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A systematic approach to correlators in $T\bar{T}$ deformed CFTs

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Abstract: We investigate higher-order corrections to correlators in a general CFT with the double trace $T\bar{T}$ deformation. Traditional perturbation theory proves inadequate for addressing this issue, due to the intricate stress tensor flow induced by the deformation. To tackle this challenge, we introduce a novel technique termed the conservation equation method. This method leverages the trace relation and conservation property of the stress tensor to establish relationships between higher and lower-order corrections and subsequently determine the correlators by enforcing symmetry properties. As an illustration, we compute both first and higher-order corrections, demonstrating the impact of stress tensor deformation on correlators in a general deformed CFT. Our results align with existing calculations in the literature.

Contents

1 Introduction

The $T\bar{T}$ deformation of two-dimensional quantum field theories, as introduced by Smirnov et al. [1–3], has attracted considerable attention due to its remarkable properties. The deformation is characterized by significantly improved analytic tractability compared to generic irrelevant deformations. Several works have shown that the deformation is integrable [3–7], such that under the $T\bar{T}$ flow the deformed spectrum remains exactly solvable and an infinite tower of conserved charges as well as their associated algebra are preserved. Multiple equivalent descriptions, such as those in terms of string theory [8, 9],[1] random metrics [15, 16], and 2D gravity [17, 18], have been provided for this deformation. The holographic counterpart of the deformed CFT has been suggested as a cutoff AdS gravity [19]. An alternative holographic description imposes a mixed boundary condition at the asymptotic AdS boundary [20][21]. Further, accumulated evidence suggests a potential connection between deformed conformal field theories (CFTs) and cutoff AdS gravity, hinting at a novel example of holography beyond conventional holographic CFTs [22–28].

Extensive efforts have been devoted to computing correlators in $T\bar{T}$ deformed CFTs using various methods. Noteworthy studies on the partition function include the works of Datta et al. [29], Aharony et al. [30], and Cardy [31]. These investigations leverage the deformed spectrum and delve into the modular properties of the deformed partition function. Previous works have also computed one-point functions of KdV charges on a torus [32, 33]. Higher-point functions of the stress tensor [16, 22, 27, 34–36] and undeformed operators [37–42] have been computed perturbatively to the first few nontrivial orders in the deformation parameter λ.

Furthermore, authors of [43–45] have constructed surface charges of $T\bar{T}$ deformation to impose constraints on correlators. Harnessing integrability, the renormalized Lagrangians of the deformed massive scalar and Dirac fermion are constructed by leveraging integrability [4, 46]. It's worth highlighting that non-perturbative investigations have also been conducted. These investigations have covered aspects such as the UV divergences of correlators [47] and the large-momentum behavior of two-point correlators [48]. A recent functional renormalization group study on the $T\bar{T}$ deformed scalar field theory has uncovered the presence of a non-trivial UV fixed point [49]. Other related studies include a non-perturbative computation of two-point correlators within the context of the TsT/$T\bar{T}$ correspondence [50].

For a general 2D CFT, The $T\bar{T}$ deformation is defined via the following flow equation of action[2]

$$\begin{aligned}\partial_\lambda S^\lambda =&\frac{1}{\pi}\int d^2x \mathcal{O}_{T\bar{T}}(x),\\ \mathcal{O}_{T\bar{T}} :=&T\bar{T}-\Theta^2,\quad T:=-2\pi T_{zz},\quad \Theta:=2\pi T_{z\bar{z}}=\frac{\pi}{2}T^\mu_\mu,\end{aligned} \tag{1.1}$$

where $T_{\mu\nu}$ denotes the stress tensor defined in the *deformed* theory, thereby making the deformation non-linear. A general correlator obeys the following flow equation [2]

$$\begin{aligned}\partial_\lambda\langle\prod_i O_i(z_i)\rangle^\lambda =&\langle\partial_\lambda\Big(\prod_i O_i(z_i)\Big)\rangle^\lambda\\ &-\frac{1}{\pi}\int d^2x\langle\mathcal{O}_{T\bar{T}}(z)\prod_i O_i(z_i)\rangle^\lambda-\langle\mathcal{O}_{T\bar{T}}(z)\rangle^\lambda\langle\prod_i O_i(z_i)\rangle^\lambda,\end{aligned} \tag{1.2}$$

[1]For the single trace $T\bar{T}$ deformation [10–14], one can refer to the relevant investigation.

[2]Our convention for the flow equation differs from [22] by a factor of $\frac{1}{\pi}$, i.e. $\frac{1}{\pi}(\partial_\lambda S^\lambda)_{ours}=(\partial_\lambda S^\lambda)_{KLM}$. Note that the T_{zz} in [22] corresponds to T in our paper.

which follows from the definition of correlators[3]

$$\langle\prod_i O_i(z_i)\rangle^\lambda = \left(\prod_i \frac{\delta}{\delta J_{O_i}}\right) \frac{Z^\lambda[J]}{Z^\lambda[J=0]}\Bigg|_{J=0}. \tag{1.3}$$

The first term in the right hand side (RHS) of (1.2) accounts for the deformation of *deformed operators*, which are operators whose functional forms in terms of the fundamental fields depend on λ, such as the conserved currents. From a Hamiltonian point of view, these deformed operators differ from their CFT counterparts even on the initial time slice, for the deformed operators are not only time evolved with the deformed Hamiltonian but also undergo changes in their explicit form [51]. It is important to note that the first term vanishes for undeformed operators whose forms are independent of λ. The second term on the RHS of (1.2) is associated with the contribution from the flow of action.

The disconnected term in the integral, namely $-\langle\mathcal{O}_{T\bar{T}}(z)\rangle^\lambda\langle(\prod_i O_i(z_i))\rangle^\lambda$, will be omitted in the rest of our discussions since it vanishes on a Euclidean plane, as proved in appendix A. We may expand both sides of the equation in powers of λ to obtain a relation between higher-order and lower-order corrections:

$$\langle\prod_i O_i(z_i)\rangle^{(n)} = \sum_{m=0}^{n-1}\langle\left(\prod_i O_i(z_i)\right)^{(m)}\rangle^{(n-m)} - \frac{1}{n\pi}\int d^2x\langle\mathcal{O}_{T\bar{T}}(z)\prod_i O_i(z_i)\rangle^{(n-1)}, \tag{1.4}$$

where $A^{(n)}$ represents the coefficient of order λ^n in the series expansion of a given quantity or object within the deformed theory, expressed as $A^\lambda = \sum_i \lambda^i A^{(i)}$. Specifically, $A^{(0)}$ corresponds to the limit of the Conformal Field Theory (CFT). For instance, $\langle\prod_i O_i(z_i)\rangle^{(n)}$ represents the n-th order correction to the correlator $\langle\prod_i O_i(z_i)\rangle^\lambda$; $O^{(n)}$ signifies the n-th order correction to the functional form of the deformed operator O in terms of the fundamental fields. As an illustration, in the deformed free boson CFT, one has [52]

$$\begin{aligned}
\mathcal{O}_{T\bar{T}} &= \frac{4\lambda(\partial\phi\bar{\partial}\phi)^2 + \sqrt{1-8t(\partial\phi\bar{\partial}\phi)^2} - 1}{2\lambda^2\sqrt{1-8\lambda(\partial\phi\bar{\partial}\phi)^2}}, \\
\mathcal{O}_{T\bar{T}} &= \sum_i \lambda^i \mathcal{O}^{(i)}_{T\bar{T}}, \quad \mathcal{O}^{(0)}_{T\bar{T}} = (2\pi)^2(\partial\phi\bar{\partial}\phi)^2, \quad \mathcal{O}^{(1)}_{T\bar{T}} = 32\pi^3(\partial\phi\bar{\partial}\phi)^3.
\end{aligned} \tag{1.5}$$

While (1.4) allows for the computation of first-order corrections to correlators of undeformed operators with conformal perturbation theory, this approach has its limitations when addressing correlators with stress tensor insertions or higher-order corrections to correlators of undeformed operators. In these scenarios, solely relying on the flow equation for correlators as described above proves insufficient without knowledge of the explicit form of the deformed stress tensor $T^{(n)}(z)$.

As a possible attempt, we may try to obtain corrections to the stress tensor by its definition as the variation of the action w.r.t. the metric:

$$\partial_\lambda T_{ab} = \frac{\delta}{\delta g^{ab}}\partial_\lambda S^\lambda = \frac{\partial}{\partial g^{ab}}(\sqrt{g}\mathcal{O}_{T\bar{T}}), \quad T^{(1)}_{ab} = \frac{\delta}{\delta g^{ab}}\partial_\lambda S^{(0)} = \frac{\partial}{\partial g^{ab}}(\sqrt{g}\mathcal{O}^{(0)}_{T\bar{T}}). \tag{1.6}$$

[3]We will use complex coordinates for most of our discussions. For simplicity, a field's coordinate dependence will be indicated by its holomorphic coordinate, while its antiholomorphic dependence will be omitted.

However, the feasibility of this method is limited to specific models with explicit classical action. For a general CFT, the dependence of the stress tensor on the metric, i.e., $\frac{\partial T_{cd}}{\partial g^{ab}}$, is unknown. Moreover, we have to assume that the stress tensor can be obtained as the response of the action to an arbitrary change in the background metric, while as discussed in [47], one may have to confront the problem of generalizing the $T\bar{T}$ deformation to curved space.

In the present work, we develop a systematic approach to compute correlators in the deformed theory to higher orders in perturbation theory without relying on the explicit form of the deformed stress tensor. Our method is based on general principles of the $T\bar{T}$ deformed CFTs, including the trace relation and the conservation of the stress tensor. As a check, the results are shown to be consistent with computation in deformed free field theories and existing results in the literature.

The paper is structured as follows. Section 2 introduces the computational method and outlines the computing correlators' procedures. In Section 3, we calculate first-order corrections for three types of correlators: those involving undeformed operators, stress tensor correlators, and mixed correlators featuring both the stress tensor and undeformed operators. These examples illustrate the execution of specific steps within our method, particularly the resolution of certain undetermined terms. In Section 4, we extend our analysis to compute second-order corrections for nearly the same set of correlators examined in Section 3. The paper concludes with a summary of our findings and outlines potential areas for further investigation.

2 The setup and prescription

In this section we present the procedures employed to compute a deformed correlator to an arbitrary order, relying on the trace relation and conservation of the deformed stress tensor. Consider the undeformed theory $S^{\lambda=0}$ to be a CFT on a Euclidean plane. The flow equation of action (1.1) implies that the stress tensor obeys the trace relation

$$\Theta = \lambda \mathcal{O}_{T\bar{T}}. \tag{2.1}$$

This relation crucially connects higher-order corrections to lower-order ones and, ultimately, CFT correlators. Its validity has been demonstrated for the deformed free boson in [3] and proved with the variational principle in [21].

Let us begin by promoting the classical trace relation (2.1) into an operator equation valid inside correlators

$$\langle \Theta(z) X\rangle^{\lambda} = \lambda \langle \mathcal{O}_{T\bar{T}}(z) X\rangle^{\lambda}. \tag{2.2}$$

Expanding both sides of equation (2.2) yields

$$\langle \Theta(z) X\rangle^{(n)} = \langle \mathcal{O}_{T\bar{T}}(z) X\rangle^{(n-1)}, \quad n \neq 0. \tag{2.3}$$

Further, using the conservation equations of the stress tensor gives

$$\begin{aligned} \langle T(z) X\rangle^{(n)} =& \langle \left(\partial_{\bar{z}}^{-1} \partial_z \Theta(z)\right) X\rangle^{(n)}, \\ \langle \bar{T}(z) X\rangle^{(n)} =& \langle \left(\partial_{z}^{-1} \partial_{\bar{z}} \Theta(z)\right) X\rangle^{(n)}. \end{aligned} \tag{2.4}$$

The equations formally replace all insertions of T and $\bar{T}$ inside correlators by $\partial_{\bar{z}}^{-1}\partial_z\Theta, \partial_z^{-1}\partial_{\bar{z}}\Theta$, respectively. It is important to emphasize that $\partial_{\bar{z}}^{-1}$ formally denotes the inverse of $\partial_{\bar{z}}$. Adding a term holomorphic in z to the following equation[4]

$$T = \partial_{\bar{z}}^{-1}\partial_z\Theta + f(z), \tag{2.5}$$

still preserves the conservation equation of T. Therefore, the latter two equations of (2.3) are understood to hold up to holomorphic/antiholomorphic terms. These terms are typically determined by symmetries and other properties of the correlators, as we shall demonstrate in the subsequent sections. Introducing an anti-derivative on other geometries, such as a torus, would result in a nontrivial constant term that cannot be determined solely from the conservation equations. However, in this study, we restrict our discussions to the Euclidean plane so such issues do not arise.

Equipped with the trace relation and conservation equations, the key insight is that higher-order corrections can be expressed in terms of lower-order ones as long as stress tensor insertions are present within the correlator under consideration. For correlators of undeformed operators, the stress tensor can always be introduced using the flow equation (1.4).

We now summarize the procedure for computing a correlator of undeformed operators up to an arbitrary order: computing $\langle X\rangle^{(n)}$. The procedure is as follows:

1. Utilizing the flow equation (1.4) and the expansion of correlator (2.4), insert the vertex $\int d^2z\mathcal{O}_{T\bar{T}}$ to lower the order of the correction by one,

$$\langle X\rangle^{(n)} = -\frac{1}{n\pi}\int d^2x\langle\mathcal{O}_{T\bar{T}}(z)X\rangle^{(n-1)}. \tag{2.6}$$

2. Rewrite the stress tensor components T or $\bar{T}$ in terms of Θ. The RHS of (2.6) becomes

$$\int d^2x\langle(T\bar{T}-(\Theta)^2)(z)X\rangle^{(n-1)} = \int d^2x\langle\left(\partial_{\bar{z}}^{-1}\partial_z\Theta\cdot\bar{T}-(\Theta)^2\right)(z)X\rangle^{(n-1)}. \tag{2.7}$$

3. Insert the trace relation to further lower the order. (2.7) becomes

$$\int d^2x\langle\left(\partial_{\bar{z}}^{-1}\partial_z\mathcal{O}_{T\bar{T}}\cdot\bar{T}-\Theta\mathcal{O}_{T\bar{T}}\right)(z)X\rangle^{(n-2)}. \tag{2.8}$$

4. Repeat steps 2 and 3 until the resulting expression contains only zeroth-order corrections or CFT correlators.

5. Perform the antiderivatives $\partial_{z_i}^{-1}$'s and fix the (anti)holomorphic terms and integration constants by the symmetries or specific properties of $\langle X\rangle^{\lambda}$.

A notable aspect of our prescription is the use of conservation equations (step 2) in dealing with the stress tensor. This approach allows us to avoid relying on the explicit form of the deformed stress tensor, which is not available in a general deformed theory.

[4]More accurately, this term can be a meromorphic function.

