Hamiltonian charges on light cones for linear field theories on (A)dS backgrounds*

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Abstract

We analyse the Noether charges for scalar and Maxwell fields on light cones on a de Sitter, Minkowski, and anti-de Sitter backgrounds. Somewhat surprisingly, under natural asymptotic conditions all charges for the Maxwell fields on both the de Sitter and anti-de Sitter backgrounds are finite. On the other hand, one needs to renormalise the charges for the conformally-covariant scalar field when the cosmological constant does not vanish. In both cases well-defined renormalised charges, with well-defined fluxes, are obtained. Again surprisingly, a Hamiltonian analysis of a suitably rescaled scalar field leads to finite charges, without the need to renormalise. Last but not least, we indicate natural phase spaces where the Poisson algebra of charges is well defined.

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1 Introduction and summary

In field theory it is commonplace to identify the total energy of a field configuration with the Hamiltonian charge, also known as the Noether charge, associated with time translations. Consider, then, a field theory on a Minkowski, de Sitter, or anti-de Sitter background. When the cosmological constant Λ is negative the notion of time translation is somewhat muddled by the fact that there are no globally timelike Killing vector fields. However, in all the above spacetimes, given a light-cone, there exists a family of Killing vectors which are timelike at its tip, generating flows which move isometrically the whole light-cone to its future. The associated Hamiltonian provides a good candidate for the definition of total energy contained in the light-cone; the resulting formula coincides with the usual definition of energy when the cosmological constant vanishes.

It should be kept in mind that the problem of real interest is the full nonlinear theory, including the gravitational field, in the presence of a cosmological constant. While some progress towards the understanding of that problem has been done [5, 7, 9, 13], there remain ambiguities which are far from understood. Therefore a systematic analysis of the simpler problem, of linear fields on a fixed background, appears in order.

In recent work [4] we analysed the Hamiltonian charge associated with a timelike translation of tips of light cones for the scalar field and the linearised gravitational field on the backgrounds just listed. Much to our surprise, we found that the charge integrals diverge when the cosmological constant does not vanish. We proposed a renormalisation procedure that led to finite charges, with well defined flux integrals. The aim of this work is to analyse similar charges associated with the flow of the remaining Killing vector fields for the conformally-covariant scalar field and for the Maxwell field on these backgrounds.

Somewhat surprisingly, we find that all resulting charges for the Maxwell field are finite. On the other hand, when $\Lambda \neq 0$ the charges for the scalar field need to be renormalised again, except for angular momentum where the divergent terms in the integrand integrate-out to zero on spheres. After renormalisation one obtains a well defined set of charges, with well defined flux formulae.

As a byproduct, we find an alternative Lagrangian for the scalar field which leads to finite charges, without need for renormalisation. The alternative Lagrangian depends explicitly upon the coordinates, and leads to different global charges. This raises the question of physical significance and relevance of the resulting expressions, and we do not have an answer for this. The point of view advocated by Kijowski [10, 11], that different energy expressions correspond to different sets of boundary conditions, does not seem to be helpful for radiating systems.

Given a full set of charges of the scalar field and the Maxwell field, it is tempting to enquire about their algebra. One is then faced with the problem of boundary terms in the variational formulae, which appear to obstruct a meaningful definition of a Poisson bracket. One way out is to work in phase-space sectors where the boundary terms vanish by choice of boundary conditions. But then the charges are defined only up to a functional which depends upon the boundary data, and there does not exist a clear principle to single-out a preferred one. Here we propose a simple solution, to extend the phase space to include the boundary degrees of freedom. For yet another proposal, see [14].

We now pass to a more detailed summary of our results.

1.1 Scalar fields

On Minkowski, de Sitter and Anti-de Sitter spacetime we consider a scalar field with Lagrangian

$$\mathscr{L} = -\frac{1}{2}\sqrt{|\det g|}\big(g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + \frac{2\Lambda}{3}\phi^2\big)\,, \tag{1.1}$$

where Λ is the cosmological constant. The mass term is chosen so that the resulting field equation is conformally covariant. We consider fields with the following asymptotic behaviour, for large r,

$$\phi(u,r,x^A) = \frac{\overset{\text{(-1)}}{\phi}(u,x^A)}{r} + \frac{\overset{\text{(-2)}}{\phi}(u,x^A)}{r^2} + \frac{\overset{\text{(-3)}}{\phi}(u,x^A)}{r^3} + \dots\,, \tag{1.2}$$

which can be justified by an analysis of the Cauchy problem for the field equations. Indeed, if the asymptotic expansion (1.2) is imposed on an initial light cone, it is preserved by evolution when $\Lambda \geq 0$. This is also the case when $\Lambda < 0$ after requiring that the associated solutions vanish at the conformal boundary at infinity.

Consider the Noether charge associated with a vector field X and a hypersurface $\mathscr{S}$ (we follow the formalism of [12]):

$$\mathscr{H}[X,\mathscr{S}] := \int_{\mathscr{S}} \big(\underbrace{\pi^\mu \mathcal{L}_X\phi - X^\mu\mathscr{L}}_{=:\mathscr{H}^\mu[X]}\big) dS_\mu \tag{1.3}$$

$$= \frac{1}{2}\left(\int_{\mathscr{S}} \omega^\mu(\phi,\mathcal{L}_X\phi)\, dS_\mu - \int_{\partial\mathscr{S}} X^{[\sigma}\pi^{\mu]}\phi dS_{\sigma\mu}\right)\,, \tag{1.4}$$

with

$$\pi^\mu = \frac{\partial \mathscr{L}}{\partial \phi_{,\mu}}\,, \qquad \omega^\mu(\phi, \mathcal{L}_X\phi) = \pi^\mu \mathcal{L}_X\phi - \phi\, \mathcal{L}_X\pi^\mu\,.$$

Here (1.3) is a definition while (1.4) is an identity for linear field theories, compare [4, Proposition 1]; the reader is referred to the text below for notation that has not been defined so far.

1.1.1 The energy

Let us denote by $\mathscr{C}_u$ the light cone of constant retarded time u. We use the symbol $\mathscr{C}_{u,R}$ to denote the truncation of $\mathscr{C}_u$ to $r \le R$:

$$\mathscr{C}_{u,R} = \mathscr{C}_u \cap \{r \le R\}\,.$$

It turns out that the Noether charge on $\mathscr{C}_{u,R}$, associated with translations in u, diverges as R tends to infinity. A direct analysis of the integrand gives (cf. (4.25) below)

$$\begin{aligned} E_{\mathscr{H}}[\mathscr{C}_{u,R}] &:= \int_{\mathscr{C}_{u,R}} \mathscr{H}^\mu[\partial_u] dS_\mu \\ &= \frac{1}{2}\int_{\mathscr{C}_{u,R}} \Big(\mathring{\gamma}^{AB}\mathring{D}_A\phi\mathring{D}_B\phi + m^2 r^2 \phi^2 \\ &\quad + \partial_r\Big[\big(r^2-\alpha^2 r^4\big)\phi\big(\partial_r\phi\big)\Big] - \phi\partial_r\big[\big(r^2-\alpha^2r^4\big)\big(\partial_r\phi\big)\big]\Big)\, dr\, d\mu_{\mathring{\gamma}} \\ &= \frac{\alpha^2 R}{2}\int_{S_R} (\overset{(-1)}{\phi})^2\, d\mu_{\mathring{\gamma}} + O(1)\,, \end{aligned} \tag{1.5}$$

where S_R denotes a sphere $r = R$ within $\mathscr{C}_u$, with

$$\alpha^2 = \frac{\Lambda}{3}\,,$$

and where $O(1)$ here denotes a volume integral which has a finite limit as $R \to \infty$. A finite renormalised Noether charge can then be obtained by discarding the divergent boundary integral:

$$\hat{E}_{\mathscr{H}}[\mathscr{C}_u] := \lim_{R\to\infty}\left\{ \int_{\mathscr{C}_{u,R}} \mathscr{H}^\mu[\partial_u] dS_\mu - \frac{\alpha^2 R}{2}\int_{S_R} (\overset{(-1)}{\phi})^2\, d\mu_{\mathring{\gamma}} \right\}\,. \tag{1.6}$$

The divergent term evolves on its own, so that the renormalised Noether charge follows a well defined evolution law, as derived below in (4.43):

$$\frac{d\hat{E}_{\mathscr{H}}[\mathscr{C}_u]}{du} \;=\; \int_{S_R}\Big[\alpha^2 \overset{(-1)}{\phi}\partial_u \overset{(-2)}{\phi} + \Big(2\alpha^2\overset{(-2)}{\phi} - \partial_u\overset{(-1)}{\phi}\Big)\partial_u\overset{(-1)}{\phi}\Big] d\mu_{\mathring{\gamma}}\,. \tag{1.7}$$

Now, some authors discard the divergence term in (1.4) and use directly

$$E_\omega[\mathscr{S}] := \frac{1}{2}\int_{\mathscr{S}} \omega^\mu(\phi, \mathcal{L}_X\phi)\, dS_\mu \tag{1.8}$$

as a definition of Noether charge; this gives of course the same total energy as the original formula (1.3) for field configurations for which the boundary term vanishes. In [4] we observed that the integral (1.8) is finite, leading to a rewriting (see Equation (2.64) there):[1]

$$E_{\mathscr{H}}[\mathscr{C}_{u,R}] \;=\; E_\omega[\mathscr{C}_{u,R}] - \frac{1}{2}\int_{S^2} \overset{\text{(-1)}}{\phi}\big(\partial_u \overset{\text{(-1)}}{\phi} - \alpha^2 R \overset{\text{(-1)}}{\phi}\big)\, d\mu_{\mathring{\gamma}} + o(1)\,. \tag{1.9}$$

So, the divergent part of (1.5) appears directly in the boundary term in (1.4). Equation (1.9) implies

$$\hat{E}_{\mathscr{H}}[\mathscr{C}_u] \;=\; E_\omega[\mathscr{C}_u] - \frac{1}{2}\int_{S^2} \overset{\text{(-1)}}{\phi}\,\partial_u \overset{\text{(-1)}}{\phi}\, d\mu_{\mathring{\gamma}}\,. \tag{1.10}$$

The energy $E_\omega[\mathscr{C}_u]$ satisfies a flux formula (see (4.47), p. 36 below)

$$\frac{dE_\omega[\mathscr{C}_u]}{du} = \frac{1}{2}\int_{S^2} \big(-\alpha^2 \overset{\text{(-1)}}{\phi}\,\partial_u \overset{\text{(-2)}}{\phi} + \alpha^2 \overset{\text{(-2)}}{\phi}\,\partial_u \overset{\text{(-1)}}{\phi} + \overset{\text{(-1)}}{\phi}\,\partial_u^2 \overset{\text{(-1)}}{\phi} - (\partial_u \overset{\text{(-1)}}{\phi})^2\big)\, d\mu_{\mathring{\gamma}}\,. \tag{1.11}$$

The fact that $E_\omega[\mathscr{C}_{u,R}]$ has a finite limit as $R \to \infty$ suggests that the resulting Noether charge $E_\omega[\mathscr{C}_u]$ is more fundamental than $E_{\mathscr{H}}$. But one should keep in mind that the equality between (1.3) and (1.4) is only guaranteed for linear theories. In fact, (1.3) is defined for any theory, whether linear or not, while (1.4) does not make sense for nonlinear theories, such as Yang-Mills or metric gravity. Last but not least, $E_\omega[\mathscr{C}_u]$ is not monotonically decreasing in asymptotically Minkowskian spacetimes, as it should; see (4.47), p. 36 below. Therefore we view (1.3) as a more fundamental equation.

To make things even more confusing, it turns out that the field equations for the field

$$\widetilde{\phi} = r\phi\,,$$

where r is an affine parameter along the generators of the light cone, can be derived from the Lagrangian[2]

$$\mathscr{L} = -\frac{1}{2r^2}\sqrt{|\det g|}\, g^{\mu\nu}\nabla_\mu\widetilde{\phi}\,\nabla_\nu\widetilde{\phi}\,, \tag{1.12}$$

[1] References to numbering in [4] are to the arXiv version.

[2] Note that the singularity at $r = 0$ in (1.12) is integrable for fields ϕ which are smooth at the origin. While the presence of an unbounded integrand might be aesthetically unpleasant, it does not present difficulties as far as calculus of variations is concerned.

which differs from (1.1) by a boundary term, compare (5.2) below. Somewhat surprisingly, again under the asymptotic conditions (1.2). the Noether charge associated with translations in u turns out to be finite (cf. (5.14) below):

$$\tilde{E}_{\mathscr{H}}[\mathscr{C}_u] := \tilde{E}[\partial_u, \mathscr{C}_u] = \frac{1}{2}\int_{\mathscr{C}_u} \underbrace{\Big(\frac{1}{r^2}\mathring{\gamma}^{AB}\mathring{D}_A\widetilde{\phi}\mathring{D}_B\widetilde{\phi} + \big(1-\alpha^2 r^2\big)\big(\partial_r\widetilde{\phi}\big)^2\Big)}_{O(r^{-2})} \, dr\, d\mu_{\mathring{\gamma}}\,. \tag{1.13}$$

So we have a third candidate for the energy of the conformally-covariant scalar field, with flux (cf. (5.24) below)

$$\frac{d\tilde{E}_{\mathscr{H}}[\mathscr{C}_u]}{du} = \int_S \big(\alpha^2 \overset{(-1)}{\widetilde{\phi}} - \partial_u \overset{(0)}{\widetilde{\phi}}\big)\partial_u \overset{(0)}{\widetilde{\phi}} d\mu_{\mathring{\gamma}}\,. \tag{1.14}$$

The fact that the numerical value of $\tilde{E}_{\mathscr{H}}$ differs from that of both $\hat{E}_{\mathscr{H}}$ and E_ω when $\alpha \neq 0$ is made clear by comparing (1.14) with (1.7) and (1.11): all three fluxes differ.

The question then arises, whether the analysis of the Poisson algebra might give a hint, which of the energy-type expressions above have better properties. This is addressed in Section 6. To answer this question one needs to have a well defined Poisson algebra, which seems to be a problem when "charges are leaky", i.e. when the variations of functionals lead to nonvanishing boundary integrals. We emphasise that in our setup such boundary terms are unavoidable, because the fields under consideration radiate along light cones.

Now, it was observed in [6] that the charge-leaking can be remedied, in the case of (fully nonlinear) gravitational fields with $\Lambda = 0$, by extending the phase space of data on the light cone by adding suitable data on the portion of $\mathscr{I}^+$ to the past of the intersection of the light cone $\mathscr{C}_u$ with $\mathscr{I}^+$. In Section 7 we show how to generalise the procedure from [6] to the conformally-covariant scalar field with $\Lambda > 0$. For this it is convenient to introduce a coordinate system in which the de Sitter metric takes the form

$$\begin{aligned} g &= \underbrace{\cosh^2(\alpha\tau)}_{=:x^{-2}}\big(-\cosh^{-2}(\alpha\tau)d\tau^2 + \alpha^{-2}\underbrace{\big(d\psi^2 + \sin^2\psi\,\big(d\theta^2 + \sin^2\theta d\varphi^2\big)\big)}_{=:\breve{\gamma}}\big) \\ &= (\alpha x)^{-2}\big(-\frac{dx^2}{1-x^2} + \breve{\gamma}\big)\,, \end{aligned} \tag{1.15}$$

with x vanishes on $\mathscr{I}^+$. Under our asymptotic conditions above the field

$$\chi := \frac{\phi}{x} \tag{1.16}$$

extends smoothly to $\mathscr{I}^+$, and the expansion (1.2) translates to, for small x,

$$\phi = \overset{(1)}{\phi} x + \overset{(2)}{\phi} x^2 + \dots \qquad \Longleftrightarrow \qquad \chi = \overset{(0)}{\chi} + \overset{(1)}{\chi} x + \dots\,, \tag{1.17}$$

Using the fact that $\mathscr{H}^\mu$ has vanishing divergence under the current conditions, together with (7.33) and (7.36), the following equivalent equations hold

$$\begin{aligned}\hat{E}_{\mathscr{H}}[\mathscr{C}_u] &= \alpha^{-1} \int_{\mathscr{I}^+ \cap I^+(\mathscr{C}_u)} \overset{(1)}{\chi} \partial_\psi (\sin\psi \overset{(0)}{\chi}) d\mu_{\breve{\gamma}} \\ &\quad - \frac{1}{\alpha \cosh^3(\alpha u)} \int_{S_{0,u}} \overset{(0)}{\chi} \Big(\overset{(0)}{\chi} \sinh(\alpha u) - \overset{(1)}{\chi} \Big) d\mu_{\mathring{\gamma}}\,. \end{aligned} \tag{1.18}$$

In the phase space of Section 7.2 the dynamical system induced by translating in u the tip of the light-cone is Hamiltonian, with Hamiltonian equal to (see (7.48) with $\mathfrak{h}^x$ given by (7.25), $\mathfrak{h}^u$ by (7.37) and $\Delta\mathcal{B}_{0,u}$ by (7.47)):

$$\begin{aligned}\mathbf{H} &= \int_{\mathscr{C}_u} \frac{1}{2r^2} \Big(\mathring{\gamma}^{AB} \mathring{D}_A \phi \mathring{D}_B \phi + m^2 r^2 \phi^2 - \phi \partial_r \big[\big(r^2 - \alpha^2 r^4 \big) \big(\partial_r \phi \big) \big] \Big) r^2 \, dr \, d\mu_{\mathring{\gamma}} \\ &\quad + \frac{1}{2\alpha \cosh^3(\alpha u)} \int_{S_{0,u}} \overset{(0)}{\chi} \Big(\overset{(1)}{\chi} - \partial_\psi \overset{(0)}{\chi} - \overset{(0)}{\chi} \sinh(\alpha u) \Big) \sin\theta \, d\theta \, d\varphi \\ &\quad + \alpha^{-1} \int_{\mathscr{I}^+_u} \Big\{ (1 - x^2) \sin\psi \partial_x \chi \partial_\psi \chi \\ &\quad + \frac{1}{2} \cos\psi \Big((1 - x^2)(2\chi \partial_x \chi + x(\partial_x \chi)^2) + 2x\chi^2 + x|\check{D}\chi|^2_{\breve{\gamma}} \Big) \Big\} d\mu_{\breve{\gamma}} \qquad (1.19) \\ &=: \check{E}_{\mathscr{H}}[\mathscr{C}_u] + \check{E}_{\mathscr{H}}[\mathscr{I}^+_u]\,, \qquad (1.20)\end{aligned}$$

where $\check{E}_{\mathscr{H}}[\mathscr{I}^+_u]$ is the volume integral over $\mathscr{I}^+_u \equiv \mathscr{I}^+ \setminus I^+(\mathscr{C}_u)$ in (1.19), with all integrals finite.

Our analysis of the scalar field can be summarised as follows, compare Table 1.1:

1. The defining equation (1.5) for $E_{\mathscr{H}}$ makes sense for any theory, including non-linear ones, has the right properties when $\Lambda = 0$, but does not lead to convergence integrals when $\Lambda \neq 0$. It needs to be "renormalised", with ambiguities concerning the finite part of the renormalising corrections.

2. The "energy" E_ω defined in (1.8) leads directly to finite integrals for all Λ. However, it does not lead to a monotonously decreasing quantity when $\Lambda = 0$. Moreover it does not have any obvious generalisation to nonlinear fields.

	$\Lambda > 0$	$\Lambda = 0$	$\Lambda < 0$	
$E_{\mathscr{H}}$	∞	$< \infty$, $\frac{dE_{\mathscr{H}}}{du} \leq 0$	∞	
E_ω	$< \infty$	$< \infty$, $\frac{dE_\omega}{du}$ can have any sign	$< \infty$	
$\hat{E}_{\mathscr{H}}$	$< \infty$	$= E_{\mathscr{H}}$	$< \infty$	corner term ad hoc
$\check{E}_{\mathscr{H}}$	$< \infty$	(already considered in [6])	-	corner term ad hoc
$\tilde{E}$	$< \infty$	$= E_{\mathscr{H}}$	$< \infty$	Lagrangian explicitly coordinate dependent

Table 1.1: Various energies for the conformally-covariant scalar field. $E_{\mathscr{H}}$ is defined in (1.5); E_ω is defined in (1.8) and differs from (1.5) by a total divergence; similarly for $\hat{E}_{\mathscr{H}}$, with yet another boundary term; $\check{E}_{\mathscr{H}}$ is defined in (1.20) based on phase-space considerations and differs again from $E_{\mathscr{H}}$ by a further boundary term; $\tilde{E}$ uses the canonical definition as in (1.3) but with the alternative Lagrangian (1.12).

3. The "energy" $\hat{E}_{\mathscr{H}}[\mathscr{C}_u]$ of (1.6) has several desirable properties:

   (a) It is finite for all Λ.

   (b) It is non-increasing when $\Lambda = 0$ (since it coincides with $E_{\mathscr{H}}$ then), and is conserved when $\Lambda < 0$ and the standard boundary condition $\overset{(-1)}{\phi} = 0$ is imposed (cf. (1.7)).

   (c) It has a reasonably natural derivation, namely one removes a manifestly divergent term in Bondi coordinates.

   However, the choice of Bondi coordinates is ad-hoc, and other similar prescriptions using different coordinate systems will lead to different expressions.

4. The energy $\tilde{E}$ given by (1.13) has properties similar to $\hat{E}$ (cf. (1.14)) but arises from a coordinate-dependent Lagrangian, which does not have any obvious generalisations to non-conformally-covariant theories.

5. The energy $\mathbf{H}$ of (1.19) appears naturally when extending the phase space to include the degrees of freedom at $\mathscr{I}^+$, its numerical value

is the same for all cones $\mathscr{C}_u$, and thus carries only global information about the field. It splits into a volume integral on a subset of $\mathscr{I}^+$ and a remainder which is determined by the fields on $\mathscr{C}_u$. However, the uniqueness of this splitting is not clear.

1.1.2 Further charges

The total angular-momentum is obtained from the following integral:

$$J[\mathscr{C}_u] \quad := \quad \int_{\mathscr{C}_u} \mathscr{H}^\mu[\mathcal{R}] dS_\mu \equiv R_i J^i[\mathscr{C}_u]\,, \tag{1.21}$$

with (cf. (4.28) below)

$$J^i[\mathscr{C}_u] \quad = \quad \int_{\mathscr{C}_u} r\varepsilon^{AB}\mathring{D}_B x^i \mathring{D}_A \phi \partial_r \phi \, dr\, d\mu_{\mathring{\gamma}}\,, \tag{1.22}$$

which converges because a potentially divergent terms in the asymptotics of the integrand integrates out to zero.

The alternative Lagrangian leads to the same integral, in a form which is manifestly convergent, as determined in (5.16) below:

$$\tilde{J}^i[\mathscr{C}_u] \quad = \quad \int_{\mathscr{C}_u} \underbrace{r^{-1}\varepsilon^{AB}\mathring{D}_B x^i \mathring{D}_A \widetilde{\phi} \partial_r \widetilde{\phi}}_{O(r^{-2})} \, dr\, d\mu_{\mathring{\gamma}} = J^i[\mathscr{C}_u]\,, \tag{1.23}$$

Explicit expressions for the remaining charges associated with Killing vector fields of the background, as well as their fluxes, can be found in Sections 4 and 5.

1.2 Maxwell fields

We consider Maxwell fields on Minkowski, de Sitter and Anti-de Sitter spacetime. Each of these spacetimes has a conformal boundary at infinity, and we consider fields which smoothly extend through that boundary; a large class of such solutions of the sourceless Maxwell equations exists, which can be justified by an analysis of the Maxwell equations on the conformally rescaled manifolds. An elegant explicit family of such solutions is presented in Appendix B, essentially due to [2].

We use the field equations to derive the asymptotic behaviour of various components of the field along light cones in Section 2; in Bondi coordinates (cf. Equation (2.1), p. 15):

$$F_{ur} = \frac{\overset{(-2)}{F}_{ur}}{r^2} - \frac{\mathring{D}^A \overset{(-2)}{F}_{Ar}}{r^3} - \frac{\mathring{D}^A \overset{(-3)}{F}_{Ar}}{2r^4} + \dots , \tag{1.24}$$

$$F_{AB} = \overset{(0)}{F}_{AB} + \frac{\partial_A \overset{(-2)}{F}_{Br} - \partial_B \overset{(-2)}{F}_{Ar}}{r} + \frac{\partial_A \overset{(-3)}{F}_{Br} - \partial_B \overset{(-3)}{F}_{Ar}}{2r^2} + \dots , \tag{1.25}$$

$$F_{uA} = \overset{(0)}{F}_{uA} + \frac{\alpha^2 \overset{(-3)}{F}_{Ar} - \mathring{D}_A \overset{(-2)}{F}_{ur} - \mathring{D}^B \overset{(0)}{F}_{BA}}{2r} + \dots , \tag{1.26}$$

see (2.5) and below for details.

Section 3 starts with an analysis of Noether currents and their flux for Maxwell theory in a general background. In order to obtain a gauge-independent Hamiltonian, following [12] we use a notion of Lie-derivatives of the Maxwell potential arising from the $U(1)$-principal-bundle formulation of the theory. The results are applied to the de Sitter background in Section 3.1. Recalling that $\mathscr{C}_u$ denotes the light cone of constant u, a calculation leads to the following formula for the Noether charge on light cones associated with u-translations of $\mathscr{C}_u$ (cf. (3.39), p. 23):

$$\begin{aligned} E_{\mathscr{H}}[\mathscr{C}_u] &= \int_{\mathscr{C}_u} \mathscr{H}^\mu[\partial_u] dS_\mu \\ &= \frac{1}{16\pi} \int_{\mathscr{C}_u} \Big(\frac{1}{r^2} \mathring{\gamma}^{AC} \mathring{\gamma}^{BD} F_{AB} F_{CD} + 2F_{ur}^2 - 2\epsilon N^2 \mathring{\gamma}^{AB} F_{rA} F_{rB} \Big) \, dr \, d\mu_{\mathring{\gamma}} \, , \end{aligned} \tag{1.27}$$

where the convergence of the integral follows from (1.24)-(1.26).

Likewise the components of the total angular-momentum vector are given by convergent integrals:

$$J[\mathcal{R}] := \int_{\mathscr{C}_u} \mathscr{H}^\mu[\mathcal{R}] dS_\mu \equiv R_i J^i \, , \tag{1.28}$$

with

$$J^i = \frac{1}{4\pi} \int_{\mathscr{C}_u} \varepsilon^{AB} \mathring{D}_B n^i \Big(r^2 F_{ur} F_{Ar} + \mathring{\gamma}^{BC} F_{Br} F_{CA} \Big) \, dr \, d\mu_{\mathring{\gamma}} \, . \tag{1.29}$$

Explicit formulae for the momentum and center of mass of the field can be found in (3.42)-(3.43).

In Section 3.2 we apply the formalism to light-cones in Minkowski spacetime, while Section 3.3 is concerned with anti-de Sitter spacetime.

In Section 3.4 we consider the time-evolution of the charges, by which we mean the evolution of the charges when the tips of the light-cones are moved along the Killing vector ∂_u. In particular we find the following formulae for the flux of the energy,

$$\frac{dE_{\mathscr{H}}[\mathscr{C}_u]}{du} = -\frac{1}{4\pi}\int_{S_\infty}\Big[\mathring{\gamma}^{AB}\big(\alpha^2 \overset{(-2)}{F}_{Ar}\overset{(0)}{F}_{Bu} + \overset{(0)}{F}_{Au}\overset{(0)}{F}_{Bu}\big)\Big]d\mu_{\mathring{\gamma}}\,, \quad (1.30)$$

and for that of angular-momentum:

$$\frac{dJ^i}{du} = -\frac{1}{4\pi}\int_{S_\infty}\Big[\varepsilon^{AB}\mathring{D}_B(n^i)\Big(\mathring{\gamma}^{BC}\big(\alpha^2\overset{(-2)}{F}_{Br} + \overset{(0)}{F}_{Bu}\big)\overset{(0)}{F}_{CA} - \overset{(-2)}{F}_{ur}\overset{(0)}{F}_{Au}\Big)\Big]d\mu_{\mathring{\gamma}}\,. \quad (1.31)$$

Similarly to the scalar field case, one can avoid phase-space leakage for the Maxwell field by considering jointly fields on $\mathscr{C}_u$ and $\mathscr{I}^+ \setminus I^+(\mathscr{C}_u)$. This leads to a Hamiltonian dynamics, with u-independent Hamiltonian (cf. (7.57), with $\mathscr{H}^u[\partial_u]$ given by (3.39) and $\mathscr{H}^x[\partial_u]$ by (7.54)

$$\begin{aligned}\mathbf{H} = {} & \frac{1}{16\pi}\int_{\mathscr{C}_u}\Big(\frac{1}{r^2}\mathring{\gamma}^{AC}\mathring{\gamma}^{BD}F_{AB}F_{CD} + 2F_{ur}^2 - 2\epsilon N^2\mathring{\gamma}^{AB}F_{rA}F_{rB}\Big)\,dr\,d\mu_{\mathring{\gamma}} \\ & -\frac{\alpha}{4\pi}\int_{\mathscr{I}^+\setminus I^+(\mathscr{C}_u)}\Big\{\frac{1}{2}x(1-x^2)\cos\psi F_{xk}F_{xl}\check{\gamma}^{kl} \\ & \qquad + (1-x^2)\sin\psi F_{xk}F_{\psi l}\check{\gamma}^{kl} + \frac{1}{4}x\cos\psi F_{mk}F_{nl}\check{\gamma}^{mn}\check{\gamma}^{kl}\Big\}d\mu_{\check{\gamma}}\,, \end{aligned} \quad (1.32)$$

where all the integrals are finite, without the need for any corrections. Since $\mathbf{H}$ is u-independent, formula (1.30) describes the flow of energy between $\mathscr{C}_u$ and $\mathscr{I}^+ \setminus I^+(\mathscr{C}_u)$.

In absence of a clear guiding principle for adding boundary terms to the Noether charges, we have not attempted to repeat the analysis of various alternative energies, as done for the scalar field, in the Maxwell case.

1.3 Poisson brackets

Section 6 is devoted to an analysis of the Poisson brackets for unconstrained fields. As already pointed-out, a direct calculation of Poisson brackets associated to initial data on characteristic surfaces is tricky. We circumvent this problem by using the fact that, for conserved quantities, the relevant Poisson brackets can be calculated by evolving the field to a spacelike hypersurface

$\mathscr{S}$ and calculating the brackets there, using the formula advocated in [3]: for functionals of the form

$$F = \int_{\mathscr{S}} f(\phi^A, \partial_i \phi^A, \pi^A)\, dS_0 \,, \quad G = \int_{\mathscr{S}} g(\phi^A, \partial_i \phi^A, \pi^A)\, dS_0 \,, \tag{1.33}$$

one sets

$$\{F, G\}_{\mathscr{S}} := \int_{\mathscr{S}} \left(\frac{\delta f}{\delta \phi^A} \frac{\delta g}{\delta \pi_A} - \frac{\delta f}{\delta \pi_A} \frac{\delta g}{\delta \phi^A} \right) dS_0 \,. \tag{1.34}$$

In Proposition 6.2, p. 48, we list a series of conditions that guarantee the equality

$$\{H_X, H_Y\}_{\mathscr{S}} = H_{[X,Y]} \,. \tag{1.35}$$

This leads to another problem, of boundary terms arising in variational identities, which might affect equations such as (1.35), and leads us to propose alternative phase spaces for the problem at hand, already mentioned above.

