Nicolai maps with four-fermion interactions

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Abstract

Nicolai maps offer an alternative description of supersymmetric theories via nonlinear and nonlocal transformations characterized by the so-called 'free-action' and 'determinant-matching' conditions. The latter expresses the equality of the Jacobian determinant of the transformation with the one obtained by integrating out the fermions, which so far have been considered only to quadratic terms. We argue that such a restriction is not substantial, as Nicolai maps can be constructed for arbitrary nonlinear sigma models, which feature four-fermion interactions. The fermionic effective one-loop action then gets generalized to higher loops and the perturbative tree expansion of such Nicolai maps receives quantum corrections in the form of fermion loop decorations. The 'free-action condition' continues to hold for the classical map, but the 'determinant-matching condition' is extended to an infinite hierarchy in fermion loop order. After general considerations for sigma models in four dimensions, we specialize to the case of $\mathbb{C}\mathrm{P}^N$ symmetric spaces and construct the associated Nicolai map. These sigma models admit a formulation with only quadratic fermions via an auxiliary vector field, which however does not simplify the construction of the map.

Contents

1 Introduction and summary

The Nicolai map T [1, 2, 3] is a (generically nonlocal and nonlinear) field transformation that relates supersymmetric theories at different values of their parameters, say coupling constants g. In particular, it allows one to compute the expectation value of any operator Y built from the bosonic fields ϕ at coupling g by evaluating a free-field (g=0) correlator of the inversely transformed operator $T^{-1}Y$,[1]

$$\big\langle Y[\phi]\big\rangle_g \;=\; \big\langle (T_g^{-1}Y)[\phi]\big\rangle_0 \;=\; \big\langle Y[T_g^{-1}\phi]\big\rangle_0 \;. \tag{1.1}$$

Here, we indicated the value of the coupling as a subscript on the correlator and also on the symbol of the map $T_g : \phi \mapsto \phi'[g,\phi] = T_g\phi$. The second equality expresses the distributivity of the map, i.e. $T(\phi_1\phi_2) = (T\phi_1)\,(T\phi_2)$. In the expectation values of (1.1) it is understood that all anticommuting degrees of freedom ψ have been integrated out.[2] This means that the Nicolai map operates in a nonlocal bosonic theory with an action

$$S_g[\phi] \;=\; S_g^{\mathrm{b}}[\phi] + \hbar\, S_g^{\mathrm{f}}[\phi] \;, \tag{1.2}$$

where the local part S_g^{b} is the bosonic piece of the original supersymmetric action $S_{\text{SUSY}}[\phi,\psi]$ and $\exp\{\mathrm{i}S_g^{\mathrm{f}}\}$ arises from the path integral over the anticommuting fields in the partition function, both at coupling g. In particular, S_g^{f} is in general nonlocal and suppressed by $\hbar$. So far in the literature only theories with quadratic fermions have been considered, for which the integration over fermions can be formally carried out. As a result, changing path-integral variables $Y \mapsto T_gY$ on the right-hand side of (1.1) and sorting powers of $\hbar$, one recovers the original defining properties of the Nicolai map,

$$S_0^{\mathrm{b}}[T_g\phi] \;=\; S_g^{\mathrm{b}}[\phi] \qquad \text{and} \qquad S_0^{\mathrm{f}} - \mathrm{i}\;\mathrm{tr}\ln\tfrac{\delta T_g\phi}{\delta\phi} \;=\; S_g^{\mathrm{f}}[\phi] \;, \tag{1.3}$$

which are called the 'free-action' and 'determinant-matching' property, respectively. We note that $S_0^{\mathrm{f}} = S_0^{\mathrm{f}}[T_g\phi]$ is a constant since this functional does not depend on the bosonic fields. The name 'determinant-matching' was indeed coined because in theories with a flat target space integrating the quadratic fermions out produces a fermionic (or Matthews–Salam–Seiler) determinant, which has to be matched by the Jacobian determinant $\det\frac{\delta T_g\phi}{\delta\phi}$. Powers of $\hbar$ are fully explicit because the fermionic determinant sums all fermionic loops in the bosonic background which is a one-loop exact operation. As a consequence, in the language of the 'Nicolai rules' to construct the map, only tree diagrams appear, and for this reason it is sometimes considered a classical construction.

In this paper we will show that the assumption of quadratic fermions is not necessary; Nicolai maps can be constructed for supersymmetric actions with higher-order fermion terms as well. The price to pay is quantum

[1] The vanishing of the vacuum energy in supersymmetric theories properly normalizes $\langle 1\rangle_g = 1$.

[2] Auxiliary fields F may be kept as part of ϕ or eliminated, see below.

corrections to the Nicolai map and a more general dependence on $\hbar$, which upsets the conditions (1.3). In other words, the Nicolai map is a formal power series not only in g but also in $\hbar$,

$$T_g\phi \;=\; T_g^{(0)}\phi \;+\sum_{r=1}^{\infty}\hbar^r T_g^{(r)}\phi\;, \tag{1.4}$$

where r counts the number of fermion loops in the diagrammatic representation. With higher-order fermionic terms in the action, the path integral over the anticommuting fields will produce an effective nonlocal action $\sum_r \hbar^r S_g^{(r)}[\phi]$ extending the previous one-loop result $\hbar\, S_g^{\rm f}[\phi]$ to all orders in $\hbar$. Revising the argument leading from (1.1) to (1.3) we then find the combined identity

$$S_0^{\rm b}[T_g\phi] \;+\; \hbar\, S_0^{\rm f} \;-\; {\rm i}\hbar\,{\rm tr}\ln\tfrac{\delta T_g\phi}{\delta\phi} \;=\; S_g^{\rm b}[\phi] \;+\; \sum_r \hbar^r S_g^{(r)}[\phi]\;. \tag{1.5}$$

Inserting (1.4) into (1.5) and comparing powers in $\hbar$ one obtains an infinite hierarchy of 'Nicolai-map conditions'. The leading two represent the tree-level and one-loop contributions and read

$$S_0^{\rm b}[T_g^{(0)}\phi] \;=\; S_g^{\rm b}[\phi] \qquad\text{and}\qquad S_0^{\rm b}[T_g\phi]\big|_{O(\hbar)} + S_0^{\rm f} - {\rm i}\;{\rm tr}\ln\tfrac{\delta T_g^{(0)}\phi}{\delta\phi} \;=\; S_g^{(1)}[\phi]\;. \tag{1.6}$$

Clearly, the 'free-action condition' is still valid for the classical part of the Nicolai map, but the 'determinant-matching condition' receives a free-action contribution from the one-loop correction to the map, and there are further conditions, each one balancing expressions of a fixed loop order.

For any off-shell supersymmetric theory, there exists a formalism and a universal formula which provides a formal power series expansion (in g) of the map and its inverse [4, 5, 6, 7, 8, 9, 10, 11].[3] The key player is the 'coupling flow operator'

$$R_g[\phi] \;=\; \int\!{\rm d}x\;\big(\partial_g T_g^{-1}\circ T_g\big)\phi(x)\,\frac{\delta}{\delta\phi(x)}\;, \tag{1.7}$$

where 'x' stands for all coordinates the fields depend on. This functional differential operator governs the infinitesimal Nicolai map,

$$\partial_g\big\langle Y[\phi]\big\rangle_g \;=\; \big\langle\big(\partial_g + R_g[\phi]\big)Y[\phi]\big\rangle_g\;. \tag{1.8}$$

The first step towards its construction is the observation that, in off-shell supersymmetric chiral theories,

$$S_{\rm SUSY}[\phi,\psi] \;=\; \delta_\alpha\Delta_\alpha[\phi,\psi] \qquad\Rightarrow\qquad \partial_g S_{\rm SUSY}[\phi,\psi] \;=\; \delta_\alpha\partial_g\Delta_\alpha[\phi,\psi]\;, \tag{1.9}$$

where α is a spinor index (we are being schematic here on the notation of spinors) and Δ_α is a certain anticommuting local functional. Super Yang–Mills theories are more complicated because $[\partial_g,\delta_\alpha]\neq 0$, and we exclude them in the following. Employing (1.9) and the supersymmetric Ward identity we integrate out the anticommuting variables to read off the coupling flow operator

$$R_g[\phi] \;=\; \frac{\rm i}{\hbar}\,\partial_g\underbrace{\Delta_\alpha[\phi]\;\delta_\alpha}_{} \;=\; \frac{\rm i}{\hbar}\int\!{\rm d}x\;\partial_g\underbrace{\Delta_\alpha[\phi]\;\delta_\alpha}_{}\phi(x)\;\frac{\delta}{\delta\phi(x)}\;, \tag{1.10}$$

where the contraction indicates a fermionic expectation value to be taken.