Before proceeding with examples, we clarify certain aspects regarding the field $\mathcal{O}_{T\bar{T}}$. $\mathcal{O}_{T\bar{T}}$ is defined through the operator product expansion as follows [1]

$$T(z)\bar{T}(z') - \Theta(z)\Theta(z') = \mathcal{O}_{T\bar{T}}(z) + \text{derivative terms.} \tag{2.9}$$

The presence of arbitrary total derivative terms on the RHS reflects the ambiguity in the definition of $\mathcal{O}_{T\bar{T}}$. This does not pose a problem when the above definition defines the deformed action. However, it does inflict deformed correlators with ambiguities [53]. Consequently, the quantum trace relation is also subject to ambiguities

$$\Theta = \lambda \mathcal{O}_{T\bar{T}} + \partial_\mu W^\mu. \tag{2.10}$$

To date, the resolution of such ambiguities remains elusive. In our treatment, we work in the gauge where all these improvement terms vanish. This means we set the total derivative terms in the trace relation to zero, such that the $T\bar{T}$ operator is defined as

$$\mathcal{O}_{T\bar{T}}(z) = \lim_{z'\to z} T(z)\bar{T}(z') - \Theta(z)\Theta(z'). \tag{2.11}$$

3 First-order corrections

3.1 Correlators of undeformed operators

In this section, we apply the formalism developed in the previous section to derive an expression for the first-order correction to $T\bar{T}$ deformed correlators of undeformed operators. We assume the undeformed ($\lambda = 0$) theory is a CFT on an Euclidean plane. A similar approach was taken in a prior study [47]. We find that the first-order correction to a correlator of undeformed operators can be expressed as a sum of correlators of descendant operators.

At first-order, the procedure is rather straightforward. Let $\langle X\rangle^\lambda$ be a correlator of undeformed operators, and then its first-order is given by

$$\begin{aligned}\langle X\rangle^{(1)} =& -\frac{1}{\pi}\int d^2x \langle \mathcal{O}_{T\bar{T}}(z) X\rangle^{(0)} \\ =& -\frac{1}{\pi}\lim_{\varepsilon\to 0}\int d^2x \langle \big(T(z+\varepsilon)\bar{T}(z) - \Theta(z+\varepsilon)\Theta(z)\big)\, X\rangle^{(0)} - \langle T(z)\bar{T}(z) - \Theta^2(z)\rangle^{(0)}\langle X\rangle^{(0)},\end{aligned} \tag{3.1}$$

where a point-splitting regulator ε is introduced. By the conformal Ward identity

$$\langle X\rangle^{(1)} = -\frac{1}{\pi}\left(\int d^2x \sum_{m,n}\sum_{r,s\geq 1}\frac{1}{(z-z_m+\varepsilon)^r(\bar{z}-\bar{z}_n)^s}\right)\langle L_{r-2,m}\bar{L}_{s-2,n}X\rangle^{(0)}, \tag{3.2}$$

with indices m, n running over all field insertions, and indices r, s running over all positive integers. $L_{r-2,m}$ is a shorthand notation for the Virasoro generator at z_m, $L_{r-2}(z_m)$, formally defined in terms of a contour integral of the undeformed stress tensor $T^{(0)}$. These Virasoro generators are undeformed operators as well. The details of the calculation can be found in appendix C.1, and the final result is given by

$$\langle X\rangle^{(1)} = \langle \mathrm{d}\, X\rangle^{(0)}, \quad \mathrm{d} := \sum_{m,n}\mathrm{d}_{z_m,z_n} \tag{3.3}$$

with

$$\mathrm{d}_{z_m,z_n} := \begin{cases} \log(|z_{mn}|^2/\varepsilon^2)\partial_{z_m}\partial_{\bar{z}_n} - \sum_{s\geq 2}\frac{1}{s-1}\frac{\bar{L}_{s-2,n}\partial_{z_m}}{\bar{z}_{mn}^{s-1}} - \sum_{r\geq 2}\frac{1}{r-1}\frac{L_{r-2,m}\partial_{\bar{z}_n}}{z_{nm}^{r-1}}, & \text{if } m\neq n, \\ \sum_{s\geq 2}\frac{1}{s-1}\frac{\bar{L}_{s-2,m}\partial_{z_m}}{\varepsilon^{s-1}} - \sum_{r\geq 2}\frac{1}{r-1}\frac{L_{r-2,m}\partial_{\bar{z}_m}}{\varepsilon^{r-1}}, & \text{if } m=n. \end{cases} \tag{3.4}$$

It should be noted that the subscripts of d_{z_m,z_n} are ordered–$\mathrm{d}_{z_m,z_n} \neq \mathrm{d}_{z_n,z_m}$. This expression implies that at first-order, the effect of the deformation on correlators can be viewed as the insertion of a set of fields of the form $\sum_m \partial_{z_m}^{-1}T(z_m)\partial_{\bar{z}_m} + \partial_{\bar{z}_m}^{-1}\bar{T}(z_m)\partial_{z_m}$.

If X consists purely of primaries of the undeformed CFT, then the Ward identity reduces to

$$\langle T(z)X\rangle^{(0)} = \left(\frac{\partial_{z_i}}{z-z_i} + \frac{h_i}{(z-z_i)^2}\right)\langle X\rangle^{(0)}, \tag{3.5}$$

and the expression for the first-order correction becomes

$$\begin{aligned} &\langle X\rangle^{(1)} = \langle \mathrm{d_p}\, X\rangle^{(0)}, \\ &\mathrm{d_p} := \sum_{m\neq n}\left(\log(|z_{mn}|^2/\varepsilon^2)\partial_{z_m}\partial_{\bar{z}_n} - \frac{\bar{h}_n\partial_{z_m}}{\bar{z}_{mn}} - \frac{h_m\partial_{\bar{z}_n}}{z_{nm}}\right) - \sum_i\left(\frac{\bar{h}_i\partial_{z_i}}{\varepsilon} + \frac{h_i\partial_{\bar{z}_i}}{\varepsilon}\right), \end{aligned} \tag{3.6}$$

where the subscript p signifies that this relation holds for primary operators.

The formula (3.4) exhibits both logarithmic and power divergences. In this order, it is possible to define locally renormalized fields whose correlators are finite as

$$O_R(z_i) := O(z_i) - \lambda\left(\log(\mu^2\varepsilon^2)\partial_{z_i}\partial_{\bar{z}_i} - \sum_{s\geq 2}\frac{1}{s-1}\frac{\bar{L}_{s-2,i}\partial_{z_i} + L_{s-2,i}\partial_{\bar{z}_i}}{\varepsilon^{s-1}}\right)O(z_i) \tag{3.7}$$

with μ an arbitrary renormalization scale. After incorporating this procedure, we arrive at a renormalized expression for the first-order correction

$$\langle X_R\rangle^{(1)} = \langle \mathrm{d}_R\, X\rangle^{(0)} \tag{3.8}$$

with

$$\mathrm{d}_R := \sum_{m\neq n}\left(\log(\mu^2|z_{mn}|^2)\partial_{z_m}\partial_{\bar{z}_n} - \sum_{s\geq 2}\frac{1}{s-1}\frac{\bar{L}_{s-2,n}\partial_{z_m}}{\bar{z}_{mn}^{s-1}} - \sum_{r\geq 2}\frac{1}{r-1}\frac{L_{r-2,m}\partial_{\bar{z}_n}}{z_{nm}^{r-1}}\right). \tag{3.9}$$

In subsequent discussions, when referring to the first-order correction to correlators of undeformed operators, we typically refer to (3.4), the unrenormalized expression, without incorporating the locally renormalized fields.

As a first example, we examine the case of a two-point correlator of primary fields of conformal dimensions $(h,\bar{h})$. Substituting $X = \mathcal{O}(z)\mathcal{O}(w)$ into the formula (3.6), we obtain

$$\langle \mathcal{O}(z)\mathcal{O}(w)\rangle^{(1)} = \frac{8h\bar{h}\log(|z-w|^2/\varepsilon^2)}{(z-w)^{2h+1}(\bar{z}-\bar{w})^{2\bar{h}+1}}, \tag{3.10}$$

where we have used the CFT two-point function

$$\langle \mathcal{O}(z)\mathcal{O}(w)\rangle^{(0)} = \frac{1}{(z-w)^{2h}(\bar{z}-\bar{w})^{2\bar{h}}}. \tag{3.11}$$

Note that the power terms in (3.6) vanish by translational invariance and the fact that the conformal dimensions of the two fields are equal. This result (3.10) is consistent with the findings in [38] [16].

As a different case, we examine a scenario where the correlator vanishes in the undeformed theory and involves non-primary fields. We consider the deformed free boson CFT and calculate $\langle \mathcal{O}^{(0)}_{T\bar{T}}(z_1)\phi(z_2)\phi(z_3)\rangle^{(1)}$. Applying the formula (3.4), we find

$$\langle \mathcal{O}^{(0)}_{T\bar{T}}(z_1)\phi(z_2)\phi(z_3)\rangle^{(1)} = -\frac{1}{24\pi}\left[\left(\frac{1}{\bar{z}_{12}^3 z_{12}^2 z_{13}} + \frac{1}{\bar{z}_{13}^3 z_{12} z_{13}^2}\right) + \left(\frac{1}{z_{13}^3 \bar{z}_{13}^2 \bar{z}_{12}} + \frac{1}{z_{12}^3 \bar{z}_{13} \bar{z}_{12}^2}\right)\right], \tag{3.12}$$

where we have used the fact that $(L_{-n}\phi)(w)$ equals $\partial\phi$ when $n=1$ and is zero for all other values of n.

3.2 Stress tensor correlators

In this section, we examine the deformed stress tensor correlators, utilizing mainly the trace relation and conservation equations, as discussed earlier.

3.2.1 Two-point and three-point functions

In the first order, the stress tensor two-point functions remain unaltered. However, delving into the reasons behind this sheds light on a fundamental constraint imposed on the holomorphic terms stemming from the conservation equations, known as the spin constraint. It's worth mentioning that similar results regarding the two-point functions have been obtained in other studies [22]. Nevertheless, our approach offers nuanced insights and applies to various scenarios.

By the trace relation, the two-point function $\langle\Theta(z_1)T(z_2)\rangle$ has a vanishing first-order correction

$$\langle\Theta(z_1)T(z_2)\rangle^{(1)} = \langle (T\bar{T}-\Theta^2)(z_1)T(z_2)\rangle^{(1)} = 0. \tag{3.13}$$

It follows from the conservation equations that $\langle T(z_1)T(z_2)\rangle^{(1)}$ can only be a term holomorphic in z_1 (and by symmetry, it is also holomorphic in z_2). Then, the question is how to constrain this holomorphic term.

To tackle this problem, we can use the symmetries inherent in the deformed theory, specifically rotational invariance. To facilitate this discussion, we shall revisit the concept of spin. The spin, denoted as s_i, of a field $\phi_i(x)$ corresponds to the eigenvalue of the spin operator [54]

$$S = J(0) - z\partial_z + \bar{z}\partial_{\bar{z}}, \tag{3.14}$$

when it acts on that field, $J(0)$ is the total angular momentum operator that generates rotations around the point $z = 0$.[5] In the context of a correlator, the spin can be determined by employing the Ward identity associated with global rotational symmetry, as given by[6]

$$\sum_i \left(-z_i\partial_{z_i} + \bar{z}_i\partial_{\bar{z}_i}\right)\langle X\rangle = \sum_i s_i\langle X\rangle, \tag{3.15}$$

where s_i is the spin of the field at $z = z_i$, and the sum $\sum_i s_i$ is identified as the spin of the correlator.

To illustrate, consider the massive free boson two-point correlator

$$\begin{aligned}\langle\partial\phi(z,\bar{z})\partial\phi(w,\bar{w})\rangle = -\frac{m^2}{4(z-w)^2}\Big(&|z|^2K_0(m|z-w|) + 2(|z-w|/m)K_1(m|z-w|)\\ &+|z|^2K_2(m|z-w|)\Big).\end{aligned} \tag{3.16}$$

One may verify that this is an eigenfunction of the differential operator $-z\partial_z + \bar{z}\partial_{\bar{z}} - w\partial_w + \bar{w}\partial_{\bar{w}}$, and the corresponding eigenvalue, or its spin, can be found to be $+2$, which is the sum of spins of the two bosonic fields. Moreover, the expanded forms of fields, such as the Laurant expansion of the holomorphic stress tensor in a CFT, namely $T(z) = \sum_n \frac{L_n(0)}{z^{n+2}}$, can also be shown to have the same spin as the original fields by using the commutation relations between the rotation generator and the Virasoro generators.

Given the preservation of rotational symmetry in the deformed theory, the spin of all fields remains unchanged under deformation, and any corrections to a correlator must exhibit the same spin as the undeformed correlator. This constraint on deformed correlators is called the spin constraint. For example, note that the expression for the first-order correction of correlators of undeformed operators (3.4) satisfies this requirement. Schematically, d can be written as $\sum_m \partial_{z_m}^{-1}T(z_m)\partial_{\bar{z}_m} + \partial_{\bar{z}_m}^{-1}\bar{T}(z_m)\partial_{z_m}$, which leaves the spin of the fields at points $z = z_m$ unaffected. In particular, the action of $\partial_{\bar{z}_m}$ lower the spin of the field at point z_m by one, but this is offset by $\partial_{z_m}^{-1}T(z_m)$. The similar precedure happened in the second term of $\sum_m \partial_{z_m}^{-1}T(z_m)\partial_{\bar{z}_m} + \partial_{\bar{z}_m}^{-1}\bar{T}(z_m)\partial_{z_m}$.

We turn our focus back on the first-order corrections to stress tensor two-point correlators. Recall that for two-dimensional quantum field theories, the zz component of the stress tensor has a spin of 2, as determined by its transformation properties under rotations. Consequently, the two-point correlator $\langle T(z_1)T(z_2)\rangle^\lambda$ should have a spin of 4, and the form of the full correlator $\langle T(z_1)T(z_2)\rangle^\lambda$ is constrained to be

$$\langle T(z_1)T(z_2)\rangle^\lambda = \frac{f(\lambda,|z_{12}|,\varepsilon)}{z_{12}^4}, \tag{3.17}$$

with the deformation effect only manifesting as a spin-neutral factor $f(\lambda,|z_{12}|,\varepsilon)$. We note that if $\langle T(z_1)T(z_2)\rangle^{(1)}$ is holomorphic in z_{12} and is nonzero, then to have the correct

[5] while $z = 0$ is often chosen as a point of reference, this choice is arbitrary, and one can opt for a different reference point z_0, expressing S as $J(z_0) - (z - z_0)\partial_z + (\bar{z} - \bar{z}_0)\partial_{\bar{z}}$ if it proves more convenient. This flexibility can be particularly advantageous when expanding fields around the point z_0, as it may simplify the commutation relation with $J(z_0)$, while the commutation with $J(0)$ remains more intricate.

[6] The contribution from $J(0)$ vanishes since it annihilates the vacuum.

dimension, it must be proportional to $1/z_{12}^6$, which has a spin of 6, violating the spin constraint. This holomorphic term is then fixed to zero. The same reasoning applies to $\langle \bar{T}(z_1)\bar{T}(z_2)\rangle^{(1)}$ and $\langle \bar{T}(z_1)T(z_2)\rangle^{(1)}$, which are also fixed to zero.