We turn our attention to Poisson brackets for Maxwell field in Section 6.3. The considerations of Section 6.2 do not apply without further due because of gauge-invariance, and the resulting constraints. We start with an ab-initio analysis, on a general spacelike hypersurface in a general spacetime, using ADM notation: in adapted coordinates such that $\mathscr{S} = \{x^0 = 0\}$,

$$\gamma_{ij} := g_{ij} \,, \quad N := \frac{1}{\sqrt{-g^{00}}} \,, \quad N_k := g_{0k} \,. \tag{1.36}$$

We define the electric field on $\mathscr{S}$ as

$$E^k = F^{k\mu} T_\mu \,, \tag{1.37}$$

where T^μ is the field of unit normals to $\mathscr{S}$, with the orientation chosen so that

$$T_\mu dx^\mu = -N dt \quad \Longleftrightarrow \quad T = N^{-1}(\partial_t - N^k \partial_k) \,. \tag{1.38}$$

The canonical momentum is defined by the usual formula,

$$\pi^\mu := \pi^{\mu 0} = \frac{\partial \mathscr{L}}{\partial\left(\partial_0 A_\mu\right)} \,. \tag{1.39}$$

When the Lagragian depends only upon $F_{\mu\nu}$ the zero-component of π^μ vanishes, so only its space-part π^k remains of interest. In the standard Maxwell electrodynamics the field π^k is the densitised equivalent of the electric field E^k:

$$\pi^k \equiv -\frac{1}{4\pi} \sqrt{\det \gamma_{ij}} N F^{0k} = -\frac{1}{4\pi} \sqrt{\det \gamma_{ij}} E^k \,, \tag{1.40}$$

Now, functionals which depend only upon $F_{\mu\nu}$, such as the Noether current, can be expressed in terms of the space-part A_i of the four-potential A_μ and of the electric field. For instance, in the standard Maxwell electrodynamics we have, using the ADM notation for the metric (cf. (6.95), p. 63)

$$\begin{aligned} H[\mathscr{S},X] &= \int_{\mathscr{S}} \mathscr{H}^0 dS_0 \\ &= \frac{1}{8\pi}\int_{\mathscr{S}} \Big[N X^0 \big(E^k E^l \gamma_{lk} + \frac{1}{2}\gamma^{km}\gamma^{ln} F_{mn}F_{kl} + 2N^{-1}E^k N^l F_{lk} \big) \\ &\quad + 2E^k X^l F_{lk} \Big] \sqrt{\det\gamma}\, dS_0 \,. \end{aligned} \tag{1.41}$$

Since π^0 vanishes by antisymmetry of $F^{\mu\nu}$, we cannot define the Poisson bracket using (1.34) with $(\phi^A) = (A_\mu)$. Instead we set

$$\{F,G\}_{\mathscr{S}} := \int_{\mathscr{S}} \left(\frac{\delta f}{\delta A_l}\frac{\delta g}{\delta \pi^l} - \frac{\delta f}{\delta \pi^l}\frac{\delta g}{\delta A_l} \right) dS_0 \,. \tag{1.42}$$

In this formula A_0 has become irrelevant, though it has neither been gauged away nor discarded, being part of the $U(1)$-gauge potential $A_\mu dx^\mu$.

When deriving the Hamilton equations for the Maxwell field, or indeed when considering (1.42), there arises a difficulty related to the fact that the Maxwell momenta are not arbitrary, but satisfy the Gauss constraint equation $\partial_i \pi^i = 0$. This is addressed in Section 6.3, both in an approach where the Lie derivative of the Maxwell potential is that of a covector field on spacetime, and where the Maxwell potential is treated as a connection form on a $U(1)$-bundle. One can implement the Gauss constraint by writing

$$\delta\pi^k = \epsilon^{k\ell m} D_\ell \delta Y_m \,, \tag{1.43}$$

where δY_m is an arbitrary covector density, leading to the following variational identity on the set of solutions of the field equations (cf. (6.50) and (6.61) with $\mathscr{E}^\mu = 0$)

$$\begin{aligned} 0 &= \int_{\mathscr{S}} \Big[\epsilon^{k\ell m} D_\ell \Big(\frac{\delta \mathscr{H}^0}{\delta \pi^k} - \mathbf{L}_X A_k \Big) \delta Y_m + \Big(\frac{\delta \mathscr{H}^0}{\delta A_k} + \mathcal{L}_X \pi^k \Big) \delta A_k \Big] dS_0 \\ &\quad + \int_{\partial\mathscr{S}} \Big[\Big(\frac{\partial \mathscr{H}^0}{\partial A_{k,\ell}} - (X^\ell \pi^k - X^0 \pi^{k\ell} - X^k \pi^\ell) \Big) \delta A_k \\ &\qquad + \Big(\frac{\delta \mathscr{H}^0}{\delta \pi^k} - \mathbf{L}_X A_k \Big) \epsilon^{k\ell m} \delta Y_m \Big] dS_{0\ell} \,, \end{aligned} \tag{1.44}$$

which can be seen to reproduce the standard form of Maxwell equations in Minkowski spacetime.

Section 6.3.2 is devoted to the Poisson algebra of Hamiltonian charges. We prove the identity (cf. (6.82), p. 61)

$$\begin{aligned}\{H_X,H_Y\} &= H_{[X,Y]}\\ &\quad+\int_{\mathscr{S}}\Big\{Y^\beta\Big(\mathscr{E}^\kappa\mathbf{L}_X A_\kappa-\pi^{\lambda\kappa}[\mathcal{L}_X,\nabla_\kappa]A_\lambda-\frac{\partial\mathscr{L}}{\partial g_{\kappa\lambda}}\mathcal{L}_X g_{\kappa\lambda}\Big)\\ &\quad-X^\beta\Big(\mathscr{E}^\kappa\mathbf{L}_Y A_\kappa-\pi^{\lambda\kappa}[\mathcal{L}_Y,\nabla_\kappa]A_\lambda-\frac{\partial\mathscr{L}}{\partial g_{\kappa\lambda}}\mathcal{L}_Y g_{\kappa\lambda}\Big)\\ &\quad+\big(E^k_X Y^\mu-E^k_Y X^\mu\big)F_{\mu k}\Big\}dS_\beta\\ &\quad+2\int_{\partial\mathscr{S}}\Big(X^{[\alpha}\mathscr{H}^{\beta]}_Y-Y^{[\alpha}\mathscr{H}^{\beta]}_X+X^{[\alpha}Y^{\beta]}\mathscr{L}\Big)dS_{\alpha\beta}\,. \end{aligned}\tag{1.45}$$

This makes clear what fields have to vanish to obtain a closed subalgebra.

We now pass to the details of the above.

2 Asymptotics of Maxwell fields along light cones

In the next section we will apply the formalism developed in [4] to Maxwell fields on Minkowski, de Sitter and anti-de Sitter spacetimes. For this it is first necessary to derive the asymptotic behaviour of the fields under natural conditions arising from conformal invariance of the equations.

We consider simultaneously the Minkowski space-time, the de Sitter and the anti-de Sitter space-times in Bondi coordinates. In these the metric takes the form

$$g\equiv g_{\alpha\beta}dx^\alpha dx^\beta=\epsilon N^2du^2-2du\,dr+r^2\underbrace{(d\theta^2+\sin^2\theta\,d\varphi^2)}_{=:\mathring{\gamma}}\,,\tag{2.1}$$

where

$$N:=\sqrt{|(1-\alpha^2r^2)|}\,,\quad \alpha\in\Big\{0,\sqrt{\frac{\Lambda}{3}}\Big\}\subset\mathbb{R}\cup i\mathbb{R}\,,\quad \epsilon\in\{\pm1\}\,,$$

with ϵ equal to one if $1-\alpha^2r^2<0$, and minus one otherwise; note that any $\Lambda\in\mathbb{R}$, is allowed, and hence $\alpha\in\mathbb{C}$ but $\alpha^2\in\mathbb{R}$.

We have

$$g^{\alpha\beta}\partial_\alpha\partial_\beta=-2\partial_u\partial_r-\epsilon N^2(\partial_r)^2+r^{-2}\mathring{\gamma}^{AB}\partial_A\partial_B\,.$$

For $r \to \infty$ we replace the coordinate r by a new coordinate

$$x := r^{-1} . \tag{2.2}$$

In this coordinate system the de Sitter metric (2.1) becomes

$$\begin{aligned} g &= -(1-\alpha^2 r^2)du^2 - 2du\,dr + r^2 \underbrace{(d\theta^2 + \sin^2\theta d\varphi^2)}_{=:\mathring{\gamma}} , \\ &= x^{-2}\big(-(x^2-\alpha^2)du^2 + 2du\,dx + \mathring{\gamma}\big) . \end{aligned} \tag{2.3}$$

The volume element is equal to

$$\sqrt{-g} = x^{-4}\sqrt{\det\mathring{\gamma}} . \tag{2.4}$$

Conformal invariance of the Maxwell equations shows that, for solutions that evolve out of smooth initial data on some spacelike Cauchy surface in de Sitter spacetime, the (u, x, x^A)-components of the Maxwell field are smooth functions of (u, x, x^A):

$$\begin{aligned} F &= F_{xu} dx \wedge du + F_{xA} dx \wedge dx^A + F_{uA} du \wedge dx^A + \frac{1}{2} F_{AB} dx^A \wedge dx^B \\ &= -r^{-2}(F_{xu} dr \wedge du + F_{xA} dr \wedge dx^A) + F_{uA} du \wedge dx^A \\ &\quad + \frac{1}{2} F_{AB} dx^A \wedge dx^B , \end{aligned} \tag{2.5}$$

with F_{xu}, etc., having full Taylor expansions in $x \equiv 1/r$ around $x = 0$. In particular the fields F_{Ar} which are associated with a conformally smooth Maxwell field have expansions of the form

$$F_{Ar} = -\overset{(0)}{F}_{Ax} r^{-2} + \ldots = \overset{(-2)}{F}_{Ar} r^{-2} + \ldots , \tag{2.6}$$

where the expansion coefficients are functions of u and x^A.

Those sourceless Maxwell equations which involve r-derivatives read

$$\partial_r (r^2 \sqrt{\det\mathring{\gamma}} F^{r\mu}) = -r^2 \sqrt{\det\mathring{\gamma}} \partial_u F^{u\mu} - \partial_A (r^2 \sqrt{\det\mathring{\gamma}} F^{A\mu}) , \tag{2.7}$$

$$\partial_r F_{\mu\nu} = -\partial_\mu F_{\nu r} - \partial_\nu F_{r\mu} . \tag{2.8}$$

Using

$$F^{ru} = F_{ur} , \quad F^{rA} = -r^{-2}\mathring{\gamma}^{AB}(F_{uB} + \epsilon N^2 F_{rB}) , \quad F^{uA} = -r^{-2}\mathring{\gamma}^{AB} F_{rB} , \tag{2.9}$$

we find

$$\begin{aligned}\partial_r(r^2F_{ur}) &= \mathring{D}^A F_{Ar}\,, &(2.10)\\ \partial_r\left(F_{uA}+\epsilon N^2 F_{rA}\right) &= \partial_u F_{Ar} + r^{-2}\mathring{D}^B F_{BA}\,, &(2.11)\\ \partial_r F_{AB} &= -\partial_A F_{Br} + \partial_B F_{Ar}\,, &(2.12)\\ \partial_u F_{Ar} &= -\partial_r F_{uA} - \partial_A F_{ru}\,, &(2.13)\end{aligned}$$

where (2.10) and (2.11) are special cases of (2.7) with $\mu = u$ and $\mu = A$. Here, and elsewhere, $\mathring{D}$ denotes the covariant derivative of the metric $\mathring{\gamma}$.

Inserting (2.13) in (2.11) one obtains

$$\partial_r\left(2F_{uA} - \epsilon N^2 F_{Ar}\right) = -\partial_A F_{ru} + r^{-2}\mathring{D}^B F_{BA}\,, \tag{2.14}$$

We conclude that prescribing $F_{Ar}dx^A$ on a cone $\{u = \text{const}\}$ allows one to determine the remaining fields on this cone by successive integrations of (2.10), (2.12) and (2.14). We will refer to these equations as the *characteristic constraint equations*. One can then view (2.13) as an equation which determines F_{Ar} "on the next cone".

The remaining Maxwell equations have an evolution character:

$$\begin{aligned}\partial_u F_{ru} &= r^{-2}\mathring{D}^A\left(F_{Au}+\epsilon N^2 F_{Ar}\right)\,, &(2.15)\\ \partial_u F_{AB} &= -\partial_A F_{Bu} + \partial_B F_{Au}\,. &(2.16)\end{aligned}$$

Another evolution equation can be obtained by subtracting (2.11) from (2.13):

$$2\partial_u F_{Ar} = -\partial_r\left(\epsilon N^2 F_{Ar}\right) - \partial_A F_{ru} - r^{-2}\mathring{D}^B F_{BA}\,. \tag{2.17}$$

Integrating (2.10) in r one obtains

$$F_{ur} = r^{-2}\int_0^r \mathring{D}^A F_{Ar}\mathrm{d}s\,, \tag{2.18}$$

so that

$$F_{ur} = \frac{\overset{(-2)}{F}{}_{ur}}{r^2} - \frac{\mathring{D}^A\overset{(-2)}{F}{}_{Ar}}{r^3} - \frac{\mathring{D}^A\overset{(-3)}{F}{}_{Ar}}{2r^4} + \ldots\,, \tag{2.19}$$

where

$$\overset{(-2)}{F}{}_{ur} = \int_0^\infty \mathring{D}^A F_{Ar}\mathrm{d}s\,. \tag{2.20}$$

Integrating (2.12) we have

$$F_{AB} = \overset{(0)}{F}{}_{AB} + \frac{\partial_A\overset{(-2)}{F}{}_{Br} - \partial_B\overset{(-2)}{F}{}_{Ar}}{r} + \frac{\partial_A\overset{(-3)}{F}{}_{Br} - \partial_B\overset{(-3)}{F}{}_{Ar}}{2r^2} + \ldots\,, \tag{2.21}$$

where

$$\overset{(0)}{F}_{AB} = \int_0^\infty (\partial_B F_{Ar} - \partial_A F_{Br})\, ds\,. \tag{2.22}$$

Substituting (2.6), (2.19), and (2.21) into (2.14), after integration one finds

$$F_{uA} = \overset{(0)}{F}_{uA} + \frac{\alpha^2 \overset{(-3)}{F}_{Ar} - \mathring{D}_A \overset{(-2)}{F}_{ur} - \mathring{D}^B \overset{(0)}{F}_{BA}}{2r} + \ldots\,. \tag{2.23}$$

Here the "integration constant" $\overset{(0)}{F}_{uA}$ equals

$$\overset{(0)}{F}_{uA} = \frac{1}{2}\left[\alpha^2 \overset{(-2)}{F}_{Ar} + \int_0^\infty \Big(\partial_A F_{ur} + s^{-2}\mathring{D}^B F_{BA}\Big)\, ds\right]. \tag{2.24}$$

Inserting (2.6), (2.19), and (2.21) into (2.17), one obtains

$$\partial_u F_{Ar} = \frac{\alpha^2 \overset{(-3)}{F}_{Ar} + \mathring{D}_A \overset{(-2)}{F}_{ur} - \mathring{D}^B \overset{(0)}{F}_{BA}}{2r^2} + \ldots\,. \tag{2.25}$$

Inserting (2.6) and (2.23) into (2.15) leads to

$$\partial_u F_{ru} = \frac{\mathring{D}^A \overset{(0)}{F}_{Ar} + \alpha^2 \mathring{D}^A \overset{(-2)}{F}_{Ar}}{r^2} + \ldots\,. \tag{2.26}$$

Substituting (2.23) into (2.16), one finds

$$\partial_u F_{AB} = -2\mathring{D}_{[A} \overset{(0)}{F}_{B]u} + \frac{\alpha^2 D_{[A} \overset{(-3)}{F}_{B]r} + \mathring{D}_{[A}\mathring{D}^C \overset{(0)}{F}_{B]C}}{r} + \ldots\,. \tag{2.27}$$

3 Noether charges in Maxwell theory

We are ready to pass to the analysis of Noether-type currents for Maxwell fields in Minkowski, de Sitter and anti-de Sitter spacetimes. In our signature the Lagrangian reads

$$\mathscr{L}(A_\mu, \partial A_\mu) = -\frac{1}{16\pi}\sqrt{|-\det g|}\, g^{\mu\nu} g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta}\,. \tag{3.1}$$

The theory is linear so there is no need to make a distinction, in the notation of [4], between the fields $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$. Denoting $\partial_\nu A_\mu$ by $A_{\mu,\nu}$, the canonical momentum density reads

$$\pi^{\alpha\beta} = \frac{\partial \mathscr{L}}{\partial\big(A_{\alpha,\beta}\big)} = \frac{1}{4\pi}\mathcal{F}^{\alpha\beta}\,, \tag{3.2}$$

where $\mathcal{F}^{\alpha\beta}$ is a density of Maxwell tensor

$$\mathcal{F}^{\alpha\beta} = \sqrt{|-\det g|} F^{\alpha\beta} . \tag{3.3}$$

The standard Noether currents, which we will denote by $\mathscr{H}_c^{\ \mu}$, is defined as

$$\begin{aligned} \mathscr{H}_c^{\ \mu}[X] &:= \frac{\partial \mathscr{L}}{\partial A_{\beta,\mu}} \mathcal{L}_X A_\beta - \mathscr{L} X^\mu \\ &= -\frac{1}{4\pi}\sqrt{|-\det g|}\big(F^{\mu\beta}\mathcal{L}_X A_\beta - \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}X^\mu\big) , \end{aligned} \tag{3.4}$$

where $\mathcal{L}_X A$ denotes Lie derivative of a covector field.

It holds that

$$\nabla_\mu(\mathscr{H}_c^{\ \mu}[X]) = 0 , \tag{3.5}$$

when A satisfies the field equations and X is a Killing field of the background metric. This follows of course from a theorem of Noether, but a direct proof can be given starting with the identity

$$\begin{aligned} \Delta_{\beta\delta}(V,A) &:= [\nabla_\delta, \mathcal{L}_V]A_\beta \\ &= A_\gamma \nabla_\delta \nabla_\beta V^\gamma - V^\gamma R^\sigma{}_{\beta\delta\gamma} A_\sigma , \end{aligned} \tag{3.6}$$

where V is an arbitrary vector field and A is an arbitrary one-form. Next, if V is a conformal Killing field of the background metric,

$$\nabla_{(\alpha} V_{\beta)} = \lambda g_{\alpha\beta} , \tag{3.7}$$

we have

$$\nabla_\gamma \nabla_\alpha V_\beta = R^\sigma{}_{\gamma\alpha\beta} V_\sigma + \nabla_\gamma \lambda g_{\alpha\beta} + \nabla_\alpha \lambda g_{\beta\gamma} - \nabla_\beta \lambda g_{\alpha\gamma} . \tag{3.8}$$

Substituting (3.8) into (3.6), we obtain for any conformal Killing field V

$$\Delta_{\beta\delta}(V,A) = \Delta_{(\beta\delta)}(V,A) = A_\beta \nabla_\delta \lambda + \nabla_\beta \lambda A_\delta - g_{\beta\delta} A^\gamma \nabla_\gamma \lambda . \tag{3.9}$$

Let

$$j^\mu := \frac{1}{4\pi} \nabla_\nu F^{\mu\nu} , \tag{3.10}$$

which of course vanishes when A satisfies the field equations. As is well known, a consequence of the definition (3.10) is

$$\nabla_\mu j^\mu = 0 . \tag{3.11}$$

We are ready now to calculate, for any vector field X, as follows:

$$
\begin{aligned}
-\frac{4\pi}{\sqrt{|-\det g|}}\nabla_\mu(\mathscr{H}_c^\mu[X]) &= \nabla_\mu(F^{\mu\nu}\mathcal{L}_X A_\nu) - \frac{1}{4}\nabla_\mu(X^\mu F^{\alpha\beta}F_{\alpha\beta}) \\
&= -4\pi j^\nu \mathcal{L}_X A_\nu + F^{\mu\nu}(\mathcal{L}_X \nabla_\mu A_\nu + \Delta_{\nu\mu}(X,A)) \\
&\quad -\frac{1}{4}[F^{\alpha\beta}F_{\alpha\beta}\nabla_\mu X^\mu + X^\mu \nabla_\mu(F^{\alpha\beta}F_{\alpha\beta})] \\
&= -4\pi j^\nu \mathcal{L}_X A_\nu + \frac{1}{4}\mathcal{L}_X\big(F^{\mu\nu}F_{\mu\nu}\big) - \frac{1}{2}F_\mu{}^\beta F_{\nu\beta}\underbrace{\mathcal{L}_X g^{\mu\nu}}_{=-2\nabla^{(\mu}X^{\nu)}} \\
&\quad +F^{\mu\nu}\Delta_{\nu\mu}(X,A) - \frac{1}{4}[F^{\alpha\beta}F_{\alpha\beta}\nabla_\mu X^\mu + X^\mu\nabla_\mu(F^{\alpha\beta}F_{\alpha\beta})] \\
&= -4\pi j^\nu \mathcal{L}_X A_\nu + F^{\mu\nu}\Delta_{\nu\mu}(X,A) + F_\mu{}^\beta F_{\nu\beta}\Big(\nabla^{(\mu}X^{\nu)} - \frac{1}{4}g^{\mu\nu}\nabla_\alpha X^\alpha\Big)\,. \quad (3.12)
\end{aligned}
$$

The last line of (3.12) vanishes for all sourceless field configurations if X is a conformal Killing vector field of the background metric.

The problem with the Hamiltonian (3.4) is its gauge dependence. This can be fixed by replacing $\mathcal{L}_X A$ by

$$
\mathbf{L}_X A_\mu := X^\nu F_{\nu\mu} \quad (3.13)
$$

(which, by the way, is a natural definition for the Lie derivative of a connection one form on a $U(1)$ principal bundle), and defining

$$
\begin{aligned}
\mathscr{H}^\mu[X] &:= \frac{\partial\mathscr{L}}{\partial A_{\beta,\mu}}\mathbf{L}_X A_\beta - \mathscr{L}X^\mu \\
&= -\frac{1}{4\pi}\sqrt{|-\det g|}\Big(F^{\mu\beta}\mathbf{L}_X A_\beta - \frac{1}{4}\big(F^{\nu\beta}F_{\nu\beta}\big)X^\mu\Big) \\
&= -\frac{1}{4\pi}\sqrt{|-\det g|}\Big(F^{\mu\beta}X^\alpha F_{\alpha\beta} - \frac{1}{4}\big(F^{\nu\beta}F_{\nu\beta}\big)X^\mu\Big)\,. \quad (3.14)
\end{aligned}
$$

Let us set

$$
\begin{aligned}
\Delta\mathscr{H}^\mu[X] &:= \mathscr{H}^\mu[X] - \mathscr{H}_c^\mu[X] \\
&= -\sqrt{|-\det g|}j^\mu X^\sigma A_\sigma + \frac{1}{4\pi}\partial_\beta\big(\mathcal{F}^{\mu\beta}X^\sigma A_\sigma\big)\,. \quad (3.15)
\end{aligned}
$$

From (3.11) and (3.15) we immediately find

$$
\partial_\mu(\Delta\mathscr{H}^\mu[X]) = -\sqrt{|-\det g|}j^\mu\nabla_\mu(X^\sigma A_\sigma)\,. \quad (3.16)
$$

so that we again have $\partial_\mu\mathscr{H}^\mu = 0$ when the field equation $j^\mu \equiv 0$ is satisfied and when X^μ is a Killing vector field.

Now, $\mathscr{H}_c^{\mu}[X]$ is of the form considered in [4]. There an alternative form of Hamiltonian density has been derived [4, Proposition 1], which in our case reads

$$\mathscr{H}_c^{\mu}[X] \;=\; \frac{1}{2}\omega^{\mu}(A,\mathcal{L}_X A) + \partial_\sigma\Big(X^{[\sigma}\pi^{\mu]\nu}A_\nu\Big)\,, \tag{3.17}$$

with

$$\omega^{\mu}(A,\mathcal{L}_X A) = \mathcal{L}_X A_\beta\,\pi^{\mu\beta} - \mathcal{L}_X \pi^{\mu\beta} A_\beta\,. \tag{3.18}$$

and where $\pi^{\mu\beta}$ is given by (3.2). This rewriting does not seem to very enlightening in the case of the Maxwell field, with a gauge behaviour even more cumbersome than that of (3.4).

In order to determine the flux of energy, we continue by calculating the Lie derivative of the Hamiltonian density in the direction of an arbitrary vector field Y. Recall the formula for the Lie derivative of a vector density Z^μ:

$$\mathcal{L}_X Z^\mu = \partial_\sigma(X^\sigma Z^\mu) - Z^\sigma\partial_\sigma X^\mu \equiv \nabla_\sigma(X^\sigma Z^\mu) - Z^\sigma\nabla_\sigma X^\mu\,. \tag{3.19}$$

In order to calculate $\mathcal{L}_Y\mathscr{H}^\mu[X]$ we use this formula to obtain

$$\begin{aligned}\mathcal{L}_Y\mathscr{H}^\mu[X] &= \nabla_\sigma\Big(Y^\sigma\mathscr{H}^\mu\Big) - \mathscr{H}^\sigma\nabla_\sigma Y^\mu\\ &= 2\nabla_\sigma\Big(Y^{[\sigma}\mathscr{H}^{\mu]}\Big) + Y^\mu\nabla_\sigma\mathscr{H}^\sigma\\ &= 2\nabla_\sigma\Big(Y^{[\sigma}\mathscr{H}^{\mu]}\Big) + Y^\mu\Big[\nabla_\sigma\mathscr{H}_c^\sigma + \nabla_\mu(\Delta\mathscr{H}^\mu)\Big]\,,\end{aligned} \tag{3.20}$$

where $\Delta\mathscr{H}^\mu$ has been defined in (3.15). Keeping in mind that if Z^α is a vector density then $\nabla_\alpha Z^\alpha = \partial_\alpha Z^\alpha$, and substituting (3.12), (3.14) and (3.16) into (3.20) we find

$$\begin{aligned}\frac{4\pi}{\sqrt{|-\det g|}}\mathcal{L}_Y\mathscr{H}^\mu[X] &= -2\nabla_\sigma\left[Y^{[\sigma}F^{\mu]\alpha}X^\kappa F_{\kappa\alpha} - \frac{1}{4}Y^{[\sigma}X^{\mu]}F^{\alpha\beta}F_{\alpha\beta}\right]\\ &-Y^\mu\Big\{-4\pi j^\nu\big(\mathcal{L}_X A_\nu - \nabla_\nu(X^\sigma A_\sigma)\big) + F^{\mu\nu}\Delta_{\nu\mu}(X,A)\\ &+F_\mu{}^\beta F_{\nu\beta}\Big(\nabla^{(\mu}X^{\nu)} - \frac{1}{4}g^{\mu\nu}\nabla_\alpha X^\alpha\Big)\Big\}\,.\end{aligned} \tag{3.21}$$

3.1 Noether charges in de Sitter spacetime

We wish to determine the Noether charges associated with the Killing fields (A.1)-(A.4). Since the Hamiltonian density (3.4) is linear in the Hamiltonian vector field X, each charge is given by an integral of a linear combination of the following four functionals

$$\begin{aligned}
\mathscr{H}^\mu[\partial_u] &= \mathtt{h}^\mu_I + \mathcal{T}^\mu \mathtt{F}^2 \,, && (3.22)\\
\mathscr{H}^\mu[\mathcal{R}] &= \varepsilon^{AB}\mathring{D}_B(R_i n^i)\mathtt{h}^\mu_A + \mathcal{R}^\mu \mathtt{F}^2 \,, && (3.23)\\
\mathscr{H}^\mu[\mathcal{P}_{dS}] &= e^{\alpha u}\Big[p_i n^i \mathtt{h}^\mu_I - \big(\alpha r + 1\big) p_i n^i \mathtt{h}^\mu_{II} \\
&\quad -\frac{\alpha r + 1}{r}\mathring{D}^A(p_i n^i)\mathtt{h}^\mu_A\Big] + \mathcal{P}^\mu_{dS}\mathtt{F}^2 \,, && (3.24)\\
\mathscr{H}^\mu[\mathcal{L}_{dS}] &= e^{-\alpha u}\Big[l_i n^i \mathtt{h}^\mu_I + \big(\alpha r - 1\big) l_i n^i \mathtt{h}^\mu_{II} \\
&\quad +\frac{\alpha r - 1}{r}\mathring{D}^A(l_i n^i)\mathtt{h}^\mu_A\Big] + \mathcal{L}^\mu_{dS}\mathtt{F}^2 \,, && (3.25)
\end{aligned}$$

where ε^{AB} is a two-dimensional Levi-Civita tensor (in spherical coordinates (θ,ϕ) we take the sign $\varepsilon^{\theta\phi} = \frac{1}{\sin\theta}$.), and

$$\begin{aligned}
\mathtt{h}^\mu[X] &= -\frac{1}{4\pi}\sqrt{|-\det g|}F^{\mu\beta}X^\alpha F_{\alpha\beta} \,, && (3.26)\\
\mathtt{h}^\mu_I &= \mathtt{h}^\mu[\partial_u] \,, && (3.27)\\
\mathtt{h}^\mu_{II} &= \mathtt{h}^\mu[\partial_r] \,, && (3.28)\\
\mathtt{h}^\mu_A &= \mathtt{h}^\mu[\partial_A] \,, && (3.29)\\
\mathtt{F}^2 &= \frac{1}{16\pi}\sqrt{|-\det g|}F^{\nu\beta}F_{\nu\beta} \,. && (3.30)
\end{aligned}$$

Written-out in detail, the functionals (3.27)-(3.30) read

$$\begin{aligned}
\mathtt{h}^u_I &= \frac{1}{4\pi}\big(r^2 F_{ur}^2 + \mathring{\gamma}^{AB}F_{uA}F_{rB}\big)\sqrt{\det\mathring{\gamma}} \,, && (3.31)\\
\mathtt{h}^r_I &= \frac{1}{4\pi}\big(\epsilon N^2\mathring{\gamma}^{AB}F_{rA}F_{uB} + \mathring{\gamma}^{AB}F_{uA}F_{uB}\big)\sqrt{\det\mathring{\gamma}} \,, && (3.32)\\
\mathtt{h}^u_{II} &= \frac{1}{4\pi}\mathring{\gamma}^{AB}F_{rA}F_{rB}\sqrt{\det\mathring{\gamma}} \,, && (3.33)\\
\mathtt{h}^r_{II} &= \frac{1}{4\pi}\Big(r^2 F_{ur}^2 + \mathring{\gamma}^{AB}F_{uA}F_{rB} + \epsilon N^2\mathring{\gamma}^{AB}F_{rA}F_{rB}\Big)\sqrt{\det\mathring{\gamma}} \,, && (3.34)\\
\mathtt{h}^u_A &= \frac{1}{4\pi}\Big(r^2 F_{ur}F_{Ar} + \mathring{\gamma}^{BC}F_{Br}F_{CA}\Big)\sqrt{\det\mathring{\gamma}} \,, && (3.35)\\
\mathtt{h}^r_A &= \frac{1}{4\pi}\Big(r^2 F_{ur}F_{uA} - \epsilon N^2\mathring{\gamma}^{BC}F_{rB}F_{CA} - \mathring{\gamma}^{BC}F_{uB}F_{CA}\Big)\sqrt{\det\mathring{\gamma}} \,, && (3.36)\\
\mathtt{F}^2 &= \frac{1}{16\pi}\Big(\frac{1}{r^2}\mathring{\gamma}^{AC}\mathring{\gamma}^{BD}F_{AB}F_{CD} - 2F_{ur}^2 - 2\epsilon N^2\mathring{\gamma}^{AB}F_{rA}F_{rB} \\
&\quad -4\mathring{\gamma}^{AB}F_{uA}F_{rB}\Big)\sqrt{\det\mathring{\gamma}} \,. && (3.37)
\end{aligned}$$

As in [4] we denote by $\mathscr{C}_u$ the light cone of constant u. One checks that all charge integrals over $\mathscr{C}_u$ are convergent. The most interesting charge is the energy-like integral associated with the motion of the tip of the light cone to the future along the flow of the Killing vector $\mathcal{T} \equiv \partial_u$; recall that ∂_u is timelike at the tip of the light cone so that each subsequent cone so obtained lies to the future of the preceding one. Letting

$$dS_\mu := \partial_\mu \rfloor dx^0 \wedge \cdots \wedge dx^n \,, \quad dS_{\mu\nu} := \partial_\mu \wedge \partial_\nu \rfloor dx^0 \wedge \cdots \wedge dx^n \equiv -\partial_\mu \rfloor dS_\nu \,,$$

and

$$d\mu_{\mathscr{C}} = \sqrt{\det g_{AB}}\, dr \wedge dx^2 \wedge dx^3 \,, \qquad d\mu_{\mathring{\gamma}} = \sqrt{\det \mathring{\gamma}_{AB}}\, dx^2 \wedge dx^3 \,, \tag{3.38}$$

we find

$$\begin{aligned} E_{\mathscr{H}}[\mathscr{C}_u] &:= \int_{\mathscr{C}_u} \mathscr{H}^\mu[\partial_u] dS_\mu = \int_{\mathscr{C}_u} \mathscr{H}^u[\partial_u] dS_u = \int_{\mathscr{C}_u} \left(\mathtt{h}^u_I + \mathtt{F}^2 \right) dr\, dx^2 dx^3 \\ &= \frac{1}{16\pi} \int_{\mathscr{C}_u} \Big(\frac{1}{r^2} \mathring{\gamma}^{AC} \mathring{\gamma}^{BD} F_{AB} F_{CD} + 2F_{ur}^2 - 2\epsilon N^2 \mathring{\gamma}^{AB} F_{rA} F_{rB} \Big) dr\, d\mu_{\mathring{\gamma}} \,. \end{aligned} \tag{3.39}$$

Likewise the total angular-momentum is obtained from the following integral:

$$J[\mathcal{R}] \;:=\; \int_{\mathscr{C}_u} \mathscr{H}^\mu[\mathcal{R}] dS_\mu \equiv R_i J^i \,, \tag{3.40}$$

where

$$\begin{aligned} J^i &:= \int_{\mathscr{C}_u} \varepsilon^{AB} \mathring{D}_B n^i \mathtt{h}^u_A \, dr\, dx^2 dx^3 \\ &= \frac{1}{4\pi} \int_{\mathscr{C}_u} \varepsilon^{AB} \mathring{D}_B n^i \Big(r^2 F_{ur} F_{Ar} + \mathring{\gamma}^{BC} F_{Br} F_{CA} \Big) dr\, d\mu_{\mathring{\gamma}} \,. \end{aligned} \tag{3.41}$$

For completeness we give the formulae for the remaining charges

$$\begin{aligned} P[\mathcal{P}_{dS}] &:= \int_{\mathscr{C}_u} \mathscr{H}^\mu[\mathcal{P}_{dS}] dS_\mu \\ &= p_i \int_{\mathscr{C}_u} \Big(e^{\alpha u} \Big[n^i \mathtt{h}^u_I - \big(\alpha r + 1\big) n^i \mathtt{h}^u_{II} - \frac{\alpha r + 1}{r} \mathring{D}^A (n^i) \mathtt{h}^u_A + n^i \mathtt{F}^2 \Big] \Big) dr\, dx^2 dx^3 \\ &= \frac{1}{16\pi} p_i \int_{\mathscr{C}_u} e^{\alpha u} \Big[n^i \Big(\frac{1}{r^2} \mathring{\gamma}^{AC} \mathring{\gamma}^{BD} F_{AB} F_{CD} + 2F_{ur}^2 \\ &\quad -2(\alpha r + 1)^2 \mathring{\gamma}^{AB} F_{Ar} F_{Br} \Big) - 4 \frac{\alpha r + 1}{r} \mathring{D}^A n^i \left(r^2 F_{ur} F_{Ar} + \mathring{\gamma}^{BC} F_{Br} F_{CA} \right) \Big] \, dr\, d\mu_{\mathring{\gamma}} \,, \end{aligned} \tag{3.42}$$

and

$$
\begin{aligned}
C[\mathcal{L}_{dS}] &:= \int_{\mathscr{C}_u} \mathscr{H}^\mu[\mathcal{L}_{dS}] dS_\mu \\
&= l_i \int_{\mathscr{C}_u} \left(e^{-\alpha u} \Big[n^i \mathtt{h}^u_I + \big(\alpha r - 1\big) n^i \mathtt{h}^u_{II} + \frac{\alpha r - 1}{r} \mathring{D}^A (n^i) \mathtt{h}^u_A + n^i \mathtt{F}^2 \Big] \right) dr\, dx^2 dx^3 \,. \quad (3.43)
\end{aligned}
$$

A more detailed formula for $C[\mathcal{L}_{dS}]$ can be obtained from (3.42) by replacing α by $-\alpha$ and p_i by l_i.

