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QED radiative corrections in a strong plane-wave background field

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Zusammenfassung

Diese Arbeit beschäftigt sich mit Strahlungskorrekturen zu den Wahrscheinlichkeiten zweier elementarer Prozesse der Quantenelektrodynamik (QED) in Präsenz eines Hintergrundfeldes in Form einer starken elektromagnetischen planaren Welle. Bei den untersuchten Prozessen handelt es sich um die nichtlineare Comptonstreuung (die Emission eines Photons durch ein Elektron) und die nichtlineare Breit-Wheeler Paarproduktion (der Zerfall eines Photons in ein Elektron-Positron-Paar).
Unter Berücksichtigung von Stahlungskorrekturen sind die Wellenfunktionen von Elektronen, Positronen und Photonen in einer planaren Welle nicht stabil, sondern "zerfallen" in dem Sinne, dass Elektronen und Positronen Photonen emittieren und Photonen wiederum in Elektron-Positron-Paare zerfallen. Mit Hilfe dieser Wellenfunktionen wurden die Wahrscheinlichkeiten für nichtlineare Comptonstreuung und nichtlineare Breit-Wheeler Paarproduktion unter der lokal-konstantes-Feld-Näherung analytisch hergeleitet. Der Zerfall der Wellenfunktionen führt zum Auftauchen eines exponentiellen Dämpfungsterms in diesen Wahrscheinlichkeiten, welcher diese, auch für planare Wellenpulse mit großen Phasenlängen und Intensitäten, auf Werten unter Eins beschränkt.
Im Anschluss wurden separat dazu die Korrekturen zur Wahrscheinlichkeit für nichtlineare Comptonstreuung, die von der Selbstwechselwirkung des Elektrons in der planaren Welle stammen, in erster Ordnung der Feinstrukturkonstante α untersucht. Es wird gezeigt, dass diese, unter Berücksichtigung gleicher Näherungen, in der zuvor hergeleiteten Wahrscheinlichkeit enthalten sind.

Abstract

In this thesis radiative corrections to the probabilities of two basic processes in Quantum Electrodynamics (QED) in the presence of a strong electromagnetic plane wave background field are investigated. The considered two processes are nonlinear Compton scattering (the emission of a single photon by an electron) and nonlinear Breit-Wheeler pair production (the decay of a photon into an electron-positron pair).
Taking radiative corrections into account, the electron, positron, and photon states inside a plane wave are not stable, but "decay" in the sense that electrons and positrons emit photons and photons decay into electron-positron pairs. Employing these states, the probabilities for nonlinear Compton scattering and nonlinear Breit-Wheeler pair production are derived analytically within the local constant field approximation. The particles states decay leads to the appearance of an exponential damping term in those probabilities, limiting them to values below unity even for plane wave pulses with large phase duration and intensity.
Afterwards, leading order corrections in the fine-structure constant α to the probability of nonlinear Compton scattering, stemming from the self-interaction of the electron inside a plane wave, are investigated separately. It is shown that those corrections are included in the previously obtained probability within the same approximations.

List of publications

Publications covered by this thesis:

Nonlinear Compton scattering and nonlinear Breit-Wheeler pair production including the damping of particle states,
T. Podszus, V. Dinu, and A. Di Piazza;
Physical Review D **106**, 056014 (2022);
DOI: 10.1103/PhysRevD.106.056014; arXiv: 2206.10345

First-order strong-field QED processes including the damping of particle states,
T. Podszus and A. Di Piazza;
Physical Review D **104**, 016014 (2021);
DOI: 10.1103/PhysRevD.104.016014, arXiv: 2103.14637

Note that the main part of the dissertation is based on this two publications and that the structure, equations, and text in this dissertation can be similar or identical to the publications.

Contents

1. Introduction

It is known that there is much in the universe that we do not know [1]. Searching for new physics beyond the standard model has become a big field and one way of doing so is to test the known theory under extreme conditions. Here Quantum Electrodynamics (QED) might be the best proven quantum field theory. Although it is not a true fundamental theory, in the sense that it is confined with the weak force at energies of the order of 10^2 GeV [2, 3], in the regime of validity experiments and theory agree to a astonishing precision (see e.g. Refs. [4,5]). Due to this precision there are for example attempts to search for new particles, like Axion-like particles or minicharged particles, via weak interactions of these hypothetical particles with strong electromagnetic fields [2, 6, 7]. This and other research in the field is facilitated by the development of new laser technologies leading to lasers with higher and higher peak intensities, making them a favorable tool to test QED under the conditions of extreme electromagnetic fields [2]. However, to be able to find new physics it is important to understand the current theory well enough and to make precise calculations of the known effects. In this manner one is able to detect small possible deviations in the comparison of theory with experiment, which can finally hint to new physics.

On the other hand, as a product of the past research also new technologies were invented, for example the particle acceleration via laser pulses, which is promising to bring useful applications in medicine, industry and science for the future [2, 8]. Again, for the development of such technologies and their applications a deep understanding of the underlying physics is necessary.

This work should contribute to the research process by investigating corrections to the probabilities of the two fundamental quantum processes of QED in strong fields, which are nonlinear Compton scattering (the emission of a photon by an electron) and nonlinear Breit-Wheeler pair production (the decay of a photon into an electron-positron pair).

Indeed, although QED is a well proven theory, QED in the presence of strong electromagnetic fields still offers untested regions to explore. Here "strong" means that the intensity of the electromagnetic background field is so high, that the interaction of electrons and positrons with the field has to be taken into account exactly in the calculations and further it is assumed that a background field is not influenced significantly by these interactions [6].

This behavior is commonly quantified by the so-called classical nonlinearity parameter $\xi_0 = |e|E_0/m\omega_0$, where e and m are the electron charge and mass, respectively, E_0 is the electric field amplitude, ω_0 the central angular frequency of the background laser pulse, and we use units where $\epsilon_0 = \hbar = c = 1$ [2, 9]. The parameter corresponds to the energy an electron/positron gains by acceleration in one wavelength of the background field in units of the rest energy of the particle. If the quantity $\xi_0 \gtrsim 1$, the particles become relativistic within one wavelength and classical nonlinear effect occur. In this case the interaction of electrons and positrons with the background field has to be taken into account exactly in the calculations. For optical lasers $\xi_0 \gtrsim 1$ corresponds to intensities of the order of $10^8 \mathrm{W/cm}^2$, which are reached today by several laser facilities [2]. The problem of including the interaction of electrons and positrons with the background field exactly in the calculations is commonly solved by working in the so-called Furry picture [10]. Here the photon field is splitted into a classical background part and a quantized photon part. The electron and positron states are then quantized in the presence of the background field by solving a modified Dirac equation which includes the interaction with the background field [11, 12]. For a plane wave background field this Dirac equation is analytically solvable and the solution is known as Volkov-state.

However, the field strength of an electromagnetic field can even become larger up to the order of the so-called critical field of QED $F_{cr} = m^2/|e| = 1.3 \times 10^{16}\mathrm{V/cm} = 4.4 \times 10^{13}\mathrm{G}$ [9]. In an electric field of this field strength the vacuum would become unstable under electron-positron pair production and in a magnetic field of strength F_{cr} the interaction energy of a Bohr magneton with the field is of the order of m [2, 13]. This regime is very interesting since it is dominated by quantum effects. However it is mainly undiscovered, as the critical field corresponds to a critical laser intensity of $I_{cr} \sim 10^{29}\mathrm{W/cm^2}$, which is far from being reached by today available lasers. Indeed the today's record peak intensity is about $I_0 \sim 1.1 \times 10^{23}\mathrm{W/cm^2}$ [14], and even upcoming laser facilities are aiming only for intensities of the order of $I_0 \sim 10^{23} - 10^{24}\mathrm{W/cm^2}$ [15–18]. Although this means that we can not observe the interesting regime in the lab frame, due to the Lorentz-invariance of QED, it can be explored in a Lorentz-boosted frame already with our current technology. In fact physical observables like transition probabilities depend only via Lorentz- and gauge-invariant parameters on the background field. Therefore, instead of observing the vacuum in strong laser pulses, for example the behavior of ultra-relativistic particles in strong electromagnetic fields can be studied to enter the quantum regime.

Considering an electron (photon) of four-momentum $p^\mu = (\varepsilon, \boldsymbol{p})$ [$q^\mu = (\omega, \boldsymbol{q})$] with energy $\varepsilon = \sqrt{m^2 + \boldsymbol{p}^2}$ ($\omega = |\boldsymbol{q}|$), moving in a background field, represented by the field tensor $F_0^{\mu\nu} = (\boldsymbol{E}_0, \boldsymbol{B}_0)$ in the laboratory frame, the probability of a physical process depends on the so-called quantum nonlinearity parameter $\chi_0 = \sqrt{|(F_0^{\mu\nu} p_\nu)^2|}/(mF_{cr})$ ($\kappa_0 = \sqrt{|(F_0^{\mu\nu} q_\nu)^2|}/(mF_{cr})$), with the metric tensor $\eta^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$ [2, 9, 19–21]. For an electron or positron this parameter corresponds to the field strength that the particle experiences in its rest frame in units of the critical field. By for example combining a high intensity laser with an ultrarelativistic electron beam, the interesting regime $\chi_0 \sim 1$, where quantum effects start to play a role, can be effectively entered already with today available technology. Indeed already in the late 1990s first experiments in the regime $\chi_0 \lesssim 1$ were performed at the Stanford Linear Accelerator Center (SLAC) [22–24], and more recently experiments using an all-optical setup were carried out reaching $\chi_0 \lesssim 1$ [25, 26]. For the future new experiments are planned at the Deutsches Elektronen-Synchrotron (DESY) [27] and at SLAC [28] aiming to reach values of $\chi_0 \gtrsim 1$.

Two elementary processes in strong field QED that can be probed in such experiments are the emission of a photon by an electron, which is called nonlinear Compton scattering, and the decay of a photon into an electron-positron pair, called nonlinear Breit-Wheeler pair production. Both processes were intensively studied over the past [2, 29–37], and the expressions of the leading-order in the fine-structure constant $\alpha = e^2/4\pi \sim 1/137$ total probabilities for nonlinear Compton scattering and nonlinear Breit-Wheeler pair production can be found for example in Refs. [9, 38–40]. Now it turned out that the probabilities for both processes exceed unity for a sufficiently long phase duration Φ_L of the background field laser pulse (or a sufficiently large laser intensity) [31, 41], which is of course unphysical and stands in contradiction to the unitarity of the S-matrix. From the point of QED this behavior indicates, that in the above mentioned limit the leading order probability is not sufficient and higher order loop corrections to the probability have to be included.

The goal of the first part of this thesis is to identify and compute those corrections, needed in the limit of a long phase duration Φ_L of the laser pulse, and to present probabilities for nonlinear Compton scattering and nonlinear Breit-Wheeler pair production which stay valid within this limit.

Indeed, several other groups have investigated this problem in the past. Already in 1951 Glauber showed that in the classical limit, when the photon recoil is negligible, the leading order probability of nonlinear Compton scattering corresponds to the mean number of photons emitted, rather than to a probability. Also he presented that in this case the probability of single photon emission corresponds to a Poissonian distribution instead [42]. Similar investigations but in the framework of strong field QED were performed in Ref. [41]. Here also the recoil of the photon was taken into account and a "renormalization" was used to ensure that the total probability of either emitting no or an arbitrary number of photons is unity. A kinetic approach in Ref. [43] verified these findings and in the same manner results for nonlinear Breit-Wheeler pair production were presented in Ref. [44]. In Ref. [45] then the probability of an electron emitting an arbitrary number of photons was calculated via a recursive equation. By including that the electron state is not stable inside the background field, but decays exponentially (by means of the electron emitting a photon), it was ensured that the total probability of emitting no or an arbitrary number of photons is unity. The exponential damping depended on the time and the energy of the electron, and it corresponded to the probability of an electron emitting no photon in an certain time interval. Also the photon recoil was considered at each emission step.

In this thesis we will derive the probabilities for nonlinear Compton scattering and nonlinear Breit-Wheeler pair production instead from first principles. For this the S-matrix is computed using the exact states for the electron/positron and photon instead of the Volkov-state and the free photon state, respectively. These exact states include radiative corrections and with them the S-matrix corresponds to the resummation of all one-particle reducible diagrams containing corrections by the mass- and polarization operator to the electron/positron and photon states, respectively. However, to obtain analytical solutions the expressions of the one-loop mass- and polarization operator in the case of a constant-crossed field have to be taken, such that we performed the computation of the probabilities in the so-called locally-constant field approximation (LCFA).

The LCFA was also applied in Refs. [41, 43–45] and it states that for a physical process the background field can locally be assumed to be constant and crossed, if the formation length of this physical process is much smaller than the wavelength of the background field [2, 9, 39]. In this case the amplitudes in the limit of a constant-crossed field are taken and finally be averaged over the phase dependent plane wave profile. This assumption is valid if $\xi_0 \gg 1$ at $\chi_0, \kappa_0 \sim 1$ [39], which we assumed throughout the derivation.

With that both probabilities obtain automatically an exponential damping term corresponding to the decay of the electron, positron, and photon state inside the plane wave background field, where again electrons and positrons "decay" by emitting photons and photons decay into electron-positron pairs. In contrast to Ref. [45] this decay also depends on the spin and polarization of the electron/positron and photon, respectively, such that the spin- and polarization resolved probabilities are presented in this thesis. We finally proof analytically that this probabilities stay below unity for long phase durations of the laser pulse.

However, this investigations were performed considering a linear polarized plane wave and by employing the LCFA. Without these assumptions we were not able to obtain the radiatively corrected particle states, which are needed for the computation of the probabilities. In a next step we therefore calculated the leading-order in α corrections stemming from the mass operator to the probability of nonlinear Compton scattering separately. Here we consider to have an arbitrary transverse polarized plane wave background field and do not employ the LCFA. The leading-order in α corrections are derived again from first principles by using the S-matrix elements of the leading order nonlinear Compton scattering process and its correction by one mass operator to the incoming and outgoing electron state. In order to simplify the calculations, this time the total probability summed over the spin and polarization indices of the outgoing particles and averaged over the initial electron spin was investigated. Finally these corrections were compared to the previously obtained probability including the particles state decay and we show that the new corrections are implicitly included in this probability after employing the same assumptions. In this step also the expressions of the new corrections within the LCFA were presented.

The thesis is structured the following: In the next Chapter the basics of Strong field QED are explained and notation is introduced. In Chapter 3 we will then elaborate which Feynman diagrams have to be considered for the probabilities of nonlinear Compton scattering and nonlinear Breit-Wheeler pair production in the case of background fields with long pulse duration. Further the derivation of the exact electron, positron, and photon states is presented and we show that they suitable are to perform the needed resummation. The probabilities for nonlinear Compton scattering and nonlinear Breit-Wheeler pair production including the decay of the states are investigated then in Chapter 4 and 5, respectively. The next part dealing with the leading-order in α correction stemming from the mass operator to the probability of nonlinear Compton scattering can be found in Chapter 6. Finally the conclusion is given in Chapter 7. In the Appendix useful relations for γ-matrices and Airy-functions, and a short summary of the notation can be found.

2. Strong field QED

As mentioned in the introduction, Strong Field QED considers electrodynamic processes in the presence of a strong electromagnetic background field. The interaction of electrons and positrons with a strong electromagnetic field has to be treated exactly in the calculations, due to the large intensity of the field. Instead the field itself is assumed to be not influenced significantly by those interactions such that it is described as a background field [6, 19]. Nowadays laser pulses are commonly used to generate such background fields in experiments [2]. For this theoretical work we consider to have a plane wave laser pulse with central photon four-momentum k_0^μ colliding with an electron of four-momentum p^μ or a photon of four-momentum q^μ as depicted in Fig. 2.1. Inside the laser pulse the electron can then emit a photon and the photon can decay into an electron-positron pair.

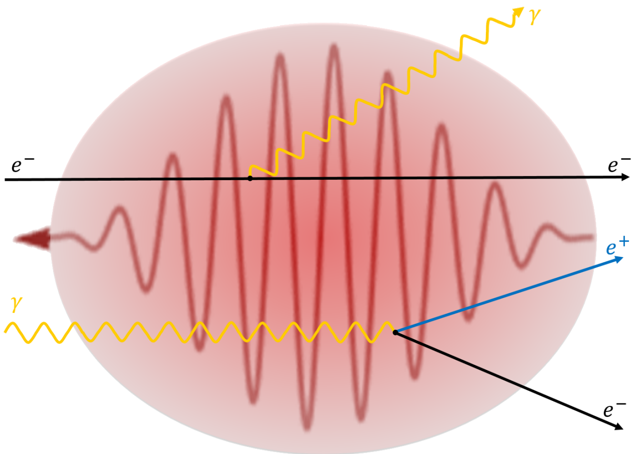

Figure 2.1.: Sketch of an electron and a photon colliding with a laser pulse. Inside the pulse the electron is emitting a photon and the photon decays into an electron-positron pair.

The behavior of fermions (electrons and positrons) and photons is thereby described by the Lagrangian of QED which is [3, 46]

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F^{\mu\nu}_{\text{total}}(x)F_{\text{total},\mu\nu}(x) - \frac{\rho}{2}(\partial A_{\text{total}}(x))^2 + \bar{\psi}(x)(i\hat{\partial} - m)\psi(x) - e\bar{\psi}(x)\hat{A}_{\text{total}}(x)\psi(x), \tag{2.0.1}$$

where $\psi(x)$ and $A^\mu_{\text{total}}(x)$ describe the fermion field and the total photon field, respectively. The gauge fixing parameter ρ ensures that the Lorentz-gauge condition $(\partial A_{\text{total}}(x)) = 0$ is considered when quantizing the photon field. Here and in the following the notation $(\partial A_{\text{total}}(x)) = \partial_\mu A^\mu_{\text{total}}(x)$ is introduced which also holds for any other arbitrary four-vectors. Further we used the electromagnetic field tensor $F^{\mu\nu}_{\text{total}}(x) = \partial^\mu A^\nu_{\text{total}}(x) - \partial^\nu A^\mu_{\text{total}}(x)$, introduced the notation $\bar{\psi}(x) = \psi^\dagger(x)\gamma^0$, and the notation $\hat{v} = \gamma^\mu v_\mu$, where γ^μ is the Dirac-matrix (see Appendix A on page 98) and v_μ is an arbitrary four-vector. By the variation of the Lagrangian the equations of motion for the fermion and photon field can be obtained. For the fermion field one obtains the Euler-Lagrange equation [3, 46]

$$(i\hat{\partial} - e\hat{A}_{\text{total}}(x) - m)\psi(x) = 0 \tag{2.0.2}$$

and for the photon field (we are working in Feynman Gauge with $\rho = 1$ at the end) [46]

$$\partial^2 A^\mu_{\text{total}}(x) - (1-\rho)\partial^\mu(\partial A_{\text{total}}(x)) = e\bar{\psi}(x)\gamma^\mu\psi(x). \tag{2.0.3}$$

2.1. The classical nonlinearity parameter ξ_0

Since QED is Lorentz- and gauge-invariant, physical observables like probabilities or transition amplitudes only depend via Lorentz- and gauge-invariant parameters on the background field [9]. Two important Lorentz- and gauge-invariant parameters in strong field QED are the classical and the quantum nonlinearity parameter.

As mentioned already, the classical nonlinearity parameter is defined as [2, 9]

$$\xi_0 = \frac{|e|E_0}{m\omega_0}, \tag{2.1.1}$$

where E_0 is the electric-field amplitude and ω_0 the central angular frequency of the laser pulse. It is related to the intensity of the background field and corresponds to the energy an electron at rest gains by the Lorenz force in one laser cycle in units of m. For $\xi_0 \gtrsim 1$ the electron becomes relativistic within one laser cycle and the interaction of the electron with the background field cannot be treated perturbatively in the calculations anymore [see next Section]. In this thesis we assume that $\xi_0 \gg 1$.

The discussion to the quantum nonlinearity parameter can be found in Section 2.8.

2.2. The Furry picture

As we discussed before, in strong field QED the interaction of the fermion with the electromagnetic field has to be taken into account exactly when the parameter $\xi_0 \gtrsim 1$. This is because the interaction term $e\bar{\psi}(x)\hat{A}_{\text{total}}(x)\psi(x)$ in the Lagrangian scales with ξ_0, such that it finally can not be treated perturbatively anymore [9]. To however still be able to obtain analytical solutions one usually works in the so-called Furry picture [10], where the total photon field is splitted into two parts, i.e.

$$A^\mu_{\text{total}}(x) = A^\mu(x) + A^\mu_{\text{rad}}(x). \tag{2.2.1}$$

Here $A^\mu(x)$ is the four-potential of the background field and resembles the vacuum expectation value of the total photon field. Since it is not influenced significantly by the interaction with the fermion field we can treat it as classical and it should obey the equation of motion of a free field, i.e. $\partial^2 A^\mu(x) = 0$ [10, 11, 19]. The radiation field $A^\mu_{\text{rad}}(x)$ instead describes the quantum fluctuations around the background field $A^\mu(x)$ and represents incoming, outgoing, or virtual photons. This field has to be quantized where the quantization is performed similar to QED in vacuum. Now for the fermion field the interaction with the background field is considered in the Furry picture already in the quantization procedure by solving the modified Dirac equation [11]

$$(i\hat{\partial} - e\hat{A}(x) - m)\psi(x) = 0. \tag{2.2.2}$$

In that way the obtained fermion states already take the interaction of the fermion field with the background field exactly into account. The interaction of the fermion field with the radiation field, however, can still be treated perturbatively, since it only scales with the small fine-structure constant $\alpha = e^2/(4\pi) \sim 1/137$ [9,11].

2.3. The background field

In the calculations we always consider a plane wave background field with on-shell photons ($k_0^2 = 0$) propagating along the direction $\boldsymbol{n}$, where $\boldsymbol{n}$ is a unit vector, i.e. $\boldsymbol{n}^2 = 1$. With that the central photon four-momentum of the plane wave is $k_0^\mu = \omega_0 n^\mu$, with the four-vector $n^\mu = (1, \boldsymbol{n})$. The four-potential of the background field is given by $A^\mu(x)$, where $x^\mu = (t, \boldsymbol{x})$ is the four-space-time vector with time t and space-vector $\boldsymbol{x}$. Since we consider the background field to be a plane wave, the four-potential only depends on the phase/light-cone time $\phi = (nx)$, i.e. $A^\mu(x) = A^\mu(\phi) = (A^0(\phi), \boldsymbol{A}(\phi))$. As mentioned already, the four-potential fulfills further the free-wave equation $\partial^2 A^\mu(\phi) = 0$ and the Lorentz gauge condition $\partial_\mu A^\mu(\phi) = 0$. Since we consider the plane wave to be generated by a laser pulse with final extend, we have $A^\mu(\phi) \to 0$ for $\phi \to \pm\infty$. We fix the gauge to $A^0(\phi) = 0$, such that the vector potential becomes perpendicular to the direction of propagation of the background field, i.e. $\boldsymbol{n} \cdot \boldsymbol{A}(\phi) = 0$.

Further we introduce the four-vector $\tilde{n}^\mu = (1, -\boldsymbol{n})/2$, with $(n\tilde{n}) = 1$, and the two four-vectors $a_j^\mu = (0, \boldsymbol{a}_j)$ with $j = 1, 2$, which obey the relations $(na_j) = -2(\tilde{n}a_j) = -\boldsymbol{n} \cdot \boldsymbol{a}_j = 0$ and $(a_j a_{j'}) = -\boldsymbol{a}_j \cdot \boldsymbol{a}_{j'} = -\delta_{jj'}$ with $j, j' = 1, 2$. The plane spanned by the two unit-vectors $\boldsymbol{a}_j$ is named in the following the transverse ($\perp$) plane. With the four-vectors n^μ, $\tilde{n}^\mu$, and a_j^μ with $j = 1, 2$ the metric can be expressed as $\eta^{\mu\nu} = n^\mu \tilde{n}^\nu + \tilde{n}^\mu n^\nu - a_1^\mu a_1^\nu - a_2^\mu a_2^\nu$ [19,47].

In this way the phase dependent part of the vector-potential can be extracted and it can be rewritten into the form $\boldsymbol{A}(\phi) = \psi_1(\phi)\boldsymbol{a}_1 + \psi_2(\phi)\boldsymbol{a}_2$. Here $\psi_j(\phi)$ denotes the jth pulse shape function and, according to $A^\mu(\phi) \to 0$ for $\phi \to \pm\infty$, it vanishes for $\phi \to \pm\infty$. Since the four-potential always occurs multiplied by the electron charge, we introduce the notation $\mathcal{A}^\mu(\phi) = eA^\mu(\phi)$.

The electromagnetic field tensor is given by $F^{\mu\nu}(\phi) = n^\mu A'^\nu(\phi) - n^\nu A'^\mu(\phi)$, where here and in the following a prime at a function denotes the derivative of the function with respect to its argument. We can rewrite the field tensor into the form $F^{\mu\nu}(\phi) = f_1^{\mu\nu}\psi_1'(\phi) + f_2^{\mu\nu}\psi_2'(\phi)$, with $f_1^{\mu\nu} = n^\mu a_1^\nu - n^\nu a_1^\mu$ and $f_2^{\mu\nu} = (1/2)\epsilon^{\mu\nu\rho\tau} f_{1,\rho\tau} = n^\mu a_2^\nu - n^\nu a_2^\mu$, where $\epsilon^{\mu\nu\rho\tau}$ is the anti-symmetric unit four-tensor with $\epsilon^{0123} = +1$. Since the field tensor is always multiplied by the electron charge, we also introduce the notation $\mathcal{F}^{\mu\nu}(\phi) = eF^{\mu\nu}(\phi)$.

In the cases where we assume to have a linearly polarized background field we choose without loss of generality $\psi_2(\phi) = 0$ and $\psi_1(\phi) = A_0\psi(\phi)$, with $A_0 < 0$ being related to the amplitude of the electric field of the plane wave. For a monochromatic plane wave we would have then $A_0 = -E_0/\omega_0$ and the pulse function $\psi(\phi)$ would be a function of $\omega_0\phi$. Again we introduce the notation $\mathcal{A}_0 = eA_0$. Further we can rewrite $\mathcal{F}^{\mu\nu}(\phi) = \mathcal{F}^{\mu\nu}\psi'(\phi)$ with $\mathcal{F}^{\mu\nu} = \mathcal{A}_0 f_1^{\mu\nu}$.

2.4. Light cone coordinates

Since the plane wave only depends on the phase $\phi = (nx)$, it is useful to apply light-cone coordinates [19, 47, 48]. This simplifies the integration over space-time coordinates and the particles momenta later. For this we use the four above introduced four-vectors $n^\mu = (1, \boldsymbol{n})$, $\tilde{n}^\mu = (1, -\boldsymbol{n})/2$, and $a_j^\mu = (0, \boldsymbol{a}_j)$ with $j = 1, 2$. Those four-vectors fulfill the conditions

$$n^2 = 0 = \tilde{n}^2, \qquad (n\tilde{n}) = 1, \qquad (na_j) = 0 = (\tilde{n}a_j), \qquad (a_j a_{j'}) = -\delta_{jj'}, \tag{2.4.1}$$

with $j, j' = 1, 2$, and the metric can be written with them via

$$\eta^{\mu\nu} = n^\mu \tilde{n}^\nu + \tilde{n}^\mu n^\nu - a_1^\mu a_1^\nu - a_2^\mu a_2^\nu. \tag{2.4.2}$$

The light-cone components of an arbitrary four-vector $v^\mu = (v^0, \boldsymbol{v})$ are now defined by

$$v_- = (vn) = v^0 - \boldsymbol{n} \cdot \boldsymbol{v}, \tag{2.4.3}$$

$$v_+ = (v\tilde{n}) = (v^0 + \boldsymbol{n} \cdot \boldsymbol{v})/2, \tag{2.4.4}$$

$$\boldsymbol{v}_\perp = (v_{\perp,1}, v_{\perp,2}) = -((va_1), (va_2)) = (\boldsymbol{a}_1 \cdot \boldsymbol{v}, \boldsymbol{a}_2 \cdot \boldsymbol{v}). \tag{2.4.5}$$

Using the metric, the product of two arbitrary four-vectors v^μ and w^μ, in light-cone coordinates, is therefore given by

$$(vw) = v_- w_+ + v_+ w_- - \boldsymbol{v}_\perp \cdot \boldsymbol{w}_\perp. \tag{2.4.6}$$

Note that the light-cone time of the space-time coordinate x^μ obtains a special notation in light-cone coordinates and is usually indicated by $\phi = x_- = t - \boldsymbol{n} \cdot \boldsymbol{x}$. The four dimensional space-time integral in Cartesian coordinates becomes in light-cone coordinates

$$\int d^4x = \int dx_- dx_+ d^2\boldsymbol{x}_\perp = \int d\phi dx_+ d^2\boldsymbol{x}_\perp, \tag{2.4.7}$$

where $d^2\boldsymbol{x}_\perp = dx_{\perp,1} dx_{\perp,2}$. Considering the anti-symmetric unit four-tensor $\epsilon^{\mu\nu\rho\tau}$ with $\epsilon^{0123} = +1$, the light-cone four-vectors obey the relation

$$\epsilon^{\mu\nu\rho\tau} n_\mu \tilde{n}_\nu a_{1,\rho} a_{2,\tau} = +1. \tag{2.4.8}$$

2.5. Radiation photon state

The quantization of the radiation field is like in vacuum QED. As a result one obtains the plane wave solution $A_{j,q}^{\text{rad},\mu}(x) = e^{-i(qx)} \epsilon_j^\mu / \sqrt{2\omega}$, where the photon has the on-shell four-momentum $q^\mu = (\omega, \boldsymbol{q})$, with angular frequency $\omega = |\boldsymbol{q}|$, and where the quantization volume was set for simplicity equal to unity [11, 46]. Here the transverse polarization indicated by the index $j = 1, 2$ is along the direction of the polarization four-vector ϵ_j^μ, which obeys $(\epsilon_j \epsilon_{j'}^*) = -\delta_{jj'}$ for $j, j' = 1, 2$. Considering linear polarization we choose the two polarization states to be represented by the two four-vectors

$$\Lambda_1^\mu(q) = \frac{f_1^{\mu\nu} q_\nu}{q_-}, \qquad \Lambda_2^\mu(q) = \frac{f_2^{\mu\nu} q_\nu}{q_-}, \tag{2.5.1}$$

which are pointing, in the case of a linearly polarized background field, along a direction corresponding to the direction of the electric and magnetic field of the background plane wave, respectively. Both four-vectors fulfill the relations $(\Lambda_j(q)\Lambda_{j'}(q)) = -\delta_{jj'}$, $(n\Lambda_j(q)) = 0$, and $(q\Lambda_j(q)) = 0$, for $j, j' = 1, 2$. (Note that in Chapter 6 also the short notation $\Lambda^\mu_j = \Lambda^\mu_j(q)$ is used.)

The photon propagator is given by [3, 46, 49]

$$D^{\mu\nu}(x-y) = \int \frac{d^4q}{(2\pi)^4} \frac{\eta^{\mu\nu}}{q^2 - \lambda^2 + i0} e^{-i(q(x-y))}, \tag{2.5.2}$$

where a fictitious photon mass λ was added to avoid infrared divergences in the calculations and which will be finally set equal to zero. Further we introduced the notation $1/(\cdots \pm i0) = \lim_{d\to+0} 1/(\cdots \pm id)$, where d is a small positive real number.

2.5.1. Light-cone related coordinates

Similar to the four-vectors n^μ, $\tilde{n}^\mu$, a^μ_1, and a^μ_2, the set of four-vectors n^μ, q^μ/q_-, $\Lambda^\mu_1(q)$, and $\Lambda^\mu_2(q)$ fulfill the relations $n^2 = 0 = q^2/q_-^2$, $(nq)/q_- = 1$, $(n\Lambda_j(q)) = 0 = (q\Lambda_j(q))/q_-$, and $(\Lambda_j(q)\Lambda_{j'}(q)) = -\delta_{jj'}$, with $j, j' = 1, 2$ and q^μ being the on-shell for momentum of a photon. Hence the metric can also be written into the form [50]

$$\eta^{\mu\nu} = \frac{q^\mu n^\nu + n^\mu q^\nu}{q_-} - \Lambda^\mu_1(q)\Lambda^\nu_1(q) - \Lambda^\mu_2(q)\Lambda^\nu_2(q). \tag{2.5.3}$$

2.6. Volkov-States

For an exact description of the electron and positron in the Furry picture one has to solve the Dirac-equation (2.2.2) in the presence of the background field. Considering a plane wave background field the Dirac-equation becomes [11]

$$(i\hat{\partial} - \hat{\mathcal{A}}(\phi) - m)\psi^V(x) = 0, \tag{2.6.1}$$

which is analytically solvable and its solution is known as Volkov state $\psi^V(x)$. Since the plane wave background field vanishes in our considerations for $x_- \to \pm\infty$, the solution should obey additionally the following boundary conditions: For an incoming (outgoing) electron and positron with momentum $p^\mu = (\varepsilon, \boldsymbol{p})$ and spin $s = \pm 1$ the solutions should correspond for $x_- \to -\infty$ ($x_- \to +\infty$) to the free electron state $\psi^{\text{free}}_{e^-,s,p}(x) = e^{-i(px)}u_s(p)/\sqrt{2\varepsilon}$ and positron state $\psi^{\text{free}}_{e^+,s,p}(x) = e^{i(px)}v_s(p)/\sqrt{2\varepsilon}$, respectively [11]. Here the free positive-energy spinor $u_s(p)$ and the free negative-energy spinor $v_s(p)$ were introduced, which fulfill, according to the free Dirac-equation, the equations [11, 46]

$$(\hat{p} - m)u_s(p) = 0 \quad \text{and} \quad (\hat{p} + m)v_s(p) = 0, \tag{2.6.2}$$

respectively, and their conjugates $\bar{u}_s(p)(\hat{p}-m) = 0$ and $\bar{v}_s(p)(\hat{p}+m) = 0$. They are normalized as $u^\dagger_s(p)u_{s'}(p) = 2\varepsilon\delta_{ss'}$ and $v^\dagger_s(p)v_{s'}(p) = 2\varepsilon\delta_{ss'}$, and further obey the following relations [11,46]

$$\begin{aligned} \bar{u}_s(p)u_{s'}(p) &= 2m\delta_{ss'}, & \bar{u}_s(p)\gamma^\mu u_{s'}(p) &= 2p^\mu\delta_{ss'}, \\ \bar{v}_s(p)v_{s'}(p) &= -2m\delta_{ss'}, & \bar{v}_s(p)\gamma^\mu v_{s'}(p) &= 2p^\mu\delta_{ss'}. \end{aligned} \tag{2.6.3}$$

Note that again the quantization volume was set to unity.