The three-point functions are also explored in [27] using almost the same method applied to exact results in the large c limit. However, in their prior work, it was suggested that $\langle T(z_1)T(z_2)T(z_3)\rangle^{(1)}$, or $\langle T(z_1)T(z_2)T(z_3)\rangle^{\lambda}_{c\to\infty}$ in their context, which is holomorphic, cannot be determined. In this context, we resolve this term based on the spin constraint. We find

$$\begin{aligned}
&\langle \Theta(z_1)\bar{T}(z_2)T(z_3)\rangle^{(1)} = \langle \mathcal{O}_{T\bar{T}}(z_1)\bar{T}(z_2)T(z_3)\rangle^{(0)} = \frac{c^2/4}{z_{12}^4\bar{z}_{13}^4},\\
&\langle T(z_1)T(z_2)\bar{T}(z_3)\rangle^{(1)} = \frac{c^2/3}{z_{12}^5\bar{z}_{13}^3} + (1\leftrightarrow 2),\\
&\langle T(z_1)T(z_2)T(z_3)\rangle^{(1)} = 0.
\end{aligned} \tag{3.18}$$

Some correlators are simply related by complex conjugation, such as $\langle T(z_1)T(z_2)\bar{T}(z_3)\rangle^{\lambda}$ and $\langle \bar{T}(z_1)\bar{T}(z_2)T(z_3)\rangle^{\lambda}$ or $\langle T(z_1)T(z_2)T(z_3)\rangle^{\lambda}$ and $\langle \bar{T}(z_1)\bar{T}(z_2)\bar{T}(z_3)\rangle^{\lambda}$. This thus serves as a complete list of three-point functions. In the above, we have used the conservation equations and fixed the integration constant by the interchange symmetry $(1\leftrightarrow 2)$. Note that the resulting expression for $\langle \Theta(z_1)\bar{T}(z_2)T(z_3)\rangle^{(1)}$ has a spin of zero, consistent with the sum of the fields' spins totaling zero. $\langle T(z_1)T(z_2)T(z_3)\rangle^{(1)}$, which is holomorphic in z_1, z_2, and z_3 and has a mass dimension of 8, is found to vanish due to the spin constraint. If it were non-zero, this term would possess a spin of 8, which does not align with the required spin of 6. This analysis completes the computation for the stress tensor two-point and three-point functions.

3.2.2 Four-point functions

To provide insight into the first-order correction to higher-point functions of the stress tensor and to highlight the emergence of logarithmic corrections, as in (3.3), in both approaches, we will compute$\langle T(z_1)\bar{T}(z_2)T(z_3)\bar{T}(z_4)\rangle^{(1)}$ with both standard perturbation theory and the conservation equation method.

We start by computing the correlator with the conservation equation method. We write

$$\begin{aligned}
\langle T(z_1)\bar{T}(z_2)T(z_3)\bar{T}(z_4)\rangle^{(1)} =&\langle (\partial_{\bar{z}_1}^{-1}\partial_{z_1}\Theta)(z_1)\bar{T}(z_2)T(z_3)\bar{T}(z_4)\rangle^{(1)}\\
=&\partial_{\bar{z}_1}^{-1}\partial_{z_1}\langle \mathcal{O}_{T\bar{T}}(z_1)\bar{T}(z_2)T(z_3)\bar{T}(z_4)\rangle^{(0)},
\end{aligned} \tag{3.19}$$

where the RHS is given by[7]

$$\partial_{\bar{z}_1}^{-1}\partial_{z_1}\langle\left(\left(\frac{L_{2,3}}{z_{13}^4}\left(\frac{\partial_{\bar{z}_4}}{\bar{z}_{14}}+\frac{2}{\bar{z}_{14}^2}\right)\right)+(4\leftrightarrow 2)\right)\bar{T}(z_2)T(z_3)\bar{T}(z_4)\rangle^{(0)} \tag{3.21}$$

$$=\frac{c^2}{z_{13}^4\bar{z}_{24}^4}\left[\left(-\frac{4\log(\bar{z}_{14})}{z_{13}\bar{z}_{24}}+\frac{2}{z_{13}\bar{z}_{14}}\right)+(4\leftrightarrow 2)\right]+\text{holomorphic in } z_1. \tag{3.22}$$

The next step is to address the arbitrary holomorphic terms in z_1 that arise from the antiderivative, which is found to be highly constrained by the symmetries and other properties of the correlator.

The correlator $\langle T(z_1)\bar{T}(z_2)T(z_3)\bar{T}(z_4)\rangle^{\lambda}$ exhibits two key symmetries. The first one is the invariance under $z_1 \leftrightarrow z_3$ or $z_2 \leftrightarrow z_4$, which follows from the interchange symmetry $T(z_1) \leftrightarrow T(z_3)$ and $\bar{T}(z_2) \leftrightarrow \bar{T}(z_4)$. The other is the invariance under complex conjugation followed by an interchange of coordinates $(z_1, z_3) \leftrightarrow (z_2, z_4)$, which results from the fact that $\bar{T}$ is the complex conjugate of T. After adding the required holomorphic terms to preserve these symmetries, (3.21) becomes

$$\partial_{\bar{z}_1}^{-1}\partial_{z_1}\langle\left(\left(\frac{L_{2,3}}{z_{13}^4}\left(\frac{\partial_{\bar{z}_4}}{\bar{z}_{14}}+\frac{2}{\bar{z}_{14}^2}\right)\right)+(4\leftrightarrow 2)\right)\bar{T}(z_2)T(z_3)\bar{T}(z_4)\rangle^{(0)} \tag{3.23}$$

$$=\frac{c^2}{z_{13}^4\bar{z}_{24}^4}\left[\left(-\frac{4\log(|z_{14}|^2)}{z_{13}\bar{z}_{24}}+\frac{2}{z_{13}\bar{z}_{14}}+\frac{2}{z_{23}\bar{z}_{24}}\right)+(1\leftrightarrow 3)+(4\leftrightarrow 2)+(1\leftrightarrow 3, 4\leftrightarrow 2)\right]. \tag{3.24}$$

Additionally, it is important to ensure that the argument of the logarithm is dimensionless. This can be achieved by inserting a term $\sim \log(\mu^2)/z_{13}\bar{z}_{24}$ with μ an arbitrary renormalization scale such that the combined term $-\frac{\log(\mu^2|z_{14}|^2)}{z_{13}\bar{z}_{24}}$ satisfies the required property.[8] After incorporating these terms, the result is[9]

$$\langle T(z_1)\bar{T}(z_2)T(z_3)\bar{T}(z_4)\rangle^{(1)}$$
$$=\frac{c^2}{z_{13}^4\bar{z}_{24}^4}\left[\left(-\frac{4\log(\mu^2|z_{14}|^2)}{z_{13}\bar{z}_{24}}+\frac{2}{z_{13}\bar{z}_{14}}+\frac{2}{z_{23}\bar{z}_{24}}\right)+(1\leftrightarrow 3)+(4\leftrightarrow 2)+(1\leftrightarrow 3, 4\leftrightarrow 2)\right]. \tag{3.25}$$

No further terms are allowable under the constraints imposed by the symmetries, with the sole exception being a term of the form $\frac{a}{z_{13}^5\bar{z}_{24}^5}$, where a is an arbitrary real number. This term, however, could be absorbed into the scale μ, which reflects the arbitrariness in the choice of μ.

[7]Note that the CFT correlator in (3.19), namely $\langle\mathcal{O}_{T\bar{T}}(z_1)\bar{T}(z_2)T(z_3)\bar{T}(z_4)\rangle^{(0)}$, can also be straightforwardly computed as

$$\langle\mathcal{O}_{T\bar{T}}(z_1)\bar{T}(z_2)T(z_3)\bar{T}(z_4)\rangle^{(0)}=\langle T(z_1)T(z_3)\rangle^{(0)}\langle\bar{T}(z_1)\bar{T}(z_2)\bar{T}(z_4)\rangle^{(0)}=\frac{c/2}{z_{13}^4}\frac{c}{\bar{z}_{12}^2\bar{z}_{24}^2\bar{z}_{41}^2}. \tag{3.20}$$

In (3.21), we opt for an alternative approach, namely, the use of the Ward identity to address this correlator. We make this choice due to its potential for generalization to more complex scenarios, as discussed later.

[8]Such a renormalization scale also appears in stress tensor correlators computed in cutoff 3D gravity [36].

[9]The conservation equation was applied to $T(z_1)$ in the first step of (3.20). It is worth noting that through the application of the conservation equation to either $T(z_1), \bar{T}(z_2), T(z_3)$ or $\bar{T}(z_4)$, and following the aforementioned procedures, the same result (3.25) can be obtained.

This result is validated by employing standard perturbation theory to compute the first-order correction in the deformed free boson CFT; the relevant details are included in appendix B.1. The same expression was obtained from the random geometry approach [16] as well.

3.3 Mixed correlators

We now focus on a different class of correlators, known as mixed correlators, which involve stress tensors and undeformed operators. This type of correlator has received limited attention thus far. Conceptually, the corrections to these correlators can be viewed as deformations of the conformal Ward identity. In the context of our study, examining these mixed correlators holds particular significance. This is because a higher-order correction to a correlator of undeformed operators can be expressed as an integral of a lower-order correction to a mixed correlator. We will delve deeper into this aspect in the next section.

As an initial exploration of this type of correlator, we compute the first-order correction to the mixed correlator $\langle \mathcal{O}_{T\bar{T}} X\rangle^{\lambda}$. We have

$$\begin{aligned}\langle \mathcal{O}_{T\bar{T}}(z) X\rangle^{(1)} &= \lim_{z'\to z} \langle \big(T(z)\bar{T}(z') - \Theta(z)\Theta(z')\big)\, X\rangle^{(1)} \\ &= \lim_{z'\to z} \langle \big(\partial_{\bar{z}}^{-1}\partial_z \mathcal{O}_{T\bar{T}}(z)\bar{T}(z') - \mathcal{O}_{T\bar{T}}(z)\Theta(z')\big)\,(z) X\rangle^{(0)}.\end{aligned} \tag{3.26}$$

Here we used the point-splitting definition of $\mathcal{O}_{T\bar{T}}(z)$ and assumed that $T(z)$ and $\bar{T}(z')$ do not act on each other. The symmetry property we leverage here is the invariance under the replacement of $T \leftrightarrow \bar{T}$ followed by $z \leftrightarrow z'$, a symmetry property of the correlator $\langle \big(T(z)\bar{T}(z') - \Theta(z)\Theta(z')\big) X\rangle^{\lambda}$. Using the conformal Ward identity and following similar procedures as in the previous section, we obtain

$$\begin{aligned}\langle \mathcal{O}_{T\bar{T}}(z) X\rangle^{(1)} =& \log(\mu^2|z-z_n|^2)(\partial_z\partial_{\bar{z}_n} + \partial_{\bar{z}}\partial_{z_n})\langle \mathcal{O}_{T\bar{T}}(z)X\rangle^{(0)} \\ &+ \left(\frac{1}{s-1}\frac{\bar{L}_{s-2,n}\partial_z}{(\bar{z}-\bar{z}_n)^{s-1}} + \frac{1}{r-1}\frac{L_{r-2,n}\partial_{\bar{z}}}{(z-z_n)^{r-1}}\right)\langle \mathcal{O}_{T\bar{T}}(z)X\rangle^{(0)} \\ &+ \langle \mathcal{O}_{T\bar{T}}(z)\,\mathrm{d}\,X\rangle^{(0)}.\end{aligned} \tag{3.27}$$

where the holomorphic terms are fixed by demanding the invariance under the replacement of $T \leftrightarrow \bar{T}$ followed by $z \leftrightarrow z'$, a symmetry property of $\langle \big(T(z)\bar{T}(z') - \Theta(z)\Theta(z')\big)X\rangle^{\lambda}$. It's worth noting that the spin constraint and the Ward identities associated with translational and rotational invariance play a crucial role in uniquely determining this correlator. For details, please refer to appendix B.2.1.

This result is one of the main outcomes of our work. It resembles the previous equation (3.4). However, it's important to note that it is not precisely d_{z,z_n} or $\mathrm{d}_{z_n,z}$ acting on the undeformed correlator. This distinction is reasonable since in (3.4), X does not flow under the $T\bar{T}$ deformation, whereas in (3.26) the operator $\mathcal{O}_{T\bar{T}}$ inside $\langle \mathcal{O}_{T\bar{T}}X\rangle^{(1)}$ does undergo a flow. Nevertheless, the resemblance implies a potential connection between perturbation theory and the conservation equation method.

We can isolate the contribution from the stress tensor's flow by subtracting $-\frac{1}{\pi}\int d^2x' \langle \mathcal{O}_{T\bar{T}}(z')\mathcal{O}_{T\bar{T}}(z)X\rangle^{(0)}$ from (3.27), yielding

$$\langle \mathcal{O}^{(1)}_{T\bar{T}}(z)X\rangle^{(0)} = -\left(\frac{1}{s-1}\frac{\bar{L}_{s-2,z}\partial_{z_n}}{(\bar{z}_n-\bar{z})^{s-1}} + \frac{1}{r-1}\frac{L_{r-2,z}\partial_{\bar{z}_n}}{(z_n-z)^{r-1}}\right)\langle \mathcal{O}^{(0)}_{T\bar{T}}(z)X\rangle^{(0)}. \tag{3.28}$$

As a way of illustration, we take X to be $(\partial\phi\bar{\partial}\phi)^3(z_2)$ in the deformed free boson CFT and compute the first-order correction to the mixed correlator $\langle \mathcal{O}_{T\bar{T}}(z_1)(\partial\phi\bar{\partial}\phi)^3(z_2)\rangle^{(1)}$ and the contribution from the flow of the stress tensor $\langle \mathcal{O}^{(1)}_{T\bar{T}}(z_1)(\partial\phi\bar{\partial}\phi)^3(z_2)\rangle^{(0)}$, using the formulae(3.27), (3.28). This is a particularly nontrivial example as it receives a nonzero correction from the flow of the stress tensor, where much of the mysteries about $T\bar{T}$ deformed correlators lie. Applying the formulae yields

$$\langle \mathcal{O}_{T\bar{T}}(z_1)(\partial\phi\bar{\partial}\phi)^3(z_2)\rangle^{(1)} = 0, \tag{3.29}$$

and

$$\langle \mathcal{O}^{(1)}_{T\bar{T}}(z_1)(\partial\phi\bar{\partial}\phi)^3(z_2)\rangle^{(0)} = \frac{9}{32\pi^3}\frac{1}{z_{12}^6\bar{z}_{12}^6}. \tag{3.30}$$

Notably, these results are consistent with those obtained using standard perturbation theory, as shown in the appendix B.2.2.

4 Higher-order corrections

4.1 Correlators of undeformed operators

We now proceed to consider the second-order correction to a correlator of undeformed operators on an Euclidean plane. Up to this point, no explicit computation has been conducted at this order, except for the stress tensor two-point functions [22]. By the relation (2.6), the second-order correction is given by an integral of (3.27), namely

$$\langle X\rangle^{(2)} = -\frac{1}{2\pi}\int d^2x \langle \mathcal{O}_{T\bar{T}}(z)X\rangle^{(1)}. \tag{4.1}$$

Given that we are dealing with an integration involving functions with poles, we employ a regularization technique by introducing point splitting for each stress tensor insertion. To facilitate this, we introduce an additional UV regulator, denoted as $\varepsilon' \ll \varepsilon$, while maintaining the condition $\varepsilon + \varepsilon' \approx \varepsilon$. Consequently, the integrand takes the form

$$\lim_{z\to z'} \partial_{\bar{z}}^{-1}\partial_z \langle \big(T(z+\varepsilon')\bar{T}(z)\big)\,\bar{T}(z'-\varepsilon)X\rangle^{(0)}. \tag{4.2}$$

The detailed calculations of the integrals are presented in appendix C.2. The result is as follows

$$\begin{aligned}
&\langle X\rangle^{(2)}\\
=&\frac{1}{2}\langle \mathrm{d}^2 X\rangle^{(0)} + \sum_{m,n,i}\sum_{r\geq 1}\sum_{(s,t)\in(\mathbb{Z}^+)^2-\{(1,1)\}}\\
&\frac{1}{2}\langle\Big\{\Big[\frac{\log(\mu^2|z_{in}|^2)}{z_{im}^r}+\Big(\frac{1}{z_{nm}^r}\log\frac{\bar z_{im}}{\bar z_{in}}+\frac{1}{z_{im}^r}\Phi(\frac{z_{nm}}{z_{im}},1,r)\Big)\Big]L_{r-2,m}\bar L_{-1,n}\bar L_{-1,i}+(c.c.)\Big\}X\rangle^{(0)}\\
&+\frac{1}{2}\langle\Big\{\frac{(1-s)_{\underline{t-2}}}{(t-1)!}\Big[\frac{1}{\bar z_{in}^{s+t-2}}\Big(\frac{1}{z_{nm}^r}-\frac{1}{z_{im}^r}\Big)+\frac{1}{z_{nm}^r\bar z_{mi}^{t-1}}\delta_{s,1}\Big]L_{r-2,m}\bar L_{s-2,n}\bar L_{t-2,i}+(c.c.)\Big\}X\rangle^{(0)}.
\end{aligned} \tag{4.3}$$

Here, $\Phi(\frac{z_{nm}}{z_{im}},1,r)$ represents the Lerch transcendent [55]. The expression above exhibits both power and logarithmic divergences when two or more indices m,n,i coincide. These divergences are regularized by ε if i coincides with either one of m,n and are regularized by ε' if m coincides with n.