3.2 Noether charges in Minkowski spacetime

All the equations in Section 3.1 apply in Minkowski spacetime by taking the limit $\alpha \to 0$. Indeed, the Killing fields for Minkowski spacetime can be obtained as a limit of those for de Sitter spacetime. In the notation of Appendix A, the equations (A.3), (A.4) and (A.36) give

$$
\mathcal{P} = -\frac{1}{2} \lim_{\alpha \to 0} \Big(\mathcal{P}_{dS} + \mathcal{L}_{dS} \Big) \,, \quad (3.44)
$$

where in (A.36)-(A.37) we set $P_i = p_i = l_i$. Similarly, (A.3), (A.4) and (A.37) leads to

$$
\mathcal{L} = \frac{1}{2} \lim_{\alpha \to 0} \Big(\frac{\mathcal{L}_{dS} - \mathcal{P}_{dS}}{\alpha} \Big) \,, \quad (3.45)
$$

where in (A.36)-(A.37) we set $L_i = p_i = l_i$. This shows that for Minkowski spacetime, the linear momentum is given by

$$
P_M = -\frac{1}{2} \lim_{\alpha \to 0} \big(P[\mathcal{P}_{dS}] + C[\mathcal{L}_{dS}] \big) \,, \quad (3.46)
$$

while the center of mass

$$
C_M = \frac{1}{2} \lim_{\alpha \to 0} \frac{\big(C[\mathcal{L}_{dS}] - P[\mathcal{P}_{dS}] \big)}{\alpha} \,. \quad (3.47)
$$

Finally the equations for angular momentum and energy are obvious. One checks that all the limits exist.

3.3 Noether charges in anti-de Sitter spacetime

All the equations in Section 3.1 apply in anti-de Sitter spacetime under the resplacement $\alpha \mapsto \sqrt{-1}\alpha$. We note that under this replacement both the energy and the angular momentum remain real, while P and C become linear combinations of two linearly independent real-valued charges:

$$\begin{aligned} P[\mathcal{P}_{adS}] &:= \int_{\mathscr{C}_u} \mathscr{H}^\mu[\mathcal{P}_{adS}] dS_\mu \\ &= \widetilde{p}_i \int_{\mathscr{C}_u} \Big[n^i \cos(\widetilde{\alpha} u) \mathtt{h}^u_I + n^i (\widetilde{\alpha} r \sin(\widetilde{\alpha} u) - \cos(\widetilde{\alpha} u)) \mathtt{h}^u_{II} \\ &\quad + \frac{(\widetilde{\alpha} r \sin(\widetilde{\alpha} u) - \cos(\widetilde{\alpha} u))}{r} \mathring{D}^A n^i \mathtt{h}^u_A + n^i \cos(\widetilde{\alpha} u) \mathtt{F}^2 \Big] \, dr \, dx^2 dx^3 \,, \end{aligned} \tag{3.48}$$

and

$$\begin{aligned} C[\mathcal{L}_{adS}] &:= \int_{\mathscr{C}_u} \mathscr{H}^\mu[\mathcal{L}_{adS}] dS_\mu \\ &= \widetilde{l}_i \int_{\mathscr{C}_u} \Big[n^i \sin(\widetilde{\alpha} u) \mathtt{h}^u_I - n^i (\sin(\widetilde{\alpha} u) + \widetilde{\alpha} r \cos(\widetilde{\alpha} u)) \mathtt{h}^u_{II} \\ &\quad - \frac{(\sin(\widetilde{\alpha} u) + \widetilde{\alpha} r \cos(\widetilde{\alpha} u))}{r} \mathring{D}^A n^i \mathtt{h}^u_A + n^i \sin(\widetilde{\alpha} u) \mathtt{F}^2 \Big] \, dr \, dx^2 dx^3 \,. \end{aligned} \tag{3.49}$$

3.4 The evolution of Noether charges

In this section we address the question of the rate of change of the charge integrals as the tip of the light cone is moved to the future along the flow of the Killing vector $\partial_u \equiv \mathcal{T}$:

$$\frac{dH[X, \mathscr{C}_u]}{du} \equiv \frac{d}{du} \int_{\mathscr{C}_u} \mathscr{H}^\mu[X] dS_\mu = \int_{\mathscr{C}_u} \mathcal{L}_{\partial_u} \mathscr{H}^\mu[X] dS_\mu \,. \tag{3.50}$$

Assuming sourceless Maxwell fields, (3.21) with two Killing vector fields X and Y reads

$$\mathcal{L}_Y \mathscr{H}^\mu[X] = -\frac{\sqrt{|-\det g|}}{2\pi} \nabla_\sigma \left[Y^{[\sigma} F^{\mu]\alpha} X^\kappa F_{\kappa\alpha} - \frac{1}{4} Y^{[\sigma} X^{\mu]} F^{\alpha\beta} F_{\alpha\beta} \right] \,. \tag{3.51}$$

Using the fields (3.26)-(3.30) one finds

$$\begin{aligned}
\mathcal{L}_{\partial_u}\mathscr{H}^\mu[\partial_u] &= 2\nabla_\sigma\Big[\mathcal{T}^{[\sigma}\mathtt{h}_I^{\mu]}\Big]\,, &(3.52)\\
\mathcal{L}_{\partial_u}\mathscr{H}^\mu[\mathcal{R}] &= 2\nabla_\sigma\Big\{\mathcal{T}^{[\sigma}\Big[\varepsilon^{AB}\mathring{D}_B(R_in^i)\mathtt{h}_A^{\mu]}+\mathcal{R}^{\mu]}\mathtt{F}^2\Big]\Big\}\,, &(3.53)\\
\mathcal{L}_{\partial_u}\mathscr{H}^\mu[\mathcal{P}_{dS}] &= 2\nabla_\sigma\Big\{e^{\alpha u}\mathcal{T}^{[\sigma}\Big[p_in^i\mathtt{h}_I^{\mu]}-\big(\alpha r+1\big)p_in^i\mathtt{h}_{II}^{\mu]}\\
&\quad -\frac{\alpha r+1}{r}\mathring{D}^A(p_in^i)\mathtt{h}_A^{\mu]}+\mathcal{P}_{dS}^{\mu]}\mathtt{F}^2\Big]\Big\}\,, &(3.54)\\
\mathcal{L}_{\partial_u}\mathscr{H}^\mu[\mathcal{L}_{dS}] &= 2\nabla_\sigma\Big\{e^{-\alpha u}\mathcal{T}^{[\sigma}\Big[l_in^i\mathtt{h}_{II}^{\mu]}+\big(\alpha r-1\big)l_in^i\mathtt{h}_{II}^{\mu]}\\
&\quad +\frac{\alpha r-1}{r}\mathring{D}^A(l_in^i)\mathtt{h}_A^{\mu]}+\mathcal{L}_{dS}^{\mu]}\mathtt{F}^2\Big]\Big\}\,. &(3.55)
\end{aligned}$$

In particular we obtain a formula for the flux of energy:

$$\begin{aligned}
\frac{dE_{\mathscr{H}}[\mathscr{C}_u]}{du} &= -2\int_{\partial\mathscr{S}_\tau}\mathcal{T}^{[\sigma}\mathtt{h}_I^{\mu]}\,dS_{\sigma\mu}\\
&= -\lim_{R\to\infty}\int_{S_R}\mathtt{h}_I^r\Big|_{r=R}dx^2dx^3\\
&= -\lim_{R\to\infty}\frac{1}{4\pi}\int_{S_R}\Big[r^2F_{ur}^2+\mathring{\gamma}^{AB}F_{uA}F_{rB}+\epsilon N^2\mathring{\gamma}^{AB}F_{rA}F_{rB}\Big]_{r=R}d\mu_{\mathring{\gamma}}\\
&= -\frac{1}{4\pi}\int_{S_\infty}\Big[\mathring{\gamma}^{AB}\big(\alpha^2\overset{(-2)}{F}{}_{Ar}\overset{(0)}{F}{}_{Bu}+\overset{(0)}{F}{}_{Au}\overset{(0)}{F}{}_{Bu}\big)\Big]d\mu_{\mathring{\gamma}}\,. \qquad (3.56)
\end{aligned}$$

The u-derivative of angular momentum is given by

$$\begin{aligned}
\frac{dJ[\mathscr{C}_{u,R}]}{du} &= -2\int_{\partial\mathscr{S}_\tau}\Big[\mathcal{T}^{[\sigma}\Big(\mathtt{h}_A^{\mu]}\varepsilon^{AB}\mathring{D}_B(R_in^i)+\mathcal{R}^{\mu]}\mathtt{F}^2\Big)\Big]\,dS_{\sigma\mu}\\
&= -R_i\lim_{R\to\infty}\int_{S_R}\Big[\mathtt{h}_A^r\varepsilon^{AB}\mathring{D}_Bn^i\Big]_{r=R}dx^2dx^3 =: R_i\frac{dJ^i}{du}\,, \qquad (3.57)
\end{aligned}$$

where

$$\begin{aligned}
\frac{dJ^i}{du} &= -\frac{1}{4\pi}\lim_{R\to\infty}\int_{S_R}\varepsilon^{AB}\mathring{D}_Bn^i\Big[r^2F_{ur}F_{uA}-\epsilon N^2\mathring{\gamma}^{BC}F_{rB}F_{CA}\\
&\quad -\mathring{\gamma}^{BC}F_{uB}F_{CA}\Big]_{r=R}d\mu_{\mathring{\gamma}}\\
&= -\frac{1}{4\pi}\int_{S_\infty}\Big[\varepsilon^{AB}\mathring{D}_B(n^i)\Big(\mathring{\gamma}^{BC}\big(\alpha^2\overset{(-2)}{F}{}_{Br}+\overset{(0)}{F}{}_{Bu}\big)\overset{(0)}{F}{}_{CA}\\
&\quad -\overset{(-2)}{F}{}_{ur}\overset{(0)}{F}{}_{Au}\Big)\Big]d\mu_{\mathring{\gamma}}\,. \qquad (3.58)
\end{aligned}$$

Finally

$$
\begin{aligned}
\frac{dP[\mathcal{P}_{dS},\mathscr{C}_u]}{du} &= -2\int_{\partial\mathscr{S}_\tau} \Big[e^{\alpha u}\mathcal{T}^{[\sigma}\Big(p_i n^i \mathtt{h}_I^{\mu]} - (\alpha r+1)p_i n^i \mathtt{h}_{II}^{\mu]} \\
&\quad -\frac{\alpha r+1}{r}\mathring{D}^A(p_i n^i)\mathtt{h}_A^{\mu]} + \mathcal{P}_{dS}^{\mu]}\mathtt{F}^2\Big)\Big]\, dS_{\sigma\mu} \\
&= -\lim_{R\to\infty}\int_{S_R} e^{\alpha u}\Big[\Big(p_i n^i \mathtt{h}_I^r - (\alpha r+1)p_i n^i \mathtt{h}_{II}^r \\
&\quad -\frac{\alpha r+1}{r}\mathring{D}^A(p_i n^i)\mathtt{h}_A^r + \mathcal{P}_{dS}^r\mathtt{F}^2\Big)\Big]_{r=R} dx^2 dx^3 \\
&= -\frac{1}{16\pi}\lim_{R\to\infty}\int_{S_R} e^{\alpha u}\Big[p_i n^i\Big(4\epsilon N^2\mathring{\gamma}^{AB}F_{uA}F_{rB} + 4\mathring{\gamma}^{AB}F_{uA}F_{uB} \\
&\quad -(\alpha r+1)\big(\frac{1}{r^2}\mathring{\gamma}^{AC}\mathring{\gamma}^{BD}F_{AB}F_{CD} + 2r^2F_{ur}^2 + 2\varepsilon N^2\mathring{\gamma}^{AB}F_{rA}F_{rB}\big)\Big) \\
&\quad +4p_i\mathring{D}^A n^i\Big(r^2F_{ur}F_{uA} - \epsilon N^2\mathring{\gamma}^{BC}F_{rB}F_{CA} - \mathring{\gamma}^{BC}F_{uB}F_{CA}\Big)\Big]_{r=R} d\mu_{\mathring{\gamma}} \\
&= -\frac{1}{4\pi}\int_{S_\infty}\Big\{e^{\alpha u}\Big[p_i n^i\mathring{\gamma}^{AB}\big(\overset{(0)}{F}_{Au}\overset{(0)}{F}_{Bu} + \alpha^2\overset{(-2)}{F}_{Ar}\overset{(0)}{F}_{Bu}\big) \\
&\quad +\alpha\mathring{D}^A(p_i n^i)\Big(\overset{(-2)}{F}_{ur}\overset{(0)}{F}_{Au} + \overset{(0)}{F}_{AB}\mathring{\gamma}^{BC}\big(\alpha^2\overset{(-2)}{F}_{Cr} + \overset{(0)}{F}_{Cu}\big)\Big)\Big]\Big\} d\mu_{\mathring{\gamma}}\,, \qquad (3.59)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{dC[\mathcal{L}_{dS},\mathscr{C}_u]}{du} &= -2\int_{\partial\mathscr{S}_\tau}\Big\{e^{-\alpha u}\mathcal{T}^{[\sigma}\Big[l_i n^i\mathtt{h}_I^{\mu]} + (\alpha r-1)l_i n^i\mathtt{h}_{II}^{\mu]} \\
&\quad +\frac{\alpha r-1}{r}\mathring{D}^A(l_i n^i)\mathtt{h}_A^{\mu]} + \mathcal{L}_{dS}^{\mu]}\mathtt{F}^2\Big]\Big\}\, dS_{\sigma\mu} \\
&= -\lim_{R\to\infty}\int_{S_R} e^{-\alpha u}\Big[l_i n^i\mathtt{h}_I^r + (\alpha r-1)l_i n^i\mathtt{h}_{II}^r \\
&\quad +\frac{\alpha r-1}{r}\mathring{D}^A(l_i n^i)\mathtt{h}_A^r + (\alpha r-1)l_i n^i\mathtt{F}^2\Big]_{r=R} dx^2dx^3 \\
&= -\frac{1}{4\pi}\int_{S_\infty} e^{-\alpha u}\Big\{l_i n^i\mathring{\gamma}^{AB}\big(\overset{(0)}{F}_{Au}\overset{(0)}{F}_{Bu} + \alpha^2\overset{(-2)}{F}_{Ar}\overset{(0)}{F}_{Bu}\big) \\
&\quad -\alpha\mathring{D}^A(l_i n^i)\Big[\overset{(-2)}{F}_{ur}\overset{(0)}{F}_{Au} + \overset{(0)}{F}_{AB}\mathring{\gamma}^{BC}\Big(\alpha^2\overset{(-2)}{F}_{Cr} + \overset{(0)}{F}_{Cu}\Big)\Big]\Big\} d\mu_{\mathring{\gamma}}\,. \qquad (3.60)
\end{aligned}
$$

4 Noether charges for scalar fields

In [4] we found that the canonical energy on light cones for a natural class of linear scalar fields in de Sitter spacetime was generically infinite, and had to be renormalised. The aim of this section is to address this question for the remaining canonical charges.

In our signature the Lagrangian reads

$$\mathscr{L} = -\frac{1}{2}\sqrt{|\det g|}\big(g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi+m^2\phi^2\big)\,, \tag{4.1}$$

for a constant m.

The theory coincides with its linearisation and we will therefore not make a distinction between the fields φ and its linearised counterpart $\tilde\varphi$, as done in [4].

The canonical energy-momentum current $\mathscr{H}^\mu$ equals

$$\mathscr{H}^\mu[X] = -\sqrt{|\det g|}\Big(\nabla^\mu\phi\,\mathcal{L}_X\phi - \frac{1}{2}\big(\nabla^\alpha\phi\nabla_\alpha\phi+m^2\phi^2\big)X^\mu\Big)\,. \tag{4.2}$$

Analogously to our analysis of the Maxwell field, we start by considering simultaneously the Minkowski space-time and the de Sitter space-time in coordinates as in (2.1).

The Lie derivative of the Hamiltonian (4.2) reads:

$$\begin{aligned}
-\frac{\mathcal{L}_Y\mathscr{H}^\mu[X]}{\sqrt{|\det g|}} &= \nabla_\sigma\big(Y^\sigma\nabla^\mu\phi\,\mathcal{L}_X\phi\big) - \nabla_\sigma Y^\mu\nabla^\sigma\phi\,\mathcal{L}_X\phi \\
&\quad -\frac{1}{2}\nabla_\sigma\Big[Y^\sigma\big(\nabla^\alpha\phi\nabla_\alpha\phi+m^2\phi^2\big)X^\mu\Big] \\
&\quad +\frac{1}{2}\nabla_\sigma Y^\mu X^\sigma\big(\nabla^\alpha\phi\nabla_\alpha\phi+m^2\phi^2\big) \\
&= 2\nabla_\sigma\big(Y^{[\sigma}\nabla^{\mu]}\phi\,\mathcal{L}_X\phi\big) + Y^\mu\nabla_\sigma\big(\nabla^\sigma\phi\,\mathcal{L}_X\phi\big) \\
&\quad -\frac{1}{2}X^\mu Y^\sigma\nabla_\sigma\big(\nabla^\alpha\phi\nabla_\alpha\phi+m^2\phi^2\big) \\
&\quad +\frac{1}{2}[X,Y]^\mu\big(\nabla^\alpha\phi\nabla_\alpha\phi+m^2\phi^2\big) \\
&\quad -\frac{1}{2}\nabla_\sigma Y^\sigma X^\mu\big(\nabla^\alpha\phi\nabla_\alpha\phi+m^2\phi^2\big)\,.
\end{aligned} \tag{4.3}$$

We combine the second and third terms with the equation of motion:

$$\begin{aligned}
&Y^\mu\nabla_\sigma\big(\nabla^\sigma\phi\,\mathcal{L}_X\phi\big) - \frac{1}{2}X^\mu Y^\sigma\nabla_\sigma\big(\nabla^\alpha\phi\nabla_\alpha\phi+m^2\phi^2\big) \\
&= \frac{1}{2}Y^\mu X^\sigma\nabla_\sigma\big(\nabla^\alpha\phi\nabla_\alpha\phi+m^2\phi^2\big) + Y^\mu\nabla^\alpha\phi\nabla_\alpha X^\sigma\nabla_\sigma\phi \\
&\quad -\frac{1}{2}X^\mu Y^\sigma\nabla_\sigma\big(\nabla^\alpha\phi\nabla_\alpha\phi+m^2\phi^2\big) \\
&= \nabla_\sigma\big[Y^{[\mu}X^{\sigma]}\big(\nabla^\alpha\phi\nabla_\alpha\phi+m^2\phi^2\big)\big] + Y^\mu\nabla^\alpha\phi\nabla_\alpha X^\sigma\nabla_\sigma\phi \\
&\quad -\frac{1}{2}\Big([X,Y]^\mu + Y^\mu\nabla_\sigma X^\sigma - X^\mu\nabla_\sigma Y^\sigma\Big)\big(\nabla^\alpha\phi\nabla_\alpha\phi+m^2\phi^2\big)\,.
\end{aligned} \tag{4.4}$$

Equations (4.3) and (4.4) lead to

$$
\begin{aligned}
-\frac{\mathcal{L}_Y \mathscr{H}^\mu[X]}{\sqrt{|\det g|}} &= 2\nabla_\sigma\Big(Y^{[\sigma}\nabla^{\mu]}\phi X^\alpha\nabla_\alpha\phi - \frac{1}{2}Y^{[\sigma}X^{\mu]}\big(\nabla^\alpha\phi\nabla_\alpha\phi + m^2\phi^2\big)\Big) \\
&\quad + Y^\mu\nabla_\alpha X^\sigma\nabla^\alpha\phi\nabla_\sigma\phi - \frac{1}{2}Y^\mu\nabla_\sigma X^\sigma\big(\nabla^\alpha\phi\nabla_\alpha\phi + m^2\phi^2\big) \\
&= 2\nabla_\sigma\Big(Y^{[\sigma}\nabla^{\mu]}\phi X^\alpha\nabla_\alpha\phi - \frac{1}{2}Y^{[\sigma}X^{\mu]}\big(\nabla^\alpha\phi\nabla_\alpha\phi + m^2\phi^2\big)\Big) \\
&\quad + Y^\mu\Big(\nabla_\alpha X_\sigma - \frac{1}{2}\nabla_\kappa X^\kappa g_{\alpha\sigma}\Big)\nabla^\alpha\phi\nabla^\sigma\phi \\
&\quad - \frac{1}{2}Y^\mu\nabla_\sigma X^\sigma m^2\phi^2\,.
\end{aligned}
\tag{4.5}
$$

4.1 Charges in (anti)-de Sitter spacetime

We only consider here a massive scalar field, with the mass chosen so that the equation is conformally covariant,

$$
\Box_g\phi - \underbrace{\frac{(d-2)R(g)}{4(d-1)}}_{=:m^2}\phi = 0\,, \tag{4.6}
$$

where d is the dimension of spacetime and $R(g)$ is the scalar curvature of g. In the four-dimensional case, it leads to

$$
m^2 = 2\alpha^2 \tag{4.7}
$$

After a conformal transformation $g \mapsto \Omega^2 g$ the field $\Omega^{d/2-1}\phi$ satisfies again (4.6), with g there replaced by $\Omega^2 g$. This is useful in that solutions of (4.6) with smooth initial data on a Cauchy surface in de Sitter spacetime extend smoothly, afer the rescaling above, in local coordinates on the conformally completed manifold, across the conformal boundary at infinity. This translates to the following asymptotic behaviour of ϕ, for large r, in spacetime dimension four:

$$
\phi(u,r,x^A) = \frac{\overset{\text{(-1)}}{\phi}(u,x^A)}{r} + \frac{\overset{\text{(-2)}}{\phi}(u,x^A)}{r^2} + \frac{\overset{\text{(-3)}}{\phi}(u,x^A)}{r^3} + \ldots\,. \tag{4.8}
$$

See [4, Section 2.2.1] for a discussion. Here we simply note that the functions $\overset{\text{(-1)}}{\phi}$ and $\overset{\text{(-2)}}{\phi}$ are freely prescribable, with all remaining expansion coefficients determined uniquely by these two.

We wish to construct the Noether charges associated with the Killing fields (A.1)-(A.4). For this let

$$\begin{aligned} \mathtt{h}^\mu[X] &= -\sqrt{|\det g|}\nabla^\mu\phi X^\alpha\nabla_\alpha\phi\,, && (4.9)\\ \mathtt{h}_L &= \frac{1}{2}\sqrt{|\det g|}\big(\nabla^\alpha\phi\nabla_\alpha\phi+m^2\phi^2\big)\,, && (4.10)\end{aligned}$$

and

$$\begin{aligned} \mathtt{h}^\mu_I &= \mathtt{h}^\mu[\partial_u]\,, && (4.11)\\ \mathtt{h}^\mu_{II} &= \mathtt{h}^\mu[\partial_r]\,, && (4.12)\\ \mathtt{h}^\mu_A &= \mathtt{h}^\mu[\partial_A]\,. && (4.13)\end{aligned}$$

Since the Hamiltonian density (4.2) is linear in the Hamiltonian vector field X, each charge is given by an integral of a linear combination of the following four functionals

$$\begin{aligned} \mathscr{H}^\mu[\partial_u] &= \mathtt{h}^\mu_I+\mathcal{T}^\mu\mathtt{h}_L\,, && (4.14)\\ \mathscr{H}^\mu[\mathcal{R}] &= \varepsilon^{AB}\mathring{D}_B(R_in^i)\mathtt{h}^\mu_A+\mathcal{R}^\mu\mathtt{h}_L\,, && (4.15)\\ \mathscr{H}^\mu[\mathcal{P}_{dS}] &= e^{\alpha u}\Big[p_in^i\mathtt{h}^\mu_I-\big(\alpha r+1\big)p_in^i\mathtt{h}^\mu_{II} \\ &\quad -\frac{\alpha r+1}{r}\mathring{D}^A(p_in^i)\mathtt{h}^\mu_A\Big]+\mathcal{P}^\mu_{dS}\mathtt{h}_L\,, && (4.16)\\ \mathscr{H}^\mu[\mathcal{L}_{dS}] &= e^{-\alpha u}\Big[l_in^i\mathtt{h}^\mu_I+\big(\alpha r-1\big)l_in^i\mathtt{h}^\mu_{II} \\ &\quad +\frac{\alpha r-1}{r}\mathring{D}^A(l_in^i)\mathtt{h}^\mu_A\Big]+\mathcal{L}^\mu_{dS}\mathtt{h}_L\,, && (4.17)\end{aligned}$$

Written-out in detail, the functionals (4.11)-(4.13) read

$$\begin{aligned} \mathtt{h}^u_I &= \big(r^2\partial_r\phi\partial_u\phi\big)\sqrt{\det\mathring{\gamma}}\,, && (4.18)\\ \mathtt{h}^r_I &= r^2\big(\partial_u\phi+\big(\alpha^2r^2-1\big)\partial_r\phi\big)\partial_u\phi\sqrt{\det\mathring{\gamma}}\,, && (4.19)\\ \mathtt{h}^u_{II} &= r^2\big(\partial_r\phi\big)^2\sqrt{\det\mathring{\gamma}}\,, && (4.20)\\ \mathtt{h}^r_{II} &= r^2\big(\partial_u\phi+\big(\alpha^2r^2-1\big)\partial_r\phi\big)\partial_r\phi\sqrt{\det\mathring{\gamma}}\,, && (4.21)\\ \mathtt{h}^u_A &= r^2\partial_r\phi\mathring{D}_A\phi\sqrt{\det\mathring{\gamma}}\,, && (4.22)\\ \mathtt{h}^r_A &= r^2\big(\partial_u\phi+\big(\alpha^2r^2-1\big)\partial_r\phi\big)\mathring{D}_A\phi\sqrt{\det\mathring{\gamma}}\,, && (4.23)\\ \mathtt{h}_L &= \frac{1}{2}\big(\mathring{\gamma}^{AB}\mathring{D}_A\phi\mathring{D}_B\phi+m^2r^2\phi^2 \\ &\quad -2r^2\partial_r\phi\partial_u\phi+\big(1-\alpha^2r^2\big)r^2\big(\partial_r\phi\big)^2\big)\sqrt{\det\mathring{\gamma}}\,. && (4.24)\end{aligned}$$

Recall that we denote by $\mathscr{C}_u$ the light cone of constant u, and $\mathscr{C}_{u,R} = \mathscr{C}_u \cap \{r \le R\}$ its truncation to $r = R$. It turns out that, generically, all charge integrals over $\mathscr{C}_u$ diverge as R tends to infinity, and therefore need to be renormalised. Therefore we first calculate the charges on $\mathscr{C}_{u,R}$ and exhibit their divergent parts, for large r. We use the asymptotics (4.8) which applies both to the $\alpha = 0$ case with $m = 0$ and to the case $\alpha^2 = m^2/2$ with $m \neq 0$:

$$
\begin{aligned}
E_{\mathscr{H}}[\mathscr{C}_{u,R}] &:= \int_{\mathscr{C}_{u,R}} \mathscr{H}^\mu[\partial_u] dS_\mu = \int_{\mathscr{C}_{u,R}} \mathscr{H}^u[\partial_u] dS_u \\
&= \int_{\mathscr{C}_{u,R}} \big(\mathtt{h}^u_I + \mathtt{h}_L\big)\, dr\, dx^2 dx^3 \\
&= \frac{1}{2}\int_{\mathscr{C}_{u,R}} \Big(\mathring{\gamma}^{AB}\mathring{D}_A\phi\mathring{D}_B\phi + m^2 r^2\phi^2 + \big(r^2 - \alpha^2 r^4\big)\big(\partial_r\phi\big)^2\Big)\, dr\, d\mu_{\mathring{\gamma}} \\
&= \frac{1}{2}\int_{\mathscr{C}_{u,R}} \Big(\mathring{\gamma}^{AB}\mathring{D}_A\phi\mathring{D}_B\phi + m^2 r^2\phi^2 \\
&\quad +\partial_r\Big[\big(r^2 - \alpha^2 r^4\big)\phi\big(\partial_r\phi\big)\Big] - \phi\partial_r\big[\big(r^2-\alpha^2 r^4\big)\big(\partial_r\phi\big)\big]\Big)\, dr\, d\mu_{\mathring{\gamma}} \\
&= \frac{\alpha^2 R}{2}\int_{S_R} (\overset{\text{(-1)}}{\phi})^2\, d\mu_{\mathring{\gamma}} + \int_{\mathscr{C}_{u,R}} O(r^{-2})\, dr\, d\mu_{\mathring{\gamma}}\,, \qquad (4.25)
\end{aligned}
$$

where we have used

$$
\big(r^2 - \alpha^2 r^4\big)\phi\big(\partial_r\phi\big) = \frac{r}{2}\alpha^2(\overset{\text{(-1)}}{\phi})^2 + O(r^{-1}) \qquad (4.26)
$$