As mentioned the analytical solutions of Eq. (2.6.1) for a plane-wave background field are already known and with our boundary conditions the corresponding Volkov in- and out-states are given for the electron by [11]

$$\psi^{V,(\text{in/out})}_{e^-,s,p}(x) = e^{i\Phi^{(\text{in/out})}(p)} E(p,x) \frac{u_s(p)}{\sqrt{2\varepsilon}} \tag{2.6.4}$$

and for the positron by

$$\psi^{V,(\text{in/out})}_{e^+,s,p}(x) = e^{i\Phi^{(\text{in/out})}(-p)} E(-p,x) \frac{v_s(p)}{\sqrt{2\varepsilon}}. \tag{2.6.5}$$

Here we introduced the so-called Ritus-matrix

$$E(p,x) = \left[1 + \frac{\hat{n}\hat{\mathcal{A}}(x_-)}{2p_-}\right] e^{-iS_p(x)} \tag{2.6.6}$$

with the phase

$$-iS_p(x) = -i(px) - i\int_0^{x_-} d\varphi \left[\frac{(p\mathcal{A}(\varphi))}{p_-} - \frac{\mathcal{A}^2(\varphi)}{2p_-}\right]. \tag{2.6.7}$$

For $l^\mu = (l^0, \boldsymbol{l})$ and $l'^\mu = (l'^0, \boldsymbol{l'})$ being two off-shell four-momenta, and with introducing the notation $\bar{M} = \gamma^0 M^\dagger \gamma^0$ where M is an arbitrary matrix, the Ritus-matrix $E(l,x)$ fulfills the following identities [9, 19, 51]:

$$\int d^4x\ \bar{E}(l,x)E(l',x) = (2\pi)^4\delta^4(l-l'), \tag{2.6.8}$$

$$\int \frac{d^4l}{(2\pi)^4}\ \bar{E}(l,x)E(l,y) = \delta^4(x-y), \tag{2.6.9}$$

$$\gamma^\mu[i\partial_\mu - \mathcal{A}_\mu(\phi)]E(l,x) = E(l,x)\hat{l}. \tag{2.6.10}$$

To obey the boundary conditions also an overall phase $\exp\{i\Phi^{(\text{in/out})}(p)\}$ had to be introduced into the Volkov in- and out-states, where

$$\Phi^{(\text{in/out})}(p) = -\int_{\mp\infty}^0 d\varphi \left[\frac{(p\mathcal{A}(\varphi))}{p_-} - \frac{\mathcal{A}^2(\varphi)}{2p_-}\right]. \tag{2.6.11}$$

The Greens function of the Dirac equation in a plane wave background field, solving the equation

$$(i\hat{\partial} - \hat{\mathcal{A}}(x_-) - m)G^V(x,y) = \delta^4(x-y), \tag{2.6.12}$$

is known as the Volkov-propagator [9, 51]. The Volkov-propagator describes intermediate off-shell electrons and positrons inside the plane wave and its expression is given by [9, 51]

$$iG^V(x,y) = i\int \frac{d^4l}{(2\pi)^4} E(l,x) \frac{\hat{l}+m}{l^2 - m^2 + i0} \bar{E}(l,y). \tag{2.6.13}$$

An operator representation of the Volkov-propagator can be found in Eq. (2.9.1) in Section 2.9.1 (see also Ref. [51]).

In Feynman diagrams both, the electron and positron Volkov-states as well as the Volkov-propagator, are indicated by double lines.

Finally we choose the spin quantization axis for the electron and positron to be along the four-vector

$$\zeta_p^\mu = -\frac{f_2^{\mu\nu} p_\nu}{p_-} = -\frac{1}{p_-}\epsilon^{\mu\nu\lambda\rho} p_\nu n_\lambda a_{1,\rho}. \tag{2.6.14}$$

The spin four-vector fulfills the relations $(n\zeta_p) = 0$, $(p\zeta_p) = 0$, and [11]

$$\gamma^5 \hat{\zeta}_p u_s(p) = s u_s(p). \tag{2.6.15}$$

2.7. Local constant field approximation (LCFA)

The local constant field approximation (LCFA) is valid in the limit of low-frequency plane waves with fixed electric-field amplitude where the wavelength of the background field becomes much larger than the formation length of the physical process. Since the formation length usually scales with $1/\xi_0$, the approximation is applicable if $\xi_0 \gg 1$ considering additionally that $\chi_0, \kappa_0 \sim 1$ [9,39]. In the LCFA one assumes that in this case the background field can be treated locally for a physical process as a constant crossed field. Hence probabilities of physical processes reduce to probabilities in a constant crossed field, which are then averaged at the end over the phase dependent plane wave profile. However, in the case of nonlinear Compton scattering problems occur for low photon energies which we do not consider here [39, 52].

2.8. The quantum nonlinearity parameters χ_0 and κ_0

The quantum nonlinearity parameter is the second important Lorentz- and gauge-invariant parameter and it is given by $\chi_0 = \sqrt{|(F^{\mu\nu}p_\nu)^2|}/(mF_{cr})$ for electrons and positrons with four momentum $p^\mu = (\varepsilon, \boldsymbol{p})$ and by $\kappa_0 = \sqrt{|(F^{\mu\nu}q_\nu)^2|}/(mF_{cr})$ for photons with four-momentum $q^\mu = (\omega, \boldsymbol{q})$, where $F_{cr} = m^2/|e|$ is the critical field [9]. For an electron this parameter resembles the field strength the particle experiences in its rest frame in units of the critical field [2]. If $\chi_0 \sim 1$ or $\kappa_0 \sim 1$, then nonlinear quantum effects start to become significant. For example, in the limit of a constant-crossed field, loop corrections (like the one-loop mass operator in the next Section) scale for $\chi_0, \kappa_0 \gg 1$ with $\alpha\chi_0^{2/3}$ or $\alpha\kappa_0^{2/3}$, which is also known as Ritus-Narozny conjecture [53]. When in this case χ_0 or κ_0 become so large that $\alpha\chi_0^{2/3} \gtrsim 1$ or $\alpha\kappa_0^{2/3} \gtrsim 1$, then higher order loop corrections would scale like leading order processes and the theory would become non-perturbative [53, 54]. However, in this thesis we assume that the quantum nonlinearity parameters are sufficiently small that a perturbative approach is applicable.

In the calculations where the LCFA is applied, the parameters χ_0 and κ_0 are replaced by their corresponding local phase dependent expressions

$$\chi_p(\phi) = \frac{p_-\mathcal{A}_0}{m^3}\psi'(\phi) \qquad \text{and} \qquad \kappa_q(\phi) = \frac{q_-\mathcal{A}_0}{m^3}\psi'(\phi). \tag{2.8.1}$$

2.9. Mass operator

The mass operator describes the self-interaction of electrons and positrons and leads to the correction of the electron mass [3, 11]. It consists of an infinite sum of one-particle irreducible Feynman diagrams where its leading order one-loop Feynman diagram is depicted in Fig. 2.2. For an exact description of electrons and positrons in principle, a resummation of up to an infinite number of corrections by the mass operator to the Volkov-states and Volkov-propagator has to be considered. However these corrections are usually small and can be neglected in the realm of pertubation theory.

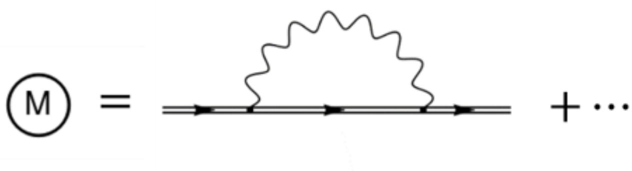

Figure 2.2.: The mass operator consists of the one-loop mass operator (Feynman diagram shown here) and higher order loop diagrams (not presented here).

An expression of the one-loop mass operator in a plane wave field can be found for example in Refs. [49, 55]. The next subsections are rather technical and present the derivation of the expression of the one-loop mass operator, which is oriented on and similar to the one presented in Ref. [49]. These expressions will be used then in Chapters 3 and 6. The reader not interested into the technical details of the derivation can proceed with Section 2.10 on page 23.

2.9.1. Renormalized expression of the mass operator depending on the outgoing electron momentum

In this subsection we want to derive the expression of the one-loop mass operator in a plane wave which depends on the electron momentum of the outgoing electron in the Feynman diagram in Fig.2.2. The one-loop mass operator depending on the electron momentum of the incoming electron was calculated in Ref. [49] and we follow their steps to calculate the one depending on the outgoing electron momentum.

The calculations in Ref. [49] start with an operator approach, thus we first have to introduce the following operators: $\varpi_P^\mu(\Phi) = P^\mu - \mathcal{A}^\mu(\Phi)$ is the kinetic four-momentum, Φ is the operator of light-cone time ϕ, X^μ is the operator of four-space, and finally the canonical four-momentum operator is $P^\mu = i\partial^\mu$, which has in light-cone coordinates the components $P_\Phi = -i\partial_\Phi = -i(\tilde{n}\partial)$, $P_\tau = -i(n\partial)$, and $\boldsymbol{P}_\perp = -i(\boldsymbol{a}_1 \cdot \boldsymbol{\nabla}, \boldsymbol{a}_2 \cdot \boldsymbol{\nabla})$.

Further, we need for the derivation an operator representation of the Volkov-propagator. For the mass operator depending on the incoming electron momentum in Ref. [49] an operator representation of the Volkov propagator, which has a $(\hat{\varpi}_P + m)$ on the right side (see Eq.(6) in [49]), was used. Instead, for the expression of the mass operator depending on the outgoing electron momentum, we need a Volkov propagator which has $(\hat{\varpi}_P + m)$ on the left side. This expression is given already in Eq.(10) in Ref. [56] and reads

$$
\begin{aligned}
G^V =&[\hat{\varpi}_P(\Phi)+m]\frac{1}{\hat{\varpi}_P^2(\Phi)-m^2+i0} \\
=&(-i)[\hat{\varpi}_P(\Phi)+m]\int_0^\infty dre^{-irm^2}e^{2irP_\tau P_\Phi}e^{-i\int_0^r dr'[\boldsymbol{P}_\perp-\boldsymbol{\mathcal{A}}_\perp(\Phi-2r'P_\tau)]^2} \\
&\times\left\{1-\frac{1}{2P_\tau}\hat{n}[\hat{\mathcal{A}}(\Phi-2rP_\tau)-\hat{\mathcal{A}}(\Phi)]\right\},
\end{aligned}
\tag{2.9.1}
$$

where the Volkov-propagator in configuration space can be obtained from this via $G^V(x,x')=\left\langle x\mid G^V\mid x'\right\rangle$.

The following calculation of the renormalized mass operator is now similar to the one in Ref. [49]. The mass operator in momentum space is given by (see Eq.(10) in [49] with substitution $q\to -q$)

$$
\tilde{M}(l,l')=-ie^2\int d^4xd^4x'\int\frac{d^4q}{(2\pi)^4}\frac{e^{i(q(x-x'))}}{q^2-\lambda^2+i0}\bar{E}(l,x)\gamma^\mu G^V(x,x')\gamma_\mu E(l',x'). \tag{2.9.2}
$$

We use now the relation $\int_0^\infty du\exp[iu(q^2-\lambda^2)]=i/(q^2-\lambda^2+i0)$ and the operator relation $\exp[i(qX)]g(P)\exp[-i(qX)]=g(P+q)$ for a generic function $g(P)$ of the momentum operator P^μ [49]. Together with $\Phi\mid x'\rangle=\phi\mid x'\rangle$ and $\langle x\mid x'\rangle=\delta^4(x-x')$ we derive to the following expression for the mass operator

$$
\begin{aligned}
\tilde{M}(l,l')=&ie^2\int d^4x\int\frac{d^4q}{(2\pi)^4}\int_0^\infty dudre^{iu(q^2-\lambda^2)-im^2r}\bar{E}(l,x)\gamma^\mu \\
&\times[\hat{\varpi}_P(\phi)+\hat{q}+m]e^{2ir(P_\tau-q_-)(P_\phi-q_+)}e^{-i\int_0^r dr'[\boldsymbol{P}_\perp+\boldsymbol{q}_\perp-\boldsymbol{\mathcal{A}}_\perp(\phi-2r'P_\tau+2r'q_-)]^2} \\
&\times\left\{1-\frac{1}{2(P_\tau-q_-)}\hat{n}[\hat{\mathcal{A}}(\phi-2rP_\tau+2rq_-)-\hat{\mathcal{A}}(\phi)]\right\}\gamma_\mu E(l',x)
\end{aligned}
\tag{2.9.3}
$$

(see Eq.(11) in [49] for comparison). Next we introduce the notation $\tilde{\phi}_r=\phi+2r(l_-+q_-)$, $\Delta\mathcal{A}^\mu(\tilde{\phi}_r)=\mathcal{A}^\mu(\tilde{\phi}_r)-\mathcal{A}^\mu(\phi)$, $\widehat{\Delta\mathcal{A}}(\tilde{\phi}_r)=\gamma_\mu\Delta\mathcal{A}^\mu(\tilde{\phi}_r)$, and the classical kinetic four-momentum of an electron in the plane wave

$$
\Pi_l^\lambda(\phi)=l^\lambda-\mathcal{A}^\lambda(\phi)+\frac{(l\mathcal{A}(\phi))}{l_-}n^\lambda-\frac{\mathcal{A}^2(\phi)}{2l_-}n^\lambda, \tag{2.9.4}
$$

with the limit $\lim_{\phi\to\pm\infty}\pi_l^\lambda(\phi)=l^\lambda$. Further we employ the relation

$$
\bar{E}(l,x)\varpi_P^\lambda(\phi)=\bar{E}(l,x)\left[\Pi_l^\lambda(\phi)+i\frac{\hat{n}\hat{\mathcal{A}}'(\phi)}{2l_-}n^\lambda\right]. \tag{2.9.5}
$$

Now three components of the integral in d^4x in Eq. (2.9.3) provide δ-functions. Only the integral in $dx_-=d\phi$ can not be taken at this stage. Considering the integrals in the momentum q, the integral in $d^2\boldsymbol{q}_\perp$ is Gaussian and can be performed via the substitution $\boldsymbol{q}_\perp=\tilde{\boldsymbol{q}}_\perp-[r\boldsymbol{l}_\perp-\int_0^r dr'\boldsymbol{\mathcal{A}}_\perp(\tilde{\phi}_{r'})]/(u+r)$. For dq_+ there are two kinds of integrals, one giving a δ-function and the other giving the derivative of a δ-function. Both can be used to solve afterwards the integral in dq_- and we obtain from the δ-functions the replacement $q_- = -l_- r/(u+r)$, such that $\tilde{\phi}_r = \phi + 2l_- ur/(u+r)$ [49]. With that we obtain for the mass operator the expression

$$\begin{aligned}
\tilde{M}(l,l') =&(2\pi)^3\delta^2(\boldsymbol{l}_\perp - \boldsymbol{l}'_\perp)\delta(l_- - l'_-)\frac{\alpha}{2\pi}\int d\phi e^{-i(l'_+ - l_+)\phi}\int_0^\infty \frac{dudr}{(u+r)^2} \\
&\times e^{-iu\lambda^2 - i\frac{r^2}{u+r}\left[m^2 - \int_0^1 dw\Delta\mathcal{A}^2(\tilde{\phi}_{wr}) + \left(\int_0^1 dw\Delta\mathcal{A}(\tilde{\phi}_{wr})\right)^2\right] + i\frac{ur}{u+r}(l^2-m^2)} \\
&\times\left\{2(m-\hat{l})\left[1 - \frac{\hat{n}\widehat{\Delta\mathcal{A}}(\tilde{\phi}_r)}{2l_-}\right] + \left[1 + \frac{r}{u}\frac{\hat{n}\widehat{\Delta\mathcal{A}}(\tilde{\phi}_r)}{2l_-}\right]\hat{\Pi}_{1,l}(\tilde{\phi}_r)\right. \\
&+ \left[1 - \frac{2u+r}{u}\frac{\hat{n}\widehat{\Delta\mathcal{A}}(\tilde{\phi}_r)}{2l_-}\right]\frac{r}{u+r}\hat{\Pi}_{2,l}(\tilde{\phi}_r) \\
&+ \frac{\hat{n}}{2l_-}\frac{r}{u}\left[\left(\int_0^1 dw\Delta\mathcal{A}(\tilde{\phi}_{wr})\right)^2 - \int_0^1 dw\Delta\mathcal{A}^2(\tilde{\phi}_{wr})\right. \\
&\left.\left. + \frac{r}{u+r}\left(\Delta\mathcal{A}(\tilde{\phi}_r) - \int_0^1 dw\Delta\mathcal{A}(\tilde{\phi}_{wr})\right)^2\right]\right\},
\end{aligned} \tag{2.9.6}$$

where we introduced the two four-vectors

$$\Pi^\lambda_{1,l}(\tilde{\phi}_r) = l^\lambda - \Delta\mathcal{A}^\lambda(\tilde{\phi}_r) + \frac{(l\Delta\mathcal{A}(\tilde{\phi}_r))}{l_-}n^\lambda - \frac{(\Delta\mathcal{A}(\tilde{\phi}_r))^2}{2l_-}n^\lambda, \tag{2.9.7}$$

$$\Pi^\lambda_{2,l}(\tilde{\phi}_r) = l^\lambda - \int_0^1 dw\Delta\mathcal{A}^\lambda(\tilde{\phi}_{wr}) + \frac{\left(\int_0^1 dw\Delta\mathcal{A}(\tilde{\phi}_{wr})\right)}{l_-}n^\lambda - \frac{\left(\int_0^1 dw\Delta\mathcal{A}(\tilde{\phi}_{wr})\right)^2}{2l_-}n^\lambda, \tag{2.9.8}$$

and the quantity $\tilde{\phi}_{wr} = \phi + 2wurl_-/(u+r)$ (compare with Eq. (15) in Ref. [49]). The mass operator has to be renormalized. Since the difference between the mass operator in a plane wave and the mass operator in vacuum is already finite, the renormalization is done by subtracting the vacuum part from the mass operator in a plane wave and adding the renormalized vacuum part to it, where the renormalization of the vacuum part is already known [55], i.e.

$$\tilde{M}_R(l,l') = \tilde{M}(l,l') - \tilde{M}(l,l')(\hat{l} = m, \mathcal{A}^\mu = 0) - (\hat{l} - m)\frac{\partial\tilde{M}(l,l')}{\partial\hat{l}}(\hat{l} = m, \mathcal{A}^\mu = 0). \tag{2.9.9}$$

Thus we finally obtain for the renormalized one-loop mass operator depending on the outgoing electron momentum the expression

$$\begin{aligned}\tilde{M}_R(l,l') = &(2\pi)^3\delta^2(\boldsymbol{l}_\perp - \boldsymbol{l}'_\perp)\delta(l_- - l'_-)\frac{\alpha}{2\pi}\int d\phi e^{-i(l'_+ - l_+)\phi}\int_0^\infty \frac{dudr}{(u+r)^2}\\
&\times e^{-iu\lambda^2 - i\frac{r^2}{u+r}m^2}\Bigg\{e^{i\frac{r^2}{u+r}\left[\int_0^1 dw\Delta\mathcal{A}^2(\tilde{\phi}_{wr}) - \left(\int_0^1 dw\Delta\mathcal{A}(\tilde{\phi}_{wr})\right)^2\right] + i\frac{ur}{u+r}(l^2 - m^2)}\\
&\times\left[2(m-\hat{l})\left[1 - \frac{\hat{n}\widehat{\Delta\mathcal{A}}(\tilde{\phi}_r)}{2l_-}\right] + M_{II}\right]\\
&- \frac{u+2r}{u+r}m - \frac{u}{u+r}\left(1 - 2i\frac{u+2r}{u+r}m^2 r\right)(m-\hat{l})\Bigg\}\end{aligned} \tag{2.9.10}$$

where

$$\begin{aligned}M_{II} = &\left[1 + \frac{r}{u}\frac{\hat{n}\widehat{\Delta\mathcal{A}}(\tilde{\phi}_r)}{2l_-}\right]\hat{\Pi}_{1,l}(\tilde{\phi}_r) + \left[1 - \frac{2u+r}{u}\frac{\hat{n}\widehat{\Delta\mathcal{A}}(\tilde{\phi}_r)}{2l_-}\right]\frac{r}{u+r}\hat{\Pi}_{2,l}(\tilde{\phi}_r)\\
&+ \frac{\hat{n}}{2l_-}\frac{r}{u}\Bigg[\left(\int_0^1 dw\Delta\mathcal{A}(\tilde{\phi}_{wr})\right)^2 - \int_0^1 dw\Delta\mathcal{A}^2(\tilde{\phi}_{wr})\\
&+ \frac{r}{u+r}\left(\Delta\mathcal{A}(\tilde{\phi}_r) - \int_0^1 dw\Delta\mathcal{A}(\tilde{\phi}_{wr})\right)^2\Bigg].\end{aligned} \tag{2.9.11}$$

With the definitions in Eqs. (2.9.7) and (2.9.8) we can further rewrite M_{II} to

$$\begin{aligned}M_{II} = &\frac{u+2r}{u+r}\hat{l} - \widehat{\Delta\mathcal{A}}(\tilde{\phi}_r) - \frac{r}{u+r}\int_0^1 dw\widehat{\Delta\mathcal{A}}(\tilde{\phi}_{wr}) + \frac{r}{u+r}\frac{\widehat{\Delta\mathcal{A}}(\tilde{\phi}_r)\hat{n}}{2l_-}\hat{l}\\
&- \frac{r}{u}\frac{2u+r}{u+r}\frac{\widehat{\Delta\mathcal{A}}(\tilde{\phi}_r)\hat{n}}{2l_-}\int_0^1 dw\widehat{\Delta\mathcal{A}}(\tilde{\phi}_{wr}) + \frac{\hat{n}}{2l_-}N_{II},\end{aligned} \tag{2.9.12}$$

where

$$\begin{aligned}N_{II} = &2\left(l\Delta\mathcal{A}(\tilde{\phi}_r)\right) - \frac{u+2r}{u+r}\left(\Delta\mathcal{A}(\tilde{\phi}_r)\right)^2 + 2\frac{r}{u+r}\left(l\int_0^1 dw\Delta\mathcal{A}(\tilde{\phi}_{wr})\right)\\
&+ 2\frac{r}{u}\frac{r}{u+r}\left(\int_0^1 dw\Delta\mathcal{A}(\tilde{\phi}_{wr})\right)^2 - \frac{r}{u}\int_0^1 dw\left(\Delta\mathcal{A}(\tilde{\phi}_{wr})\right)^2\\
&- 2\frac{r}{u}\frac{r}{u+r}\left(\Delta\mathcal{A}(\tilde{\phi}_r)\int_0^1 dw\Delta\mathcal{A}(\tilde{\phi}_{wr})\right).\end{aligned} \tag{2.9.13}$$

2.9.2. Renormalized expression of the mass operator depending on the incoming electron momentum

For completeness also the renormalized expression of the one-loop mass operator depending on the incoming electron momentum is presented, which was derived in Eq.(18) in Ref. [49],

$$\begin{aligned}
M_R(l,l') = &(2\pi)^3\delta^2(\boldsymbol{l}_\perp - \boldsymbol{l}'_\perp)\delta(l_- - l'_-)\frac{\alpha}{2\pi}\int d\phi e^{-i(l'_+ - l_+)\phi}\int_0^\infty \frac{dudr}{(u+r)^2}\\
&\times e^{-iu\lambda^2 - i\frac{r^2}{u+r}m^2}\Bigg\{ e^{i\frac{r^2}{u+r}\left[\int_0^1 dw\Delta\mathcal{A}^2(\phi_{wr}) - \left(\int_0^1 dw\Delta\mathcal{A}(\phi_{wr})\right)^2\right] + i\frac{ur}{u+r}(l'^2 - m^2)}\\
&\times\left[2\left[1 + \frac{\hat{n}\widehat{\Delta\mathcal{A}}(\phi_r)}{2l'_-}\right](m - \hat{l}') + M_I\right]\\
&-\frac{u+2r}{u+r}m - \frac{u}{u+r}\left(1 - 2i\frac{u+2r}{u+r}m^2 r\right)(m - \hat{l}')\Bigg\}.
\end{aligned} \tag{2.9.14}$$

Here the notation $\phi_r = \phi - 2url'_-/(u+r)$ and $\phi_{wr} = \phi - 2wurl'_-/(u+r)$ is used. Further we introduced the quantity M_I which is given by

$$\begin{aligned}
M_I = &\hat{\Pi}_{1,l'}(\phi_r)\left[1 - \frac{r}{u}\frac{\hat{n}\widehat{\Delta\mathcal{A}}(\phi_r)}{2l'_-}\right] + \frac{r}{u+r}\hat{\Pi}_{2,l'}(\phi_r)\left[1 + \frac{2u+r}{u}\frac{\hat{n}\widehat{\Delta\mathcal{A}}(\phi_r)}{2l'_-}\right]\\
&+\frac{\hat{n}}{2l'_-}\frac{r}{u}\Bigg[\left(\int_0^1 dw\Delta\mathcal{A}(\phi_{wr})\right)^2 - \int_0^1 dw\Delta\mathcal{A}^2(\phi_{wr})\\
&+\frac{r}{u+r}\left(\Delta\mathcal{A}(\phi_r) - \int_0^1 dw\Delta\mathcal{A}(\phi_{wr})\right)^2\Bigg].
\end{aligned} \tag{2.9.15}$$

Note that M_I and M_{II} in Eq. (2.9.11) are related to each other by $M_{II} = \gamma^0 M_I^\dagger \gamma^0|_{l'\to l,\phi_{r,wr}\to\tilde{\phi}_{r,wr}}$. With the definitions in Eqs. (2.9.7) and (2.9.8) we can rewrite M_I to

$$\begin{aligned}
M_I = &\frac{u+2r}{u+r}\hat{l}' - \widehat{\Delta\mathcal{A}}(\phi_r) - \frac{r}{u+r}\int_0^1 dw\widehat{\Delta\mathcal{A}}(\phi_{wr}) + \frac{r}{u+r}\hat{l}'\frac{\hat{n}\widehat{\Delta\mathcal{A}}(\phi_r)}{2l'_-}\\
&-\frac{r}{u}\frac{2u+r}{u+r}\int_0^1 dw\widehat{\Delta\mathcal{A}}(\phi_{wr})\frac{\hat{n}\widehat{\Delta\mathcal{A}}(\phi_r)}{2l'_-} + \frac{\hat{n}}{2l'_-}N_I,
\end{aligned} \tag{2.9.16}$$

where

$$\begin{aligned}
N_I = &2\left(l'\Delta\mathcal{A}(\phi_r)\right) - \frac{u+2r}{u+r}\left(\Delta\mathcal{A}(\phi_r)\right)^2 + 2\frac{r}{u+r}\left(l'\int_0^1 dw\Delta\mathcal{A}(\phi_{wr})\right)\\
&+2\frac{r}{u}\frac{r}{u+r}\left(\int_0^1 dw\Delta\mathcal{A}(\phi_{wr})\right)^2 - \frac{r}{u}\int_0^1 dw\left(\Delta\mathcal{A}(\phi_{wr})\right)^2\\
&-2\frac{r}{u}\frac{r}{u+r}\left(\Delta\mathcal{A}(\phi_r)\int_0^1 dw\Delta\mathcal{A}(\phi_{wr})\right).
\end{aligned} \tag{2.9.17}$$

2.9.3. The one-loop mass operator in the LCFA

In this subsection the expression of the renormalized one-loop mass operator in a linear polarized plane wave and within the LCFA is derived, which we will need later in Section 3.2. For this we start from the renormalized expression of the mass operator depending on the incoming electron momentum in Eq. (2.9.14).

We use that due to the δ-functions the momenta $l'_- = l_-$ and $\boldsymbol{l}'_\perp = \boldsymbol{l}_\perp$. Since we deal with a linearly polarized background field M_I in Eq. (2.9.16) reduces to

$$\begin{aligned} M_I =& \frac{u+2r}{u+r}\hat{l}' - \widehat{\Delta\mathcal{A}}(\phi_r) - \frac{r}{u+r}\int_0^1 dw \widehat{\Delta\mathcal{A}}(\phi_{wr}) + \frac{r}{u+r}\hat{l}'\frac{\hat{n}\widehat{\Delta\mathcal{A}}(\phi_r)}{2l_-} \\ &+ \frac{\hat{n}}{2l_-}\left[N_I + \frac{r}{u}\,\frac{2u+r}{u+r}\left(\Delta\mathcal{A}(\phi_r)\int_0^1 dw\Delta\mathcal{A}(\phi_{wr})\right)\right], \end{aligned} \tag{2.9.18}$$

In the next step we want to perform the LCFA for the mass operator. This means that we assume the formation length of the mass operator to be much smaller than the wavelength of the background field, and hence we can approximate that the phase difference $\phi_r - \phi \ll 1$. Therefore we can expand the potential of the background field around ϕ up to first order [9,38], such that

$$\begin{aligned} \Delta\mathcal{A}^\mu(\phi_r) &= \mathcal{A}^\mu(\phi_r) - \mathcal{A}^\mu(\phi) \approx \mathcal{A}^\mu(\phi) - \frac{2url_-}{u+r}\mathcal{A}'^\mu(\phi) - \mathcal{A}^\mu(\phi) \\ &\approx -\frac{2url_-}{u+r}\mathcal{A}'^\mu(\phi) \end{aligned} \tag{2.9.19}$$

and similar

$$\Delta\mathcal{A}^\mu(\phi_{wr}) \approx -w\frac{2url_-}{u+r}\mathcal{A}'^\mu(\phi). \tag{2.9.20}$$

With this expansion we have to make the following replacements

$$\int_0^1 dw\Delta\mathcal{A}^\mu(\phi_{wr}) \approx -\frac{url_-}{u+r}\mathcal{A}'^\mu(\phi), \tag{2.9.21}$$

$$\int_0^1 dw\Delta\mathcal{A}^2(\phi_{wr}) \approx \frac{4u^2r^2l_-^2}{3(u+r)^2}\mathcal{A}'^2(\phi), \tag{2.9.22}$$

$$\left(\int_0^1 dw\Delta\mathcal{A}^\mu(\phi_{wr})\right)^2 \approx \frac{u^2r^2l_-^2}{(u+r)^2}\mathcal{A}'^2(\phi), \tag{2.9.23}$$

$$\Delta\mathcal{A}^2(\phi_r) \approx \frac{4u^2r^2l_-^2}{(u+r)^2}\mathcal{A}'^2(\phi), \tag{2.9.24}$$

$$\left(\Delta\mathcal{A}(\phi_r)\int_0^1 dw\Delta\mathcal{A}(\phi_{wr})\right) \approx \frac{2u^2r^2l_-^2}{(u+r)^2}\mathcal{A}'^2(\phi), \tag{2.9.25}$$

and the exponential becomes

$$e^{i\frac{r^2}{u+r}\left[\int_0^1 dw\Delta\mathcal{A}^2(\phi_{wr}) - \left(\int_0^1 dw\Delta\mathcal{A}(\phi_{wr})\right)^2\right]} \approx e^{i\frac{u^2r^4l_-^2}{3(u+r)^3}\mathcal{A}'^2(\phi)}. \tag{2.9.26}$$

Inserting the above replacements into Eq. (2.9.18) we derive to the following expression of the mass operator in the LCFA

$$
\begin{aligned}
M_R^{\mathrm{LCFA}}(l,l') = {} & (2\pi)^3\delta^2(\boldsymbol{l}_\perp - \boldsymbol{l}'_\perp)\delta(l_- - l'_-)\frac{\alpha}{2\pi}\int d\phi e^{-i(l'_+ - l_+)\phi}\int_0^\infty \frac{dudr}{(u+r)^2} \\
& \times e^{-iu\lambda^2 - i\frac{r^2}{u+r}m^2}\Bigg\{ e^{i\frac{u^2r^4l_-^2}{3(u+r)^3}\mathcal{A}'^2(\phi) + i\frac{ur}{u+r}(l'^2 - m^2)} \\
& \times \left[2\left[1 - \frac{ur}{u+r}\hat{n}\hat{\mathcal{A}}'(\phi)\right](m - \hat{l}') + M_I^{\mathrm{LCFA}}\right] \\
& - \frac{u+2r}{u+r}m - \frac{u}{u+r}\left(1 - 2i\frac{u+2r}{u+r}m^2 r\right)(m - \hat{l}')\Bigg\},
\end{aligned} \tag{2.9.27}
$$

where

$$
\begin{aligned}
M_I^{\mathrm{LCFA}} = {} & \frac{u+2r}{u+r}\hat{l}' + \frac{(2u+3r)url_-}{(u+r)^2}\hat{\mathcal{A}}'(\phi) - \frac{ur^2}{(u+r)^2}\hat{l}'\hat{n}\hat{\mathcal{A}}'(\phi) \\
& - \frac{\hat{n}}{2l_-}\left[\frac{(2u+3r)ur2l_-}{(u+r)^2}(l\mathcal{A}'(\phi)) + \left(4 + \frac{4r}{3u}\right)\frac{u^2r^2l_-^2}{(u+r)^2}\mathcal{A}'^2(\phi)\right].
\end{aligned} \tag{2.9.28}
$$

Now we commute $\hat{l}'$ in Eq. (2.9.27) to the left side and use, for terms with three γ matrices, the relation in Eq. (A.0.4) in the Appendix, which gives us $\hat{l}'\hat{n}\hat{\mathcal{A}}'(\phi) = l_-\hat{\mathcal{A}}'(\phi) - (l\mathcal{A}'(\phi))\hat{n} - i\epsilon^{\tau\mu\nu\rho}\gamma_\tau\gamma^5 l'_\mu n_\nu \mathcal{A}'_\rho(\phi)$. Further, we perform two substitutions, first $v = r/u$ and then $\tilde{u} = um^2/(1+v)$ [49]. Since the background field is assumed to be linear polarized, we can write the derivative of the potential $\mathcal{A}'^\mu(\phi) = (0, \boldsymbol{\mathcal{A}}'(\phi))$ as $\boldsymbol{\mathcal{A}}'(\phi) = \mathcal{A}_0\psi'(\phi)\boldsymbol{a}_1$. In the exponential functions we express the derivative of the potential in terms of the local quantum nonlinearity parameter $\chi_l(\phi) = (l_-/m^3)\mathcal{A}_0\psi'(\phi)$. With these changes we obtain the following expression

$$
\begin{aligned}
M_R^{\mathrm{LCFA}}(l,l') = {} & (2\pi)^3\delta^2(\boldsymbol{l}_\perp - \boldsymbol{l}'_\perp)\delta(l_- - l'_-)\frac{\alpha}{2\pi}\int d\phi e^{-i(l'_+ - l_+)\phi}\int_0^\infty \frac{d\tilde{u}}{\tilde{u}}\int_0^\infty \frac{dv}{(1+v)^2} \\
& \times e^{-i\tilde{u}\left[(1+v)\frac{\lambda^2}{m^2} + v^2 + v\left(1 - \frac{l'^2}{m^2}\right)\right]}\Bigg\{\left(2m - \frac{\hat{l}'}{1+v}\right)\left[e^{-\frac{i}{3}\tilde{u}^3v^4\chi_l^2(\phi)} - 1\right] \\
& - e^{-\frac{i}{3}\tilde{u}^3v^4\chi_l^2(\phi)}\Bigg[2\frac{\tilde{u}v}{m}\hat{n}\hat{\mathcal{A}}'(\phi) + i\frac{(2+v)\tilde{u}v}{(1+v)m^2}\epsilon^{\tau\mu\nu\rho}\gamma_\tau\gamma^5 l'_\mu n_\nu \mathcal{A}'_\rho(\phi) \\
& + 2\hat{n}\left(1 + \frac{v}{3}\right)\frac{\tilde{u}^2v^2l_-}{m^4}\mathcal{A}'^2(\phi)\Bigg] + \left(2m^2 - \frac{\hat{l}'}{1+v}\right)\left[1 - e^{i\tilde{u}v\left(1 - \frac{l'^2}{m^2}\right)}\right] \\
& - 2i\tilde{u}v\frac{1+2v}{1+v}(\hat{l}' - m)e^{i\tilde{u}v\left(1 - \frac{l'^2}{m^2}\right)}\Bigg\}.
\end{aligned} \tag{2.9.29}
$$

Now we change in the pre-exponent the notation to

$$2\hat{n}\hat{\mathcal{A}}'(\phi) = -i\sigma_{\mu\nu}\mathcal{F}^{\mu\nu}(\phi), \tag{2.9.30}$$

$$l_-\mathcal{A}'^2(\phi)\hat{n} = -(l\mathcal{F}^2(\phi)\gamma), \tag{2.9.31}$$

$$\frac{i}{m^2}\epsilon^{\tau\mu\nu\rho}\gamma_\tau\gamma^5 l'_\mu n_\nu \mathcal{A}'_\rho(\phi) = im\chi_l(\phi)\gamma^5\hat{\zeta}_{l'}, \tag{2.9.32}$$

where we introduced the matrix $\sigma^{\mu\nu} = (i/2)(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$ and used the spin four-vector in Eq. (2.6.14). Notice that the spin four-vector only depends on l'_- and $\boldsymbol{l}'_\perp$, such that here $\zeta^\mu_{l'} = \zeta^\mu_l$. The final expression of the one-loop mass operator in a linear polarized plane wave and within the LCFA is then

$$\begin{aligned} M_R^{\mathrm{LCFA}}(l,l') = &(2\pi)^3\delta^2(\boldsymbol{l}_\perp - \boldsymbol{l}'_\perp)\delta(l_- - l'_-)\frac{\alpha}{2\pi}\int d\phi e^{-i(l'_+ - l_+)\phi}\int_0^\infty \frac{d\tilde{u}}{\tilde{u}}\int_0^\infty \frac{dv}{(1+v)^2} \\ &\times e^{-i\tilde{u}\left[(1+v)\frac{\lambda^2}{m^2}+v^2+v\left(1-\frac{l'^2}{m^2}\right)\right]}\left\{\left(2m - \frac{\hat{l}'}{1+v}\right)\left[e^{-\frac{i}{3}\tilde{u}^3v^4\chi_l^2(\phi)} - 1\right]\right. \\ &+ e^{-\frac{i}{3}\tilde{u}^3v^4\chi_l^2(\phi)}\left[\frac{2\tilde{u}^2v^2}{m^4}\left(1+\frac{v}{3}\right)(l\mathcal{F}^2(\phi)\gamma) + i\frac{\tilde{u}v}{m}\sigma_{\mu\nu}\mathcal{F}^{\mu\nu}(\phi)\right. \\ &\left. - i\tilde{u}vm\frac{2+v}{1+v}\chi_l(\phi)\gamma^5\hat{\zeta}_l\right] + \left(2m^2 - \frac{\hat{l}'}{1+v}\right)\left[1 - e^{i\tilde{u}v\left(1-\frac{l'^2}{m^2}\right)}\right] \\ &\left. - 2i\tilde{u}v\frac{1+2v}{1+v}(\hat{l}'-m)e^{i\tilde{u}v\left(1-\frac{l'^2}{m^2}\right)}\right\}, \end{aligned} \tag{2.9.33}$$

which is in agreement with the one presented in Ref. [48].