Exponentiating the action of $\partial_g{+}R_g$ generates the (finite-flow) inverse Nicolai map. Alternatively, a g-ordered exponential of $-\!\int_0^g {\rm d}g'\,R_{g'}$ directly yields the Nicolai map. In any case, the R_g action has to be iterated, $R_{g_1}R_{g_2}\cdots R_{g_s}\phi$, which grafts full fermionic $2k$-point functions onto previously generated diagrams. For the Wess–Zumino model (quadratic in the fermions) this produces fermionic tree diagrams, dressed with bosonic 'leaves'. Still, fermion loops remain absent. For nonlinear sigma models, however, the graphical expansion of the Nicolai map will feature fermionic trees with all kinds of fermion loops embedded. Thus it

[3] Under certain conditions the construction works also in the absence of off-shell supersymmetry, e.g. for super Yang–Mills theory in dimensions 6 and 10 in the Landau gauge [8, 12, 13, 14].

can no longer be considered a classical map. Nevertheless, at any power of the coupling g a finite number of diagrams contributes to the Nicolai map, and we may still employ the universal formula to write it down. In the following, this will be demonstrated for four-dimensional supersymmetric sigma models, first in general and second for supersymmetric $\mathbb{C}\mathrm{P}^N$ models.

For the scope of this paper, relating the perturbative Nicolai map to the standard perturbative (Feynman) expansion solely concerns the generation of diagrams. These are in general divergent and in most applications require regularization. We assume that it has been been done in an appropriate way, e.g. via dimensional regularization or with a UV cutoff.[4] These aspects, and more generally the interplay between the Nicolai map and regularization or renormalization, deserve further study.

There are several ways in which the work presented here can be further expanded or generalized. It would be interesting to work out explicitly additional orders of the Nicolai map for $\mathbb{C}\mathrm{P}^N$ models presented here and study in detail the higher-loop identities (1.5). Conceivably, then, the formalism presented here can be naturally extended to gauge theories. Finally, a more ambitious goal is the application of the Nicolai map to supergravity, which has been excluded so far owing to the four-fermion contributions, with the potential application of shedding light on its UV behaviour.

The rest of the paper is organized as follows. Section 2 collects some general expressions for supersymmetric nonlinear sigma models and its coupling flow operator. In Section 3 we specialize to $\mathbb{C}\mathrm{P}^N$ and construct the associated Nicolai map in perturbation theory. A possibility of eliminating the four-fermion interactions via auxiliary (Hubbard–Stratonovich) scalars or vectors is discussed in Section 4. We then comment on the superfield origin of an auxiliary vector in $\mathbb{C}\mathrm{P}^N$ models and its relevance for the construction of the perturbative Nicolai map.

2 Supersymmetric nonlinear sigma model

The prototypical supersymmetric field theory with higher-than-quadratic fermion terms in the action is the supersymmetric nonlinear sigma model in (3+1)-dimensional Minkowski space [15], which is characterized by a Kähler potential $\mathrm{K}(\Phi_a^\dagger, \Phi^a)$ and a superpotential $\mathrm{W}(\Phi^a)$ for a collection

$$\{\Phi^a\} \;=\; \{\Phi^0, \Phi^A\} \;=\; \{\Phi^0, \Phi^1, \Phi^2, \ldots, \Phi^N\} \tag{2.1}$$

of $N{+}1$ chiral superfields, i.e. $a, b, c, \ldots = 0, 1, \ldots, N$ and $A, B, C, \ldots = 1, \ldots, N$. Complex conjugation raises or lowers target-space indices. Adopting the Wess–Bagger notation [16] their component expansion (in x coordinates) reads

$$\Phi^a \;=\; \phi^a + \mathrm{i}\theta\sigma^m\bar\theta\,\partial_m\phi^a + \tfrac14\theta^2\bar\theta^2\,\Box\phi^a + \sqrt{2}\theta\,\psi^a - \tfrac{\mathrm{i}}{\sqrt2}\theta^2\partial_m\psi^a\sigma^m\bar\theta + \theta^2 F^a \;=:\; \phi^a + \Xi^a\ , \tag{2.2}$$

where Weyl spinor indices $\alpha, \dot\alpha = 1, 2$ are suppressed. We note that $\Xi^3 = 0$. For the purpose of this paper the superpotential is not essential (although it can easily be added), and thus we omit any F-term in the action and restrict ourselves to the Kähler potential in the supersymmetric D-term action

$$S_{\text{SUSY}} \;=\; \int \mathrm{d}^4x\;\mathcal{L} \qquad \text{with} \qquad \mathcal{L} \;=\; \int \mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta\;\mathrm{K}(\Phi_a^\dagger, \Phi^a)\ . \tag{2.3}$$

We further define the functional Δ given by

$$\Delta \;=\; \int \mathrm{d}^4x\;\mathcal{M} \qquad \text{with} \qquad \mathcal{M} \;=\; \int \mathrm{d}\theta\,\mathrm{d}^2\bar\theta\;\mathrm{K}(\Phi_a^\dagger, \Phi^a)\ , \tag{2.4}$$

so that

$$\mathcal{L} \;=\; -\tfrac14\big(\tfrac{1+\kappa}{2}\,\delta^\alpha\mathcal{M}_\alpha + \tfrac{1-\kappa}{2}\,\bar\delta_{\dot\alpha}\bar{\mathcal{M}}^{\dot\alpha}\big) \;+\; \text{total derivative}\ , \tag{2.5}$$

[4] In particular, the perturbative non-renormalizability of the nonlinear sigma models described here is not of concern.

with a free parameter κ controlling the relative weight of the chiral against antichiral contribution.[5]

We introduce the notation

$$K := \mathrm{K}(\phi^*_a, \phi^a) \qquad \text{and} \qquad K^{cd\ldots}_{ab\ldots} := \tfrac{\partial}{\partial\phi^a}\tfrac{\partial}{\partial\phi^b}\cdots\tfrac{\partial}{\partial\phi^*_c}\tfrac{\partial}{\partial\phi^*_d}\cdots K\ . \tag{2.6}$$

Expanding

$$\begin{aligned}\mathrm{K} \;&=\; K + K_a\,\Xi^a + K^a\,\bar\Xi_a + K^a_b\,\Xi^b\bar\Xi_a + \tfrac12 K_{ab}\,\Xi^a\Xi^b + \tfrac12 K^{ab}\,\bar\Xi_a\bar\Xi_b \\ &\quad + \tfrac12 K^a_{bc}\,\Xi^b\Xi^c\bar\Xi_a + \tfrac12 K^{bc}_a\,\Xi^a\bar\Xi_b\bar\Xi_c + \tfrac14 K^{ab}_{cd}\,\Xi^a\Xi^b\bar\Xi_c\bar\Xi_d \\ &=\; \ldots + \bar\theta^2\theta\,\mathcal{M} + \theta^2\bar\theta\,\bar{\mathcal{M}} + \theta^2\bar\theta^2\,\mathcal{L}\end{aligned} \tag{2.7}$$

and employing the identity

$$\Box\,K \;=\; K^{AB}\,\partial_m\phi^*_A\partial^m\phi^*_B + K_{AB}\,\partial_m\phi^A\partial^m\phi^B + 2K^A_B\,\partial_m\phi^*_A\partial^m\phi^B + K^A\,\Box\phi^*_A + K_A\,\Box\phi^A\ , \tag{2.8}$$

