_p^{(m)}$ has the unique maximal ideal

$$\mathbb{M}_p^{(m)} = \{a \in \mathbb{Q}_p^{(m)} \mid |a|_p < 1\},$$

such that $\mathbb{Z}_p^{(m)}/\mathbb{M}_p^{(m)}$ is isomorphic to the finite field $\mathbb{F}_{p^m}$.

For every $u \in \mathbb{F}_{p^m}$ there is a unique $\tilde{u} \in \mathbb{Z}_p^{(m)}$ that is a lift of u and such that $\tilde{u}^{p^m} = \tilde{u}$. The element $\tilde{u}$ is called the Teichmuller lift of u.

3.2. **Domain $\mathfrak{D}_B$.** For $u \in \mathbb{F}_{p^m}$ and $r > 0$ denote

$$D_{u,r} = \{a \in \mathbb{Z}_p^{(m)} \mid |a - \tilde{u}|_p < r\}\,.$$

We have the partition

$$\mathbb{Z}_p^{(m)} = \bigcup_{u \in \mathbb{F}_{p^m}} D_{u,1}\,.$$

Recall $z = (z_1, \dots, z_n)$. For $B(z) \in \mathbb{Z}[z]$, define

$$\mathfrak{D}_B \ = \ \{a \in (\mathbb{Z}_p^{(m)})^n \ \mid \ |B(a)|_p = 1\}.$$

Let $\bar{B}(z)$ be the projection of $B(z)$ to $\mathbb{F}_p[z] \subset \mathbb{F}_{p^m}[z]$. Then $\mathfrak{D}_B$ is the union of unit polydiscs,

$$\mathfrak{D}_B = \bigcup_{\substack{u_1, \dots, u_n \in \mathbb{F}_{p^m} \\ \bar{B}(u_1, \dots, u_n) \neq 0}} D_{u_1,1} \times \dots \times D_{u_n,1}\,.$$

For any k we have

$$\begin{aligned} \{a \in (\mathbb{Z}_p^{(m)})^n \mid \ |B(a^{p^k})|_p = 1\} &= \bigcup_{\substack{u_1, \dots, u_n \in \mathbb{F}_{p^m} \\ \sigma^k(\bar{B}(u_1, \dots, u_n)) \neq 0}} D_{u_1,1} \times \dots \times D_{u_n,1} = \\ &= \bigcup_{\substack{u_1, \dots, u_n \in F_{p^m} \\ \bar{B}(u_1, \dots, u_n) \neq 0}} D_{u_1,1} \times \dots \times D_{u_n,1} = \mathfrak{D}_B\,. \end{aligned}$$

Lemma 3.1 ([VZ2, Lemma 6.1])**.** *Let $\bar{B}_1(z), \dots, \bar{B}_k(z) \in \mathbb{F}_p[z]$ be nonzero polynomials such that $\deg \bar{B}_j(z) \leqslant d$, $j = 1, \dots, k$, for some d. If $kd + 1 < p^m$, then the set*

$$\{a \in (\mathbb{F}_{p^m})^n \mid \bar{B}_j(a) \neq 0,\ j = 1, \dots, f\}$$

is nonempty.

3.3. **Uniqueness theorem.** Let $\mathfrak{D} \subset (\mathbb{Z}_p^{(m)})^n$ be the union of some of the unit polydiscs $D_{u_1,1} \times \cdots \times D_{u_n,1}$, where $u_1, \dots, u_n \in \mathbb{F}_{p^m}$.

Let $(F_i(z))_{i=1}^\infty$ and $(G_i(z))_{i=1}^\infty$ be two sequences of rational functions on $(\mathbb{F}_{p^m})^n$. Assume that each of the rational functions has the form $P(z)/Q(z)$, where $P(z), Q(z) \in \mathbb{Z}[z]$, and for any polydisc $D_{u_1,1} \times \cdots \times D_{u_n,1} \subset \mathfrak{D}$, we have $|Q(\tilde{u}_1, \dots, \tilde{u}_n)|_p = 1$, which implies that

$$|Q(a_1, \dots, a_n)|_p = 1, \qquad \forall\ (a_1, \dots, a_n) \in \mathfrak{D}.$$

Assume that the sequences $(F_i(z))_{i=1}^\infty$ and $(G_i(z))_{i=1}^\infty$ uniformly converge on $\mathfrak{D}$ to analytic functions, which we denote by $F(z)$ and $G(z)$, respectively.

Theorem 3.2 ([VZ2])**.** *Under these assumptions, if $F(z) = G(z)$ on an open nonempty subset of $\mathfrak{D}$. Then $F(z) = G(z)$ on $\mathfrak{D}$.*

3.4. **Infinite tuples.** Let $\mathbf{e} = (e_1, e_2, \dots)$ be an infinite tuple of positive integers. Let $\Delta = (\Delta_0, \dots, \Delta_l)$ be an infinite tuple of nonempty finite subsets of $\mathbb{Z}^r$ of the same size $\#\Delta_j = g$ for some positive integer g. Let $\Lambda = (\Lambda_0(x), \Lambda_1(x), \dots)$ be an infinite tuple of Laurent polynomials in $\mathbb{Z}_p[x^{\pm1}] = \mathbb{Z}_p[t^{\pm1}, z^{\pm1}]$.

Assume that the tuple Λ is $(\Delta, \mathbf{e})$-admissible.

Assume that each of the tuples $\mathbf{e}, \Delta, \Lambda$ have only finitely many distinct elements. This means that there is a finite set of 4-tuples

$$\mathcal{T} = \{(e^j, \bar{\Delta}^j, \tilde{D}^j, \Lambda^j) \mid j = 1, \dots, k\} \tag{3.1}$$

such that for any $l \geqslant 0$ the 4-tuple $(e_{l+1}, \Delta_l, \Delta_{l+1}, \Lambda_l)$ equals one of the 4-tuples in $\mathcal{T}$.

Definition 3.3. *The $(\Delta, \mathbf{e})$-admissible tuple Λ is called* nondegenerate*, if for any $i = 1, \dots, k$, the Laurent polynomial*

$$\det A\big(e^j, \bar{\Delta}^j, \tilde{\Delta}^j, \Lambda^j\big) \ \in\ \mathbb{Z}_p[z^{\pm1}]$$

is nonzero modulo p.