As before, the total angular-momentum is obtained from the following integral:

$$
J[\mathscr{C}_{u,R}] \ := \ \int_{\mathscr{C}_{u,R}} \mathscr{H}^\mu[\mathcal{R}] dS_\mu \equiv R_i J^i[\mathscr{C}_{u,R}]\,, \qquad (4.27)
$$

where now

$$
\begin{aligned}
J^i[\mathscr{C}_{u,R}] &:= \int_{\mathscr{C}_{u,R}} \varepsilon^{AB}\mathring{D}_B n^i \mathtt{h}^u_A\, dr\, dx^2 dx^3 \\
&= \int_{\mathscr{C}_{u,R}} r^2\varepsilon^{AB}\mathring{D}_B n^i \mathring{D}_A\phi\partial_r\phi\, dr\, d\mu_{\mathring{\gamma}} \\
&= \int_{\mathscr{C}_{u,R}} \Big(-\frac{\overset{\text{(-1)}}{\phi}\,\varepsilon^{AB}\mathring{D}_B n^i \mathring{D}_A \overset{\text{(-1)}}{\phi}}{r} + O(r^{-2})\Big)\, dr\, d\mu_{\mathring{\gamma}} \\
&= \int_{\mathscr{C}_{u,R}} O(r^{-2})\, dr\, d\mu_{\mathring{\gamma}}\,, \qquad (4.28)
\end{aligned}
$$

where we have used

$$\int_{S^2} \overset{(-1)}{\phi}\, \varepsilon^{AB} \mathring{D}_B n^i \mathring{D}_A \overset{(-1)}{\phi}\, d\mu_{\mathring{\gamma}} = \frac{1}{2}\int_{S^2} \mathring{D}_A \left(\varepsilon^{AB} \mathring{D}_B n^i (\overset{(-1)}{\phi})^2 \right) d\mu_{\mathring{\gamma}} = 0\,. \quad (4.29)$$

We further have

$$\begin{aligned}
P[\mathcal{P}_{dS}, \mathscr{C}_{u,R}] &:= \int_{\mathscr{C}_{u,R}} \mathscr{H}^\mu[\mathcal{P}_{dS}] dS_\mu \\
&= p_i \int_{\mathscr{C}_{u,R}} e^{\alpha u} \Big[n^i \mathtt{h}^u_I - (\alpha r + 1) n^i \mathtt{h}^u_{II} - \frac{\alpha r + 1}{r} \mathring{D}^A (n^i) \mathtt{h}^u_A + n^i \mathtt{h}_L \Big] \, dr\, dx^2 dx^3 \\
&= p_i \int_{\mathscr{C}_{u,R}} e^{\alpha u} \Big[\frac{1}{2} n^i \Big(\mathring{\gamma}^{AB} \mathring{D}_A \phi \mathring{D}_B \phi + m^2 \phi^2 r^2 \\
&\quad - (\alpha^2 r^4 + 2\alpha r^3 + r^2) (\partial_r \phi)^2 \Big) - (\alpha r + 1) r \mathring{D}^A n^i \, (\partial_r \phi)\, \mathring{D}_A \phi \Big] \, dr \, d\mu_{\mathring{\gamma}} \\
&= p_i \int_{\mathscr{C}_{u,R}} \mathrm{e}^{\alpha u} \Big[\frac{1}{2} \alpha^2 n^i \overset{(-1)}{\phi}{}^2 + \frac{1}{r} \overset{(-1)}{\phi} \left(\alpha \mathring{\gamma}^{AB} \mathring{D}_A n^i \mathring{D}_B \overset{(-1)}{\phi} - n^i \alpha \overset{(-1)}{\phi} \right) + O(r^{-2}) \Big] \, dr \, d\mu_{\mathring{\gamma}} \\
&= p_i \Big[\frac{R \alpha^2 \mathrm{e}^{\alpha u}}{2} \int_{S_R} n^i \overset{(-1)}{\phi}{}^2 \, d\mu_{\mathring{\gamma}} + \int_{\mathscr{C}_{u,R}} O(r^{-2}) \, dr \, d\mu_{\mathring{\gamma}} \Big] \,,
\end{aligned} \quad (4.30)$$

and note that the second and third terms in the before-last line integrate out to zero. Finally,

$$\begin{aligned}
C[\mathcal{L}_{dS}, \mathscr{C}_{u,R}] &:= \int_{\mathscr{C}_{u,R}} \mathscr{H}^\mu[\mathcal{L}_{dS}] dS_\mu \\
&= l_i \int_{\mathscr{C}_{u,R}} \Big(e^{-\alpha u} \Big[n^i \mathtt{h}^u_I + (\alpha r - 1) n^i \mathtt{h}^u_{II} + \frac{\alpha r - 1}{r} \mathring{D}^A (n^i) \mathtt{h}^u_A + n^i \mathtt{h}_L \Big] \Big) \, dr\, dx^2 dx^3 \,.
\end{aligned} \quad (4.31)$$

Similarly to the case of the Maxwell field, a more detailed formula version of (4.31) can be obtained from (4.30) by replacing there α by $-\alpha$ and p_i by l_i.

4.2 Noether charges in Minkowski spacetime

All the equations in Section 4.1 apply to the massless scalar field in Minkowski spacetime by passing to the limit $m = \alpha = 0$. In that case we clearly have a finite energy. This is also clear for the total momentum, which we denote by P_M, using (4.30)-(4.31) with P_i replaced by p_i for consistency of notation with (A.27) (see (A.36)), compare (A.38)):

$$\begin{aligned} P_M[\mathcal{P},\mathscr{C}_{u,R}] &= -\frac{1}{2}\lim_{\alpha\to 0}\big(P[\mathcal{P}_{dS},\mathscr{C}_{u,r}]+C[\mathcal{L}_{dS},\mathscr{C}_{u,r}]_{l_i:=P_i}\big)\,,\\ &= P_i\int_{\mathscr{C}_{u,R}}\Big(n^i\,(\mathtt{h}^u_{II}-\mathtt{h}^u_{I})+\frac{1}{r}\mathring{D}^A n^i\mathtt{h}^u_A-n^i\mathtt{h}_L\Big)\Big|_{\alpha=0}\,dr\,d\mu_{\mathring{\gamma}}\\ &= \int_{\mathscr{C}_{u,R}} O(r^{-2})\,dr\,d\mu_{\mathring{\gamma}}\,. \end{aligned} \tag{4.32}$$

Consider, next, the formula for the center of mass, which can be similarly obtained from (A.39) and (4.30)-(4.31): in the notation of (A.30),

$$\begin{aligned} C_M[\mathcal{L},\mathscr{C}_{u,R}] &= \frac{1}{2}\lim_{\alpha\to 0}\frac{\big(C[\mathcal{L}_{dS},\mathscr{C}_{u,r}]-P[\mathcal{P}_{dS},\mathscr{C}_{u,r}]_{p_i:=L_i}\big)}{\alpha}\\ &= L_i\int_{\mathscr{C}_{u,R}}\Big(-un^i\mathtt{h}^u_I+(u+r)n^i\mathtt{h}^u_{II}+\big(1+\frac{u}{r}\big)\mathring{D}^A n^i\mathtt{h}^u_A-un^i\mathtt{h}_L\Big)\,dr\,d\mu_{\mathring{\gamma}}\\ &= \int_{\mathscr{C}_{u,R}} O(r^{-2})\,dr\,d\mu_{\mathring{\gamma}}\,, \end{aligned} \tag{4.33}$$

Finally, the total angular momentum is finite,

$$\begin{aligned} \hat{J}^i[\mathscr{C}_u] \;:=\; \lim_{R\to\infty}\Big[&\int_{\mathscr{C}_{u,R}} r^2\varepsilon^{AB}\mathring{D}_B n^i\mathring{D}_A\phi\partial_r\phi\,dr\,d\mu_{\mathring{\gamma}}\\ &+\ln R\underbrace{\int_{S^2}\varepsilon^{AB}\mathring{D}_B n^i\overset{(-1)}{\phi}\mathring{D}_A\overset{(-1)}{\phi}\,d\mu_{\mathring{\gamma}}}_{0}\Big]\,, \end{aligned} \tag{4.34}$$

as the boundary integral in (4.34) is a total divergence.

4.3 Noether charges in anti-de Sitter spacetime

All the equations in Section 4.1 apply in anti-de Sitter spacetime under the replacement $\alpha\mapsto\sqrt{-1}\alpha$. Then both the energy and the angular momentum remain real, while P and C become linear combinations of two linearly independent real-valued charges. Indeed, using (A.25)-(A.26) one finds

$$\begin{aligned} &P[\mathcal{P}_{adS},\mathscr{C}_{u,R}] := \int_{\mathscr{C}_{u,R}}\mathscr{H}^\mu[\mathcal{P}_{adS}]dS_\mu\\ &= \widetilde{p}_i\int_{\mathscr{C}_{u,R}}\Big[n^i\cos(\widetilde{\alpha}u)\mathtt{h}^u_I+n^i(\widetilde{\alpha}r\sin(\widetilde{\alpha}u)-\cos(\widetilde{\alpha}u))\mathtt{h}^u_{II}\\ &\quad+\frac{(\widetilde{\alpha}r\sin(\widetilde{\alpha}u)-\cos(\widetilde{\alpha}u))}{r}\mathring{D}^A n^i\mathtt{h}^u_A+n^i\cos(\widetilde{\alpha}u)\mathtt{h}_L\Big]\,dr\,dx^2dx^3\,, \end{aligned} \tag{4.35}$$

and

$$
\begin{aligned}
C[\mathcal{L}_{adS}, \mathscr{C}_{u,R}] &:= \int_{\mathscr{C}_{u,R}} \mathscr{H}^\mu[\mathcal{L}_{adS}] dS_\mu \\
&= \widetilde{l}_i \int_{\mathscr{C}_{u,R}} \Big[n^i \sin(\widetilde{\alpha} u) \mathtt{h}_I^u - n^i (\sin(\widetilde{\alpha} u) + \widetilde{\alpha} r \cos(\widetilde{\alpha} u)) \mathtt{h}_{II}^u \\
&\quad - \frac{(\sin(\widetilde{\alpha} u) + \widetilde{\alpha} r \cos(\widetilde{\alpha} u))}{r} \mathring{D}^A n^i \mathtt{h}_A^u + n^i \sin(\widetilde{\alpha} u) \mathtt{h}_L \Big] \, dr \, dx^2 dx^3 \,. \qquad (4.36)
\end{aligned}
$$

4.4 The time-evolution of Noether charges

A question of interest is the rate of change of the charge integrals as the tip of the light cone is moved to the future along the flow of the Killing vector $\partial_u \equiv \mathcal{T}$:

$$
\frac{dH[X, \mathscr{C}_{u,R}]}{du} \equiv \frac{d}{du} \int_{\mathscr{C}_{u,R}} \mathscr{H}^\mu[X] dS_\mu = \int_{\mathscr{C}_{u,R}} \mathcal{L}_{\partial_u} \mathscr{H}^\mu[X] dS_\mu \,. \qquad (4.37)
$$

Assuming two Killing vector fields X and Y, we have

$$
\begin{aligned}
\mathcal{L}_Y \mathscr{H}^\mu[X] &= -2\sqrt{|\det g|} \nabla_\sigma \Big(Y^{[\sigma} \nabla^{\mu]} \phi X^\alpha \nabla_\alpha \phi \\
&\quad - \frac{1}{2} Y^{[\sigma} X^{\mu]} \big(\nabla^\alpha \phi \nabla_\alpha \phi + m^2 \phi^2 \big) \Big) \,. \qquad (4.38)
\end{aligned}
$$

Using the fields (4.9)-(4.13) one finds

$$
\begin{aligned}
\mathcal{L}_{\partial_u} \mathscr{H}^\mu[\partial_u] &= 2\nabla_\sigma \Big[\mathcal{T}^{[\sigma} \mathtt{h}_I^{\mu]} \Big] \,, \qquad (4.39) \\
\mathcal{L}_{\partial_u} \mathscr{H}^\mu[\mathcal{R}] &= 2\nabla_\sigma \Big\{ \mathcal{T}^{[\sigma} \Big[\varepsilon^{AB} \mathring{D}_B (R_i n^i) \mathtt{h}_A^{\mu]} + \mathcal{R}^{\mu]} \mathtt{h}_L \Big] \Big\} \,, \qquad (4.40) \\
\mathcal{L}_{\partial_u} \mathscr{H}^\mu[\mathcal{P}_{dS}] &= 2\nabla_\sigma \Big\{ e^{\alpha u} \mathcal{T}^{[\sigma} \Big[p_i n^i \mathtt{h}_I^{\mu]} - \big(\alpha r + 1 \big) p_i n^i \mathtt{h}_{II}^{\mu]} \\
&\quad - \frac{\alpha r + 1}{r} \mathring{D}^A (p_i n^i) \mathtt{h}_A^{\mu]} + \mathcal{P}_{dS}^{\mu]} \mathtt{h}_L \Big] \Big\} \,, \qquad (4.41) \\
\mathcal{L}_{\partial_u} \mathscr{H}^\mu[\mathcal{L}_{dS}] &= 2\nabla_\sigma \Big\{ e^{-\alpha u} \mathcal{T}^{[\sigma} \Big[l_i n^i \mathtt{h}_I^{\mu]} + \big(\alpha r - 1 \big) l_i n^i \mathtt{h}_{II}^{\mu]} \\
&\quad + \frac{\alpha r - 1}{r} \mathring{D}^A (l_i n^i) \mathtt{h}_A^{\mu]} + \mathcal{L}_{dS}^{\mu]} \mathtt{h}_L \Big] \Big\} \,. \qquad (4.42)
\end{aligned}
$$

In particular we obtain the energy-flux: [3]

$$\begin{aligned}
\frac{dE_{\mathscr{H}}[\mathscr{C}_{u,R}]}{du} &= -\int_{\partial\mathscr{C}_{u,R}} \mathcal{T}^{[\sigma}\mathtt{h}_I^{\mu]}\, dS_{\sigma\mu} \\
&= -\int_{S_R} \mathtt{h}_I^r\Big|_{r=R} dx^2 dx^3 \\
&= -\int_{S_R} \Big[r^2\left(\partial_u\phi + \left(\alpha^2 r^2 - 1\right)\partial_r\phi\right)\partial_u\phi\Big]_{r=R} d\mu_{\mathring{\gamma}} \\
&= \int_{S_R} \Big[\alpha^2 \overset{(-1)}{\phi}\, \partial_u \overset{(-1)}{\phi}\, R \\
&\quad + \alpha^2 \overset{(-1)}{\phi}\, \partial_u \overset{(-2)}{\phi} + \left(2\alpha^2 \overset{(-2)}{\phi} - \partial_u \overset{(-1)}{\phi}\right)\partial_u \overset{(-1)}{\phi}\Big] d\mu_{\mathring{\gamma}} + o(1)\,, \qquad (4.43)
\end{aligned}$$

where $o(1)$ tends to zero as R tends to infinity.

We turn now our attention to the rate of change of the functional $E_\omega[\mathscr{S}]$ of (1.8), when the tip of the light cone is moved to the future along the flow of the Killing vector ∂_u:

$$\frac{dE_\omega[\mathscr{C}_{u,R}]}{du} \equiv \frac{1}{2}\frac{d}{du}\int_{\mathscr{C}_{u,R}} \omega^\mu(\phi, \mathcal{L}_{\partial_u}\phi) dS_\mu = \frac{1}{2}\int_{\mathscr{C}_{u,R}} \mathcal{L}_{\partial_u}\omega^\mu(\phi, \mathcal{L}_{\partial_u}\phi) dS_\mu\,. \qquad (4.44)$$

We find

$$\begin{aligned}
\mathcal{L}_Y\omega^\mu(\phi, \mathcal{L}_{\partial_u}\phi) &= \partial_\sigma\big(Y^\sigma\omega^\mu\big) - \omega^\sigma\partial_\sigma Y^\mu \\
&= 2\partial_\sigma\big(Y^{[\sigma}\omega^{\mu]}\big) + Y^\mu\partial_\sigma\omega^\sigma\,. \qquad (4.45)
\end{aligned}$$

Assuming that the field equations hold, we have $\partial_\sigma\big(\omega^\sigma(\phi, \mathcal{L}_{\partial_u}\phi)\big) = 0$. The flux of $E_\omega[\mathscr{C}_{u,R}]$ reads

$$\begin{aligned}
\frac{dE_\omega[\mathscr{C}_{u,R}]}{du} &= \frac{1}{2}\int_{\mathscr{C}_{u,R}} 2\partial_\sigma\big(\mathcal{T}^{[\sigma}\omega^{\mu]}\big) dS_\mu \\
&= -\frac{1}{2}\int_{S_R} \mathcal{T}^{[\sigma}\omega^{\mu]} dS_{\sigma\mu} \\
&= \frac{1}{2}\int_{S_R} r^2\Big(\phi(\alpha^2 r^2 - 1)\partial_u\partial_r\phi + \phi\partial_u^2\phi \\
&\quad - \big(\partial_u\phi + (\alpha^2 r^2 - 1)\partial_r\phi\big)\partial_u\phi\Big)_{r=R} d\mu_{\mathring{\gamma}}\,, \qquad (4.46)
\end{aligned}$$

[3] We take this opportunity to correct a misprint in [4, Equation (2.68)], where the terms involving $\overset{(-2)}{\phi}$ are missing.

with asymptotic expansion

$$\frac{dE_\omega[\mathscr{C}_{u,R}]}{du} = \frac{1}{2}\int_{S_R}\Big(-\alpha^2\overset{(-1)}{\phi}\partial_u\overset{(-2)}{\phi}+\alpha^2\overset{(-2)}{\phi}\partial_u\overset{(-1)}{\phi}+\overset{(-1)}{\phi}\partial_u^2\overset{(-1)}{\phi}-(\partial_u\overset{(-1)}{\phi})^2\Big)d\mu_{\mathring{\gamma}} + O(\frac{1}{R})\,. \tag{4.47}$$

Note that when $\alpha = 0$ the limit of dE_ω/du as R tends to infinity is *not* negative, which makes questionable the interpretation of E_ω as the right functional for a physically significant definition of energy.

We continue with the u-derivative of angular momentum, given by

$$\begin{aligned}\frac{dJ[\mathscr{C}_{u,R}]}{du} &= -\int_{\mathscr{C}_{u,R}}\Big[\mathcal{T}^{[\sigma}\Big(\mathtt{h}^{\mu]}_A\varepsilon^{AB}\mathring{D}_B(R_in^i)+\mathcal{R}^{\mu]}\mathtt{h}_L\Big)\Big]\,dS_{\sigma\mu}\\ &= -R_i\int_{S_R}\Big[\mathtt{h}^r_A\varepsilon^{AB}\mathring{D}_Bn^i\Big]_{r=R}dx^2dx^3 =: R_i\frac{dJ^i}{du}\,,\end{aligned} \tag{4.48}$$

with

$$\begin{aligned}\frac{dJ^i}{du} &= -\int_{S_R}\sqrt{\det\mathring{\gamma}}\varepsilon^{AB}\mathring{D}_Bn^i\Big[r^2\big(\partial_u\phi+\big(\alpha^2r^2-1\big)\partial_r\phi\big)\mathring{D}_A\phi\Big]_{r=R}d\mu_{\mathring{\gamma}}\\ &= \int_{S_R}\Big[\varepsilon^{AB}\mathring{D}_Bn^i\Big(\alpha^2\overset{(-1)}{\phi}\mathring{D}_A\overset{(-1)}{\phi}R\\ &\quad+\alpha^2\overset{(-1)}{\phi}\mathring{D}_A\overset{(-2)}{\phi}+\Big(2\alpha^2\overset{(-2)}{\phi}-\partial_u\overset{(-1)}{\phi}\Big)\mathring{D}_A\overset{(-1)}{\phi}\Big)\Big]d\mu_{\mathring{\gamma}}+o(1)\\ &= -\int_{S_R}\Big[\alpha^2\overset{(-1)}{\phi}\mathring{D}_A\overset{(-2)}{\phi}+\partial_u\overset{(-1)}{\phi}\mathring{D}_A\overset{(-1)}{\phi}\Big)\Big]d\mu_{\mathring{\gamma}}+o(1)\,,\end{aligned} \tag{4.49}$$

where the terms proportional to R integrated-out to zero. Finally

$$
\begin{aligned}
\frac{dP[\mathcal{P}_{dS},\mathscr{C}_{u,R}]}{du} &= -2\int_{\mathscr{C}_{u,R}} \Big[e^{\alpha u}\mathcal{T}^{[\sigma}\Big(p_i n^i \mathtt{h}^{\mu]}_I - \big(\alpha r+1\big)p_i n^i \mathtt{h}^{\mu]}_{II} \\
&\quad -\frac{\alpha r+1}{r}\mathring{D}^A(p_i n^i)\mathtt{h}^{\mu]}_A + \mathcal{P}^{\mu]}_{dS}\mathtt{h}_L\Big)\Big]\, dS_{\sigma\mu} \\
&= -\int_{\mathscr{C}_{u,R}} \Big[e^{\alpha u}\Big(p_i n^i \mathtt{h}^{r}_I - \big(\alpha r+1\big)p_i n^i \mathtt{h}^{r}_{II} \\
&\quad -\frac{\alpha r+1}{r}\mathring{D}^A(p_i n^i)\mathtt{h}^{r}_A - \big(\alpha r+1\big)\mathtt{h}_L\Big)\Big]\, dx^2 dx^3 \\
&= -\int_{S_R} e^{\alpha u}p_i\Big\{n^i\Big[r^2\Big(\partial_u\phi + \big(\alpha^2 r^2-1\big)\,\partial_r\phi\Big)\partial_u\phi \\
&\quad -(\alpha r+1)r^2\left(\partial_u\phi + \big(\alpha^2 r^2-1\big)\,\partial_r\phi\right)\partial_r\phi \\
&\quad -\frac{1}{2}(\alpha r+1)\Big(\mathring{\gamma}^{AB}\mathring{D}_A\phi\mathring{D}_B\phi - 2r^2\left(\partial_r\phi\right)\left(\partial_u\phi\right) \\
&\quad +m^2r^2\phi^2 - \big(\alpha^2r^2-1\big)\,r^2\left(\partial_r\phi\right)^2\Big)\Big] \\
&\quad -(\alpha r+1)r\mathring{D}^A n^i\partial_r\phi\mathring{D}_A\phi\Big\}_{r=R} d\mu_{\mathring{\gamma}} \\
&= \int_{S_R} e^{\alpha u}p_i n^i\Big\{\alpha \overset{(-1)}{\phi}\Big[\alpha^2\overset{(-1)}{\phi} + m^2\overset{(-1)}{\phi} + 2\alpha\partial_u\overset{(-1)}{\phi}\Big]R \\
&\quad +\Big[4\alpha^3\overset{(-2)}{\phi} + 2\alpha m^2\overset{(-2)}{\phi} + 2\alpha^2\partial_u\overset{(-2)}{\phi} + \alpha^2\overset{(-1)}{\phi} + m^2\overset{(-1)}{\phi}\Big]\overset{(-1)}{\phi} \\
&\quad +4\alpha^2\partial_u\overset{(-1)}{\phi}\overset{(-2)}{\phi} - 2\left(\partial_u\overset{(-1)}{\phi}\right)^2\Big\}d\mu_{\mathring{\gamma}} + o(1)\,. \qquad (4.50)
\end{aligned}
$$

Comparing (A.3) with (A.4), we see that an analogous formula for $dC[\mathcal{L}_{dS},\mathscr{C}_{u,R}]/du$ can be obtained from (4.50) by replacing α by $-\alpha$ and p_i by l_i:

$$
\begin{aligned}
\frac{dC[\mathcal{L}_{dS},\mathscr{C}_{u,R}]}{du} &= -2\int_{\mathscr{C}_{u,R}} \Big\{e^{-\alpha u}\mathcal{T}^{[\sigma}\Big[l_i n^i \mathtt{h}^{\mu]}_I + \big(\alpha r-1\big)l_i n^i \mathtt{h}^{\mu]}_{II} \\
&\quad +\frac{\alpha r-1}{r}\mathring{D}^A(l_i n^i)\mathtt{h}^{\mu]}_A + \mathcal{L}^{\mu]}_{dS}\mathtt{h}_L\Big]\Big\}\, dS_{\sigma\mu} \\
&= \int_{S_R} e^{-\alpha u}l_i n^i\Big\{-\alpha \overset{(-1)}{\phi}\Big[\alpha^2\overset{(-1)}{\phi} + m^2\overset{(-1)}{\phi} - 2\alpha\partial_u\overset{(-1)}{\phi}\Big]R \\
&\quad +\Big[2\alpha^2\partial_u\overset{(-2)}{\phi} + \alpha^2\overset{(-1)}{\phi} + m^2\overset{(-1)}{\phi} - 4\alpha^3\overset{(-2)}{\phi} - 2\alpha m^2\overset{(-2)}{\phi}\Big]\overset{(-1)}{\phi} \\
&\quad +4\alpha^2\partial_u\overset{(-1)}{\phi}\overset{(-2)}{\phi} - 2\left(\partial_u\overset{(-1)}{\phi}\right)^2\Big\}d\mu_{\mathring{\gamma}} + o(1)\,. \qquad (4.51)
\end{aligned}
$$

5 An alternative Lagrangian for the scalar field

The Lagrangian for a conformally-covariant scalar field theory on the de Sitter background reads

$$\mathscr{L} = -\frac{1}{2}\sqrt{|\det g|}\big(g^{\mu\nu}\nabla_\mu\phi\,\nabla_\nu\phi + 2\alpha^2\phi^2\big)\,, \tag{5.1}$$

which coincides with (4.1) with $m^2 = 2\alpha^2$. With some work this can be rewritten as

$$\frac{\mathscr{L}}{\sqrt{|\det g|}} = -\frac{1}{2r^2}\Big(g^{\mu\nu}\nabla_\mu(r\phi)\,\nabla_\nu(r\phi)\Big) - \frac{1}{4}\nabla_\nu\Big(\big(r\phi\big)^2\nabla^\nu r^{-2}\Big)\,. \tag{5.2}$$

Since boundary terms in a Lagrangian do not change the Euler–Lagrange equations, after neglecting the boundary term in (5.2) we obtain a Lagrangian which leads us to an equivalent theory

$$\widetilde{\mathscr{L}} = -\frac{1}{2r^2}\sqrt{|\det g|}g^{\mu\nu}\nabla_\mu\widetilde{\phi}\,\nabla_\nu\widetilde{\phi}\,, \tag{5.3}$$

where

$$\widetilde{\phi} = r\phi\,.$$

As already announced, all Noether charges turn out to be finite, no renormalisation is required. The price is that the time-derivatives of some charges are not boundary integrals anymore, because both the Lagrangian and the Hamiltonian depend explicitly on the coordinate r now.

The canonical momentum for (5.3) reads

$$\widetilde{\pi}^\alpha = \frac{\partial\widetilde{\mathscr{L}}}{\partial(\nabla_\alpha\widetilde{\phi})} = -\sqrt{|\det g|}\frac{1}{r^2}\nabla^\alpha\widetilde{\phi}\,. \tag{5.4}$$

The canonical energy-momentum current equals

$$\widetilde{\mathscr{H}}^\mu[X] = -\sqrt{|\det g|}\Big(\frac{1}{r^2}\nabla^\mu\widetilde{\phi}\,\mathcal{L}_X\widetilde{\phi} - \frac{1}{2r^2}\big(\nabla^\alpha\widetilde{\phi}\,\nabla_\alpha\widetilde{\phi}\big)X^\mu\Big)\,. \tag{5.5}$$

The large-r asymptotic behaviour of ϕ of (4.8) translates into the following asymptotics for $\widetilde{\phi}$:

$$\widetilde{\phi}(u,r,x^A) = \overset{(0)}{\widetilde{\phi}}(u,x^A) + \frac{\overset{(-1)}{\widetilde{\phi}}(u,x^A)}{r} + \frac{\overset{(-2)}{\widetilde{\phi}}(u,x^A)}{r^2} + \dots\,. \tag{5.6}$$

5.1 Noether charges

The analysis of the Noether charges associated with the Killing fields (A.1)-(A.4) proceeds now in a way completely analogous to that for ϕ. The charges are again of the form (4.14)-(4.17), where now

$$\begin{aligned}
\mathtt{h}^u_I &= \big(\partial_r\widetilde\phi\partial_u\widetilde\phi\big)\sqrt{\det\mathring\gamma}\,, &(5.7)\\
\mathtt{h}^r_I &= \big(\partial_u\widetilde\phi+\big(\alpha^2r^2-1\big)\partial_r\widetilde\phi\big)\partial_u\widetilde\phi\sqrt{\det\mathring\gamma}\,, &(5.8)\\
\mathtt{h}^u_{II} &= \big(\partial_r\widetilde\phi\big)^2\sqrt{\det\mathring\gamma}\,, &(5.9)\\
\mathtt{h}^r_{II} &= \big(\partial_u\widetilde\phi+\big(\alpha^2r^2-1\big)\partial_r\widetilde\phi\big)\partial_r\widetilde\phi\sqrt{\det\mathring\gamma}\,, &(5.10)\\
\mathtt{h}^u_A &= \partial_r\widetilde\phi\mathring D_A\widetilde\phi\sqrt{\det\mathring\gamma}\,, &(5.11)\\
\mathtt{h}^r_A &= \big(\partial_u\widetilde\phi+\big(\alpha^2r^2-1\big)\partial_r\widetilde\phi\big)\mathring D_A\widetilde\phi\sqrt{\det\mathring\gamma}\,, &(5.12)\\
\mathtt{h}_L &= \frac12\Big(\frac{1}{r^2}\mathring\gamma^{AB}\mathring D_A\widetilde\phi\mathring D_B\widetilde\phi-2\partial_r\widetilde\phi\partial_u\widetilde\phi\\
&\quad+\big(1-\alpha^2r^2\big)\big(\partial_r\widetilde\phi\big)^2\Big)\sqrt{\det\mathring\gamma}\,. &(5.13)
\end{aligned}$$

The asymptotic behaviour (5.6) leads to

$$\begin{aligned}
&\widetilde E_{\mathscr H}[\mathscr C_{u,R}] := \int_{\mathscr C_{u,R}}\widetilde{\mathscr H}^\mu[\partial_u]dS_\mu=\int_{\mathscr C_{u,R}}\widetilde{\mathscr H}^u[\partial_u]dS_u\\
&= \int_{\mathscr C_{u,R}}\big(\mathtt{h}^u_I+\mathtt{h}_L\big)\,dr\,dx^2dx^3\\
&= \frac12\int_{\mathscr C_{u,R}}\underbrace{\Big(\frac{1}{r^2}\mathring\gamma^{AB}\mathring D_A\widetilde\phi\mathring D_B\widetilde\phi+\big(1-\alpha^2r^2\big)\big(\partial_r\widetilde\phi\big)^2\Big)}_{O(r^{-2})}\,dr\,d\mu_{\mathring\gamma}\,, \qquad (5.14)
\end{aligned}$$

hence the volume integral has a finite limit as R tends to infinity, resulting in a finite total energy.