we read off the component lagrangian

$$\begin{aligned}\mathcal{L} \;&=\; K^b_a\left[-\partial^m\phi^a\partial_m\phi^*_b - \tfrac{\mathrm{i}}{2}\psi^a\sigma^m\partial_m\bar\psi_b + \tfrac{\mathrm{i}}{2}\partial_m\psi^a\sigma^m\bar\psi_b + F^aF^*_b\right] \\ &\quad + \tfrac12 K^c_{ab}\left[\mathrm{i}\psi^b\sigma^m\bar\psi_c\,\partial_m\phi^a - \psi^a\psi^bF^*_c\right] + \tfrac12 K^{ab}_c\left[\mathrm{i}\bar\psi_b\,\bar\sigma^m\psi^c\partial_m\phi_a - \bar\psi_a\bar\psi_b\,F^c\right] \\ &\quad + \tfrac14 K^{ab}_{cd}\ \bar\psi_a\bar\psi_b\ \psi^c\psi^d \\ &=\; K^b_a\left[-\partial^m\phi^a\partial_m\phi^*_b - \tfrac{\mathrm{i}}{2}\psi^a\sigma^m\nabla_m\bar\psi_b + \tfrac{\mathrm{i}}{2}\nabla_m\psi^a\sigma^m\bar\psi_b + F^aF^*_b\right] \\ &\quad - \tfrac12 K^c_{ab}\ \psi^a\psi^bF^*_c - \tfrac12 K^{ab}_c\ \bar\psi_a\bar\psi_b\,F^c + \tfrac14 K^{ab}_{cd}\ \bar\psi_a\bar\psi_b\ \psi^c\psi^d\end{aligned} \tag{2.9}$$

with target-space covariant derivative on spinors defined as

$$K^b_a\ \nabla_m\psi^a = K^b_a\,\partial_m\psi^a + K^b_{cd}\,\partial_m\phi^c\psi^d\ , \tag{2.10}$$

and we also obtain the penultimate component

$$\mathcal{M} \;=\; \sqrt2\,K^b_a\big[\psi^aF^*_b - \mathrm{i}\sigma^m\bar\psi_b\,\partial_m\phi^a\big] + \tfrac{1}{\sqrt2}K^{bc}_a\ \psi^a\ \bar\psi_b\bar\psi_c\ , \tag{2.11}$$

which are both manifestly invariant under Kähler transformations $\mathrm{K}\mapsto\mathrm{K}+\Lambda+\Lambda^\dagger$.

In order to simplify the coupling-flow operator, it is advisable to also integrate out the auxiliary F^a by inserting its classical value

$$K^b_a\,F^*_b \;=\; \tfrac12 K^{cd}_a\ \bar\psi_c\bar\psi_d \qquad\Rightarrow\qquad F^a \;=\; \tfrac12(K^{-1})^a_b\,K^b_{cd}\ \psi^c\psi^d \tag{2.12}$$

back into $\mathcal{L}$ and $\mathcal{M}$, which produces

$$\mathcal{L} \;=\; K^b_a\left[-\partial^m\phi^a\partial_m\phi^*_b - \tfrac{\mathrm{i}}{2}\psi^a\sigma^m\nabla_m\bar\psi_b + \tfrac{\mathrm{i}}{2}\nabla_m\psi^a\sigma_m\bar\psi_b\right] \;+\; \tfrac14 R^{ab}_{cd}\ \bar\psi_a\bar\psi_b\ \psi^c\psi^d \tag{2.13}$$

with the Riemann tensor

$$R^{ab}_{cd} \;=\; K^{ab}_{cd} - K^r_{cd}(K^{-1})^s_rK^{ab}_s \tag{2.14}$$

as well as

$$\mathcal{M} \;=\; -\mathrm{i}\sqrt2\,K^b_a\ \sigma^m\bar\psi_b\,\partial_m\phi^a \;+\; \sqrt2\,K^{bc}_a\ \psi^a\ \bar\psi_b\bar\psi_c\ . \tag{2.15}$$

We note that the invertibility of the Kähler metric is not needed in the final expression for the coupling-flow operator. Still, as we shall see, if redundant target coordinates are employed we must constrain them ('fix a gauge') and also introduce a coupling g into the Kähler potential K in order to obtain a perturbative expansion for the Nicolai map.

[5]This freedom is R symmetry: a D-term can be reached in two ways, by applying either δ^α to $\mathcal{M}_\alpha$ or $\bar\delta_{\dot\alpha}$ to $\bar{\mathcal{M}}^{\dot\alpha}$.

Suppose now that S_{SUSY} features terms of order $2k$ in the ψ fields, with $k = 1, 2, \ldots$. Then, $\partial_g \mathcal{M}_\alpha$ has terms of the form ψ^{2k-1}. Since δ_α acting on ϕ is linear in ψ, the contraction in

$$R_g \;=\; -\tfrac{\mathrm{i}}{4\hbar}\int\!\mathrm{d}y\!\int\!\mathrm{d}x\;\Big\{\tfrac{1+\kappa}{2}\,\partial_g\underbrace{\mathcal{M}_\alpha(y)\;\delta^\alpha}(x)\;+\;\tfrac{1-\kappa}{2}\,\partial_g\underbrace{\bar{\mathcal{M}}^{\dot\alpha}(y)\;\bar\delta_{\dot\alpha}}(x)\Big\} \tag{2.16}$$

signifies a fermionic $2k$-point function in the ϕ background, with $2k{-}1$ legs fused. For a flat target space, we look at Wess–Zumino chiral models, and fermions appear only quadratically in the action. Thus, $k{=}1$ implies only a full fermion propagator in (1.10),

$$\underbrace{\bar\psi^{\dot\beta}_b(x')\;\psi^a_\alpha}(x)\;\equiv\;\hbar\,(G_2^{(g)})^{a\dot\beta}_{b\alpha}(x',x)\;, \tag{2.17}$$

which contains all Feynman diagrams connected by a single fermion line with external ϕ legs. Expanding the full fermion propagator in powers of g, one obtains chains of free fermion propagators, with vertices encoding the coupling to the bosonic background. No fermion loops arise. In case of a curved target space, instead, we face a nonlinear sigma model, whence S_{SUSY} has terms quadratic ($k{=}1$) as well as quartic ($k{=}2$) in the fermions. In this case, a ψ^3 contribution in $\mathcal{M}_\alpha$ produces a correlator of a composite ψ^3 with another ψ in the bosonic background, which then occurs in (1.10),

$$\underbrace{\psi^b_\beta(x')\bar\psi^{\dot\gamma}_c(x')\bar\psi^{\dot\delta}_d(x')\;\psi^a_\alpha}(x)\;\equiv\;\hbar^2(G_4^{(g)})^{ba\dot\gamma\dot\delta}_{cd\beta\alpha}(x',x)\;, \tag{2.18}$$

which contains all Feynman diagrams connecting a triple fermion vertex with another fermion. We note that both $G_2^{(g)}$ and $G_4^{(g)}$ include diagrams with fermion loops generated by the four-fermion coupling. Expanding in powers of g and taking into account the ψ^4 interaction in S_{SUSY}, we encounter fermion loop diagrams in the graphical expansion of the coupling flow operator and hence of the Nicolai map.

