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COMMENT

Recurrence in Lissajous Curves and the Visual Representation of Tuning Systems

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1 Introduction

Curves in the two-dimensional plane that describe the trajectory of two perpendicular oscillators are commonly know as Bowditch or Lissajous curves. Jules Antoine Lissajous popularized these curves with his ingenious experiments in which he projected a beam of light to two vibrating tuning forks set at right angles, and with a mirror attached to their tip (Lovering, 1880; Hales, 1945; Gallozzi and Strollo, 2023). The resulting light projected on a screen described the motion of the two forks vibrating at different frequencies but at fixed ratios, producing visually appealing shapes. Similar experiments were reproduced by others using different types of vibrating systems, commonly pendulums (Lovering, 1880). In 1877, Samuel C. Tisley introduced at a Royal Society Meeting the harmonograph, a device consisting of two pendulums moving perpendicularly, designed to reproduce Lissajous curves on a piece of paper. Writing in Foundations of Science, Gallozzi and Strollo (2023) give a comprehensive introduction to the history of Lissajous curves and the harmonograph.

Lissajous curves and the harmonograph are well suited to graphically represent interval relations in music. The length of the pendulums can be adjusted to reproduce fixed frequency ratios as when two musical notes are played simultaneously. Whitty (1893) published a book presenting visually appealing curves produced by a harmonograph adjusted at specific frequency ratios emulating music intervals. One particular aspect of the harmonograph is that the amplitude of the oscillations decay due to friction of the different components of the system such as air-pendulum friction and pen-paper friction. Thus, the curves produced by the harmonograph are not recurrent, i.e. they do not return to previous positions due to the decay of the amplitude of the oscillations. In their article, Gallozzi and Strollo (2023) refer to this issue with the following statement *"...only if a harmonograph works in the absence of friction the curve would be always the same and the writing tip would pass indefinitely on the same line"*. I believe this statement is a misinterpretation of the role of friction in the production of curves by the harmonograph. This statement is only true for a finite set of frequency ratios as I will show below. Most frequency ratios actually lead to curves that are not recurrent, and if a harmonograph would work indefinitely in the absence of friction, the pen would pass through all points in the plane defined by the amplitude of the oscillations.

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Table 1 Frequency ratios among of the varios intervals for the just intonation system and the 12-tone equal temperament system

Interval	Just interval	12-tone interval	Difference
Unison (C)	$1/1 = 1$	$2^{0/12} = 1$	0
Minor second (D♭)	$16/15 = 1.0\bar{6}$	$2^{1/12} = 1.0594\ldots$	−0.0072...
Major second (D)	$9/8 = 1.125$	$2^{2/12} = 1.1224\ldots$	−0.0025...
Minor third (E♭)	$6/5 = 1.2$	$2^{3/12} = 1.1892\ldots$	−0.0107...
Major third (E)	$5/4 = 1.25$	$2^{4/12} = 1.2599\ldots$	+ 0.0099...
Perfect fourth (F)	$4/3 = 1.\bar{3}$	$2^{5/12} = 1.3348\ldots$	+ 0.0015...
Tritone (G♭)	$64/45 = 1.4\bar{2}$	$2^{6/12} = 1.4142\ldots$	−0.0080...
Perfect fifth (G)	$3/2 = 1.5$	$2^{7/12} = 1.4983\ldots$	−0.0016...
Minor sixth (A♭)	$8/5 = 1.6$	$2^{8/12} = 1.5874\ldots$	−0.0125...
Major sixth (A)	$5/3 = 1.\bar{6}$	$2^{9/12} = 1.6817\ldots$	+ 0.0151...
Minor seventh (B♭)	$16/9 = 1.\bar{7}$	$2^{10/12} = 1.7817\ldots$	+ 0.0040...
Major seventh (B)	$15/8= 1.875$	$2^{11/12} = 1.8877\ldots$	+ 0.0127...
Octave (C)	$2/1 = 2$	$2^{12/12} = 2$	0

As a reference, intervals are expressed with respect to the tonic C note

The interval ratios used by Whitty (1893) were those from the just intonation system, in which the music intervals are expressed as whole number ratios (rational frequency ratios) (Table 1). Tuning systems have evolved over time and space (historically and geographically) due to a combination of cultural, aesthetic, economic and technical reasons (Polansky et al., 2009). Although the Pythagorean and just intonation systems were predominantly used in Western music for centuries, they provided difficulties to instrument builders, musicians and composers because intervals among whole notes were not all equal in terms of their frequency ratio. Although some intervals such as the perfect fifth or the perfect fourth are considered ‘pure’, ‘concord’ or ‘consonant’ in these systems, other intervals are slightly off or dissonant, mostly because it is not possible to split a twelve tone scale into equally consonant intervals (see Gann, 2019, for details).