As before, the total angular-momentum is obtained from the following integral:

$$\widetilde J[\mathscr C_{u,R}] := \int_{\mathscr C_{u,R}}\widetilde{\mathscr H}^\mu[\mathcal R]dS_\mu\equiv R_i\widetilde J^i[\mathscr C_{u,R}]\,, \qquad (5.15)$$

where now

$$\begin{aligned}
\widetilde J^i[\mathscr C_{u,R}] &:= \int_{\mathscr C_{u,R}}\varepsilon^{AB}\mathring D_Bn^i\mathtt{h}^u_A\,dr\,dx^2dx^3\\
&= \int_{\mathscr C_{u,R}}\underbrace{\varepsilon^{AB}\mathring D_Bn^i\mathring D_A\widetilde\phi\partial_r\widetilde\phi}_{O(r^{-2})}\,dr\,d\mu_{\mathring\gamma}\,, \qquad (5.16)
\end{aligned}$$

again an integral which converges to a finite value as R tends to infinity.

We further have

$$\begin{aligned}
\widetilde{P}[\mathcal{P}_{dS},\mathscr{C}_{u,R}] &:= \int_{\mathscr{C}_{u,R}} \widetilde{\mathscr{H}}^{\mu}[\mathcal{P}_{dS}]dS_{\mu} \\
&= p_i \int_{\mathscr{C}_{u,R}} e^{\alpha u}\Big[n^i \mathtt{h}^u_I - \big(\alpha r+1\big)n^i\mathtt{h}^u_{II} - \frac{\alpha r+1}{r}\mathring{D}^A(n^i)\mathtt{h}^u_A + n^i\mathtt{h}_L\Big]\, dr\, dx^2 dx^3 \\
&= p_i \int_{\mathscr{C}_{u,R}} e^{\alpha u}\Big[\frac{1}{2}n^i\Big(\frac{1}{r^2}\mathring{\gamma}^{AB}\mathring{D}_A\widetilde{\phi}\mathring{D}_B\widetilde{\phi} - \big(\alpha^2 r^2+2\alpha+1\big)\big(\partial_r\widetilde{\phi}\big)^2\Big) \\
&\quad -\frac{(\alpha r+1)}{r}\mathring{D}^A n^i\left(\partial_r\widetilde{\phi}\right)\mathring{D}_A\widetilde{\phi}\Big]\, dr\, d\mu_{\mathring{\gamma}} \\
&= \int_{\mathscr{C}_{u,R}} O(r^{-2})\, dr\, d\mu_{\mathring{\gamma}}\,. \qquad (5.17)
\end{aligned}$$

Finally,

$$\begin{aligned}
\widetilde{C}[\mathcal{L}_{dS},\mathscr{C}_{u,R}] &:= \int_{\mathscr{C}_{u,R}} \widetilde{\mathscr{H}}^{\mu}[\mathcal{L}_{dS}]dS_{\mu} \\
&= l_i \int_{\mathscr{C}_{u,R}} \big(e^{-\alpha u}\Big[n^i \mathtt{h}^u_I + \big(\alpha r-1\big)n^i\mathtt{h}^u_{II} + \frac{\alpha r-1}{r}\mathring{D}^A(n^i)\mathtt{h}^u_A + n^i\mathtt{h}_L\Big]\big)\, dr\, dx^2 dx^3\,. \qquad (5.18)
\end{aligned}$$

A more detailed version of the integral (5.18), which is again finite in the limit $R\to\infty$, can be obtained from (5.17) by replacing there α by $-\alpha$ and p_i by l_i.

5.2 Time derivatives

Recall that the Lie derivatives of the Noether current read

$$\begin{aligned}
\mathcal{L}_Y\widetilde{\mathscr{H}}^{\mu} &= \nabla_\alpha\left(Y^\alpha\widetilde{\mathscr{H}}^{\mu}\right) - \widetilde{\mathscr{H}}^{\alpha}\nabla_\alpha Y^\mu \\
&= 2\nabla_\alpha\left(Y^{[\alpha}\widetilde{\mathscr{H}}^{\mu]}\right) + Y^\mu\nabla_\alpha\widetilde{\mathscr{H}}^{\alpha}\,. \qquad (5.19)
\end{aligned}$$

We associate Hamiltonian density with the canonical energy-momentum tensor through the formula

$$\widetilde{\mathscr{H}}^{\mu}[X] = \widetilde{T}^{\mu}{}_{\alpha}X^\alpha\,, \qquad (5.20)$$

where

$$\widetilde{T}^{\mu}{}_{\alpha} = \tilde{\pi}^\mu\nabla_\alpha\widetilde{\phi} - \delta^\mu_\alpha\widetilde{\mathscr{L}}\,. \qquad (5.21)$$

Since the alternative Lagrangian (5.2) depends explicitly upon the coordinate r, for solutions of the field equations we find

$$\nabla_\mu \widetilde{\mathscr{H}}^\mu = \nabla_\mu \Big(\widetilde{T}^\mu{}_\alpha X^\alpha \Big) = \widetilde{T}^\mu{}_\alpha \nabla_\mu X^\alpha - \frac{\partial \widetilde{\mathscr{L}}}{\partial r} X^\alpha \partial_\alpha r \,. \tag{5.22}$$

Using (5.19), (5.22), and assuming that X is Killing vector field, the Lie derivative of the Noether current reads

$$\mathcal{L}_Y \widetilde{\mathscr{H}}^\mu = 2\nabla_\alpha \left[Y^{[\alpha} \widetilde{\mathscr{H}}^{\mu]} \right] - Y^\mu \frac{\partial \widetilde{\mathscr{L}}}{\partial r} X^\alpha \partial_\alpha r \,.$$

Using (5.3) and (5.5) one obtains

$$\begin{aligned} \mathcal{L}_Y \widetilde{\mathscr{H}}^\mu \;\; &= \;\; -2\sqrt{|\det g|}\nabla_\alpha \left[Y^{[\alpha} \Big(\frac{1}{r^2} \nabla^{\mu]} \widetilde{\phi}\, \mathcal{L}_X \widetilde{\phi} - \frac{1}{2r^2} \nabla^\beta \widetilde{\phi}\, \nabla_\beta \widetilde{\phi} X^{\mu]} \Big) \right] \\ &\quad\; - Y^\mu \frac{1}{r^3} \sqrt{|\det g|} g^{\nu\rho} \nabla_\nu \widetilde{\phi}\, \nabla_\rho \widetilde{\phi} X^\alpha \partial_\alpha r \,. \end{aligned} \tag{5.23}$$

The "non-divergence term" (...)$Y^\mu X^\alpha \partial_\alpha r$ in this equation implies that some charges might have volume terms in their evolution formulae. No such terms will certainly occur when either Y is tangent to $\mathscr{S}$ (which will be the case for rotations), or when r is invariant under the flow of X (which will be the case for u-translations and rotations).

For instance, consider (5.23) with $X = Y = \partial_u \equiv \mathcal{T}$. In this case (4.39) applies, with the relevant component given by (5.8). Passing to the limit $R \to \infty$ in (4.37) with (4.39) and (5.8) one obtains

$$\begin{aligned} \frac{d\widetilde{E}_{\mathscr{H}}[\mathscr{C}_u]}{du} &= -2 \int_{\partial \mathscr{S}_\tau} \mathcal{T}^{[\sigma} \mathtt{h}_I^{\mu]} \, dS_{\sigma\mu} \\ &= - \lim_{R\to\infty} \int_{S_R} \mathtt{h}_I^r \Big|_{r=R} dx^2 dx^3 \\ &= - \lim_{R\to\infty} \int_{S_R} \Big[\Big(\partial_u \widetilde{\phi} + \big(\alpha^2 r^2 - 1 \big) \partial_r \widetilde{\phi} \Big) \partial_u \widetilde{\phi} \Big]_{r=R} d\mu_{\mathring{\gamma}} \\ &= \int_S \big(\alpha^2 \overset{(-1)}{\widetilde{\phi}} - \partial_u \overset{(0)}{\widetilde{\phi}} \big) \partial_u \overset{(0)}{\widetilde{\phi}} \, d\mu_{\mathring{\gamma}} \,. \end{aligned} \tag{5.24}$$

As another example, the u-derivative of angular momentum is obtained from (4.37), and (4.40) with $X = \mathcal{R}$ and $Y = \partial_u \equiv \mathcal{T}$:

$$\begin{aligned} \frac{d\widetilde{J}[\mathscr{C}_u]}{du} &= -2 \int_{\partial \mathscr{S}_\tau} \mathcal{T}^{[\sigma} \Big(\mathtt{h}_A^{\mu]} \varepsilon^{AB} \mathring{D}_B (R_i n^i) + \mathcal{R}^{\mu]} \mathtt{h}_L \Big) dS_{\sigma\mu} \\ &= -R_i \lim_{R\to\infty} \int_{S_R} \Big[\mathtt{h}_A^r \varepsilon^{AB} \mathring{D}_B n^i \Big]_{r=R} dx^2 dx^3 =: R_i \frac{d\widetilde{J}^i}{du} \,. \end{aligned} \tag{5.25}$$

Using (5.12) we find

$$\frac{d\widetilde{J}^i}{du} = -\lim_{R\to\infty}\int_{S_R}\sqrt{\det\mathring{\gamma}}\varepsilon^{AB}\mathring{D}_B n^i\Big[\big(\partial_u\widetilde{\phi}+\big(\alpha^2r^2-1\big)\,\partial_r\widetilde{\phi}\big)\mathring{D}_A\widetilde{\phi}\Big]_{r=R}d\mu_{\mathring{\gamma}}$$
$$= \int_S \varepsilon^{AB}\mathring{D}_B n^i\Big(2\alpha^2\overset{(-1)}{\widetilde{\phi}}-\partial_u\overset{(0)}{\widetilde{\phi}}\Big)\mathring{D}_A\overset{(0)}{\widetilde{\phi}}\,d\mu_{\mathring{\gamma}}\,, \tag{5.26}$$

which does not coincide with (4.49).

The remaining u-derivatives have both fluxes and volume integrals. For instance, calculating similarly to (4.50) and taking into account the volume term in (5.23),

$$
\begin{aligned}
\frac{d\widetilde{P}[\mathcal{P}_{dS},\mathscr{C}_u]}{du} &= -2\int_{\partial\mathscr{S}_\tau}\Big[e^{\alpha u}\mathcal{T}^{[\sigma}\Big(p_in^i\mathtt{h}_I^{\mu]}-(\alpha r+1)p_in^i\mathtt{h}_{II}^{\mu]}\\
&\quad -\frac{\alpha r+1}{r}\mathring{D}^A(p_in^i)\mathtt{h}_A^{\mu]}+\mathcal{P}_{dS}^{\mu]}\mathtt{h}_L\Big)\Big]\,dS_{\sigma\mu}\\
&\quad -\int_{\mathscr{S}_\tau}\frac{2}{r}\Big[\mathcal{T}^\mu\mathtt{h}_L\mathcal{P}^r_{dS}\Big]dS_\mu\\
&= -\int_{\partial\mathscr{S}_\tau}\Big[e^{\alpha u}\Big(p_in^i\mathtt{h}^r_I-(\alpha r+1)p_in^i\mathtt{h}^r_{II}\\
&\quad -\frac{\alpha r+1}{r}\mathring{D}^A(p_in^i)\mathtt{h}^r_A-(\alpha r+1)\mathtt{h}_L\Big)\Big]\,dS_{\sigma\mu}\\
&\quad +\int_{\mathscr{S}_\tau}\frac{2}{r}\Big[\mathtt{h}_L(\alpha r+1)p_in^i\Big]dr\,dx^2dx^3\\
&= -\int_{S_R}e^{\alpha u}p_i\Big\{n^i\Big[\Big(\partial_u\widetilde{\phi}+(\alpha^2r^2-1)\,\partial_r\widetilde{\phi}\Big)\partial_u\widetilde{\phi}\\
&\quad -(\alpha r+1)\left(\partial_u\widetilde{\phi}+(\alpha^2r^2-1)\,\partial_r\widetilde{\phi}\right)\partial_r\widetilde{\phi}\\
&\quad -\frac{1}{2}(\alpha r+1)\Big(\frac{1}{r^2}\mathring{\gamma}^{AB}\mathring{D}_A\widetilde{\phi}\mathring{D}_B\widetilde{\phi}-2\left(\partial_r\widetilde{\phi}\right)\left(\partial_u\widetilde{\phi}\right)\\
&\quad -\left(\alpha^2r^2-1\right)\left(\partial_r\widetilde{\phi}\right)^2\Big)\Big]\\
&\quad -\frac{(\alpha r+1)}{r}\mathring{D}^An^i\partial_r\widetilde{\phi}\mathring{D}_A\widetilde{\phi}\Big\}_{r=R}d\mu_{\mathring{\gamma}}\\
&\quad +\int_{\mathscr{S}_\tau}\frac{1}{r}\Big[\Big(\frac{1}{r^2}\mathring{\gamma}^{AB}\mathring{D}_A\widetilde{\phi}\mathring{D}_B\widetilde{\phi}-2\partial_r\widetilde{\phi}\partial_u\widetilde{\phi}\\
&\quad +\big(1-\alpha^2r^2\big)\big(\partial_r\widetilde{\phi}\big)^2\Big)\sqrt{\det\mathring{\gamma}}(\alpha r+1)p_in^i\Big]dS_\mu\\
&= \int_{\partial\mathscr{S}_\tau}e^{\alpha u}p_in^i\Big[\alpha^2\overset{(-1)}{\widetilde{\phi}}-\partial_u\overset{(0)}{\widetilde{\phi}}\Big]\partial_u\overset{(0)}{\widetilde{\phi}}d\mu_{\mathring{\gamma}}\\
&\quad +\int_{\mathscr{S}_\tau}O(r^{-2})\,dr\,d\mu_{\mathring{\gamma}}\,. \qquad (5.27)
\end{aligned}
$$

It is not clear whether a meaningful comparison to (4.50) is possible because of the volume term appearing here.

A formula for $\frac{dC[\mathcal{L}_{dS},\mathscr{C}_{u,R}]}{du}$ can be obtained from (5.27) by replacing there α by $-\alpha$ and p_i by l_i.

6 Poisson algebras

Having obtained a set of global charges, either directly or after renormalisation, the question arises whether the charges satisfy a well-defined Poisson algebra. As we will see, the question is far from clear, because of the boundary integrals arising when varying the charges.

Quite generally, we consider two Hamiltonian functionals, $H[\mathscr{S},X]$ and $H[\mathscr{S},Y]$, defined as integrals on a hypersurface $\mathscr{S}$ with boundary $\partial\mathscr{S}$, with two vector field X and Y. Here the boundary might be at finite distance, before a limit to infinity is taken, or it can be a boundary at infinity in the conformally compactified spacetime. We take an approach similar to that of [3] to define the Poisson algebra of charges through the Poisson algebra of fields on $\mathscr{S}$. When there are no constraints, as is the case of the scalar field, *and* when there are no boundary terms in the variations, *and* when $\mathscr{S}$ is spacelike, the algebra is straightforward. When the hypersurface is null the algebra of the fields is more demanding. We avoid the work associated with the last problem by deforming $\mathscr{S}$ to a hypersurface which is spacelike, and calculating the Poisson brackets on the deformed hypersurface. We expect this to give a correct answer in situations where the charges are independent of the hypersurface, within the family of hypersurfaces sharing the same boundary.

The problem of boundary integrals that remain after a variation of the charges has been carried-out, which arises in the situations of interest in this work, will be addressed in Section 7.

Let us pass now to an analysis of the Poisson algebra of Noether charges associated with diffeomorphisms generated by two vector fields X and Y. We consider first order Lagrangian densities depending upon the fields, the metric, and possibly upon coordinates: $\mathscr{L} = \mathscr{L}(\phi^A, \partial_\mu\phi^A, g_{\alpha\beta}, x^\sigma)$. The key assumption in this section is that there are no Hamiltonian constraints; thus some of the calculations that follow do not apply to Maxwell fields, which will be discussed elsewhere.

As elsewhere in this work, the Noether current associated with a vector field X reads

$$\mathscr{H}^\mu = \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi^A)}\mathcal{L}_X\phi^A - X^\mu\mathscr{L}\,. \tag{6.1}$$

Given a hypersurface $\mathscr{S} = \{x^0 = 0\}$, one thus obtains a charge integral

$$H[\mathscr{S},X] = \int_{\mathscr{S}} \mathscr{H}^0 dS_0\,. \tag{6.2}$$

The canonical momentum on $\mathscr{S}$ is defined as

$$\pi_A \equiv \pi_A{}^0 := \frac{\partial \mathscr{L}}{\partial\left(\partial_0 \phi^A\right)}\,. \tag{6.3}$$

6.1 Hamilton equations

To avoid ambiguities, variations of fields are defined as follows: given a one parameter family of fields $\lambda \mapsto \phi^A(\lambda)$ one sets

$$\delta\phi^A := \frac{d\phi^A}{d\lambda}\,, \qquad \delta\pi_A{}^k := \frac{d\pi_A{}^k}{d\lambda}\,,$$

etc.

The calculations that follow are standard. We carry them out in detail in order to keep track of the boundary terms that arise in the process. We assume that δX vanishes, in particular $[\mathcal{L}_X, \delta]\phi^A = 0$. The variation of the functional (6.2) is defined as

$$\delta H := \frac{d}{d\lambda}\int_{\mathscr{S}} \mathscr{H}^0 dS_0 = \int_{\mathscr{S}} \frac{d\mathscr{H}^0}{d\lambda} dS_0 \equiv \int_{\mathscr{S}} \delta\mathscr{H}^0 dS_0\,, \tag{6.4}$$

assuming that differentiation under the integral is justified, with

$$\begin{aligned}
\delta\mathscr{H}^0 &= \delta\left(\pi_A \mathcal{L}_X \phi^A - X^0\mathscr{L}\right)\\
&= \mathcal{L}_X\phi^A\delta\pi_A + \pi_A\mathcal{L}_X\delta\phi^A - X^0\left(\frac{\partial\mathscr{L}}{\partial\phi^A}\delta\phi^A + \frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\phi^A\right)}\partial_\mu\delta\phi^A\right)\\
&= \mathcal{L}_X\phi^A\delta\pi_A - \mathcal{L}_X\pi_A\delta\phi^A + \mathcal{L}_X\left(\pi_A\delta\phi^A\right)\\
&\quad -X^0\Big\{\underbrace{\Big[\frac{\partial\mathscr{L}}{\partial\phi^A} - \partial_\mu\Big(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi^A)}\Big)\Big]}_{\mathcal{E}_A}\delta\phi^A + \partial_\mu\Big(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi^A)}\delta\phi^A\Big)\Big\}\,,
\end{aligned} \tag{6.5}$$

where the vanishing of $\mathcal{E}_A$ is the contents of the Euler–Lagrange equations. If we assume additionally that $\pi_A\delta\phi^A \equiv \pi_A{}^0\delta\phi^A$ is the 0-component of a vector density, using the definition of the Lie derivative of a vector density we find

$$\mathcal{L}_X(\pi_A\delta\phi^A) = \mathcal{L}_X(\pi_A{}^0\delta\phi^A) = \partial_\alpha(X^\alpha\pi_A{}^0\delta\phi^A) - \delta\phi^A\pi_A{}^\alpha\partial_\alpha X^0\,. \tag{6.6}$$

Inserting (6.6) into (6.5) we obtain

$$\begin{aligned}
\delta\mathscr{H}^0 &= \mathcal{L}_X\phi^A\delta\pi_A - \mathcal{L}_X\pi_A\delta\phi^A + \partial_\alpha(X^\alpha\pi_A{}^0\delta\phi^A) - \delta\phi^A\pi_A{}^\alpha\partial_\alpha X^0\\
&\quad -X^0\Big\{\mathcal{E}_A\delta\phi^A + \partial_\mu\Big(\pi_A{}^\mu\delta\phi^A\Big)\Big\}\\
&= \mathcal{L}_X\phi^A\delta\pi_A - \left[\mathcal{L}_X\pi_A + X^0\mathcal{E}_A\right]\delta\phi^A\\
&\quad +\partial_\mu\Big[(X^\mu\pi_A - X^0\pi_A{}^\mu)\delta\phi^A\Big]\,.
\end{aligned} \tag{6.7}$$

Here $\mathcal{L}_X\pi_A$ is understood as the $\alpha = 0$-component of the field $\mathcal{L}_X{\pi_A}^\alpha$.

Recall that by assumption the Lagrangian, and therefore also $\mathscr{H}$, is a functional of the fields and their first derivatives. *Let us suppose* that the equation defining π_A can be inverted to express the x^0-derivative of ϕ^A as a function of π_A, ϕ^A, and of the derivatives of fields ϕ^A in directions tangential to $\mathscr{S}$ (which we denote by $\partial_k\phi^A$); we emphasise that (6.7) holds regardless of whether or not this assumption is true. Reexpressing $\mathscr{H}^0$ as a functional of π_A and ϕ^A we can then calculate as

$$\begin{aligned}\delta\mathscr{H}^0 &= \underbrace{\frac{\partial\mathscr{H}^0}{\partial\pi_A}}_{=:\frac{\delta\mathscr{H}^0}{\delta\pi_A}}\delta\pi_A + \frac{\partial\mathscr{H}^0}{\partial\phi^A}\delta\phi^A + \frac{\partial\mathscr{H}^0}{\partial\left(\partial_k\phi^A\right)}\delta\partial_k\phi^A \\ &= \frac{\delta\mathscr{H}^0}{\delta\pi_A}\delta\pi_A + \underbrace{\left(\frac{\partial\mathscr{H}^0}{\partial\phi^A} - \partial_k\left(\frac{\partial\mathscr{H}^0}{\partial\left(\partial_k\phi^A\right)}\right)\right)}_{=:\frac{\delta\mathscr{H}^0}{\delta\phi^A}}\delta\phi^A + \partial_k\left(\frac{\partial\mathscr{H}^0}{\partial\left(\partial_k\phi^A\right)}\delta\phi^A\right).\end{aligned} \tag{6.8}$$

In this equation the notation $\frac{\delta\mathscr{H}^0}{\delta\pi_A}$ is somewhat of an overkill: by assumption $\mathscr{L}$ depends only on the first derivatives of ϕ^A, thus $\mathscr{H}^0$ does *not* depend on the derivatives of $\partial_0\phi^A$, and $\frac{\delta\mathscr{H}^0}{\delta\pi_A}$ is simply a partial derivative with respect to π_A.

Comparing (6.5) with (6.8), for variations $\delta\pi_A$ and $\delta\phi^A$ of compact support, and supported away from the boundaries of $\mathscr{S}$ if any, we find

$$\int_{\mathscr{S}}\left((\mathcal{L}_X\phi^A - \frac{\delta\mathscr{H}^0}{\delta\pi_A})\delta\pi_A + (\frac{\delta\mathscr{H}^0}{\delta\phi^A} - [\mathcal{L}_X\pi_A + X^0\mathcal{E}_A])\delta\phi^A\right)dS_0\,. \tag{6.9}$$

If all the variations $\delta\pi_A$ and $\delta\phi^A$ are independent and arbitrary we can conclude that

$$\frac{\delta\mathscr{H}^0}{\delta\pi_A} = \mathcal{L}_X\phi^A\,, \tag{6.10}$$

$$\frac{\delta\mathscr{H}^0}{\delta\phi^A} = -\mathcal{L}_X\pi_A - X^0\mathcal{E}_A\,. \tag{6.11}$$

We emphasise that the assumptions above are satisfied for a scalar field, but *are not* for a Maxwell field.

It further holds that, for variations that do not necessarily vanish on $\partial\mathscr{S}$,

$$\int_{\partial\mathscr{S}}\Big(\frac{\partial\mathscr{H}^0}{\partial{\phi^A}_{,k}} - \big(X^k\pi_A - X^0{\pi_A}^k\big)\Big)\delta\phi^A dS_{0k} = 0\,, \tag{6.12}$$

where the integration over $\partial\mathscr{S}$, when not compact, is understood by exhausting $\mathscr{S}$ with a family of compact domains with smooth boundary, and passing to the limit. In situations where both $\partial\mathscr{S}$ and the variations of the fields on $\partial\mathscr{S}$ are arbitrary, we conclude that

$$\frac{\partial \mathscr{H}^0}{\partial \phi^A{}_{,k}} = X^k \pi_A - X^0 \pi_A{}^k \,. \tag{6.13}$$

6.2 Algebra of charges

Consider two functionals F and G depending upon the fields π_A, ϕ^A and the tangential derivatives $\partial_k \phi^A$, which in the adapted coordinates as above take the form

$$F = \int_{\mathscr{S}} f(\phi^A, \partial_i \phi^A, \pi^A)\, dS_0 \,, \quad G = \int_{\mathscr{S}} g(\phi^A, \partial_i \phi^A, \pi^A)\, dS_0 \,. \tag{6.14}$$

Following [3] we set

$$\{F, G\}_{\mathscr{S}} := \int_{\mathscr{S}} \left(\frac{\delta f}{\delta \phi^A} \frac{\delta g}{\delta \pi_A} - \frac{\delta f}{\delta \pi_A} \frac{\delta g}{\delta \phi^A} \right) dS_0 \,, \tag{6.15}$$

with

$$\frac{\delta f}{\delta \phi^A} := \frac{\partial f}{\partial \phi^A} - \partial_i \left(\frac{\partial f}{\partial \phi^A{}_{,i}} \right) , \quad \frac{\delta f}{\delta \pi_A} \equiv \frac{\partial f}{\partial \pi_A} \,,$$

similarly for g.

We note that there is no reason for $\{F, G\}_{\mathscr{S}}$ to be independent of $\mathscr{S}$, e.g. when the original functionals F or G depend upon $\mathscr{S}$. We will, however, see shortly that $\{H_X, H_Y\}_{\mathscr{S}}$ will be independent of $\mathscr{S}$, within its homology class, in situations of interest.

We note that the question of boundary terms in the variations of Noether charges arising from flows in spacetime can be shuffled under the carpet by defining instead

$$\{H_X, H_Y\}_{\mathscr{S}} := -\Omega_{\mathscr{S}}(\mathscr{L}_X \phi, \mathscr{L}_Y \phi) \,, \tag{6.16}$$

where

$$\Omega_{\mathscr{S}} := \int_{\mathscr{S}} \omega^\mu dS_\mu \equiv \int_{\mathscr{S}} \delta\pi_A \wedge \delta\phi^A \, dS_\mu \,. \tag{6.17}$$

Equation 6.16 is a special case of (6.15) *whenever* no boundary terms arise in the variations of H_X and H_Y. We will, however, use the more fundamental equation (6.15) in our calculation of the left-hand side of (6.16).

Using (6.10) and (6.11), the Poisson bracket of two Hamiltonian functionals H_X and H_Y thus equals

$$
\begin{aligned}
\{H_X, H_Y\}_{\mathscr{S}} &= \int_{\mathscr{S}} \left[\frac{\delta \mathscr{H}^0_X}{\delta \phi^A} \frac{\delta \mathscr{H}^0_Y}{\delta \pi_A} - \frac{\delta \mathscr{H}^0_X}{\delta \pi_A} \frac{\delta \mathscr{H}^0_Y}{\delta \phi^A} \right] dS_0 \\
&= -\int_{\mathscr{S}} \Big[\mathcal{L}_Y \phi^A \left(\mathcal{L}_X \pi_A + X^0 \mathcal{E}_A \right) - \mathcal{L}_X \phi^A \left(\mathcal{L}_Y \pi_A + Y^0 \mathcal{E}_A \right) \Big] dS_0 \\
&= -\int_{\mathscr{S}} \Big\{ \mathcal{L}_X \Big(\pi_A \mathcal{L}_Y \phi^A \Big) - \mathcal{L}_Y \Big(\pi_A \mathcal{L}_X \phi^A \Big) + \pi_A \left(\mathcal{L}_Y \mathcal{L}_X \phi^A - \mathcal{L}_X \mathcal{L}_Y \phi^A \right) \\
&\quad + \mathcal{E}_A \left(X^0 \mathcal{L}_Y \phi^A - Y^0 \mathcal{L}_X \phi^A \right) \Big\} dS_0 \\
&= -\int_{\mathscr{S}} \{ \mathcal{L}_X \mathscr{H}^0_Y - \mathcal{L}_Y \mathscr{H}^0_X + \mathcal{L}_X \Big(Y^0 \mathscr{L} \Big) - \mathcal{L}_Y \Big(X^0 \mathscr{L} \Big) + \pi_A \mathcal{L}_{[Y,X]} \phi^A \\
&\quad + \mathcal{E}_A \left(X^0 \mathcal{L}_Y \phi^A - Y^0 \mathcal{L}_X \phi^A \right) \} dS_0 \,. \qquad (6.18)
\end{aligned}
$$

The following relations hold

$$
\begin{aligned}
\mathcal{L}_X \mathscr{H}^\beta_Y - \mathcal{L}_Y \mathscr{H}^\beta_X &= 2\partial_\alpha \Big(X^{[\alpha} \mathscr{H}^{\beta]}_Y - Y^{[\alpha} \mathscr{H}^{\beta]}_X \Big) \\
&\quad + X^\beta \partial_\alpha \mathscr{H}^\alpha_Y - Y^\beta \partial_\alpha \mathscr{H}^\alpha_X \,, \qquad (6.19) \\
\mathcal{L}_X \Big(Y^\beta \mathscr{L} \Big) - \mathcal{L}_Y \Big(X^\beta \mathscr{L} \Big) &= 2\partial_\alpha \Big(X^{[\alpha} Y^{\beta]} \mathscr{L} \Big) + [X,Y]^\beta \mathscr{L} \,. \qquad (6.20)
\end{aligned}
$$

Inserting (6.20) and (6.19) into (6.18) results in

$$
\begin{aligned}
\{H_X, H_Y\}_{\mathscr{S}} &= \int_{\mathscr{S}} \Big\{ \mathscr{H}^\beta_{[X,Y]} - \mathcal{E}_A \Big(X^\beta \mathcal{L}_Y \phi^A - Y^\beta \mathcal{L}_X \phi^A \Big) \\
&\quad + Y^\beta \partial_\alpha \mathscr{H}^\alpha_X - X^\beta \partial_\alpha \mathscr{H}^\alpha_Y \\
&\quad - 2\partial_\alpha \Big(X^{[\alpha} \mathscr{H}^{\beta]}_Y - Y^{[\alpha} \mathscr{H}^{\beta]}_X + X^{[\alpha} Y^{\beta]} \mathscr{L} \Big) \Big\} dS_\beta \,. \qquad (6.21)
\end{aligned}
$$

We conclude that:

PROPOSITION 6.1 *If* $\partial_\alpha \mathscr{H}^\alpha_Y = \partial_\alpha \mathscr{H}^\alpha_Y = \partial_\alpha \mathscr{H}^\alpha_{[X,Y]} = 0$, *if the field equations are satisfied, and if* $\partial \mathscr{S}_1 = \partial \mathscr{S}_2$, *then*

$$
\{H_X, H_Y\}_{\mathscr{S}_1} = \{H_X, H_Y\}_{\mathscr{S}_2} \,. \qquad (6.22)
$$

PROOF: Under the conditions listed the right-hand side of (6.21) does not depend upon $\mathscr{S}$. □

Another immediate consequence of (6.21) is:

PROPOSITION 6.2 *If*

1.

$$\int_{\partial\mathscr{S}} \Big(X^{[\alpha}Y^{\beta]}\mathscr{L} + \big(X^{[\alpha}\mathcal{L}_Y\phi^A - Y^{[\alpha}\mathcal{L}_X\phi^A\big)\pi_A{}^{\beta]}\Big)dS_{\alpha\beta} = 0\,, \quad (6.23)$$

and if

2. X is tangent to $\mathscr{S}$ or *the field equations are satisfied and $\partial_\alpha\mathscr{H}^\alpha_Y$ vanishes, and if*

3. Y is tangent to $\mathscr{S}$ or *the field equations are satisfied and $\partial_\alpha\mathscr{H}^\alpha_X$ vanishes,*

then it holds that

$$\{H_X, H_Y\}_{\mathscr{S}} = H_{[X,Y]}\,. \quad (6.24)$$

□

A comment on the vanishing of $\partial_\alpha\mathscr{H}^\alpha$ is in order. For this, we recall that in [4] theories satisfying the following were considered:

H1. $\mathscr{L}$ is a scalar density.