3 Supersymmetric $\mathbb{C}\mathrm{P}^N$ model

Let us become more concrete and specialize to a maximally symmetric Kähler target, namely the complex projective space $\mathbb{C}\mathrm{P}^N \simeq \frac{\mathrm{SU}(N+1)}{\mathrm{SU}(N)\times\mathrm{U}(1)}$. It is embedded into $\mathbb{C}^{N+1}\ni\phi^a$ by the identification $\phi^a\sim\lambda\,\phi^a$ with $\lambda\in\mathbb{C}^*$, which yields the Kähler potential

$$K(\phi^*_a,\phi^a)\;=\;\frac{\mu^2}{g}\,\log\Big[\frac{g}{\mu^2}\,\phi^*\phi\Big]\qquad\text{with}\qquad\phi^*\phi\equiv\phi^*_a\phi^a\;, \tag{3.1}$$

where g is a dimensionless coupling, and μ is a mass parameter accounting for the dimensionality of ϕ.[6] The superfield extension identifies $\Phi^a\sim\Lambda\,\Phi^a$ with a complex chiral superfield Λ. For later use, we introduce the abbreviations

$$f_g^{-1}\;:=\;\tfrac{g}{\mu^2}\,\phi^*\phi\qquad\qquad\text{and}\qquad\qquad\Pi^b_a\;:=\;\delta^b_a-\tfrac{\phi^b\phi^*_a}{\phi^*\phi}\;, \tag{3.2}$$

in terms of which the derivatives of K are as follows,

$$K^b_a=f_g\,\Pi^b_a\,,\qquad K^{bc}_a=-2f_g^2\tfrac{g}{\mu^2}\,\phi^{(b}\Pi^{c)}_a\,,\qquad K^{cd}_{ab}=4f_g^3\tfrac{g^2}{\mu^4}\,\phi^{(c}\Pi^{d)}_{(a}\phi^*_{b)}-2f_g^2\tfrac{g}{\mu^2}\,\Pi^{(c}_{(a}\Pi^{d)}_{b)}\;. \tag{3.3}$$

We remark that $\Pi^b_a\,\phi^a=0=\phi^*_b\,\Pi^b_a$ implements the projection transversal to the 'radial direction' of the identification $\phi^a\sim\lambda\,\phi^a$.

As it stands, this Kähler potential is singular in the $g\to0$ limit. In order to set up a perturbation theory around flat $\mathbb{C}^N$, we need to select a point on $\mathbb{C}\mathrm{P}^N$ and pick coordinates $\phi^A\in\mathbb{C}^N$ centered around it. An appropriate superfield choice fixing the above redundancy is

$$\Phi^0=\sqrt{\tfrac{\mu^2}{g}}\qquad\Leftrightarrow\qquad\big\{\phi^0,\psi^0,F^0\big\}\;=\;\big\{\sqrt{\tfrac{\mu^2}{g}},0,0\big\}\;, \tag{3.4}$$

[6]There really is only one (dimensionful) parameter $M^2=\mu^2/g$; we introduce the dimensionless coupling g only for later convenience. Also, μ^2 need not be positive.

which leaves us with N proper (super)coordinates

$$\Phi^A \;=\; \phi^A + \mathrm{i}\theta\sigma^m\bar\theta\,\partial_m\phi^A + \tfrac14\theta^2\bar\theta^2\,\Box\phi^A + \sqrt2\theta\,\psi^A - \tfrac{\mathrm{i}}{\sqrt2}\theta^2\partial_m\psi^A\sigma^m\bar\theta + \theta^2 F^A \tag{3.5}$$

and a well-behaved Kähler potential

$$K(\phi^*_A,\phi^A) \;=\; \frac{\mu^2}{g}\,\log\Big[1+\frac{g}{\mu^2}\,\hat\phi^*\hat\phi\Big] \;=\; \hat\phi^*\hat\phi \;-\; \tfrac12\tfrac{g}{\mu^2}\big(\hat\phi^*\hat\phi\big)^2 +\; O(g^2) \qquad \text{with} \qquad \hat\phi^*\hat\phi \equiv \phi^*_A\phi^A\;. \tag{3.6}$$

Observing that (3.4) effectively reduces the summation ranges in (2.9) and (2.11) to $1,\dots,N$, i.e. $a,b,\ldots \to A,B,\ldots$, we only need to insert

$$K_A^B = f_g\,\Pi_A^B\;,\quad K_A^{BC} = -2f_g^2\tfrac{g}{\mu^2}\,\phi^{(B}\Pi_A^{C)}\;,\quad K_{AB}^{CD} = 4f_g^3\tfrac{g^2}{\mu^4}\,\phi^{(C}\Pi_{(A}^{D)}\phi^*_{B)} - 2f_g^2\tfrac{g}{\mu^2}\,\Pi_{(A}^{(C}\Pi_{B)}^{D)} \tag{3.7}$$

with $f_g^{-1} = 1+\frac{g}{\mu^2}\,\hat\phi^*\hat\phi$ and

$$\Pi_A^B \;=\; \delta_A^B - f_g\,\tfrac{g}{\mu^2}\,\phi^B\phi^*_A \qquad\Rightarrow\qquad \Pi_A^B\,\phi^A = f_g\,\phi^B \qquad \text{and} \qquad \phi^*_B\,\Pi_A^B = f_g\,\phi^*_A\;. \tag{3.8}$$

The elimination of the auxiliary F via (2.12) commutes with the coordinate choice for $\mathcal{M}$,

$$\begin{aligned}\mathcal{M} \;&=\; -\mathrm{i}\sqrt2\,K_A^B\;\partial\!\!\!/\phi^A\,\bar\psi_B \;+\; \sqrt2\,K_A^{BC}\;\psi^A\;\bar\psi_B\bar\psi_C \\ &=\; -\mathrm{i}\sqrt2\,f_g\,\Pi_A^B\,\partial\!\!\!/\phi^A\,\bar\psi_B \;-\; 2\sqrt2\,f_g^2\tfrac{g}{\mu^2}\,\phi^C\,\Pi_A^B\,\psi^A\;\bar\psi_B\bar\psi_C\;,\end{aligned} \tag{3.9}$$

but not so for the lagrangian (2.9) because (2.14) requires invertibility of the Kähler metric K_a^b, which is degenerate in the redundant coordinates. However, after choosing (3.4) we simply obtain

$$K_A^B\,F^*_B \;=\; \tfrac12 K_A^{CD}\;\bar\psi_C\bar\psi_D \qquad\Rightarrow\qquad F^A \;=\; \tfrac12(K^{-1})_B^A\,K_{CD}^B\;\psi^C\psi^D \quad \text{with} \quad (K^{-1})_B^A\,K_C^B = \delta_C^A \tag{3.10}$$

with an inverse K^{-1} on $\mathbb{C}\mathrm{P}^N$ and hence a four-fermion interaction

$$\tfrac14 R_{CD}^{AB}\;\bar\psi_A\bar\psi_B\;\psi^C\psi^D \qquad \text{with} \qquad R_{CD}^{AB} \;=\; K_{CD}^{AB} - K_{CD}^R(K^{-1})_R^S K_S^{AB}\;. \tag{3.11}$$

The $\mathbb{C}\mathrm{P}^N$ Riemann tensor computes to (c.f. (3.7))

$$R_{CD}^{AB} \;=\; -2f_g^2\tfrac{g}{\mu^2}\,\Pi_{(C}^{(A}\Pi_{D)}^{B)} \;=\; -\tfrac{g^2}{\mu^4}\big(K_C^AK_D^B + K_D^AK_C^B\big)\;. \tag{3.12}$$

Working out the details one arrives at

$$\begin{aligned}\mathcal{L} \;=\; & f_g\,\Pi_A^B\big[-\partial_m\phi^A\,\partial^m\phi^*_B - \tfrac{\mathrm{i}}{2}(\psi^A\partial\!\!\!/\bar\psi_B + \bar\psi_B\bar\partial\!\!\!/\psi^A) + \tfrac{\mathrm{i}}{2}\,f_g\tfrac{g}{\mu^2}\,(\psi^A\phi^C\partial\!\!\!/\phi^*_B\bar\psi_C + \bar\psi_B\phi^*_C\bar\partial\!\!\!/\phi^A\psi^C)\big] \\ & +\tfrac{\mathrm{i}}{2}\,f_g^2\tfrac{g}{\mu^2}\,\Pi_A^B\,\psi^A(\phi^C\partial\!\!\!/\phi^*_C - \phi^*_C\partial\!\!\!/\phi^C)\,\bar\psi_B + \tfrac14\,f_g^2\tfrac{g}{\mu^2}\,[\psi^C\sigma_m\Pi_C^A\bar\psi_A]\,[\psi^D\sigma^m\Pi_D^B\bar\psi_B]\;,\end{aligned} \tag{3.13}$$

where we have employed the Fierz identity

$$[\bar\psi_A\bar\psi_B]\,[\psi^C\psi^D] \;=\; -\tfrac12[\psi^C\sigma_m\bar\psi_A]\,[\psi^D\sigma^m\bar\psi_B]\;. \tag{3.14}$$