Since the 18th century, the preferred tuning system is based on equally spaced intervals on a logarithmic scale that divides an octave in 12 equally spaced steps. This system is called the 12-tone equal temperament, and the frequency ratio among each note is equal to $2^{1/12}$ (or $\sqrt[12]{2} \approx 1.05946$). Although the 12-tone equal system approximates the just intonation system, only the unison and the octave intervals agree exactly among the two systems (Table 1).

Lissajous curves can be used to represent graphically differences between the two tuning systems. In the following, I will present a simple analysis of Lissajous curves to show that in the just intonation system Lissajous curves show recurrent periodic patterns, while in the 12-tone equal temperament system, only the unison and the octave show such a recurrent pattern.

2 Lissajous Curves for the Two Tuning Systems

Lissajous curves are expressed mathematically as the coordinates x and y in the two dimensional plane described by the equations

$$
\begin{aligned}
x &= A_1 \sin\left(\omega_1 t\right), \\
y &= A_2 \sin\left(\omega_2 t + \delta\right),
\end{aligned}
\tag{1}
$$

where A_1 and A_2 are amplitudes, ω_1 and ω_2 angular frequencies, t is time or the number of oscillations per frequency, and δ is a phase shift. We assume here no damping of the oscillations so to represent the case of a harmonograph with no friction. However, friction can be represented by multiplying x and y in Eq. (1) by an exponential decay term of the form $e^{(-d\ t)}$, with d as a damping coefficient.

Lissajous curves produce recurrent patterns, i.e. they return to the same points in the plane, only if the ratio ω_1/ω_2 is a rational number (Weisstein, 2023). Since frequency ratios in the just intonation system are defined only by rational frequency ratios (Table 1), we can conclude that Lissajous curves are recurrent for all frequency ratios defined in this system.

Equation (1) can be used to draw Lissajous curves of musical intervals, with a tonic note vibrating at frequency ω_1 and a second note that defines the size of the interval vibrating at a frequency ω_2. Assuming that the two notes are played simultaneously and at the same loudness, then we can assume $\delta = 0$ and $A_1 = A_2$ in Eq. (1). These assumptions make it easy to use the interval ratios from Table 1 to draw Lissajous curves for each interval and for each tuning system. For example, to draw the curve corresponding to the Major third in the just intonation system, we can make $\omega_1 = \pi$ and $\omega_2 = \frac{5\pi}{4}$ in Eq. (1) and plot the curves for a t number of cycles.

Figure 1 shows Lissajous curves for each of the intervals in the just intonation system, with the exception of the unison. In all cases, the curves were drawn for 300 cycles ($t \in [0, 300]$), but they always followed the same trajectory passing for certain points in the plane recurrently. We see that each curve is unique and defined only by the differences in their frequency ratio. The octave, which appears in row 4 column 3 of Fig. 1, has one of the simplest shapes and the curve passes through the center ($x = 0, y = 0$) multiple times. At the vertical line defined by $x = 0$, the curve of the octave passes only through a single point at the center, $y = 0$, and not through any other point along this vertical line. For the Major fifth, located at row 3 column 1 in Fig. 1, the curve passes recurrently along the vertical line at $x = 0$ at only three different values of y, -1, 0, and 1. Similarly for other interval curves in Fig. 1, the trajectories pass through only a limited set of points in the plane. At the vertical line defined by $x = 0$, all curves pass recurrently through this vertical line only at a finite number of points, which are defined by the rational frequency ratios presented in Table 1.[1]

An interesting characteristic of the Lissajous curves for the just intonation system in Fig. 1 is that the perfect fifth and fourth, together with the octave, have simple shapes and a small number of recurrent points on the plane. These intervals are considered harmonically consonant in music theory, and they played a major role in 15th and 16th century music. Minor and major thirds and sixths, are also harmonically consonant, and they have also relatively simple Lissajous curves with few recurrent points. In contrast, dissonant intervals such as the minor second or minor and major sevenths, have a larger number of recurrent points. In particular, the tritone, often referred to as the 'devil's interval', has the largest

[1] The procedure I use here to demonstrate the recurrence of the points is based on the idea of creating a Poincaré map. I do not provide here a rigorous treatment of this procedure, but this would be the formal method to mathematically prove recurrence of the Lissajous curves.