Several noteworthy aspects regarding the divergence structure of the formula (4.3) are worth mentioning. Firstly, the Lerch transcendent $\Phi(z,1,r)$ yields logarithmic divergences at the branch point at the branch point $z=1$, introducing no new types of divergences. Furthermore, while $\log\frac{\bar z_{im}}{\bar z_{in}}$ may seem to introduce branch cuts into the expression, the combination $\frac{1}{z_{nm}^r}\log\frac{\bar z_{im}}{\bar z_{in}}+\frac{1}{z_{im}^r}\Phi(\frac{z_{nm}}{z_{im}},1,r)$ does not. This is since for all values of r, $\frac{1}{z_{im}^r}\Phi(\frac{z_{nm}}{z_{im}},1,r)$ is equal to $\frac{1}{z_{nm}^r}\log\frac{z_{im}}{z_{in}}$ plus terms meromorphic in z_i,z_n,z_m. Additionally, the formula contains double logarithms, introduced by the term $\frac{1}{2}\langle\mathrm{d}^2 X\rangle^{(0)}$. This observation is consistent with the findings in [47], where it is shown that a correlator of undeformed operators displays divergences of the form $(\log\varepsilon)^n$ at order λ^n.

In Section 3, we demonstrated that the first-order correction to correlators of undeformed operators can be made finite by employing a local field redefinition. Unfortunately, this approach is not feasible in the second-order. This is because when two indices m, n or i coincide, the third index may not coincide with any of them, resulting in divergent terms with dynamic behavior. These dynamic divergent terms manifest as divergences multiplied by nontrivial functions of the coordinates. This implies that the divergences cannot be eliminated by local field redefinitions alone.

In contrast to the first-order correction, where we provided specific examples, the expression for the second-order correction is generally intricate and not substantially simplified for two-point functions or other straightforward cases.

4.2 Stress tensor correlators

In this section, we will delve into the analysis of higher-order corrections to stress tensor correlators. The cutoff AdS-$T\bar T$ CFT duality has seen relatively few examinations at the level of correlators thus far. Previous work, such as [22] and [27], have computed two-point and three-point correlators of the stress tensor to the lowest nontrivial order. More recently, authors of [36] obtained the two-point functions to two-loop order in G from 3D gravity, extending the previous classical computations.

Stress tensor correlators are of particular interest in the tests of cutoff AdS-$T\bar{T}$ CFT duality for several reasons. They exhibit consistent structural properties across different conformal field theories, enabling us to draw universal conclusions. Additionally, the finite cutoff gravity dual for $T\bar{T}$ deformed CFTs is expected to hold in the pure gravity sector without turning on matter fields. Consequently, correlators of matter fields may not be the most appropriate quantities to consider within the context of this duality proposal.

4.2.1 Two-point and three-point functions

We employ our method to compute the second-order correction to the correlator $\langle T(z)T(w)\rangle^{\lambda}$. We write

$$\begin{aligned}&\langle T(z)T(w)\rangle^{(2)} = \partial_{\bar{z}}^{-1}\partial_z\partial_{\bar{w}}^{-1}\partial_w\langle\Theta(z)\Theta(w)\rangle^{(2)}\\ =&\partial_{\bar{z}}^{-1}\partial_z\partial_{\bar{w}}^{-1}\partial_w\left[\langle\mathcal{O}_{T\bar{T}}(z)\mathcal{O}_{T\bar{T}}(w)\rangle^{(0)}+\langle\Theta(z)\Theta^{(0)}(w)\rangle^{(0)}\right]=\frac{5c^2}{6}\frac{1}{(z-w)^6(\bar{z}-\bar{w})^2},\end{aligned} \tag{4.4}$$

the holomorphic term resulting from taking the antiderivative w.r.t. $\bar{z}$ is fixed to zero by the spin constraint. This result is consistent with the findings presented in [4], which were obtained through a direct perturbation theory computation.

We further extend our analysis to higher orders and consider the third-order correction to the two-point correlator

$$\langle\Theta(z)\Theta(w)\rangle^{(3)} = \langle\mathcal{O}_{T\bar{T}}(z)\mathcal{O}_{T\bar{T}}(w)\rangle^{(1)} = \langle(T\bar{T}-(\Theta)^2)(z)(T\bar{T}-(\Theta)^2)(w)\rangle^{(1)}. \tag{4.5}$$

Here, such terms as $\langle(\Theta)^2(z)(\Theta)^2(w)\rangle^{(1)}$ and $\langle(T\bar{T})(z)(\Theta)^2(w)\rangle^{(1)}$ on the RHS vanish, as can be seen by inserting the trace relation. The remaining term $\langle(T\bar{T})(z)(T\bar{T})(w)\rangle^{(1)}$ can be extracted from the first-order correction to $\langle T(z_1)\bar{T}(z_2)T(z_3)\bar{T}(z_4)\rangle^{(1)}$ by taking the limits $z_1\to z_2, z_3\to z_4$ then subtracting the divergent terms that arise from these limits

$$\langle\Theta(z)\Theta(0)\rangle^{(3)} =\langle(T\bar{T})(z)(T\bar{T})(0)\rangle^{(1)} = \frac{c^2}{(z\bar{z})^5}\left(8\log(\mu^2|z|^2)-8\right). \tag{4.6}$$

Applying the conservation equations, we determine the third-order corrections for the other stress tensor two-point functions as

$$\begin{aligned}\langle T(z)\Theta(0)\rangle^{(3)} =&\frac{c^2}{z^6\bar{z}^4}\left(10\log(\mu^2|z|^2)-19/2\right),\\ \langle T(z)T(0)\rangle^{(3)} =&\frac{c^2}{z^7\bar{z}^3}\left(20\log(\mu^2|z|^2)-47/3\right).\end{aligned} \tag{4.7}$$

Moving on to second-order, we find that three-point functions begin to receive nontrivial corrections. As an example, we examine $\langle T(z_1)T(z_2)\Theta(z_3)\rangle^{\lambda}$. Following standard procedures, we find

$$\langle T(z_1)T(z_2)\Theta(z_3)\rangle^{(2)} = \frac{c^2}{3\bar{z}_{13}^3}\left[\frac{1}{z_{23}^2}\left(\frac{1}{z_{12}^2z_{13}^3}+\frac{1}{z_{12}^3z_{13}^2}\right)\right]+(1\leftrightarrow 2), \tag{4.8}$$

where a symmetry under the interchanges $(1\leftrightarrow 2)$ and $(1\leftrightarrow 3)$ and $(2\leftrightarrow 3)$ is demanded. The result exhibits a spin of 4, which is the sum of the spins of two T operators and one Θ operator. These second-order corrections, as expected, comply with the spin constraint discussed in Section 3.2.1.

4.2.2 Four-point functions

In the subsequent analysis, we extend the computation of $\langle T(z_1)\bar{T}(z_2)T(z_3)\bar{T}(z_4)\rangle^\lambda$ to the second-order. This four-point function comprises two components, one of order c^2, the other of order c^3, expressed as $\langle T(z_1)\bar{T}(z_2)T(z_3)\bar{T}(z_4)\rangle^{(2)} = c^2 f_2(z_i,\bar{z}_i) + c^3 f_3(z_i,\bar{z}_i)$. The dependencies on the coordinates of the four fields are collectively denoted as z_i. The specific expressions for these coefficients are

$$
\begin{aligned}
&f_2(z_i,\bar{z}_i)\\
=&\frac{1}{z_{13}^4\bar{z}_{24}^4}\Bigg\{-10\Bigg[\frac{10\log(\mu^2|z_{12}|^2)\left[\log(\mu^2|z_{23}|^2)-\log(\mu^2|z_{34}|^2)\right]}{z_{13}^2\bar{z}_{24}^2}\\
&+\left(\frac{4}{z_{13}^2\bar{z}_{24}\bar{z}_{34}}+(c.c., z_1\leftrightarrow z_2, z_3\leftrightarrow z_4)\right)\log(\mu^2|z_{12}|^2)\\
&+\left(-\frac{4}{z_{13}^2\bar{z}_{12}\bar{z}_{24}}+(c.c., z_1\leftrightarrow z_2, z_3\leftrightarrow z_4)\right)\log(\mu^2|z_{34}|^2)\\
&+\left(\left(\frac{2\bar{z}_{12}-\bar{z}_{24}}{z_{13}^2\bar{z}_{12}^2\bar{z}_{24}}+\frac{2\bar{z}_{23}+\bar{z}_{24}}{z_{13}^2\bar{z}_{23}^2\bar{z}_{24}}+\frac{\bar{z}_{24}^2}{z_{13}^2\bar{z}_{12}^2\bar{z}_{14}^2}\right)+(c.c., z_1\leftrightarrow z_2, z_3\leftrightarrow z_4)\right)\log(\mu^2|z_{13}|^2)\\
&+\left(\frac{6\bar{z}_{12}-\bar{z}_{24}}{z_{13}^2\bar{z}_{12}^2\bar{z}_{24}}+(c.c., z_1\leftrightarrow z_2, z_3\leftrightarrow z_4)\right)\log(\mu^2|z_{23}|^2)\\
&+\left(-\frac{6\bar{z}_{23}+\bar{z}_{24}}{z_{13}^2\bar{z}_{23}^2\bar{z}_{24}}+(c.c., z_1\leftrightarrow z_2, z_3\leftrightarrow z_4)\right)\log(\mu^2|z_{12}|^2)\Bigg]\\
&+(1\leftrightarrow 3)+(4\leftrightarrow 2)+(1\leftrightarrow 3, 4\leftrightarrow 2)\Bigg\},
\end{aligned}
\tag{4.9}
$$

and

$$
\begin{aligned}
&f_3(z_i,\bar{z}_i)\\
=&\frac{c^2}{z_{13}^4\bar{z}_{24}^4}\left[-\frac{5c}{12}\left(\frac{1}{z_{13}^2\bar{z}_{13}^2}+\frac{1}{z_{24}^2\bar{z}_{24}^2}\right)-\frac{5c}{18}\left(\left(\frac{\bar{z}_{24}^4}{z_{13}^2\bar{z}_{12}^3\bar{z}_{34}^3}+\frac{z_{13}^4}{\bar{z}_{24}^2 z_{12}^3 z_{34}^3}\right)+(2\leftrightarrow 4)\right)\right].
\end{aligned}
\tag{4.10}
$$

A detailed derivation can be found in appendix B.3.

5 Conclusions

In this study, we have computed various correlators in $T\bar{T}$ deformed CFTs up to both first and higher orders in λ. The calculation is particularly intricate, even at the first-order level. The primary challenge stems from the absence of explicit expressions for the deformed operators, especially the stress tensor, in a general deformed CFT. This renders conventional conformal perturbation theory impractical, a complication that persists when dealing with higher-order computations involving undeformed operators.

To address this issue, we have introduced an innovative approach that we refer to as conservation equation method, which relies on the trace relation and the conservation properties of the stress tensor. This method enables us to represent higher-order corrections in terms of lower-order corrections, albeit introducing certain unknown functions. In our examples, we have successfully constrained these unknown functions by leveraging the symmetries and characteristics of the correlators and the underlying deformed theory.

As an application, we studied first-order corrections to correlators of several types, namely those undeformed operators only, those with the stress tensor only, and those combining $\mathcal{O}_{T\bar{T}}$ and undeformed operators, i.e. $\langle O_{T\bar{T}} X\rangle^{(1)}$. From these results, we have extracted the correlator involving the first-order deformation of $\mathcal{O}_{T\bar{T}}$, namely $\langle \mathcal{O}^{(1)}_{T\bar{T}} X\rangle^{(0)}$, which is unaccessible in standard perturbation theory. As for the second-order, we have investigated correlators of undeformed operators, or $\langle X\rangle^{(2)}$ based on the result for $\langle O_{T\bar{T}} X\rangle^{(1)}$, and we have also considered the case with stress tensor insertions only. Our discussion applies to a general CFT, and we have cross-checked our results with examples in the deformed free boson CFT.

The techniques used for calculating first-order corrections to stress tensor correlators and the mixed correlator $\langle O_{T\bar{T}} X\rangle^{\lambda}$ are extensible to higher orders, allowing for calculations of arbitrary precision. However, challenges arise when addressing holomorphic terms arising from conservation equations, and whether these terms can be consistently handled in all cases remains an open question. Moreover, as higher orders are considered, issues related to the proliferation of logarithmic and power divergences become apparent. These divergences intertwine with finite terms and cannot be eliminated through a local field redefinition. To tackle these concerns, a non-local renormalization scheme may be necessary, offering a path for future research.

One promising avenue for future investigation involves extending this approach to deformed finite-size or finite-temperature CFTs. In these scenarios, the trace relation no longer holds, and the conservation equations lack essential information, such as one-point functions. Alternative principles, like modular covariance, may provide constraints for correlators in these contexts. Additionally, verifying the AdS/$T\bar{T}$ duality cutoff by calculating stress tensor correlators in a deformed CFT on a torus and comparing them with results from the gravity side, using the newly proposed prescription [28], holds significant promise. Insights from the gravitational perspective could lead to the formulation of a generalized trace relation.

A persistent challenge involves devising a method for computing holographic mixed correlators and correlators of undeformed operators, where undeformed operators correspond to matter fields in bulk. It has been suggested that in the presence of matter fields, the deformed CFT corresponds to a gravitational theory with mixed boundary conditions [21]. If general expressions for the first-order corrections to mixed correlators could be obtained, it would offer robust support for the holographic proposal.