H2. There exists a notion of derivation with respect to a family of vector fields X, which we will denote by $\mathcal{L}_X$, which coincides with the usual Lie derivative on vector densities, and which we will call *Lie derivative* regardless of whether or not this is the usual Lie derivative on the remaining fields, such that the following holds:

   a) $\mathcal{L}_X$ preserves the type of a field, thus $\mathcal{L}_X$ of a scalar density is a scalar density, etc.;
   b) the field $\pi_A{}^\mu\mathcal{L}_X\phi^A$ is a vector density;
   c) in a coordinate system in which $X = \partial_0$ we have $\mathcal{L}_X = \partial_0$;
   d) $\mathcal{L}_X$ satisfies the Leibniz rule.

In our case the Lagrangian also depends on a background structure, namely the background metric. Let us collectively denote background fields by ψ^I, with the understanding that if the Lagrangian depends upon both a background field χ and its derivatives, then these derivatives appear as a separate entry in $\psi^I = (\chi, \partial_\mu\chi, \ldots)$. As will be seen shortly, under H1-H2 one then has the identity

$$\partial_\mu\mathscr{H}^\mu_X = \mathscr{E}_A\mathcal{L}_X\phi^A - \frac{\partial\mathscr{L}}{\partial\psi^I}\mathcal{L}_X\psi^I\,. \quad (6.25)$$

Hence the divergence of $\mathscr{H}^\mu$ vanishes when the field equations are satisfied and the background quantities are invariant under $\mathscr{L}$. For the scalar field, or for linearised gravity, this requires X to be a Killing vector field of the background. In the Maxwell case, the divergence of $\mathscr{H}$ also vanishes for conformal Killing vector fields of the background metric.

The proof of (6.25) is simplest in adapted coordinates as in [4]. An "explicitly covariant" proof can be given for tensor fields, in which case we can write

$$\nabla_\mu \phi^A = \partial_\mu \phi^A + \Gamma^A{}_{B\mu}\phi^B ,$$

where ∇ is the covariant derivative operator of $g_{\mu\nu}$. Let us assume, for simplicity, that the Lagrangian $\mathscr{L}\big(\phi^A, \partial_\mu\phi^A, g_{\mu\nu}, \partial_\sigma g_{\mu\nu}\big)$ depends upon the derivatives of the metric through the connection coefficients only:

$$\mathscr{L}\big(\phi^A, \partial_\mu\phi^A, g_{\mu\nu}, \partial_\sigma g_{\mu\nu}\big) = \overline{\mathscr{L}}\big(\phi^A, \nabla_\mu\phi^A, g_{\mu\nu}\big) .$$

Then

$$\frac{\partial\overline{\mathscr{L}}}{\partial(\nabla_\mu\phi^A)} = \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi^A)} \equiv \pi_A{}^\mu , \tag{6.26}$$

$$\frac{\partial\overline{\mathscr{L}}}{\partial\phi^A} = \frac{\partial\mathscr{L}}{\partial\phi^A} - \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi^B)}\Gamma^B{}_{A\mu} = \frac{\partial\mathscr{L}}{\partial\phi^A} - \pi_B{}^\mu\Gamma^B{}_{A\mu} , \tag{6.27}$$

$$\frac{\partial\overline{\mathscr{L}}}{\partial(\partial_\sigma g_{\alpha\beta})} = \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi^A)}\frac{\partial\Gamma^A{}_{B\mu}}{\partial(\partial_\sigma g_{\alpha\beta})}\phi^B \equiv \pi_A{}^\mu\frac{\partial\Gamma^A{}_{B\mu}}{\partial(\partial_\sigma g_{\alpha\beta})}\phi^B . \tag{6.28}$$

Assuming that $\mathscr{L}$ is a scalar density, it follows from (6.27) that the Euler-Lagrange equations

$$\partial_\mu\pi_A{}^\mu = \frac{\partial\mathscr{L}}{\partial\phi^A} = \frac{\partial\overline{\mathscr{L}}}{\partial\phi^A} + \pi_B{}^\mu\Gamma^B{}_{A\mu}$$

can be equivalently written as

$$\nabla_\mu\pi_A{}^\mu = \frac{\partial\overline{\mathscr{L}}}{\partial\phi^A} ,$$

since $\pi_A{}^\alpha\mathcal{L}_X\phi^A\partial_\alpha$ is a vector density. Further:

$$\begin{aligned}&\mathcal{L}_X\overline{\mathscr{L}}\big(\phi^A, \nabla_\mu\phi^A, g_{\mu\nu}, \partial_\sigma g_{\mu\nu}\big)\\ &= \frac{\partial\overline{\mathscr{L}}}{\partial\phi^A}\mathcal{L}_X\phi^A + \pi_A{}^\alpha\mathcal{L}_X\nabla_\alpha\phi^A + \frac{\partial\overline{\mathscr{L}}}{\partial g_{\mu\nu}}\mathcal{L}_X g_{\mu\nu} + \pi_A{}^\mu\phi^B\mathcal{L}_X\Gamma^A{}_{B\mu} ;\end{aligned} \tag{6.29}$$

recall that

$$\mathcal{L}_X\Gamma^\alpha_{\beta\gamma} = \nabla_\beta\nabla_\gamma X^\alpha + R^\alpha{}_{\beta\sigma\gamma}X^\sigma\,, \tag{6.30}$$

which carries over to $\mathcal{L}_X\Gamma^A{}_{B\gamma}$ according to the rank of the tensor field ϕ^A.

The divergence of the Noether currents reads

$$\begin{aligned}\partial_\alpha\mathscr{H}^\alpha &= \partial_\alpha\big(\pi_A{}^\alpha\mathcal{L}_X\phi^A - X^\alpha\mathscr{L}\big) = \partial_\alpha\big(\pi_A{}^\alpha\mathcal{L}_X\phi^A - X^\alpha\overline{\mathscr{L}}\big)\\ &= \partial_\alpha\big(\pi_A{}^\alpha\mathcal{L}_X\phi^A\big) - \mathcal{L}_X\overline{\mathscr{L}}\,,\end{aligned} \tag{6.31}$$

where we used the fact that the Lagrangian is a scalar density:

$$\mathcal{L}_X\overline{\mathscr{L}} = \partial_\alpha\big(X^\alpha\overline{\mathscr{L}}\big)\,. \tag{6.32}$$

Using again the fact that the vector field $\pi_A{}^\alpha\mathcal{L}_X\phi^A\partial_\alpha$ is a vector density, we note that

$$\partial_\alpha\big(\pi_A{}^\alpha\mathcal{L}_X\phi^A\big) = \nabla_\alpha\big(\pi_A{}^\alpha\mathcal{L}_X\phi^A\big)\,.$$

From (6.31) and (6.29) we conclude that

$$\begin{aligned}\partial_\alpha\mathscr{H}^\alpha &= \nabla_\alpha\big(\pi_A{}^\alpha\mathcal{L}_X\phi^A\big)\\ &\quad -\Big(\frac{\partial\overline{\mathscr{L}}}{\partial\phi^A}\mathcal{L}_X\phi^A + \pi_A{}^\alpha\mathcal{L}_X\nabla_\alpha\phi^A + \frac{\partial\overline{\mathscr{L}}}{\partial g_{\mu\nu}}\mathcal{L}_Xg_{\mu\nu} + \frac{\partial\overline{\mathscr{L}}}{\partial\phi^A{}_{,\mu}}\phi^B\mathcal{L}_X\Gamma^A{}_{B\mu}\Big)\\ &= \mathscr{E}_A\mathcal{L}_X\phi^A - \pi_A{}^\alpha[\mathcal{L}_X,\nabla_\alpha]\phi^A - \frac{\partial\overline{\mathscr{L}}}{\partial g_{\mu\nu}}\mathcal{L}_Xg_{\mu\nu} - \frac{\partial\overline{\mathscr{L}}}{\partial\phi^A{}_{,\mu}}\phi^B\mathcal{L}_X\Gamma^A{}_{B\mu}\,,\end{aligned} \tag{6.33}$$

which provides an explicitly covariant derivation of (6.25).

The above treatment applies to any theories of tensor fields with a coordinate-invariant Lagrangian and without constraints, e.g. for a scalar field. This does, however, fail for theories where constraints are present, which require further considerations.

6.3 The Maxwell field

We turn now our attention to Maxwell fields. Unless explicitly indicated otherwise we consider a general Lagrangian

$$\mathscr{L}(A_\mu, \partial_\alpha A_\beta, g_{\rho\sigma}) \equiv \mathscr{L}(\partial_{[\alpha}A_{\beta]}, g_{\rho\sigma})\,, \tag{6.34}$$

thus $\mathscr{L}$ neither involves the undifferentiated potential A_μ nor derivatives of the metric, and the canonical momentum is antisymmetric:

$$\pi^{\mu\nu} = \pi^{[\mu\nu]}\,.$$

We start by noting that there are several ways to proceed:

1. We view A_μ as a covector field on spacetime, with the Noether currents $\mathscr{H}_c$ of (3.4) providing a starting point of further analysis; or
2. we view A_μ as a $U(1)$-gauge field, using the Noether currents $\mathscr{H}$ instead; and
3. in either case we may, or we may not, gauge fix, to address issues arising from the vanishing of the momentum conjugate to A_0.
4. Yet another approach is presented in [1, Chapter 3].

We continue by noting that the Lagrangian (3.1) is a scalar density, so that the condition H1, p. 49 is satisfied in all cases.

Next, while the replacement of $\mathcal{L}_X A$ in the Noether currents by $\mathbf{L}_X A_\mu$ as given by (3.13) renders the current $\mathscr{H}^\mu$ given by (3.14) manifestly gauge invariant, it leads to problems with point c) of H2. For instance, if $X = \partial_0$ the partial derivative $\partial_0 A_\beta \equiv \mathcal{L}_{\partial_0} A_\beta$ will be equal to

$$\mathbf{L}_{\partial_0} A_\beta = F_{0\beta} = \partial_0 A_\beta - \partial_\beta A_0 \tag{6.35}$$

only in a gauge where

$$\partial_\beta A_0 \equiv 0\,. \tag{6.36}$$

But we do not wish to gauge-fix, and therefore we need to revisit the scheme.

6.3.1 Hamilton's equations

For future reference we calculate on a general hypersurface $\mathscr{S}$, in a general metric, for a general vector field X, but using adapted coordinates in which $\mathscr{S} = \{x^0 = 0\}$.

Choosing the covector-field approach leads to the Noether charge integral

$$H_c[\mathscr{S}, X] = \int_{\mathscr{S}} \mathscr{H}_c^0 dS_0\,, \tag{6.37}$$

cf. (3.4), while the $U(1)$-gauge field approach leads instead to

$$H[\mathscr{S}, X] = \int_{\mathscr{S}} \mathscr{H}^0 dS_0\,, \tag{6.38}$$

where $\mathscr{H}^0$ is defined by (3.13).

The variation of $H_c[\mathscr{S}, X]$ is obtained immediately by setting $(\phi^A) = (A_\alpha)$ in (6.7):

$$\begin{aligned} \delta\mathscr{H}_c^0 &= \mathcal{L}_X A_\alpha \delta\pi^{\alpha 0} - \left[\mathcal{L}_X \pi^{\alpha 0} + X^0 \mathcal{E}^\alpha\right] \delta A_\alpha \\ &\quad + \partial_\mu \Big[(X^\mu \pi^{\alpha 0} - X^0 \pi^{\alpha\mu})\delta A_\alpha\Big]\,, \end{aligned} \tag{6.39}$$

where $\mathcal{L}_X \pi^{\alpha 0}$ is understood as the $\alpha = 0$-component of the field $\mathcal{L}_X \pi^{\alpha\mu}$, and where $\mathscr{E}^\alpha$ denotes the field equations operator,

$$\mathcal{E}^\alpha \equiv \frac{\delta \mathscr{L}}{\delta A_\alpha} := \frac{\partial \mathscr{L}}{\partial A_\alpha} - \partial_\mu \Big(\frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\alpha)} \Big) \equiv -\partial_\mu \pi^{\alpha\mu} \,. \tag{6.40}$$

Integration gives

$$\begin{aligned} \int_{\mathscr{S}} \delta \mathscr{H}_c^0 &= \int_{\mathscr{S}} \Big[\mathcal{L}_X A_i \delta \pi^{i0} - \big[\mathcal{L}_X \pi^{i0} + X^0 \mathcal{E}^i \big] \delta A_i - X^0 \mathcal{E}^0 \delta A_0 \Big] dS_0 \\ &\quad + \int_{\partial \mathscr{S}} \Big[(X^i \pi^{j0} - X^0 \pi^{ji}) \delta A_j - X^0 \pi^{0i} \delta A_0 \Big] dS_{0i} \,. \end{aligned} \tag{6.41}$$

For the variations of $H[\mathscr{S}, X]$, one can recycle the calculations leading to (6.39) by rewriting $\mathscr{H}^\mu[X]$ as

$$\begin{aligned} \mathscr{H}^\mu[X] &= \pi^{\beta\mu} \mathbf{L}_X A_\beta - \mathscr{L} X^\mu \\ &= \pi^{\beta\mu} \big(\mathcal{L}_X A_\beta + \underbrace{X^\alpha F_{\alpha\beta} - \mathcal{L}_X A_\beta}_{-\partial_\beta (X^\alpha A_\alpha)} \big) - \mathscr{L} X^\mu \\ &= \underbrace{\pi^{\beta\mu} \mathcal{L}_X A_\beta - \mathscr{L} X^\mu}_{\mathscr{H}_c^\mu[X]} - \partial_\beta \Big(\pi^{\beta\mu} X^\alpha A_\alpha \Big) + X^\alpha A_\alpha \underbrace{\partial_\beta \pi^{\beta\mu}}_{\mathscr{E}^\mu} \,. \end{aligned} \tag{6.42}$$

(Setting $\mu = 0$, we observe the well known fact that $\mathscr{H}^0$ and $\mathscr{H}_c^0$ differ by a divergence when the constraint equation $\mathscr{E}^0 = 0$ holds.) We can apply (6.39) to the first two terms in the right-hand side of (6.42), obtaining thus

$$\begin{aligned} \delta \mathscr{H}^0 &= \underbrace{\mathcal{L}_X A_\alpha}_{\mathbf{L}_X A_\alpha + \partial_\alpha (X^\beta A_\beta)} \delta \pi^{\alpha 0} - \big[\mathcal{L}_X \pi^{\alpha 0} + X^0 \mathcal{E}^\alpha \big] \delta A_\alpha \\ &\quad + \partial_\alpha \Big[(X^\alpha \pi^{\beta 0} - X^0 \pi^{\beta\alpha}) \delta A_\beta \Big] - \delta \partial_\alpha \Big(\pi^{\alpha 0} X^\beta A_\beta \Big) - \delta \big(X^\alpha A_\alpha \mathscr{E}^0 \big) \\ &= \mathbf{L}_X A_\alpha \delta \pi^{\alpha 0} - \big[\mathcal{L}_X \pi^{\alpha 0} + X^0 \mathcal{E}^\alpha - X^\alpha \mathcal{E}^0 \big] \delta A_\alpha \\ &\quad + \partial_\alpha \Big[(X^\alpha \pi^{\beta 0} - X^0 \pi^{\beta\alpha} - X^\beta \pi^{\alpha 0}) \delta A_\beta \Big] \\ &= \mathbf{L}_X A_i \delta \pi^{i0} - \big[\mathcal{L}_X \pi^{i0} + X^0 \mathcal{E}^i - X^i \mathcal{E}^0 \big] \delta A_i \\ &\quad + \partial_i \Big[(X^i \pi^{j0} - X^0 \pi^{ji} - X^j \pi^{i0}) \delta A_j \Big] \,, \end{aligned} \tag{6.43}$$

where we use lower-case latin indices for coordinates on $\mathscr{S}$. Note that δA_0 vanished from this formula. Equation (6.43) is the $U(1)$-gauge-field-equivalent of (6.39), and leads to

$$\begin{aligned}\int_{\mathscr{S}} \delta\mathscr{H}^0 &= \int_{\mathscr{S}} \Big[\mathbf{L}_X A_i \delta\pi^{i0} - \left[\mathcal{L}_X \pi^{i0} + X^0\mathcal{E}^i - X^i\mathcal{E}^0\right]\delta A_i\Big] dS_0 \\ &\quad + \int_{\partial\mathscr{S}} \big(X^i\pi^{j0} - X^0\pi^{ji} - X^j\pi^{i0}\big)\delta A_j \, dS_{0i}\,, \end{aligned} \tag{6.44}$$

where the integration over $\partial\mathscr{S}$ might be understood by exhausting $\mathscr{S}$ with a family of compact domains with smooth boundary, and passing to the limit.

To continue we set

$$\pi^\mu := \pi^{\mu 0} \tag{6.45}$$

Anti-symmetry of $\pi^{\mu\nu}$ leads to

$$\pi^0 \equiv \pi^{00} = 0\,, \tag{6.46}$$

and so the variations of the momenta are *not* arbitrary. Further, the field equations give in particular

$$\partial_k \pi^k = 0 \tag{6.47}$$

which gives a constraint on the π^k's when the field equations are assumed.

In view of (6.46), and keeping in mind that δA_0 does not appear in (6.43), one could be tempted to drop the field A_0 from the Hamiltonian formalism altogether. But then treating A_μ as a vector field on spacetime, or a gauge-field on a $U(1)$-principal bundle over spacetime, will not make sense. Likewise neither (6.35) would not make sense, nor the usual expression for the Lie derivative

$$X^\mu \partial_\mu A_\nu + \partial_\nu X^\mu A_\mu\,,$$

should one wish to use this expression instead of (6.35) in the definition of the Noether charge. So it is natural to keep the field A_0 as part of the variables, even though it does not appear in some equations below.

To continue, recall that the variation δH is defined as follows: given any one parameter family of fields $\lambda \mapsto A_\mu(\lambda)$ one sets

$$\delta A_k := \frac{dA_k}{d\lambda}\,, \qquad \delta\pi^k := \frac{d\pi^k}{d\lambda}\,,$$

etc. Now, in the $U(1)$-bundle Hamiltonian picture both the time derivatives of A_k and the space derivatives of A_0 are eliminated in terms of π^k. We can therefore calculate as follows

$$\begin{aligned}
\delta H[\mathscr{S},X] &:= \int_{\mathscr{S}} \frac{d\mathscr{H}^0}{d\lambda} dS_0 \equiv \int_{\mathscr{S}} \delta\mathscr{H}^0 dS_0 \\
&= \int_{\mathscr{S}} \Big[\frac{\partial\mathscr{H}^0}{\partial\pi^k}\delta\pi^k + \frac{\partial\mathscr{H}^0}{\partial A_k}\delta A_k + \frac{\partial\mathscr{H}^0}{\partial A_{k,\ell}}\delta\partial_\ell A_k\Big] dS_0 \\
&= \int_{\mathscr{S}} \Big[\frac{\partial\mathscr{H}^0}{\partial\pi^k}\delta\pi^k + \Big(\frac{\partial\mathscr{H}^0}{\partial A_k} - \partial_\ell\big(\frac{\partial\mathscr{H}^0}{\partial A_{k,\ell}}\big)\Big)\delta A_k\Big] dS_0 \\
&\quad + \int_{\partial\mathscr{S}} \frac{\partial\mathscr{H}^0}{\partial A_{k,\ell}}\delta A_k \, dS_{0k} \\
&\equiv \int_{\mathscr{S}} \Big[\frac{\delta\mathscr{H}^0}{\delta\pi^k}\delta\pi^k + \frac{\delta\mathscr{H}^0}{\delta A_k}\delta A_k\Big] dS_0 + \int_{\partial\mathscr{S}} \frac{\partial\mathscr{H}^0}{\partial A_{k,\ell}}\delta A_k \, dS_{0\ell} \, .
\end{aligned} \tag{6.48}$$

On the other hand, in the covector-field Hamiltonian approach we find

$$\begin{aligned}
\delta H_c[\mathscr{S},X] &:= \int_{\mathscr{S}} \frac{d\mathscr{H}_c^0}{d\lambda} dS_0 \equiv \int_{\mathscr{S}} \delta\mathscr{H}_c^0 dS_0 \\
&= \int_{\mathscr{S}} \Big[\frac{\delta\mathscr{H}_c^0}{\delta\pi^k}\delta\pi^k + \frac{\delta\mathscr{H}_c^0}{\delta A_\mu}\delta A_\mu\Big] dS_0 + \int_{\partial\mathscr{S}} \frac{\partial\mathscr{H}_c^0}{\partial A_{\mu,\ell}}\delta A_\mu \, dS_{0\ell} \, .
\end{aligned} \tag{6.49}$$

Hamilton's equations of motion will be obtained after comparing (6.44) with (6.48), or (6.41) with (6.49).

Now, comparison of (6.44) with (6.48) leads to

$$\begin{aligned}
0 &= \int_{\mathscr{S}} \Big[\Big(\frac{\delta\mathscr{H}^0}{\delta\pi^k} - \mathbf{L}_X A_k\Big)\delta\pi^k + \Big(\frac{\delta\mathscr{H}^0}{\delta A_k} + \mathcal{L}_X\pi^k + X^0\mathcal{E}^k - X^k\mathcal{E}^0\Big)\delta A_k\Big] dS_0 \\
&\quad + \int_{\partial\mathscr{S}} \Big[\frac{\partial\mathscr{H}^0}{\partial A_{k,\ell}} - (X^\ell\pi^k - X^0\pi^{k\ell} - X^k\pi^\ell)\Big]\delta A_k \, dS_{0\ell} \, .
\end{aligned} \tag{6.50}$$

The question then arises whether or not, and if so how, to take into account the Gauss constraint (6.47). (Strictly speaking, this is the Gauss constraint when the Lagrangian for the standard Maxwell electrodynamics is considered, but we will keep using this terminology for the more general theories considered here.) We emphasise that (6.50), as well as (6.64) below, are identities which hold for all variations, whether or not the constraints are satisfied. So we have now at least two options:

1. We allow any variations, perhaps but not necessarily assuming that the Gauss constraint is satisfied at the field configuration at which the variation is carried-out; or

2. we assume that the Gauss constraint is satisfied, and we restrict ourselves to variations which satisfy this constraint.

Consider, then, Equation (6.50). In the first case, both δA_i and $\delta\pi^k$ are arbitrary. Restricting to variations which vanish at $\partial\mathscr{S}$ we obtain

$$\mathbf{L}_X A_k = \frac{\delta\mathscr{H}^0}{\delta\pi^k}\,, \tag{6.51}$$

$$\mathcal{L}_X \pi^k = -\frac{\delta\mathscr{H}^0}{\delta A_k} - X^0\mathcal{E}^k + X^k\mathcal{E}^0\,. \tag{6.52}$$

It then also follows that

$$\int_{\partial\mathscr{S}} \Big[\frac{\partial\mathscr{H}^0}{\partial A_{k,\ell}} - (X^\ell\pi^k - X^0\pi^{k\ell} - X^k\pi^\ell)\Big]\delta A_k\, dS_{0\ell} = 0\,, \tag{6.53}$$

(If δA_k is arbitrary on $\partial\mathscr{S}$ we can further conclude that

$$\big(\frac{\partial\mathscr{H}^0}{\partial A_{k,\ell}} - (X^\ell\pi^k - X^0\pi^{k\ell} - X^k\pi^\ell)\big)\big|_{\partial\mathscr{S}} n_\ell = 0\,,$$

where n_ℓ is the field of conormals to $\partial\mathscr{S}$, but in some situations it might be appropriate to restrict the class of field variations allowed at $\partial\mathscr{S}$. If moreover δA_k is arbitrary on $\partial\mathscr{S}$ and we allow $\partial\mathscr{S}$ to vary we further find

$$\frac{\partial\mathscr{H}^0}{\partial A_{k,\ell}} = X^\ell\pi^k - X^0\pi^{k\ell} - X^k\pi^\ell\,.) \tag{6.54}$$

Next, we return to (6.50) in the second case where the variations of A_k remain arbitrary but those of π^k are subject to the constraint

$$\delta\mathscr{E}^0 \equiv \partial_k\delta\pi^k = 0\,. \tag{6.55}$$

Now, the vanishing of the divergence of $\delta\pi^k$ implies that for any function λ we have

$$\int_{\mathscr{S}} \delta\pi^i\partial_i\lambda dS_0 = -\underbrace{\int_{\mathscr{S}} \delta\partial_i\pi^i\lambda dS_0}_{0} + \int_{\partial\mathscr{S}} \delta\pi^i\lambda\, dS_{0i}\,. \tag{6.56}$$

The right-hand side vanishes when λ or when the normal component of π^i vanish on $\partial\mathscr{S}$. We expect therefore that (6.51) should be replaced by

$$X^\mu(A_{k,\mu} - A_{\mu,k}) \equiv \mathbf{L}_X A_k = \frac{\delta\mathscr{H}^0}{\delta\pi^k} + \partial_k\lambda\,, \tag{6.57}$$

and it is conceivable that this equation can be justified for classes of fields with restricted boundary conditions, but we have not attempted to do this.

The apparent discrepancy between the last equation and (6.51) is easiest to understand in Minkowski spacetime, on the standard slices $t = \text{const}$, with $X = \partial_t$. Then (6.57) reads

$$\partial_t A_k - \partial_k A_0 \quad = \quad \pi_k + \partial_k \lambda \,, \tag{6.58}$$

so that in this case the function λ can be absorbed in a redefinition of A_0.

More generally, (6.57) can be rewritten as

$$\mathcal{L}_X A_k - \partial_k (X^\mu A_\mu) = \frac{\delta \mathscr{H}^0}{\delta \pi^k} + \partial_k \lambda \,, \tag{6.59}$$

which makes it clear that the freedom in the choice of λ is closely related to the gauge freedom of the theory.

Regardless of whether or not (6.57) provides the correct way to proceed in whole generality, one can take into account the constraint (6.55) by using variations of the form

$$\delta \pi^k = \epsilon^{k\ell m} D_\ell \delta Y_m \,, \tag{6.60}$$

which have vanishing divergence for all vector fields δY. For such variations, and after taking into account (6.51)-(6.53), Equation (6.50) becomes

$$\begin{aligned} 0 \quad &= \quad \int_{\mathscr{S}} \Big(\frac{\delta \mathscr{H}^0}{\delta \pi^k} - \mathbf{L}_X A_k \Big) \epsilon^{k\ell m} D_\ell \delta Y_m dS_0 \\ &= \quad \int_{\mathscr{S}} \epsilon^{k\ell m} D_\ell \Big(\frac{\delta \mathscr{H}^0}{\delta \pi^k} - \mathbf{L}_X A_k \Big) \delta Y_m dS_0 \\ &\quad + \int_{\partial\mathscr{S}} \Big(\frac{\delta \mathscr{H}^0}{\delta \pi^k} - \mathbf{L}_X A_k \Big) \epsilon^{k\ell m} \delta Y_m dS_{0\ell} \,. \end{aligned} \tag{6.61}$$

As this holds in particular for all vector fields δY vanishing at $\partial\mathscr{S}$, we conclude that we must have

$$\epsilon^{k\ell m} D_\ell \Big(\frac{\delta \mathscr{H}^0}{\delta \pi^k} - \mathbf{L}_X A_k \Big) = 0 \,. \tag{6.62}$$

In the Minkowskian case as in (6.58), this is the usual Maxwell equation

$$\partial_t \vec{B} = -\text{rot}\, \vec{E} \,. \tag{6.63}$$

We finish this section by comparing (6.41) with (6.49):

$$
\begin{aligned}
0 \;=\; & \int_{\mathscr{S}} \Big[\Big(\frac{\delta \mathscr{H}^0_c}{\delta \pi^k} - \mathcal{L}_X A_k\Big)\delta\pi^k + \Big(\frac{\delta \mathscr{H}^0_c}{\delta A_k} + \mathcal{L}_X \pi^k + X^0\mathcal{E}^k\Big)\delta A_k \\
& \quad + \Big(\frac{\delta \mathscr{H}^0_c}{\delta A_0} + X^0\mathcal{E}^0\Big)\delta A_0\Big]\, dS_0 \\
& + \int_{\partial\mathscr{S}} \Big\{\Big[\frac{\partial \mathscr{H}^0_c}{\partial A_{k,\ell}} - (X^\ell \pi^k - X^0\pi^{k\ell})\Big]\delta A_k \\
& \quad + \Big(\frac{\partial \mathscr{H}^0_c}{\partial A_{0,\ell}} + X^0\pi^\ell\Big)\delta A_0\Big\}\, dS_{0\ell}\,. \qquad (6.64)
\end{aligned}
$$

A discussion similar to the one for $\mathscr{H}^0$ applies, we leave the details to the reader.

6.3.2 Noether charge algebra

In this section, unless explicitly indicated otherwise we consider a theory with a general Lagrangian density of the form (6.34). As before we set

$$
\mathscr{H}^\mu_X = \pi^{\nu\mu}\mathbf{L}_X A_\nu - X^\mu \mathscr{L}\,. \qquad (6.65)
$$

Given two functionals F and G of the form

$$
F = \int_{\mathscr{S}} f(A_k, \partial_i A_k, \pi^\ell)\, dS_0\,, \quad G = \int_{\mathscr{S}} g(A_k, \partial_i A_k, \pi^\ell)\, dS_0\,, \qquad (6.66)
$$

following [3] we set

$$
\{F, G\}_{\mathscr{S}} := \int_{\mathscr{S}} \left(\frac{\delta f}{\delta A_l}\frac{\delta g}{\delta \pi^l} - \frac{\delta f}{\delta \pi^l}\frac{\delta g}{\delta A_l}\right) dS_0\,. \qquad (6.67)
$$

In this formula the operator $\delta/\delta\pi^i$ is defined by ignoring the fact that $\delta\pi^i$ should satisfy the Gauss constraint, so that (6.51)-(6.52) apply.