Recall the notation:

$$\begin{aligned} W_s(x) \ &:=\ \Lambda_0(x)\Lambda_1(x)^{p^{e_1}} \cdots \Lambda_s(x)^{p^{e_1+\cdots+e_s}}, \\ W_s^{(j)}(x) \ &:=\ \Lambda_j(x)\Lambda_{j+1}(x)^{p^{e_{j+1}}} \cdots \Lambda_s(x)^{p^{e_{j+1}+\cdots+e_s}}. \end{aligned}$$

If a $(\Delta, \mathbf{e})$-admissible tuple Λ is nondegenerate, then for any $0 \leqslant j \leqslant s$, the Laurent polynomials $\det A\big(\sum_{a=j+1}^{s+1} e_a, \Delta_j, \Delta_{s+1}, W_s^{(j)}\big) \in \mathbb{Z}_p[z^{\pm1}]$ are not congruent to zero modulo p and we may consider congruences involving the inverse matrices $A\big(\sum_{a=j+1}^{s+1} e_a, \Delta_j, \Delta_{s+1}, W_s^{(j)}\big)^{-1}$.

3.5. **Domain of convergence.** Assume that Λ is an infinite nondegenerate $(\Delta, \mathbf{e})$-admissible tuple and m is a positive integer. Denote

$$\mathfrak{D}^{(m)} = \{a \in (\mathbb{Z}_p^{(m)})^n \ \mid\ |\det A\big(e^j, \bar{\Delta}^j, \tilde{\Delta}^j, \Lambda^j(t,a)\big)|_p = 1,\ j = 1, \dots, k\}.$$

Lemma 3.4. *For any $0 \leqslant j \leqslant s$ and $a \in \mathfrak{D}^{(m)}$ we have*

$$\Big|\det A\Big(\sum\nolimits_{a=j+1}^{s+1} e_a, \Delta_j, \Delta_{s+1}, W_s^{(j)}(t,a)\Big)\Big|_p = 1.$$

□

Corollary 3.5. *All entries of* $A\big(\sum_{a=j+1}^{s+1} e_a, \Delta_j, \Delta_{s+1}, W_s^{(j)}(t,z)\big)^{-1}$ *are rational functions in* z *regular on* $\mathfrak{D}^{(m)}$*. For every* $a \in \mathfrak{D}^{(m)}$ *all entries of* $A\big(\sum_{a=j+1}^{s+1} e_a, \Delta_j, \Delta_{s+1}, W_s^{(j)}(t,a)\big)$ *and* $A\big(\sum_{a=j+1}^{s+1} e_a, \Delta_j, \Delta_{s+1}, W_s^{(j)}(t,a)\big)^{-1}$ *are elements of* $\mathbb{Z}_p^{(m)}$*.* □

Theorem 3.6. *Let* Λ *be an infinite nondegenerate* $(\Delta, \mathbf{e})$*-admissible tuple. Consider the sequence of* $g \times g$ *matrices*

$$\Big(A\Big(\sum\nolimits_{a=1}^{s+1} e_a, \Delta_0, \Delta_{s+1}, W_s(t,z)\Big) \cdot \sigma^{e_1}\Big(A\Big(\sum\nolimits_{a=2}^{s+1} e_a, \Delta_1, \Delta_{s+1}, W_s^{(1)}(t,z)\Big)\Big)^{-1}\Big)_{s\geqslant 0} \tag{3.2}$$

whose entries are rational functions in z *regular on the domain* $\mathfrak{D}^{(m)}$*. This sequence uniformly converges on* $\mathfrak{D}^{(m)}$ *as* $s \to \infty$ *to an analytic* $g \times g$ *matrix with values in* $\mathbb{Z}_p^{(m)}$*. Denote this matrix by* $\mathcal{A}_\Lambda(z)$*. For* $a \in \mathfrak{D}^{(m)}$ *we have*

$$\Big|\det \mathcal{A}_\Lambda(a)\Big|_p = 1 \tag{3.3}$$

and the matrix $\mathcal{A}_\Lambda(a)$ *is invertible.*

Proof. By part (i) of Theorem 2.6 we have $|\det \sigma^{e_1}\big(A\big(\sum_{a=2}^{s+1} e_a, \Delta_1, \Delta_{s+1}, W_s^{(1)}(t,a)\big)\big)|_p = 1$ for $a \in \mathfrak{D}^{(m)}$. Hence the matrix in (3.2) is a matrix of rational functions in z regular on $\mathfrak{D}^{(m)}$. Moreover, if $a \in \mathfrak{D}^{(m)}$, then every entry of this matrix is an element of $\mathbb{Z}_p^{(m)}$. The uniform convergence on $\mathfrak{D}^{(m)}$ of the sequence (3.2) is a corollary of part (ii) of Theorem 2.6. Equation (3.3) follows from part (i) of Theorem 2.6. The theorem is proved. □

Theorem 3.7. *Let* Λ *be an infinite nondegenerate* $(\Delta, \mathbf{e})$*-admissible tuple, and* $D = D_v$*,* $v = 1, \dots, n$*. Given* $\ell \geqslant 0$ *consider the sequence of* $g \times g$ *matrices*

$$\Big(D\Big(\sigma^\ell\Big(A\Big(\sum\nolimits_{a=1}^{s+1} e_a, \Delta_0, \Delta_{s+1}, W_s\Big)\Big)\Big) \cdot \sigma^\ell\Big(A\Big(\sum\nolimits_{a=1}^{s+1} e_a, \Delta_0, \Delta_{s+1}, W_s\Big)\Big)^{-1}\Big)_{s\geqslant 0}$$

whose entries are rational functions in z *regular on the domain* $\mathfrak{D}$*. This sequence uniformly converges on* $\mathfrak{D}$ *as* $s \to \infty$ *to an analytic* $g \times g$ *matrix with values in* $\mathbb{Z}_p^{(m)}$*. Denote this matrix by* $\mathcal{A}_{\Lambda, D\sigma^\ell}(z)$*.*