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A Proof of the vanishing of the disconnected term

We demonstrate that the disconnected term $-\langle\mathcal{O}_{T\bar{T}}(z)\rangle^{\lambda}\langle\left(\prod_i O_i^{\lambda}(z_i)\right)\rangle^{\lambda}$ in the flow equation (1.2) vanishes. Since this term is proportional to $\langle\mathcal{O}_{T\bar{T}}(z)\rangle^{\lambda}$, it suffices to show that this one-point function vanishes. In early works on this operator [1], it was shown that on a Euclidean plane, the relation $\langle\mathcal{O}_{T\bar{T}}\rangle^{\lambda} = (\langle\Theta\rangle^{\lambda})^2$ holds. Expanding both sides of the equation in powers of λ, we obtain

$$\langle\mathcal{O}_{T\bar{T}}\rangle^{(n)} = \sum_{i+j=n}\langle\Theta\rangle^{(i)}\langle\Theta\rangle^{(j)}. \tag{A.1}$$

Inserting the trace relation gives

$$\langle\Theta\rangle^{(n+1)} = \langle\mathcal{O}_{T\bar{T}}\rangle^{(n)} = \sum_{i+j=n}\langle\Theta\rangle^{(i)}\langle\Theta\rangle^{(j)}. \tag{A.2}$$

Namely, higher-order corrections to $\langle\Theta\rangle^{\lambda}$ can always be expressed as a sum of products of its lower-order corrections. Conformal invariance of the undeformed theory implies the vanishing of $\langle\Theta\rangle^{(0)}$. Therefore, by mathematical induction, we can conclude that $\langle\Theta\rangle^{\lambda}$ vanishes to all orders. This, in turn, implies that $\langle\mathcal{O}_{T\bar{T}}\rangle^{\lambda}$ vanishes as well.

B Consistency checks and details

B.1 First-order correction to the stress tensor four-point function

As a check for the result in Section 3.2.2, we compute $\langle T(z_1)\bar{T}(z_2)T(z_3)\bar{T}(z_4)\rangle^{(1)}$ using standard perturbation theory. It's important to note that this approach is typically impractical for a generic deformed CFT due to the unavailability of a prescription for constructing the deformed stress tensor. However, for certain seed theories, such as free theories, explicit expressions for the deformed stress tensor are available, allowing us to perform this calculation. In the following discussion, we consider the deformed free boson CFT, for which we have

$$T^{(0)} = -2\pi(\partial\phi)^2,\quad \mathcal{L}^{(1)} = \frac{1}{\pi}T^{(0)}\bar{T}^{(0)},\quad T^{(1)} = -8\pi^2(\partial\phi)^3\bar{\partial}\phi,\quad \mathcal{O}^{(1)}_{T\bar{T}} = 32\pi^3(\partial\phi\bar{\partial}\phi)^3. \tag{B.1}$$

In this example, contributions from the flow of action, such as

$$\langle T^{(1)}(z_1)\bar{T}^{(0)}(z_2)T^{(0)}(z_3)\bar{T}^{(0)}(z_4)\rangle^{(0)} + (1\leftrightarrow 3),$$

vanish upon Wick contraction. Therefore, we have

$$\begin{aligned}
&\langle T(z_1)\bar{T}(z_2)T(z_3)\bar{T}(z_4)\rangle^{(1)} = -\frac{1}{\pi}\int d^2z\langle\mathcal{O}_{T\bar{T}}(z)T(z_1)\bar{T}(z_2)T(z_3)\bar{T}(z_4)\rangle^{(0)}\\
=&-\frac{1}{\pi}\int d^2z\left(\frac{\partial_{z_1}}{(z-z_1)}+\frac{2}{(z-z_1)^2}+(1\leftrightarrow 3)\right)\left(\frac{\partial_{\bar{z}_4}}{(\bar{z}-\bar{z}_4)}+\frac{2}{(\bar{z}-\bar{z}_4)^2}+(4\leftrightarrow 2)\right)\\
&\times\langle T(z_1)\bar{T}(z_2)T(z_3)\bar{T}(z_4)\rangle^{(0)}\\
=&\frac{c^2}{z_{13}^4\bar{z}_{24}^4}\left[\left(-\frac{4\log(|z_{14}|^2/\epsilon^2)}{z_{13}\bar{z}_{24}}+\frac{2}{z_{13}\bar{z}_{14}}+\frac{2}{z_{23}\bar{z}_{24}}\right)+(1\leftrightarrow 3)+(4\leftrightarrow 2)+(1\leftrightarrow 3, 4\leftrightarrow 2)\right],
\end{aligned} \tag{B.2}$$

we find perfect agreement with the previous calculation (3.25) based on the conservation equation method. Here, the point-splitting regulator is identified with the renormalization scale μ.

B.2 First-order correction to mixed correlators

B.2.1 Deriving the general expression

Here, we present the details involved in deriving the expression for the first-order correction to a mixed correlator, (3.27). Inserting the conformal Ward identity into (3.26) yields

$$\begin{aligned}&\langle \mathcal{O}_{T\bar{T}}(z)X\rangle^{(1)}\\ =&\lim_{z'\to z}\langle\left(T(z)\bar{T}(z')-\Theta(z)\Theta(z')\right)X\rangle^{(1)}=\lim_{z'\to z}\partial_{\bar{z}}^{-1}\partial_z\langle T(z)\bar{T}(z)\bar{T}(z')X\rangle^{(0)}\\ =&\lim_{z\to z'}\sum_{m,n}\sum_{t,r,s\geq 1}\langle\frac{\bar{L}_{t-2,i}}{(\bar{z}'-\bar{z}_i)^t}\partial_{\bar{z}}^{-1}\partial_z\frac{L_{r-2,m}\bar{L}_{s-2,n}}{(z-z_m)^r(\bar{z}-\bar{z}_n)^s}X\rangle^{(0)},\end{aligned}\tag{B.3}$$

where the terms in the summation fall into two types after working out $\partial_{\bar{z}}^{-1}$

$$\begin{cases}\lim\limits_{z'\to z}\langle\frac{\bar{L}_{t-2,i}}{(\bar{z}'-\bar{z}_i)^t}\left(\frac{r}{s-1}\frac{L_{r-2,m}\bar{L}_{s-2,n}}{(z-z_m)^{r+1}(\bar{z}-\bar{z}_n)^{s-1}}\right)X\rangle^{(0)}+\text{holomorphic in }z, & \text{for } s>1;\\ \lim\limits_{z'\to z}\langle\frac{\bar{L}_{t-2,i}}{(\bar{z}'-\bar{z}_i)^t}\left(-\frac{rL_{r-2,m}\bar{L}_{-1,n}}{(z-z_m)^{r+1}}\log(\mu(\bar{z}-\bar{z}_n))\right)X\rangle^{(0)}+\text{holomorphic in }z, & \text{for } s=1.\end{cases}\tag{B.4}$$

To ensure invariance under the replacement of $T\leftrightarrow\bar{T}$ followed by $z\leftrightarrow z'$, we introduce an additional term arising from taking the complex conjugate of all factors in the above terms except for X and then swapping z and z'. Adding this term, we obtain

$$\begin{cases}\lim\limits_{z'\to z}\langle\frac{\bar{L}_{t-2,i}}{(\bar{z}'-\bar{z}_i)^t}\left(\frac{r}{s-1}\frac{L_{r-2,m}\bar{L}_{s-2,n}}{(z-z_m)^{r+1}(\bar{z}-\bar{z}_n)^{s-1}}+\frac{t}{s-1}\frac{L_{r-2,m}L_{s-2,n}}{(z-z_m)^r(z'-z_n)^{s-1}(\bar{z}'-\bar{z}_i)}\right)X\rangle^{(0)}, & \text{for } s>1;\\ \lim\limits_{z'\to z}\langle\frac{\bar{L}_{t-2,i}}{(\bar{z}'-\bar{z}_i)^t}\left(-\frac{rL_{r-2,m}\bar{L}_{-1,n}}{(z-z_m)^{r+1}}\log(\mu^2|z-z_n|^2)-\frac{tL_{r-2,m}L_{-1,n}}{(z-z_m)^r(\bar{z}'-\bar{z}_i)}\log(\mu^2|z'-z_n|^2)\right)X\rangle^{(0)}, & \text{for } s=1.\end{cases}\tag{B.5}$$

After taking the limit, the terms in the summation in equation (B.3) become

$$\begin{cases}\langle\left(\frac{r}{s-1}\frac{L_{r-2,m}\bar{L}_{s-2,n}\bar{L}_{t-2,i}}{(z-z_m)^{r+1}(\bar{z}-\bar{z}_n)^{s-1}(\bar{z}-\bar{z}_i)^t}+(c.c., L\leftrightarrow\bar{L})\right)X\rangle^{(0)}, & \text{for s}>1;\\ \langle\left(-\frac{rL_{r-2,m}\bar{L}_{-1,n}\bar{L}_{t-2,i}}{(z-z_m)^{r+1}(\bar{z}-\bar{z}_i)^t}\log(\mu^2|z-z_n|^2)+(c.c., L\leftrightarrow\bar{L})\right)X\rangle^{(0)}, & \text{for s}=1.\end{cases}\tag{B.6}$$

The above can be written in a way that resembles the expression of the first-order correction to correlators of undeformed operators:

$$\begin{cases}\left(\frac{1}{s-1}\frac{\bar{L}_{s-2,n}\partial_z}{(\bar{z}-\bar{z}_n)^{s-1}}+\frac{1}{r-1}\frac{L_{r-2,n}\partial_{\bar{z}}}{(z-z_n)^{r-1}}\right)\langle\mathcal{O}_{T\bar{T}}(z)X\rangle^{(0)}, & \text{for s}>1;\\ \log(\mu^2|z-z_n|^2)(\bar{L}_{-1,n}\partial_z+L_{-1,n}\partial_{\bar{z}})\langle\mathcal{O}_{T\bar{T}}(z)X\rangle^{(0)}, & \text{for s}=1.\end{cases}\tag{B.7}$$

In addition to the complex conjugation symmetry, the holomorphic terms are also required to have $\frac{\bar{L}_{t-2,i}}{(\bar{z}'-\bar{z}_i)^t}$ as one of their factors. Apart from what we have already included, the only term that possesses the correct dimension and is consistent with the complex conjugation symmetry takes the form

$$\sim\langle\frac{L_{k-2,j}\bar{L}_{t-2,i}}{(z-z_j)^k(\bar{z}'-\bar{z}_i)^t}f(z_{mn})X\rangle^{(0)},\tag{B.8}$$

where $\sim$ denotes equality up to a possible prefactor consisting of some dimensionless functions of the coordinates that are consistent with the symmetries. Here $f(z_{mn})$ has dimensions of length^{-2} and depends only on the coordinates of the fields in X. Note that the term

$$\sim \langle \frac{L_{k-2,j}\bar{L}_{t-2,i}}{(z-z_j)^{k+1}(\bar{z}'-\bar{z}_i)^{t+1}} X\rangle^{(0)} \tag{B.9}$$

is not allowed, although it has the correct dimension and is consistent with complex conjugation symmetry. This is because $T(z)$ and $\bar{T}(z')$ are supposed to have spins of $+2$ and -2, respectively. Yet the above term can be written as $\sim \langle \partial_z T(z)\partial_{\bar{z}'}\bar{T}(z')X\rangle^{(0)}$, in which the field at z and z' has a spin of $+3$ and -3, respectively, thus violating the spin constraint.

To determine $f(z_m)$, we utilize the stress tensor's properties. In a 2D QFT on a Euclidean plane with translation and rotation symmetry, a correlator with a stress tensor insertion must contain a term of the form [47]

$$\langle T_{ij}(\vec{x})X\rangle = \sum_m \left(\frac{(x_j - x_{j,m})\partial_i}{|\vec{x}-\vec{x}_m|^2} + \xi \frac{\epsilon_{jk}(x^k - x^k_m)\epsilon_{ia}\partial^a}{|\vec{x}-\vec{x}_m|^2} \right) \langle X\rangle + \cdots , \tag{B.10}$$

where the first term is fixed by the Ward identity associated with translational invariance $\langle \partial^i T_{ij}(\vec{x})X\rangle = \sum_m \delta^2(\vec{x}-\vec{x}_m)\langle \partial_{j,m}X\rangle$ and the remainder is fixed by rotational symmetry and parity. Symmetry under $i \leftrightarrow j$ further fixes ξ to -1. In complex coordinates, we have

$$\langle T(z)X\rangle = \sum_m \frac{\partial_{z_m}}{z-z_m}\langle X\rangle + \cdots \tag{B.11}$$

The ellipsis represents unknown terms not $O(1/z)$. Subsequently, the only $O(1/z\bar{z}')$ term in the mixed correlator $\langle T(z)\bar{T}(z')X\rangle^\lambda$ is

$$\langle T(z)\bar{T}(z')X\rangle^\lambda = \sum_{m,n} \frac{\partial_{z_m}\partial_{\bar{z}_n}}{(z-z_m)(\bar{z}'-\bar{z}_n)}\langle X\rangle^\lambda + \cdots \tag{B.12}$$

Expanding in λ on both sides, we obtain a constraint on the first-order correction to this mixed correlator (and, more generally, on n-th-order corrections as well)

$$\langle T(z)\bar{T}(z')X\rangle^{(1)} = \sum_{m,n} \frac{\partial_{z_m}\partial_{\bar{z}_n}}{(z-z_m)(\bar{z}-\bar{z}_n)}\langle X\rangle^{(1)} + \cdots \tag{B.13}$$

This constraint leads us to uniquely determine that $f(z_{mn})$ is d and the prefactor of the extra term (B.8) is 1, ensuring the presence of a term

$$\frac{\partial_{z_i}\partial_{\bar{z}_j}}{(z-z_j)(\bar{z}'-\bar{z}_i)}\langle \mathrm{d}\, X\rangle^{(0)} = \frac{\partial_{z_i}\partial_{\bar{z}_j}}{(z-z_j)(\bar{z}'-\bar{z}_i)}\langle X\rangle^{(1)}. \tag{B.14}$$

The full expression for $\langle \mathcal{O}_{T\bar{T}}(z)X\rangle^{(1)}$ is thus given by formula (3.27).

B.2.2 Examples

In Section 3.3, we have derived a general expression for the first-order correction to mixed correlators and the contribution from the flow of the stress tensor. As an example, we have applied the formulae to the case where $X = (\partial\phi\bar{\partial}\phi)^3(z_2)$. The computation proceeds as follows.

We begin by applying the formula (3.27). The first line with the logarithm vanishes due to the vanishing of the undeformed correlator. The remaining terms are given by:

$$\begin{aligned}
&\langle \mathcal{O}_{T\bar{T}}(z) L_n \mathcal{O}^{(1)}_{T\bar{T}}(z_2)\rangle^{(0)} \\
=&\frac{1}{2\pi i}\oint_{z_2} dz' \ (z'-z_2)^{n+1}\langle \mathcal{O}_{T\bar{T}}(z)T(z')\mathcal{O}^{(1)}_{T\bar{T}}(z_2)\rangle^{(0)} \\
=&-\frac{1}{2\pi i}\oint_{z} dz' \ (z'-z_2)^{n+1}\langle \mathcal{O}_{T\bar{T}}(z)T(z')\mathcal{O}^{(1)}_{T\bar{T}}(z_2)\rangle^{(0)} \ \text{(Reversing the contour)} \\
=&-\frac{1}{2\pi i}\oint_{z} dz' \ (z'-z_2)^{n+1}\left(\frac{\partial_z}{z'-z}+\frac{2}{(z'-z)^2}\right)\langle \mathcal{O}_{T\bar{T}}(z)\mathcal{O}^{(1)}_{T\bar{T}}(z_2)\rangle^{(0)} \\
&-\frac{1}{2\pi i}\oint_{z} dz' \ (z'-z_2)^{n+1}\frac{c/2}{(z'-z)^4}\langle \bar{T}(z)\mathcal{O}^{(1)}_{T\bar{T}}(z_2)\rangle^{(0)} = 0,
\end{aligned} \tag{B.15}$$

where in the first line, we used the definition of L_n. Obviously, the last equality vanishes, since the correlators $\langle \mathcal{O}_{T\bar{T}}(z)\mathcal{O}^{(1)}_{T\bar{T}}\rangle^{(0)}$ and $\langle \bar{T}(z)\mathcal{O}^{(1)}_{T\bar{T}}(z_2)\rangle^{(0)}$ vanish upon Wick contraction.