The Poisson bracket of two Hamiltonian functionals H_X and H_Y with integrands $\mathscr{H}^\mu_X$ and $\mathscr{H}^\mu_Y$ thus equals

$$
\{H_X, H_Y\}_{\mathscr{S}} = \int_{\mathscr{S}} \left(\frac{\delta \mathscr{H}^0_X}{\delta A_l}\frac{\delta \mathscr{H}^0_Y}{\delta \pi^l} - \frac{\delta \mathscr{H}^0_X}{\delta \pi^l}\frac{\delta \mathscr{H}^0_Y}{\delta A_l}\right) dS_0\,, \qquad (6.68)
$$

Let

$$
E^k_X := X^k \mathscr{E}^0 - X^0 \mathscr{E}^k\,, \quad E^k_Y := Y^k \mathscr{E}^0 - Y^0 \mathscr{E}^k\,.
$$

Recall that $\pi^0 = 0$ and that $\mathcal{L}_X \pi^0 := \mathcal{L}_X \pi^{00} = 0$ for any vector field. Using (6.51)-(6.52) we find

$$
\begin{aligned}
&\frac{\delta \mathscr{H}_X^0}{\delta A_l}\frac{\delta \mathscr{H}_Y^0}{\delta \pi^l} - \frac{\delta \mathscr{H}_X^0}{\delta \pi^l}\frac{\delta \mathscr{H}_Y^0}{\delta A_l} = \\
&= \big(- \mathcal{L}_X \pi^k + E_X^k \big) \mathbf{L}_Y A_k - \big(- \mathcal{L}_Y \pi^k + E_Y^k \big) \mathbf{L}_X A_k \\
&= -\mathcal{L}_X \big(\pi^\nu \mathbf{L}_Y A_\nu \big) + \pi^\nu \mathcal{L}_X \big(\mathbf{L}_Y A_\nu \big) + \mathcal{L}_Y \big(\pi^\nu \mathbf{L}_X A_\nu \big) \\
&\quad -\pi^\nu \mathcal{L}_Y \big(\mathbf{L}_X A_\nu \big) + \big(E_X^k Y^\mu - E_Y^k X^\mu \big) F_{\mu k} \\
&= \pi^\nu \big(\mathcal{L}_X \mathbf{L}_Y A_\nu - \mathcal{L}_Y \mathbf{L}_X A_\nu \big) - \mathcal{L}_X \big(\pi^\nu \mathbf{L}_Y A_\nu - Y^0 \mathscr{L} \big) \\
&\quad + \mathcal{L}_Y \big(\pi^\nu \mathbf{L}_X A_\nu - X^0 \mathscr{L} \big) - \mathcal{L}_X \big(Y^0 \mathscr{L} \big) + \mathcal{L}_Y \big(X^0 \mathscr{L} \big) \\
&\quad + \big(E_X^k Y^\mu - E_Y^k X^\mu \big) F_{\mu k} \,.
\end{aligned} \tag{6.69}
$$

Additionally, we have

$$
\begin{aligned}
&\big(\mathcal{L}_X \mathbf{L}_Y A_\nu - \mathcal{L}_Y \mathbf{L}_X A_\nu \big) = \\
&= \mathcal{L}_X \big(Y^\mu F_{\mu\nu} \big) - \mathcal{L}_Y \big(X^\mu F_{\mu\nu} \big) \\
&= X^\alpha \partial_\alpha \big(Y^\mu F_{\mu\nu} \big) + Y^\mu F_{\mu\alpha} \partial_\nu X^\alpha - Y^\alpha \partial_\alpha \big(X^\mu F_{\mu\nu} \big) - X^\mu F_{\mu\alpha} \partial_\nu Y^\alpha \\
&= \underbrace{\big(X^\alpha \partial_\alpha Y^\mu - Y^\alpha \partial_\alpha X^\mu \big) F_{\mu\nu}}_{\mathbf{L}_{[X,Y]} A_\nu} + \big(X^\alpha Y^\mu - Y^\alpha X^\mu \big) \partial_\alpha F_{\mu\nu} \\
&\quad + \big(Y^\mu \partial_\nu X^\alpha - X^\mu \partial_\nu Y^\alpha \big) F_{\mu\alpha} \\
&= \mathbf{L}_{[X,Y]} A_\nu + X^\alpha Y^\mu \big(\partial_\alpha F_{\mu\nu} - \partial_\mu F_{\alpha\nu} \big) + F_{\mu\alpha} \partial_\nu \big(Y^\mu X^\alpha \big)
\end{aligned} \tag{6.70}
$$

Using the identity

$$\partial_\alpha F_{\mu\nu} - \partial_\mu F_{\alpha\nu} = \partial_\nu F_{\mu\alpha} \tag{6.71}$$

we continue as follows:

$$
\begin{aligned}
&\big(\mathcal{L}_X \mathbf{L}_Y A_\nu - \mathcal{L}_Y \mathbf{L}_X A_\nu \big) = \\
&= \mathbf{L}_{[X,Y]} A_\nu + X^\alpha Y^\mu \partial_\nu F_{\mu\alpha} + F_{\mu\alpha} \partial_\nu \big(Y^\mu X^\alpha \big) \\
&= \mathbf{L}_{[X,Y]} A_\nu + \partial_\nu \big(F_{\mu\alpha} Y^\mu X^\alpha \big) \,.
\end{aligned} \tag{6.72}
$$

Recall the identities (6.19)-(6.20) for vector densities,

$$
\begin{aligned}
\mathcal{L}_X \mathscr{H}_Y^\beta - \mathcal{L}_Y \mathscr{H}_X^\beta &= 2\partial_\alpha \Big(X^{[\alpha} \mathscr{H}_Y^{\beta]} - Y^{[\alpha} \mathscr{H}_X^{\beta]} \Big) \\
&\quad + X^\beta \partial_\alpha \mathscr{H}_Y^\alpha - Y^\beta \partial_\alpha \mathscr{H}_X^\alpha \,,
\end{aligned} \tag{6.73}
$$

$$
\mathcal{L}_X \Big(Y^\beta \mathscr{L} \Big) - \mathcal{L}_Y \Big(X^\beta \mathscr{L} \Big) = 2\partial_\alpha \Big(X^{[\alpha} Y^{\beta]} \mathscr{L} \Big) + [X,Y]^\beta \mathscr{L} \,. \tag{6.74}
$$

Inserting (6.72)-(6.74) into (6.69) results in

$$\begin{aligned}\{H_X, H_Y\} &= \int_{\mathscr{S}} \Big\{ \mathscr{H}^\beta_{[X,Y]} + Y^\beta \partial_\alpha \mathscr{H}^\alpha_X - X^\beta \partial_\alpha \mathscr{H}^\alpha_Y \\ &\quad -2\partial_\alpha \Big(X^{[\alpha} \mathscr{H}^{\beta]}_Y - Y^{[\alpha} \mathscr{H}^{\beta]}_X + X^{[\alpha} Y^{\beta]} \mathscr{L} \Big) \\ &\quad + \big(E^k_X Y^\mu - E^k_Y X^\mu \big) F_{\mu k} \Big\} dS_\beta \,. \end{aligned} \tag{6.75}$$

To continue, we wish to show that the divergence of the Noether currents, which appear above, vanishes when the field equations hold and when the Lagrangian does not depend upon A_μ. For this, recall that we have assumed that the Lagrangian is a scalar density, so that

$$\mathcal{L}_X \mathscr{L} = \partial_\alpha \big(X^\alpha \mathscr{L} \big) \,. \tag{6.76}$$

Thus

$$\begin{aligned}\partial_\alpha \mathscr{H}^\alpha_X &= \partial_\alpha \big(\pi^{\nu\alpha} \mathbf{L}_X A_\nu - X^\alpha \mathscr{L} \big) \\ &= \partial_\alpha \big(\pi^{\nu\alpha} \mathbf{L}_X A_\nu \big) - \mathcal{L}_X \mathscr{L} \,, \end{aligned} \tag{6.77}$$

which can be rearranged as

$$\partial_\alpha \mathscr{H}^\alpha_X = \partial_\alpha \big(\pi^{\nu\alpha} (\mathbf{L}_X A_\nu - \mathcal{L}_X A_\nu) \big) + \partial_\alpha \mathscr{H}^\alpha_c [X] \,. \tag{6.78}$$

For $\partial_\alpha \mathscr{H}^\alpha_c[X]$ the formulae (6.33) holds with $\overline{\mathscr{L}} = \mathscr{L}$. Note that $\mathscr{L}$ does not depend on connection coefficients. Equation (6.78) becomes

$$\begin{aligned}\partial_\alpha \mathscr{H}^\alpha_X &= \partial_\alpha \big(\pi^{\nu\alpha} (\mathbf{L}_X A_\nu - \mathcal{L}_X A_\nu \big) \\ &\quad + \mathscr{E}^\kappa \mathcal{L}_X A_\kappa - \pi^{\lambda\kappa} [\mathcal{L}_X, \nabla_\kappa] A_\lambda - \frac{\partial \mathscr{L}}{\partial g_{\kappa\lambda}} \mathcal{L}_X g_{\kappa\lambda} \,. \end{aligned} \tag{6.79}$$

Now, the divergence term in (6.79) reads

$$\begin{aligned} &\partial_\alpha \big[\pi^{\nu\alpha} \big(\mathbf{L}_X A_\nu - \mathcal{L}_X A_\nu \big) \big] = \\ &= \mathscr{E}^\nu \big(\mathbf{L}_X A_\nu - \mathcal{L}_X A_\nu \big) + \pi^{\nu\alpha} \partial_\alpha \big(\mathbf{L}_X A_\nu - \mathcal{L}_X A_\nu \big) \\ &= \mathscr{E}^\nu \Big(\mathbf{L}_X A_\nu - \mathcal{L}_X A_\nu \big) \\ &\quad + \pi^{\nu\alpha} \partial_\alpha \big(\underbrace{X^\mu F_{\mu\nu} - X^\mu \partial_\mu A_\nu + X^\mu \partial_\nu A_\mu}_{=0} - \partial_\nu \big(X^\mu A_\mu \big) \Big) \\ &= \mathscr{E}^\nu \Big(\mathbf{L}_X A_\nu - \mathcal{L}_X A_\nu \big) \,. \end{aligned} \tag{6.80}$$

Summarising, we have shown that

$$\partial_\alpha \mathscr{H}^\alpha_X \;=\; \mathscr{E}^\nu \mathbf{L}_X A_\nu - \pi^{\lambda\kappa}[\mathcal{L}_X, \nabla_\kappa] A_\lambda - \frac{\partial \mathscr{L}}{\partial g_{\kappa\lambda}} \mathcal{L}_X g_{\kappa\lambda} \,. \tag{6.81}$$

Inserting this into (6.75) we conclude that

$$\begin{aligned}\{H_X, H_Y\} \;=\; & \int_{\mathscr{S}} \Big\{ \mathscr{H}^\beta_{[X,Y]} - 2\partial_\alpha \Big(X^{[\alpha} \mathscr{H}^{\beta]}_Y - Y^{[\alpha} \mathscr{H}^{\beta]}_X + X^{[\alpha} Y^{\beta]} \mathscr{L} \Big) \\ & + Y^\beta \Big(\mathscr{E}^\kappa \mathbf{L}_X A_\kappa - \pi^{\lambda\kappa} [\mathcal{L}_X, \nabla_\kappa] A_\lambda - \frac{\partial \mathscr{L}}{\partial g_{\kappa\lambda}} \mathcal{L}_X g_{\kappa\lambda} \Big) \\ & - X^\beta \Big(\mathscr{E}^\kappa \mathbf{L}_Y A_\kappa - \pi^{\lambda\kappa} [\mathcal{L}_Y, \nabla_\kappa] A_\lambda - \frac{\partial \mathscr{L}}{\partial g_{\kappa\lambda}} \mathcal{L}_Y g_{\kappa\lambda} \Big) \\ & + \big(E^k_X Y^\mu - E^k_Y X^\mu \big) F_{\mu k} \Big\} dS_\beta \,. \end{aligned} \tag{6.82}$$

where (6.79)-(6.80) have been used.

6.3.3 $\mathscr{H}^0$: $3+1$ decomposition

We continue by deriving the formula for $\mathscr{H}^0$, and its variation, assuming the standard Maxwell Lagrangean. It is convenient to introduce some notation. In the remainder of this section we will assume that $\mathscr{S}$ is spacelike. We will use the ADM parametrisation of the metric,

$$\gamma_{ij} := g_{ij} \,, \quad N := \frac{1}{\sqrt{-g^{00}}} \,, \quad N_k := g_{0k} \,, \quad N^k = \gamma^{ki} N_i \,, \tag{6.83}$$

where γ^{ij} is an inverse of the three-dimensional metric γ_{ij} induced by $g_{\mu\nu}$ on $\mathscr{S}$:

$$\begin{aligned} g_{\mu\nu} \;&=\; \begin{bmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{bmatrix} = \begin{bmatrix} -N^2 + N^k N_k & N_j \\ N_i & \gamma_{ij} \end{bmatrix} \,, \\ g^{\mu\nu} \;&=\; \begin{bmatrix} g^{00} & g^{0j} \\ g^{i0} & g^{ij} \end{bmatrix} = \begin{bmatrix} -\frac{1}{N^2} & \frac{N^j}{N^2} \\ \frac{N^i}{N^2} & \gamma^{ij} - \frac{N^i N^j}{N^2} \end{bmatrix} \,. \end{aligned} \tag{6.84}$$

It holds that

$$\sqrt{|\det g_{\mu\nu}|} = N \sqrt{\det \gamma_{ij}} \,. \tag{6.85}$$

Let T^μ denote the field of unit future-directed normals to $\mathscr{S}$, thus $T_\mu = -N\, dt$. We define the electric field E^k as

$$E^k := N F^{0k} = -F^{\mu k} T_\mu \,. \tag{6.86}$$

The canonical momentum π^k is related to the electric field as

$$\pi^k = -\frac{1}{4\pi}\sqrt{\det\gamma_{ij}}E^k\,, \tag{6.87}$$

and we note that

$$D_k E^k \equiv \frac{1}{\sqrt{\det\gamma_{ij}}}\partial_k\big(\sqrt{\det\gamma_{ij}}E^k\big) = -\frac{1}{\sqrt{\det\gamma_{ij}}}\partial_\mu\big(\sqrt{\det g_{\alpha\beta}}F^{0\mu}\big) = 0\,, \tag{6.88}$$

where D is the covariant derivative operator of the metric γ.

The decomposition of the Maxwell tensor density (3.3) associated with the (3+1)-decomposition of the metric reads

$$\begin{aligned} \mathcal{F}_{0k} &= N^l\mathcal{F}_{lk} - N^2\mathcal{F}^{0l}\gamma_{lk}\,, &(6.89)\\ \mathcal{F}^{kl} &= (N^l\mathcal{F}^{0k} - N^k\mathcal{F}^{0l}) + \gamma^{km}\gamma^{ln}\mathcal{F}_{mn}\,, &(6.90)\end{aligned}$$

where

$$\mathcal{F}^{\alpha\beta} = \sqrt{|-\det g|}F^{\alpha\beta}\,. \tag{6.91}$$

For (6.90) the following calculation is useful:

$$\begin{aligned} \mathcal{F}_{kl}\gamma^{kp}\gamma^{lq} &= \mathcal{F}^{\mu\nu}g_{\mu k}g_{\nu l}\gamma^{kp}\gamma^{lq}\\ &= \Big(\mathcal{F}^{0m}g_{0k}g_{ml} + \mathcal{F}^{m0}g_{mk}g_{0l} + \mathcal{F}^{mn}g_{mk}g_{nl}\Big)\gamma^{kp}\gamma^{lq}\\ &= \mathcal{F}^{0q}N^p + \mathcal{F}^{p0}N^q + \mathcal{F}^{pq}\,, \end{aligned} \tag{6.92}$$

and this last equation is also useful as an intermediate step for (6.89). The field equations operator $\mathscr{E}^k$ reads

$$4\pi\mathscr{E}^k = \partial_l(N^l\mathcal{F}^{0k} - N^k\mathcal{F}^{0l}) + \partial_l(\gamma^{km}\gamma^{ln}\mathcal{F}_{mn}) - \partial_0\mathcal{F}^{0k}\,, \tag{6.93}$$

which has to be supplemented with the constraint equation

$$4\pi\mathscr{E}^0 = -\partial_i\mathcal{F}^{i0}\,.$$

Using (6.89)-(6.90) the Noether current (3.14) can be rewritten as

$$\begin{aligned}
\mathscr{H}^0 &= \frac{1}{2}\pi^k X^0 F_{0k} + \pi^k X^l F_{lk} + \frac{1}{16\pi}\sqrt{|\det g|}X^0 F^{kl}F_{kl} \\
&= \frac{1}{2}\pi^k X^0 \big(N^l F_{lk} - N^2 F^{0l}\gamma_{lk}\big) + \pi^k X^l F_{lk} \\
&\quad + \frac{1}{16\pi}\sqrt{|\det g|}X^0\big((N^l F^{0k} - N^k F^{0l}) + \gamma^{km}\gamma^{ln}F_{mn}\big)F_{kl} \\
&= -\frac{1}{2}X^0 N^2 \pi^k F^{0l}\gamma_{lk} + \frac{1}{16\pi}\sqrt{|\det g|}X^0\gamma^{km}\gamma^{ln}F_{mn}F_{kl} \\
&\quad + X^0\pi^k N^l F_{lk} + \pi^k X^l F_{lk} \\
&= \frac{\sqrt{\det\gamma}}{8\pi}\Big[NX^0\big(E^k E^l\gamma_{lk} + \frac{1}{2}\gamma^{km}\gamma^{ln}F_{mn}F_{kl} + 2N^{-1}E^k N^l F_{lk}\big) \\
&\quad + 2E^k X^l F_{lk}\Big]\,. \qquad (6.94)
\end{aligned}$$

We can therefore write $\mathscr{H}^0$ in terms of the π^k's as

$$\begin{aligned}
\mathscr{H}^0 &= \frac{2\pi X^0 N}{\sqrt{\det\gamma}}\pi^k\pi^l\gamma_{lk} + \frac{1}{16\pi}N\sqrt{\det\gamma}X^0\gamma^{km}\gamma^{ln}F_{mn}F_{kl} \\
&\quad + X^0\pi^k N^l F_{lk} + \pi^k X^l F_{lk}\,. \qquad (6.95)
\end{aligned}$$

For completeness we calculate the variation of $\mathscr{H}^0$ as given by (6.94):

$$\begin{aligned}
\delta\mathscr{H}^0 &= \frac{\sqrt{\det\gamma}}{8\pi}\Big[NX^0\big(2\gamma_{lk}E^k\delta E^l + \gamma^{km}\gamma^{ln}F_{mn}\delta F_{kl} \\
&\quad + 2N^{-1}N^l(E^k\delta F_{lk} + F_{lk}\delta E^k)\big) + 2X^l\big(E^k\delta F_{lk} + F_{lk}\delta E^k\big)\Big] \\
&= \frac{\sqrt{\det\gamma}}{8\pi}\Big\{2\big[NX^0\big(\gamma_{lk}E^k\delta E^l + N^{-1}N^k F_{kl}\big) + X^k F_{kl}\big]\delta E^l \\
&\quad - 2\partial_k\big[NX^0\big(\gamma^{km}\gamma^{ln}F_{mn} - 2N^{-1}N^{[l}E^{k]}\big) - 2X^{[l}E^{k]}\big]\delta A_l \\
&\quad - 2\partial_k\big[\big(NX^0\big(\gamma^{km}\gamma^{ln}F_{mn} - 2N^{-1}N^{[l}E^{k]}\big) - 2X^{[l}E^{k]}\big)\delta A_l\big]\Big\}\,. \\
&\qquad (6.96)
\end{aligned}$$

7 Plumbing the leakage

The variational identities discussed so far suffer from the existence of "leaky boundary terms", i.e., non-zero boundary terms in the variational formulae. These create problems when attempting to define Poisson brackets. In this section we show how this can be avoided by suitably extending the phase spaces.

7.1 De Sitter background

In what follows we will need explicit formulae for Fefferman-Graham coordinates in de Sitter spacetime; the aim of this section is to address this.

In addition to the form (2.1) of the de Sitter metric, let us recall the more standard form

$$g = -(1-\alpha^2 r^2)dt^2 + \frac{dr^2}{1-\alpha^2 r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{7.1}$$

The Bondi form (2.1) of g can be obtained from (7.1) with $\alpha r > 1$ by introducing a coordinate u through the formula

$$du := dt - \frac{dr}{1-\alpha^2 r^2} \equiv d\Big(t + \frac{1}{2\alpha}\ln\big(\frac{\alpha r - 1}{\alpha r + 1}\big)\Big)\,; \tag{7.2}$$

cf., e.g., [8].

Instead of either form of the metric above, for the purpose of global Hamiltonian analysis it seems best to use a globally defined, manifestly regular representation of the metric on the cylinder $\mathbb{R}\times S^3$ for de Sitter spacetime. For instance, the apparent singularity of the metric (7.1) at $r=\alpha$ is due to a poor choice of coordinates, as can be seen by setting,

$$r = \alpha^{-1}\sin\psi\cosh(\alpha\tau)\,,\quad t = \alpha^{-1}\,\mathrm{areatanh}\left(\frac{\cos\psi}{\tanh(\alpha\tau)}\right)\,. \tag{7.3}$$

Using (7.2), we can obtain a relation between Bondi coordinates and the coordinates on the cylinder $\mathbb{R}\times S^3$,

$$u = \frac{1}{\alpha}\,\mathrm{areatanh}\left(\frac{\sinh(\alpha\tau)\cos(\psi) - \sin(\psi)}{\cosh(\alpha\tau)}\right)\,, \tag{7.4}$$

toghether with the first equation in (7.3).

After the coordinate transformation (7.3) the metric (7.1) becomes

$$g = -d\tau^2 + \frac{\cosh^2(\alpha\tau)}{\alpha^2}\left(d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right)\,, \tag{7.5}$$

with $\sqrt{\det|g|} = \alpha^{-3}\cosh(\alpha\tau)^3\sin^2\psi\sin\theta$. The Killing vector field

$$\mathcal{T} = \partial_t \equiv \partial_u\,, \tag{7.6}$$

defining the Hamiltonian energy, in the coordinates (7.3) reads

$$\mathcal{T} = \cos\psi\partial_\tau - \alpha\sin\psi\tanh(\alpha\tau)\partial_\psi\,. \tag{7.7}$$

For future reference we note

$$\partial_r = \frac{1}{1-\sin^2\psi\cosh^2(\alpha\tau)}\Big[-\sin\psi\sinh(\alpha\tau)\partial_\tau + \frac{\alpha\cos\psi}{\cosh(\alpha\tau)}\partial_\psi\Big]\,. \qquad (7.8)$$

The metric (7.5) can be rewritten in a manifestly conformally smooth form

$$\begin{aligned} g &= \underbrace{\cosh^2(\alpha\tau)}_{=:x^{-2}}\big(-\cosh^{-2}(\alpha\tau)d\tau^2 + \alpha^{-2}\underbrace{\big(d\psi^2+\sin^2\psi\,\big(d\theta^2+\sin^2\theta d\varphi^2\big)\big)}_{=:\check\gamma}\big) \\ &= (\alpha x)^{-2}\big(-\frac{dx^2}{1-x^2}+\check\gamma\big)\,, \end{aligned} \qquad (7.9)$$

so that the coordinate x (not to be confused with the coordinate x of (2.2)) is a time coordinate for $|x|<1$, with spacelike level sets there.

The conformal boundary is obtained by attaching the hypersurface

$$\mathscr{I}^+ := \{x=0\}$$

to the physical spacetime. The Killing vector (7.7) becomes

$$\begin{aligned} \mathcal{T} &= -\alpha\tanh(\alpha\tau)\big[\operatorname{sech}(\alpha\tau)\cos\psi\partial_x + \sin\psi\partial_\psi\big] \\ &= -\alpha\sqrt{1-x^2}\big[x\cos\psi\partial_x+\sin\psi\partial_\psi\big]\,, \end{aligned} \qquad (7.10)$$

and extends smoothly to $\mathscr{I}^+$. For further reference we note

$$\partial_r = -\frac{\alpha x^2}{\sin^2\psi - x^2}\Big((1-x^2)\sin\psi\partial_x + x\cos\psi\partial_\psi\Big)\,. \qquad (7.11)$$

Let $\tilde g := x^2 g$, the $\tilde g$-Lorentzian norm-squared of $\mathcal{T}$ is

$$\begin{aligned} \tilde g(\mathcal{T},\mathcal{T}) &\equiv x^2 g(\mathcal{T},\mathcal{T}) = x^2(\alpha^2r^2-1) = x^2(\sin^2(\psi)\cosh^2(\alpha\tau)-1) \\ &= \sin^2(\psi) - x^2\,, \end{aligned} \qquad (7.12)$$

thus $\mathcal{T}$ is spacelike througout $\mathscr{I}^+$.

7.2 Conformally-covariant scalar field

Using the coordinates as in (7.9), the phase space of Cauchy data on three-dimensional spheres of constant x consists of smooth fields $(\phi,\partial_x\phi)$ with symplectic form

$$\Omega = -\int_{x=\mathrm{const}} \delta\pi^x\wedge\delta\phi\, dS_x\,, \qquad (7.13)$$

where the minus sign in front of the integral comes from the fact that ∂_x is past pointing. Writing π for $-\pi^x$ (where again the negative sign is motivated by the time orientation of ∂_x), and $\mathscr{S}_c := \{x = c\}$, there is an associated Poisson bracket, without problems with boundary terms since S^3 has no boundary:

$$\{F, G\}_{\mathscr{S}_c} := \int_{x=c} \left(\frac{\delta F}{\delta \phi} \frac{\delta G}{\delta \pi} - \frac{\delta F}{\delta \pi} \frac{\delta G}{\delta \phi} \right) dS_x \,. \tag{7.14}$$

It follows from conformal invariance of the equation satisfied by ϕ that the field

$$\chi := \frac{\phi}{x} \tag{7.15}$$

extends smoothly to the boundary $\{x = 0\}$. We have the expansions, for small x,

$$\phi = \overset{(1)}{\phi} x + \overset{(2)}{\phi} x^2 + \ldots \qquad \Longleftrightarrow \qquad \chi = \overset{(0)}{\chi} + \overset{(1)}{\chi} x + \ldots \,, \tag{7.16}$$

with coefficients which are functions on S^3, where

$$\overset{(1)}{\chi} \equiv \partial_x \chi|_{x=0} \,. \tag{7.17}$$

Since Ω is x-independent when applied to variations satisfying the field equations, it is tempting to pass with x to zero. Using (7.9) and (7.15) one obtains

$$\Omega = -\frac{1}{\alpha^2} \int_{x=0} \delta \partial_x \chi \wedge \delta \chi \, d\mu_{\tilde{\gamma}} \,, \tag{7.18}$$

with Poisson bracket

$$\{F, G\}_{\mathscr{S}_0} := \int_{S^3} \left(\frac{\delta f}{\delta \chi} \frac{\delta g}{\delta \pi} - \frac{\delta f}{\delta \pi} \frac{\delta g}{\delta \chi} \right) dS_x \,. \tag{7.19}$$

where now $\pi = -\alpha^{-2} \partial_x \chi \sqrt{\det \tilde{\gamma}}$.

In order to avoid leakage for fields on light-cones $\mathscr{C}_u$, for each u we can consider the phase space consisting of the field ϕ on $\mathscr{C}_u$ and of the fields $(\chi, \partial_x \chi)$ on

$$\mathscr{I}^+_u := \{x = 0\} \setminus I^+(\mathscr{C}_u) \tag{7.20}$$

(compare Figure 7.1), equipped with the symplectic form

$$\int_{\mathscr{C}_u} \partial_r \delta \phi \wedge \delta \phi \, r^2 dr \, d\mu_{\mathring{\gamma}} - \frac{1}{\alpha^2} \int_{\mathscr{I}^+_u} \delta \partial_x \chi \wedge \delta \chi \, d\mu_{\tilde{\gamma}} \,. \tag{7.21}$$

If F and G are associated with conserved functionals, the Poisson brackets thereof can be calculated using (7.19).

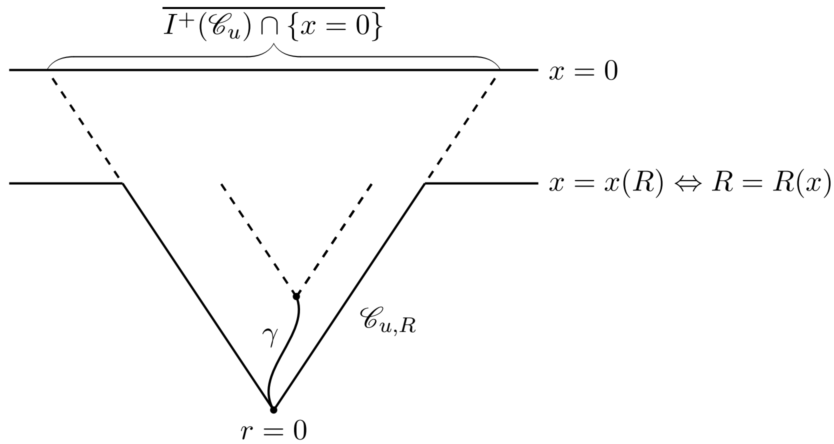

Figure 7.1: The integral curve of ∂_u passing through $r=0$ is denoted by γ.

We wish to calculate the Noether charge associated with translations of the light-cones in u for a conformally-covariant scalar field, thus with Lagrangian

$$\mathscr{L}=-\frac{1}{2}\sqrt{|\det g|}\big(g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi+\underbrace{m^2}_{2\alpha^2}\phi^2\big)\,. \tag{7.22}$$

One expects a formula of the kind

$$\begin{aligned}\mathbf{H} &= \int_{\mathscr{C}_u\cup\mathscr{I}^+_u}\mathscr{H}^\mu[\partial_u]dS_\mu\\ &= \int_{\mathscr{C}_u}\mathscr{H}^u[\partial_u]dS_u-\int_{\mathscr{I}^+_u}\mathscr{H}^x[\partial_u]\,dS_x\,,\end{aligned} \tag{7.23}$$

where the minus sign in front of the second integral arises again from the fact that ∂_x is past-directed. However, the individual integrals diverge, so some care must be taken. For instance, assuming $\alpha\neq 0$, on the level sets of

x the integrand $\mathscr{H}^x[\partial_u] \equiv \mathscr{H}^x[\mathcal{T}]$ is,

$$\begin{aligned}
\mathscr{H}^x[\partial_u] &= \sqrt{|\det g|}\Big(-\nabla^x\phi\,\mathcal{L}_{\mathcal{T}}\phi + \frac{1}{2}\big(\nabla^\alpha\phi\nabla_\alpha\phi + m^2\phi^2\big)\mathcal{T}^x\Big) \\
&= (\alpha x)^{-4}\sqrt{\frac{\det\check{\gamma}}{1-x^2}}\Big(-\alpha^3x^2(1-x^2)^{3/2}\partial_x\phi\big[x\cos\psi\partial_x\phi + \sin\psi\partial_\psi\phi\big] \\
&\quad -\frac{\alpha^3x^2}{2}\sqrt{1-x^2}\,x\cos\psi\big(-(1-x^2)(\partial_x\phi)^2 + |\check{D}\phi|^2_{\check{\gamma}} + m^2(\alpha x)^{-2}\phi^2\big)\Big) \\
&= -\alpha^{-1}x^{-2}\sqrt{\det\check{\gamma}}\Big((1-x^2)\sin\psi\partial_x\phi\partial_\psi\phi \\
&\quad +\frac{1}{2}x\cos\psi\big((1-x^2)(\partial_x\phi)^2 + |\check{D}\phi|^2_{\check{\gamma}} + m^2(\alpha x)^{-2}\phi^2\big)\Big)\,, \qquad (7.24)
\end{aligned}$$

where $\check{D}$ is the covariant derivative associated with $\check{\gamma}$. Inspection of (7.24) reveals terms which diverge as $x \to 0$ with this asymptotics and which, using $m^2 = 2\alpha^2$, can be collected into a divergence as follows:

$$\begin{aligned}
\mathscr{H}^x[\partial_u] &= -\alpha^{-1}x^{-2}\sqrt{\det\check{\gamma}}\Big\{(1-x^2)\sin\psi\partial_x(x\chi)\partial_\psi(x\chi) \\
&\quad +\frac{1}{2}x\cos\psi\big((1-x^2)(\partial_x(x\chi))^2 + |\check{D}(x\chi)|^2_{\check{\gamma}} + 2\chi^2\big)\Big\} \\
&= -\alpha^{-1}x^{-2}\sin^2\psi\sin\theta\Big\{x(1-x^2)\sin\psi(\chi + x\partial_x\chi)\partial_\psi\chi \\
&\quad +\frac{1}{2}x\cos\psi\big((1-x^2)(\chi + x\partial_x\chi)^2 + x^2|\check{D}\chi|^2_{\check{\gamma}} + 2\chi^2\big)\Big\} \\
&= -\frac{1}{2}\partial_\psi\Big\{\alpha^{-1}x^{-1}\sin^3\psi\sin\theta(1-x^2)\chi^2\Big\} \\
&\quad +\frac{3}{2}\alpha^{-1}x^{-1}\sin^2\psi\cos\psi\sin\theta(1-x^2)\chi^2 \\
&\quad -\alpha^{-1}x^{-2}\sin^2\psi\sin\theta\Big\{x^2(1-x^2)\sin\psi\partial_x\chi\partial_\psi\chi \\
&\quad +\frac{1}{2}x\cos\psi\big((1-x^2)(3\chi^2 + 2x\chi\partial_x\chi + x^2(\partial_x\chi)^2) + 2x^2\chi^2 + x^2|\check{D}\chi|^2_{\check{\gamma}}\big)\Big\} \\
&= -\partial_\psi\underbrace{\Big\{\frac{1}{2}\alpha^{-1}x^{-1}\sin^3\psi\sin\theta(1-x^2)\chi^2\Big\}}_{=:-\hat{\mathfrak{h}}^x} \\
&\quad -\alpha^{-1}\sqrt{\det\check{\gamma}}\Big\{(1-x^2)\sin\psi\partial_x\chi\partial_\psi\chi \\
&\quad +\frac{1}{2}\cos\psi\big((1-x^2)(2\chi\partial_x\chi + x(\partial_x\chi)^2) + 2x\chi^2 + x|\check{D}\chi|^2_{\check{\gamma}}\big)\Big\} \\
&=: \partial_\psi\hat{\mathfrak{h}}^x + \mathfrak{h}^x\,. \qquad (7.25)
\end{aligned}$$

It should be admitted that this way of handling the divergence is ambiguous, and could lead to a different finite part of the resulting boundary integral when, e.g., other coordinates are used.