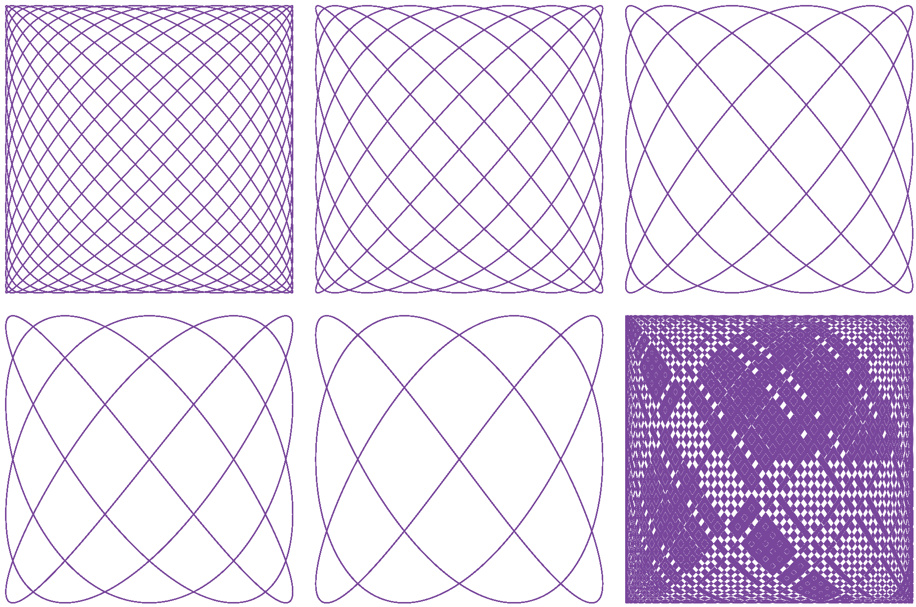

Fig. 1 Lissajous curves for the 12 intervals of the just intonation system. From top left to bottom right, the intervals are Minor second, Major second, Minor third, Major third, Perfect fourth, Tritone, Perfect fifth, Minor sixth, Major sixth, Minor seventh, Major seventh, and Octave

number of recurrent points on the plane (Fig. 1). Thus, Lissajous curves provide a visual aid for the description of consonance and dissonance of music intervals.

As opposed to the curves for the just intonation system, the Lissajous curves of the 12-tone equal temperament system do not pass recurrently through a small fixed number