2.10. Polarization operator

Similar to the mass operator, the polarization operator describes the self-interaction of the photon from the radiation field inside the background field [3, 11]. The polarization operator leads to the correction of the electron charge and its exact description requires the summation of infinitely many one-particle irreducible Feynman diagrams. Its leading order one-loop Feynman diagram is depicted in Fig. 2.3. For the exact photon state or propagator corrections by the polarization operator have to be included in principle up to an infinite order, however the corrections are usually small and can be neglected in the realm of pertubation theory.

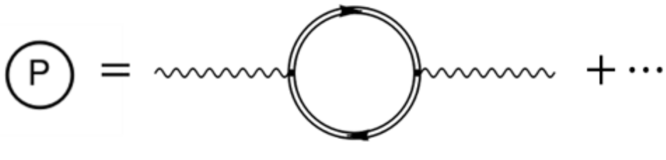

Figure 2.3.: The polarization operator consists of the one-loop polarization operator (Feynman diagram is shown here) and higher order loop diagrams (not presented here).

The expression of the one-loop polarization operator in a plane wave can be found in Refs. [57, 58]. Here the derivation of the polarization operator is not presented, since we will use its expression given in Ref. [58].

2.11. Vertex correction

The vertex correction, as the name indicates already, describes the correction to the vertex and contributes to the anomalous magnetic moment of the electron [3]. Also here the exact description requires the summation of infinitely many one-particle irreducible Feynman diagrams. Its leading order Feynman diagram is depicted in Fig. 2.4. For the exact description of processes including a vertex, the vertex correction has to be included in principle. Since the corrections are usually small, its contribution can be often neglected in the realm of pertubation theory.

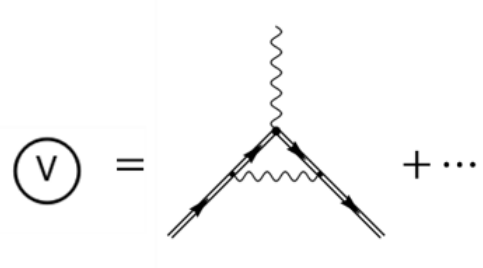

Figure 2.4.: The vertex correction consists of the one-loop vertex correction (Feynman diagram is shown here) and higher order loop diagrams (not presented here).

Since we do not need the expression of the vertex correction in this thesis, its expression is not presented here. However, a derivation of the one-loop vertex correction can be found in Ref. [50].

3. Nonlinear Compton scattering and nonlinear Breit-Wheeler pair production

Note that the content of this chapter was also published in the publication [20] and therefore the structure, the equations, and the text of this chapter are similar or identical to the one in Ref. [20].

Using Volkov-states there are two first order processes in Strong-Field QED, nonlinear Compton scattering and nonlinear Breit-Wheeler pair production [9, 38]. As mentioned nonlinear Compton scattering describes the process when an electron moving in a background field emits a single photon, whereas nonlinear Breit-Wheeler pair production describes the decay of a photon in a background field into an electron-positron pair. Both processes are not possible in vacuum since a second photon is needed for energy-momentum conservation. In a background field however energy-momentum conservation is achieved by the interaction of the electron/positron with the background field. The corresponding Feynman diagrams to this two processes are shown in Fig. 3.1, on the left hand side for nonlinear Compton scattering and on the right hand side for nonlinear Breit-Wheeler pair production.

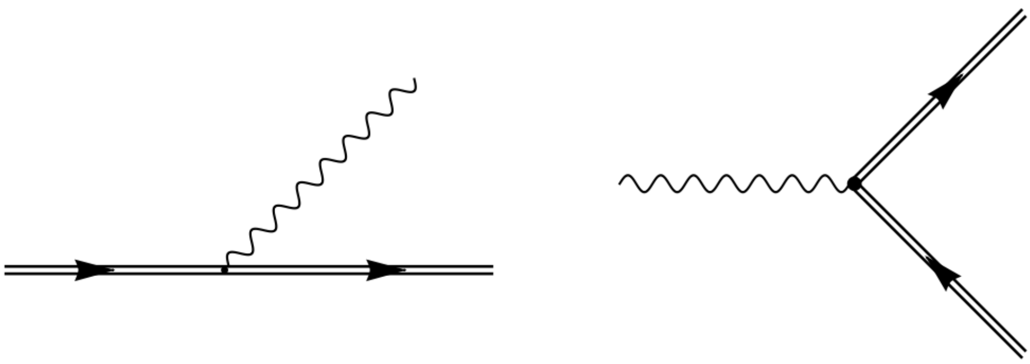

Figure 3.1.: Shown are the leading Feynman diagrams of nonlinear Compton scattering (left) and nonlinear Breit-Wheeler pair production (right).

The S-matrix elements are given by [9]

$$S_0^{\mathrm{NCS}} = -ie \int d^4x \bar{\psi}^{V,(\mathrm{out})}_{e^-,s',p'}(x) \hat{A}^{\mathrm{rad},*}_{j,q}(x) \psi^{V,(\mathrm{in})}_{e^-,s,p}(x) \tag{3.0.1}$$

for nonlinear Compton scattering with an incoming (outgoing) electron of momentum p^μ (p'^μ) and spin quantum number s (s') and with an outgoing photon of momentum q^μ and polarization j, and for nonlinear Breit-Wheeler pair production by

$$S_0^{\mathrm{NBW}} = -ie \int d^4x \bar{\psi}^{V,(\mathrm{out})}_{e^-,s',p'}(x) \hat{A}^{\mathrm{rad}}_{j,q}(x) \psi^{V,(\mathrm{out})}_{e^+,s,p}(x) \tag{3.0.2}$$

with an incoming photon of momentum q^μ and polarization j, and an outgoing positron (electron) of momentum p^μ (p'^μ) and spin quantum number s (s'). From this two S-matrices one obtains the leading order probabilities for both processes (see e.g. Refs. [9, 38, 40]).

3.1. Wrong probabilities for a large pulse length and higher order loop corrections

The leading order probabilities for nonlinear Compton scattering and nonlinear Breit-Wheeler pair production exceed unity for sufficiently large pulse length (intensities) of the background field [41, 42], which is of course unphysical and violates against the unitarity of the S-matrix. Indeed the reason behind this behavior is that pertubation theory, as it was used to calculate these probabilities, is not applicable anymore and higher order corrections have to be taken into account to obtain correct results. These corrections are in general given by the exact mass operator, polarization operator and vertex correction [11, 46]. To obtain an exact result in principle diagrams including an arbitrary number of mass- and polarization operators and combinations with the vertex correction have to be resummed. The summation of Feynman diagrams with corrections up to order α is depicted for nonlinear Compton scattering in Fig. 3.2. However, already the exact expressions of the complete mass operator, polarization operator, and vertex correction are not known. In order to get analytical results we therefore have to understand why the higher order corrections become important in the limit of large pulse length and which corrections contribute to the leading amplitude.

In order to do so we will have a brief look at a higher order strong field QED process, namely nonlinear double Compton scattering. Here the electron is emitting two photons by two following nonlinear Compton scattering processes. This process was investigated by several groups (see e.g. Refs. [59–63]) and they found out that there are two contributions to nonlinear double Compton scattering, the one-step and the two-step contribution. In the one-step contribution the intermediate electron in between the two photon emissions stays off-shell and the phases of the two photon emissions are correlated, whereas in the two-step contribution the intermediate electron also can go on-shell. Here the intermediate on-shell particles in the two-step contribution can travel macroscopic large distances and the phases of the two photon emissions are uncorrelated. Also the probabilities of both contributions scale differently. The probability of the one-step contribution scales with $\alpha^2\Phi_L/\Phi_f$ and the two-step contribution scales with $\alpha^2\Phi_L^2/\Phi_f^2$, where Φ_L is the total phase duration of the background field and Φ_f is a measure of the formation length. Hence, for a sufficiently large phase duration $\Phi_L \gg \Phi_f$ the two-step contribution dominates over the one-step contribution and when the phase duration is in the order of $\Phi_L \gtrsim \Phi_f/\alpha \approx 137\Phi_f$ the probability of the two-step contribution becomes comparable to the one of a first-order process (this condition was also identified in Ref. [41]).

Now we go back to single nonlinear Compton scattering. For this, as mentioned, in Fig. 3.2 additionally to the leading order diagram also the corrections up to linear order in α are depicted. Having a look on the corrections by the mass operator on the incoming and outgoing electron (second and third diagram in Fig. 3.2) we see that we have an intermediate electron (between mass operator and vertex) which principally can go on-shell and receives then a contribution from the plane wave that scales with Φ_L/Φ_f. For a sufficiently large pulse duration ($\Phi_L \gtrsim \Phi_f/\alpha \approx 137\Phi_f$) this contribution becomes in the order of $1/\alpha$ such that the whole diagram scales like the leading order contribution (the first Feynman diagram in Fig. 3.2). The same holds for the correction by the polarization operator to the outgoing photon. Here we have an intermediate photon (between polarization operator and vertex) which can go on-shell and receives a contribution from the plane wave. Also higher order contributions with several mass- and polarization operator corrections contribute to the leading order amplitude since for every mass- and polarization operator there is also an additional intermediate fermion and photon, respectively, that receives a contribution from the plane wave when going on-shell.

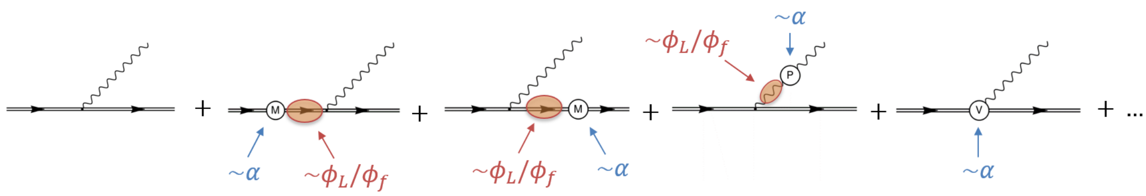

Figure 3.2.: Shown is the sum of Feynman diagrams contributing to nonlinear Compton scattering with corrections up to linear order in α (blue) by the mass operator (M), polarization operator (P), and vertex correction (V). Intermediate particles (orange circles) can go on-shell and receive a contribution from the plane wave scaling with Φ_L/Φ_f (red).

For the vertex correction the situation is however different, since here all intermediate particles are in a loop (in the vertex correction itself). If two particles go on-shell in the same loop the above discussed contributions from the plane wave do not occur, which was investigated in Ref. [64] in the case of the one-loop polarization operator. Therefore we expect that also the vertex correction does not receive a contribution from the plane wave. This makes also physically sense since the vertex correction is a local physical process whereas intermediate on-shell particles can travel macroscopic large distances. Hence contributions from the vertex correction only scale with α and can be neglected when calculating the leading contributions to the probability in the limit of large phase duration.

Now, for the same reason as for the vertex correction, we only have to consider for the mass- and polarization operators the one-loop diagram contributions, because higher loop corrections would be at least scale with α^2 and do not contain intermediate particles outside a loop. Therefore they are sub-leading in α and it is enough to consider the one-loop mass- and polarization operator for the leading order probability. But, as discussed before, we still have to take all higher order corrections into account where we have multiple corrections by one-loop mass- and polarization operators (including combinations of both) since for every mass- and polarization operator we have also an intermediate particle receiving contributions from the background field.

In conclusion, we have to compute the resummation of all one-particle reducible diagrams containing only corrections by the one-loop mass- and polarization operator depicted in Figs. 4.1 and 5.1 for nonlinear Compton scattering and nonlinear Breit-Wheeler pair production, respectively.

In order to perform this resummation we use instead of the Volcov electron/positron states and the vacuum photon state in the S-matrix elements in Eqs. (3.0.1) and (3.0.2), the exact electron and positron state $\Psi(x)$, and exact photon state $\mathscr{A}^\mu(x)$. These states are in general for a plane wave background field obtained by solving the following equations [11,13]

$$[i\hat{\partial} - \hat{\mathcal{A}}(\phi) - m]\Psi(x) = \int d^4y M_{PW}(x,y)\Psi(y), \tag{3.1.1}$$

$$-\partial^2 \mathscr{A}^\mu(x) = \int d^4y P^{\mu\nu}_{PW}(x,y)\mathscr{A}_\nu(y), \tag{3.1.2}$$

where $M_{PW}(x,y)$ and $P^{\mu\nu}_{PW}(x,y)$ are the mass and polarization operator in a plane wave, respectively. Instead of only considering the propagation of the electron/positron (photon) through a plane wave field like in the modified Dirac equation (free wave equation), Eq. (3.1.1) (Eq. (3.1.2)) takes also the self-interaction of the particle by the mass (polarization) operator into account and hence describes the exact electron and positron (photon) state. The solution of this equation intrinsically contains the resummation of all one-particle reducible corrections by the mass (polarization) operator to the electron/positron (photon) state [11], which will be showed in the next Section in the case of the electron state. Since this properties are similar to Schwinger-Dyson equations which describe exact Greens-functions [3], we will name in the following the above equations Schwinger-Dyson equations of the fermion and photon. For an exact description of the electron, positron and photon again the exact expression of the mass- and polarization operator would be needed, however as discussed above it is in our case sufficient to consider only the one-loop contributions.

In that way the resummation of the Feynman diagrams for nonlinear Compton scattering and nonlinear Breit-Wheeler pair production can be absorbed into the expressions of the exact electron, positron, and photon states. The solutions of the Schwinger-Dyson equations (3.1.1) and (3.1.2) in the LCFA and for the one-loop mass and polarization operator, respectively, are already known and can be found in Refs. [48,64,65]. However, we will see that both equations are only suitable for electron and photon in-states, respectively. In the following two Sections for completeness a derivation for the exact electron, positron, and photon in- and out-states is presented.

3.2. Exact electron and positron states

We start with the Schwinger-Dyson equation (3.1.1) for a fermion inside a plane wave. Before solving this equation, we observe that with the help of the Volkov propagator $G^V(x,y)$ and the Volkov state $\psi^V(x)$ the solution of the Schwinger-Dyson equation can be written as the series [48]

$$\begin{aligned}\Psi(x) = \psi^V(x) &+ \int d^4y d^4z\, G^V(x,y) M_{PW}(y,z)\psi^V(z) \\ &+ \int d^4y d^4z d^4r d^4s\, G^V(x,y) M_{PW}(y,z) G^V(z,r) M_{PW}(r,s)\psi^V(s) + \cdots ,\end{aligned} \tag{3.2.1}$$

which can easily be proved by plugging the series into the Schwinger-Dyson equation and using Eqs. (2.6.1) and (2.6.12) (This expansion is similar to the expansion of the exact Green's function, which is a solution of the Schwinger-Dyson equation for the Green's function, see e.g.

Ref. [11]). Interestingly, the solution of this equation precisely gives us the resummation of all one-particle reducible diagrams of the fermion state with corrections by the mass operator (see Fig. 3.3), which is exactly what we need to perform a part of the resummation for the probabilities of nonlinear Compton scattering and nonlinear Breit-Wheeler pair production.

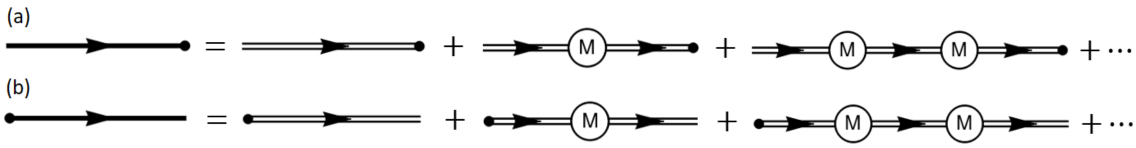

Figure 3.3.: The thick lines indicate the exact incoming electron line (a) and the exact outgoing electron line (b), and are symbolically expressed as series expansion in terms of Volkov propagators (double internal lines) and mass operators (circles with letter M inside).

Now we imagine to use this solution for the computation of S-matrix elements and we notice that it is only suitable for describing an electron in-state $\Psi_{e^-}^{(\mathrm{in})}(x)$ (see Fig. 3.3(a)), since in a S-matrix element the electron in-state acts from the right hand side to the amplitude. We therefore rewrite

$$\begin{aligned}\Psi_{e^-}^{(\mathrm{in})}(x) = \psi_{e^-}^{V,(\mathrm{in})}(x) &+ \int d^4y d^4z\, G_V(x,y) M_{PW}(y,z)\psi_{e^-}^{V,(\mathrm{in})}(z) \\ &+ \int d^4y d^4z d^4r d^4s\, G_V(x,y) M_{PW}(y,z) G_V(z,r) M_{PW}(r,s)\psi_{e^-}^{V,(\mathrm{in})}(s) + \cdots\end{aligned} \tag{3.2.2}$$

and we rewrite Eq. (3.1.1) to

$$[i\hat{\partial} - \hat{\mathcal{A}}(\phi) - m]\Psi_{e^-}^{(\mathrm{in})}(x) = \int d^4y\, M_{PW}(x,y)\Psi_{e^-}^{(\mathrm{in})}(y). \tag{3.2.3}$$

An exact electron out-state $\Psi_{e^-}^{(\mathrm{out})}(x)$ instead requires the series (see Fig. 3.3(b))

$$\begin{aligned}\bar{\Psi}_{e^-}^{(\mathrm{out})}(x) = \bar{\psi}_{e^-}^{V,(\mathrm{out})}(x) &+ \int d^4y d^4z\, \bar{\psi}_{e^-}^{V,(\mathrm{out})}(z) M_{PW}(z,y) G_V(y,x) \\ &+ \int d^4y d^4z d^4r d^4s\, \bar{\psi}_{e^-}^{V,(\mathrm{out})}(s) M_{PW}(s,r) G_V(r,z) M_{PW}(z,y) G_V(y,x) + \cdots,\end{aligned} \tag{3.2.4}$$

since in a S-matrix element the electron out-state acts from the left hand side to the amplitude. This is not simply the Dirac conjugated of Eq. (3.2.2) and we rather need to solve for the exact electron out-state $\Psi_{e^-}^{(\mathrm{out})}(x)$ the Schwinger-Dyson equation

$$\bar{\Psi}_{e^-}^{(\mathrm{out})}(x)[-i\overleftarrow{\partial}_\mu \gamma^\mu - \hat{\mathcal{A}}(\phi) - m] = \int d^4y\, \bar{\Psi}_{e^-}^{(\mathrm{out})}(y) M_{PW}(y,x). \tag{3.2.5}$$

or, equivalently, the equation

$$[i\hat{\partial} - \hat{\mathcal{A}}(\phi) - m]\Psi_{e^-}^{(\mathrm{out})}(x) = \int d^4y\, \bar{M}_{PW}(y,x)\Psi_{e^-}^{(\mathrm{out})}(y). \tag{3.2.6}$$

With similar considerations we derive the Schwinger-Dyson equations for the incoming and outgoing positron states, i.e. $\Psi^{(\text{in})}_{e^+}(x)$ and $\Psi^{(\text{out})}_{e^+}(x)$, respectively, which are

$$[i\hat{\partial} - \hat{\mathcal{A}}(\phi) - m]\Psi^{(\text{in})}_{e^+}(x) = \int d^4y\, \bar{M}_{PW}(y,x)\Psi^{(\text{in})}_{e^+}(y), \tag{3.2.7}$$

$$[i\hat{\partial} - \hat{\mathcal{A}}(\phi) - m]\Psi^{(\text{out})}_{e^+}(x) = \int d^4y\, M_{PW}(x,y)\Psi^{(\text{out})}_{e^+}(y). \tag{3.2.8}$$

In the following, only the derivation of the solution of Eq. (3.2.3) is presented as the other equations (3.2.6)-(3.2.8) can be solved in an analogous way. The solution of Eq. (3.2.3) was already found in Refs. [48, 65] and in the following an equivalent but alternative solution is presented.

To find a solution of Eq. (3.2.3) we first simplify the expression of the mass operator. For this we assume that the plane wave background field is linearly polarized, the classical nonlinearity parameter $\xi_0 \gg 1$, and the quantum nonlinearity parameter $\chi_0 \sim 1$, such that the LCFA can be applied. Also, as explained in the introduction of this Chapter, it is enough to consider the one-loop expression of the mass operator since higher order loop terms would only give subleading contributions. The derivation of the one-loop mass operator in the LCFA was presented in Section (2.9.3) and we use its expression given in Eq. (2.9.33) (see also Ref. [48]),

$$\begin{aligned}
M_R^{\text{LCFA}}(l,l') = {} & (2\pi)^3\delta^2(\boldsymbol{l}_\perp - \boldsymbol{l}'_\perp)\delta(l_- - l'_-)\int d\phi\, e^{-i(l'_+ - l_+)\phi}\frac{\alpha}{2\pi}\int_0^\infty \frac{d\tilde{u}}{\tilde{u}}\int_0^\infty \frac{dv}{(1+v)^2} \\
& \times e^{-i\tilde{u}\left[(1+v)\frac{\lambda^2}{m^2}+v^2+v\left(1-\frac{l'^2}{m^2}\right)\right]}\left\{\left(2m - \frac{\hat{l}'}{1+v}\right)\left[e^{-\frac{i}{3}v^4\chi_l^2(\phi)\tilde{u}^3} - 1\right] + e^{-\frac{i}{3}v^4\chi_l^2(\phi)\tilde{u}^3}\right. \\
& \times \left[\frac{2\tilde{u}^2v^2}{m^4}\left(1+\frac{v}{3}\right)(l\mathcal{F}^2(\phi)\gamma) + i\frac{\tilde{u}v}{m}\sigma_{\mu\nu}\mathcal{F}^{\mu\nu}(\phi) - im\tilde{u}v\frac{2+v}{1+v}\chi_l(\phi)\gamma^5\hat{\zeta}_l\right] \\
& \left. + \left(2m - \frac{\hat{l}'}{1+v}\right)\left[1 - e^{i\tilde{u}v\left(1-\frac{l'^2}{m^2}\right)}\right] - 2i\tilde{u}v\frac{1+2v}{1+v}(\hat{l}' - m)e^{i\tilde{u}v\left(1-\frac{l'^2}{m^2}\right)}\right\}.
\end{aligned} \tag{3.2.9}$$

By using Eqs. (2.6.8) and (2.6.9) we go to configuration space and obtain for the one-loop mass operator

$$M_R^{\text{LCFA}}(x,y) = \int \frac{d^4l}{(2\pi)^4}\frac{d^4l'}{(2\pi)^4}E(l,x)M_R^{\text{LCFA}}(l,l')\bar{E}(l',y) = \int \frac{d^4l'}{(2\pi)^4}E(l',x)M_R^{(1)}(l',\phi_x)\bar{E}(l',y), \tag{3.2.10}$$

where

$$\begin{aligned}
M_R^{(1)}(l',\phi) = {} & \frac{\alpha}{2\pi}\int_0^\infty \frac{d\tilde{u}}{\tilde{u}}\int_0^\infty \frac{dv}{(1+v)^2}e^{-i\tilde{u}\left[(1+v)\frac{\lambda^2}{m^2}+v^2+v\left(1-\frac{l'^2}{m^2}\right)\right]} \\
& \times \left\{\left(2m - \frac{\hat{l}'}{1+v}\right)\left[e^{-\frac{i}{3}v^4\chi_{l'}^2(\phi)\tilde{u}^3} - 1\right] + e^{-\frac{i}{3}v^4\chi_{l'}^2(\phi)\tilde{u}^3}\right. \\
& \times \left[\frac{2\tilde{u}^2v^2}{m^4}\left(1+\frac{v}{3}\right)(l'\mathcal{F}^2(\phi)\gamma) + i\frac{\tilde{u}v}{m}\sigma_{\mu\nu}\mathcal{F}^{\mu\nu}(\phi) - im\tilde{u}v\frac{2+v}{1+v}\chi_{l'}(\phi)\gamma^5\hat{\zeta}_{l'}\right] \\
& \left. + \left(2m - \frac{\hat{l}'}{1+v}\right)\left[1 - e^{i\tilde{u}v\left(1-\frac{l'^2}{m^2}\right)}\right] - 2i\tilde{u}v\frac{1+2v}{1+v}(\hat{l}' - m)e^{i\tilde{u}v\left(1-\frac{l'^2}{m^2}\right)}\right\},
\end{aligned} \tag{3.2.11}$$

and where ϕ_x is the minus light-cone coordinate of the spacetime point x.

To arrive to the second equality in Eq. (3.2.10) the integral in d^4l was taken. This is trivial for the components $d^2\boldsymbol{l}_\perp$ and dl_- which exploit the δ-functions in Eq. (3.2.9). For the integral in dl_+ we made the observation, that except of the first exponential all the dependence on l^μ in Eq. (3.2.9) is only in the components $\boldsymbol{l}_\perp$ and l_-. Hence the integral in dl_+ leads to the δ-function $\delta(\phi-\phi_x)$, which is removed by the integral in $d\phi$, and the remaining exponential $\exp(-il'_+\phi_x)$ is absorbed into the left Ritus matrix in the second equality of Eq. (3.2.10). Now it might surprise that $M^{(1)}_R(l,\phi_x)$ evaluated at ϕ_x looks "asymmetric". This only arises because we have decided to express the mass operator in terms of l'^2 and $\hat{l}'$ by starting in the derivation of Eq. (3.2.9) in Section 2.9.3 from the mass operator expression depending on the incoming electron momentum given in Eq. (2.9.14). If we would have started the derivation from the mass operator depending on the outgoing electron momentum, given in Eq. (2.9.10), we would have obtained an equivalent expression to Eq. (3.2.9) depending on l^2 and $\hat{l}$ instead and the quantity $M^{(1)}_R(l,\phi_y)$ would have appeared in the second equality of Eq. (3.2.10), with ϕ_y being the minus light-cone coordinate of the spacetime point y. In the following derivation the expression with $M^{(1)}_R(l',\phi_x)$ has been chosen.

Now we want to find a solution to the Schwinger-Dyson equation for the electron in state in Eq. (3.2.3). We observe that the left hand side of the Schwinger-Dyson equation is essentially the Dirac equation in a plane wave field which is then modified by the mass operator on the right hand side of the equation, leading to radiative corrections of the Volkov state. Similarly as in Ref. [65], thus we make the Ansatz that the solution of the Schwinger-Dyson equation can be obtained by a modification of the Volkov electron in state $\psi^{V,(\text{in})}_{s,p}(x)$, i.e.

$$\Psi^{(\text{in})}_{e^-,s,p}(x) = f^{(\text{in})}_s(p,\phi)\psi^{V,(\text{in})}_{s,p}(x) = e^{i\Phi^{(\text{in})}(p)} f^{(\text{in})}_s(p,\phi)E(p,x)\frac{u_s(p)}{\sqrt{2\varepsilon}}, \tag{3.2.12}$$

where $f^{(\text{in})}_s(p,\phi)$ is a function to be determined. According to the physical requirement that the electron in state coincides with the free state $\exp(-i(px))u_s(p)/\sqrt{2\varepsilon}$ before the electron interacts with the plane-wave field, the function $f^{(\text{in})}_s(p,\phi)$ has to fulfill the initial condition $\lim_{\phi\to-\infty} f^{(\text{in})}_s(p,\phi) = 1$. By substituting the expression of $\Psi^{(\text{in})}_e(x)$ into Eq. (3.2.3), applying the relation in Eq. (2.6.10), multiplying the resulting expression by $\bar{u}_{s'}(p)\bar{E}(p,x)$, and using Eq. (2.6.3), we obtain

$$\begin{aligned} &2ip_-\delta_{ss'}\frac{df^{(\text{in})}_s(p,\phi_x)}{d\phi_x} \\ &\quad= \int d^4y\frac{d^4l'}{(2\pi)^4}\bar{u}_{s'}(p)\bar{E}(p,x)E(l',x)M^{(1)}_R(l',\phi_x)\bar{E}(l',y)E(p,y)u_s(p)f^{(\text{in})}_s(p,\phi_y). \end{aligned} \tag{3.2.13}$$

The number of integrals on the right hand side can further be reduced. As the Ritus matrices $E(p,y)$ depend only linearly in the phase on the transverse and the plus light-cone spacetime coordinates, the integrals in $d^2\boldsymbol{y}_\perp$ and dy_+ reduce to δ-functions. These are employed to perform the integrals in $d^2\boldsymbol{l}'_\perp$ and dl'_-, enforcing the replacement $\boldsymbol{l}'_\perp = \boldsymbol{p}_\perp$ and $l'_- = p_-$. With that we obtain

$$2ip_-\delta_{ss'}\frac{df^{(\text{in})}_s(p,\phi_x)}{d\phi_x} = \int d\phi_y\frac{dl'_+}{2\pi}e^{i(p_+-l'_+)(\phi_x-\phi_y)}\bar{u}_{s'}(p)M^{(1)}_R(l',\phi_x)u_s(p)f^{(\text{in})}_s(p,\phi_y). \tag{3.2.14}$$

where now for notational simplicity the four-momentum l'^{μ} has light-cone components p_-, $\boldsymbol{p}_\perp$, and l'_+. For the component l'_+ in the quantity $M_R^{(1)}(l',\phi_x)$ we notice, that it appears only linearly in the exponents and the preexponential functions in Eq. (3.2.11) via the squared four-momentum $l'^2 = 2p_-l'_+ - \boldsymbol{p}_\perp^2$ and the matrix $\hat{l}' = l'_+\hat{n} + p_-\hat{\tilde{n}} - (pa_1)\hat{a}_1 - (pa_2)\hat{a}_2$, respectively. This observation is important as it allows us to perform the integral in l'_+ analytically. For this we first employ the substitution $r_+ = l'_+ - p_+$, which leads to the replacements $l'^2 = p^2 + 2p_-r_+$ and $\hat{l}' = \hat{p} + r_+\hat{n}$. The integral over r_+ is still complicated but it can be solved within our approximations. There are three kinds on how terms of $M_R^{(1)}(l',\phi_x)$ depend on l'_+ and then on r_+ and they can be integrated in the following way [see also Eq. (3.2.11)]:

1. Terms in the quantity $M_R^{(1)}(l',\phi_x)$ independent on r_+ can be directly integrated and yield the δ-function $\delta(\phi_y - \phi_x)$. With the integral in ϕ_y this reduces to the function $f_s^{(\mathrm{in})}(p,\phi_x)$ on the right hand side of Eq. (3.2.14).

2. Terms where r_+ only appears in the exponential function $\exp(2i\tilde{u}vp_-r_+/m^2)$ can be again directly integrated and give the δ-function $\delta(\phi_y - \phi_x + 2\tilde{u}vp_-/m^2)$. With that the integral in ϕ_y can be performed such that the function $f_s^{(\mathrm{in})}(p,\phi_x - 2\tilde{u}vp_-/m^2)$ appears on the right hand side of Eq. (3.2.14). Now, as our considerations are only valid within the LCFA and since the function $f_s^{(\mathrm{in})}(p,\phi_x - 2\tilde{u}vp_-/m^2)$ can be *a posteriori* ascertained to be sufficiently smooth, we can approximate that $f_s^{(\mathrm{in})}(p,\phi_x - 2\tilde{u}vp_-/m^2) \approx f_s^{(\mathrm{in})}(p,\phi_x)$ (see also below Eq. (3.2.20)).

3. Terms containing r_+ in the preexponential can be rewritten into a derivative in $\partial/\partial\phi_y$ of the exponential function in Eq. (3.2.14) and then, after partial integration, result to the derivative $\partial f_s(p,\phi_y)/\partial\phi_y$ appearing on the right-hand side of Eq. (3.2.14). The integral in r_+ and ϕ_y can be taken as explained in the first and second remark where now instead the δ-functions lead finally to $\partial f_s(p,\phi_x)/\partial\phi_x$. Apart from being proportional to α, this contributions have the same structure as the left hand side and we can combine both together. After dividing the resulting equation by the overall factor of $\partial f_s(p,\phi_x)/\partial\phi_x$ we observe that these additional terms only lead to higher order corrections proportional to α and hence we can neglect them within our approximations.

After these steps we obtain the equation

$$ip_-\delta_{ss'}\frac{df_s^{(\mathrm{in})}(p,\phi)}{d\phi} = mM_{ss'}(p,\phi)f_s^{(\mathrm{in})}(p,\phi), \tag{3.2.15}$$

where $M_{ss'}(p,\phi) = \bar{u}_{s'}(p)M(p,\phi)u_s(p)/\bar{u}_s(p)u_s(p)$, with

$$\begin{aligned} M(p,\phi) = \frac{\alpha}{2\pi}\int_0^\infty \frac{d\tilde{u}}{\tilde{u}}\int_0^\infty \frac{dv}{(1+v)^2}e^{-iv^2\tilde{u}}\Bigg\{\left(2m - \frac{m}{1+v}\right)\left[e^{-\frac{i}{3}v^4\chi_p^2(\phi)\tilde{u}^3} - 1\right] + e^{-\frac{i}{3}v^4\chi_p^2(\phi)\tilde{u}^3} \\ \times\left[\frac{2\tilde{u}^2v^2}{m^4}\left(1+\frac{v}{3}\right)(p\mathcal{F}^2(\phi)\gamma) + i\frac{\tilde{u}v}{m}\sigma_{\mu\nu}\mathcal{F}^{\mu\nu}(\phi) - im\tilde{u}v\frac{2+v}{1+v}\chi_p(\phi)\gamma^5\hat{\zeta}_p\right]\Bigg\}. \end{aligned} \tag{3.2.16}$$

Here the square of the fictitious photon mass λ was finally set equal to zero, because we use the mass operator on the mass shell [3, 9, 46]. Further we observe that the quantity $M(p,\phi)$

vanishes if the plane wave vanishes since it does not contain vacuum terms. This behavior was expected, as on-shell states do not undergo radiative corrections in vacuum [11].