Proof. The theorem is a corollary of Theorem 2.8. □

Theorem 3.8. *Let* $\Lambda = (\Lambda_0(x), \Lambda_1(x), \Lambda_2(x), \dots)$ *be an infinite nondegenerate* $(\Delta, \mathbf{e})$*-admissible tuple. Given* $\ell \geqslant 0$ *and* $1 \leqslant u, v \leqslant n$*, consider the sequence of* $g \times g$ *matrices*

$$\Big(D_u\Big(D_v\Big(A\Big(\sum\nolimits_{a=1}^{s+1} e_a, \Delta_0, \Delta_{s+1}, W_s\Big)\Big)\Big) \cdot A\Big(\sum\nolimits_{a=1}^{s+1} e_a, \Delta_0, \Delta_{s+1}, W_s\Big)^{-1}\Big)_{s\geqslant 0}$$

whose entries are rational functions in z *regular on the domain* $\mathfrak{D}$*. This sequence uniformly converges on* $\mathfrak{D}$ *as* $s \to \infty$ *to an analytic* $g \times g$ *matrix with values in* $\mathbb{Z}_p^{(m)}$*. Denote this matrix by* $\mathcal{A}_{\Lambda, D_u D_v}(z)$*.*

Proof. The theorem is a corollary of Theorem 2.9. □

Let $\Lambda = (\Lambda_0(x), \Lambda_1(x), \Lambda_2(x), \dots)$ be an infinite nondegenerate $(\Delta, \mathbf{e})$-admissible tuple. Consider the $g\times g$ matrix valued functions $\mathcal{A}_{\Lambda, \frac{\partial}{\partial z_u}\sigma^0}(z)$, $\mathcal{A}_{\Lambda, \frac{\partial}{\partial z_v}\sigma^0}(z)$ in Theorem 3.7 and denote them by $\mathcal{A}_u(z)$, $\mathcal{A}_v(z)$, respectively. Consider the $g\times g$ matrix valued function $\mathcal{A}_{\Lambda, \frac{\partial}{\partial z_u}\frac{\partial}{\partial z_v}}(z)$ in Theorem 3.8 and denote it by $\mathcal{A}_{u,v}(z)$. All the three functions are analytic on $\mathfrak{D}^{(m)}$.

Lemma 3.9 ([VZ2, Lemma 3.7])**.** *We have*

$$\frac{\partial}{\partial z_u}\mathcal{A}_v = \mathcal{A}_{u,v} - \mathcal{A}_v\mathcal{A}_u\,.$$

□

4. *KZ* equations and complex solutions

4.1. ***KZ* equations.** Let $\mathfrak{g}$ be a simple Lie algebra with an invariant scalar product. The Casimir element is $\Omega = \sum_i h_i\otimes h_i \in \mathfrak{g}\otimes\mathfrak{g}$, where $(h_i)\subset\mathfrak{g}$ is an orthonormal basis. Let $V = \otimes_{i=1}^n V_i$ be a tensor product of $\mathfrak{g}$-modules, $\kappa\in\mathbb{C}^\times$ a nonzero number. The *KZ equations* is the system of differential equations on a V-valued function $I(z_1,\dots,z_n)$,

$$\frac{\partial I}{\partial z_i} \;=\; \frac{1}{\kappa}\sum\nolimits_{j\neq i}\frac{\Omega_{i,j}}{z_i-z_j}I, \qquad i=1,\dots,n,$$

where $\Omega_{i,j}: V\to V$ is the Casimir operator acting in the ith and jth tensor factors, see [KZ, EFK].

This system is a system of Fuchsian first order linear differential equations. The equations are defined on the complement in $\mathbb{C}^n$ to the union of all diagonal hyperplanes.

The object of our discussion is the following particular case. Let n,q be positive integers. We consider the following system of differential and algebraic equations for a column n-vector $I=(I_1,\dots,I_n)$ depending on variables $z=(z_1,\dots,z_n)$:

$$\frac{\partial I}{\partial z_i} \;=\; \frac{1}{q}\sum_{j\neq i}\frac{\Omega_{ij}}{z_i-z_j}I, \quad i=1,\dots,n, \qquad I_1+\dots+I_n=0, \tag{4.1}$$

where $z=(z_1,\dots,z_n)$; the $n\times n$-matrices Ω_{ij} have the form

$$\Omega_{ij} \;=\; \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{c c} & \begin{array}{ccccc} & {\scriptstyle i} & & {\scriptstyle j} & \end{array} \\ \begin{array}{c} \\ {\scriptstyle i} \\ \\ {\scriptstyle j} \\ \\ \end{array} & \begin{pmatrix} & \vdots & & \vdots & \\ \cdots & -1 & \cdots & 1 & \cdots \\ & \vdots & & \vdots & \\ \cdots & 1 & \cdots & -1 & \cdots \\ & \vdots & & \vdots & \end{pmatrix}\end{array},$$

and all other entries are zero. This joint system of *differential and algebraic equations* will be called the *system of KZ equations* in this paper.

For $i=1,\dots,n$ denote

$$H_i(z) = \tfrac{1}{q}\sum\nolimits_{j\neq i}\tfrac{\Omega_{ij}}{z_i-z_j}\,, \qquad \nabla_i^{\mathrm{KZ}} = \tfrac{\partial}{\partial z_i} - H_i(z), \qquad i=1,\dots,n. \tag{4.2}$$

The linear operators $H_i(z)$ are called the Gaudin Hamiltonians. The KZ equations can be written as the system of equations,

$$\nabla^{\rm KZ}_i I = 0, \quad i=1,\dots,n, \qquad I_1+\dots+I_n=0.$$

System (4.1) is the system of the differential KZ equations with parameter $\kappa = q$ associated with the Lie algebra $\mathfrak{sl}_2$ and the subspace of singular vectors of weight $n-2$ of the tensor power $(\mathbb{C}^2)^{\otimes n}$ of two-dimensional irreducible $\mathfrak{sl}_2$-modules, up to a gauge transformation, see this example in [V2, Section 1.1], see also [V3].