We now have all the ingredients for constructing the Nicolai map. Remembering that

$$\delta_\alpha\phi^A = \sqrt2\,\psi_\alpha^A \quad,\qquad \delta_\alpha\phi^*_A = 0 \quad,\qquad \bar\delta^{\dot\alpha}\phi^A = 0 \quad,\qquad \bar\delta^{\dot\alpha}\phi^*_A = \sqrt2\,\bar\psi_A^{\dot\alpha} \tag{3.15}$$

the coupling-flow operator takes the form

$$\begin{aligned}R_g \;=\; & -\tfrac{\mathrm{i}}{4\hbar}\sqrt2\int\!\mathrm{d}y\!\int\!\mathrm{d}x\;\Big\{\tfrac{1+\kappa}{2}\,\partial_g\mathcal{M}_\alpha(y)\;\psi^{A\alpha}(x)\,\tfrac{\delta}{\delta\phi^A(x)} \;+\; \tfrac{1-\kappa}{2}\,\partial_g\bar{\mathcal{M}}^{\dot\alpha}(y)\;\bar\psi_{A\dot\alpha}(x)\,\tfrac{\delta}{\delta\phi^*_A(x)}\Big\} \\ =\; & -\tfrac{1}{2\hbar}\int\!\mathrm{d}y\!\int\!\mathrm{d}x\;\tfrac{1+\kappa}{2}\,\Big\{\partial_g(f_g\,\Pi_A^B)\,\partial\!\!\!/_{\alpha\dot\alpha}\phi^A(y)\,\bar\psi_B^{\dot\alpha}(y)\;\psi^{D\alpha}(x)\,\tfrac{\delta}{\delta\phi^D(x)} \\ & \qquad -2\mathrm{i}\,\partial_g(f_g^2\tfrac{g}{\mu^2}\,\Pi_A^B)\,\phi^C(y)\,\psi_\alpha^A(y)\,\bar\psi_{B\dot\alpha}(y)\,\bar\psi_C^{\dot\alpha}(y)\;\psi^{D\alpha}(x)\,\tfrac{\delta}{\delta\phi^D(x)}\Big\} \\ & -\tfrac{1}{2\hbar}\int\!\mathrm{d}y\!\int\!\mathrm{d}x\;\tfrac{1-\kappa}{2}\,\Big\{\partial_g(f_g\,\Pi_A^B)\,\bar\partial\!\!\!/^{\dot\alpha\alpha}\,\phi^*_B(y)\,\psi_\alpha^A(y)\;\bar\psi_{D\dot\alpha}(x)\,\tfrac{\delta}{\delta\phi^*_D(x)} \\ & \qquad -2\mathrm{i}\,\partial_g(f_g^2\tfrac{g}{\mu^2}\,\Pi_A^B)\,\phi^*_C(y)\,\bar\psi_B^{\dot\alpha}(y)\,\psi^{A\alpha}(y)\,\psi_\alpha^C(y)\;\bar\psi_{D\dot\alpha}(x)\,\tfrac{\delta}{\delta\phi^*_D(x)}\Big\}\end{aligned} \tag{3.16}$$

with full fermionic correlators indicated by the contractions, see (2.17) and (2.18).

Let us take a look at first order in the coupling g,

$$(T_g\phi)^A(x) \;=\; \phi^A(x) \;-\; g\,(R_g\phi)|^A_{g=0}(x) \;+\; O(g^2)\;. \tag{3.17}$$

We compute

$$\begin{aligned}
(R_g\phi)|^A_{g=0}(x) \;&=\; \tfrac{1+\kappa}{4\hbar\mu^2}\int \mathrm{d}y\;\Big\{\phi^*_B(\phi^B\,\partial\!\!\!/_{\alpha\dot\alpha}\phi^C+\phi^C\,\partial\!\!\!/_{\alpha\dot\alpha}\phi^B)\,\bar\psi^{\dot\alpha}_C(y)\;\psi^{A\alpha}(x)+2\mathrm{i}\,\phi^C\,\psi^B_\alpha\,\bar\psi_{B\dot\alpha}\,\bar\psi^{\dot\alpha}_C(y)\;\psi^{A\alpha}(x)\Big\}\;,\\
(R_g\phi^*)|^{g=0}_A(x) \;&=\; \tfrac{1-\kappa}{4\hbar\mu^2}\int \mathrm{d}y\;\Big\{\phi^B(\phi^*_B\,\bar\partial\!\!\!/^{\dot\alpha\alpha}\phi^*_C+\phi^*_C\,\bar\partial\!\!\!/^{\dot\alpha\alpha}\phi^*_B)\,\psi^C_\alpha(y)\;\bar\psi_{A\dot\alpha}(x)+2\mathrm{i}\,\phi^*_C\,\bar\psi^{\dot\alpha}_B\,\psi^{B\alpha}\,\psi^C_\alpha(y)\;\bar\psi_{A\dot\alpha}(x)\Big\}\;,
\end{aligned} \tag{3.18}$$

where now the contractions are free-field ones,

$$\begin{aligned}
\sigma^m_{\alpha\dot\alpha}\;\bar\psi^{\dot\alpha}_C(y)\;\psi^{A\alpha}(x) \;&=\; -2\hbar\,\delta^A_C\,\partial^m\Box^{-1}(y{-}x) \;=\; \bar\sigma^{m\dot\alpha\alpha}\;\psi^A_\alpha(y)\;\bar\psi_{B\dot\alpha}(x)\;,\\
\psi^B_\alpha\;\bar\psi_{B\dot\alpha}\;\bar\psi^{\dot\alpha}_C(y)\;\psi^{A\alpha}(x) \;&=\; \hbar^2(N{+}1)\,\delta^A_C\,\partial_m\Box^{-1}(y{-}z)\,\partial^m\Box^{-1}(y{-}x)\big|_{z=y}\;,
\end{aligned} \tag{3.19}$$

and arrive at the classical map

$$\begin{aligned}
(T^{(0)}_g\phi)^A(x) \;&=\; \phi^A(x) \;+\; \tfrac{1+\kappa}{2}\,\tfrac{g}{\mu^2}\int \mathrm{d}y\;\phi^*_B\,\partial_m(\phi^B\phi^A)(y)\,\partial^m\Box^{-1}(y{-}x) \;+\; O(g^2)\;,\\
(T^{(0)}_g\phi^*)_A(x) \;&=\; \phi^*_A(x) \;+\; \tfrac{1-\kappa}{2}\,\tfrac{g}{\mu^2}\int \mathrm{d}y\;\phi^B\,\partial_m(\phi^*_B\phi^*_A)(y)\,\partial^m\Box^{-1}(y{-}x) \;+\; O(g^2)\;,
\end{aligned} \tag{3.20}$$

while the four-fermion contraction yields the leading (or one-loop) quantum contribution

$$\begin{aligned}
(T^{(1)}_g\phi)^A(x) \;&=\; -\mathrm{i}\tfrac{g}{\mu^2}\tfrac{1+\kappa}{2}(N{+}1)\int \mathrm{d}y\;\phi^A(y)\,\partial_m\Box^{-1}(y{-}z)\,\partial^m\Box^{-1}(y{-}x)\big|_{z=y} \;+\; O(g^2)\;,\\
(T^{(1)}_g\phi^*)_A(x) \;&=\; -\mathrm{i}\tfrac{g}{\mu^2}\tfrac{1-\kappa}{2}(N{+}1)\int \mathrm{d}y\;\phi^*_A(y)\,\partial_m\Box^{-1}(y{-}z)\,\partial^m\Box^{-1}(y{-}x)\big|_{z=y} \;+\; O(g^2)\;,
\end{aligned} \tag{3.21}$$