Eq. (3.2.15) is simplified further using the properties of the free states $u_s(p)$ in Eqs. (2.6.2), (2.6.3), and (2.6.15). Additionally, with Eq. (A.0.14), the commutator $[(\gamma f_2 \gamma), \hat{p}] = 4(\gamma f_2 p)$ and by multiplying ones in the form of $\hat{p}^2/m^2$ and $(\gamma^5)^2$, one obtains the replacement $\sigma_{\mu\nu}\mathcal{F}^{\mu\nu} \to 2m^2\chi_p(\phi)\gamma^5\hat{\zeta}_p$. In this way the matrix $M_{ss'}(p,\phi)$ becomes diagonal and Eq. (3.2.15) reduces to

$$i\frac{df_s^{(\text{in})}(p,\phi)}{d\phi} = \frac{m}{p_-}M_s(p,\phi)f_s^{(\text{in})}(p,\phi), \tag{3.2.17}$$

where

$$\begin{aligned} M_s(p,\phi) = m\frac{\alpha}{2\pi}\int_0^\infty \frac{d\tilde{u}}{\tilde{u}}\int_0^\infty \frac{dv}{(1+v)^2}e^{-iv^2\tilde{u}}\left\{\frac{1+2v}{1+v}\left[e^{-\frac{i}{3}v^4\chi_p^2(\phi)\tilde{u}^3}-1\right]\right. \\ \left. +e^{-\frac{i}{3}v^4\chi_p^2(\phi)\tilde{u}^3}\left[2\tilde{u}^2v^2\left(1+\frac{v}{3}\right)\chi_p^2(\phi)+is\frac{\tilde{u}v^2}{1+v}\chi_p(\phi)\right]\right\}. \end{aligned} \tag{3.2.18}$$

After employing the substitution $u = \tilde{u}v^2$ and using the relation in Eq. (B.0.18) the quantity $M_s(p,\phi)$ reduces to

$$M_s(p,\phi) = m\frac{\alpha}{2\pi}\int_0^\infty du\int_0^\infty \frac{dv}{(1+v)^3}e^{-iu\left[1+\frac{1}{3}\frac{\chi_p^2(\phi)}{v^2}u^2\right]}\left[\frac{5+7v+5v^2}{3}\frac{\chi_p^2(\phi)}{v^2}u+is\chi_p(\phi)\right], \tag{3.2.19}$$

which turns out to exactly coincide with the spin-dependent mass correction in a constant crossed field, with the replacement $\chi_0 \to \chi_p(\phi)$ [48, 65–67].

At this point the differential equation (3.2.17) can be integrated. Taking the initial condition $\lim_{\phi\to-\infty} f_s^{(\text{in})}(p,\phi) = 1$ into account we obtain for the radiatively corrected Volkov electron in-state $\Psi^{R,(\text{in})}_{e^-,s,p}(x)$ the expression

$$\begin{aligned} \Psi^{R,(\text{in})}_{e^-,s,p}(x) &= e^{i\Phi^{(\text{in})}(p)}e^{-i\frac{m}{p_-}\int_{-\infty}^{\phi}d\varphi M_s(p,\varphi)}E(p,x)\frac{u_s(p)}{\sqrt{2\varepsilon}} \\ &= \left[1+\frac{\hat{n}\hat{\mathcal{A}}(\phi)}{2p_-}\right]e^{i\left\{-(px)-\int_{-\infty}^{\phi}d\varphi\left[\frac{(p\mathcal{A}(\varphi))}{p_-}-\frac{\mathcal{A}^2(\varphi)}{2p_-}+\frac{m}{p_-}M_s(p,\varphi)\right]\right\}}\frac{u_s(p)}{\sqrt{2\varepsilon}}. \end{aligned} \tag{3.2.20}$$

Apparently new in comparison to the ordinary Volkov electron in-state is the exponential function, which depends on the mass operator, as it can be seen especially in the first equality. For the imaginary part of the mass operator this term features an exponential damping of the state, which can be understood as the radiatively corrected state decays in the plane wave (see discussion below Eq. (3.2.23)).

This expression also coincides with the results found in Ref. [65] after applying the LCFA to the latter one. Note that the preexponential function in Ref. [65] contains an additional term proportional to $\hat{n}\hat{\mathcal{A}}(\phi)$ but scaling with $1/\xi_0$ (see also Ref. [48]). This term can be ignored within the LCFA since $\xi_0 \gg 1$. Further, regarding point two below Eq. (3.2.14) we observe, that a shift of ϕ proportional to p_-/m^2 in the integral over the quantity $M_s(p,\varphi)$ in the exponent would give a correction to the preexponential in the order of α that can be neglected within our approximations.

Now, for the positron in-state the Schwinger-Dyson equation (3.2.7) can be solved analogously. The corresponding radiatively-corrected Volkov positron in-state again includes a decaying exponential function and is given by

$$\Psi^{R,(\mathrm{in})}_{e^+,s,p}(x) = \left[1 - \frac{\hat{n}\hat{\mathcal{A}}(\phi)}{2p_-}\right] e^{i\left\{(px) - \int_{-\infty}^{\phi} d\varphi \left[\frac{(p\mathcal{A}(\varphi))}{p_-} + \frac{\mathcal{A}^2(\varphi)}{2p_-} - \frac{m}{p_-} M_s^*(-p,\varphi)\right]\right\}} \frac{v_s(p)}{\sqrt{2\varepsilon}}. \tag{3.2.21}$$

At this point we have a look onto the normalization of the radiatively-corrected electron and positron in-states

$$\frac{\bar{\Psi}^{R,(\mathrm{in})}_{e^-,s,p}(x)\Psi^{R,(\mathrm{in})}_{e^-,s,p}(x)}{\bar{\psi}^{\mathrm{free}}_{e^-,s,p}(x)\psi^{\mathrm{free}}_{e^-,s,p}(x)} = e^{\frac{2m}{p_-}\int_{-\infty}^{\phi} d\varphi\, \mathrm{Im}[M_s(p,\varphi)]}, \tag{3.2.22}$$

$$\frac{\bar{\Psi}^{R,(\mathrm{in})}_{e^+,s,p}(x)\Psi^{R,(\mathrm{in})}_{e^-,s,p}(x)}{\bar{\psi}^{\mathrm{free}}_{e^+,s,p}(x)\psi^{\mathrm{free}}_{e^+,s,p}(x)} = e^{\frac{2m}{p_-}\int_{-\infty}^{\phi} d\varphi\, \mathrm{Im}[M_s(-p,\varphi)]}. \tag{3.2.23}$$

Apparently both depend on the imaginary part of the mass operator, i.e. $-(2m/p_-)\mathrm{Im}[M_s(\pm p, \phi)]$. According to the optical theorem this quantity precisely corresponds to the total probability per unit of light-cone time ϕ that an electron/positron with initial four-momentum p^μ and spin quantum number s emits a photon [68]. Therefore, the additional corrections by the mass operator in the radiatively corrected states describe the fact, that Volkov electron and positron states are not stable inside a plane wave background field but "decay" in the sense that electrons and positrons emit photons.

Finally the radiatively-corrected Volkov electron and positron out-states can be obtained similarly from the Schwinger-Dyson equations (3.2.6) and (3.2.8):

$$\Psi^{R,(\mathrm{out})}_{e^-,s,p}(x) = \left[1 + \frac{\hat{n}\hat{\mathcal{A}}(\phi)}{2p_-}\right] e^{i\left\{-(px) + \int_{\phi}^{\infty} d\varphi \left[\frac{(p\mathcal{A}(\varphi))}{p_-} - \frac{\mathcal{A}^2(\varphi)}{2p_-} + \frac{m}{p_-} M_s^*(p,\varphi)\right]\right\}} \frac{u_s(p)}{\sqrt{2\varepsilon}}, \tag{3.2.24}$$

$$\Psi^{R,(\mathrm{out})}_{e^+,s,p}(x) = \left[1 - \frac{\hat{n}\hat{\mathcal{A}}(\phi)}{2p_-}\right] e^{i\left\{(px) + \int_{\phi}^{\infty} d\varphi \left[\frac{(p\mathcal{A}(\varphi))}{p_-} + \frac{\mathcal{A}^2(\varphi)}{2p_-} - \frac{m}{p_-} M_s(-p,\varphi)\right]\right\}} \frac{v_s(p)}{\sqrt{2\varepsilon}}. \tag{3.2.25}$$

In this case in the new exponential decaying function the mass operator is integrated from ϕ to ∞ as it is expected for outgoing particles.

3.3. Exact photon states

Now we move over to the Schwinger-Dyson equation (3.1.2) for the radiation field, which has been already solved within the LCFA in Refs. [48,64]. Again with the free photon state of the radiation field and the photon propagator (see Section 2.5) the solution can be written as a series representing the resummation of all one-particle reducible diagrams featuring corrections by the polarization operator (see Fig. 3.4 and Ref. [48]).

However, similar to the previous subsection the Schwinger-Dyson equation (3.1.2) only describes the photon in-states $\mathscr{A}^{(\mathrm{in})}_\nu(x)$ correctly (see Fig. 3.4(a)). We therefore rewrite it for a plane wave background field to

$$-\partial_\mu\partial^\mu \mathscr{A}^{(\mathrm{in})}_\nu(x) = \int d^4y\, P_{PW,\nu}^{\ \ \lambda}(x,y)\mathscr{A}^{(\mathrm{in})}_\lambda(y). \tag{3.3.1}$$

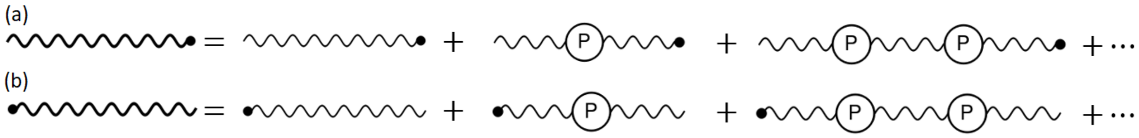

Figure 3.4.: The thick wiggly lines indicate the exact incoming photon line (a) and the exact outgoing photon line (b), and are symbolically expressed as series expansion in terms of free photon propagators (thin wiggly internal lines) and polarization operators (circles with letter P inside).

For the photon out-state $\mathscr{A}_\nu^{(\mathrm{out})}(x)$ the Schwinger-Dyson equation is instead given by (see Fig. 3.4(b))

$$-\partial_\mu\partial^\mu\mathscr{A}_\nu^{(\mathrm{out})}(x)=\int d^4y\,P^{*\,\lambda}_{PW\ \ \nu}(y,x)\mathscr{A}_\lambda^{(\mathrm{out})}(y). \tag{3.3.2}$$

Again only the derivation of the photon in-states is presented, as the expression for the photon out-states can be obtained analogously. As mentioned the Schwinger-Dyson equation (3.3.1) for the photon in-states has been solved already in Refs. [48, 64] with the one-loop polarization operator $P^{(1)\,\nu\lambda}_{\mathrm{LCFA}}(x,y)$ within the LCFA and a linearly polarized plane wave instead of $P^{\nu\lambda}_{PW}(x,y)$. Therefore we present only a few steps of the derivation for completeness.

The expression of the one-loop polarization operator was derived in Refs. [57, 64, 69]. In the following we restrict us to photons with transverse polarization, such that the expression of the one-loop polarization operator within the LCFA and a linearly polarized plane wave is given by [58, 64]

$$P^{(1)\,\mu\nu}_{\mathrm{LCFA}}(x,y)=\int\frac{d^4l}{(2\pi)^4}\frac{d^4l'}{(2\pi)^4}e^{-i(lx)}P^{(1)\,\mu\nu}_{\mathrm{LCFA}}(l,l')e^{i(l'y)}, \tag{3.3.3}$$

where

$$\begin{aligned}P^{(1)\,\mu\nu}_{\mathrm{LCFA}}(l,l')=&-(2\pi)^3\delta^2(\boldsymbol{l}_\perp-\boldsymbol{l}'_\perp)\delta(l_--l'_-)\int d\phi\,e^{-i(l'_+-l_+)\phi}\frac{\alpha}{24\pi}m^2\kappa_l^2(\phi)\\&\times\int_0^\infty du\,u\int_0^1 dv(1-v^2)e^{-iu\left[1-\frac{l'^2}{m^2}\frac{1-v^2}{4}+\frac{(1-v^2)^2}{48}\kappa_l^2(\phi)u^2\right]}\\&\times\left[(3+v^2)\Lambda_1^\mu(l)\Lambda_1^\nu(l)+(6-2v^2)\Lambda_2^\mu(l)\Lambda_2^\nu(l)\right]\end{aligned} \tag{3.3.4}$$

is the transverse part of the one-loop polarization operator in momentum space. The expression is in agreement with the corresponding one in a constant crossed field after replacing $\kappa_0\to\kappa_l(\phi)=l_-\mathcal{A}_0\psi'(\phi)/m^3$ [70–72].

We rewrite the polarization operator similar to the previous section into the form

$$P^{(1)\,\mu\nu}_{\mathrm{LCFA}}(x,y)=\int\frac{d^4l'}{(2\pi)^4}e^{-i(l'x)}P^{(1)\,\mu\nu}_{\mathrm{LCFA}}(l',\phi_x)e^{i(l'y)}, \tag{3.3.5}$$

where

$$\begin{aligned}P^{(1)\,\mu\nu}_{\mathrm{LCFA}}(l',\phi)=&-\frac{\alpha}{24\pi}m^2\kappa_{l'}^2(\phi)\int_0^\infty du\,u\int_0^1 dv(1-v^2)e^{-iu\left[1-\frac{l'^2}{m^2}\frac{1-v^2}{4}+\frac{(1-v^2)^2}{48}\kappa_{l'}^2(\phi)u^2\right]}\\&\times\left[(3+v^2)\Lambda_1^\mu(l')\Lambda_1^\nu(l')+(6-2v^2)\Lambda_2^\mu(l')\Lambda_2^\nu(l')\right].\end{aligned} \tag{3.3.6}$$

At this point we want to solve the Schwinger-Dyson equation (3.3.1). We observe that the left hand side of the equation is identical to the wave equation in vacuum, which would lead to the free photon in-state $A^{\text{rad}}_{j,q,\mu}(x) = \exp(-i(qx))\Lambda_{j,\mu}(q)/\sqrt{2\omega}$, and the right hand side contains the radiative corrections. Therefore, we make similar to the last Section (and like in Ref. [64]) the Ansatz, that these corrections modify the free photon in-state, i.e. we assume

$$\mathscr{A}^{(\text{in})}_{\mu}(x) = g^{(\text{in})}_{j}(q,\phi)e^{-i(qx)}\frac{\Lambda_{j,\mu}(q)}{\sqrt{2\omega}} \tag{3.3.7}$$

to be a solution of Eq. (3.3.1), where the function $g^{(\text{in})}_{j}(q,\phi)$ has to be determined. As the solution should correspond to the free photon in-state before interacting with the plane wave, $g^{(\text{in})}_{j}(q,\phi)$ has to fulfill the boundary condition $\lim_{\phi\to-\infty} g^{(\text{in})}_{j}(q,\phi) = 1$. With this Ansatz the Schwinger-Dyson equation (3.3.1) becomes

$$2iq_{-}\delta_{jj'}\frac{dg^{(\text{in})}_{j}(q,\phi)}{d\phi} = \int d^4y\frac{d^4l'}{(2\pi)^4}e^{-i((l'-q)x)}\Lambda_{j',\mu}(q)P^{(1)\,\mu\nu}_{\text{LCFA}}(l',\phi_x)\Lambda_{j,\nu}(q)e^{i((l'-q)y)}g^{(\text{in})}_{j}(q,\phi_y). \tag{3.3.8}$$

Similar to the previous Section we can directly perform the integrals over $d^2\boldsymbol{y}_{\perp}$ and dy_{+} leading to the δ-functions $\delta^2(\boldsymbol{l}'_{\perp} - \boldsymbol{q}_{\perp})$ and $\delta(l'_{-} - q_{-})$, respectively, which on the other hand are employed to solve the integrals in $d^2\boldsymbol{l}'_{\perp}$ and dl'_{+}, respectively. Thus we obtain

$$2iq_{-}\delta_{jj'}\frac{dg^{(\text{in})}_{j}(q,\phi)}{d\phi} = \int d\phi_y\frac{dl'_{+}}{2\pi}e^{i(q_{+}-l'_{+})(\phi_x-\phi_y)}\Lambda_{j',\mu}(q)P^{(1)\,\mu\nu}_{\text{LCFA}}(l',\phi_x)\Lambda_{j,\nu}(q)g^{(\text{in})}_{j}(q,\phi_y), \tag{3.3.9}$$

with the four-vector l'^{μ} having light-cone components q_{-}, $\boldsymbol{q}_{\perp}$, and l'_{+}. We observe that $\Lambda^{\mu}_{j}(l')$ in the polarization operator in Eq. (3.3.6) only depends on the minus and perpendicular light-cone component of the momentum and that thus $\Lambda^{\mu}_{j}(l') = \Lambda^{\mu}_{j}(q)$. Since $\Lambda^{\mu}_{j}(q)\Lambda_{\mu,j'}(q) = -\delta_{jj'}$ with $j,j' = 1,2$, the right hand side of Eq. (3.3.9) is diagonal in the polarization quantum number j and j'. Further the polarization operator only depends on l'_{+} via $l'^2 = 2q_{-}l'_{+} - \boldsymbol{q}^2_{\perp}$ in the exponent, such that the integral in l'_{+} can be performed similar to the electron case (see remark 2 below Eq. (3.2.14)). According to this, the function $g^{(\text{in})}_{j}(q,\phi)$ has to solve within the LCFA the differential equation

$$iq_{-}\frac{dg^{(\text{in})}_{j}(q,\phi)}{d\phi} = mP_j(q,\phi)g^{(\text{in})}_{j}(q,\phi), \tag{3.3.10}$$

where

$$P_1(q,\phi) = \frac{\alpha}{48\pi}m\kappa^2_q(\phi)\int_0^{\infty}du\,u\int_0^1 dv e^{-iu\left[1+\frac{(1-v^2)^2}{48}\kappa^2_q(\phi)u^2\right]}(1-v^2)(3+v^2), \tag{3.3.11}$$

$$P_2(q,\phi) = \frac{\alpha}{48\pi}m\kappa^2_q(\phi)\int_0^{\infty}du\,u\int_0^1 dv e^{-iu\left[1+\frac{(1-v^2)^2}{48}\kappa^2_q(\phi)u^2\right]}(1-v^2)(6-2v^2), \tag{3.3.12}$$

or equivalently

$$P_j(q,\phi) = \frac{\alpha}{48\pi}mj\kappa^2_q(\phi)\int_0^{\infty}du\,u\int_0^1 dv e^{-iu\left[1+\frac{(1-v^2)^2}{48}\kappa^2_q(\phi)u^2\right]}(1-v^2)[3-(-1)^j v^2]. \tag{3.3.13}$$

Note that $P_1(q,\phi)$ and $P_2(q,\phi)$ vanish for $q_- \to 0$ as $\kappa_q(\phi)$ is proportional to q_-.

At this point Eq. (3.3.10) can be easily solved. Considering the initial condition for $g_j^{(\mathrm{in})}(q,\phi)$ we finally arrive to the expression of the radiatively-corrected photon in-state

$$\mathscr{A}^{(\mathrm{in})}_{R,j,\mu}(q,x) = e^{-i(qx)-i\frac{m}{q_-}\int_{-\infty}^{\phi}d\varphi P_j(q,\varphi)}\frac{\Lambda_{j,\mu}(q)}{\sqrt{2\omega}}. \tag{3.3.14}$$

The Schwinger-Dyson equation (3.3.2) for the photon out-state can be solved analogously. We obtain for the radiatively-corrected photon out-state the expression

$$\mathscr{A}^{(\mathrm{out})}_{R,j,\mu}(q,x) = e^{-i(qx)+i\frac{m}{q_-}\int_{\phi}^{\infty}d\varphi P_j^*(q,\varphi)}\frac{\Lambda_{j,\mu}(q)}{\sqrt{2\omega}}. \tag{3.3.15}$$

In both cases the difference to the photon state without radiative corrections is the exponential function depending on the polarization operator. This can be also seen in the normalization of the exact photon state, given e.g. for the photon in-state by

$$\frac{\mathscr{A}^{(\mathrm{in}),*}_{R,j,\mu}(q,x)\mathscr{A}^{(\mathrm{in}),\mu}_{R,j}(q,x)}{A^{\mathrm{rad},*}_{j,q,\mu}(x)A^{\mathrm{rad},\mu}_{j,q}(x)} = e^{\frac{2m}{q_-}\int_{-\infty}^{\phi}d\varphi\,\mathrm{Im}[P_j(q,\varphi)]}, \tag{3.3.16}$$

and in the case of the photon out-state the normalization would be in the exponent with light-cone integration from ϕ to ∞. Now, according to the optical theorem, the quantity $-(2m/q_-)\mathrm{Im}[P_j(q,\phi)]$ in the exponent on the right hand side of the equality precisely describes the total probability per unit of light-cone time ϕ that a photon with four-momentum q^μ and polarization j decays into an electron-positron pair [57]. Hence, similar to the case of the exact electron and positron states, the new exponential damping term containing the polarization operator takes into account the fact that photons can decay in a plane wave into electron-positron pairs and the photon state is therefore not stable inside the background field.

In the next two chapters we will use these radiatively corrected electron, positron, and photon states to calculate radiatively corrected probabilities for nonlinear Compton scattering and nonlinear Breit-Wheeler pair production, which should give appropriate results in the limit of a long phase duration for the plane wave pulse.

4. Nonlinear Compton scattering including the particle states decay

Note that the content of this chapter was published in the publications [20, 21] and therefore the structure, the equations, and the text of this chapter are similar or identical to Refs. [20,21].

In the previous chapter we presented the radiatively corrected electron, positron and photon states which include the effects of the particle states decay. Now we want to use these states to compute the probabilities of the basic strong-field QED processes at leading order in α. For this the states in the S-matrices in Eqs. (3.0.1) and (3.0.2) are replaced by the corresponding radiatively corrected states. In this chapter we start with the computation of the nonlinear Compton scattering probability. In the next chapter the same computation for nonlinear Breit-Wheeler pair production is examined.

In the case of nonlinear Compton scattering, we assume the incoming (outgoing) electron to have four-momentum $p^\mu = (\varepsilon, \boldsymbol{p})$ ($p'^\mu = (\varepsilon', \boldsymbol{p}')$), with energy $\varepsilon = \sqrt{m^2 + \boldsymbol{p}^2}$ ($\varepsilon' = \sqrt{m^2 + \boldsymbol{p}'^2}$), and an asymptotic spin quantum number $s = \pm 1$ ($s' = \pm 1$). For the outgoing photon the four-momentum is $q^\mu = (\omega, \boldsymbol{q})$, with energy $\omega = |\boldsymbol{q}|$, and its asymptotic transverse polarization state is indicated by the index $j = 1, 2$.

The leading-order S-matrix amplitude in α of nonlinear Compton scattering including the states decay is then given by

$$S^{(e^-\to e^-\gamma)}_{j,s,s'} = -ie \int d^4x\, \bar{\Psi}^{R,(\text{out})}_{e^-,s',p'}(x) \hat{\mathscr{A}}^{(\text{out})\,*}_{R,j}(q,x) \Psi^{R,(\text{in})}_{e^-,s,p}(x). \tag{4.0.1}$$

This S-matrix intrinsically contains the resummation of all one-particle reducible diagrams with corrections by the one-loop mass and polarization operator, as depicted in Fig. 4.1.

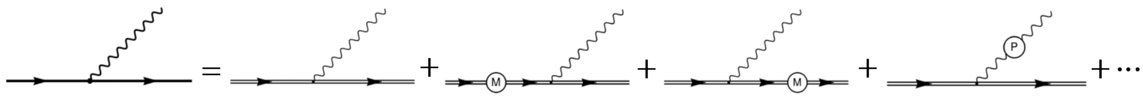

Figure 4.1.: The exact amplitude of nonlinear Compton scattering (first diagram) corresponds to the sum of all one-particle reducible diagrams with corrections by the one-loop mass and polarization operators (circles with M and P, respectively). Here the first Feynman diagrams of this series are presented.

After inserting the expressions of the radiatively corrected states presented in Eqs. (3.2.20), (3.2.24), and (3.3.15) into the S-matrix, it is easy to arrive to the amplitude depending only on the integral over the light-cone time ϕ :

$$\begin{aligned}
S^{(e^-\to e^-\gamma)}_{j,s,s'} &= -\frac{ie}{\sqrt{8\varepsilon\varepsilon'\omega}}(2\pi)^3\delta^2(\boldsymbol{p}'_\perp+\boldsymbol{q}_\perp-\boldsymbol{p}_\perp)\delta(p'_-+q_--p_-) \\
&\quad\times\int d\phi\, e^{-i\frac{m}{p_-}\int_{-\infty}^{\phi}d\varphi M_s(p,\varphi)-i\int_{\phi}^{\infty}d\varphi\left[\frac{m}{p'_-}M_{s'}(p',\varphi)+\frac{m}{q_-}P_j(q,\varphi)\right]} \\
&\quad\times e^{i\left\{(p'_++q_+-p_+)\phi-\int_{\phi}^{\infty}d\varphi\left[\frac{(p'\mathcal{A}(\varphi))}{p'_-}-\frac{\mathcal{A}^2(\varphi)}{2p'_-}\right]-\int_{-\infty}^{\phi}d\varphi\left[\frac{(p\mathcal{A}(\varphi))}{p_-}-\frac{\mathcal{A}^2(\varphi)}{2p_-}\right]\right\}} \\
&\quad\times\bar{u}_{s'}(p')\left[1-\frac{\hat{n}\hat{\mathcal{A}}(\phi)}{2p'_-}\right]\hat{\Lambda}_j(q)\left[1+\frac{\hat{n}\hat{\mathcal{A}}(\phi)}{2p_-}\right]u_s(p).
\end{aligned} \tag{4.0.2}$$

4.1. The radiatively corrected probability

Now we use the S-matrix in Eq. (4.0.2) to compute the probability of nonlinear Compton scattering,

$$\begin{aligned}
P^{(e^-\to e^-\gamma)}_{j,s,s'} &= \int\frac{d^3q}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}|S^{(e^-\to e^-\gamma)}_{j,s,s'}|^2 \\
&= \int\frac{d^3q}{16\pi^2}\frac{d^3p'}{(2\pi)^3}\frac{\alpha}{\varepsilon\varepsilon'\omega}(2\pi)^6[\delta^2(\boldsymbol{p}'_\perp+\boldsymbol{q}_\perp-\boldsymbol{p}_\perp)\delta(p'_-+q_--p_-)]^2 \\
&\quad\times\int d\phi d\phi' e^{-i\frac{m}{p_-}\int_{-\infty}^{\phi}d\varphi M_s(p,\varphi)+i\frac{m}{p_-}\int_{-\infty}^{\phi'}d\varphi M^*_s(p,\varphi)} \\
&\quad\times e^{-i\int_{\phi}^{\infty}d\varphi\left[\frac{m}{p'_-}M_{s'}(p',\varphi)+\frac{m}{q_-}P_j(q,\varphi)\right]+i\int_{\phi'}^{\infty}d\varphi\left[\frac{m}{p'_-}M^*_{s'}(p',\varphi)+\frac{m}{q_-}P^*_j(q,\varphi)\right]} \\
&\quad\times e^{i(p'_++q_+-p_+)(\phi-\phi')+i\int_{\phi'}^{\phi}d\varphi\left[\frac{(p'\mathcal{A}(\varphi))}{p'_-}-\frac{\mathcal{A}^2(\varphi)}{2p'_-}-\frac{(p\mathcal{A}(\varphi))}{p_-}+\frac{\mathcal{A}^2(\varphi)}{2p_-}\right]} \\
&\quad\times\operatorname{tr}\left\{\left[1-\frac{\hat{n}\hat{\mathcal{A}}(\phi)}{2p'_-}\right]\hat{\Lambda}_j(q)\left[1+\frac{\hat{n}\hat{\mathcal{A}}(\phi)}{2p_-}\right]u_s(p)\bar{u}_s(p)\right. \\
&\quad\times\left.\left[1-\frac{\hat{n}\hat{\mathcal{A}}(\phi')}{2p_-}\right]\hat{\Lambda}_j(q)\left[1+\frac{\hat{n}\hat{\mathcal{A}}(\phi')}{2p'_-}\right]u_{s'}(p')\bar{u}_{s'}(p')\right\}.
\end{aligned} \tag{4.1.1}$$

The bispinors $u_s(p)\bar{u}_s(p)$ can be rewritten in terms of the positive-energy electron density matrix $u_s(p)\bar{u}_s(p) = (\hat{p}+m)(1+s\gamma^5\hat{\zeta}_p)/2$ and analogously for $u_{s'}(p')\bar{u}_{s'}(p')$ [11]. Further, the probability contains the square of three δ-functions in light-cone coordinates. To remove the square, the δ-functions first have to be transformed into Cartesian coordinates. Transforming the remaining δ-function back after performing the square leads to the following replacement [39]:

$$(2\pi)^3[\delta^2(\boldsymbol{p}'_\perp+\boldsymbol{q}_\perp-\boldsymbol{p}_\perp)\delta(p'_-+q_--p_-)]^2 = \frac{\varepsilon}{p_-}\delta^2(\boldsymbol{p}'_\perp+\boldsymbol{q}_\perp-\boldsymbol{p}_\perp)\delta(p'_-+q_--p_-). \tag{4.1.2}$$

The remaining δ-functions are exploited to perform the integral in d^3p'. Also here the integration variables are in Cartesian coordinates and we first have to transform them into light-cone coordinates. This transformation yields $d^3p' = dp'_- d^2p'_\perp(\varepsilon'/p'_-)$ [39] and the integral enforces the momentum conservation $p_- = p'_- + q_-$ and $\boldsymbol{p}_\perp = \boldsymbol{p}'_\perp + \boldsymbol{q}_\perp$. For the exponential function containing the phase of the Volkov-states we can use the fact that the particles are on-shell to rewrite the plus light-cone components of the momenta and get

$$\begin{aligned} & e^{i(p'_+ + q_+ - p_+)(\phi-\phi') + i\int_{\phi'}^{\phi} d\varphi\left[\frac{(p'\mathcal{A}(\varphi))}{p'_-} - \frac{\mathcal{A}^2(\varphi)}{2p'_-} - \frac{(p\mathcal{A}(\varphi))}{p_-} + \frac{\mathcal{A}^2(\varphi)}{2p_-}\right]} \\ & \quad = e^{i\frac{m^2 q_-}{2p_-p'_-}\int_{\phi'}^{\phi} d\varphi\left[1+\left(\frac{\boldsymbol{p}_\perp}{m} - \frac{p_-}{q_-}\frac{\boldsymbol{q}_\perp}{m} - \frac{\boldsymbol{\mathcal{A}}_\perp(\varphi)}{m}\right)^2\right]}. \end{aligned} \tag{4.1.3}$$

With these transformations the probability reduces to

$$\begin{aligned} P_{j,s,s'}^{(e^-\to e^-\gamma)} = & \int \frac{d^3q}{16\pi^2}\frac{\alpha}{p_-p'_-\omega}\int d\phi d\phi' e^{-i\frac{m}{p_-}\int_{-\infty}^{\phi} d\varphi M_s(p,\varphi) + i\frac{m}{p_-}\int_{-\infty}^{\phi'} d\varphi M_s^*(p,\varphi)} \\ & \times e^{-i\int_{\phi}^{\infty} d\varphi\left[\frac{m}{p'_-}M_{s'}(p',\varphi) + \frac{m}{q_-}P_j(q,\varphi)\right] + i\int_{\phi'}^{\infty} d\varphi\left[\frac{m}{p'_-}M_{s'}^*(p',\varphi) + \frac{m}{q_-}P_j^*(q,\varphi)\right]} \\ & \times e^{i\frac{m^2 q_-}{2p_-p'_-}\int_{\phi'}^{\phi} d\varphi\left[1+\left(\frac{\boldsymbol{p}_\perp}{m} - \frac{p_-}{q_-}\frac{\boldsymbol{q}_\perp}{m} - \frac{\boldsymbol{\mathcal{A}}_\perp(\varphi)}{m}\right)^2\right]} \\ & \times \frac{1}{4}\mathrm{tr}\left\{\left[1 - \frac{\hat{n}\hat{\mathcal{A}}(\phi)}{2p'_-}\right]\hat{\Lambda}_j(q)\left[1 + \frac{\hat{n}\hat{\mathcal{A}}(\phi)}{2p_-}\right](\hat{p}+m)(1+s\gamma^5\hat{\zeta}_p)\right. \\ & \left.\times \left[1 - \frac{\hat{n}\hat{\mathcal{A}}(\phi')}{2p_-}\right]\hat{\Lambda}_j(q)\left[1 + \frac{\hat{n}\hat{\mathcal{A}}(\phi')}{2p'_-}\right](\hat{p}'+m)(1+s'\gamma^5\hat{\zeta}_{p'})\right\}. \end{aligned} \tag{4.1.4}$$

Next we apply the substitution $\phi_+ = (\phi+\phi')/2$ and $\phi_- = \phi - \phi'$, which leads to the replacements $d\phi d\phi' = d\phi_+ d\phi_-$, $\phi = \phi_+ + \phi_-/2$, and $\phi' = \phi_+ - \phi_-/2$ [29, 39].

We recall that the expressions of the mass- and polarization operator, i.e. $M_s(p,\phi)$ and $P_j(q,\phi)$, respectively, are only valid within the LCFA and a linearly polarized plane wave. To arrive to a correct probability we therefore have to apply the LCFA to the remaining expression, too, which we will do in the following.

Since in the LCFA the phase $\omega_0\phi_-$ is approximately of the order of $1/\xi_0$ and it is assumed that $\xi_0 \gg 1$, we have to expand the integrand around $\phi_- \to 0$ (see also Refs. [29,39]). In order to obtain the leading order results it is sufficient to expand the preexponential function up to the linear order in ϕ_-. The quantities $M_s(p,\phi)$ and $P_j(q,\phi)$ are expanded at the zero order in ϕ_- as they are already in the LCFA (a first-order expansion in ϕ_- would result into a correction of the order of α and is neglected within our approximations). For the remaining phase coming from the Volkov-states we first add and subtract $iq_-(\int_{\phi_+-\phi_-/2}^{\phi_++\phi_-/2} d\varphi\boldsymbol{\mathcal{A}}_\perp(\varphi))^2/(2p_-p'_-\phi_-)$ in the exponent such that we have

$$\begin{aligned} & e^{i\frac{m^2 q_-}{2p_-p'_-}\int_{\phi_+-\phi_-/2}^{\phi_++\phi_-/2} d\varphi\left[1+\left(\frac{\boldsymbol{p}_\perp}{m} - \frac{p_-}{q_-}\frac{\boldsymbol{q}_\perp}{m} - \frac{\boldsymbol{\mathcal{A}}_\perp(\varphi)}{m}\right)^2\right]} \\ & = e^{i\frac{m^2 q_-\phi_-}{2p_-p'_-}\left(\frac{\boldsymbol{p}_\perp}{m} - \frac{p_-}{q_-}\frac{\boldsymbol{q}_\perp}{m} - \frac{1}{\phi_-}\int_{\phi_+-\phi_-/2}^{\phi_++\phi_-/2} d\varphi\frac{\boldsymbol{\mathcal{A}}_\perp(\varphi)}{m}\right)^2} \\ & \quad \times e^{i\frac{m^2 q_-}{2p_-p'_-}\left[\phi_- - \frac{1}{\phi_-}\left(\int_{\phi_+-\phi_-/2}^{\phi_++\phi_-/2} d\varphi\frac{\boldsymbol{\mathcal{A}}_\perp(\varphi)}{m}\right)^2 + \int_{\phi_+-\phi_-/2}^{\phi_++\phi_-/2} d\varphi\left(\frac{\boldsymbol{\mathcal{A}}_\perp(\varphi)}{m}\right)^2\right]}. \end{aligned} \tag{4.1.5}$$

The first exponential function can be used to perform the integral in $d^2\boldsymbol{q}_\perp$ which gives Gaussian integrals and contributes to the preexponential. We do not do it at this stage, but at a later stage. However, since we expand the prefactor only up to linear order in ϕ_-, we also have to expand this part only up to linear order in ϕ_-. The second exponential instead has to be expanded up to cubic order in ϕ_- and results at a later stage to Airy-functions when solving the integral in $d\phi_-$ [9, 38, 39].