4.2. **Solutions over $\mathbb{C}$.** Define the *master function*

$$\Phi(t,z) = (t-z_1)^{-1/q}\dots(t-z_n)^{-1/q}$$

and the column n-vector

$$I^{(C)}(z) = (I_1,\dots,I_n) := \int_C \Big(\frac{\Phi(t,z)}{t-z_1},\dots,\frac{\Phi(t,z)}{t-z_n}\Big) dt\,, \tag{4.3}$$

where $C\subset\mathbb{C}-\{z_1,\dots,z_n\}$ is a contour on which the integrand takes its initial value when t encircles C.

Theorem 4.1. *The function $I^{(C)}(z)$ is a solution of system (4.1).*

This theorem is a very particular case of the results in [SV1].

Proof. The theorem follows from Stokes' theorem and the two identities:

$$-\frac1q\Big(\frac{\Phi(t,z)}{t-z_1}+\dots+\frac{\Phi(t,z)}{t-z_n}\Big)\ =\ \frac{\partial\Phi}{\partial t}(t,z)\,, \tag{4.4}$$

$$\Big(\frac{\partial}{\partial z_i}-\frac1q\sum_{j\ne i}\frac{\Omega_{i,j}}{z_i-z_j}\Big)\Big(\frac{\Phi(t,z)}{t-z_1},\dots,\frac{\Phi(t,z)}{t-z_n}\Big)\ =\ \frac{\partial\Psi^i}{\partial t}(t,z), \tag{4.5}$$

where $\Psi^i(t,z)$ is the column n-vector $(0,\dots,0,-\frac{\Phi(t,z)}{t-z_i},0,\dots,0)$ with the nonzero element at the i-th place. □

Theorem 4.2 (cf. [V1, Formula (1.3)])**.** *All solutions of system (4.1) have this form. Namely, the complex vector space of solutions of the form (4.3) is $(n-1)$-dimensional.*

4.3. **Solutions as vectors of first derivatives.** Consider the integral

$$T(z) = T^{(C)}(z) = \int_C \Phi(t,z)\,dt.$$

Then

$$I^{(C)}(z) =\ q\,\Big(\frac{\partial T^{(C)}}{\partial z_1},\dots,\frac{\partial T^{(C)}}{\partial z_n}\Big).$$

Denote $\nabla T = \Big(\frac{\partial T}{\partial z_1},\dots,\frac{\partial T}{\partial z_n}\Big)$. Then the column gradient vector ∇T of the function $T(z)$ satisfies the following system of KZ equations

$$\nabla^{\rm KZ}_i\nabla T = 0, \quad i=1,\dots,n, \qquad \frac{\partial T}{\partial z_1}+\dots+\frac{\partial T}{\partial z_n}=0.$$

This is a system of second order linear differential equations on the function $T(z)$.

5. Solutions modulo powers of p

5.1. Assumptions.

Let $p, q,\ p > q$, be prime numbers. Let e be the order of p modulo q, that is, the least positive integer such that $p^e \equiv 1 \pmod q$. Hence $(p^e - 1)/q$ is a positive integer. Let $n = gq + 1$ for some positive integer g. Assume that $p^e > n$ and $p \geqslant n + q - 2$.

In this paper we consider the system of *KZ* equations (4.1) with $n = gq + 1$ and $\kappa = q$ and study polynomial solutions of the *KZ* equations modulo powers of p.

5.2. Polynomial solutions. For an integer $s \geqslant 1$ define the *master polynomial*

$$\Phi_s(t, z) = \big((t - z_1) \dots (t - z_n)\big)^{(p^{es}-1)/q}.$$

For $\ell = 1, \dots, g$ define the column n-vector

$$I_{s,\ell}(z) = (I_{s,\ell,1}, \dots, I_{s,\ell.n})$$

as the coefficient of $t^{\ell p^{es}-1}$ in the column n-vector of polynomials $\big(\frac{\Phi_s(t,z)}{t-z_1}, \dots, \frac{\Phi_s(t,z)}{t-z_n}\big)$. Notice that

$$\deg_t \frac{\Phi_s(t, z)}{t - z_i} = (gq + 1)\frac{p^{es} - 1}{q} - 1 = gp^{es} - 1 + \frac{p^{es} - 1}{q} - g.$$

If $\ell > g$, then the polynomial $\frac{\Phi_s(t,z)}{t-z_i}$ does not have the monomial $t^{\ell p^{es}-1}$.

Theorem 5.1 (cf. [V5, VZ2])**.** *The column n-vector $I_{s,\ell}(z)$ of polynomials in z is a solution of the system of KZ equations* (4.1) *modulo p^{es}.*

We call the column n-vectors $I_{s,\ell}(z)$, $\ell = 1, \dots, g$, the p^{es}*-hypergeometric solutions* of the *KZ* equations (4.1).

Proof. We have the following modifications of identities (4.4), (4.5) :

$$\frac{p^{es} - 1}{q} \Big(\frac{\Phi_s(t, z)}{t - z_1} + \dots + \frac{\Phi_s(t, z)}{t - z_n}\Big) \;=\; \frac{\partial \Phi_s}{\partial t}(t, z)\,,$$

$$\Big(\frac{\partial}{\partial z_i} + \frac{p^{es} - 1}{q} \sum_{j \neq i} \frac{\Omega_{i,j}}{z_i - z_j}\Big)\Big(\frac{\Phi_s(t, z)}{t - z_1}, \dots, \frac{\Phi_s(t, z)}{t - z_n}\Big) \;=\; \frac{\partial \Psi^i_s}{\partial t}(t, z),$$

where $\Psi^i_s(t, z)$ is the column n-vector $(0, \dots, 0, -\frac{\Phi_s(t,z)}{t-z_i}, 0, \dots, 0)$ with the nonzero element at the i-th place. Theorem 5.1 follows from these identities. □

Consider the $n \times g$ matrix

$$I_s(z) = (I_{s,1}, \dots, I_{s,g}) = \big(I_{s,\ell,i}\big)^{i=1,\dots,n}_{\ell=1,\dots,g}\,,$$

where $I_{s,\ell,i}$ stays at the ℓ-th column and i-th row. The matrix $I_s(z)$ satisfies the *KZ* equations,

$$\nabla^{\mathrm{KZ}}_i I_s(z) = 0, \quad i = 1, \dots, n, \qquad I_{s,\ell,1} + \dots + I_{s,\ell,n}(z) = 0, \quad \ell = 1, \dots, g,$$

modulo p^{es}.