The generalized free-action condition in (1.6) means that

$$S^{\mathrm{b}}_0[T^{(0)}_g\phi] \;=\; -\int \partial_m(T_g\phi)^A\,\partial^m(T_g\phi^*)_A \;\stackrel{!}{=}\; -\int f_g\,\Pi^B_A\,\partial_m\phi^A\,\partial^m\phi^*_B \;=\; S^{\mathrm{b}}_g[\phi]\;, \tag{3.22}$$

which is met to first order in g because both sides are equal to

$$-\int\Big\{\partial_m\phi^A\,\partial^m\phi^*_A \;-\; \tfrac{g}{\mu^2}\,\phi^*_B\,(\phi^B\,\partial_m\phi^A+\phi^A\,\partial_m\phi^B)\,\partial^m\phi^*_A \;+\; O(g^2)\Big\} \tag{3.23}$$

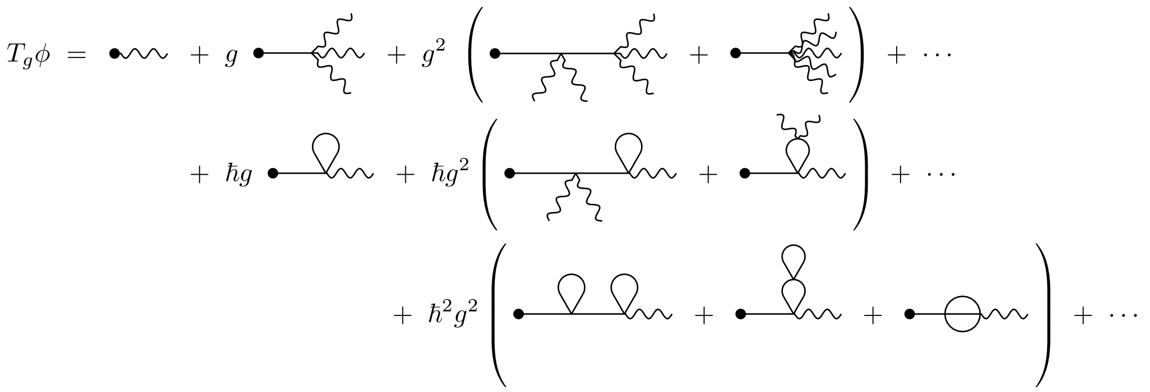
and κ cancels out. For the one-loop matching in (1.6) we obtain the $O(g)$ contributions

$$\begin{aligned}
S^{\mathrm{b}}_0[T_g\phi]\big|_{O(\hbar)} \;&=\; \mathrm{i}\tfrac{g}{\mu^2}(N{+}1)\int\Big\{\tfrac12\phi^A\phi^*_A\,\delta(0) \;+\; \tfrac{\kappa}{2}\big(\phi^A\Box\phi^*_A-\phi^*_A\Box\phi^A\big)\,\Box^{-1}(0)\Big\}\;,\\
-\mathrm{i}\;\mathrm{tr}\ln\tfrac{\delta T^{(0)}_g\phi}{\delta\phi} \;&=\; \mathrm{i}\tfrac{g}{\mu^2}(N{+}1)\int\Big\{\tfrac12\phi^A\phi^*_A\,\delta(0) \;-\; \tfrac{\kappa}{2}\big(\phi^A\Box\phi^*_A-\phi^*_A\Box\phi^A\big)\,\Box^{-1}(0)\Big\}\;,\\
S^{(1)}_g[\phi] \;&=\; \mathrm{i}\tfrac{g}{\mu^2}(N{+}1)\int\Big\{\phi^A\phi^*_A\,\delta(0)\Big\}\;,
\end{aligned} \tag{3.24}$$

which verifies the condition again with the expected cancellation of κ. In (3.24) we use the regularization-dependent quantities $\delta(0)\equiv\delta^{(4)}(x{-}x)$ and $\Box^{-1}(0)\equiv\Box^{-1}(x{-}x)$, which contribute to a mass shift and wavefunction renormalization, respectively.[7] We kept them in this form to maintain the discussion as general as possible and to illustrate the matching at first order.

[7] In dimensional regularization both contributions vanish in a massless theory such as the one under consideration. With a UV cutoff they are quartically and quadratically divergent, respectively.

By choosing $\kappa{=}{+}1$ or $\kappa{=}{-}1$ we have the freedom to shift the Nicolai map entirely to ϕ^A or ϕ^*_A alone, respectively. A graphical representation of the Nicolai map to order g^2 looks as follows,



$$+\; O(g^3)\;. \tag{3.25}$$

Here, the thick dot at the left end of each diagram stands for the argument x of the map, other vertex positions are integrated over. Solid lines are free fermion propagators $\partial\!\!\!/\,\Box^{-1}$ or $\bar{\partial\!\!\!/}\,\Box^{-1}$, and wavy lines represent bosonic field insertions ϕ or ϕ^*. One of the bosonic legs emanating from each vertex not sourcing a loop carries a derivative (not shown). For the full 'Nicolai rules', one of course needs to add target-space indices, spinor traces, and weight factors. All diagrams shown above already appear in the first application of R_g on ϕ. We see that in the $\hbar$ expansion of the map an r-loop contribution arises first at order g^r, so that at each given order in perturbation theory only a finite number of diagrams contribute, as expected.

4 Adding an auxiliary vector field

In some field theories with four-fermion interactions one can 'resolve' the latter through a coupling with an auxiliary field A: the Hubbard–Stratonovich transformation. Schematically, one adds to an action with a $(\bar\psi\psi)^2$ term an auxiliary-field coupling $(\bar\psi\psi - A)^2$, schematically

$$\bar\psi\,\mathrm{i}\partial\psi + \tfrac14 g(\bar\psi\psi)^2 \qquad\longrightarrow\qquad \bar\psi\,\mathrm{i}\partial\psi + \tfrac12 g\,A\,\bar\psi\psi - \tfrac14 gA^2 \;=\; \bar\psi\,\mathrm{i}(\partial - \tfrac{\mathrm{i}}{2}gA)\psi - \tfrac14 gA^2\;. \tag{4.1}$$

The four-fermion term has been cancelled, but eliminating A brings it back. Hence, the only price is an additional auxiliary field (or several of them). Filling in the indices in our schematic argument and allowing for the fierzing (3.14), we see that the transformation requires our four-fermion term (3.11) to be 'factorizable', i.e.

$$R^{AB}_{CD} \;=\; \lambda_{\mathrm{s}}\,(\bar m_i)^{AB}\,(m^i)_{CD} \;+\; \lambda_{\mathrm{v}}\,(\ell_I)^{A}_{(C}\,(\ell^I)^{B}_{D)} \tag{4.2}$$

with some coefficients λ_{s} and λ_{v} and indices i and I counting several such terms. For target geometries with this property we can remove the four-fermion interaction by introducing a bunch of complex scalar and real vector auxiliary fields

$$A^i \;=\; (m^i)_{CD}\,\psi^C\psi^D \qquad\text{and}\qquad A^I_m \;=\; (\ell^I)^B_D\,\psi^D\sigma_m\bar\psi_B\;, \tag{4.3}$$

respectively.