Following these steps we end up with the probability of nonlinear Compton scattering including the particle states decay within the LCFA and a linearly polarized plane wave:

$$\begin{aligned} P^{(e^-\to e^-\gamma)}_{j,s,s'} =& \int \frac{d^3q}{16\pi^2}\frac{\alpha}{p_-p'_-\omega}\int d\phi_+ e^{D^{\rm NC}_{j,s,s'}} \\ &\times \int d\phi_- e^{i\frac{m^2}{2p_-}\frac{q_-}{p'_-}\left\{[1+\boldsymbol{\pi}^2_{\perp,e}(\phi_+)]\phi_- + \frac{\boldsymbol{\mathcal{E}}^2(\phi_+)}{m^2}\frac{\phi_-^3}{12}\right\}} T_{j,s,s'}, \end{aligned} \tag{4.1.6}$$

where we introduced the exponential damping function

$$D^{\rm NC}_{j,s,s'} = 2{\rm Im}\left\{\frac{m}{p_-}\int_{-\infty}^{\phi_+} d\varphi M_s(p,\varphi) + \int_{\phi_+}^{\infty} d\varphi\left[\frac{m}{p'_-}M_{s'}(p',\varphi) + \frac{m}{q_-}P_j(q,\varphi)\right]\right\}, \tag{4.1.7}$$

the trace

$$\begin{aligned} T_{j,s,s'} =& \frac{1}{4}{\rm tr}\left\{\left[1 - \frac{\hat{n}[\hat{\mathcal{A}}(\phi_+) + \hat{\mathcal{A}}'(\phi_+)\phi_-/2]}{2p'_-}\right]\hat{\Lambda}_j(q)\left[1 + \frac{\hat{n}[\hat{\mathcal{A}}(\phi_+) + \hat{\mathcal{A}}'(\phi_+)\phi_-/2]}{2p_-}\right]\right. \\ &\times(\hat{p}+m)(1+s\gamma^5\hat{\zeta}_p)\left[1 - \frac{\hat{n}[\hat{\mathcal{A}}(\phi_+) - \hat{\mathcal{A}}'(\phi_+)\phi_-/2]}{2p_-}\right]\hat{\Lambda}_j(q) \\ &\times\left.\left[1 + \frac{\hat{n}[\hat{\mathcal{A}}(\phi_+) - \hat{\mathcal{A}}'(\phi_+)\phi_-/2]}{2p'_-}\right](\hat{p}'+m)(1+s'\gamma^5\hat{\zeta}_{p'})\right\}, \end{aligned} \tag{4.1.8}$$

the transverse momentum

$$\boldsymbol{\pi}_{\perp,e}(\phi) = \frac{\boldsymbol{p}_\perp}{m} - \frac{p_-}{q_-}\frac{\boldsymbol{q}_\perp}{m} - \frac{\boldsymbol{\mathcal{A}}_\perp(\phi)}{m}, \tag{4.1.9}$$

and the plane-wave electric field (times the electron charge) $\boldsymbol{\mathcal{E}}(\phi) = -\boldsymbol{\mathcal{A}}'(\phi)$. Due to energy-momentum conservation the minus component and the perpendicular component of the outgoing electron momentum are fixed to $p'_- = p_- - q_-$ and $\boldsymbol{p}'_\perp = \boldsymbol{p}_\perp - \boldsymbol{q}_\perp$, respectively.

As expected, the probability comprises a damping term due to the particle states decay, which is given by $\exp(D^{\rm NC}_{j,s,s'})$ in Eq. (4.1.6). This term is new in comparison to the probability obtained without radiatively corrected states, which expression can be found for example in Refs. [9, 38–40]. The damping function $D^{\rm NC}_{j,s,s'}$ contains the imaginary part of the one-loop mass and polarization operator (see Eqs. (3.2.19) and (3.3.13), respectively) which are, as already mentioned, related to the probabilities of an electron emitting a photon and a photon decaying into an electron-positron pair, respectively.

Indeed, for $P^{\rm NC}_{s,p}$ being the probability of nonlinear Compton scattering without radiative corrections of an electron with momentum p and spin quantum number s, and $P^{\rm NBW}_{j,q}$ being the probability of nonlinear Breit-Wheeler pair production without radiative corrections of a photon with momentum q and polarization quantum number j, according to the optical theorem it is $-(2m/p_-)\mathrm{Im}[M_s(p,\phi)] = \partial P^{\mathrm{NC}}_{s,p}/\partial\phi$ and $-(2m/q_-)\mathrm{Im}[P_j(q,\phi)] = \partial P^{\mathrm{NBW}}_{j,q}/\partial\phi$ [3, 57, 68]. Therefore we can rewrite the damping function to

$$D^{\mathrm{NC}}_{j,s,s'} = -\int_{-\infty}^{\phi_+} d\varphi \frac{\partial P^{\mathrm{NC}}_{s,p}}{\partial\varphi} - \int_{\phi_+}^{\infty} d\varphi \left(\frac{\partial P^{\mathrm{NC}}_{s',p'}}{\partial\varphi} + \frac{\partial P^{\mathrm{NBW}}_{j,q}}{\partial\varphi} \right). \tag{4.1.10}$$

We see that these probabilities per unit of light-cone time φ are integrated over the light-cone time φ from $\varphi \to -\infty$ to $\varphi = \phi_+$ (from $\varphi = \phi_+$ to $\varphi \to \infty$) for an incoming (outgoing) state, such that the damping term scales with the phase length of the plane wave. Then the exponential damping term itself corresponds to the probability that the electron and photon states do not decay in the plane wave which is also sketched in Fig 4.2. Therefore we conclude that the probability in Eq. (4.1.6) describes the radiatively corrected probability of an electron with momentum p and spin s emitting only one single photon in a linearly polarized plane wave pulse. Processes where multi photon emissions occur or emitted photons decay into electron-positron pairs are excluded in this probability.

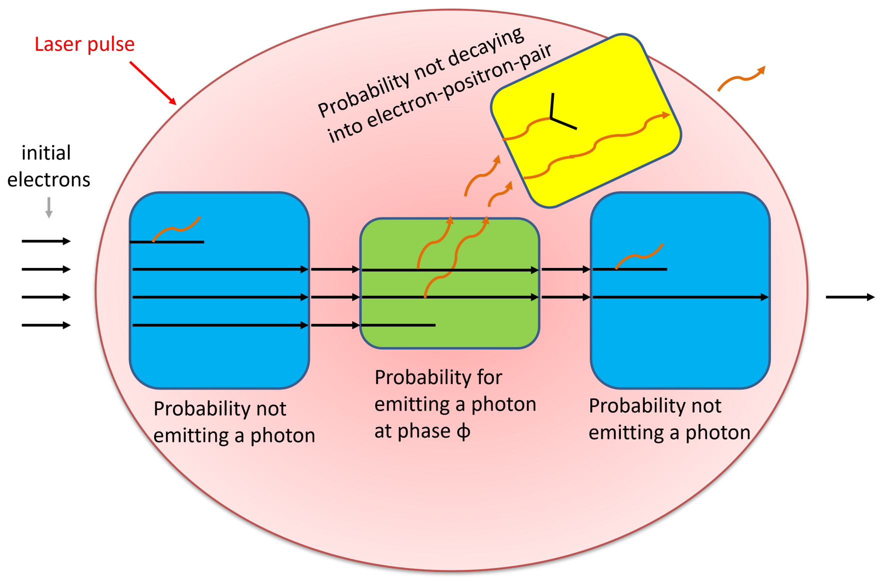

Figure 4.2.: Sketch of an interpretation of the new radiatively corrected probability. Considering an electron bunch colliding with a laser pulse, the new probability with damping at a certain phase ϕ_+ corresponds to the product of the probability without damping at a certain phase ϕ_+ (green box) times the probability of an electron emitting not a photon (blue boxes, left for incoming electrons and right for outgoing electrons) times the probability that the produced photons do not decay into electron-positron pairs (yellow box).

Now we observe that the mass and polarization operators in $D^{\mathrm{NC}}_{j,s,s'}$ in Eq. (4.1.7) scale at least with $\alpha m\chi_p(\phi)$ (analogously for p') and $\alpha m\kappa_q(\phi)$, respectively (see Eqs. (3.2.19) and (3.3.13)). In a monochromatic plane wave with $\mathcal{A}_0 = |e|E_0/\omega_0$ therefore the function $D^{\mathrm{NC}}_{j,s,s'}$ scales as $\alpha\chi_p(\phi)m^2/p_- = \alpha\kappa_q(\phi)m^2/q_- = \alpha\mathcal{A}_0\psi'(\phi)/m = \alpha\xi_0\psi'(\phi)$. Considering that we assume to have $\chi_0 \sim 1$ and $\kappa_0 \sim 1$ for additional powers of these parameters and that the pulse function is integrated over the phase, this leads to the conclusion that damping effects become important for pulse phase lengths Φ_L when the product $\alpha\xi_0\Phi_L \gtrsim 1$, which is in agreement with the discussion in Chapter 3.

For the computation of the trace in Eq. (4.1.8) complications arise due to the fact that the damping depends not only on the momentum of the particles, but also on the electrons spin and photons polarization. This prevents us using well-known spin and polarization summation rules for the trace [3, 11] and the spin and polarization resolved trace has to be computed.

Ignoring these spin- and polarization-effects and neglecting the damping of the photon state, the probability is, after averaging (summing) over the quantum numbers of the initial (final) states, in agreement with the result obtained by the probabilistic approach in Ref. [45]. In the next two sections, first an analytical computation of the trace is presented, and then the integrals in $d^2\boldsymbol{q}_\perp$ and $d\phi_-$ are performed.

4.2. Solving the trace

As mentioned already, the exponential damping term depends on the spin of the incoming and outgoing electrons and on the polarization of the outgoing photon. This prevents one from employing the commonly used spin and polarization summation rules when solving the trace in Eq. (4.1.8), such that the spin- and polarization-resolved trace has to be calculated. This trace was computed already in Refs. [40, 73] (see also Refs. [38, 74]), however in the following an alternative and more detailed analytical computation of the trace for nonlinear Compton scattering given in Eq. (4.1.8) is presented. For this we first introduce the quantity

$$\begin{aligned} Q_{p,s}(\phi_+,\phi_-) = &\left[1 + \frac{\hat{n}[\hat{\mathcal{A}}(\phi_+) + \hat{\mathcal{A}}'(\phi_+)\phi_-/2]}{2p_-}\right](\hat{p}+m)(1+s\gamma^5\hat{\zeta}_p) \\ &\times \left[1 - \frac{\hat{n}[\hat{\mathcal{A}}(\phi_+) - \hat{\mathcal{A}}'(\phi_+)\phi_-/2]}{2p_-}\right]. \end{aligned} \tag{4.2.1}$$

The trace of nonlinear Compton scattering in Eq. (4.1.8) is then with this new quantity given by

$$T_{j,s,s'} = \frac{1}{4}\mathrm{tr}\left\{\hat{\Lambda}_j(q)Q_{p,s}(\phi_+,\phi_-)\hat{\Lambda}_j(q)Q_{p',s'}(\phi_+,-\phi_-)\right\}. \tag{4.2.2}$$

Now every 4×4 matrix can be decomposed into a linear combination of the matrices $1_{4\times 4}$, γ^5, γ^μ, $i\gamma^\mu\gamma^5$, $\sigma^{\mu\nu} = (i/2)(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$ (see Appendix A and Refs. [3, 48]). Using this fact the function $Q_{p,s}(\phi_+,\phi_-)$ can be rewritten into the form

$$Q_{p,s}(\phi_+,\phi_-) = c_1 1_{4\times 4} + c_5\gamma^5 + c_\mu\gamma^\mu + c_{5\mu}i\gamma^\mu\gamma^5 + c_{\mu\nu}\sigma^{\mu\nu}, \tag{4.2.3}$$

where the coefficients are defined as

$$
\begin{aligned}
c_1 =& \frac{1}{4}\mathrm{Tr}\left[1_{4\times 4} Q_{p,s}(\phi_+,\phi_-)\right], && (4.2.4)\\
c_5 =& \frac{1}{4}\mathrm{Tr}\left[\gamma^5 Q_{p,s}(\phi_+,\phi_-)\right], && (4.2.5)\\
c_\mu =& \frac{1}{4}\mathrm{Tr}\left[\gamma_\mu Q_{p,s}(\phi_+,\phi_-)\right], && (4.2.6)\\
c_{5\mu} =& \frac{1}{4}\mathrm{Tr}\left[i\gamma_\mu\gamma^5 Q_{p,s}(\phi_+,\phi_-)\right], && (4.2.7)\\
c_{\mu\nu} =& \frac{1}{8}\mathrm{Tr}\left[\sigma_{\mu\nu} Q_{p,s}(\phi_+,\phi_-)\right]. && (4.2.8)
\end{aligned}
$$

The traces defining the coefficients are calculated using relations for γ-matrices presented in Appendix A on page 98f. To reduce the number of indices we introduce for the following steps the short notation $\zeta^\mu = \zeta_p^\mu$ and $\zeta'^\mu = \zeta_{p'}^\mu$. For a linear polarized plane wave background field we obtain

$$c_1 = m - i\frac{s}{4p_-}\epsilon^{\alpha\beta\gamma\delta}\zeta_\alpha \mathcal{F}_{\beta\gamma}\psi'(\phi_+)\phi_- p_\delta, \tag{4.2.9}$$

$$c_5 = 0, \tag{4.2.10}$$

$$
\begin{aligned}
c_\mu =& p_\mu - \mathcal{A}_\mu(\phi_+) - \frac{1}{p_-}n_\mu \boldsymbol{p}_\perp\cdot\boldsymbol{\mathcal{A}}_\perp(\phi_+)\\
&+\frac{1}{2p_-}n_\mu\left[\boldsymbol{\mathcal{A}}_\perp^2(\phi_+) - \boldsymbol{\mathcal{A}}_\perp'^2(\phi_+)\frac{\phi_-^2}{4}\right]\\
&- i\frac{ms}{4p_-}\eta_{\mu\nu}\epsilon^{\alpha\beta\delta\nu}\zeta_\alpha\mathcal{F}_{\beta\delta}\psi'(\phi_+)\phi_-,
\end{aligned}
\tag{4.2.11}
$$

$$c_{5\mu} = ims\zeta_\mu + \frac{1}{4p_-}\eta_{\mu\nu}\epsilon^{\alpha\beta\delta\nu}p_\alpha\mathcal{F}_{\beta\delta}\psi'(\phi_+)\phi_-, \tag{4.2.12}$$

and

$$
\begin{aligned}
c_{\mu\nu} =& \frac{1}{2}\Bigg\{ - i\frac{m}{2p_-}\mathcal{F}_{\mu\nu}\psi'(\phi_+)\phi_- - s\epsilon_{\mu\nu\rho\sigma}p^\rho\zeta^\sigma\\
&+\frac{s}{2p_-}(\eta_{\nu\rho}\epsilon_{\mu\sigma\tau\delta} - \eta_{\mu\rho}\epsilon_{\nu\sigma\tau\delta})p^\rho\zeta^\sigma\mathcal{F}^{\tau\delta}\psi(\phi_+)\\
&+\frac{s}{2p_-}\epsilon_{\mu\nu\tau\rho}n^\tau\zeta^\rho\left[\mathcal{A}_\delta(\phi_+)\mathcal{A}^\delta(\phi_+) - \mathcal{A}'_\delta(\phi_+)\mathcal{A}'^\delta(\phi_+)\frac{\phi_-^2}{4}\right]\Bigg\}.
\end{aligned}
\tag{4.2.13}
$$

Now the trace of nonlinear Compton scattering in Eq. (4.2.2) further depends on the quantity $Q_{p',s'}(\phi_+,-\phi_-)$ for which we introduce the following notation

$$Q_{p',s'}(\phi_+,-\phi_-) = c_1' 1_{4\times 4} + c_5'\gamma^5 + c_\tau'\gamma^\tau + c_{5\tau}' i\gamma^\tau\gamma^5 + c_{\tau\lambda}'\sigma^{\tau\lambda}, \tag{4.2.14}$$

where the primed coefficients are defined analogously to Eqs. (4.2.4)-(4.2.8). Using Eqs. (4.2.3) and (4.2.14) the trace for nonlinear Compton scattering reduces to

$$
\begin{aligned}
T_{j,s,s'} =& - c_1c_1' + c_5c_5' + (c_\mu c_\tau' - c_{5\mu}c_{5\tau}')(2\Lambda_j^\mu(q)\Lambda_j^\tau(q) + \eta^{\mu\tau})\\
&- 2c_{\mu\nu}c_{\tau\lambda}'\eta^{\nu\lambda}(4\Lambda_j^\mu(q)\Lambda_j^\tau(q) + \eta^{\mu\tau}),
\end{aligned}
\tag{4.2.15}
$$

where again relations from the Appendix A were used to derive to this expression. We observe that the trace only depends on contractions of the primed and the corresponding not primed coefficients, which we can calculate:

$$\begin{aligned} c_1 c_1' =& m^2 - i\frac{m}{4}\epsilon^{\alpha\beta\gamma\delta}\mathcal{F}_{\beta\gamma}\psi'(\phi_+)\phi_-\left[\frac{s}{p_-}\zeta_\alpha p_\delta - \frac{s'}{p'_-}\zeta'_\alpha p'_\delta\right] \\ &+ \frac{1}{16}\frac{ss'}{p_- p'_-}\epsilon^{\alpha\beta\gamma\delta}\mathcal{F}_{\beta\gamma}\zeta_\alpha p_\delta \epsilon^{\alpha'\beta'\gamma'\delta'}\mathcal{F}_{\beta'\gamma'}\zeta'_{\alpha'}p'_{\delta'}\psi'^2(\phi_+)\phi_-^2, \end{aligned} \tag{4.2.16}$$

$$c_5 c_5' = 0, \tag{4.2.17}$$

$$\begin{aligned} c_\mu c'_\tau &\left(2\Lambda_j^\mu(q)\Lambda_j^\tau(q) + \eta^{\mu\tau}\right) = (pp') + (\boldsymbol{p}_\perp + \boldsymbol{p}'_\perp)\cdot\boldsymbol{\mathcal{A}}_\perp(\phi_+) \\ &- \left[\frac{p'_-}{p_-}\boldsymbol{p}_\perp + \frac{p_-}{p'_-}\boldsymbol{p}'_\perp\right]\cdot\boldsymbol{\mathcal{A}}_\perp(\phi_+) + \frac{1}{2}\left(\frac{p'_-}{p_-} + \frac{p_-}{p'_-}\right)\left[\boldsymbol{\mathcal{A}}_\perp^2(\phi_+) - \boldsymbol{\mathcal{A}}'^2_\perp(\phi_+)\frac{\phi_-^2}{4}\right] \\ &+ \mathcal{A}_\mu(\phi_+)\mathcal{A}^\mu(\phi_+) - i\frac{m}{4}\epsilon^{\alpha\beta\gamma\delta}\mathcal{F}_{\beta\gamma}\psi'(\phi_+)\phi_-\left[\frac{s}{p_-}\zeta_\alpha p'_\delta - \frac{s'}{p'_-}\zeta'_\alpha p_\delta\right] \\ &+ 2\left(p\Lambda_j(q)\right)\left(p'\Lambda_j(q)\right) - 2[p_\mu + p'_\mu]\Lambda_j^\mu(q)\left(\Lambda_j(q)\mathcal{A}(\phi_+)\right) \\ &+ 2\left(\Lambda_j(q)\mathcal{A}(\phi_+)\right)^2, \end{aligned} \tag{4.2.18}$$

$$\begin{aligned} c_{5\mu} c'_{5\tau} &\left(2\Lambda_j^\mu(q)\Lambda_j^\tau(q) + \eta^{\mu\tau}\right) = -m^2 ss'\left[2\left(\zeta\Lambda_j(q)\right)\left(\zeta'\Lambda_j(q)\right) + (\zeta\zeta')\right] \\ &+ i\frac{m}{4}\epsilon^{\alpha\beta\gamma\delta}\mathcal{F}_{\beta\gamma}\psi'(\phi_+)\phi_-\left[\frac{s'}{p_-}p_\alpha\zeta'_\delta - \frac{s}{p'_-}p'_\alpha\zeta_\delta\right] \\ &+ i\frac{m}{2}\epsilon^{\alpha\beta\gamma\delta}\mathcal{F}_{\beta\gamma}\psi'(\phi_+)\phi_-\left[\frac{s'}{p_-}p_\alpha\left(\zeta'\Lambda_j(q)\right) - \frac{s}{p'_-}p'_\alpha\left(\zeta\Lambda_j(q)\right)\right]\Lambda_{j,\delta}(q) \\ &- \frac{1}{16 p_- p'_-}\epsilon^{\alpha\beta\gamma\delta}p_\alpha\mathcal{F}_{\beta\gamma}\epsilon^{\alpha'\beta'\gamma'\delta'}p'_{\alpha'}\mathcal{F}_{\beta'\gamma'}\left[\eta_{\delta\delta'} + 2\Lambda_{j,\delta}(q)\Lambda_{j,\delta'}(q)\right]\psi'^2(\phi_+)\phi_-^2, \end{aligned} \tag{4.2.19}$$

and

$$\begin{aligned} 2c_{\mu\nu}c'_{\tau\lambda}\eta^{\nu\lambda} &\left(4\Lambda_j^\mu(q)\Lambda_j^\tau(q) + \eta^{\mu\tau}\right) = -ss'\left[(p\zeta')(p'\zeta) - (pp')(\zeta\zeta')\right] \\ &- 2ss'\Big[(p\zeta')(p'\Lambda_j(q))(\zeta\Lambda_j(q)) + (\zeta p')(\zeta'\Lambda_j(q))(p\Lambda_j(q)) \\ &\qquad - (pp')(\zeta\Lambda_j(q))(\zeta'\Lambda_j(q)) - (\zeta\zeta')(p\Lambda_j(q))(p'\Lambda_j(q))\Big] \\ &+ i\frac{m}{4}\psi'(\phi_+)\phi_-\epsilon^{\mu\nu\rho\sigma}\left[\mathcal{F}_{\mu\nu} + 4\Lambda_{j,\mu}(q)\left(\Lambda_j^\tau(q)\mathcal{F}_{\tau\nu}\right)\right]\left[\frac{s'}{p_-}p'_\rho\zeta'_\sigma - \frac{s}{p'_-}p_\rho\zeta_\sigma\right] \\ &- \frac{ss'}{2}\left[\mathcal{A}_\delta(\phi_+)\mathcal{A}^\delta(\phi_+) - \mathcal{A}'_\delta(\phi_+)\mathcal{A}'^\delta(\phi_+)\frac{\phi_-^2}{4}\right] \\ &\qquad \times\left[(\zeta\zeta') + 2(\zeta\Lambda_j(q))(\zeta'\Lambda_j(q))\right]\left(\frac{p_-}{p'_-} + \frac{p'_-}{p_-}\right) \\ &+ ss'\psi(\phi_+)(\zeta\zeta')\left[\frac{p_- - p'_-}{p_- p'_-}(p'\mathcal{F}p) - 2\frac{(p'\Lambda_j(q))}{p'_-}(p\mathcal{F}\Lambda_j(q)) - 2\frac{(p\Lambda_j(q))}{p_-}(p'\mathcal{F}\Lambda_j(q))\right] \\ &- \frac{ss'}{4p_- p'_-}\psi^2(\phi_+)(\epsilon^{\rho'\sigma\gamma\delta}p'_{\rho'}\zeta_\sigma\mathcal{F}_{\gamma\delta})(\epsilon^{\rho\sigma'\gamma'\delta'}p_\rho\zeta'_{\sigma'}\mathcal{F}_{\gamma'\delta'}). \end{aligned} \tag{4.2.20}$$

Now we insert Eqs. (4.2.16)-(4.2.20) into Eq. (4.2.15) and simplify the expression for the two polarization states $j = 1$ and $j = 2$. With our definitions of the spin and polarization four-vectors (see Sections 2.6 and 2.5, respectively), we obtain for the trace in a linearly-polarized field in the two polarization states $j = 1, 2$:

$$
\begin{aligned}
T_{1,s,s'} = (1+ss')\Bigg[&(pp') - m^2 - \frac{1}{2}\frac{q_-}{p_-}\frac{q_-}{p_- - q_-}\left(p_1 - \frac{p_-}{q_-}q_1\right)^2 \\
&- \left(2 + \frac{1}{2}\frac{q_-}{p_-}\frac{q_-}{p_- - q_-}\right)\mathcal{A}_0^2\psi'^2(\phi_+)\frac{\phi_-^2}{4} \\
&+ \left(2 + \frac{1}{2}\frac{q_-}{p_-}\frac{q_-}{p_- - q_-}\right)\left(p_1 - \frac{p_-}{q_-}q_1 + \mathcal{A}_0\psi(\phi_+)\right)^2\Bigg] \\
&- i(s+s')\frac{m}{2}\mathcal{A}_0\psi'(\phi_+)\phi_-\frac{q_-}{p_-}\left(2 + \frac{q_-}{p_- - q_-}\right) - ss'\frac{q_-}{p_-}\frac{q_-}{p_- - q_-}\left(p_2 - \frac{p_-}{q_-}q_2\right)^2
\end{aligned}
\tag{4.2.21}
$$

and

$$
\begin{aligned}
T_{2,s,s'} = (1-ss')\Bigg[&(pp') - m^2 - \frac{1}{2}\frac{q_-}{p_-}\frac{q_-}{p_- - q_-}\left(p_1 - \frac{p_-}{q_-}q_1\right)^2 \\
&- \frac{1}{2}\frac{q_-}{p_-}\frac{q_-}{p_- - q_-}\mathcal{A}_0^2\psi'^2(\phi_+)\frac{\phi_-^2}{4} \\
&+ \frac{1}{2}\frac{q_-}{p_-}\frac{q_-}{p_- - q_-}\left(p_1 - \frac{p_-}{q_-}q_1 + \mathcal{A}_0\psi(\phi_+)\right)^2\Bigg] \\
&+ (1+ss')2\left(p_2 - \frac{p_-}{q_-}q_2\right)^2 + ss'\frac{q_-}{p_-}\frac{q_-}{p_- - q_-}\left(p_2 - \frac{p_-}{q_-}q_2\right)^2 \\
&+ i(s-s')\frac{m}{2}\mathcal{A}_0\psi'(\phi_+)\phi_-\frac{q_-}{p_-}\frac{q_-}{p_- - q_-}.
\end{aligned}
\tag{4.2.22}
$$

4.3. Final integrals

In this Section we want to perform the integrals in $d^2\boldsymbol{q}_\perp$ and $d\phi_-$ in Eq. (4.1.6). This procedure is well known and was carried out e.g. in Refs. [29, 39, 40]. It turns out that the integral in $d^2\boldsymbol{q}_\perp$ is Gaussian and the integral in $d\phi_-$ leads to Airy-functions (see Appendix B).

But before performing the integrals we observe that the traces in Eqs. (4.2.21) and (4.2.22) depend on the pulse shape function $\psi(\phi_+)$. However, we should be able to remove the dependence on $\psi(\phi_+)$ and write the trace in a manifestly gauge-invariant way. For this we first expand the four-product (pp') and observe that the remaining dependence on the pulse function $\psi(\phi_+)$ and q_1 can be rewritten into a dependence on the square of the electron quasi-momentum $\boldsymbol{\pi}_{\perp,e}(\phi_+)$ defined in Eq. (4.1.9). Now it is worth noticing that by adding and subtracting suitable terms these dependence turns out to be part of a term proportional to the derivative in ϕ_- of the second exponential function in Eq. (4.1.6), i.e. proportional to $\left[1 + \boldsymbol{\pi}^2_{\perp,e}(\phi_+) + (\mathcal{A}_0^2\psi'^2(\phi_+)/m^2)(\phi_-^2/4)\right]$. This is important as those contributions vanish when performing the integral over ϕ_-, which follows from properties of the Airy-function and a proof thereof is presented in Eq. (B.0.6) on page 100.

Therefore the trace only depends on the derivative $\psi'(\phi_+)$ of the pulse shape function and the probability is manifestly gauge-invariant. Without these vanishing terms the trace becomes

$$
\begin{aligned}
T_{1,s,s'} = & -2(1+ss')m^2 - (1+ss')\left(4+\frac{q_-}{p_-}\frac{q_-}{p_- - q_-}\right)\mathcal{A}_0^2\psi'^2(\phi_+)\frac{\phi_-^2}{4} \\
& - i(s+s')\frac{m}{2}\mathcal{A}_0\psi'(\phi_+)\phi_-\frac{q_-}{p_-}\left(2+\frac{q_-}{p_- - q_-}\right) \\
& - \left(2+2ss'+ss'\frac{q_-}{p_-}\frac{q_-}{p_- - q_-}\right)\left(p_2 - \frac{p_-}{q_-}q_2\right)^2
\end{aligned}
\tag{4.3.1}
$$

and

$$
\begin{aligned}
T_{2,s,s'} = & -(1-ss')\frac{q_-}{p_-}\frac{q_-}{p_- - q_-}\mathcal{A}_0^2\psi'^2(\phi_+)\frac{\phi_-^2}{4} \\
& + i(s-s')\frac{m}{2}\mathcal{A}_0\psi'(\phi_+)\phi_-\frac{q_-}{p_-}\frac{q_-}{p_- - q_-} \\
& + \left(2+2ss'+ss'\frac{q_-}{p_-}\frac{q_-}{p_- - q_-}\right)\left(p_2 - \frac{p_-}{q_-}q_2\right)^2 .
\end{aligned}
\tag{4.3.2}
$$

Now we want to perform the integral in $d^2\boldsymbol{q}_\perp$ and $d\phi_-$ in Eq. (4.1.6). For the former one we have to transform the integral in d^3q from Cartesian into light-cone coordinates which is done via the relation $d^3q = (\omega/q_-)dq_- d^2\boldsymbol{q}_\perp$ [39]. Further we introduce the notation

$$
\tilde{T}_{j,s,s'} = -\frac{1}{4\pi^2 m^2}\frac{p_-}{q_- p'_-}\int d\phi_- \int d^2\boldsymbol{q}_\perp e^{i\frac{m^2 q_-}{2p_- p'_-}\left\{[1+\boldsymbol{\pi}^2_{\perp,e}(\phi_+)]\phi_- + \frac{\boldsymbol{\mathcal{E}}^2(\phi_+)}{m^2}\frac{\phi_-^3}{12}\right\}} T_{j,s,s'} . \tag{4.3.3}
$$

As mentioned, the integral in the perpendicular photon momentum $\boldsymbol{q}_\perp$ is Gaussian and can be computed analytically with the following two basic integrals [75]

$$
\int d^2\boldsymbol{q}_\perp e^{i\frac{m^2 q_-}{2p_- p'_-}\boldsymbol{\pi}^2_{\perp,e}(\phi_+)\phi_-} = 2\pi i \frac{q_- p'_-}{p_-(\phi_- + i0)}, \tag{4.3.4}
$$

$$
\int d^2\boldsymbol{q}_\perp \left(p_2 - \frac{p_-}{q_-}q_2\right)^2 e^{i\frac{m^2 q_-}{2p_- p'_-}\boldsymbol{\pi}^2_{\perp,e}(\phi_+)\phi_-} = -2\pi\frac{p'^2_-}{(\phi_- + i0)^2}. \tag{4.3.5}
$$

The integral in ϕ_- is related to the Airy-function (see Appendix B) which has the integral representation $\mathrm{Ai}(z) = \int_{-\infty}^{\infty}(d\tilde{\phi})/(2\pi)\exp[iz\tilde{\phi} + i\tilde{\phi}^3/3]$ (see Eq. (B.0.2) or Ref. [75]). It can be solved using the four basic integrals given in Eqs. (B.0.2)-(B.0.5) on page 100 after applying the substitutions $\tilde{\phi} = \left[q_-\boldsymbol{\mathcal{E}}^2(\phi_+)/(8p_- p'_-)\right]^{1/3}\phi_-$ and $z = \left[q_-/(p'_-\chi_p(\phi_+))\right]^{2/3}$. Further, since $p_-, p'_-, q_- \geq 0$ for on-shell particles and due to momentum conservation, the integration boundary in dq_- becomes finite. Finally we derive to the following probability of nonlinear Compton scattering including the damping of particle states

$$
P^{(e^-\to e^-\gamma)}_{j,s,s'} = -\frac{\alpha m^2}{4p_-^2}\int_0^{p_-} dq_- \int d\phi_+ \, e^{D^{\mathrm{NC}}_{j,s,s'}}\tilde{T}_{j,s,s'}, \tag{4.3.6}
$$

where

$$\begin{aligned}\tilde{T}_{1,s,s'} &= \left[1 + ss'\left(1 - \frac{q_-^2}{2p_-(p_- - q_-)}\right)\right]\mathrm{Ai}_1(z)\\ &+ \left[3 + \frac{q_-^2}{p_-(p_- - q_-)} + ss'\left(3 + \frac{q_-^2}{2p_-(p_- - q_-)}\right)\right]\frac{\mathrm{Ai}'(z)}{z}\\ &+ (s+s')\left(2\frac{q_-}{p_-} + \frac{q_-^2}{p_-(p_- - q_-)}\right)\frac{\mathrm{Ai}(z)}{\sqrt{z}}\mathrm{sgn}(\psi'(\phi_+))\end{aligned} \tag{4.3.7}$$

and

$$\begin{aligned}\tilde{T}_{2,s,s'} &= \left[1 + ss'\left(1 + \frac{q_-^2}{2p_-(p_- - q_-)}\right)\right]\mathrm{Ai}_1(z)\\ &+ \left[1 + \frac{q_-^2}{p_-(p_- - q_-)} + ss'\left(1 - \frac{q_-^2}{2p_-(p_- - q_-)}\right)\right]\frac{\mathrm{Ai}'(z)}{z}\\ &+ (s'-s)\frac{q_-^2}{p_-(p_- - q_-)}\frac{\mathrm{Ai}(z)}{\sqrt{z}}\mathrm{sgn}(\psi'(\phi_+)),\end{aligned} \tag{4.3.8}$$

with $\mathrm{Ai}_1(z) = \int_z^\infty dx \mathrm{Ai}(x)$ and with $\mathrm{sgn}(\psi'(\phi_+))$ denoting the sign of $\psi'(\phi_+)$. It is worth noticing that without the exponential damping the probability is in agreement with the spin- and polarization resolved expression of nonlinear Compton scattering presented in Ref. [40]. Therefore, neglecting the exponential damping and summing over the final quantum numbers yields $P_{s,p}^{\mathrm{NC}} = -(\alpha m^2)/(4p_-^2)\sum_{j,s'}\int_0^{p_-} dq_- \int d\phi_+ \, \tilde{T}_{j,s,s'}$.

4.4. Proof that the probability stays below unity

Our original motivation for the computation of the probability in Eq. (4.3.6) was, that the probability of nonlinear Compton scattering without radiative corrections exceeds unity for a sufficiently large phase length of the plane wave. At this point we want to proof that the total probability obtained in Eq. (4.3.6) summed and averaged over the quantum numbers of the final and initial states, respectively, indeed stays below unity. As the average of two numbers, which are smaller than unity, is smaller than unity as well, it is sufficient to proof that the probability $P_s^{(e^-\to e^-\gamma)} = \sum_{j,s'} P_{j,s,s'}^{(e^-\to e^-\gamma)}$ stays always below unity. In order of doing so we recall the following observations:

1. The damping function $D_{j,s,s'}^{\mathrm{NC}}$ is the sum of three functions where the exponential of each part itself is smaller or equal to unity, such that when neglecting part of them the expression is larger than the original one.