We now reproduce the result with standard perturbation theory. The contribution from the flow of $\mathcal{O}_{T\bar{T}}(z_1)$, i.e., the first term on the RHS of (1.4) is

$$\langle \mathcal{O}^{(1)}_{T\bar{T}}(z_1)(\partial\phi\bar{\partial}\phi)^3(z_2)\rangle^{(0)} = 32\pi^3\langle(\partial\phi\bar{\partial}\phi)^3(z_1)(\partial\phi\bar{\partial}\phi)^3(z_2)\rangle^{(0)} = \frac{9}{32\pi^3}\frac{1}{z_{12}^6\bar{z}_{12}^6}. \tag{B.16}$$

The 2nd term on the RHS of (1.4), which is the contribution from the flow of action, is given by

$$-\frac{1}{\pi}\int d^2x \langle \mathcal{O}_{T\bar{T}}(z)\mathcal{O}_{T\bar{T}}(z_1)(\partial\phi\bar{\partial}\phi)^3(z_2)\rangle^{(0)} \tag{B.17}$$

$$= -\frac{1}{\pi}\int d^2x \langle (2\pi)^2(\partial\phi\bar{\partial}\phi)^2(z)(2\pi)^2(\partial\phi\bar{\partial}\phi)^2(z_1)(\partial\phi\bar{\partial}\phi)^3(z_2)\rangle^{(0)}, \tag{B.18}$$

where the only contribution comes from the term involving the contact term contraction $\langle\bar{\partial}\phi(z)\partial\phi(z_1)\rangle = \frac{1}{4\pi}\bar{\partial}\frac{1}{z-z_1} = \frac{1}{4\pi}\partial\frac{1}{\bar{z}-\bar{z}_1} = \frac{1}{4}\delta^2(x-x_1)$. All else contractions, such as $\langle\bar{\partial}\phi(z)\partial\phi(z_3)\rangle$, lead to zero. Thus, (B.17) becomes

$$\begin{aligned}
&-\frac{1}{\pi}(2\pi)^4\cdot 2\cdot 2^2\int d^2x \frac{1}{4}\delta^2(x-x_1)\langle\left((\partial\phi)^2\bar{\partial}\phi\right)(z)\left(\partial\phi(\bar{\partial}\phi)^2\right)(z_1)(\partial\phi\bar{\partial}\phi)^3(z_2)\rangle^{(0)} \\
=&-32\pi^3\langle(\partial\phi\bar{\partial}\phi)^3(z_1)(\partial\phi\bar{\partial}\phi)^3(z_2)\rangle^{(0)}.
\end{aligned} \tag{B.19}$$

Now adding up (B.16) and (B.17) leads to

$$\langle \mathcal{O}_{T\bar{T}}(z_1)(\partial\phi\bar{\partial}\phi)^3(z_2)\rangle^{(1)} = 0. \tag{B.20}$$

Namely, the contribution from the flow of action and the flow of stress tensor cancel out.

As another example, we compute $\langle\mathcal{O}^{(1)}_{T\bar{T}}(z_1)(\partial\phi\bar{\partial}\phi)^3(z_2)\rangle^{(0)}$, i.e., the contribution from the flow of action, using the formula (3.28). For the deformed free boson CFT, the undeformed $T\bar{T}$ operator is given by $\mathcal{O}^{(0)}_{T\bar{T}}(z_1) = (2\pi)^2(\partial\phi\bar{\partial}\phi)^2(z_1)$. We can write $(\bar{\partial}\phi)^3(z_2)$ as

$$\lim_{5\to 2,4\to 2,3\to 2}\partial_{\bar{z}_3}\partial_{\bar{z}_4}\partial_{\bar{z}_5}[\phi(z_3)\phi(z_4)\phi(z_5)],$$

then evaluate $\langle\mathcal{O}^{(1)}_{T\bar{T}}(z_1)(\partial\phi)^3(z_2)[\phi(z_3)\phi(z_4)\phi(z_5)]\rangle^{(0)}$, and take the derivatives and limits at the end of the calculation (without taking the contractions between fields at z_2, z_3, z_4, z_5 act on each other).[10] Applying the formula (3.28), we find the relevant terms to be

$$\begin{aligned}&\langle\mathcal{O}^{(1)}_{T\bar{T}}(z_1)(\partial\phi)^3(z_2)[\phi(z_3)\phi(z_4)\phi(z_5)]\rangle^{(0)}\\ =&(2\pi)^2\sum_{n=3,4,5}\langle\frac{\bar{L}_{s-2,1}\partial_{z_n}}{\bar{z}_{n1}^{s-1}}(\partial\phi\bar{\partial}\phi)^2(z_1)(\partial\phi)^3(z_2)[\phi(z_3)\phi(z_4)\phi(z_5)]\rangle^{(0)}.\end{aligned}\tag{B.21}$$

As a trick, $(\partial\phi)^3(z_2)$ can be rewritten as $(\partial\phi)^2(z_2)\partial\phi(z_2) = -\frac{1}{2\pi}T(z_2)\partial\phi(z_2)$, allowing us to use the conformal Ward identity to evaluate this term. The contributing terms are the connected part acting on $(\partial\phi\bar{\partial}\phi)^2(z_1)$

$$\begin{aligned}&\langle\mathcal{O}^{(1)}_{T\bar{T}}(z_1)(\partial\phi)^3(z_2)[\phi(z_3)\phi(z_4)\phi(z_5)]\rangle^{(0)}\\ =&(-2\pi)\sum_{n=3,4,5}\langle\frac{\bar{L}_{s-2,1}\partial_{z_n}}{\bar{z}_{n1}^{s-1}}\left(\frac{\partial_{z_1}}{z_{21}}+\frac{2}{z_{21}^2}\right)(\partial\phi\bar{\partial}\phi)^2(z_1)\partial\phi(z_2)[\phi(z_3)\phi(z_4)\phi(z_5)]\rangle^{(0)}.\end{aligned}\tag{B.22}$$

The above is nonzero only when $s = 2$, or $s = 4$; in the latter case, $(\bar{\partial}\phi)^2(z_1)$ is annihilated, and the whole correlator vanishes upon Wick contraction. The surviving terms are

$$\begin{aligned}&\langle\mathcal{O}^{(1)}_{T\bar{T}}(z_1)(\partial\phi)^3(z_2)[\phi(z_3)\phi(z_4)\phi(z_5)]\rangle^{(0)}\\ =&(-2\pi)\sum_{n=3,4,5}\langle\frac{2\partial_{z_n}}{\bar{z}_{n1}}\left(\frac{\partial_{z_1}}{z_{21}}+\frac{2}{z_{21}^2}\right)(\partial\phi\bar{\partial}\phi)^2(z_1)\partial\phi(z_2)[\phi(z_3)\phi(z_4)\phi(z_5)]\rangle^{(0)}.\end{aligned}\tag{B.23}$$

Performing Wick contractions gives

$$\begin{aligned}&\langle\mathcal{O}^{(1)}_{T\bar{T}}(z_1)(\partial\phi)^3(z_2)[\phi(z_3)\phi(z_4)\phi(z_5)]\rangle^{(0)}\\ =&4(-2\pi)\frac{1}{(4\pi)^4}\frac{2}{\bar{z}_{31}}\left(\frac{\partial_{z_1}}{z_{21}}+\frac{2}{z_{21}^2}\right)\frac{1}{z_{12}^2}\left(\frac{1}{z_{13}\bar{z}_{14}\bar{z}_{15}}+\text{perm.}(3,4,5)\right).\end{aligned}\tag{B.24}$$

Taking the $\partial_{\bar{z}_3}\partial_{\bar{z}_4}\partial_{\bar{z}_5}$ derivatives and the $z_3 \to z_2, z_4 \to z_2, z_5 \to z_2$ limits results in

$$\langle\mathcal{O}^{(1)}_{T\bar{T}}(z_1)(\partial\phi\bar{\partial}\phi)^3(z_2)\rangle^{(0)} = \frac{9}{32\pi^3}\frac{1}{z_{12}^6\bar{z}_{12}^6},\tag{B.25}$$

which agrees with the result obtained using contractions and the explicit form of $\mathcal{O}^{(1)}_{T\bar{T}}(z_1)$:

$$\langle\mathcal{O}^{(1)}_{T\bar{T}}(z_1)(\partial\phi\bar{\partial}\phi)^3(z_2)\rangle^{(0)} = 32\pi^3\langle(\partial\phi\bar{\partial}\phi)^3(z_1)(\partial\phi\bar{\partial}\phi)^3(z_2)\rangle^{(0)} = \frac{9}{32\pi^3}\frac{1}{z_{12}^6\bar{z}_{12}^6}.\tag{B.26}$$

[10] $\lim_{5\to 2,4\to 2,3\to 2}\bar{\partial}\phi(z_3)\bar{\partial}\phi(z_4)\bar{\partial}\phi(z_5)$ is nothing but the definition of the product of fields $(\bar{\partial}\phi)^3(z_2)$; Here the only trick is taking the derivatives at the end of the calculation, which is justified by the rule $\partial_{z_m}\langle\mathcal{O}^{(1)}_{T\bar{T}}(z)X\rangle^{(0)} = \langle\mathcal{O}^{(1)}_{T\bar{T}}(z)(\partial_{z_m}X)\rangle^{(0)}$. Further explanation of such tricks is given at the end of this section.

In computing these examples, we have relied on the rule $\langle \partial_{z_m} X\rangle^\lambda = \partial_{z_m}\langle X\rangle^\lambda$, which follows from translational invariance of the vacuum state. Its use is not entirely justified, for the related expressions(3.4)(3.28) are inconsistent with the above rule. This may be attributed to the regularization process involved in evaluating the integrals. This discrepancy can be attributed to the regularization process of evaluating these integrals. Such irregularities are inherent to the regularization procedure and affect all results that rely on regularization.

On the other hand, results that do not require regularization are exempt from these issues. For instance, in the case of the first two lines of (3.27), where the rule $\langle \partial_{z_m} X\rangle^\lambda = \partial_{z_m}\langle X\rangle^\lambda$ is trivially satisfied.

In the context of the deformed free boson CFT, we work with the assumption that the fields ϕ and their correlators are the basic objects and correlators of their derivatives, specifically those involving $\partial\phi$ and $\bar\partial\phi$ can ultimately be derived from correlators of ϕ itself.

B.3 Second-order correction to the stress tensor four-point function

Here we present in detail the computation of the second-order correction to the stress tensor four-point function $\langle T(z_1)\bar T(z_2)T(z_3)\bar T(z_4)\rangle^\lambda$. We have

$$\begin{aligned}
&\langle T(z_1)\bar T(z_2)T(z_3)\bar T(z_4)\rangle^{(2)}\\
=&\langle(\partial_{\bar z_1}^{-1}\partial_{z_1}\Theta)(z_1)\bar T(z_2)(\partial_{\bar z_3}^{-1}\partial_{z_3}\Theta)(z_2)\bar T(z_4)\rangle^{(2)}\\
=&\partial_{\bar z_1}^{-1}\partial_{z_1}\partial_{\bar z_3}^{-1}\partial_{z_3}\left(\langle T(z_1)T(z_3)\rangle^{(0)}\langle\bar T(z_1)\bar T(z_2)\bar T(z_3)\bar T(z_4)\rangle^{(0)}\right)\\
=&-\frac{10c}{z_{13}^6}\partial_{\bar z_1}^{-1}\partial_{\bar z_3}^{-1}\left(\langle\bar T(z_1)\bar T(z_2)\bar T(z_3)\bar T(z_4)\rangle^{(0)}_{\text{connected}}\right)\\
&-\frac{10c}{z_{13}^6}\partial_{\bar z_1}^{-1}\partial_{\bar z_3}^{-1}\left(\langle\bar T(z_1)\bar T(z_2)\rangle^{(0)}\langle\bar T(z_3)\bar T(z_4)\rangle^{(0)}+(2\leftrightarrow3)+(2\leftrightarrow4)\right),
\end{aligned}\tag{B.27}$$

where we have used the CFT stress tensor four-point function

$$\langle\bar T(z_1)\bar T(z_2)\bar T(z_3)\bar T(z_4)\rangle^{(0)}_{\text{connected}}=\frac{c}{\bar z_{12}^2\bar z_{13}^2\bar z_{24}^2\bar z_{34}^2}+\frac{c}{\bar z_{12}^2\bar z_{14}^2\bar z_{23}^2\bar z_{34}^2}+\frac{c}{\bar z_{13}^2\bar z_{14}^2\bar z_{23}^2\bar z_{24}^2}.\tag{B.28}$$

Next, we account for the contribution from the connected term (B.27). Performing the $\bar z_1$ anti-derivative of the connected term gives

$$\begin{aligned}
&\partial_{\bar z_1}^{-1}\left(\langle\bar T(z_1)\bar T(z_2)\bar T(z_3)\bar T(z_4)\rangle^{(0)}_{\text{connected}}\right)\\
=&-\frac{2c}{\bar z_{23}^2\bar z_{24}^2\bar z_{34}^2}\left[\left(\log(\bar z_{12})\left(\frac{1}{\bar z_{23}}+\frac{1}{\bar z_{24}}\right)+\frac{1}{\bar z_{12}}\right)+(2\leftrightarrow3)+(2\leftrightarrow4)\right]\\
=&-\frac{2c}{\bar z_{23}^2\bar z_{24}^2\bar z_{34}^2}\left[\left(\log(\mu^2|\bar z_{12}|^2)\left(\frac{1}{\bar z_{23}}+\frac{1}{\bar z_{24}}\right)+\frac{1}{\bar z_{12}}\right)+(2\leftrightarrow3)+(2\leftrightarrow4)\right].
\end{aligned}\tag{B.29}$$

Further performing the $\bar z_3$ anti-derivative gives

$$\begin{aligned}
&\frac{c}{\bar z_{24}^4}\Bigg(\Bigg[\frac{10\log(\bar z_{12})(\log(\bar z_{23})-\log(\bar z_{34}))}{\bar z_{24}^2}+\frac{4}{\bar z_{24}\bar z_{34}}\log(\bar z_{12})-\frac{4}{\bar z_{12}\bar z_{24}}\log(\bar z_{34})\\
&+\left(\frac{(2\bar z_{12}-\bar z_{24})}{\bar z_{12}^2\bar z_{24}}+\frac{(2\bar z_{23}+\bar z_{24})}{\bar z_{23}^2\bar z_{24}}+\frac{\bar z_{24}^2}{\bar z_{12}^2\bar z_{14}^2}\log(\bar z_{13})\right)\\
&+\frac{6\bar z_{12}-\bar z_{24}}{\bar z_{12}^2\bar z_{24}}\log(\bar z_{23})-\frac{6\bar z_{23}+\bar z_{24}}{\bar z_{23}^2\bar z_{24}}\log(\bar z_{12})\Bigg]+(2\leftrightarrow4)\Bigg).
\end{aligned}$$