5.3. **Coefficients of solutions.** Consider the lexicographical ordering of monomials $z_1^{d_1}\dots z_n^{d_n}$. We have $z_1 > \dots > z_n$ and so on. For a nonzero Laurent polynomial $f(z) = \sum_{d_1,\dots,d_n} a_{d_1,\dots,d_n} z_1^{d_1}\dots z_n^{d_n}$ with coefficients in $\mathbb{Z}$, the nonzero summand $a_{d_1,\dots,d_n} z_1^{d_1}\dots z_n^{d_n}$ with the largest monomial $z_1^{d_1}\dots z_n^{d_n}$ is called the leading term of $f(z)$.

If $f(z)$ and $g(z)$ are two nonzero Laurent polynomials, then the leading term of $f(z)g(z)$ equals the product of the leading terms of $f(z)$ and $g(z)$.

Lemma 5.2. *For $l = 1,\dots,g$, the leading term of the vector-polynomial $I_{1,\ell}$ equals $C_\ell\cdot(z_1\dots z_{q(g-\ell)+1})^{(p^e-1)/q}/z^\ell_{q(g-\ell)+1}$,*

$$C_\ell = \pm\binom{(p^e-1)/q-1}{\ell-1}\Big(0,\dots,0,1,\frac{p^e-1}{q\ell},\dots,\frac{p^e-1}{q\ell}\Big), \tag{5.1}$$

where $\frac{p^e-1}{q\ell}$ is repeated $q\ell$ times, and

$$\binom{(p^e-1)/q-1}{\ell-1}\not\equiv 0 \pmod{p}. \tag{5.2}$$

Proof. Formula (5.1) is obtained by inspection. To prove (5.2) consider the p-ary presentation $(p^e-1)/q-1 = a_0+a_1p+\dots$ with $0\leqslant a_i\leqslant p-1$. The inequality (5.2) follows from the inequality $a_0\geqslant g-1$ and Lucas theorem.

We prove that $a_0\geqslant g-1$ under our assumption $p\geqslant n+q-2$. Indeed, $p^e = 1+q(1+a_0)+qa_1p+\dots$. Hence $1+q(1+a_0)\geqslant p$. Let $p = qk+r$ for some integers k,r, $1\leqslant r\leqslant q-1$. Then $1+q(1+a_0)\geqslant qk+r$ or $q(1+a_0)\geqslant qk+r-1\geqslant kq$ or $a_0\geqslant k-1$. Hence $a_0\geqslant g-1$ if $k\geqslant g$.

The inequality $p\geqslant n+q-2$ can be written as $kq+r\geqslant gq+1+q-2$ or $kq\geqslant gq+q-r-1$. Hence $k\geqslant g$. The lemma is proved. □

Lemma 5.3. *Consider the $n\times g$ matrix $I_1(z) = (I_{1,1},\dots,I_{1,g})$ and its $g\times g$ minor $M(z)$ in rows with indices $q(g-\ell)+1$ where $\ell = 1,\dots,g$. Then $M(z)$ is a homogeneous polynomial of degree*

$$d_M = \frac{p^e-1}{q}\cdot\frac{qg^2+2g-qg}{2} - \frac{g(g+1)}{2}\,, \tag{5.3}$$

and the polynomial $M(z)$ is nonzero modulo p.

Proof. Every column of $I_{1,\ell}$ is a homogeneous polynomial. Hence $M(z)$ is a homogeneous polynomial. By Lemma 5.2 the leading term of $M(z)$ equals

$$\pm\prod_{\ell=1}^{g}\binom{(p^e-1)/q-1}{\ell-1}(z_1\dots z_{q(g-\ell)+1})^{(p^e-1)/q}/z^\ell_{q(g-\ell)+1}\,. \tag{5.4}$$

This expression is nonzero modulo p by Lemma 5.2. Formula (5.4) implies (5.3). □

6. Congruences for solutions of KZ equations

6.1. **Congruences for Hasse–Witt matrices of KZ equations.** Let $r=1$, $n=gq+1$, $\mathbf{e}=(e,e,\dots)$, where e is defined in Section 5.1. Let

$$\Gamma=\{1,\dots,g\}\subset\mathbb{Z},\qquad \Delta=(\Gamma,\Gamma,\dots),$$
$$N=[0,gp^e+(p^e-1)/q-g]\subset\mathbb{R}. \tag{6.1}$$

The infinite tuple $(N,N,\dots)$ of intervals is $(\Delta,\mathbf{e})$-admissible, see Definition 2.3.

Recall the polynomial

$$\Phi_1(t,z)=\big((t-z_1)\dots(t-z_n)\big)^{(p^e-1)/q}.$$

The Newton polytope of $\Phi_1(t,z)$ with respect to variable t is the interval $N=[0,gp+(p-1)/q-g]$. We also have

$$\Phi_s(t,z)=\Phi_1(t,z)\cdot\Phi_1(t,z)^{p^e}\dots\Phi_1(t,z)^{p^{e(s-1)}}.$$

The infinite tuple $(\Phi_1(t,z),\Phi_1(t,z),\dots)$ is $(\Delta,\mathbf{e})$-admissible, see Definition 2.4.

For $s\geqslant 1$ consider the Hasse–Witt $g\times g$ matrix

$$A(\Phi_s(t,z)):=A(es,\Gamma,\Gamma,\Phi_s(t,z))=\big(\operatorname{Coeff}_{p^{es}v-u}(\Phi_s(t,z))\big)_{u,v=1,\dots,g},$$

see (2.6). The entries of this matrix are polynomials in z.

Theorem 6.1. *The determinant* $\det A(\Phi_1(t,z))$ *is a homogeneous polynomial in z of degree*

$$d_\Phi=\frac{p^e-1}{q}\cdot\frac{qg^2+2g-qg}{2}, \tag{6.2}$$

and the determinant is nonzero modulo p.