This is actually the case for all hermitian symmetric spaces [17, 18].[8] Therefore, these geometries allow for an auxiliary-field reformulation. The complex projective spaces $\mathbb{C}\mathrm{P}^N$ treated above are the maximally symmetric compact examples, and indeed (3.12) shows that a single real vector auxiliary A_m suffices. Actually, the Hubbard–Stratonovich trick can be slightly generalized by shifting A by an arbitrary function of

[8] See [19] for a more general review on nonlinear realization and hidden local symmetries.

bosonic fields. We make use of this option and choose

$$A_m \;=\; \tfrac{1}{\mu^2}\, f_g \left[\mathrm{i}\,\phi^C \partial_m \phi^*_C - \mathrm{i}\,\phi^*_C\, \partial_m \phi^C + \psi^A\, \Pi_A^B\, \sigma_m\, \bar\psi_B\right] , \tag{4.4}$$

where the form of the (arbitrary) bosonic contribution will be justified later on. With some algebra this yields the enhanced lagrangian

$$\begin{aligned}\widetilde{\mathcal{L}} \;=\; &-f_g\, \Pi_A^B \partial_m \phi^A \partial^m \phi^*_B + \tfrac14 f_g^2 \tfrac{g}{\mu^2} (\phi^A \partial_m \phi^*_A - \phi^*_A \partial_m \phi^A)^2 + \tfrac{\mathrm{i}}{2} f_g\, g\, (\phi^A \partial_m \phi^*_A - \phi^*_A \partial_m \phi^A)\, A^m - \tfrac14 g \mu^2 A_m A^m \\ &- \tfrac{\mathrm{i}}{2} f_g\, \Pi_A^B (\psi^A \partial\!\!\!/ \bar\psi_B + \bar\psi_B \bar{\partial\!\!\!/} \psi^A) - \tfrac12 f_g\, g\, \Pi_A^B\, \psi^A A\!\!\!/\, \bar\psi_B + \tfrac{\mathrm{i}}{2} f_g^2 \tfrac{g}{\mu^2}\, \Pi_A^B (\psi^A \phi^C \partial\!\!\!/ \phi^*_B \bar\psi_C + \bar\psi_B \phi^*_C \bar{\partial\!\!\!/} \phi^A \psi^C) \,. \end{aligned} \tag{4.5}$$

There exists an instructive superfield formulation [17, 18],

$$\widetilde{S}_{\text{SUSY}} \;=\; \int \mathrm{d}^4 x\; \widetilde{\mathcal{L}} \qquad \text{with} \qquad \widetilde{\mathcal{L}} \;=\; \int \mathrm{d}^2\theta\, \mathrm{d}^2\bar\theta\, \left\{ \mathrm{e}^{gV}\; \Phi^\dagger_a \Phi^a \;-\; \mu^2\, V \right\} , \tag{4.6}$$

and corresponding expressions for $\widetilde{\Delta}$ and $\widetilde{\mathcal{M}}$, where the auxiliary real vector superfield V with components $(C, L, A_m, \lambda, \chi, D)$ expands in x coordinates as

$$V \;=\; -C - \mathrm{i}\theta\chi + \mathrm{i}\bar\theta\bar\chi - \tfrac{\mathrm{i}}{2}\theta^2 L + \tfrac{\mathrm{i}}{2}\bar\theta^2 L^* + \theta\sigma^m\bar\theta A_m - \mathrm{i}\theta^2\bar\theta\big[\bar\lambda + \tfrac{\mathrm{i}}{2}\bar{\partial\!\!\!/}\chi\big] + \mathrm{i}\bar\theta^2\theta\big[\lambda + \tfrac{\mathrm{i}}{2}\partial\!\!\!/\bar\chi\big] - \tfrac12\theta^2\bar\theta^2\big[D + \tfrac12 \Box C\big] \,. \tag{4.7}$$

The action (4.6) enjoys a complexified local U(1) invariance under

$$V \;\mapsto\; V \,+\, \mathrm{i}\Lambda \,-\, \mathrm{i}\Lambda^\dagger \qquad\qquad \text{and} \qquad\qquad \Phi^a \;\mapsto\; \mathrm{e}^{-\mathrm{i}g\Lambda}\, \Phi^a \tag{4.8}$$

with a chiral superfield parameter Λ. The coordinate choice (3.4) is not compatible with the Wess–Zumino gauge, and it completely breaks the gauge symmetry. Indeed, eliminating the vector superfield by its algebraic equation of motion,

$$V \;=\; -\frac{1}{g}\, \log\Big[\frac{g}{\mu^2}\Phi^\dagger_a \Phi^a\Big] \;=\; -\frac{1}{\mu^2}\mathrm{K}[\Phi^a, \Phi^\dagger_a] \,, \tag{4.9}$$

brings back the original action (2.3) with the $\mathbb{C}\mathrm{P}^N$ Kähler potential (3.1) (up to a constant). We can interpret the action (4.6) and the gauge transformation (4.8) as a way of performing the $\mathbb{C}\mathrm{P}^N$ identification of scalar superfields under complex chiral parameters: the supergauge transformation realizes in a supersymmetric fashion a complex $\mathrm{U}(1) \ni e^{i\alpha(x)+\beta(x)}$ local transformation which extends the standard gauge invariance to a symmetry under conformal rescaling.

Let us go now to the component level and eliminate auxiliary fields F, L, χ, λ, C and D from the action by their equations of motion, e.g.

$$F^a \;=\; -\psi^a\, \tfrac{\psi^b\phi^*_b}{\phi^*\phi} \quad , \qquad \chi = -\tfrac{\mathrm{i}\sqrt{2}}{g}\, \tfrac{\psi^a\phi^*_a}{\phi^*\phi} \quad , \qquad g\, \mathrm{e}^{-gC}\, \phi^*\phi \;=\; \mu^2 \,. \tag{4.10}$$

After lengthy but straightforward computations we arrive at

$$\begin{aligned}\widetilde{\mathcal{L}} \;=\; & f_g\, \Pi_a^b \Big\{ -D_m\phi^a D^m \phi^*_b - \tfrac{\mathrm{i}}{2}\big[\psi^a D\!\!\!\!/\, \bar\psi_b + \bar\psi_b \bar{D\!\!\!\!/}\, \psi^a\big] + \tfrac{\mathrm{i}}{2}\tfrac{1}{\phi^*\phi}\big[\psi^a\phi^c D\!\!\!\!/\, \phi^*_b \bar\psi_c + \bar\psi^b \phi^*_c \bar{D\!\!\!\!/}\, \phi^a \psi^c\big]\Big\} \\ &+ \tfrac14 f_g^2 \tfrac{g}{\mu^2}\, (\phi^a D_m \phi^*_a - \phi^*_a D_m \phi^a)^2 \end{aligned} \tag{4.11}$$

where $D_m = \partial_m - \tfrac{\mathrm{i}}{2}\, g\, A_m$ is a U(1)-covariant derivative. We note that $\Pi_a^b D_m \phi^a = \Pi_a^b \partial_m \phi^a$. For the penultimate components we find

$$\widetilde{\mathcal{M}} \;=\; -\mathrm{i}\sqrt{2}\, f_g\, \Pi_a^b\, D\!\!\!\!/\, \phi^a\, \bar\psi_b + \mathrm{i}\sqrt{2} f_g^2 \tfrac{g}{\mu^2}\, (\phi^b D\!\!\!\!/\, \phi^*_b - \phi^*_b D\!\!\!\!/\, \phi^b)\, \phi^a \bar\psi_a \,. \tag{4.12}$$

Both $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{M}}$ are manifestly gauge invariant. We are left with the fields ϕ^a, ψ^a and A_m. Eliminating the latter via its equation of motion

$$\mu^2 A_m \;=\; f_g \big[\mathrm{i}\phi^c \partial_m \phi^*_c - \mathrm{i}\phi^*_c \partial_m \phi^c + \Pi_a^b\, \psi^a \sigma_m \bar\psi_b\big] \qquad \Leftrightarrow \qquad \mathrm{i}\phi^c D_m \phi^*_c - \mathrm{i}\phi^*_c D_m \phi^c \;=\; -\Pi_a^b\, \psi^a \sigma_m \bar\psi_b \tag{4.13}$$

reverts to the four-fermion interaction, so we keep A_m in the lagrangian. Notice that the value of A_m in (4.13) is exactly (4.4).