2. The probability without damping is in agreement with the probability of nonlinear Compton scattering without radiative corrections (see remark below Eq. (4.3.8)).

3. The exponents in the damping terms contain the probabilities of nonlinear Compton scattering and nonlinear Breit-Wheeler pair production without radiative corrections (see Eq. (4.1.10)).

With these observation the analytical proof that the probability $P_s^{(e^-\to e^-\gamma)}$ is always smaller than unity can be performed within a few steps:

$$\begin{aligned}
P_s^{(e^-\to e^-\gamma)} &< -\frac{\alpha m^2}{4p_-^2}\sum_{j,s'}\int_0^{p_-} dq_- \int d\phi_+ \, \tilde{T}_{j,s,s'} e^{2\mathrm{Im}\frac{m}{p_-}\int_{-\infty}^{\phi_+} d\varphi M_s(p,\varphi)} \\
&= \int d\phi_+ \, \frac{\partial P^{\mathrm{NC}}_{s,p}}{\partial\phi_+} e^{-\int_{-\infty}^{\phi_+} d\varphi \frac{\partial P^{\mathrm{NC}}_{s,p}}{\partial\varphi}} = -\int d\phi_+ \, \frac{\partial}{\partial\phi_+} e^{-\int_{-\infty}^{\phi_+} d\varphi \frac{\partial P^{\mathrm{NC}}_{s,p}}{\partial\varphi}} \\
&= 1 - e^{-\int_{-\infty}^{\infty} d\varphi \frac{\partial P^{\mathrm{NC}}_{s,p}}{\partial\varphi}} < 1.
\end{aligned} \tag{4.4.1}$$

Of course this proof also implies that the probability in Eq. (4.3.6) is smaller than unity.

4.5. Limits

In this Section we investigate the asymptotic behavior in the two regions $q_- \ll p_-$ and $p_- - q_- \ll p_-$ of the differential probability

$$\frac{\partial P^{(e^-\to e^-\gamma)}_{j,s,s'}}{\partial q_-} = -\frac{\alpha m^2}{4p_-^2}\int d\phi_+ e^{D^{\mathrm{NC}}_{j,s,s'}} \tilde{T}_{j,s,s'}, \tag{4.5.1}$$

which we obtained from Eq. (4.3.6).

Asymptotic expression for $q_- \ll p_-$

In the region $q_- \ll p_-$ the recoil of the outgoing photon is negligible. Starting with this case, we assume that the quantum nonlinearity parameter $\chi_p(\varphi)$ of the electron is fixed, and the absolute value of the quantum nonlinearity parameter $\kappa_q(\varphi) = (q_-/p_-)\chi_p(\varphi)$ of the photon is much smaller than unity (if $|\chi_p(\varphi)|$ is larger than unity, we presume that the ratio q_-/p_- is sufficiently small that $|\kappa_q(\varphi)| \ll 1$). In this way, the corresponding asymptotic expansion of the nonlinear Breit-Wheeler pair production probability in the damping function can be used, i.e. [9]

$$\frac{\partial P^{\mathrm{NBW}}_{j,q}}{\partial\varphi} \overset{\kappa_q(\varphi)\ll 1}{\approx} \sqrt{\frac{3}{2}}\frac{\alpha m^2|\kappa_q(\varphi)|j}{8q_-} e^{-\frac{8}{3|\kappa_q(\varphi)|}}. \tag{4.5.2}$$

We observe that for $\kappa_q(\varphi) \to 0$ this probability is exponentially suppressed and can be neglected within the damping function. As the recoil of the outgoing photon is negligible, we can further estimate that $p'_- \approx p_-$, such that the damping function in Eq. (4.1.10) is approximately given by

$$D^{\mathrm{NC}}_{j,s,s'} \overset{q_-\ll p_-}{\approx} -\int_{-\infty}^{\phi_+} d\varphi \frac{\partial P^{\mathrm{NC}}_{s,p}}{\partial\varphi} - \int_{\phi_+}^{\infty} d\varphi \frac{\partial P^{\mathrm{NC}}_{s',p}}{\partial\varphi}. \tag{4.5.3}$$

Moving over to the preexponential of the differential probability in Eq. (4.5.1), we can expand the Airy functions in the functions $\tilde{T}_{j,s,s'}$ for $z = \big(q_-/(p'_-\chi_p(\phi_+))\big)^{2/3} \approx \big(q_-/(p_-\chi_p(\phi_+))\big)^{2/3} \ll 1$ (we assume that $\chi_p(\phi_+)$ is fixed and that the ratio q_-/p_- is much smaller than $1/|\chi_p(\varphi)|$ if

$|\chi_p(\varphi)| < 1$). The asymptotic expressions of the Airy functions are presented in Eqs. (B.0.7)-(B.0.9) and with these the functions $\tilde{T}_{j,s,s'}$ reduce to

$$\tilde{T}_{1,s,s} \overset{z \ll 1}{\approx} -\frac{2 \times 3^{2/3}}{\Gamma\left(\frac{1}{3}\right) z}, \qquad \tilde{T}_{1,s,-s} \overset{z \ll 1}{\approx} -\frac{q_-^2}{2p_-^2} \frac{1}{3^{1/3}\Gamma\left(\frac{1}{3}\right) z}, \tag{4.5.4}$$

$$\tilde{T}_{2,s,s} \overset{z \ll 1}{\approx} -\frac{2}{3^{1/3}\Gamma\left(\frac{1}{3}\right) z}, \qquad \tilde{T}_{2,s,-s} \overset{z \ll 1}{\approx} -\frac{q_-^2}{2p_-^2} \frac{3^{2/3}}{\Gamma\left(\frac{1}{3}\right) z}, \tag{4.5.5}$$

where $\Gamma(x)$ is the Gamma-function. We observe that the quantities $\tilde{T}_{j,s,s'}$ are independent of the initial electron spin and that the probability with spin flip ($s = -s'$) is significantly suppressed in comparison to the probability without spin flip ($s = s'$) within the region $q_- \ll p_-$. This is in agreement with the expectations for the classical limit where the recoil of the photon is negligible (see Ref. [38]).

Now, considering identical spin quantum numbers, the damping function in Eq. (4.5.3) becomes independent of the phase and reduces to a constant, corresponding to minus the probability of an electron with momentum p and spin s emitting a photon between phase $-\infty$ and ∞. However, in the classical limit this quantity rather corresponds to the mean number of photons emitted by the electron than to a probability [42]. Therefore and according to the dominance of the terms with same spins $s = s'$, we can approximate in the classical limit the total probability with damping summed over the quantum numbers of the final states to be

$$P_s^{(e^- \to e^- \gamma)} \overset{\text{c.l.}}{\approx} P_{s,p}^{\text{NC}} e^{-P_{s,p}^{\text{NC}}}, \tag{4.5.6}$$

which describes a Poissonian distribution and is in agreement with the result found in Ref. [42].

Asymptotic expression for $p_- - q_- \ll p_-$

We move over to the case where $p'_- = p_- - q_- \ll p_-$. In this region the final photon takes up almost all the light-cone energy of the incoming electron. To obtain an asymptotic expression of the differential probability in Eq. (4.5.1), again we assume that the quantum nonlinearity parameter $\chi_p(\varphi)$ of the incoming electron is fixed. In this way the absolute value of the quantum nonlinearity parameter $\chi_{p'}(\varphi) = (p'_-/p_-)\chi_p(\varphi)$ of the outgoing electron can be considered as being smaller than unity (again, if $|\chi_p(\varphi)|$ is larger than unity, we presume that the ratio p'_-/p_- is sufficiently small that $|\chi_{p'}(\varphi)| \ll 1$). Similar to the previous case we can use for the decay of the outgoing electron the corresponding asymptotic expression for the probability of nonlinear Compton scattering in the damping function in Eq. (4.1.10). This one is independent of p' and given by [9]

$$\frac{\partial P_{s',p'}^{\text{NC}}}{\partial \varphi} \overset{\chi_{p'}(\varphi) \ll 1}{\approx} \frac{5}{2\sqrt{3}} \frac{\alpha m^2 |\chi_p(\varphi)|}{p_-}. \tag{4.5.7}$$

Since the photon takes up almost all the light-cone energy we approximate $q_- \approx p_-$ and the damping function becomes

$$D_{j,s,s'}^{\text{NC}} \overset{p'_- \ll p_-}{\approx} -\int_{-\infty}^{\phi_+} d\varphi \frac{\partial P_{s,p}^{\text{NC}}}{\partial \varphi} - \int_{\phi_+}^{\infty} d\varphi \left(\frac{\partial P_{j,p}^{\text{NBW}}}{\partial \varphi} + \frac{5}{\sqrt{3}} \frac{\alpha m^2 |\chi_p(\varphi)|}{p_-} \right). \tag{4.5.8}$$

Further, with a sufficiently small ratio p'_-/p_- we can expand the Airy-functions in the pre-exponential quantities $\tilde{T}_{j,s,s'}$ for $z = \left(q_-/(p'_-\chi_p(\phi_+))\right)^{2/3} \approx \left(p_-/(p'_-\chi_p(\phi_+))\right)^{2/3} \gg 1$. The asymptotic expressions of the Airy-functions are presented in Eqs. (B.0.10)-(B.0.12). In the following we consider each combination of quantum numbers separately. Starting with a photon polarization $j = 1$ and identical spin quantum numbers ($s = s'$) of the electrons, the asymptotic expression of $\tilde{T}_{1,s,s}$ is

$$\begin{aligned}\tilde{T}_{1,s,s} \overset{z\gg 1}{\approx} & -\frac{1}{\sqrt{\pi}} z^{-3/4} e^{-\frac{2}{3}z^{3/2}} \left[\frac{p_-}{p'_-}\left(1 - s\,\mathrm{sgn}(\psi'(\phi_+))\right) + 2\frac{p'_-}{p_-} s\,\mathrm{sgn}(\psi'(\phi_+))\right] \\ & - \frac{1}{96\sqrt{\pi}} z^{-9/4} e^{-\frac{2}{3}z^{3/2}} \left(124 + 20s\,\mathrm{sgn}(\psi'(\phi_+))\right) \\ & - \frac{1}{9216\sqrt{\pi}} z^{-15/4} e^{-\frac{2}{3}z^{3/2}} \frac{p_-}{p'_-} \left(3938 - 770s\,\mathrm{sgn}(\psi'(\phi_+))\right).\end{aligned} \tag{4.5.9}$$

Due to compensations which occur in the case $s = \mathrm{sgn}(\psi'(\phi_+))$ higher-orders have been reported here. However, considering an oscillating laser wave, the quantity $\mathrm{sgn}(\psi'(\phi_+))$ switches between the two values $+1$ and -1 for different phases ϕ_+. Since the functions $\tilde{T}_{j,s,s'}$ are finally integrated over ϕ_+ in order to obtain the total probability, the scaling of the probability is determined by the term in $\tilde{T}_{1,s,s}$ scaling with $z^{-3/4}/p'_-$ and we can approximate

$$\tilde{T}_{1,s,s} \overset{z\gg 1}{\approx} -\frac{2}{\sqrt{\pi}} z^{-3/4} e^{-\frac{2}{3}z^{3/2}} \frac{p_-}{p'_-}, \quad \text{for } s = -\mathrm{sgn}(\psi'(\phi_+)). \tag{4.5.10}$$

Moving over to the next case of opposite spin quantum numbers ($s = -s'$) the asymptotic expression is

$$\tilde{T}_{1,s,-s} \overset{z\gg 1}{\approx} -\frac{1}{4\sqrt{\pi}} z^{-9/4} e^{-\frac{2}{3}z^{3/2}} \frac{p_-}{p'_-}. \tag{4.5.11}$$

Considering outgoing photons with polarization $j = 2$, we obtain for identical spin quantum numbers ($s = s'$) the asymptotic expression

$$\tilde{T}_{2,s,s} \overset{z\gg 1}{\approx} -\frac{1}{4\sqrt{\pi}} z^{-9/4} e^{-\frac{2}{3}z^{3/2}} \frac{p_-}{p'_-} \tag{4.5.12}$$

and for opposite spin quantum numbers ($s = -s'$)

$$\begin{aligned}\tilde{T}_{2,s,-s} \overset{z\gg 1}{\approx} & -\frac{1}{\sqrt{\pi}} z^{-3/4} e^{-\frac{2}{3}z^{3/2}} \frac{p_-}{p'_-} \left[1 + s\,\mathrm{sgn}(\psi'(\phi_+))\right] \\ & - \frac{1}{9216\sqrt{\pi}} z^{-15/4} e^{-\frac{2}{3}z^{3/2}} \frac{p_-}{p'_-} \left[3938 + 770s\,\mathrm{sgn}(\psi'(\phi_+))\right].\end{aligned} \tag{4.5.13}$$

Analogously to the case of $\tilde{T}_{1,s,s}$ above, the final emission probability will scale with the approximated expression

$$\tilde{T}_{2,s,-s} \overset{z\gg 1}{\approx} -\frac{2}{\sqrt{\pi}} z^{-3/4} e^{-\frac{2}{3}z^{3/2}} \frac{p_-}{p'_-}, \quad \text{for } s = \mathrm{sgn}(\psi'(\phi_+)). \tag{4.5.14}$$

4.6. Independence of spin and polarization basis

Up to now, all calculations in this Chapter and in the previous Chapter we preformed considering a special direction for the spin and polarization of the electron/positron and photon, respectively (in the next Chapter we will choose these directions, too). The advantage of this choice is, that the mass- and polarization operator become diagonal and the equations involving the particles states decay are relatively easy to solve. However, of course the final probabilities should be independent of the choice of the spin and polarization basis at the end, when summing the results over the corresponding spin and/or polarization quantum numbers. In the following we will prove this independence explicitly.

First we consider the direction of the spin, the independence of the choice of the polarization basis follows analogously and is treated at the end. For an electron the spin state is presented via the two free, positive-energy spinors $u_1(p)$ and $u_{-1}(p)$, which form the spin basis in our case (see also Section 2.6 and Ref. [11]). Similarly, the two free negative-energy spinors $v_1(p)$ and $v_{-1}(p)$ form the spin basis of the positron and we restrict us in the following to the case of the electron as the proof for the positron follows analogously. Now, as expected for a basis, an arbitrary spinor $u_+(p)$, corresponding to an arbitrary direction of the electron spin, can be decomposed into a linear combination of the two basis spinors $u_1(p)$ and $u_{-1}(p)$, i.e.

$$u_+(p) = \beta_1 u_1(p) + \beta_{-1} u_{-1}(p). \tag{4.6.1}$$

Here both coefficients, β_1 and β_{-1}, are complex numbers fulfilling the relation $|\beta_1|^2+|\beta_{-1}|^2 = 1$ and they are related to the polar and the azimuthal angle between the spin vector $\boldsymbol{\zeta}$ and the new spin axis corresponding to the spinor $u_+(p)$ [11]. We further introduce the spinor

$$u_-(p) = \beta^*_{-1} u_1(p) - \beta^*_1 u_{-1}(p), \tag{4.6.2}$$

which is perpendicular to $u_+(p)$. Together the spinors $u_+(p)$ and $u_-(p)$ form again a new basis. Now we consider the radiatively corrected probability of nonlinear Compton scattering in Eq. (4.3.6), which was calculated from the S-matrix by using the exact electron and photon states. These states were obtained by solving the corresponding Schwinger-Dyson equations, which is, for example, for the electron out-state $\Psi^{(\text{out})}_{e^-}(x)$ given by $[i\hat{\partial} - \hat{\mathcal{A}}(\phi) - m]\Psi^{(\text{out})}_{e^-}(x) = \int d^4y\, \bar{M}_{PW}(y,x)\Psi^{(\text{out})}_{e^-}(y)$ (see Eq. (3.2.6)). We observed in Section 3.2 that this equation is linear in the spin basis $u_1(p)$ and $u_{-1}(p)$, such that we can construct the states $\Psi^{(\text{out})}_{e^-,1}(x)$ and $\Psi^{(\text{out})}_{e^-,-1}(x)$ via those spinors, respectively, which are solutions of the Schwinger-Dyson equation for the corresponding spinor. Therefore the electron out-state can be decomposed into the sum $\Psi^{(\text{out})}_{e^-}(x) = \Psi^{(\text{out})}_{e^-,1}(x) + \Psi^{(\text{out})}_{e^-,-1}(x)$.

Now, if we consider to have an arbitrary spin direction, where the spinors $u_b(p)$ with spin quantum numbers $b = \{+,-\}$ form a basis, the electron out-state $\Psi^{(\text{out})}_{e^-,b}(x)$, which is a solution of the Schwinger-Dyson equation corresponding to those new spinors, can then be expressed by the linear combination

$$\begin{aligned}\Psi^{(\text{out})}_{e^-,+}(x) =& \beta_1 \Psi^{(\text{out})}_{e^-,1}(x) + \beta_{-1}\Psi^{(\text{out})}_{e^-,-1}(x),\\ \Psi^{(\text{out})}_{e^-,-}(x) =& \beta^*_{-1} \Psi^{(\text{out})}_{e^-,1}(x) - \beta^*_1\Psi^{(\text{out})}_{e^-,-1}(x).\end{aligned} \tag{4.6.3}$$

Note that, as the electron out-states $\Psi^{(\text{out})}_{e^-,1}(x)$ and $\Psi^{(\text{out})}_{e^-,-1}(x)$ evolve differently under the Schwinger-Dyson equation, the spin axis of the state $\Psi^{(\text{out})}_{e^-,b}(x)$ with the above choice of β_1 and β_{-1} is intended to point along the chosen axis at $x^0 \to \infty$ in the rest frame of the electron. With the above relations we can write the S-matrix element of nonlinear Compton scattering for an incoming electron of spin quantum number s, an outgoing photon of polarization j, and an outgoing electron with spin quantum number b, into the form

$$\begin{aligned} S^{(e^-\to e^-\gamma)}_{j,s,+} &\sim \int d^4x\ \bar{\Psi}^{(\text{out})}_{e^-,+}(p',x)\hat{\mathcal{A}}^{(\text{out})*}_j(q,x)\Psi^{(\text{in})}_{e^-,s}(p,x) \\ &\sim \beta_1^* S^{(e^-\to e^-\gamma)}_{j,s,1} + \beta_{-1}^* S^{(e^-\to e^-\gamma)}_{j,s,-1} \end{aligned} \tag{4.6.4}$$

and

$$\begin{aligned} S^{(e^-\to e^-\gamma)}_{j,s,-} &\sim \int d^4x\ \bar{\Psi}^{(\text{out})}_{e^-,-}(p',x)\hat{\mathcal{A}}^{(\text{out})*}_j(q,x)\Psi^{(\text{in})}_{e^-,s}(p,x) \\ &\sim \beta_{-1} S^{(e^-\to e^-\gamma)}_{j,s,1} - \beta_1 S^{(e^-\to e^-\gamma)}_{j,s,-1}. \end{aligned} \tag{4.6.5}$$

Now it is straight forward to express the probability of nonlinear Compton scattering for the two final electron spin quantum numbers $+$ and $-$ in terms of S-matrix elements. They are given by

$$\begin{aligned} P^{(e^-\to e^-\gamma)}_{j,s,+} &= \int \frac{d^3q}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}|S^{(e^-\to e^-\gamma)}_{j,s,+}|^2 \\ &= \int \frac{d^3q}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\Big[|\beta_1|^2|S^{(e^-\to e^-\gamma)}_{j,s,1}|^2 + |\beta_{-1}|^2|S^{(e^-\to e^-\gamma)}_{j,s,-1}|^2 \\ &\qquad + \beta_1\beta_{-1}^* S^{(e^-\to e^-\gamma)*}_{j,s,1} S^{(e^-\to e^-\gamma)}_{j,s,-1} + \beta_1^*\beta_{-1} S^{(e^-\to e^-\gamma)}_{j,s,1} S^{(e^-\to e^-\gamma)*}_{j,s,-1}\Big] \end{aligned} \tag{4.6.6}$$

and

$$\begin{aligned} P^{(e^-\to e^-\gamma)}_{j,s,-} &= \int \frac{d^3q}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}|S^{(e^-\to e^-\gamma)}_{j,s,-}|^2 \\ &= \int \frac{d^3q}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\Big[|\beta_{-1}|^2|S^{(e^-\to e^-\gamma)}_{j,s,1}|^2 + |\beta_1|^2|S^{(e^-\to e^-\gamma)}_{j,s,-1}|^2 \\ &\qquad - \beta_1\beta_{-1}^* S^{(e^-\to e^-\gamma)*}_{j,s,1} S^{(e^-\to e^-\gamma)}_{j,s,-1} - \beta_1^*\beta_{-1} S^{(e^-\to e^-\gamma)}_{j,s,1} S^{(e^-\to e^-\gamma)*}_{j,s,-1}\Big]. \end{aligned} \tag{4.6.7}$$

Summing the probability now over the spin quantum number $b = \{+,-\}$ of the outgoing electron, i.e. taking the sum of Eqs. (4.6.6) and (4.6.7), we derive to

$$\begin{aligned} \sum_{b=\{+,-\}} P^{(e^-\to e^-\gamma)}_{j,s,b} &= \int \frac{d^3q}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\Big[|S^{(e^-\to e^-\gamma)}_{j,s,+}|^2 + |S^{(e^-\to e^-\gamma)}_{j,s,-}|^2\Big] \\ &= \int \frac{d^3q}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\Big[|S^{(e^-\to e^-\gamma)}_{j,s,1}|^2 + |S^{(e^-\to e^-\gamma)}_{j,s,-1}|^2\Big] = \sum_{s'=\{1,-1\}} P^{(e^-\to e^-\gamma)}_{j,s,s'}, \end{aligned} \tag{4.6.8}$$

which is identical to the analogous result obtained for the original spin axis with spin quantum numbers $s = \{1,-1\}$. Therefore, the probability is independent of the choice of the spin quantization axis after summing over the corresponding spin quantum numbers. This is true for electron in-states and for the positron in- and out-states, too, and the proof thereof is analogous to the one of the electron out-state.

Finally we discuss the photon polarization. For a transverse polarized photon we had chosen the two four-vectors $\Lambda_1^\mu(q)$ and $\Lambda_2^\mu(q)$ in order to describe the photon polarization state. They form a basis for the plane perpendicular to the photon momentum and any other polarization basis, e.g. given by the two four-vectors $\Lambda_+^\mu(q)$ and $\Lambda_-^\mu(q)$, can be decomposed into the linear combination

$$\begin{aligned}\Lambda_+^\mu(q) &= b_1\Lambda_1^\mu(q) + b_2\Lambda_2^\mu(q),\\ \Lambda_-^\mu(q) &= b_2\Lambda_1^\mu(q) - b_1\Lambda_2^\mu(q),\end{aligned} \tag{4.6.9}$$

where b_1, b_2 are real numbers obeying $b_1^2 + b_2^2 = 1$. Following similar steps as for the electron spin we can finally conclude, that the probability is also independent of the choice of the polarization basis after summing over the corresponding polarization index.

4.7. Numerical results

Finally some numerical results for the probability of nonlinear Compton scattering including the decay of the states in Eq. (4.3.6) will be presented. The numerical results shown in this Section are taken from the reference [21] and were all performed by Victor Dinu. However they are presented here to visualize some properties of the probability which we also found in the analytical investigations.

For the numerical calculations the vector potential of the background field was chosen as

$$\boldsymbol{A}(\phi) = A_0 e^{-(\phi/\tau)^2}\sin(\omega_0\phi)\boldsymbol{a}_1, \tag{4.7.1}$$

describing a linearly polarized plane wave laser pulse with Gaussian envelope. Here, the carrier frequency ω_0 is fixed to 1.55 eV in our units and the parameter τ describes the length of the pulse.

In the plots in Figs. 4.3 and 4.4 the probabilities are presented in a color code and are plotted over the classical nonlinearity parameter ξ_0 on the horizontal axis and the energy of the incoming electron in MeV (and the parameter $\eta_0 = \chi_0/\xi_0 = (k_0p)/m^2$) on the vertical axis. As the probabilities are valid within the LCFA the parameter ξ_0 is here restricted to values $\xi_0 > 5$. Further, since we assumed that $\chi_0 \sim 1$, the parameter η_0 is restricted to be smaller than unity, which corresponds to incoming electron energies smaller than 100 GeV.

Now in Fig. 4.3 the top panel presents the probability in Eq. (4.3.6) summed over the final quantum numbers, i.e. $P_s^{(e^-\to e^-\gamma)} = \sum_{j,s'} P_{j,s,s'}^{(e^-\to e^-\gamma)}$, whereas the bottom panel presents a Poissonian distribution of the form $P_{s,p}^{\mathrm{NC}}\exp[-P_{s,p}^{\mathrm{NC}}]$, both for spin quantum number $s = 1$ and a pulse length of $\tau = 5$ fs.

We see that the probability of nonlinear Compton scattering is in agreement with that of a Poissonian distribution in the case of low initial electron energies. We observed this behavior also in the classical limit of the analytical expressions presented in Eq. (4.5.6) (see also Ref. [42]). Important deviations occur however for increasing incoming electron energies.

In Fig. 4.4 again the probability of nonlinear Compton scattering is presented as in Fig. 4.3 top panel, but now for a pulse length of $\tau = 20$ fs.

Now if we compare the top panel of Fig. 4.3 with Fig. 4.4 we observe the following: In the top panel of Fig. 4.3 the probability first increases with increasing ξ_0 and reaches its maximum for $\xi_0 \lesssim 10$ within the presented range of η_0. Here the maximum of the probability lies around $e^{-1} \approx 0.367$ for low values of η_0, as expected from the Poissonian distribution. After the maximum the probability is decreasing monotonically with growing ξ_0. Instead in Fig. 4.4 the probability is monotonically decreasing with increasing ξ_0 within the whole presented range of η_0. Further the probability is staying well below the values of Fig. 4.3 (note the different scales of ξ_0 on the horizontal axis). This indicates that the exponential damping of the probability is stronger in the case of a larger pulse length in Fig. 4.4, which we also expected from our analytical investigations and discussed below Eq. (4.1.10).

Finally we observe in both cases, Fig. 4.3 and Fig. 4.4, that the probability always stays well below unity, like we presented in the analytical proof in Eq. (4.4.1).

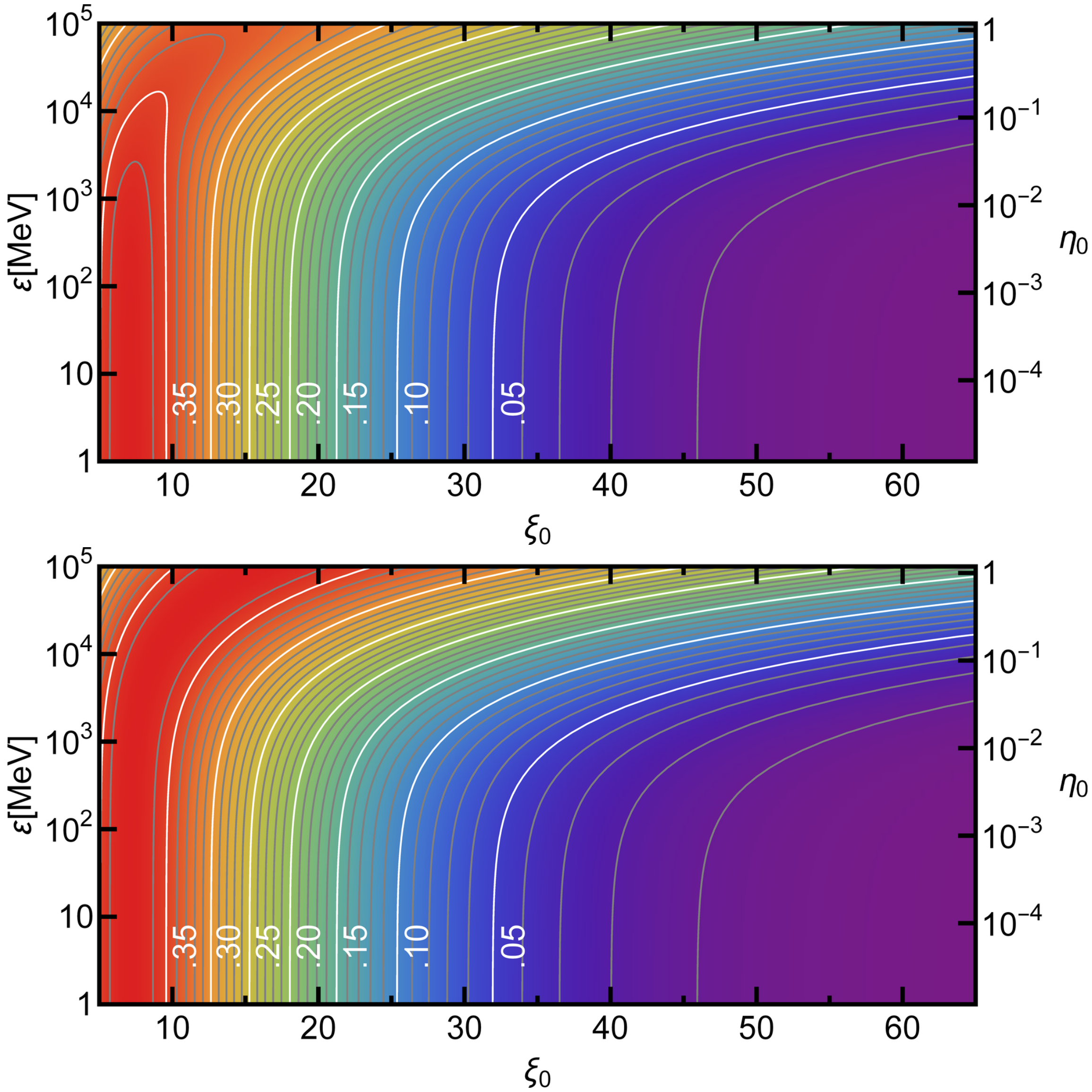

Figure 4.3.: The total probability of nonlinear Compton scattering with damping $P_s^{(e^- \to e^- \gamma)}$ (top panel), as compared to the result obtained from a Poissonian distribution whose average photon number is the total "undamped emission probability" $P_{s,p}^{\mathrm{NC}}$ (bottom panel). The pulse length corresponds to $\tau = 5$ fs and the initial electron spin corresponds to $s = 1$. The plots are taken from Ref. [21] and were done by Victor Dinu.

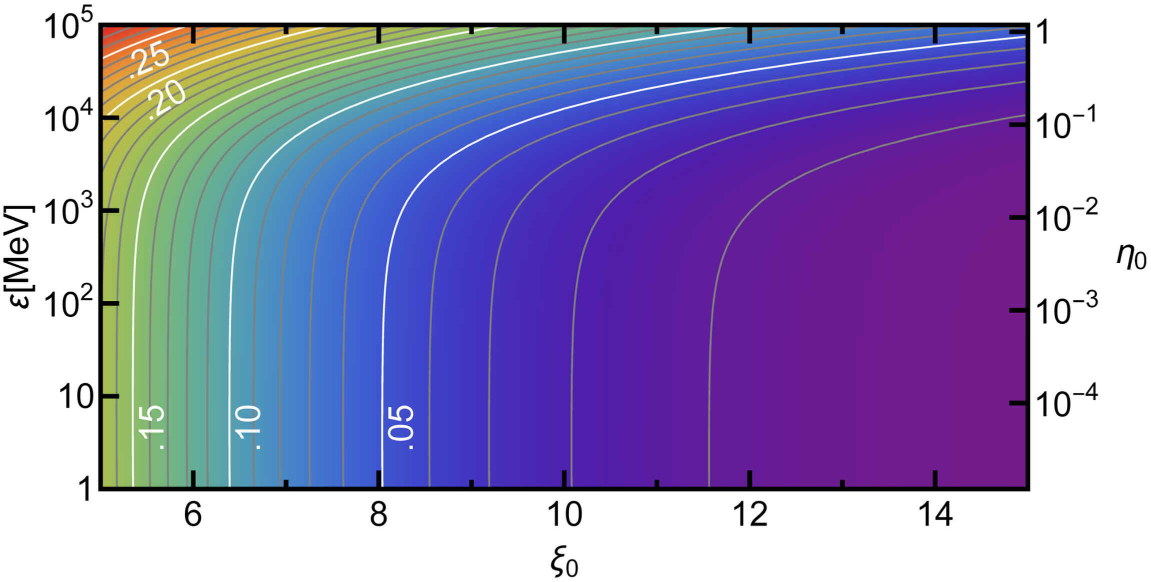

Figure 4.4.: Nonlinear Compton scattering total probability $P_s^{(e^- \to e^- \gamma)}$ including the decay of the wave functions, as in Fig. 4.3 top panel, but here for $\tau = 20$ fs. Again the plot is taken from Ref. [21] and was done by Victor Dinu.

5. Nonlinear Breit-Wheeler pair production including particle states decay

Note that the content of this chapter was published in the publications [20, 21] and therefore the structure, the equations, and the text of this chapter are similar or identical to Refs. [20,21].

Similar as to the last chapter, we calculate in this chapter the radiatively corrected probability but now for nonlinear Breit-Wheeler pair production. For this the states in the S-matrix in Eq. (3.0.2) are replaced by the corresponding electron, positron and photon states including the effects of the states decay, which we obtained in Sections 3.2 and 3.3.

In the case of nonlinear Breit-Wheeler pair production, we assume the incoming photon to have four-momentum $q^\mu = (\omega, \boldsymbol{q})$, with energy $\omega = |\boldsymbol{q}|$, and its asymptotic transverse polarization state is indicated by the index $j = 1, 2$. For the outgoing positron (electron) the four-momentum is $p^\mu = (\varepsilon, \boldsymbol{p})$ ($p'^\mu = (\varepsilon', \boldsymbol{p}')$), with energy $\varepsilon = \sqrt{m^2 + \boldsymbol{p}^2}$ ($\varepsilon' = \sqrt{m^2 + \boldsymbol{p}'^2}$), and it is supposed to have the asymptotic spin quantum number $s = \pm 1$ ($s' = \pm 1$). With this choice of symbols for the particles quantum numbers it is easy to exploit the symmetry between nonlinear Compton scattering and nonlinear Breit-Wheeler pair production. The leading-order S-matrix amplitude in α of nonlinear Breit-Wheeler pair production including the states decay is then given by (compare Eq. (3.0.2))

$$S^{(\gamma\to e^-e^+)}_{j,s,s'} = -ie\int d^4x\, \bar{\Psi}^{R,(\mathrm{out})}_{e^-,s',p'}(x)\hat{\mathscr{A}}^{(\mathrm{in})}_{R,j}(q,x)\Psi^{R,(\mathrm{out})}_{e^+,s,p}(x). \tag{5.0.1}$$

This S-matrix intrinsically contains the resummation of all one-particle reducible diagrams with corrections by the one-loop mass and polarization operator, as depicted in Fig. 5.1.

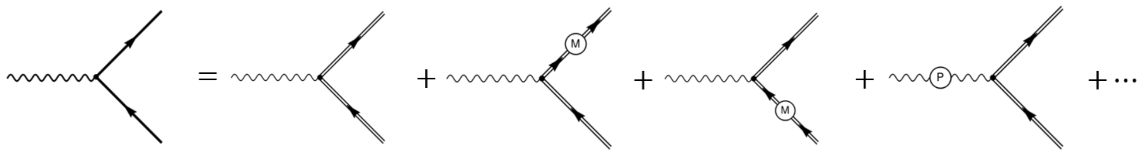

Figure 5.1.: The exact amplitude of nonlinear Breit-Wheeler pair production (first diagram) corresponds to the sum of all one-particle reducible diagrams with corrections by the one loop mass and polarization operators (circles with M and P, respectively). Shown are the first Feynman diagrams of this series.