Proof. Denote $A(\Phi_1(t,z))=:(A_{u,v}(z))_{u,v=1,\dots,g}$.

Lemma 6.2. *The leading term of $A_{u,v}(z)$ equals*

$$\pm\binom{(p^e-1)/q}{v-u}(z_1z_2\dots z_{qg+1-qv})^{(p^e-1)/q}/z_{qg+1-qv}^{v-u},\qquad \textit{if } v\geqslant u,$$
$$\pm\binom{(p^e-1)/q}{u-v}(z_1z_2\dots z_{qg+1-qv})^{(p^e-1)/q}z_{qg+2-qv}^{u-v},\qquad \textit{if } v\leqslant u.$$

For example, for $g=2$ the matrix of leading terms is

$$\begin{pmatrix}\pm(z_1\dots z_{g+1})^{(p^e-1)/q} & \pm\binom{(p^e-1)/q}{1}z_1^{(p^e-1)/q}/z_1\\ \pm\binom{(p^e-1)/q}{1}(z_1\dots z_{q+1})^{(p^e-1)/q}z_{q+2} & \pm z_1^{(p^e-1)/q}\end{pmatrix}. \tag{6.3}$$

Proof. The proof is by inspection. $\square$

The fact that $\det A(\Phi_1(t,z))$ is a homogeneous polynomial easily follows from the definition of $A(\Phi_1(t,z))$. It is also easy to see that the leading term of the determinant of the matrix of leading terms of $A_{u,v}(z)$ equals the product of diagonal elements,

$$\pm\prod_{v=1}^{g}(z_1\dots z_{qg+1-qv})^{(p^e-1)/q}. \tag{6.4}$$

This expression is not congruent to zero modulo p. Counting the degree of the monomial in (6.4) we obtain (6.2). This proves Theorem 6.1. ☐

Corollary 6.3. *The infinite nondegenerate $(\Delta, \mathbf{e})$-admissible tuple $(\Phi_1(t,z), \Phi_1(t,z), \dots)$ satisfies the assumptions of Theorem 2.6. Therefore,*

(i) *for $s \geqslant 1$ we have*

$$A(\Phi_s(t,z)) \equiv A(\Phi_1(t,z)) \cdot \sigma^e(A(\Phi_1(t,z))) \cdots \sigma^{e(s-1)}(A\big(\Phi_1(t,z)\big)) \pmod{p}\,; \tag{6.5}$$

(ii) *for $s \geqslant 1$ the determinant of the matrix $A(\Phi_s(t,z))$ is a polynomial, which is nonzero modulo p, and we have modulo p^s:*

$$A(\Phi_{s+1}(t,z)) \cdot \sigma^e(A(\Phi_s(t,z)))^{-1} \equiv A(\Phi_s(t,z)) \cdot \sigma^e(A(\Phi_{s-1}(t,z)))^{-1},$$

where for $s = 1$ we understand the second factor on the right-hand side as the $g \times g$ identity matrix.

Proof. The corollary follows from Theorems 6.1 and 2.6. ☐

6.2. Congruences for frames of solutions of *KZ* equations.

Theorem 6.4. *We have the following congruences of $n \times g$ matrices.*

(i) *For $s \geqslant 1$,*

$$I_{s+1}(z) \cdot A(\Phi_{s+1}(t,z))^{-1} \equiv I_s(z) \cdot A(\Phi_s(t,z))^{-1} \pmod{p^s}\,.$$

(ii) *For $s \geqslant 1$ and $j = 1, \dots, n$,*

$$\frac{\partial I_{s+1}}{\partial z_j}(z) \cdot A(\Phi_{s+1}(t,z))^{-1} \equiv \frac{\partial I_s}{\partial z_j}(z) \cdot A(\Phi_s(t,z))^{-1} \pmod{p^s}\,.$$

Proof. Consider the first row of the Hasse–Witt matrix $A(\Phi_s(t,z))$,

$$\big(A_{1,1}(\Phi_s(t,z)), \dots, A_{1,g}(\Phi_s(t,z))\big), \quad A_{1,\ell}(\Phi_s(t,z)) = \mathrm{Coeff}_{\ell p^s - 1}(\Phi_s(t,z)).$$

For each $A_{1,\ell}(\Phi_s(t,z))$ we view the gradient

$$\nabla A_{1,\ell}(\Phi_s(t,z)) = \Big(\frac{\partial A_{1,\ell}(s)}{\partial z_1}, \dots, \frac{\partial A_{1,\ell}(s)}{\partial z_n}\Big)$$

as a column n-vector. The resulting $n \times g$ matrix of gradients

$$\nabla A(s,z) := (\nabla A_{1,1}(\Phi_s(t,z)), \dots, \nabla A_{1,g}(\Phi_s(t,z)))$$

is proportion to the matrix $I_s(z)$, $\nabla A(s,z) = \frac{1-p^{es}}{q} I_s(z)$. By Theorems 2.8 and 2.9 we have modulo p^s,

$$\nabla A(s+1,z) \cdot A(\Phi_{s+1}(t,z))^{-1} \equiv \nabla A(s,z) \cdot A(\Phi_s(t,z))^{-1},$$
$$\tfrac{\partial}{\partial z_j}\big(\nabla A(s+1,z)\big) \cdot A(\Phi_{s+1}(t,z))^{-1} \equiv \tfrac{\partial}{\partial z_j}\big(\nabla A(s,z)\big) \cdot A(\Phi_s(t,z))^{-1}.$$

These congruences imply the theorem. ☐

Corollary 6.5. *For $s \geqslant 1$ we have*

$$I_s(z) \cdot A(\Phi_s(t,z))^{-1} \equiv I_1(z) \cdot A(\Phi_1(t,z))^{-1} \pmod{p}.$$

6.3. **Domain of convergence.** By Theorem 6.1 the polynomial $\det A(\Phi_1(t,z)) \in \mathbb{Z}[z]$ is of degree d_Φ and this polynomial is nonzero modulo p. For a positive integer m define

$$\mathfrak{D}^{(m)}_{\mathrm{KZ}} = \{a \in (\mathbb{Z}_p^{(m)})^n \mid \ |\det A(\Phi_1(t,a))|_p = 1\}\,.$$

By Lemma 3.1 the domain $\mathfrak{D}^{(m)}_{\mathrm{KZ}}$ is nonempty if $p^m > d_\Phi$. In what follows we assume that $p^m > d_\Phi$.