This expression is invariant under $(2 \leftrightarrow 4)$, but not under $(1 \leftrightarrow 3)$ nor $z \leftrightarrow \bar{z}$ followed by $(z_1 \leftrightarrow z_2, z_3 \leftrightarrow z_4)$. As usual, we demand these symmetries and add holomorphic terms as appropriate. The full result for the order c^2 part is given by

$$\begin{aligned}
&\frac{1}{z_{13}^4 \bar{z}_{24}^4}\Bigg\{-10\Bigg[\frac{10\log(\mu^2|z_{12}|^2)\left[\log(\mu^2|z_{23}|^2)-\log(\mu^2|z_{34}|^2)\right]}{z_{13}^2\bar{z}_{24}^2} \\
&+\left(\frac{4}{z_{13}^2\bar{z}_{24}\bar{z}_{34}}+(c.c., z_1\leftrightarrow z_2, z_3\leftrightarrow z_4)\right)\log(\mu^2|z_{12}|^2) \\
&+\left(-\frac{4}{z_{13}^2\bar{z}_{12}\bar{z}_{24}}+(c.c., z_1\leftrightarrow z_2, z_3\leftrightarrow z_4)\right)\log(\mu^2|z_{34}|^2) \\
&+\left(\left(\frac{2\bar{z}_{12}-\bar{z}_{24}}{z_{13}^2\bar{z}_{12}^2\bar{z}_{24}}+\frac{2\bar{z}_{23}+\bar{z}_{24}}{z_{13}^2\bar{z}_{23}^2\bar{z}_{24}}+\frac{\bar{z}_{24}^2}{z_{13}^2\bar{z}_{12}^2\bar{z}_{14}^2}\right)+(c.c., z_1\leftrightarrow z_2, z_3\leftrightarrow z_4)\right)\log(\mu^2|z_{13}|^2) \\
&+\left(\frac{6\bar{z}_{12}-\bar{z}_{24}}{z_{13}^2\bar{z}_{12}^2\bar{z}_{24}}+(c.c., z_1\leftrightarrow z_2, z_3\leftrightarrow z_4)\right)\log(\mu^2|z_{23}|^2) \\
&+\left(-\frac{6\bar{z}_{23}+\bar{z}_{24}}{z_{13}^2\bar{z}_{23}^2\bar{z}_{24}}+(c.c., z_1\leftrightarrow z_2, z_3\leftrightarrow z_4)\right)\log(\mu^2|z_{12}|^2)\Bigg] \\
&+(1\leftrightarrow 3)+(4\leftrightarrow 2)+(1\leftrightarrow 3, 4\leftrightarrow 2)\Bigg\},
\end{aligned} \tag{B.30}$$

The order c^3 part in (B.27), namely the contribution from the disconnected terms, is much simpler, as given by

$$\begin{aligned}
&-\frac{10c}{z_{13}^6}\partial_{\bar{z}_1}^{-1}\partial_{\bar{z}_3}^{-1}\left(\langle\bar{T}(z_1)\bar{T}(z_3)\rangle^{(0)}\langle\bar{T}(z_2)\bar{T}(z_4)\rangle^{(0)}\right) \\
&-\frac{10c}{z_{13}^6}\partial_{\bar{z}_1}^{-1}\partial_{\bar{z}_3}^{-1}\left(\langle\bar{T}(z_1)\bar{T}(z_2)\rangle^{(0)}\langle\bar{T}(z_3)\bar{T}(z_4)\rangle^{(0)}+(2\leftrightarrow 4)\right) \\
=&-\frac{10c}{z_{13}^6}\partial_{\bar{z}_1}^{-1}\partial_{\bar{z}_3}^{-1}\left(\frac{c^2/4}{\bar{z}_{13}^4\bar{z}_{24}^4}\right)-\frac{10c}{z_{13}^6}\partial_{\bar{z}_1}^{-1}\partial_{\bar{z}_3}^{-1}\left(\frac{c^2/4}{\bar{z}_{12}^4\bar{z}_{34}^4}+(2\leftrightarrow 4)\right) \\
=&\frac{c^2}{z_{13}^4\bar{z}_{24}^4}\left[-\frac{5c}{12}\frac{1}{z_{13}^2\bar{z}_{13}^2}-\frac{5c}{18}\left(\frac{\bar{z}_{24}^4}{z_{13}^2\bar{z}_{12}^3\bar{z}_{34}^3}+(2\leftrightarrow 4)\right)\right]+\text{holomorphic in } z_1 \text{ or } z_3 \\
=&\frac{c^2}{z_{13}^4\bar{z}_{24}^4}\left[-\frac{5c}{12}\left(\frac{1}{z_{13}^2\bar{z}_{13}^2}+\frac{1}{z_{24}^2\bar{z}_{24}^2}\right)-\frac{5c}{18}\left(\left(\frac{\bar{z}_{24}^4}{z_{13}^2\bar{z}_{12}^3\bar{z}_{34}^3}+\frac{z_{13}^4}{\bar{z}_{24}^2z_{12}^3z_{34}^3}\right)+(2\leftrightarrow 4)\right)\right].
\end{aligned} \tag{B.31}$$

We have thus completed the computation of the second-order correction to $\langle T(z_1)\bar{T}(z_2)T(z_3)\bar{T}(z_4)\rangle^\lambda$.

C Useful Integrals

Here we compute some integrals used in constructing the expressions for the first and second-order correction to a correlator of undeformed operators. Divergent integrals will be regularized by cutting an infinitesimal disk of radius ε around the poles, evaluating, then taking the limit $\varepsilon \to 0$.

C.1 Integrals for the first-order correction of undeformed operators

In the following, we will evaluate the integrals for computing the first-order correction to a correlator of undeformed operators, namely integrals of the form

$$I^{r\bar{s}}_{m\bar{n}} = \int \frac{d^2x}{(z-z_m)^r(\bar{z}-\bar{z}_n)^s} = \frac{i}{2}\int \frac{d^2z}{(z-z_m)^r(\bar{z}-\bar{z}_n)^s}. \tag{C.1}$$

For the case of $r=1, s=1$

$$I^{1\bar{1}}_{m\bar{n}} = \frac{i}{2}\int \frac{d^2z}{(z-z_m)(\bar{z}-\bar{z}_n)} = \frac{i}{2}\int \frac{d^2z}{(z-z_{mn})\,\bar{z}} = \frac{i}{2}\int d^2z\partial_{\bar{z}}\left(\frac{\log\bar{z}}{(z-z_{mn})\,\bar{z}}\right) = -\frac{i}{2}\oint dz\frac{\log\bar{z}}{z-z_{mn}}. \tag{C.2}$$

The integrand has a branch cut $z = 0 \to \infty$ and a pole $z = z_{mn}$, see Fig. 1

$$-\frac{i}{2}\oint dz\frac{\log\bar{z}}{z-z_{mn}} = I_\Lambda + I_{l_1+l_2} + I_{\epsilon_0} + I_{\epsilon_1}, \tag{C.3}$$

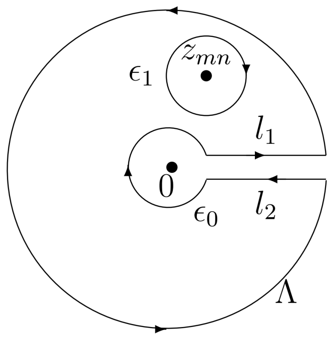

Figure 1: Contour for the $r=1, s=1$ integral.

$$\begin{aligned}
I_\Lambda &= \frac{1}{2}\int_0^{2\pi}\Lambda e^{i\theta}d\theta\frac{\log\Lambda - i\theta}{\Lambda e^{i\theta}-z_{mn}} = \frac{1}{2}\int_0^{2\pi}d\theta(\log\Lambda - i\theta) + \frac{z_{mn}}{2}\int_0^{2\pi}d\theta\frac{\log\Lambda - i\theta}{\Lambda e^{i\theta}-z_{mn}} \\
&= \pi\log\Lambda - \pi^2 i + 0; \\
I_{\epsilon_0} &= -\frac{1}{2}\int_0^{2\pi}\epsilon e^{i\theta}d\theta\frac{\log\epsilon - i\theta}{\epsilon e^{i\theta}-z_{mn}} \to 0; \\
I_{\epsilon_1} &= \frac{i}{2}\oint_{|z-z_{mn}|=\epsilon}dz\frac{\log\left(\bar{z}_{mn}+\frac{\epsilon^2}{z-z_{mn}}\right)}{z-z_{mn}} = \frac{i}{2}\oint_{|z'|=\epsilon}\frac{dz'}{z'}\log\left(\bar{z}_{mn}+\frac{\epsilon^2}{z'}\right) \\
&= \frac{i}{2}\oint_{|z'|=\epsilon_1}\frac{dz'}{z'}\left[\log\bar{z}_{mn} + \log\left(1+\frac{\epsilon_1^2}{z'\bar{z}_{mn}}\right)\right] = -\pi\log\bar{z}_{mn} - \frac{i}{2}\sum_{n=1}^{\infty}\frac{1}{n}\oint_0\frac{dz}{z^{n+1}}\left(\frac{\epsilon_1^2}{\bar{z}_{nm}}\right)^n \\
&= -\pi\log\bar{z}_{mn} - 0; \\
I_{l_1+l_2} &= -\frac{i}{2}\int_\epsilon^\Lambda dx\frac{\log x}{x-z_{mn}} + \frac{i}{2}\int_\epsilon^\Lambda dx\frac{\log x - 2\pi i}{x-z_{mn}} = \pi\int_\epsilon^\Lambda\frac{dx}{x-z_{mn}} = \pi\log\Lambda - \pi\log z_{nm}.
\end{aligned} \tag{C.4}$$

Summing up

$$\begin{aligned} I^{1\bar{1}}_{m\bar{n}} &= 2\pi \log \Lambda - \pi \log\left(-\left|z_{mn}\right|^2\right) - \pi^2 i = 2\pi \log\Lambda - \pi\left[\log\left(-\left|z_{mn}\right|^2\right) - \pi i\right] \\ &= 2\pi\log\Lambda - \pi\log\left(-\left|z_{mn}\right|^2 e^{i\pi}\right) = 2\pi\left(\log\Lambda - \log\left|z_{mn}\right|\right) = -\pi\log(|z_{mn}|^2/\Lambda^2), \end{aligned} \tag{C.5}$$

where Λ is an IR cutoff. For $r=1, s\geq 1$

$$\begin{aligned} I^{1\bar{s}}_{m\bar{n}} =& \frac{i}{2}\int\frac{d^2z}{\left(z-z_{mn}\right)\bar{z}^s} = \frac{i}{2}\int_{|z|>|z_{mn}|}\frac{d^2z}{\left(1-\frac{z_{mn}}{z}\right)z\bar{z}^s} - \frac{i}{2}\int_{|z|<|z_{mn}|}\frac{d^2z}{\left(1-\frac{z}{z_{mn}}\right)z_{mn}\bar{z}^s} \\ =& \frac{i}{2}\int_{|z|>|z_{mn}|}\frac{d^2z}{z\bar{z}^s}\sum_{i=0}^{\infty}\left(\frac{z_{mn}}{z}\right)^i - \int_{|z|<|z_{mn}|}\frac{d^2z}{z_{mn}\bar{z}^s}\sum_{i=0}^{\infty}\left(\frac{z}{z_{mn}}\right)^i \\ =& \frac{i}{2}\int_{|z|>|z_{mn}|}d^2z\frac{z^{s-1}_{mn}}{z^s\bar{z}^s} - 0 = \int_0^{2\pi}d\theta\int_{|z_{mn}|}^{\infty}\rho d\rho\frac{z^{s-1}_{mn}}{\rho^{2s}} \\ =& \frac{\pi}{1-s}\left.\frac{z^{s-1}_{mn}}{\rho^{2s-2}}\right|_{\rho=|z_{mn}|}^{\infty} = \frac{\pi}{s-1}\frac{1}{\bar{z}^{s-1}_{mn}}. \end{aligned}$$

Integrals with $r\geq 2, s\geq 2$ vanish because they are $\partial^{r-1}_{z_n}\partial^{s-1}_{\bar{z}_m}$ derivatives of the $r=1, s=1$ integral. We may also evaluate these by using the Stokes' theorem

$$\begin{aligned} I^{1\bar{s}}_{m\bar{n}} =& \frac{i}{2}\int\frac{d^2z}{\left(z-z_{mn}\right)\bar{z}^s} = -\frac{i}{2}\frac{1}{s-1}\int d^2z\partial_{\bar{z}}\left(\frac{1}{\left(z-z_{mn}\right)\bar{z}^{s-1}}\right) \\ =& \frac{i}{2}\frac{1}{s-1}\left[\oint_{|z|=\Lambda}\frac{dz}{\left(z-z_{mn}\right)\bar{z}^{s-1}} - \oint_{|z-z_{mn}|=\epsilon}\frac{dz}{\left(z-z_{mn}\right)\bar{z}^{s-1}}\right] \\ =& 0 - \frac{i}{2}\frac{1}{s-1}\oint_{|z-z_{mn}|=\epsilon}\frac{dz}{\left(z-z_{mn}\right)\left(\frac{\epsilon^2}{z-z_{mn}}+\bar{z}\right)^{s-1}} \\ =& -\frac{i}{2}\frac{2\pi i}{s-1}\frac{1}{\bar{z}^{s-1}_{mn}} + \mathcal{O}(\epsilon) = \frac{\pi}{s-1}\frac{1}{\bar{z}^{s-1}_{mn}}, \end{aligned} \tag{C.6}$$

where the residue term $\mathcal{O}(\epsilon)$ vanishes

$$\mathcal{O}(\epsilon) \sim \oint_{|z-z_{mn}|=\epsilon}\frac{dz}{\left(z-z_{mn}\right)^n}, n>2 \rightarrow \mathcal{O}(\epsilon)=0. \tag{C.7}$$

One may also obtain the same results by taking derivatives of the $r=s=1$ integral w.r.t. z_m, z_n

$$I^{1\bar{s}}_{m\bar{n}} = \frac{\partial^{s-1}_{\bar{z}_n}}{(s-1)!}\int\frac{d^2x}{(z-z_n)(\bar{z}-\bar{z}_m)} = -\frac{\partial^{s-1}_{\bar{z}_m}}{(s-1)!}\pi\log(|z_{mn}|^2/\Lambda^2) = \frac{\pi}{s-1}\frac{1}{\bar{z}^{s-1}_{mn}}. \tag{C.8}$$

Collecting results gives

$$\begin{aligned} \langle X\rangle^{(1)} =& \sum_{m\neq n}\langle\left(\log(|z_{mn}|^2/\Lambda^2)\partial_{z_m}\partial_{\bar{z}_n} - \sum_{s\geq 2}\frac{1}{s-1}\frac{\bar{L}_{s-2,n}\partial_{z_m}}{\bar{z}^{s-1}_{mn}} - \sum_{r\geq 2}\frac{1}{r-1}\frac{L_{r-2,m}\partial_{\bar{z}_n}}{z^{r-1}_{nm}}\right)X\rangle^{(0)} \\ &+ \sum_{m=n}\langle\left(\log(\varepsilon^2/\Lambda^2)\partial_{z_m}\partial_{\bar{z}_n} - \sum_{s\geq 2}\frac{1}{s-1}\frac{\bar{L}_{s-2,n}\partial_{z_m}}{\varepsilon^{s-1}} - \sum_{r\geq 2}\frac{1}{r-1}\frac{L_{r-2,m}\partial_{\bar{z}_n}}{\varepsilon^{r-1}}\right)X\rangle^{(0)}. \end{aligned} \tag{C.9}$$

The dependence on the IR cutoff cancels upon summation over all fields due to translational invariance $\sum_m \partial_{z_m}\langle X\rangle = \sum_n \partial_{\bar{z}_n}\langle X\rangle = 0$. This can be seen by rewriting $\sum_{m=n}\log(\varepsilon^2/\Lambda^2)\partial_{z_m}\partial_{\bar{z}_n}$ as $\sum_{m\neq n}\log(\varepsilon^2/\Lambda^2)\partial_{z_m}\partial_{\bar{z}_n}$.