7.2.1 Spacelike Cauchy surfaces

Consider, first, the Hamiltonian charge obtained by integrating (7.25) over a three-dimensional sphere of constant x. Then there is no boundary term, and choosing the exterior orientation of the slices appropriately one is led to a finite charge equal to

$$\alpha^{-1}\int_{S^3}\Big\{(1-x^2)\sin\psi\partial_x\chi\partial_\psi\chi + \frac{1}{2}\cos\psi\big((1-x^2)(2\chi\partial_x\chi+x(\partial_x\chi)^2)+2x\chi^2+x|\check D\chi|^2_{\check\gamma}\big)\Big\}d\mu_{\check\gamma}\,, \quad (7.26)$$

independently of x. In particular we can pass to the limit $x\to 0$ to define

$$\begin{aligned}\check E_{\mathscr{H}}[\mathscr{I}^+] &:= \alpha^{-1}\int_{S^3}\big(\sin\psi\partial_x\chi\partial_\psi\chi+\cos\psi\chi\partial_x\chi\big)|_{x=0}\,d\mu_{\check\gamma}\\ &= \alpha^{-1}\int_{S^3}\big(\sin\psi\overset{(1)}{\chi}\partial_\psi\overset{(0)}{\chi}+\cos\psi\overset{(0)}{\chi}\overset{(1)}{\chi}\big)d\mu_{\check\gamma}\\ &= \alpha^{-1}\int_{S^3}\overset{(1)}{\chi}\partial_\psi(\sin\psi\overset{(0)}{\chi})d\mu_{\check\gamma}\,. \end{aligned}\quad (7.27)$$

Note that ∂_u is tangent to $\{x=0\}$, and equals there

$$\partial_u|_{x=0}=-\alpha\sin\psi\partial_\psi\,.$$

Writing $\pi^x=-p^x\sqrt{\det\check\gamma}$, the symplectic form Ω is independent of x on solutions of the field equations and reads

$$\Omega=\int_{S^3}\delta p^x\wedge\delta\chi\,d\mu_{\check\gamma} \quad (7.28)$$

The Legendre transformation leads to $p^x=-\alpha^{-2}\partial_x\chi$ and a Hamiltonian on $\mathscr{I}$ equal to

$$\begin{aligned}H &:= \check E_{\mathscr{H}}[\mathscr{I}^+]\\ &= \alpha\int_{S^3}p^x\partial_\psi(\sin\psi\overset{(0)}{\chi})d\mu_{\check\gamma}\\ &\equiv \alpha\int_{S^3}p^x\partial_\psi(\sin\psi\overset{(0)}{\chi})\sin^2(\psi)\sin(\theta)d\psi d\theta d\varphi\,.\end{aligned}\quad (7.29)$$

The resulting Hamilton equations take the simple form:

$$\frac{dp^x}{du} = -\frac{\delta H}{\delta \overset{(0)}{\chi}} = \frac{1}{\sin\psi}\alpha\partial_\psi(\sin^2\psi p^x)\,, \quad \frac{d\overset{(0)}{\chi}}{du} = \frac{\delta H}{\delta p^x} = \alpha\partial_\psi(\sin\psi \overset{(0)}{\chi})\,, \quad (7.30)$$

with p^x and $\overset{(0)}{\chi}$ evolving independently of each other.

7.2.2 Corner terms

We pass now to a Cauchy surface which is the union of a light-cone and the "complementary part of $\mathscr{I}^+$".

Recall that $\mathscr{C}_u$ is a light-cone in $\mathscr{M}$ with vertex at $r = 0$. For $\alpha R > 1$ we set

$$\mathscr{S}_{x,u} := \mathscr{S}_x \setminus I^+(\mathscr{C}_u)\,, \qquad \mathscr{I}^+_u := \mathscr{I}^+ \setminus I^+(\mathscr{C}_u)\,,$$

see Figure 7.1. The hypersurface $\mathscr{I}^+_u$ can be viewed as the limit, as x tends to 0^+, of the $\mathscr{S}_{x,u}$'s. Let $S_{x,u}$ be the intersection of $\mathscr{C}_u$ with the $\mathscr{S}_x$. Let $S_{0,u}$ be the intersection of $\mathscr{C}_u$ with the conformal boundary at infinity $\mathscr{I}^+$; thus the surface $S_{0,u}$ is a limit of $S_{x,u}$ in which x tends to 0^+.

Since the Noether current $\mathscr{H}^\mu = \mathscr{H}^\mu[\partial_u]$ has vanishing divergence, we have for $x > x' > 0$

$$\int_{\mathscr{C}_{u,R(x)}\cup\mathscr{S}_{x,u}} \mathscr{H}^\mu dS_\mu = \int_{\mathscr{C}_{u,R(x')}\cup\mathscr{S}_{x',u}} \mathscr{H}^\mu dS_\mu\,. \quad (7.31)$$

We can thus pass to the limit $x' \to 0$ to obtain

$$\int_{\mathscr{C}_{u,R(x)}\cup\mathscr{S}_{x,u}} \mathscr{H}^\mu dS_\mu = \lim_{x\to 0}\int_{\mathscr{C}_{u,R(x)}\cup\mathscr{S}_{x,u}} \mathscr{H}^\mu dS_\mu\,, \quad (7.32)$$

in particular the limit exists and is finite.

Recall that (see (7.25))

$$\mathscr{H}^x = \mathfrak{h}^x + \partial_\psi \hat{\mathfrak{h}}^x\,. \quad (7.33)$$

so that

$$\int_{\mathscr{S}_{x,u}} \mathscr{H}^\mu dS_\mu = \int_{\mathscr{S}_{x,u}} \mathfrak{h}^x dS_x + \mathcal{B}_1\,, \quad (7.34)$$

where

$$\begin{aligned}\mathcal{B}_1 &:= -\frac{1}{2}\int_{S^3}\partial_\psi\Big\{\alpha^{-1}x^{-1}\sin^3\psi(1-x^2)\chi^2\Big\}\,d\psi\,d\theta\,d\varphi\\ &= -\frac{1}{2}\int_{S_{x,u}}\Big\{\alpha^{-1}x^{-1}\sin^3\psi(1-x^2)\chi^2\Big\}\,d\theta\,d\varphi\\ &= -\frac{1}{2\alpha\cosh(\alpha u)^3}\int_{S_{x,u}}\Big[\frac{(\overset{(0)}{\chi})^2}{x}+2\overset{(0)}{\chi}\overset{(1)}{\chi}-3\sinh(\alpha u)(\overset{(0)}{\chi})^2\\ &\quad +O(x)\Big]\sin\theta\,d\theta\,d\varphi\,. \qquad (7.35)\end{aligned}$$

Similarly, setting

$$\mathscr{H}^u=\mathfrak{h}^u+\partial_r\hat{\mathfrak{h}}^u\,, \qquad (7.36)$$

we can rewrite (4.25) as

$$\int_{\mathscr{C}_{u,R(x)}}\mathscr{H}^\mu dS_\mu=\int_{\mathscr{C}_{u,R(x)}}\mathfrak{h}^u dS_u+\mathcal{B}_2\,, \qquad (7.37)$$

where

$$\begin{aligned}\mathcal{B}_2 &:= \frac{1}{2}\int_{\mathscr{C}_{u,R(x)}}\partial_r\Big[\big(r^2-\alpha^2r^4\big)\phi\big(\partial_r\phi\big)\sin\theta\Big]\,dr\,d\theta\,d\varphi\\ &= \frac{1}{2}\int_{S_{x,u}}\big(r^2-\alpha^2r^4\big)\phi\partial_r\phi\sin\theta\,d\theta\,d\varphi\,. \qquad (7.38)\end{aligned}$$

Thus

$$\int_{\mathscr{C}_{u,R(x)}\cup\mathscr{S}_{x,u}}\mathscr{H}^\mu dS_\mu=\int_{\mathscr{S}_{x,u}}\mathfrak{h}^x dS_x+\int_{\mathscr{C}_{u,R(x)}}\mathfrak{h}^u dS_u+\mathcal{B}_1+\mathcal{B}_2\,. \qquad (7.39)$$

Both volume integrals have a finite limit as $x\to 0$. It remains to analyze the divergent boundary terms in the energy on the family $S_{x,u}$'s. This requires some changes of coordinates. Equations (7.3)-(7.4) together with $x=1/\cosh(\alpha\tau)$ can be inverted as

$$\begin{aligned}r(x,u) &= \alpha^{-1}\frac{\sqrt{1-x^2}-x\sinh(\alpha u)}{x\cosh(\alpha u)}\\ &= \frac{1-x\sinh(\alpha u)+O(x^2)}{\alpha x\cosh(\alpha u)}\,, \qquad (7.40)\\ \sin\psi(x,u) &= \frac{\sqrt{1-x^2}-x\sinh(\alpha u)}{\cosh(\alpha u)}\\ &= \frac{1-x\sinh(\alpha u)}{\cosh(\alpha u)}+O(x^2)\,. \qquad (7.41)\end{aligned}$$

Since we need to calculate derivatives of the fields along $\mathscr{C}_u$ in the new variables, we need instead (x, ψ) as a function of (r, u). For $x > 0$ and $\alpha r > 0$ one finds:

$$x(r,u) = \sqrt{\frac{1-\tanh(\alpha u)^2}{1+\alpha^2 r^2 + 2\alpha r \tanh(\alpha u)}}\,, \tag{7.42}$$

$$\sin\big(\psi(r,u)\big) = \alpha r \sqrt{\frac{1-\tanh(\alpha u)^2}{1+\alpha^2 r^2 + 2\alpha r \tanh(\alpha u)}}\,. \tag{7.43}$$

This leads to

$$\left.\frac{\partial \psi}{\partial r}\right|_{u=\mathrm{const}} = \alpha x^2 \cosh(\alpha u)\,, \tag{7.44}$$

$$\left.\frac{\partial x}{\partial r}\right|_{u=\mathrm{const}} = -\alpha x^2 \sqrt{1-x^2}\cosh(\alpha u)\,. \tag{7.45}$$

One then finds

$$\begin{aligned}
\mathcal{B}_2 &= \frac{1}{2}\int_{S_{x,u}} \big(r^2-\alpha^2 r^4\big)\phi\Big(\frac{\partial\psi}{\partial r}\partial_\psi + \frac{\partial x}{\partial r}\partial_x\Big)\phi \sin\theta \, d\theta \, d\varphi \\
&= \frac{1}{2}\int_{S_{x,u}} \big(\alpha^{-2}x^{-2}\sin^2\psi - \alpha^2(\alpha^{-1}x^{-1}\sin\psi)^4\big)x\chi \times \\
&\qquad \alpha x^2\cosh(\alpha u)\bigg(\partial_\psi - \sqrt{1-x^2}\partial_x\bigg)x\chi \sin\theta\, d\theta\, d\varphi \\
&= \frac{1}{2\alpha\cosh^3(\alpha u)}\int_{S_{x,u}} \overset{(0)}{\chi}\Big[\frac{\overset{(0)}{\chi}}{x} - \partial_\psi\overset{(0)}{\chi} + 3\overset{(-1)}{\chi} - 4\sinh(\alpha u)\overset{(0)}{\chi} \\
&\qquad + O(x)\Big]\sin\theta\, d\theta\, d\varphi\,.
\end{aligned} \tag{7.46}$$

We are ready now to compare (7.35) and (7.46). Note that each diverges when x tends to 0^+, but their sum $\Delta\mathcal{B}_{x,u} := \mathcal{B}_1 + \mathcal{B}_2$,, relevant for the total energy, is finite. Indeed, passing to the limit $x \to 0^+$ one finds

$$\begin{aligned}
\Delta\mathcal{B}_{0,u} &:= \lim_{x\to 0^+}\Delta\mathcal{B}_{x,u} \\
&= \frac{1}{2\alpha\cosh^3(\alpha u)}\int_{S_{0,u}} \overset{(0)}{\chi}\Big(\overset{(1)}{\chi} - \partial_\psi\overset{(0)}{\chi} - \overset{(0)}{\chi}\sinh(\alpha u)\Big)\sin\theta\, d\theta\, d\varphi\,.
\end{aligned} \tag{7.47}$$

Summarising: in the phase space described above the dynamical system induced by translating in u the tip of the light-cone is Hamiltonian, with

Hamiltonian equal to (compare (7.23) and (7.33)):

$$\begin{aligned}\mathbf{H} &:= \lim_{x\to 0}\int_{\mathscr{C}_{u,R(x)}\cup\mathscr{S}_{x,u}} \mathscr{H}^\mu dS_\mu \\ &= \underbrace{\int_{\mathscr{C}_u}\mathfrak{h}^u dS_u + \Delta\mathcal{B}_{0,u}}_{=:\check{E}_{\mathscr{H}}[\mathscr{C}_u]} - \int_{\mathscr{I}^+_u}\mathfrak{h}^x dS_x \end{aligned} \tag{7.48}$$

(with the minus sign in the last integral arising from the fact that ∂_x is past-directed), where now all the terms are finite. In this picture the "leaking terms" correspond to an exchange of energy between the subsystem consisting of the field on the light cone and the field on $\mathscr{I}^+_u$.

7.3 Maxwell fields

The analysis for Maxwell fields is quite simpler than that for the conformally-covariant scalar field. The phase space of Cauchy data on three-dimensional spheres of constant x consists of smooth fields $(A_\mu, \partial_\nu A_\mu)$ with, loosely speaking, symplectic form

$$\begin{aligned}\Omega &= -\int_{x=\text{const}} \delta\pi^{\mu x}\wedge\delta A_\mu\, dS_x \\ &= -\int_{x=\text{const}} \delta\pi^{kx}\wedge\delta A_k\, dS_x\,, \end{aligned} \tag{7.49}$$

where we used $(x^\mu) = (x, x^k)$, where the x^k's are local coordinates on $\mathscr{I}^+$, as well as the fact that $\pi^{xx} = 0$. There exists a gauge in which all fields extend smoothly through $x = 0$, so that we can write

$$A_k = \overset{(0)}{A}_k + x\overset{(1)}{A}_k + \cdots\,, \tag{7.50}$$

$$F_{xk} = \overset{(0)}{F}_{xk} + x\overset{(1)}{F}_{xk} + x^2\overset{(2)}{F}_{xk} + \ldots\,, \tag{7.51}$$

where the expansion coefficients are functions of x^k. Since Ω is conserved for variations satisfying the field equations it holds that

$$\Omega = \frac{1}{4\pi}\int_{x=0}\breve{\gamma}^{kl}\delta\overset{(0)}{F}_{xl}\wedge\delta\overset{(0)}{A}_l\, d\mu_{\breve{\gamma}}\,. \tag{7.52}$$

The dynamics generated by the flow of ∂_u is Hamiltonian, with

$$E_{\mathscr{H}}[S^3] := -\int_{S^3}\mathscr{H}^x[\partial_u]dS_x\,. \tag{7.53}$$

Assuming $\alpha \neq 0$, using (7.9)-(7.10) we find

$$\begin{aligned}
\mathscr{H}^x[\partial_u] &:= \frac{\partial \mathscr{L}}{\partial A_{\beta,x}} \mathbf{L}_{\mathcal{T}} A_\beta - \mathscr{L}\mathcal{T}^x \\
&= -\frac{1}{4\pi}\sqrt{|-\det g|}\Big(F^{x\beta}\mathcal{T}^\alpha F_{\alpha\beta} - \frac{1}{4}\big(F^{\nu\beta}F_{\nu\beta}\big)\mathcal{T}^x\Big) \\
&= \frac{(\alpha x)^{-4}}{4\pi}\sqrt{\frac{\det\check\gamma}{1-x^2}}\Big\{F^{x\beta}\alpha\sqrt{1-x^2}\big[x\cos\psi F_{x\beta} + \sin\psi F_{\psi\beta}\big] \\
&\quad -\frac{1}{4}\alpha\sqrt{1-x^2}x\cos\psi\big(F^{\nu\beta}F_{\nu\beta}\big)\Big\} \\
&= -\frac{1}{4\pi}\alpha\sqrt{\det\check\gamma}\Big\{\frac{1}{2}x(1-x^2)\cos\psi F_{xk}F_{xl}\check\gamma^{kl} \\
&\quad +(1-x^2)\sin\psi F_{xk}F_{\psi l}\check\gamma^{kl} + \frac{1}{4}x\cos\psi F_{mk}F_{nl}\check\gamma^{mn}\check\gamma^{kl}\Big\}. \qquad (7.54)
\end{aligned}$$

Hence

$$\begin{aligned}
E_{\mathscr{H}}[S^3] &= -\frac{\alpha}{4\pi}\int_{S^3}\Big[\sin\psi\,\check\gamma^{kl}\overset{(0)}{F}_{xk}\overset{(0)}{F}_{\psi l} + O(x)\Big]\,d\mu_{\check\gamma} \\
&= -\frac{\alpha}{4\pi}\int_{S^3}\sin\psi\,\check\gamma^{kl}\overset{(0)}{F}_{xk}\overset{(0)}{F}_{\psi l}\,d\mu_{\check\gamma}\,, \qquad (7.55)
\end{aligned}$$

where in the last equality we used the fact that $E_{\mathscr{H}}[S^3]$ does not depend upon x.

To take care of the leakage, for each u we can consider the phase space consisting of the fields A_μ on $\mathscr{C}_u$, and $(A_k, \partial_x A_k)$ on the set $\mathscr{I}^+_u$ of (7.20), equipped with the symplectic form

$$\begin{aligned}
\Omega &= \frac{1}{4\pi}\int_{\mathscr{C}_u}\Big(r^2\delta F_{ur}\wedge\delta A_r + \mathring{\gamma}^{AB}\delta F_{rB}\wedge\delta A_A\Big)\,dr\,d\mu_{\mathring\gamma} \\
&\quad -\frac{1}{4\pi}\int_{\mathscr{I}^+_u}\check\gamma^{kl}\delta\overset{(0)}{F}_{xk}\wedge\delta\overset{(0)}{A}_l\,d\mu_{\check\gamma}\,. \qquad (7.56)
\end{aligned}$$

The Hamiltonian charge associated with moving the light-cones along the flow of the Killing vector $\partial_u \equiv \mathcal{T}$ decomposes as in (7.23),

$$\mathbf{H} = \int_{\mathscr{C}_u}\mathscr{H}^u[\partial_u]dS_u - \int_{\mathscr{I}^+_u}\mathscr{H}^x[\partial_u]\,dS_x\,, \qquad (7.57)$$

(where the minus sign in the second integral is again motivated by orientation considerations) but now each integrand is finite without further due; hence no corner contributions arise.

A Killing fields in Minkowski, de Sitter and anti-de Sitter spacetimes

In order to determine the Noether charges in our formalism we will need the explicit form of the Killing vector fields in Bondi coordinates on the de Sitter, and anti-de Sitter and Minkowski spacetimes.

A.1 Killing fields in de Sitter spacetime

We use the following basis of the space of Killing vectors in de Sitter spacetime

$$\mathcal{T} = \partial_u \,, \tag{A.1}$$

$$\mathcal{R} = \varepsilon^{BA}\mathring{D}_A\big(R_i n^i\big)\partial_B \,, \tag{A.2}$$

$$\mathcal{P}_{dS} = e^{\alpha u}\Big[p_i n^i\partial_u - \big(\alpha r+1\big)p_i n^i\partial_r - \frac{\alpha r+1}{r}\mathring{D}^A(p_i n^i)\partial_A\Big] \,, \tag{A.3}$$

$$\mathcal{L}_{dS} = e^{-\alpha u}\Big[l_i n^i\partial_u + \big(\alpha r-1\big)l_i n^i\partial_r + \frac{\alpha r-1}{r}\mathring{D}^A(l_i n^i)\partial_A\Big] \,, \tag{A.4}$$

where R_i, p_i and l_i are constants. Using the following coordinate change

$$x^0 = \frac{\big(\sinh(\alpha u) - r^2\alpha^2\sinh(\alpha u) - \alpha r\big)\cosh(\alpha u) + \alpha r - \sinh(\alpha u)}{\alpha\big(\big(\alpha^2 r^2-1\big)\cosh(\alpha u)^2 + 2\cosh(\alpha u) - \alpha^2 r^2 - 1\big)} \,, \tag{A.5}$$

$$x^1 = \frac{\big(\cosh(\alpha u) - \sinh(u\alpha)\alpha r - 1\big)r\sin\theta\cos\phi}{\big(\alpha^2 r^2-1\big)\cosh(\alpha u)^2 + 2\cosh(\alpha u) - \alpha^2 r^2 - 1} \,, \tag{A.6}$$

$$x^2 = \frac{\big(\cosh(\alpha u) - \sinh(u\alpha)\alpha r - 1\big)r\sin\theta\sin\phi}{\big(\alpha^2 r^2-1\big)\cosh(\alpha u)^2 + 2\cosh(\alpha u) - \alpha^2 r^2 - 1} \,, \tag{A.7}$$

$$x^3 = \frac{\big(\cosh(\alpha u) - \sinh(u\alpha)\alpha r - 1\big)r\cos\theta}{\big(\alpha^2 r^2-1\big)\cosh(\alpha u)^2 + 2\cosh(\alpha u) - \alpha^2 r^2 - 1} \,, \tag{A.8}$$

the de Sitter metric (7.1) transforms into conformally Minkowskian form

$$g = \frac{4}{\big(1+s^2\alpha^2\big)^2}\eta_{\mu\nu}dx^\mu dx^\nu \,, \tag{A.9}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ and $s^2 = \eta_{\mu\nu}x^\mu x^\nu$. Defining

$$S_\mu = -\frac{1}{2}\partial_\mu - \frac{\alpha^2}{2}\big(2\eta_{\mu\lambda}x^\lambda x^\nu - s^2\delta_\mu{}^\nu\big)\partial_\nu \,, \tag{A.10}$$

$$L_{\mu\nu} = \eta_{\mu\lambda}x^\lambda\partial_\nu - \eta_{\nu\lambda}x^\lambda\partial_\mu \,, \tag{A.11}$$

one finds

$$\begin{aligned}
\mathcal{T} &= S_0\,, && \text{(A.12)}\\
\mathcal{R} &= \widetilde{R}_i\epsilon^{ijk}L_{jk}\,, && \text{(A.13)}\\
\mathcal{P}_{dS} &= \widetilde{p}_i\eta^{ij}(\alpha L_{0j}-S_j)\,, && \text{(A.14)}\\
\mathcal{L}_{dS} &= \widetilde{l}_i\eta^{ij}(\alpha L_{0j}+S_j)\,, && \text{(A.15)}
\end{aligned}$$

where $\{i,j,k\}\in\{1,2,3\}\,,\epsilon^{123}=1$. $\widetilde{R}_i,\widetilde{p}_i,\widetilde{l}_i$ are respectively linear combinations of R_i,. The following commutation relations hold

$$\begin{aligned}
\big[L_{\alpha\beta},L_{\rho\sigma}\big] &= \eta_{\beta\rho}L_{\alpha\sigma}+\eta_{\alpha\sigma}L_{\beta\rho}-\eta_{\beta\sigma}L_{\alpha\rho}-\eta_{\alpha\rho}L_{\beta\sigma}\,, && \text{(A.16)}\\
\big[L_{\mu\nu},S_\rho\big] &= \big(\eta_{\mu\lambda}\eta_{\rho\nu}-\eta_{\mu\rho}\eta_{\lambda\nu}\big)\eta^{\lambda\sigma}S_\sigma\,, && \text{(A.17)}\\
\big[S_\beta,S_\rho\big] &= \alpha^2L_{\rho\beta}\,, && \text{(A.18)}
\end{aligned}$$

which leads to

$$\begin{aligned}
\big[\mathcal{T},\mathcal{R}\big] &= 0\,, && \text{(A.19)}\\
\big[\mathcal{T},\mathcal{P}_{dS}\big] &= \alpha\mathcal{P}_{dS}\,, && \text{(A.20)}\\
\big[\mathcal{T},\mathcal{L}_{dS}\big] &= -\alpha\mathcal{L}_{dS}\,, && \text{(A.21)}\\
\big[\mathcal{R}_I,\mathcal{R}_{II}\big] &= \mathcal{R}_{III}\,, && \text{(A.22)}
\end{aligned}$$

where $\mathcal{R}_I,\mathcal{R}_{II},\mathcal{R}_{III}$are given by (A.13).

A.2 Killing fields in anti-de Sitter spacetime

Using for anti-de Sitter $\widetilde{\alpha}:=-\imath\alpha$, while simultaneously keeping p_i,l_i real, the real and imaginary parts of $\mathcal{P}_{dS}$ and $\mathcal{L}_{dS}$ are

$$\begin{aligned}
\mathcal{P}_{dS}(p_i) &= \mathcal{P}_{adS}(p_i)+\imath\mathcal{L}_{adS}(p_i) && \text{(A.23)}\\
\mathcal{L}_{dS}(l_i) &= \mathcal{P}_{adS}(l_i)+\imath\mathcal{L}_{adS}(-l_i) && \text{(A.24)}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{P}_{adS}(\widetilde{p}_i) &= \widetilde{p}_i\Big[n^i\mathrm{cos}(\widetilde{\alpha}u)\partial_u+n^i(\widetilde{\alpha}r\mathrm{sin}(\widetilde{\alpha}u)-\mathrm{cos}(\widetilde{\alpha}u))\partial_r\\
&\quad+\frac{(\widetilde{\alpha}r\mathrm{sin}(\widetilde{\alpha}u)-\mathrm{cos}(\widetilde{\alpha}u))}{r}\mathring{D}^An^i\partial_A\Big]\,, && \text{(A.25)}\\
\mathcal{L}_{adS}(\widetilde{l}_i) &= \widetilde{l}_i\Big[n^i\mathrm{sin}(\widetilde{\alpha}u)\partial_u-n^i(\mathrm{sin}(\widetilde{\alpha}u)+\widetilde{\alpha}r\mathrm{cos}(\widetilde{\alpha}u))\partial_r\\
&\quad-\frac{(\mathrm{sin}(\widetilde{\alpha}u)+\widetilde{\alpha}r\mathrm{cos}(\widetilde{\alpha}u))}{r}\mathring{D}^An^i\partial_A\Big]\,. && \text{(A.26)}
\end{aligned}$$

A.3 Killing fields in Minkowski spacetime

The Killing fields in Minkowski spacetime will be labelled as

$$\begin{aligned} \mathcal{T} &= \partial_t\,, & \text{(A.27)}\\ \mathcal{R} &= \epsilon^{ijk}R_i\delta_{jl}x^l\partial_k\,, & \text{(A.28)}\\ \mathcal{P} &= P^k\partial_k\,, & \text{(A.29)}\\ \mathcal{L} &= L_ix^i\partial_t+tL^i\partial_i\,, & \text{(A.30)} \end{aligned}$$

where $P_i \equiv P^i$, $L_i \equiv L^i$ and R_i are all constants.

The coordinate transformation between Minkowskian and Bondi coordinates

$$\big(u=t-r,r,x^A\big) \tag{A.31}$$

gives

$$\partial_t=\partial_u\,,\quad \partial_i=n^i\left(\partial_r-\partial_u\right)+\frac{1}{r}\mathring{D}^An^i\partial_A\,, \tag{A.32}$$

where the fields

$$n^i:=\frac{x^i}{r} \tag{A.33}$$

form a basis of the space of $\ell=1$ spherical harmonics, and thus n^i is viewed as a scalar on S^2 in formulae such as $\mathring{D}^An^i$. Under (A.31) the Killing vectors (A.27)-(A.30) become

$$\begin{aligned} \mathcal{T} &= \partial_u\,, & \text{(A.34)}\\ \mathcal{R} &= \varepsilon^{AB}\mathring{D}_B(R_in^i)\partial_A\,, & \text{(A.35)}\\ \mathcal{P} &= P_i\Big(n^i\left(\partial_r-\partial_u\right)+\frac{1}{r}\mathring{D}^An^i\partial_A\Big)\,, & \text{(A.36)}\\ \mathcal{L} &= L_i\Big(-un^i\partial_u+(u+r)n^i\partial_r+\big(1+\frac{u}{r}\big)\mathring{D}^An^i\partial_A\big)\Big)\,. & \text{(A.37)} \end{aligned}$$

where ε^{AB} is a two-dimensional Levi-Civita tensor; in spherical coordinates (θ,ϕ) we take the sign $\varepsilon^{\theta\phi}=\frac{1}{\sin\theta}$.

The Killing fields for Minkowski spacetime can be obtained as a limit of de Sitter spacetime. Equations (A.3)-(A.4) and (A.36) give

$$\mathcal{P}=-\frac{1}{2}\lim_{\alpha\to 0}\Big(\mathcal{P}_{dS}+\mathcal{L}_{dS}\Big)\,; \tag{A.38}$$

where one has to set $l_i=p_i=P_i$. Analogically, (A.3)-(A.4) and (A.37) lead to

$$\mathcal{L}=\frac{1}{2}\lim_{\alpha\to 0}\Big(\frac{\mathcal{L}_{dS}-\mathcal{P}_{dS}}{\alpha}\Big)\,, \tag{A.39}$$

where the parameters should be taken as $p_i=l_i=L_i$.

B An example: Blanchet-Damour-type solutions of the Maxwell equations

An elegant class of linearised solutions of the Maxwell equations with $\Lambda = 0$ can be constructed in analogy to the Blanchet–Damour solution for linearized gravity, introduced in [2]. The electromagnetic potential $A_\mu dx^\mu$ in Lorenz gauge satisfies

$$\Box_\eta A_\nu = 0\,, \quad \partial_\mu A^\mu = 0\,. \tag{B.1}$$

Here η is the Minkowski metric, taken to be $-(dx^0)^2+(dx^1)^2+(dx^2)^2+(dx^3)^2$ in the coordinates of (B.1), and $\Box_\eta$ the associated wave operator. As in [2] we start with an ansatz for the electromagnetic potential in Lorenz gauge: given a collection of smooth functions $I_i : \mathbb{R} \to \mathbb{R}$, the one-form

$$\begin{aligned} A_t &= \partial_j\big(\frac{\dot I_j(t-r) - \dot I_j(t+r)}{r}\big) \\ &= -\big(\ddot I_j(t-r) + \ddot I_j(t+r)\big)\frac{x^j}{r^2} + O(r^{-2})\,, \end{aligned} \tag{B.2}$$

$$A_j = \frac{\ddot I_j(t-r) - \ddot I_j(t+r)}{r}\,, \tag{B.3}$$

where each dot represents a derivative with respect to the argument of I_i, is a smooth tensor field on Minkowski spacetime solving (B.1).

Since the operators appearing in (B.1) commute with partial differentiation, further solutions can be constructed by applying $\partial_{\mu_1}\cdots\partial_{\mu_\ell}$ to A_μ, and by applying Poincaré transformations.

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