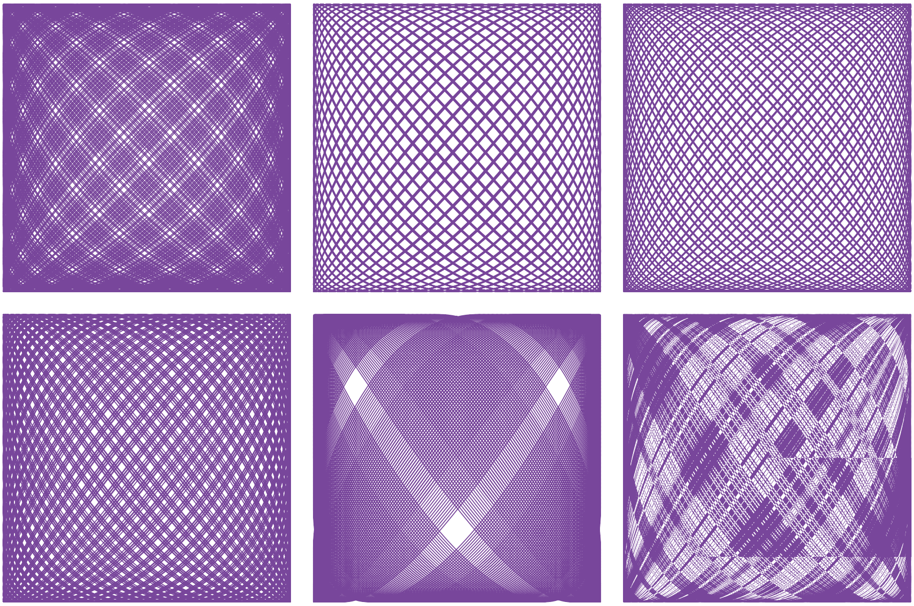

Fig. 2 Lissajous curves for the 12 intervals of the 12-tone equal temperament system. From top left to bottom right, the intervals are Minor second, Major second, Minor third, Major third, Perfect fourth, Tritone, Perfect fifth, Minor sixth, Major sixth, Minor seventh, Major seventh, and Octave

of points on the plane (Fig. 2). In all cases, except for the unison and the octave, the curves pass through each point in the plane only once and the curves tend to fill all the space in the plane. At the vertical line defined by $x = 0$, the curves pass through 301 different points in the y axis because we draw the curves through 301 cycles, from $t = 0$ to $t = 300$. If we would draw these curves for a larger number of cycles N, then they would cross the $x = 0$ line N number of times and eventually all the empty space in the plane would be visually occupied with lines. Although some of the intervals in Fig. 2 show some patterns that may indicate recurrence, this is only because the difference of these intervals from the whole number interval of the just intonation system is small (Table 1), but in every cycle the curve deviates slightly from the previous trajectory.

The whole number ratios that define the just intonation system are the key element that determines the recurrence of the Lissajous curves through the same points in the plane. A defined rational ratio $\frac{a\pi}{b}$, with a and b as integer numbers, guarantees that the curve would pass through $x = \sin \pi = 0$ after every number of b cycles, establishing the recurrent pattern.

Although it has been said in the past that Lissajous curves have no practical use (Newton, 1884), I believe they can actually play an important role in music education and other fields. For example, the curves nicely illustrate the tradeoff between splitting a vibrating string in harmonic ratios, versus splitting it in fixed proportions that facilitate composition with fixed intervals. They can also be used to represent the concept of consonance and dissonance of intervals.

I only included here examples of two tuning systems, but many other systems have been developed throughout the history of music such as the Pythagorean, mean tone, well temperament, or other equal-temperament systems. Lissajous curves could be produced for any tuning system to visually analyze differences of intervals among systems. Our current 12-tone equal temperament system is an approximate solution to the problem of uniform and consonant relations among intervals that can be used in harmonic compositions in all keys, but with a tradeoff in terms of mathematically accurate agreement among frequency ratios.

In addition, one can easily produce harmonograph curves such as those published by Whitty (1893) by adding friction to a Lissajous curve, i.e. by multiplying Eq. (1) by an exponential decay term. Figures 3 and 4 show how the original shape of the Lissajous curves for the two tuning systems are transformed by reducing their size at each cycle, shrinking the original pattern towards the center of each curve. In this case, each curve is a simple representation of an interval which loudness reduces over time as when one would play two notes on a piano. Again, given their rich visual patterns, these curves could be very useful for the teaching of basic concepts of intervals and tuning systems in music education.

Lissajous curves and harmonographs can also play an important role in graphic design as some variants of them are already part of the corporate logo of private companies (Wikipedia contributors, 2023). They can also be used as teaching resource to introduce concepts such as sine waves, state space, harmonic oscillators, among others. In the supplementary material I provide code in the R computing language to reproduce the curves in Figs. 1 to 4, which can be used by interested readers to draw any other harmonograph shape. For example, adding a phase shift to the y coordinate with a value of $\delta \neq 0$.

In summary, this commentary clarifies a statement provided by Gallozzi and Strollo (2023) regarding the recurrence of Lissajous curves. A harmonograph working in the absence of friction would only produce Lissajous curves with recurrent patterns if the frequency ratio among oscillating pendulums is rational. In all other frequency ratios, the curves would pass through all points in the plane enclosed by the amplitude of the oscillations if the harmonograph works indefinitely without friction. This difference in the shapes of Lissajous curves can serve as an engaging method to teach differences between tuning systems in musical education.