After inserting the expressions of the radiatively corrected states presented in Eqs. (3.2.24), (3.2.25), and (3.3.14) into the S-matrix, we arrive to:

$$\begin{aligned}
S^{(\gamma\to e^-e^+)}_{j,s,s'} &= -\frac{ie}{\sqrt{8\varepsilon\varepsilon'\omega}}(2\pi)^3\delta^2(\boldsymbol{p}'_\perp-\boldsymbol{q}_\perp+\boldsymbol{p}_\perp)\delta(p'_--q_-+p_-)\\
&\times\int d\phi\, e^{-i\frac{m}{q_-}\int_{-\infty}^{\phi}d\varphi P_j(q,\varphi)-i\int_\phi^\infty d\varphi\left[\frac{m}{p'_-}M_{s'}(p',\varphi)+\frac{m}{p_-}M_s(-p,\varphi)\right]}\\
&\times e^{i\left\{(p'_+-q_++p_+)\phi-\int_\phi^\infty d\varphi\left[\frac{(p'\mathcal{A}(\varphi))}{p'_-}-\frac{\mathcal{A}^2(\varphi)}{2p'_-}\right]+\int_\phi^\infty d\varphi\left[\frac{(p\mathcal{A}(\varphi))}{p_-}+\frac{\mathcal{A}^2(\varphi)}{2p_-}\right]\right\}}\\
&\times\bar{u}_{s'}(p')\left[1-\frac{\hat{n}\hat{\mathcal{A}}(\phi)}{2p'_-}\right]\hat{\Lambda}_j(q)\left[1-\frac{\hat{n}\hat{\mathcal{A}}(\phi)}{2p_-}\right]v_s(p).
\end{aligned}\tag{5.0.2}$$

5.1. The radiatively corrected probability

With the S-matrix in Eq. (5.0.2) we can compute the probability of nonlinear Breit-Wheeler pair production,

$$\begin{aligned}
P^{(\gamma\to e^-e^+)}_{j,s,s'} &= \int\frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}|S^{(\gamma\to e^-e^+)}_{j,s,s'}|^2\\
&= \int\frac{d^3p}{16\pi^2}\frac{d^3p'}{(2\pi)^3}\frac{\alpha}{\varepsilon\varepsilon'\omega}(2\pi)^6[\delta^2(\boldsymbol{p}'_\perp-\boldsymbol{q}_\perp+\boldsymbol{p}_\perp)\delta(p'_--q_-+p_-)]^2\\
&\times\int d\phi d\phi' e^{-i\frac{m}{q_-}\int_{-\infty}^{\phi}d\varphi P_j(q,\varphi)+i\frac{m}{q_-}\int_{-\infty}^{\phi'}d\varphi P^*_j(q,\varphi)}\\
&\times e^{-i\int_\phi^\infty d\varphi\left[\frac{m}{p'_-}M_{s'}(p',\varphi)+\frac{m}{p_-}M_s(-p,\varphi)\right]+i\int_{\phi'}^\infty d\varphi\left[\frac{m}{p'_-}M^*_{s'}(p',\varphi)+\frac{m}{p_-}M^*_s(-p,\varphi)\right]}\\
&\times e^{i(p'_+-q_++p_+)(\phi-\phi')+i\int_{\phi'}^{\phi}d\varphi\left[\frac{(p'\mathcal{A}(\varphi))}{p'_-}-\frac{\mathcal{A}^2(\varphi)}{2p'_-}-\frac{(p\mathcal{A}(\varphi))}{p_-}-\frac{\mathcal{A}^2(\varphi)}{2p_-}\right]}\\
&\times\mathrm{tr}\left\{\left[1-\frac{\hat{n}\hat{\mathcal{A}}(\phi)}{2p'_-}\right]\hat{\Lambda}_j(q)\left[1-\frac{\hat{n}\hat{\mathcal{A}}(\phi)}{2p_-}\right]v_s(p)\bar{v}_s(p)\right.\\
&\times\left.\left[1+\frac{\hat{n}\hat{\mathcal{A}}(\phi')}{2p_-}\right]\hat{\Lambda}_j(q)\left[1+\frac{\hat{n}\hat{\mathcal{A}}(\phi')}{2p'_-}\right]u_{s'}(p')\bar{u}_{s'}(p')\right\}.
\end{aligned}\tag{5.1.1}$$

Now we can perform similar steps as in the case of nonlinear Compton scattering, where in this case the negative-energy electron density matrix can be replaced by $v_s(p)\bar{v}_s(p)=(\hat{p}-m)(1+s\gamma^5\hat{\zeta}_p)/2$ [11]. Alternatively the probability can be obtained from Eq. (4.1.6) by symmetry considerations. In both cases one arrives to the following probability of nonlinear Breit-Wheeler pair production including the decay of particle states within the LCFA and a linearly polarized plane wave

$$\begin{aligned}
P^{(\gamma\to e^-e^+)}_{j,s,s'} &= \int\frac{d^3p}{16\pi^2}\frac{\alpha}{q_-p'_-\varepsilon}\int d\phi_+ e^{D^{\mathrm{NBW}}_{j,s,s'}}\\
&\times\int d\phi_- e^{i\frac{m^2}{2p_-}\frac{q_-}{p'_-}\left\{[1+\boldsymbol{\pi}^2_{\perp,p}(\phi_+)]\phi_-+\frac{\boldsymbol{\varepsilon}^2(\phi_+)}{m^2}\frac{\phi_-^3}{12}\right\}}G_{j,s,s'},
\end{aligned}\tag{5.1.2}$$

with the exponential damping function

$$D^{\mathrm{NBW}}_{j,s,s'} = 2\mathrm{Im}\left\{\frac{m}{q_-}\int_{-\infty}^{\phi_+} d\varphi P_j(q,\varphi) + \int_{\phi_+}^{\infty} d\varphi\left[\frac{m}{p'_-}M_{s'}(p',\varphi) + \frac{m}{p_-}M_s(-p,\varphi)\right]\right\}, \tag{5.1.3}$$

the trace

$$\begin{aligned} G_{j,s,s'} = \frac{1}{4}\mathrm{tr}\Bigg\{&\left[1-\frac{\hat{n}[\hat{\mathcal{A}}(\phi_+)+\hat{\mathcal{A}}'(\phi_+)\phi_-/2]}{2p'_-}\right]\hat{\Lambda}_j(q)\left[1-\frac{\hat{n}[\hat{\mathcal{A}}(\phi_+)+\hat{\mathcal{A}}'(\phi_+)\phi_-/2]}{2p_-}\right] \\ &\times(\hat{p}-m)(1+s\gamma^5\hat{\zeta}_p)\left[1+\frac{\hat{n}[\hat{\mathcal{A}}(\phi_+)-\hat{\mathcal{A}}'(\phi_+)\phi_-/2]}{2p_-}\right]\hat{\Lambda}_j(q) \\ &\times\left[1+\frac{\hat{n}[\hat{\mathcal{A}}(\phi_+)-\hat{\mathcal{A}}'(\phi_+)\phi_-/2]}{2p'_-}\right](\hat{p}'+m)(1+s'\gamma^5\hat{\zeta}_{p'})\Bigg\}, \end{aligned} \tag{5.1.4}$$

and the transverse momentum

$$\boldsymbol{\pi}_{\perp,p}(\phi) = \frac{\boldsymbol{p}_\perp}{m} - \frac{p_-}{q_-}\frac{\boldsymbol{q}_\perp}{m} + \frac{\boldsymbol{\mathcal{A}}_\perp(\phi)}{m}. \tag{5.1.5}$$

Here, due to energy-momentum conservation the minus component and the perpendicular component of the outgoing electron momentum are fixed to $p'_- = q_- - p_-$ and $\boldsymbol{p}'_\perp = \boldsymbol{q}_\perp - \boldsymbol{p}_\perp$, respectively.

New in comparison to the expression of the probability without the decay of the states (see for example Refs. [29, 40]), is the exponential damping in Eq. (5.1.2) with the damping function $D^{\mathrm{NBW}}_{j,s,s'}$. Similar to the case of nonlinear Compton scattering, the damping function is here, according to the optical theorem, equal to minus the sum of the total probability of nonlinear Breit-Wheeler pair production between $-\infty$ and ϕ_+ for the incoming photon with light-cone energy q_- and polarization quantum number j and the total probabilities of nonlinear Compton scattering between ϕ_+ and $+\infty$ for the outgoing electron with light-cone energy p'_- and spin quantum number s' and for the outgoing positron with light-cone energy p_- and spin quantum number s [57, 68], i.e.

$$D^{\mathrm{NBW}}_{j,s,s'} = -\int_{-\infty}^{\phi_+} d\varphi\frac{\partial P^{\mathrm{NBW}}_{j,q}}{\partial\varphi} - \int_{\phi_+}^{\infty} d\varphi\left(\frac{\partial P^{\mathrm{NC}}_{s',p'}}{\partial\varphi} + \frac{\partial P^{\mathrm{NC}}_{s,-p}}{\partial\varphi}\right). \tag{5.1.6}$$

Analogous remarks about the damping as for the case of nonlinear Compton scattering can be made here, too: The damping becomes significant when the product $\alpha\xi_0\Phi_L \gtrsim 1$, with Φ_L being the pulse phase length and assuming that $\chi_0 \sim 1$ and $\kappa_0 \sim 1$. Again the dependence of the damping on the quantum numbers prevents us using summation rules [3, 11] and forces us to employ the spin and polarization resolved trace. This trace will be computed in the next Section.

5.2. Solving the trace

In this Section we are going to compute the spin and polarization resolved trace for nonlinear Breit-Wheeler pair production in Eq. (5.1.4). This trace was already solved in Ref. [40] (see also Refs. [38, 74]), here however a short alternative analytical derivation based on the results for nonlinear Compton scattering in Section 4.2 is presented.

Using the notation in Eq. (4.2.1) the trace for nonlinear Breit-Wheeler pair production in Eq. (5.1.4) can be expressed as

$$G_{j,s,s'} = -\frac{1}{4}\mathrm{tr}\left\{\hat{\Lambda}_j(q)Q_{-p,s}(\phi_+,\phi_-)\hat{\Lambda}_j(q)Q_{p',s'}(\phi_+,-\phi_-)\right\}. \tag{5.2.1}$$

Now comparing this trace with the one of nonlinear Compton scattering in Eq. (4.2.2) we observe that we can derive the final expression of the trace directly from the results for nonlinear Compton scattering in Eqs. (4.2.15)-(4.2.20). For this we have to multiply Eq. (4.2.15) by an overall minus sign and change the sign of the four-momentum p^μ in the Eqs. (4.2.16)-(4.2.20). With our choice of the spin and polarization four-vectors and the conservation laws for the minus and perpendicular light-cone components of the four-momenta we obtain for the two polarization states $j = 1$ and $j = 2$

$$\begin{aligned} G_{1,s,s'} = (1+ss')\Bigg[&\frac{1}{2}\frac{q_-^2}{p_-p'_-}\left(m^2 + m^2\boldsymbol{\pi}^2_{\perp,p}(\phi_+) - \mathcal{A}_0^2\psi'^2(\phi_+)\frac{\phi_-^2}{4}\right) \\ &- 2\left(m^2\boldsymbol{\pi}^2_{\perp,p}(\phi_+) - \mathcal{A}_0^2\psi'^2(\phi_+)\frac{\phi_-^2}{4}\right)\Bigg] \\ + i(s+s')&\frac{m}{2}\mathcal{A}_0\psi'(\phi_+)\phi_-\frac{q_-}{p_-}\left(2-\frac{q_-}{p'_-}\right) \\ + \Bigg(2+2ss'&-ss'\frac{q_-^2}{p_-p'_-}\Bigg)\left(p_2-\frac{p_-}{q_-}q_2\right)^2 \end{aligned} \tag{5.2.2}$$

and

$$\begin{aligned} G_{2,s,s'} = (1-ss')&\frac{1}{2}\frac{q_-^2}{p_-p'_-}\left(m^2 + m^2\boldsymbol{\pi}^2_{\perp,p}(\phi_+) - \mathcal{A}_0^2\psi'^2(\phi_+)\frac{\phi_-^2}{4}\right) \\ + i(s-s')&\frac{m}{2}\mathcal{A}_0\psi'(\phi_+)\phi_-\frac{q_-^2}{p_-p'_-} \\ - \Bigg(2+2ss'&-ss'\frac{q_-^2}{p_-p'_-}\Bigg)\left(p_2-\frac{p_-}{q_-}q_2\right)^2, \end{aligned} \tag{5.2.3}$$

where the pulse shape function was already expressed in terms of $\boldsymbol{\pi}_{\perp,p}(\phi_+)$ (see Eq. (5.1.5)). Note that here the conservation laws for the minus and perpendicular light-cone components of the four-momenta are different than for nonlinear Compton scattering.

5.3. Final integrals

In the following we will perform the integration over the variables $d\phi_-$ and $d^2\boldsymbol{p}_\perp$ analogue to the case of nonlinear Compton scattering (see also Refs. [29, 39, 40]). For this we first recall that, similar as in the case of nonlinear Compton scattering, the dependence of the traces on $\boldsymbol{\pi}_{\perp,p}(\phi_+)$ can be removed by using properties of the Airy function (see Appendix B). In fact, according to Eq. (B.0.6), terms proportional to $\left[1+\boldsymbol{\pi}^2_{\perp,p}(\phi_+) + (\mathcal{A}_0^2\psi'^2(\phi_+)/m^2)(\phi_-^2/4)\right]$ do not contribute to the probability as they vanish after the integration in ϕ_-. Therefore, after adding and subtracting suitable terms, the traces are manifestly gauge invariant. Continuing with the integral in $d^2\boldsymbol{p}_\perp$, we use the relation $d^3p = (\varepsilon/p_-)dp_-d^2\boldsymbol{p}_\perp$ to transform the integral in the positron momentum into light-cone coordinates and introduce the notation

$$\tilde{G}_{j,s,s'} = -\frac{1}{4\pi^2 m^2}\frac{q_-}{p_-p'_-}\int d\phi_- \int d^2\boldsymbol{p}_\perp e^{i\frac{m^2 q_-}{2p_-p'_-}\left\{[1+\boldsymbol{\pi}^2_{\perp,p}(\phi_+)]\phi_- + \frac{\boldsymbol{\varepsilon}^2(\phi_+)}{m^2}\frac{\phi_-^3}{12}\right\}} G_{j,s,s'}. \tag{5.3.1}$$

Similar as to the previous Chapter the integral in $d^2\boldsymbol{p}_\perp$ turns out to be Gaussian and is solved by [75]

$$\int d^2\boldsymbol{p}_\perp e^{i\frac{m^2 q_-}{2p_-p'_-}\boldsymbol{\pi}^2_{\perp,p}(\phi_+)\phi_-} = 2\pi i\frac{p_-p'_-}{q_-(\phi_- + i0)}, \tag{5.3.2}$$

$$\int d^2\boldsymbol{p}_\perp \left(p_2 - \frac{p_-}{q_-}q_2\right)^2 e^{i\frac{m^2 q_-}{2p_-p'_-}\boldsymbol{\pi}^2_{\perp,p}(\phi_+)\phi_-} = -2\pi\left[\frac{p_-p'_-}{q_-(\phi_- + i0)}\right]^2. \tag{5.3.3}$$

The remaining integral in ϕ_- is solved by Airy functions (see Eqs. (B.0.2)-(B.0.5)). With that we derive to the final expression of the probability of nonlinear Breit-Wheeler pair production including the damping due to the particle states decay:

$$P^{(\gamma\to e^-e^+)}_{j,s,s'} = -\frac{\alpha m^2}{4q_-^2}\int_0^{q_-} dp_- \int d\phi_+ e^{D^{\text{NBW}}_{j,s,s'}}\tilde{G}_{j,s,s'}, \tag{5.3.4}$$

with

$$\begin{aligned}\tilde{G}_{1,s,s'} &= \left[-(1+ss') - ss'\frac{q_-^2}{2p_-p'_-}\right]\text{Ai}_1(z) \\ &+ \left[-3(1+ss') + \left(1+\frac{ss'}{2}\right)\frac{q_-^2}{p_-p'_-}\right]\frac{\text{Ai}'(z)}{z} \\ &- (s+s')\left(\frac{q_-}{p_-} - \frac{q_-}{p'_-}\right)\frac{\text{Ai}(z)}{\sqrt{z}}\text{sgn}(\psi'(\phi_+))\end{aligned} \tag{5.3.5}$$

and

$$\begin{aligned}\tilde{G}_{2,s,s'} &= \left[-(1+ss') + ss'\frac{q_-^2}{2p_-p'_-}\right]\text{Ai}_1(z) \\ &+ \left[-(1+ss') + \left(1-\frac{ss'}{2}\right)\frac{q_-^2}{p_-p'_-}\right]\frac{\text{Ai}'(z)}{z} \\ &+ (s'-s)\frac{q_-^2}{p_-p'_-}\frac{\text{Ai}(z)}{\sqrt{z}}\text{sgn}(\psi'(\phi_+)).\end{aligned} \tag{5.3.6}$$

This expression describes the radiatively corrected probability of a photon with momentum q and polarization j decaying into an electron-positron pair in a linearly polarized plane wave background field. Processes where the final electrons and positrons emit additional photons are not included in this probability. Note that, due to momentum conservation, the minus component of the electron momentum has to be replaced by $p'_- = q_- - p_-$ before performing the remaining integral in p_-.

The above probability, ignoring the exponential damping term, is in agreement with the spin- and polarization-resolved probability of nonlinear Breit-Wheeler pair production presented in Ref. [40]. Moreover, neglecting the exponential damping and summing over the final quantum numbers yields $P^{\mathrm{NBW}}_{j,q} = -(\alpha m^2)/(4q_-^2)\sum_{s,s'}\int_0^{q_-} dp_- \int d\phi_+\, \tilde{G}_{j,s,s'}$.

5.4. Proof that the probability stays below unity

Analogously as in the case of nonlinear Compton scattering, it can be proved analytically that the probability in Eq. (5.3.4) summed over the final quantum numbers, i.e. $P_j^{(\gamma\to e^-e^+)} = \sum_{s,s'} P_{j,s,s'}^{(\gamma\to e^-e^+)}$, is always smaller than unity (see also Section 4.4). The explicit proof is given by

$$\begin{aligned} P_s^{(\gamma\to e^-e^+)} &< -\frac{\alpha m^2}{4q_-^2}\sum_{s,s'}\int_0^{q_-} dp_- \int d\phi_+\, \tilde{G}_{j,s,s'} e^{2\mathrm{Im}\frac{m}{q_-}\int_{-\infty}^{\phi_+} d\varphi P_j(q,\varphi)} \\ &= \int d\phi_+\, \frac{\partial P^{\mathrm{NBW}}_{j,q}}{\partial \phi_+} e^{-\int_{-\infty}^{\phi_+} d\varphi \frac{\partial P^{\mathrm{NBW}}_{j,q}}{\partial\varphi}} = -\int d\phi_+\, \frac{\partial}{\partial\phi_+} e^{-\int_{-\infty}^{\phi_+} d\varphi \frac{\partial P^{\mathrm{NBW}}_{j,q}}{\partial\varphi}} \\ &= 1 - e^{-\int_{-\infty}^{\infty} d\varphi \frac{\partial P^{\mathrm{NBW}}_{j,q}}{\partial\varphi}} < 1. \end{aligned} \tag{5.4.1}$$

This proof implies that the average over the initial quantum number as well as Eq. (5.3.4) stay below unity, too.

5.5. Limits

In this Section we consider the two cases where $q_- - p_- \ll q_-$ and $p_- \ll q_-$, and study the asymptotic behavior for the differential probability of nonlinear Breit-Wheeler pair production, i.e.

$$\frac{\partial P_{j,s,s'}^{(\gamma\to e^-e^+)}}{\partial p_-} = -\frac{\alpha m^2}{4q_-^2}\int d\phi_+ e^{D^{\mathrm{NBW}}_{j,s,s'}}\tilde{G}_{j,s,s'}, \tag{5.5.1}$$

which we obtained from Eq. (5.3.4).

Asymptotic expression for $q_- - p_- \ll q_-$

We start with the region $p'_- = q_- - p_- \ll q_-$. Here almost all the light-cone energy of the photon goes into the positron such that $p_- \approx q_-$. Similar to the case of nonlinear Compton scattering we first assume that the quantum nonlinearity parameter $\kappa_q(\varphi)$ of the photon is fixed. The absolute value of the quantum nonlinearity parameter of the electron $\chi_{p'}(\varphi) = (p'_-/q_-)\kappa_q(\varphi)$ is estimated to be much smaller than unity (if $\kappa_q(\varphi)$ is larger than unity, we suppose the ratio p'_-/q_- to be such small that $|\chi_{p'}(\varphi)| \ll 1$). With this assumptions the probability of nonlinear Compton scattering of the electron in the damping function in Eq.

(5.1.6) can be replaced by the corresponding asymptotic expression, which is independent of p' and presented in Eq. (4.5.7). Therefore the damping function reduces to

$$D_{j,s,s'}^{\mathrm{NBW}} \overset{p'_-\ll q_-}{\approx} -\int_{-\infty}^{\phi_+} d\varphi \frac{\partial P_{j,q}^{\mathrm{NBW}}}{\partial \varphi} - \int_{\phi_+}^{\infty} d\varphi \left(\frac{\partial P_{s,-q}^{\mathrm{NC}}}{\partial \varphi} + \frac{5}{\sqrt{3}} \frac{\alpha m^2 |\kappa_q(\varphi)|}{q_-} \right), \tag{5.5.2}$$

where we additionally used that $\chi_p(\varphi) = (p_-/q_-)\kappa_q(\varphi)$. In the preexponent of the differential probability the quantities $\tilde{G}_{j,s,s'}$ contain Airy functions which we can expand for $z = \left(q_-/(p'_-\chi_p(\phi_+))\right)^{2/3} \approx \left(q_-/(p'_-\kappa_q(\phi_+))\right)^{2/3} \gg 1$ (see Eqs. (B.0.10)-(B.0.12) for the asymptotic expressions of the Airy-functions). Starting with the case of photon polarization $j = 1$ and identical spin quantum numbers ($s = s'$), the asymptotic behavior of $\tilde{G}_{1,s,s}$ is given by

$$\begin{aligned} \tilde{G}_{1,s,s} \overset{z\gg 1}{\approx} & -\frac{1}{\sqrt{\pi}} z^{-3/4} e^{-\frac{2}{3}z^{3/2}} \left[\frac{q_-}{p'_-} \left(1 - s\ \mathrm{sgn}(\psi'(\phi_+))\right) + 2\frac{p'_-}{q_-} s\ \mathrm{sgn}(\psi'(\phi_+)) \right] \\ & + \frac{1}{96\sqrt{\pi}} z^{-9/4} e^{-\frac{2}{3}z^{3/2}} \left[124 + 20s\ \mathrm{sgn}(\psi'(\phi_+))\right] \\ & - \frac{1}{9216\sqrt{\pi}} z^{-15/4} e^{-\frac{2}{3}z^{3/2}} \frac{q_-}{p'_-} \left[3938 - 770s\ \mathrm{sgn}(\psi'(\phi_+))\right]. \end{aligned} \tag{5.5.3}$$

Similar to the case of nonlinear Compton scattering (see discussion below Eq. (4.5.9)), the case $s = -\mathrm{sgn}(\psi'(\phi_+))$ determines the scaling of the probability, i.e.

$$\tilde{G}_{1,s,s} \overset{z\gg 1}{\approx} -\frac{2}{\sqrt{\pi}} z^{-3/4} e^{-\frac{2}{3}z^{3/2}} \frac{q_-}{p'_-}. \tag{5.5.4}$$

Considering opposite spin quantum numbers ($s = -s'$) we obtain the expression

$$\tilde{G}_{1,s,-s} \overset{z\gg 1}{\approx} -\frac{1}{4\sqrt{\pi}} z^{-9/4} e^{-\frac{2}{3}z^{3/2}} \frac{q_-}{p'_-}. \tag{5.5.5}$$

For the photon polarization $j = 2$ and identical spin quantum numbers ($s = s'$) we have

$$\tilde{G}_{2,s,s} \overset{z\gg 1}{\approx} -\frac{1}{4\sqrt{\pi}} z^{-9/4} e^{-\frac{2}{3}z^{3/2}} \frac{q_-}{p'_-} \tag{5.5.6}$$

and finally for opposite spin quantum numbers ($s = -s'$)

$$\begin{aligned} \tilde{G}_{2,s,-s} \overset{z\gg 1}{\approx} & -\frac{1}{\sqrt{\pi}} z^{-3/4} e^{-\frac{2}{3}z^{3/2}} \frac{q_-}{p'_-} \left[1 + s\ \mathrm{sgn}(\psi'(\phi_+))\right] \\ & - \frac{1}{9216\sqrt{\pi}} z^{-15/4} e^{-\frac{2}{3}z^{3/2}} \frac{q_-}{p'_-} \left[3938 + 770s\ \mathrm{sgn}(\psi'(\phi_+))\right]. \end{aligned} \tag{5.5.7}$$

Here the case $s = \mathrm{sgn}(\psi'(\phi_+))$ determines the scaling of the probability of nonlinear Breit-Wheeler pair production and we obtain

$$\tilde{G}_{2,s,-s} \overset{z\gg 1}{\approx} -\frac{2}{\sqrt{\pi}} z^{-3/4} e^{-\frac{2}{3}z^{3/2}} \frac{q_-}{p'_-}. \tag{5.5.8}$$

Asymptotic expression for $p_- \ll q_-$

In the region $p_- \ll q_-$ almost all the light-cone energy of the photon goes into the electron. Instead of performing the same steps as before for this limit, we notice that the probability of nonlinear Breit-Wheeler pair production is symmetric under the exchanges $p_- \leftrightarrow p'_-$, $s \leftrightarrow s'$, and $\psi'(\phi_+) \leftrightarrow -\psi'(\phi_+)$ (see Eqs. (5.3.4)-(5.3.6) and (5.1.3) and (3.2.19)). By employing these substitution rules it is straight forward to obtain the asymptotic expressions in the limit $p_- \ll q_-$ for the differential probability of nonlinear Breit-Wheeler pair production from those in the case $p'_- \ll q_-$, which we investigated in the previous Subsection.

5.6. Independence of spin and polarization basis

The proof that the probability summed over a spin or polarization quantum number stays unchanged when using a different basis for that spin or polarization index is similar to the one presented for nonlinear Compton scattering. The latter one can be found in Section 4.6 on page 52ff. and for this reason the proof is not presented here again.

5.7. Numerical results

Finally some numerical results for the probability of nonlinear Breit-Wheeler pair production including the decay of the states in Eq. (5.3.4) will be presented. Again the numerical results shown in this Section are taken from the Ref. [21] and were all performed by Victor Dinu. However they are presented here to visualize some properties of the probability which we found in the analytical investigations, too.

As in the case of nonlinear Compton scattering in Section 4.7 a linearly polarized plane wave laser pulse with Gaussian envelope described by the vector potential in Eq. (4.7.1) was chosen, with carrier frequency $\omega_0 = 1.55$ eV and the length of the pulse is described by the parameter τ.

Now in Figs. 5.2 and 5.3 the total probability of nonlinear Breit-Wheeler pair production including the states decay and summed over the final quantum numbers, i.e. $P_j^{(\gamma\to e^-e^+)} = \sum_{s,s'} P_{j,s,s'}^{(\gamma\to e^-e^+)}$, is plotted for a pulse length of $\tau = 5$ fs and $\tau = 20$ fs, respectively. In both Figures the left plot presents the result for polarization $j = 1$ and the right one for $j = 2$. Here the probability is presented in a color code and is plotted over the classical nonlinearity parameter ξ_0 on the horizontal axis and on the vertical axis over the incoming photon energy in GeV which is also related to the parameter $\rho_0 = \kappa_0/\xi_0 = (k_0 q)/m^2$. Analogously as for nonlinear Compton scattering, we restrict the parameter $\xi_0 \geq 5$ due to the LCFA and set an upper bound of 100 GeV on the photons energy corresponding to the parameter ρ_0 not exceeding (approximately) unity.

First we observe that in all plots the probability stays well below unity as we expected from the analytical proof in Section 5.4. Now, comparing both Figures, we observe that within the same parameter range the probability in the longer pulse (Fig. 5.3) is significantly smaller and reaches only about half of the maximal value from the one in the shorter pulse (Fig. 5.2). Also the maximum value is moved towards smaller ξ_0. This indicates that the damping due

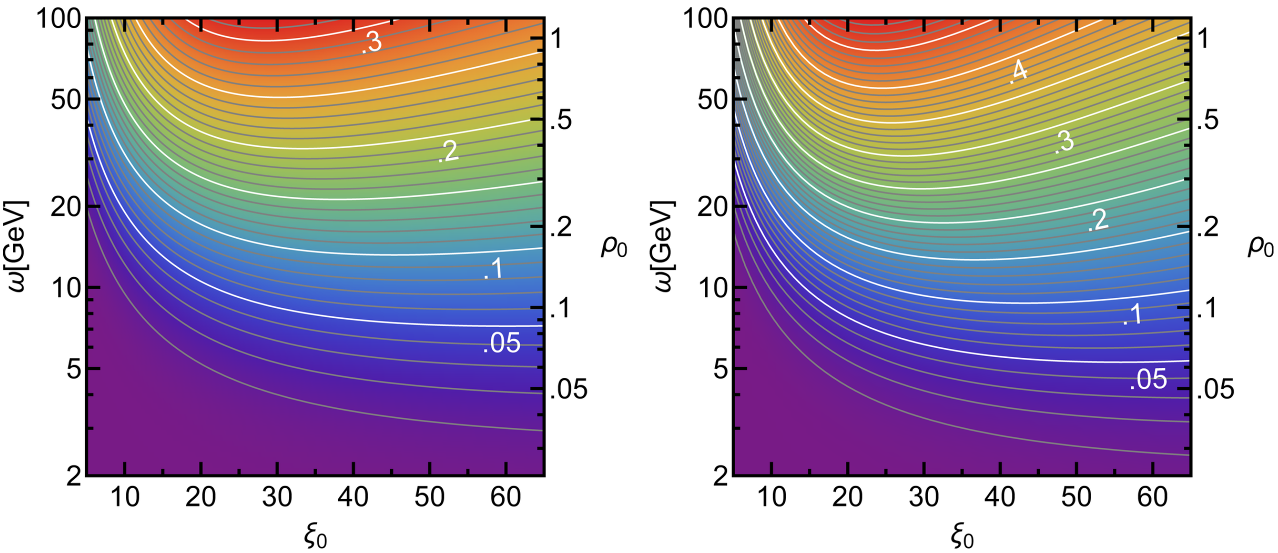

Figure 5.2.: The probability $P_j^{(\gamma\to e^- e^+)}$ of nonlinear Breit-Wheeler pair production in a short, $\tau = 5$ fs pulse, by a photon with polarization quantum number $j = 1$ (left panel) and $j = 2$ (right panel). The plots are taken from Ref. [21] and were done by Victor Dinu.

to the particles states decay is stronger in the longer pulse as we expected from or analytical results (see discussion below Eq. (5.1.6)).

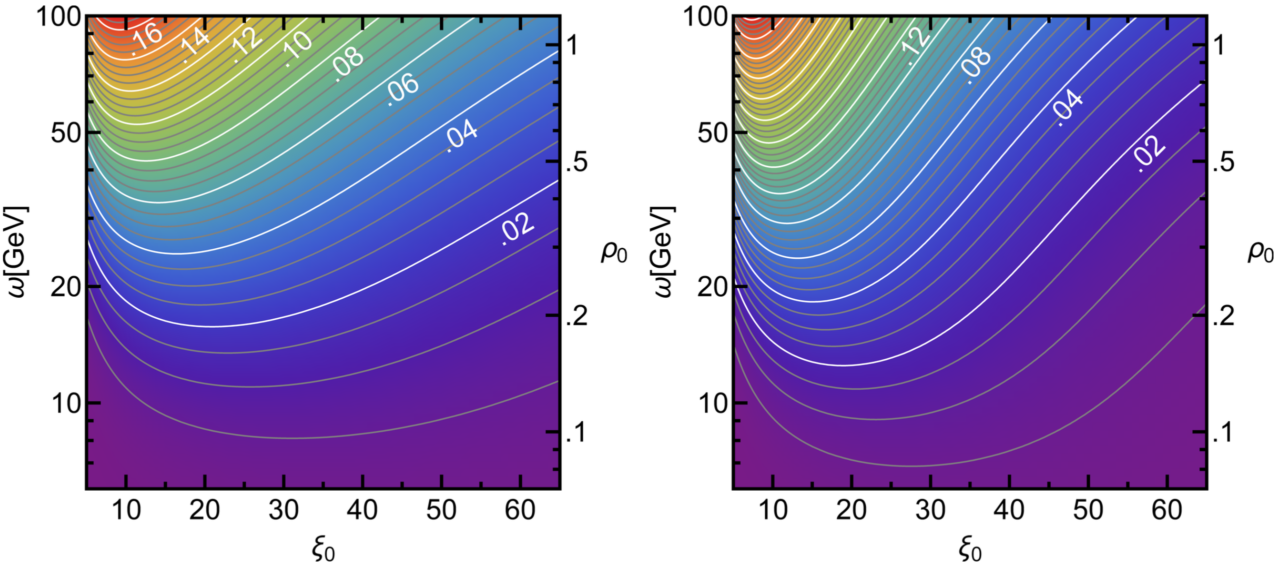

Figure 5.3.: Same as in Fig. 5.2, but for $\tau = 20$ fs. Again the plots are taken from Ref. [21] and were done by Victor Dinu.

6. Corrections to nonlinear Compton scattering of order α^2

In the last chapters we investigated how radiative corrections modify the probabilities of nonlinear Compton scattering and nonlinear Breit-Wheeler pair production and obtained probabilities which are also valid in the limit of a large pulse phase length Φ_L of the plane wave background field. For this we assumed that the product $\alpha\Phi_L/\Phi_f \gtrsim 1$, with Φ_f being the formation length (see discussion in Chapter 3), which allowed us to neglect the Vertex correction and only consider an infinite series of mass- and polarization operator corrections to the electron and photon states for both processes. This series of Feynman diagrams was implicitly resummed by using radiatively corrected states for the electron, positron and photon which we obtained from their corresponding Schwinger-Dyson equations (3.1.1) and (3.1.2). However, to obtain analytical solutions, we had to apply the LCFA and considered only a linearly polarized background field.

In a more general case this resummation is not trivial. The Vertex correction has to be considered and also the expressions of the mass- and polarization operator become more complicated. Our next goal is to calculate (a part of) the leading order in α corrections to the probability of nonlinear Compton scattering in a more general case. Here the background field is considered to be a plane wave field with an arbitrary transverse polarization and no LCFA is applied.