Instead, we now employ the local supersymmetric $\mathrm{U}(1)_{\mathbb{C}}$ invariance to fix one of the chiral superfields, $\Phi^0=\sqrt{\frac{\mu^2}{g}}$, which explicitly connects the auxiliary-superfield formulation with our coordinate choice (3.4) for the nonlinear sigma model. It is not hard to see that this gauge fixing comes with a trivial Faddeev–Popov determinant. The gauge-fixed lagrangian then indeed agrees with (4.5). Using the identities $\Pi^B_A\phi^A=f_g\phi^B$ and $\frac{g}{\mu^2}\phi^A\phi^*_A=f_g^{-1}-1$ as well as

$$\begin{aligned} \phi^c D\phi^*_c-\phi^*_c D\phi^c &= \phi^C\partial\phi^*_C-\phi^*_C\partial\phi^C+\mathrm{i}f_g^{-1}\mu^2 A\ , \\ \Pi^b_a D\phi^a\bar\psi_b &= \Pi^B_A D\phi^A\bar\psi_B+\tfrac{\mathrm{i}}{2}f_g\,g\,A\,\phi^A\bar\psi_A\ , \end{aligned} \tag{4.14}$$

we also obtain

$$\widetilde{\mathcal{M}} \;=\; -\mathrm{i}\sqrt{2}f_g\,\Pi^B_A\,\not\partial\phi^A\,\bar\psi_B+\mathrm{i}\sqrt{2}f_g^2\tfrac{g}{\mu^2}\,(\phi^C\not\partial\phi^*_C-\phi^*_C\not\partial\phi^C)\,\phi^A\bar\psi_A-\sqrt{2}f_g\,g\,\not{A}\,\phi^A\bar\psi_A\ . \tag{4.15}$$

Again, we can set up the Nicolai map. Employing

$$\delta_\alpha Y \;=\; \sqrt{2}\,\psi_\alpha\,\tfrac{\delta Y}{\delta\phi}\;+\;\delta^\alpha A_m\,\tfrac{\delta Y}{\delta A_m}\ , \tag{4.16}$$

with a rather complicated expression for $\delta^\alpha A_m$, which we shall not reproduce here, and

$$\tfrac{\mu^2}{\sqrt{2}}\,\partial_g\widetilde{\mathcal{M}}_\alpha \;=\; \mathrm{i}f_g^2\phi^B\phi^*_B\,\not\partial_{\alpha\dot\alpha}\phi^A\,\bar\psi^{\dot\alpha}_A+\mathrm{i}f_g^3(1-\tfrac{g}{\mu^2}\phi^C\phi^*_C)\,\phi^B\not\partial_{\alpha\dot\alpha}\phi^*_B\,\phi^A\bar\psi^{\dot\alpha}_A-\mu^2 f_g^2\not{A}_{\alpha\dot\alpha}\phi^A\bar\psi^{\dot\alpha}_A\ , \tag{4.17}$$

we may compose the coupling flow operator

$$\widetilde{R}_g \;=\; -\tfrac{\mathrm{i}}{4}\tfrac{1+\kappa}{2}\int\!\mathrm{d}y\;\partial_g\underbracket{\widetilde{\mathcal{M}}(y)_\alpha\;\delta^\alpha}\;-\;\tfrac{\mathrm{i}}{4}\tfrac{1-\kappa}{2}\int\!\mathrm{d}y\;\partial_g\underbracket{\bar{\widetilde{\mathcal{M}}}(y)^{\dot\alpha}\;\bar\delta_{\dot\alpha}}\ , \tag{4.18}$$

with the contractions now being defined in a (ϕ,A) background.

It appears that we can arrive at a purely classical Nicolai map, in the sense described in the Introduction. After all, the fermions appear just quadratically in the lagrangian (4.11). To leading order in the coupling g, it looks as follows,

$$(T_g\phi)^A(x) \;=\; \phi^A(x)\;-\;\tfrac{1+\kappa}{2}\tfrac{g}{\mu^2}\int\!\mathrm{d}y\;\big[\phi^B(\phi^*_B\partial_m\phi^A-\phi^A\partial_m\phi^*_B)+\mathrm{i}\mu^2A_m\phi^A\big](y)\,\partial^m\Box^{-1}(y-x)+O(g^2)\ , \tag{4.19}$$

and similarly for $(T_gA)_m$. Checking the free-action condition

$$\begin{aligned} \int\partial_m(T_g\phi)^A\,\partial^m(T_g\phi^*)_A \;\stackrel{!}{=}\; \int\big\{ & f_g\Pi^B_A\,\partial_m\phi^A\,\partial^m\phi^*_B-\tfrac14 f_g^2\tfrac{g}{\mu^2}(\phi^C\partial_m\phi^*_C-\phi^*_C\partial_m\phi^C)^2 \\ & -\tfrac{\mathrm{i}}{2}f_g\,g\,(\phi^C\partial_m\phi^*_C-\phi^*_C\partial_m\phi^C)\,A^m+\tfrac14 g\mu^2A_mA^m\big\} \end{aligned} \tag{4.20}$$

we obtain the first-order requirement

$$\begin{aligned} &\int\Big\{(\phi^*\phi)(\partial\phi^*\!\cdot\partial\phi)+\tfrac{1+\kappa}{2}(\phi\,\partial\phi^*)\!\cdot\!(\phi\,\partial\phi^*)+\tfrac{1-\kappa}{2}(\phi^*\partial\phi)\!\cdot\!(\phi^*\partial\phi)+\tfrac{1+\kappa}{2}\mathrm{i}\mu^2A\!\cdot\!(\phi\,\partial\phi^*)-\tfrac{1-\kappa}{2}\mathrm{i}\mu^2A\!\cdot\!(\phi^*\partial\phi)\Big\} \\ &\quad\stackrel{!}{=}\;\int\Big\{(\phi^*\phi)(\partial\phi^*\!\cdot\partial\phi)+(\phi^*\partial\phi)\!\cdot\!(\phi\,\partial\phi^*)-\tfrac14\big[\mu^2A-\mathrm{i}(\phi\,\partial\phi^*-\phi^*\partial\phi)\big]\!\cdot\!\big[\mu^2A-\mathrm{i}(\phi\,\partial\phi^*-\phi^*\partial\phi)\big]\Big\}\ , \end{aligned} \tag{4.21}$$

where the dots indicate Lorentz contractions, and the round brackets enclose target coordinate contractions. We observe that the left-hand side is linear in A while the right-hand side is quadratic. Matching both sides (and cancelling κ) requires putting

$$\mu^2A_m \;\stackrel{!}{=}\; \mathrm{i}\,\big(\phi^C\partial_m\phi^*_C-\phi^*_C\partial_m\phi^C\big)\;+\;O(g)\;+\;O(\hbar)\ , \tag{4.22}$$

which agrees with the equation of motion (4.13) after integrating out the fermions! The crux of the mismatch, however, lies in the absence of A in the free action, which renders the $g \to 0$ limit singular for the auxiliary field. In other words, the propagator for A is proportional to $\frac{1}{g}$, which upsets the perturbative expansion in g (not in powers of the fields) of correlation functions, with or without the Nicolai map. The remedy is to integrate out the auxiliary vector A and work with an effective theory of ϕ alone. This, however, revives the four-fermion interaction: the classical solution (4.13) shows that A yields a fermion loop, and the ultralocal propagator $\langle A(x)\, A(y)\rangle \sim \delta(x{-}y)$ glues two such loops together, effectively reproducing the four-fermion interaction. Therefore, the auxiliary-field reformulation of the supersymmetric nonlinear sigma models does not simplify the Nicolai map in the end.

Let us conclude with a comment on the $N{=}1$ case. We have the accidental isomorphism $\mathbb{C}\mathrm{P}^1 \simeq S^2$, the real 2-sphere, which is maximally symmetric in the real sense and whose Riemann tensor admits therefore the standard decomposition Riem $= gg - gg$ in real coordinates. In this case indices $A, B, \ldots$ have only one value and $\Pi \equiv f_g$. Correspondingly, the four-fermion term of the lagrangian (3.13) takes the form

$$\mathcal{L}_{\psi^4} \;=\; -\tfrac{1}{2} f_g^3 \tfrac{g}{\mu^2}\, \psi\psi\, \bar{\psi}\bar{\psi} \;, \tag{4.23}$$

which can be resolved by means of a standard Hubbard–Stratonovich auxiliary complex scalar $A \sim \psi\psi$. Of course, the clash with the perturbative expansion in g described in (4.20)-(4.21) applies also in this case.

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