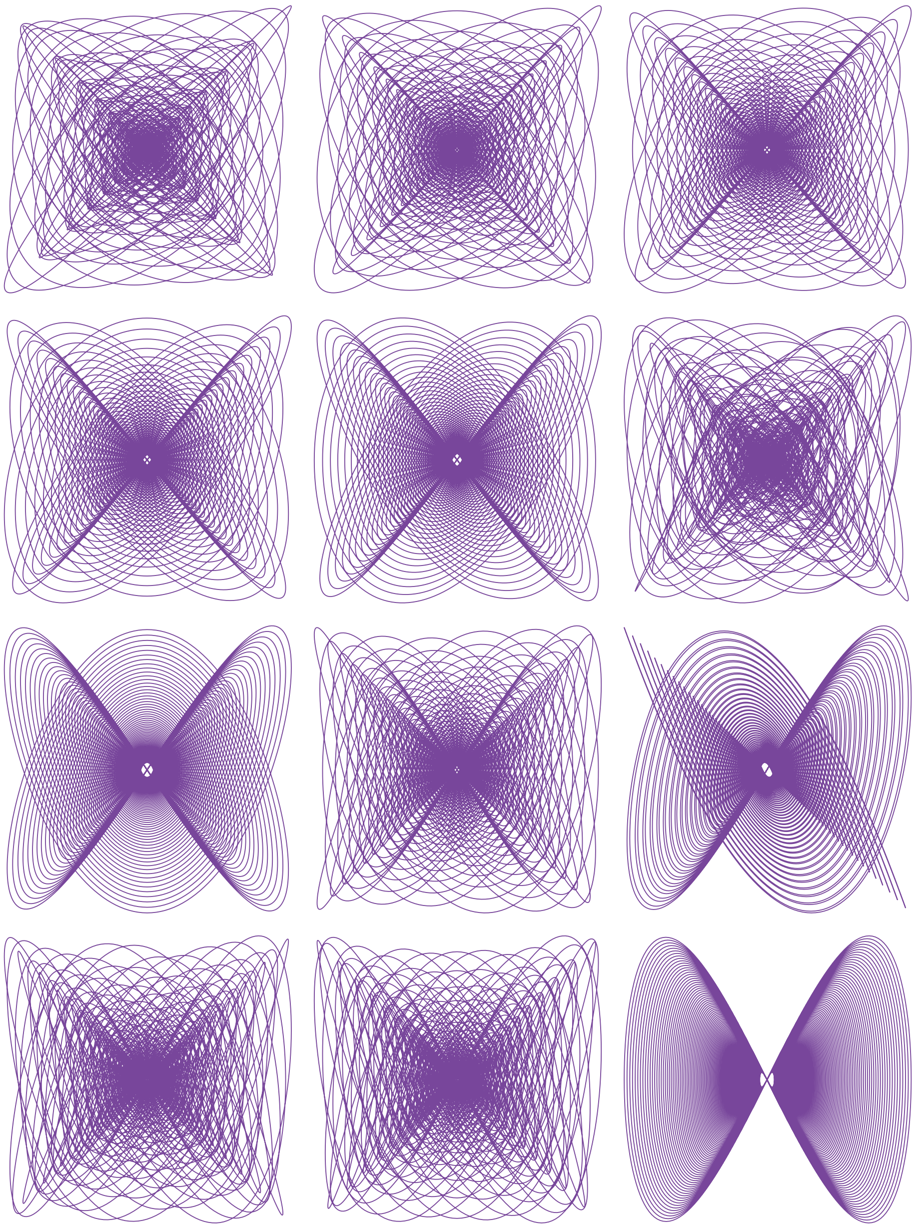

Fig. 3 Lissajous curves with decay of the oscillations (friction) for the 12 intervals of the just intonation system imitating the curves produced by a Harmonograph. From top left to bottom right, the intervals are Minor second, Major second, Minor third, Major third, Perfect fourth, Tritone, Perfect fifth, Minor sixth, Major sixth, Minor seventh, Major seventh, and Octave. The decay of the oscillations was obtained multiplying x and y from Eq. (1) by $e^{(-d\ t)}$ with $d = 0.01$