We have $\big|\det A(\Phi_s(t,a))\big|_p = 1$ for $a \in \mathfrak{D}^{(m)}_{\mathrm{KZ}}$. All entries of $A(\Phi_s(t,z))^{-1}$ are rational functions in z regular on $\mathfrak{D}^{(m)}_{\mathrm{KZ}}$. For every $a \in \mathfrak{D}^{(m)}_{\mathrm{KZ}}$ all entries of $A(\Phi_s(t,a))$ and $A(\Phi_s(t,a))^{-1}$ are elements of $\mathbb{Z}_p^{(m)}$.

Theorem 6.6. *The sequence of $g \times g$ matrices*

$$\big(A\big(\Phi_s(t,z)\big)\cdot \sigma^e\big(A\big(\Phi_{s-1}(t,z)\big)\big)^{-1}\big)_{s\geqslant 1}\,,$$

whose entries are rational functions in z regular on $\mathfrak{D}^{(m)}_{\mathrm{KZ}}$, uniformly converges on $\mathfrak{D}^{(m)}_{\mathrm{KZ}}$ as $s \to \infty$ to an analytic $g\times g$ matrix which will be denoted by $\mathcal{A}(z)$. For $a \in \mathfrak{D}^{(m)}_{\mathrm{KZ}}$ we have

$$\big|\det \mathcal{A}(a)\big|_p = 1$$

and the matrix $\mathcal{A}(a)$ is invertible.

Proof. The theorem follows from Theorem 3.6. □

Theorem 6.7. *For $i = 1,\dots,n$ the sequence of $g\times g$ matrices*

$$\Big(\Big(\frac{\partial}{\partial z_i}A\big(\Phi_s(t,z)\big)\Big)\cdot A\big(\Phi_s(t,z)\big)^{-1}\Big)_{s\geqslant 1}\,,$$

whose entries are rational functions in z regular on $\mathfrak{D}^{(m)}_{\mathrm{KZ}}$, uniformly converges on $\mathfrak{D}^{(m)}_{\mathrm{KZ}}$ as $s \to \infty$ to an analytic $g\times g$ matrix, which will be denoted by $\mathcal{A}^{(i)}(z)$.

The sequence of $n \times g$ matrices

$$\big(I_s(z)\cdot A\big(\Phi_s(t,z)\big)^{-1}\big)_{s\geqslant 1}\,,$$

whose entries are rational functions in z regular on $\mathfrak{D}^{(m)}_{\mathrm{KZ}}$, uniformly converges on $\mathfrak{D}^{(m)}_{\mathrm{KZ}}$ as $s \to \infty$ to an analytic $n\times g$ matrix which will be denoted by $\mathcal{I}(z)$.

For $i = 1,\dots,n$ the sequence of $n\times g$ matrices

$$\Big(\frac{\partial I_s}{\partial z_i}(z)\cdot A\big(s,\Phi_s(t,z)\big)^{-1}\Big)_{s\geqslant 1}\,,$$

whose entries are rational functions in z regular on $\mathfrak{D}^{(m)}_{\mathrm{KZ}}$, uniformly converges on $\mathfrak{D}^{(m)}_{\mathrm{KZ}}$ as $s \to \infty$ to an analytic $n\times g$ matrix which will be denoted by $\mathcal{I}^{(i)}(z)$.

We have

$$\frac{\partial \mathcal{I}}{\partial z_i} = \mathcal{I}^{(i)} - \mathcal{I}\cdot\mathcal{A}^{(i)}\,.$$

Proof. The theorem follows from Theorems 3.7, 3.8, and Lemma 3.9. □

Theorem 6.8. *We have the following system of equations on $\mathfrak{D}^{(m)}_{\mathrm{KZ}}$:*

$$\mathcal{I}^{(i)} = H_i \cdot \mathcal{I}, \qquad i = 1, \dots, n,$$

where H_i are the Gaudin Hamiltonians defined in (4.2).

Proof. The theorem is a corollary of Theorem 5.1. □

Corollary 6.9. *For $a \in \mathfrak{D}^{(m)}_{\mathrm{KZ}}$ we have*

$$\mathcal{I}(a) \equiv I_1(a) \cdot A\big(\Phi_1(t,a)\big)^{-1} \pmod{p}.$$

Proof. The corollary follows from Corollary 6.5 and Theorem 6.7. □

6.4. **Vector bundle $\mathcal{L} \to \mathfrak{D}^{(m),o}_{\mathrm{KZ}}$.** Denote

$$W = \{(I_1, \dots, I_n) \in (\mathbb{Q}^{(m)}_p)^n \mid I_1 + \dots + I_n = 0\}.$$

We consider vectors $(I_1, \dots, I_n)$ as column vectors. The differential operators ∇^{KZ}_i, $i = 1, \dots, n$, define a connection on the trivial bundle $W \times \mathfrak{D}^{(m)}_{\mathrm{KZ}} \to \mathfrak{D}^{(m)}_{\mathrm{KZ}}$, called the *KZ* connection. The connection has singularities at the diagonal hyperplanes in $(\mathbb{Z}^{(m)}_p)^n$ and is well-defined over

$$\mathfrak{D}^{(m),o}_{\mathrm{KZ}} = \{a = (a_1, \dots, a_n) \in (\mathbb{Z}^{(m)}_p)^n \mid |\det A(\Phi_1(t,a))|_p = 1,\, a_i \neq a_j \ \ \forall i, j\}.$$

The *KZ* connection is flat,

$$\big[\nabla^{\mathrm{KZ}}_i, \nabla^{\mathrm{KZ}}_j\big] = 0 \qquad \forall\, i, j,$$

see [EFK]. The flat sections of the *KZ* connection are solutions of system (4.1) of *KZ* equations.