C.2 Integrals for the second-order correction of undeformed operators

To obtain the second-order correction, we need to evaluate the following integral[11]

$$\begin{cases} \left(-\frac{1}{2}\right)\frac{1}{\pi}\int d^2x\langle\left(\frac{r}{s-1}\frac{L_{r-2,m}\bar{L}_{s-2,n}\bar{L}_{t-2,i}}{(z-z_m)^{r+1}(\bar{z}-\bar{z}_n)^{s-1}(\bar{z}-\bar{z}_i)^t} + (c.c.)\right) X\rangle^{(0)}, & \text{for } s>1; \\ \left(-\frac{1}{2}\right)\frac{1}{\pi}\int d^2x\langle\left(r\frac{L_{r-2,m}\bar{L}_{-1,n}\bar{L}_{t-2,i}}{(z-z_m)^{r+1}(\bar{z}-\bar{z}_i)^t}\log(\mu^2|z-z_n|^2) + (c.c.)\right) X\rangle^{(0)}, & \text{for } s=1. \end{cases} \tag{C.10}$$

We assume that $z_m \neq z_n \neq z_i$ since the cases where they coincide can be obtained by taking limits. Here, we only have to evaluate the $s=1$ integral since the rest can be obtained by taking derivatives.

Note that $\log(\mu^2|z-z_n|^2) = \log(\mu(z-z_n)) + \log(\mu(\bar{z}-\bar{z}_n))$. We will first evaluate the terms with $\log(\mu(z-z_n))$. They can be further divided into two groups: those with $t=1$ and those with $t>1$. As usual, we only have to evaluate the former; the rest is obtained by taking derivatives.

The integral involving $\log(\mu(z-z_n))$ can be evaluated via Stokes' theorem:

$$\begin{aligned} &\frac{1}{\pi}\int d^2x\frac{\log(\mu(z-z_n))}{(z-z_m)^{r+1}(\bar{z}-\bar{z}_i)} \\ =&\frac{1}{\pi(-r)}\frac{i}{2}\int d^2z\partial_z\left(\frac{\log(\mu(z-z_n))}{(z-z_m)^r(\bar{z}-\bar{z}_i)}\right) - \frac{1}{\pi(-r)}\frac{i}{2}\int d^2z\frac{\partial_z\log(\mu(z-z_n))}{(z-z_m)^r(\bar{z}-\bar{z}_i)} \\ =&-\frac{1}{2\pi i r}\left(-\oint_{|z-z_i|=\epsilon} - \oint_{|z-z_n|=\epsilon} - \oint_{|z-z_m|=\epsilon} + \oint_{l_n} + \oint_{|z|=\Lambda}\right)d\bar{z}\frac{\log(\mu(z-z_n))}{(z-z_m)^r(\bar{z}-\bar{z}_i)} \\ &+\frac{1}{\pi r}\frac{i}{2}\int d^2z\frac{1}{(z-z_m)^r(z-z_n)(\bar{z}-\bar{z}_i)}, \end{aligned} \tag{C.11}$$

where the contour of the $\oint_{l_n}$ integral runs along the branch cut of $\log(\mu(z-z_n))$. We will start by evaluating the contour integrals. The $\oint_{|z-z_m|=\epsilon}$, $\oint_{|z-z_n|=\epsilon}$ integrals vanish because they are proportional to integrals around infinitesimal circles surrounding the singularities of $\log(\mu(z-z_n))$ and $1/(z-z_m)^r$ at $z=z_n$ and $z=z_m$, respectively. These are given by

$$\begin{aligned} &\oint_{|z-z_n|=\epsilon} d\bar{z}\log(\mu(z-z_n)) = \lim_{\epsilon\to 0}\int_0^{2\pi} d\theta\,\epsilon e^{-i\theta}\log(\mu\epsilon e^{i\theta}) \\ =&\lim_{\epsilon\to 0}\int_0^{2\pi} d\theta\,\epsilon e^{-i\theta}(\log(\mu\epsilon)+i\theta) = \lim_{\epsilon\to 0}2\pi\epsilon = 0, \end{aligned}$$

and

$$\oint_{|z-z_n|=\epsilon} d\bar{z}\frac{1}{(z-z_m)^r} = \lim_{\epsilon\to 0}\oint_{|z-z_n|=\epsilon} d\theta\,\epsilon e^{-i\theta}\frac{1}{(\epsilon e^{i\theta})^r} = 0. \tag{C.12}$$

[11] There is also a trivial integral, namely $-\frac{1}{2\pi}\int d^2x\langle\mathcal{O}_{T\bar{T}}(z)\,\mathrm{d}\,X\rangle^{(0)}$, which is simply $\frac{1}{2}\left(\sum_{a\neq b}\mathrm{d}_{z_a,z_b}\right)^2\langle X\rangle^{(0)} = \frac{1}{2}\langle\mathrm{d}\,X\rangle^{(0)}$.

The $\oint_{|z-z_i|=\epsilon}$ integral is straightforward

$$\frac{1}{2\pi i r}\oint_{|z-z_i|=\epsilon} d\bar{z}\frac{\log(\mu(z-z_n))}{(z-z_m)^r(\bar{z}-\bar{z}_i)} = -\frac{1}{r}\frac{\log(\mu z_{in})}{z_{im}^r}. \tag{C.13}$$

The integral along the branch cut of $\log(\mu(z-z_n))$ is given by

$$-\frac{1}{2\pi i r}\oint_{l_n} d\bar{z}\frac{\log(\mu(z-z_n))}{(z-z_m)^r(\bar{z}-\bar{z}_i)} = \frac{1}{r}\int_{x_n}^{\Lambda} dx\frac{1}{(z-z_m)^r(\bar{z}-\bar{z}_i)}\bigg|_{y=y_n}. \tag{C.14}$$

This term cancels out a term that appears in the computation of $\frac{1}{\pi}\int d^2x\frac{\log(\mu(\bar{z}-\bar{z}_n))}{(z-z_m)^{r+1}(\bar{z}-\bar{z}_i)}$. Next, we evaluate the integral around the large circle at infinity

$$-\frac{1}{2\pi i r}\oint_{|z|=\Lambda} dz\frac{\log(\mu(z-z_n))}{(z-z_m)^r(\bar{z}-\bar{z}_i)} \sim -\frac{1}{2\pi i r}\int_0^{2\pi}\Lambda e^{i\theta}d\theta\frac{\log(\mu\Lambda)+i\theta}{(\Lambda e^{i\theta})^r\Lambda e^{-i\theta}} \to 0, \tag{C.15}$$

which vanishes for all values of r after taking the $\Lambda\to\infty$ limit.

Now, we proceed to the three-pole integral in the second line of (C.11). We only have to calculate the integral for $r=1$, as the general result can be obtained by taking ∂_{z_m} derivatives. Using the decomposition rule

$$\frac{1}{(z-z_m)(z-z_n)} = \frac{1}{z_{mn}}\left(\frac{1}{z-z_m}-\frac{1}{z-z_n}\right), \tag{C.16}$$

we can express the three-pole integral with $r=1$ in terms of the two-pole integrals $I_{m\bar{n}}^{r\bar{s}}$ we have already encountered

$$\begin{aligned}&\frac{1}{\pi r}\frac{i}{2}\int d^2z\frac{1}{(z-z_m)(z-z_n)(\bar{z}-\bar{z}_i)}\\ =&\frac{1}{\pi r}\frac{i}{2}\frac{1}{z_{mn}}\int d^2z\left(\frac{1}{z-z_m}-\frac{1}{z-z_n}\right)\frac{1}{\bar{z}-\bar{z}_i} = -\frac{1}{r}\frac{1}{z_{mn}}\log\frac{|z_{im}|^2}{|z_{in}|^2}.\end{aligned}$$

Taking derivatives w.r.t. z_m yields

$$\frac{\partial_{z_m}^{r-1}}{(r-1)!}\frac{1}{r}\frac{1}{z_{mn}}\log\frac{|z_{im}|^2}{|z_{in}|^2} = -\frac{1}{r}\left(\frac{1}{z_{nm}^r}\log\frac{\bar{z}_{im}}{\bar{z}_{in}} - \frac{1}{z_{im}^r}\Phi(\frac{z_{nm}}{z_{im}},1,r)\right). \tag{C.17}$$

We can now proceed to the terms with $\log(\mu(\bar{z}-\bar{z}_n))$

$$\begin{aligned}&\frac{1}{\pi}\int d^2x\frac{\log(\mu(\bar{z}-\bar{z}_n))}{(z-z_m)^{r+1}(\bar{z}-\bar{z}_i)} = \frac{1}{\pi(-r)}\frac{i}{2}\int d^2z\partial_z\left(\frac{\log(\mu(\bar{z}-\bar{z}_n))}{(z-z_m)^r(\bar{z}-\bar{z}_i)}\right)\\ =&-\frac{1}{(2\pi i)r}\left(-\oint_{|z-z_n|=\epsilon}-\oint_{|z-z_i|=\epsilon}+\oint_{l_n}+\oint_{|z|=\Lambda}\right)d\bar{z}\frac{\log(\mu(\bar{z}-\bar{z}_n)))}{(z-z_m)^r(\bar{z}-\bar{z}_i)}.\end{aligned}$$

Likewise, we find the contributing terms to be

$$\frac{1}{(2\pi i)r}\oint_{|z-z_i|=\epsilon} d\bar{z}\frac{\log(\mu(\bar{z}-\bar{z}_n))}{(z-z_m)^r(\bar{z}-\bar{z}_i)} = -\frac{1}{r}\frac{\log(\mu\bar{z}_{in})}{z_{im}^r}, \tag{C.18}$$

and

$$-\frac{1}{(2\pi i)r}\oint_{l_n} d\bar{z}\frac{\log(\bar{z}-\bar{z}_n)}{(z-z_m)^r(\bar{z}-\bar{z}_i)} = -\frac{1}{r}\int_{x_n}^{\Lambda} dx\frac{1}{(z-z_m)^r(\bar{z}-\bar{z}_i)}\bigg|_{y=y_n}, \tag{C.19}$$

where the latter cancels out the term in (C.14).[12] Collecting results, we conclude that the $s=1, t=1$ integrals are given by

$$\frac{1}{\pi}\int d^2x\frac{\log(\mu^2|z-z_n|^2)}{(z-z_m)^{r+1}(\bar{z}-\bar{z}_i)} = -\frac{1}{r}\left[\frac{\log(\mu^2|z_{in}|^2)}{z_{im}^r} + \left(\frac{1}{z_{nm}^r}\log\frac{\bar{z}_{im}}{\bar{z}_{in}} + \frac{1}{z_{im}^r}\Phi(\frac{z_{nm}}{z_{im}},1,r)\right)\right]. \tag{C.20}$$

where $\Phi(\frac{z_{nm}}{z_{im}},1,r)$ denotes the Lerch transcendent. For $s=1, t>1$, this becomes

$$\begin{aligned}&\frac{1}{\pi}\int d^2x\frac{\log(\mu^2|z-z_n|^2)}{(z-z_m)^{r+1}(\bar{z}-\bar{z}_i)^t} = \frac{\partial_{\bar{z}_i}^{t-1}}{(t-1)!}\frac{1}{\pi}\int d^2x\frac{\log(\mu^2|z-z_n|^2)}{(z-z_m)^{r+1}(\bar{z}-\bar{z}_i)} \\ =&\frac{1}{r(t-1)}\left(\frac{1}{\bar{z}_{ni}^{t-1}z_{im}^r} + \frac{1}{z_{nm}^r\bar{z}_{mi}^{t-1}} - \frac{1}{z_{nm}^r\bar{z}_{ni}^{t-1}}\right).\end{aligned} \tag{C.21}$$

The $s>1, t=1$ integrals can be obtained by taking derivatives w.r.t. $\bar{z}_n$

$$\begin{aligned}&\frac{1}{\pi}\int\frac{d^2x}{(z-z_m)^{r+1}(\bar{z}-\bar{z}_n)^{s-1}(\bar{z}-\bar{z}_i)} = \frac{\partial_{\bar{z}_n}^{s-1}}{(s-2)!(-1)}\frac{1}{\pi}\int d^2x\frac{\log(\mu^2|z-z_n|^2)}{(z-z_m)^{r+1}(\bar{z}-\bar{z}_i)} \\ =&-\frac{1}{r}\frac{1}{\bar{z}_{in}^{s-1}}\left(\frac{1}{z_{im}^r} - \frac{1}{z_{nm}^r}\right).\end{aligned} \tag{C.22}$$

Further derivation gives the integrals with $s>1, t>1$, which are

$$\begin{aligned}&\frac{1}{\pi}\int\frac{d^2x}{(z-z_m)^{r+1}(\bar{z}-\bar{z}_n)^{s-1}(\bar{z}-\bar{z}_i)^t} = \frac{\partial_{\bar{z}_i}^{t-1}}{(t-1)!}\frac{1}{\pi}\int\frac{d^2x}{(z-z_m)^{r+1}(\bar{z}-\bar{z}_n)^{s-1}(\bar{z}-\bar{z}_i)} \\ =&-\frac{(1-s)^{\underline{t-1}}}{r(t-1)!}\frac{1}{\bar{z}_{in}^{s+t-2}}\left(\frac{1}{z_{im}^r} - \frac{1}{z_{nm}^r}\right),\end{aligned} \tag{C.23}$$

where $a^{\underline{b}} := \prod_{p=0}^{b-1}(a-p)$, with the exception of $0^{\underline{b}} \equiv 1$. We have completed the evaluation of all the integrals. The final step is to combine these results into the complete expression for $\langle X\rangle^{(2)}$. By attaching the corresponding Virasoro generators to the integrals, we can write down the final results for the integrals in (C.10) as

$$\begin{cases}\frac{1}{2}\langle\left\{\left[\frac{\log(\mu^2|z_{in}|^2)}{z_{im}^r} + \left(\frac{\log(\bar{z}_{im}/\bar{z}_{in})}{z_{nm}^r} + \frac{1}{z_{im}^r}\Phi(\frac{z_{nm}}{z_{im}},1,r)\right)\right]L_{r-2,m}\bar{L}_{-1,n}\bar{L}_{-1,i} + (c.c.)\right\}X\rangle^{(0)} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } (s,t)=(1,1), \\ \frac{1}{2}\langle\left\{\frac{(1-s)^{\underline{t-2}}}{(t-1)!}\left[\frac{1}{\bar{z}_{in}^{s+t-2}}\left(\frac{1}{z_{nm}^r} - \frac{1}{z_{im}^r}\right) + \frac{\delta_{s,1}}{z_{nm}^r\bar{z}_{mi}^{t-1}}\right]L_{r-2,m}\bar{L}_{s-2,n}\bar{L}_{t-2,i} + (c.c.)\right\}X\rangle^{(0)} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } (s,t)\neq(1,1) \text{ and } s>0, t>0.\end{cases} \tag{C.24}$$

These expressions constitute the second-order correction to $\langle X\rangle^{\lambda}$.

[12] If one actually calculates this integral, one will yield anomalous expressions that spoil rotational invariance, more specifically expressions containing $(\bar{z}_{in} - z_{mn})$.

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