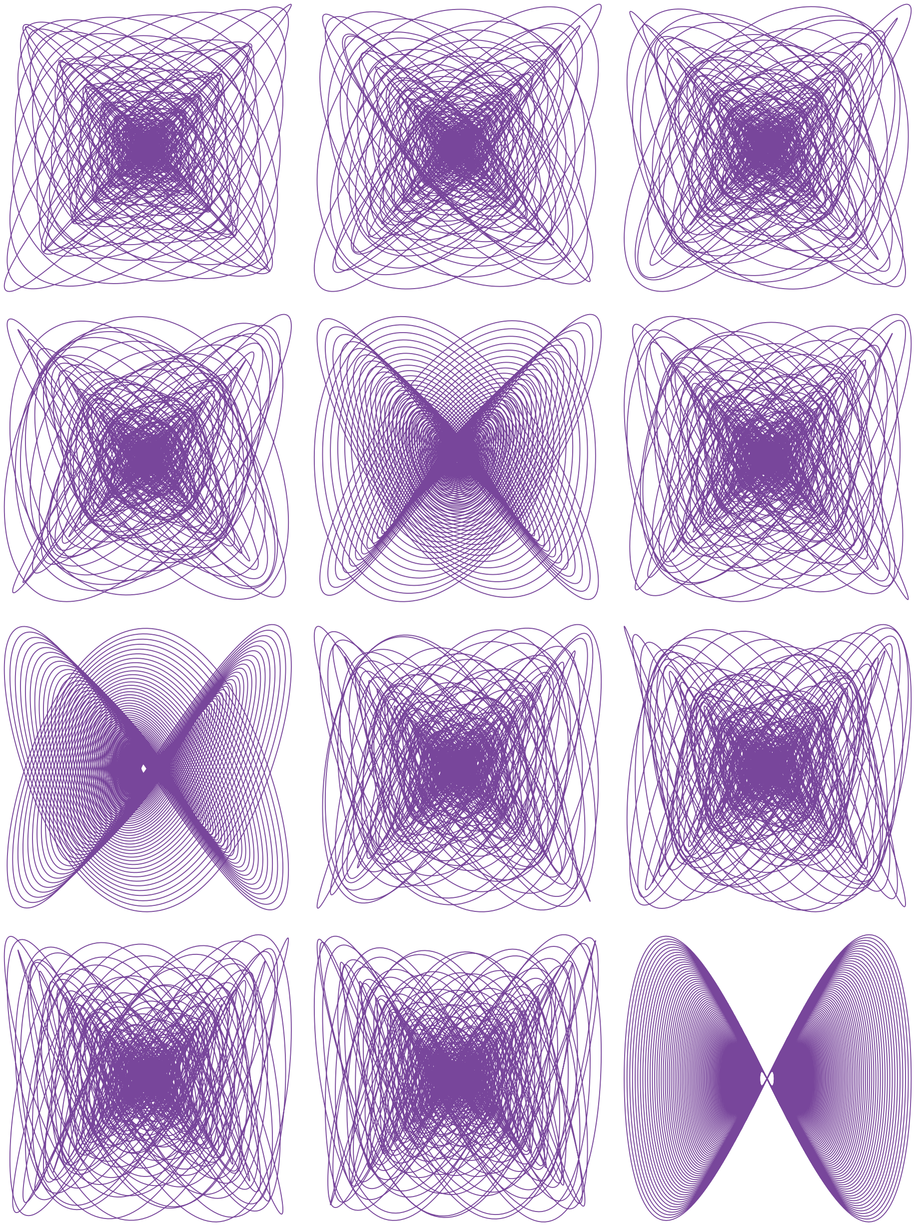

Fig. 4 Lissajous curves with decay of the oscillations (friction) for the 12 intervals of the 12-tone equal temperament system imitating the curves produced by a Harmonograph. From top left to bottom right, the intervals are Minor second, Major second, Minor third, Major third, Perfect fourth, Tritone, Perfect fifth, Minor sixth, Major sixth, Minor seventh, Major seventh, and Octave. The decay of the oscillations was obtained multiplying x and y from Eq. (1) by $e^{(-d\ t)}$ with $d = 0.01$

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Declarations

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