For any $a \in \mathfrak{D}^{(m)}_{\mathrm{KZ}}$ let $\mathcal{L}_a \subset W$ be the vector subspace generated by columns of the $n \times g$ matrix $\mathcal{I}(a)$. Then

$$\mathcal{L} := \bigcup\nolimits_{a \in \mathfrak{D}^{(m)}_{\mathrm{KZ}}} \mathcal{L}_a \to \mathfrak{D}^{(m)}_{\mathrm{KZ}}$$

is an analytic distribution of vector subspaces in the fibers of the trivial bundle $W \times \mathfrak{D}^{(m)}_{\mathrm{KZ}} \to \mathfrak{D}^{(m)}_{\mathrm{KZ}}$.

Theorem 6.10 ([VZ2, Theorem 6.7])**.** *The distribution $\mathcal{L} \to \mathfrak{D}^{(m)}_{\mathrm{KZ}}$ is invariant with respect to the KZ connection. In other words, if $s(z)$ is a local section of $\mathcal{L} \to \mathfrak{D}^{(m)}_{\mathrm{KZ}}$, then the sections $\nabla^{\mathrm{KZ}}_i s(z)$, $i = 1, \dots, n$, also are sections of $\mathcal{L} \to \mathfrak{D}^{(m)}_{\mathrm{KZ}}$.*

Proof. Let $\mathcal{I}(z) = (\mathcal{I}_1(z), \dots, \mathcal{I}_g(z))$ be columns of the $n \times g$ matrix $\mathcal{I}(z)$. Let $a \in \mathfrak{D}^{(m)}_{\mathrm{KZ}}$. Let $c(z) = (c_1(z), \dots, c_g(z))$ be a column vector of analytic functions at a. Consider a local section of the distribution $\mathcal{L} \to \mathfrak{D}^{(m)}_{\mathrm{KZ}}$, $s(z) = \sum_{j=1}^{g} c_j(z)\mathcal{I}_j(z) =: \mathcal{I}\cdot c$. Then

$$\begin{aligned}\nabla^{\mathrm{KZ}}_i s(z) &= -H_i\cdot\mathcal{I}\cdot c + \frac{\partial\mathcal{I}}{\partial z_i}\cdot c + \mathcal{I}\cdot\frac{\partial c}{\partial z_i}\\ &= -H_i\cdot\mathcal{I}\cdot c + (\mathcal{I}^{(i)} - \mathcal{I}\cdot\mathcal{A}^{(i)})\cdot c + \mathcal{I}\cdot\frac{\partial c}{\partial z_i}\\ &= -H_i\cdot\mathcal{I}\cdot c + (H_i\cdot\mathcal{I} - \mathcal{I}\cdot\mathcal{A}^{(i)})\cdot c + \mathcal{I}\cdot\frac{\partial c}{\partial z_i}\\ &= -\mathcal{I}\cdot\mathcal{A}^{(i)}\cdot c + \mathcal{I}\cdot\frac{\partial c}{\partial z_i}\,.\end{aligned}$$

Clearly, the last expression is a local section of $\mathcal{L} \to \mathfrak{D}^{(m)}_{\mathrm{KZ}}$. □

Theorem 6.11. *The function $a \mapsto \dim_{\mathbb{Q}_p^{(m)}} \mathcal{L}_a$ is constant on $\mathfrak{D}^{(m),o}_{\mathrm{KZ}}$, in other words, $\mathcal{L} \to \mathfrak{D}^{(m)}_{\mathrm{KZ}}$ is a vector bundle over $\mathfrak{D}^{(m),o}_{\mathrm{KZ}} \subset \mathfrak{D}^{(m)}_{\mathrm{KZ}}$.*

The proof coincides with the proof of Theorem 6.8 in [VZ2].

Recall that d_Φ is the degree of the polynomial $\det A(\Phi_1(t,z))$ and d_M is the degree of the minor defined in Lemma 5.3.

Theorem 6.12. *If $p^m > d_\Phi + d_M$, then the analytic vector bundle $\mathcal{L} \to \mathfrak{D}^{(m),o}_{\mathrm{KZ}}$ is of rank g.*

Proof. If $p^m > d_\Phi + d_M$, then the minor $M(z)$ defines a function on $\mathfrak{D}^{(m),o}_{\mathrm{KZ}}$ nonzero modulo p by Lemma 3.1. Then by Corollary 6.9, the $n\times g$ matrix valued function $\mathcal{I}(z)$ has a $g\times g$ minor nonzero on $\mathfrak{D}^{(m)}_{\mathrm{KZ}}$. This proves the theorem. □

6.5. **Remarks.**

6.5.1. One may expect that the subbundle $\mathcal{L} \to \mathfrak{D}^{(m),o}_{\mathrm{KZ}}$ can be extended to a rank g subbundle over $\mathfrak{D}^{(m)}_{\mathrm{KZ}} - \mathfrak{D}^{(m),o}_{\mathrm{KZ}}$, the union of the diagonal hyperplanes in $\mathfrak{D}^{(m)}_{\mathrm{KZ}}$.

6.5.2. Following Dwork we may expect that locally at any point $a \in D^{(m),o}_{\mathrm{KZ}}$, the solutions of the KZ equations with values in $\mathcal{L} \to \mathfrak{D}^{(m),o}_{\mathrm{KZ}}$ are given at a by power series in $z_i - a_i$, $i = 1,\dots,n$, bounded in their polydiscs of convergence, while any other local solution at a is given by a power series unbounded in its polydisc of convergence, cf. [Dw] and [V5, Theorem A.4].

6.5.3. The KZ connection ∇^{KZ}_i, $i = 1,\dots,n$, over $\mathbb{C}$ has no nontrivial proper invariant subbundles due to the irreducibility of its monodromy representation, see [Fo, Lemma 6]. Thus the existence of the invariant subbundle $\mathcal{L} \to \mathfrak{D}^{(m),o}_{\mathrm{KZ}}$ is a p-adic feature.

6.5.4. The invariant subbundles of the KZ connection over $\mathbb{C}$ usually are related to some additional conformal block constructions, for example see [FSV, SV2, V3, V4]. Apparently our subbundle $\mathcal{L} \to \mathfrak{D}^{(m),o}_{\mathrm{KZ}}$ is of a different p-adic nature.

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