The Feynman diagrams for the leading order in α corrections to nonlinear Compton scattering were already depicted in Fig. 3.2 on page 27 and we write this Figure in terms of S-matrix elements

$$S_{\mathrm{RC}}^{\mathrm{NCS}} = S_0^{\mathrm{NCS}} + S_{\mathrm{corr}}^{\mathrm{NCS}}, \tag{6.0.1}$$

with

$$S_{\mathrm{corr}}^{\mathrm{NCS}} = S_{\mathrm{mass\ in}}^{\mathrm{NCS}} + S_{\mathrm{mass\ out}}^{\mathrm{NCS}} + S_{\mathrm{pol}}^{\mathrm{NCS}} + S_{\mathrm{vertex}}^{\mathrm{NCS}} + ... \tag{6.0.2}$$

Here $S_{\mathrm{RC}}^{\mathrm{NCS}}$ is the complete S-matrix including all corrections, S_0^{NCS} is the S-matrix for the leading diagram without corrections (see Eq. (3.0.1)), and $S_{\mathrm{corr}}^{\mathrm{NCS}}$ is the S-matrix containing all corrections. Further $S_{\mathrm{mass\ in}}^{\mathrm{NCS}}$ and $S_{\mathrm{mass\ out}}^{\mathrm{NCS}}$ are the corrections by one mass operator acting on the incoming and outgoing electron, respectively, $S_{\mathrm{pol}}^{\mathrm{NCS}}$ is the correction by one polarization operator on the outgoing photon, and $S_{\mathrm{vertex}}^{\mathrm{NCS}}$ is the correction of the vertex. The S-matrix S_0^{NCS} scales with $\sqrt{\alpha}$, whereas the S-matrices $S_{\mathrm{mass\ in}}^{\mathrm{NCS}}$, $S_{\mathrm{mass\ out}}^{\mathrm{NCS}}$, $S_{\mathrm{pol}}^{\mathrm{NCS}}$, and $S_{\mathrm{vertex}}^{\mathrm{NCS}}$ scale with $\alpha^{3/2}$. Higher corrections would result in S-matrices scaling at least with $\alpha^{5/2}$ and we can therefore neglect them in our considerations. Now the probability of nonlinear Compton scattering is proportional to the complex square of the complete S-matrix element and we can write

$$\begin{aligned} P_{\mathrm{RC}}^{\mathrm{NCS}} \propto |S_{\mathrm{RC}}^{\mathrm{NCS}}|^2 &= |S_0^{\mathrm{NCS}}|^2 + S_0^{\mathrm{NCS},*} S_{\mathrm{corr}}^{\mathrm{NCS}} + S_0^{\mathrm{NCS}} S_{\mathrm{corr}}^{\mathrm{NCS},*} + |S_{\mathrm{corr}}^{\mathrm{NCS}}|^2 \\ &= |S_0^{\mathrm{NCS}}|^2 + 2\mathrm{Re}[S_0^{\mathrm{NCS},*} S_{\mathrm{corr}}^{\mathrm{NCS}}] + |S_{\mathrm{corr}}^{\mathrm{NCS}}|^2. \end{aligned} \tag{6.0.3}$$

We observe that the leading order correction term is given by the quantity $2\mathrm{Re}[S_0^{\mathrm{NCS},*} S_{\mathrm{corr}}^{\mathrm{NCS}}]$ which is proportional to α^2, whereas for comparison the term $|S_{\mathrm{corr}}^{\mathrm{NCS}}|^2$ scales with α^3 and of course $|S_0^{\mathrm{NCS}}|^2 \propto \alpha$.

In the following we want to investigate the leading order in α corrections which are stemming from the quantity $2\mathrm{Re}[S_0^{\mathrm{NCS},*} S_{\mathrm{corr}}^{\mathrm{NCS}}]$. However we restrict us in this thesis to only a part of them, namely corrections resulting from the mass operator acting on the incoming and outgoing electron states, i.e. $2\mathrm{Re}[S_0^{\mathrm{NCS},*} S_{\mathrm{mass\ in}}^{\mathrm{NCS}}]$ and $2\mathrm{Re}[S_0^{\mathrm{NCS},*} S_{\mathrm{mass\ out}}^{\mathrm{NCS}}]$. Further we only consider the one-loop expression of the mass operator, since higher order contributions of the mass operator scale with additional powers of α and do not contribute to the leading order in α corrections.

6.1. The S-matrix elements

To calculate the corrections to the probability of nonlinear Compton scattering stemming from $2\mathrm{Re}[S_0^{\mathrm{NCS},*} S_{\mathrm{mass\ in}}^{\mathrm{NCS}}]$ and $2\mathrm{Re}[S_0^{\mathrm{NCS},*} S_{\mathrm{mass\ out}}^{\mathrm{NCS}}]$ we need the S-matrix elements corresponding to the three Feynman diagrams depicted in Figs. 6.1-6.3.

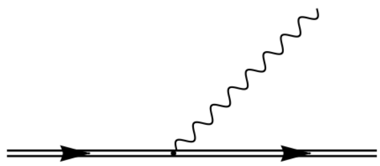

Figure 6.1.: The leading order nonlinear Compton scattering amplitude, corresponding to the S-matrix S_0^{NCS}. Here we need the complex conjugate $S_0^{\mathrm{NCS},*}$.

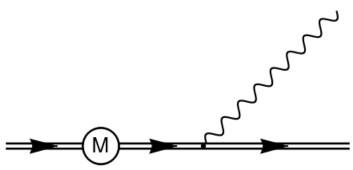

Figure 6.2.: One leading order correction where the mass operator is acting on the incoming electron, corresponding to $S_{\mathrm{mass\ in}}^{\mathrm{NCS}}$. For the computation of this term we use the one-loop mass operator $M_R(l,l')$.

All incoming and outgoing particles are assumed to be on-shell, where the incoming (outgoing) electron has four-momentum $p^\mu = (\varepsilon, \boldsymbol{p})$ ($p'^\mu = (\varepsilon', \boldsymbol{p}')$), with energy $\varepsilon = \sqrt{m^2 + \boldsymbol{p}^2}$ ($\varepsilon' = \sqrt{m^2 + \boldsymbol{p}'^2}$), and an asymptotic spin quantum number $s = \pm 1$ ($s' = \pm 1$). For the outgoing photon the four-momentum is $q^\mu = (\omega, \boldsymbol{q})$, with energy $\omega = |\boldsymbol{q}|$, and its asymptotic transverse polarization state is indicated by the index $j = 1, 2$.

To compute the S-matrix elements we use the Volkov electron state in Eq. (2.6.4), the Volkov-propagator in Eq. (2.6.12) and the photon state from Section 2.5. For the mass

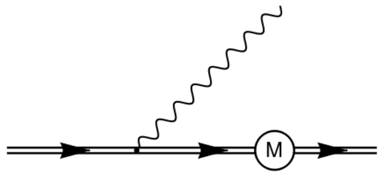

Figure 6.3.: Another leading order correction where the mass operator is acting on the outgoing electron, corresponding to $S^{\text{NCS}}_{\text{mass out}}$. For the computation of this term we use the one-loop mass operator $\tilde{M}_R(l,l')$.

operators we use for $S^{\text{NCS}}_{\text{mass in}}$ the expression of the renormalized one-loop mass operator $M_R(l,l')$ in Eq. (2.9.14) on page 20, which depends on the incoming electron momentum, and for $S^{\text{NCS}}_{\text{mass out}}$ we use the renormalized one-loop mass operator $\tilde{M}_R(l,l')$ in Eq. (2.9.10) on page 19, which depends on the outgoing electron momentum. Both mass operators were presented in momentum-space and they have to be transformed to configuration space via

$$-iM_R(x,y) = -i\int \frac{d^4l}{(2\pi)^4}\int \frac{d^4l'}{(2\pi)^4} E(l,x)M_R(l,l')\bar{E}(l',y) \tag{6.1.1}$$

and analogously for $\tilde{M}_R(x,y)$. With that the S-matrix elements can be expressed as

$$S_0^{\text{NCS},*} = ie\int d^4x'\ \bar{\psi}^{V,(\text{in})}_{e^-,s,p}(x')\hat{A}^{\text{rad}}_{j,q}(x')\psi^{V,(\text{out})}_{e^-,s',p'}(x'), \tag{6.1.2}$$

$$S^{\text{NCS}}_{\text{mass in}} = -ie\int d^4xd^4yd^4z\ \bar{\psi}^{V,(\text{out})}_{e^-,s',p'}(x)\hat{A}^{\text{rad},*}_{j,q}(x)G^V(x,y)M_R(y,z)\psi^{V,(\text{in})}_{e^-,s,p}(z), \tag{6.1.3}$$

$$S^{\text{NCS}}_{\text{mass out}} = -ie\int d^4xd^4yd^4z\ \bar{\psi}^{V,(\text{out})}_{e^-,s',p'}(z)\tilde{M}_R(z,y)G^V(y,x)\hat{A}^{\text{rad},*}_{j,q}(x)\psi^{V,(\text{in})}_{e^-,s,p}(x). \tag{6.1.4}$$

Note, as we need later the complex conjugate $S_0^{\text{NCS},*}$, its expression is presented here directly instead (see also Eq. (3.0.1)). Further different integration variables were applied in order to avoid confusion when later calculating the probabilities. To simplify computational steps we rewrite the S-matrix elements into the form

$$S_0^{\text{NCS},*} = \frac{ie}{\sqrt{8\varepsilon\varepsilon'\omega}}\int d^4x'\ \bar{u}_s(p)\Gamma^\mu_{M0}(x')u_{s'}(p')\epsilon_{j,\mu}e^{-iqx'+i\Phi^{(out)}(p')-i\Phi^{(in)}(p)}, \tag{6.1.5}$$

$$S^{\text{NCS}}_{\text{mass in}} = \frac{-ie}{\sqrt{8\varepsilon\varepsilon'\omega}}\int d^4x\ \bar{u}_{s'}(p')\Gamma^\nu_{M1}(x)u_s(p)\epsilon^*_{j,\nu}e^{iqx-i\Phi^{(out)}(p')+i\Phi^{(in)}(p)}, \tag{6.1.6}$$

$$S^{\text{NCS}}_{\text{mass out}} = \frac{-ie}{\sqrt{8\varepsilon\varepsilon'\omega}}\int d^4x\ \bar{u}_{s'}(p')\Gamma^\nu_{M2}(x)u_s(p)\epsilon^*_{j,\nu}e^{iqx-i\Phi^{(out)}(p')+i\Phi^{(in)}(p)}, \tag{6.1.7}$$

where we introduced the quantities

$$\Gamma^\mu_{M0}(x') = \bar{E}(p,x')\gamma^\mu E(p',x'), \tag{6.1.8}$$

$$\Gamma^\nu_{M1}(x) = \int \frac{d^4l}{(2\pi)^4}\ \bar{E}(p',x)\gamma^\nu E(l,x)\frac{\hat{l}+m}{l^2-m^2+i0}M_R(l,p), \tag{6.1.9}$$

$$\Gamma^\nu_{M2}(x) = \int \frac{d^4l}{(2\pi)^4}\ \tilde{M}_R(p',l)\frac{\hat{l}+m}{l^2-m^2+i0}\bar{E}(l,x)\gamma^\nu E(p,x). \tag{6.1.10}$$

Here the property of the Ritus matrix in Eq. (2.6.8) was exploited in order to perform the integrals in dy and dz in Eqs. (6.1.3) and (6.1.4) and to reduce the number of momentum integrals. Now, the propagator in the quantities $\Gamma^\nu_{M1}(x)$ and $\Gamma^\nu_{M2}(x)$ can be rewritten into the form

$$\frac{\hat{l}+m}{l^2-m^2+i0} = \frac{\hat{n}}{2l_-} + \frac{\hat{l}_- - \hat{l}_\perp + m + \frac{\hat{n}}{2l_-}(\boldsymbol{l}_\perp^2+m^2)}{2l_+l_- - \boldsymbol{l}_\perp^2 - m^2 + i0}, \tag{6.1.11}$$

where we want to stress that the term $\hat{l}_+$ was not simply replaced by $\frac{\hat{n}}{2l_-}(\boldsymbol{l}_\perp^2+m^2)$, which would be incorrect for the off-shell momentum l. Instead we added and subtracted suitable terms, i.e. $\hat{l}_+ = \hat{n}l_+2l_-/(2l_-) = \hat{n}(2l_-l_+ - \boldsymbol{l}_\perp^2 - m^2 + i0 + \boldsymbol{l}_\perp^2 + m^2 - i0)/(2l_-)$, in order to remove l_+ in the numerator by cancellation with the denominator (the remaining quantity $i0$ in the numerator was set finally equal to zero).

At this point we insert the mass operators from Eqs. (2.9.14) and (2.9.10) into Eqs. (6.1.9) and (6.1.10) and then into Eqs. (6.1.6) and (6.1.7), respectively. Using the properties $(\hat{p}-m)u_s(p)=0$ and $\bar{u}_{s'}(p')(\hat{p}'-m)=0$ it turns out that some terms of the mass operator do not contribute. Further with the δ-functions in the mass operator the integrals in dl_- and $d^2\boldsymbol{l}_\perp$ can be taken, enforcing momentum conservation in this light-cone components. The expressions become

$$\begin{aligned}
\Gamma^\nu_{M1}(x) = {}& \frac{\alpha}{2\pi}\int d\phi\, e^{-i(\phi-x_-)p_+}\int_0^\infty \frac{dudr}{(u+r)^2} e^{-iu\lambda^2 - i\frac{r^2}{u+r}m^2} \\
&\times \bar{E}(p',x)\gamma^\nu E(p,x)\int\frac{dl_+}{2\pi}\, e^{-i(x_--\phi)l_+}\left(\frac{\hat{n}}{2p_-} + \frac{\hat{p}+m}{2l_+p_- - \boldsymbol{p}_\perp^2 - m^2 + i0}\right) \\
&\times \left[e^{i\frac{r^2}{u+r}\left[\int_0^1 dw\Delta\mathcal{A}^2(\phi_{wr}) - \left(\int_0^1 dw\Delta\mathcal{A}(\phi_{wr})\right)^2\right]} M_I - \frac{u+2r}{u+r}m\right]
\end{aligned} \tag{6.1.12}$$

and

$$\begin{aligned}
\Gamma^\nu_{M2}(x) = {}& \frac{\alpha}{2\pi}\int d\phi\, e^{i(\phi-x_-)p'_+}\int_0^\infty \frac{dudr}{(u+r)^2} e^{-iu\lambda^2 - i\frac{r^2}{u+r}m^2} \\
&\times \left[e^{i\frac{r^2}{u+r}\left[\int_0^1 dw\Delta\mathcal{A}^2(\tilde{\phi}_{wr}) - \left(\int_0^1 dw\Delta\mathcal{A}(\tilde{\phi}_{wr})\right)^2\right]} M_{II} - \frac{u+2r}{u+r}m\right] \\
&\times \int\frac{dl_+}{2\pi}\, e^{-i(\phi-x_-)l_+}\left(\frac{\hat{n}}{2p'_-} + \frac{\hat{p}'+m}{2l_+p'_- - \boldsymbol{p}'^2_\perp - m^2 + i0}\right)\bar{E}(p',x)\gamma^\nu E(p,x),
\end{aligned} \tag{6.1.13}$$

where $\phi_r = \phi - 2urp_-/(u+r)$, $\phi_{wr} = \phi - 2wurp_-/(u+r)$, $\tilde{\phi}_r = \phi + 2urp'_-/(u+r)$, $\tilde{\phi}_{wr} = \phi + 2wurp'_-/(u+r)$, and M_I and M_{II} are given in Eqs. (2.9.16) and (2.9.12) on pages 20 and 19 with the replacements $l' \to p$ and $l \to p'$, respectively. Again we use the property of the free bispinor in Eq. (2.6.2) to rewrite $M_I = \tilde{M}_I + m(u+2r)/(u+r)$ and $M_{II} = \tilde{M}_{II} + m(u+2r)/(u+r)$ with

$$\begin{aligned}
\tilde{M}_I = {}& -\widehat{\Delta\mathcal{A}}(\phi_r) - \frac{r}{u+r}\int_0^1 dw\widehat{\Delta\mathcal{A}}(\phi_{wr}) + \frac{r}{u+r}\hat{p}\frac{\hat{n}\widehat{\Delta\mathcal{A}}(\phi_r)}{2p_-} \\
&- \frac{r}{u}\frac{2u+r}{u+r}\int_0^1 dw\widehat{\Delta\mathcal{A}}(\phi_{wr})\frac{\hat{n}\widehat{\Delta\mathcal{A}}(\phi_r)}{2p_-} + \frac{\hat{n}}{2p_-}N_I,
\end{aligned} \tag{6.1.14}$$

and

$$\tilde{M}_{II} = -\widehat{\Delta\mathcal{A}}(\tilde{\phi}_r) - \frac{r}{u+r}\int_0^1 dw \widehat{\Delta\mathcal{A}}(\tilde{\phi}_{wr}) + \frac{r}{u+r}\frac{\widehat{\Delta\mathcal{A}}(\tilde{\phi}_r)\hat{n}}{2p'_-}\hat{p}' \\ - \frac{r}{u}\frac{2u+r}{u+r}\frac{\widehat{\Delta\mathcal{A}}(\tilde{\phi}_r)\hat{n}}{2p'_-}\int_0^1 dw \widehat{\Delta\mathcal{A}}(\tilde{\phi}_{wr}) + \frac{\hat{n}}{2p'_-}N_{II}, \tag{6.1.15}$$

where N_I and N_{II} are given in Eqs. (2.9.17) and (2.9.13) with the replacements $l' \to p$ and $l \to p'$, respectively.

Now we are able to perform the integral in l_+ which we split for this purpose into two parts. Beginning with $\Gamma^\nu_{M1}(x)$ the first part of the integral in l_+ is

$$\int \frac{dl_+}{2\pi}\, e^{-i(x_- - \phi)l_+}\frac{\hat{n}}{2p_-} = \frac{\hat{n}}{2p_-}\delta(x_- - \phi) \tag{6.1.16}$$

which simply gives a δ-function. The second part of the integral can be solved using the relation $\Theta(x) = i\int_{-\infty}^{\infty}\frac{d\tau}{2\pi}\frac{1}{\tau+i0}e^{-ix\tau}$ [76, 77] and the substitution $\tau = l_+ - p_+$, such that the result is a Heavyside-function, i.e.

$$\int \frac{dl_+}{2\pi}\, e^{-i(x_- - \phi)l_+}\frac{1}{2l_+p_- - \boldsymbol{p}_\perp^2 - m^2 + i0} = \frac{-i}{2p_-}\Theta(x_- - \phi)e^{-i(x_- - \phi)p_+}. \tag{6.1.17}$$

Analogously the integral in $\Gamma^\nu_{M2}(x)$ can be solved. Writing both contributions of the integral separately, i.e. $\Gamma^\nu_{M1}(x) = \Gamma^{(\delta),\nu}_{M1}(x) + \Gamma^{(\theta),\nu}_{M1}(x)$ and $\Gamma^\nu_{M2}(x) = \Gamma^{(\delta),\nu}_{M2}(x) + \Gamma^{(\theta),\nu}_{M2}(x)$, we obtain the four expressions:

$$\Gamma^{(\delta),\nu}_{M1}(x) = \frac{\alpha}{2\pi}\int_0^\infty \frac{dudr}{(u+r)^2}e^{-i\frac{r^2}{u+r}m^2}\bar{E}(p',x)\gamma^\nu E(p,x) \\ \times \frac{\hat{n}}{2p_-}\left[e^{T_1^{x_-}}\tilde{M}_I^{x_-} - \frac{u+2r}{u+r}m\left(1 - e^{T_1^{x_-}}\right)\right], \tag{6.1.18}$$

$$\Gamma^{(\theta),\nu}_{M1}(x) = \frac{\alpha}{2\pi}\int_{-\infty}^{x_-} d\phi\int_0^\infty \frac{dudr}{(u+r)^2}e^{-i\frac{r^2}{u+r}m^2}\bar{E}(p',x)\gamma^\nu E(p,x) \\ \times \frac{(-i)}{2p_-}(\hat{p}+m)\left[e^{T_1}\tilde{M}_I - \frac{u+2r}{u+r}m\left(1 - e^{T_1}\right)\right], \tag{6.1.19}$$

$$\Gamma^{(\delta),\nu}_{M2}(x) = \frac{\alpha}{2\pi}\int_0^\infty \frac{dudr}{(u+r)^2}e^{-i\frac{r^2}{u+r}m^2} \\ \times \left[e^{T_2^{x_-}}\tilde{M}_{II}^{x_-} - \frac{u+2r}{u+r}m\left(1 - e^{T_2^{x_-}}\right)\right]\frac{\hat{n}}{2p'_-}\bar{E}(p',x)\gamma^\nu E(p,x), \tag{6.1.20}$$

$$\Gamma^{(\theta),\nu}_{M2}(x) = \frac{\alpha}{2\pi}\int_{x_-}^\infty d\phi\int_0^\infty \frac{dudr}{(u+r)^2}e^{-i\frac{r^2}{u+r}m^2} \\ \times \left[e^{T_2}\tilde{M}_{II} - \frac{u+2r}{u+r}m\left(1 - e^{T_2}\right)\right]\frac{(-i)}{2p'_-}(\hat{p}'+m)\bar{E}(p',x)\gamma^\nu E(p,x), \tag{6.1.21}$$

where we introduced the quantities

$$e^{T_1} = \exp\left\{ i\frac{r^2}{u+r}\left[\int_0^1 dw \Delta\mathcal{A}^2(\phi_{wr}) - \left(\int_0^1 dw \Delta\mathcal{A}(\phi_{wr})\right)^2\right]\right\}, \tag{6.1.22}$$

$$e^{T_2} = \exp\left\{ i\frac{r^2}{u+r}\left[\int_0^1 dw \Delta\mathcal{A}^2(\tilde{\phi}_{wr}) - \left(\int_0^1 dw \Delta\mathcal{A}(\tilde{\phi}_{wr})\right)^2\right]\right\}. \tag{6.1.23}$$

Note that also a new notation was employed and the upper index x_- means that in this expression $\phi \to x_-$, for example $\tilde{M}_I^{x-} = \tilde{M}_I|_{\phi\to x_-}$. In this example $\tilde{M}_I^{x-}$ depends then on ϕ_{wr}^{x-} and ϕ_r^{x-}, which again are defined by replacing $\phi \to x_-$ in their definitions. Other functions depending on ϕ have to be treated analogously. Further the square of the fictitious photon mass was set to $\lambda^2 = 0$, since all momenta are on-shell [3, 9, 46].

6.2. Leading order in α correction to the probability

Now, as we discussed already, corrections to the probability scaling with α^2 are coming from the quantity $2\mathrm{Re}[S_0^{\mathrm{NCS},*}S_{\mathrm{corr}}^{\mathrm{NCS}}]$ and we will consider here only the contributions by the one-loop mass operator, i.e. $2\mathrm{Re}[S_0^{\mathrm{NCS},*}S_{\mathrm{mass\ in}}^{\mathrm{NCS}}]$ and $2\mathrm{Re}[S_0^{\mathrm{NCS},*}S_{\mathrm{mass\ out}}^{\mathrm{NCS}}]$. Further, we will consider only the total correction to the probability, summed (averaged) over the final (initial) particles spin/polarization quantum numbers. This time we can employ the commonly used summation rules $\sum_s u_s(p)\bar{u}_s(p) = (\hat{p}+m)$ and $\sum_j \varepsilon^*_{j,\nu}\varepsilon_{j,\mu} = -\eta_{\mu\nu}$ [3, 11] to perform the summation and the average over the quantum numbers. With this and Eqs. (6.1.5)-(6.1.7) we obtain for the corrections to the probability

$$\begin{aligned} P_1 =& \frac{1}{2}\sum_{s,s',j}\int\frac{d^3q}{(2\pi)^3}\int\frac{d^3p'}{(2\pi)^3}2\mathrm{Re}\left[S_0^{\mathrm{NCS},*}S_{\mathrm{mass\ in}}^{\mathrm{NCS}}\right] \\ =& \mathrm{Re}\Bigg\{\int\frac{d^3q}{(2\pi)^3}\int\frac{d^3p'}{(2\pi)^3}\frac{-\alpha\pi}{2\varepsilon\varepsilon'\omega}\int d^4x\int d^4x'\ e^{-iq(x'-x)} \\ &\times \mathrm{Tr}\left[(\hat{p}+m)\Gamma^\mu_{M0}(x')(\hat{p'}+m)\Gamma^\nu_{M1}(x)\right]\eta_{\mu\nu}\Bigg\} \end{aligned} \tag{6.2.1}$$

and

$$\begin{aligned} P_2 =& \frac{1}{2}\sum_{s,s',j}\int\frac{d^3q}{(2\pi)^3}\int\frac{d^3p'}{(2\pi)^3}2\mathrm{Re}[S_0^{\mathrm{NCS},*}S_{\mathrm{mass\ out}}^{\mathrm{NCS}}] \\ =& \mathrm{Re}\Bigg\{\int\frac{d^3q}{(2\pi)^3}\int\frac{d^3p'}{(2\pi)^3}\frac{-\alpha\pi}{2\varepsilon\varepsilon'\omega}\int d^4x\int d^4x'\ e^{-iq(x'-x)} \\ &\times \mathrm{Tr}\left[(\hat{p}+m)\Gamma^\mu_{M0}(x')(\hat{p'}+m)\Gamma^\nu_{M2}(x)\right]\eta_{\mu\nu}\Bigg\}. \end{aligned} \tag{6.2.2}$$

Now, similar as in Ref. [50], with the representation of the metric in Eq. (2.5.3), i.e. $\eta_{\mu\nu} = (q_\mu n_\nu + n_\mu q_\nu)/q_- - \Lambda_{1,\mu}(q)\Lambda_{1,\nu}(q) - \Lambda_{2,\mu}(q)\Lambda_{2,\nu}(q)$, we can express for example P_1 into the form

$$\begin{aligned} P_1 =& \mathrm{Re}\Bigg\{ \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} \frac{-\alpha\pi}{2\varepsilon\varepsilon'\omega} \int d^4x \int d^4x'\ e^{-iq(x'-x)} \\ &\times \Bigg\{ \frac{1}{q_-} \mathrm{Tr}\left[(\hat{p}+m)(q\Gamma_{M0}(x'))(\hat{p'}+m)(n\Gamma_{M1}(x))\right] \\ &\quad + \frac{1}{q_-} \mathrm{Tr}\left[(\hat{p}+m)(n\Gamma_{M0}(x'))(\hat{p'}+m)(q\Gamma_{M1}(x))\right] \\ &\quad - \sum_{i=1}^{2} \mathrm{Tr}\left[(\hat{p}+m)(\Lambda_i(q)\Gamma_{M0}(x'))(\hat{p'}+m)(\Lambda_i(q)\Gamma_{M1}(x))\right] \Bigg\}\Bigg\}. \end{aligned} \tag{6.2.3}$$

Due to gauge invariance and the Ward-identity $(q\Gamma_{M0}(x')) = 0$ and the first trace vanishes [3]. For the second trace one can show that finally it does not contribute to any physical observable. To obtain the complete corrections in order α^2 both probabilities, P_1 and P_2, as well as the probabilities coming from the polarization operator and the vertex correction have to be added. In Ref. [56] it was shown that the sum over the two mass operator diagrams (Figs. 6.2 and 6.3) and the vertex correction diagram contracted with the photon four-momentum vanishes, i.e. $(q\Gamma_{M1}(x)) + (q\Gamma_{M2}(x)) + (q\Gamma_{Vertex}(x)) = 0$. Therefore, the second trace does not contribute to any physical observable and we ignore it in the following. Similar conclusions can be made for P_2 and we are left with the expressions

$$\begin{aligned} \tilde{P}_1 =& \mathrm{Re}\Bigg\{ \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} \frac{\alpha\pi}{2\varepsilon\varepsilon'\omega} \int d^4x \int d^4x'\ e^{-iq(x'-x)} \\ &\times \sum_{i=1}^{2} \mathrm{Tr}\left[(\hat{p}+m)(\Lambda_i(q)\Gamma_{M0}(x'))(\hat{p'}+m)(\Lambda_i(q)\Gamma_{M1}(x))\right] \Bigg\}, \end{aligned} \tag{6.2.4}$$

and

$$\begin{aligned} \tilde{P}_2 =& \mathrm{Re}\Bigg\{ \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} \frac{\alpha\pi}{2\varepsilon\varepsilon'\omega} \int d^4x \int d^4x'\ e^{-iq(x'-x)} \\ &\times \sum_{i=1}^{2} \mathrm{Tr}\left[(\hat{p}+m)(\Lambda_i(q)\Gamma_{M0}(x'))(\hat{p'}+m)(\Lambda_i(q)\Gamma_{M2}(x))\right] \Bigg\}. \end{aligned} \tag{6.2.5}$$

Inserting the expressions of the Γ quantities in Eq. (6.1.8) and in Eqs. (6.1.18)-(6.1.21) one observes that, except of the phase terms of the initial and final states, the remaining equation only depends on the minus component of the four-space-time, such that six integrals in space-time can be taken, leading to δ-functions. By rewriting the square of δ-functions as in Eq. (4.1.2) and changing the integral d^3p' from Cartesian to light-cone coordinates via $d^3p' = dp'_- d^2p'_\perp(\varepsilon'/p'_-)$ [39], these δ-functions lead to momentum conservation in the light-cone components $p_- = p'_- + q_-$ and $\boldsymbol{p}_\perp = \boldsymbol{p}'_\perp + \boldsymbol{q}_\perp$. Further with these transformations the corrections $\tilde{P}_1 = \tilde{P}_1^{(\delta)} + \tilde{P}_1^{(\theta)}$ and $\tilde{P}_2 = \tilde{P}_2^{(\delta)} + \tilde{P}_2^{(\theta)}$ become

$$\begin{aligned}
\tilde{P}_1^{(\delta)} =&\mathrm{Re}\Bigg\{\frac{\alpha^2}{p_-p'_-\omega}\int\frac{d^3q}{(2\pi)^3}\int dx_-\int dx'_-\int_0^\infty\frac{dudr}{(u+r)^2}e^{-i\frac{r^2}{u+r}m^2}\\
&\times e^{i(p_+-p'_+-q_+)(x'_--x_-)+i\int_{x_-}^{x'_-}d\varphi\left[\frac{(p\mathcal{A}(\varphi))}{p_-}-\frac{(p'\mathcal{A}(\varphi))}{p'_-}-\frac{\mathcal{A}^2(\varphi)}{2p_-}+\frac{\mathcal{A}^2(\varphi)}{2p'_-}\right]}\\
&\times\sum_{i=1}^{2}\frac{1}{4}\mathrm{Tr}\Bigg\{(\hat{p}+m)\left[1-\frac{\hat{n}\hat{\mathcal{A}}(x'_-)}{2p_-}\right]\hat{\Lambda}_i(q)\left[1+\frac{\hat{n}\hat{\mathcal{A}}(x'_-)}{2p'_-}\right](\hat{p}'+m)\\
&\times\left[1-\frac{\hat{n}\hat{\mathcal{A}}(x_-)}{2p'_-}\right]\hat{\Lambda}_i(q)\left[1+\frac{\hat{n}\hat{\mathcal{A}}(x_-)}{2p_-}\right]\frac{\hat{n}}{2p_-}\\
&\times\left[e^{T_1^{x_-}}\tilde{M}_I^{x_-}-\frac{u+2r}{u+r}m\left(1-e^{T_1^{x_-}}\right)\right]\Bigg\}\Bigg\},
\end{aligned}\tag{6.2.6}$$

$$\begin{aligned}
\tilde{P}_1^{(\theta)} =&\mathrm{Re}\Bigg\{\frac{\alpha^2}{p_-p'_-\omega}\int\frac{d^3q}{(2\pi)^3}\int dx_-\int dx'_-\int_{-\infty}^{x_-}d\phi\int_0^\infty\frac{dudr}{(u+r)^2}e^{-i\frac{r^2}{u+r}m^2}\\
&\times e^{i(p_+-p'_+-q_+)(x'_--x_-)+i\int_{x_-}^{x'_-}d\varphi\left[\frac{(p\mathcal{A}(\varphi))}{p_-}-\frac{(p'\mathcal{A}(\varphi))}{p'_-}-\frac{\mathcal{A}^2(\varphi)}{2p_-}+\frac{\mathcal{A}^2(\varphi)}{2p'_-}\right]}\\
&\times\sum_{i=1}^{2}\frac{1}{4}\mathrm{Tr}\Bigg\{(\hat{p}+m)\left[1-\frac{\hat{n}\hat{\mathcal{A}}(x'_-)}{2p_-}\right]\hat{\Lambda}_i(q)\left[1+\frac{\hat{n}\hat{\mathcal{A}}(x'_-)}{2p'_-}\right](\hat{p}'+m)\\
&\times\left[1-\frac{\hat{n}\hat{\mathcal{A}}(x_-)}{2p'_-}\right]\hat{\Lambda}_i(q)\left[1+\frac{\hat{n}\hat{\mathcal{A}}(x_-)}{2p_-}\right]\frac{(-i)}{2p_-}(\hat{p}+m)\\
&\times\left[e^{T_1}\tilde{M}_I-\frac{u+2r}{u+r}m\left(1-e^{T_1}\right)\right]\Bigg\}\Bigg\},
\end{aligned}\tag{6.2.7}$$

$$\begin{aligned}
\tilde{P}_2^{(\delta)} =&\mathrm{Re}\Bigg\{\frac{\alpha^2}{p_-p'_-\omega}\int\frac{d^3q}{(2\pi)^3}\int dx_-\int dx'_-\int_0^\infty\frac{dudr}{(u+r)^2}e^{-i\frac{r^2}{u+r}m^2}\\
&\times e^{i(p_+-p'_+-q_+)(x'_--x_-)+i\int_{x_-}^{x'_-}d\varphi\left[\frac{(p\mathcal{A}(\varphi))}{p_-}-\frac{(p'\mathcal{A}(\varphi))}{p'_-}-\frac{\mathcal{A}^2(\varphi)}{2p_-}+\frac{\mathcal{A}^2(\varphi)}{2p'_-}\right]}\\
&\times\sum_{i=1}^{2}\frac{1}{4}\mathrm{Tr}\Bigg\{(\hat{p}+m)\left[1-\frac{\hat{n}\hat{\mathcal{A}}(x'_-)}{2p_-}\right]\hat{\Lambda}_i(q)\left[1+\frac{\hat{n}\hat{\mathcal{A}}(x'_-)}{2p'_-}\right](\hat{p}'+m)\\
&\times\left[e^{T_2^{x_-}}\tilde{M}_{II}^{x_-}-\frac{u+2r}{u+r}m\left(1-e^{T_2^{x_-}}\right)\right]\\
&\times\frac{\hat{n}}{2p'_-}\left[1-\frac{\hat{n}\hat{\mathcal{A}}(x_-)}{2p'_-}\right]\hat{\Lambda}_i(q)\left[1+\frac{\hat{n}\hat{\mathcal{A}}(x_-)}{2p_-}\right]\Bigg\}\Bigg\},
\end{aligned}\tag{6.2.8}$$

and

$$
\begin{aligned}
\tilde{P}_2^{(\theta)} =& \mathrm{Re}\Bigg\{ \frac{\alpha^2}{p_- p'_- \omega} \int \frac{d^3q}{(2\pi)^3} \int dx_- \int dx'_- \int_{x_-}^{\infty} d\phi \int_0^{\infty} \frac{dudr}{(u+r)^2} e^{-i\frac{r^2}{u+r}m^2} \\
& \times e^{i(p_+ - p'_+ - q_+)(x'_- - x_-) + i\int_{x_-}^{x'_-} d\varphi \left[\frac{(p\mathcal{A}(\varphi))}{p_-} - \frac{(p'\mathcal{A}(\varphi))}{p'_-} - \frac{\mathcal{A}^2(\varphi)}{2p_-} + \frac{\mathcal{A}^2(\varphi)}{2p'_-} \right]} \\
& \times \sum_{i=1}^{2} \frac{1}{4} \mathrm{Tr} \Bigg\{ (\hat{p} + m) \left[1 - \frac{\hat{n}\hat{\mathcal{A}}(x'_-)}{2p_-} \right] \hat{\Lambda}_i(q) \left[1 + \frac{\hat{n}\hat{\mathcal{A}}(x'_-)}{2p'_-} \right] (\hat{p}' + m) \\
& \times \left[e^{T_2} \tilde{M}_{II} - \frac{u+2r}{u+r} m \left(1 - e^{T_2} \right) \right] \\
& \times \frac{(-i)}{2p'_-} (\hat{p}' + m) \left[1 - \frac{\hat{n}\hat{\mathcal{A}}(x_-)}{2p'_-} \right] \hat{\Lambda}_i(q) \left[1 + \frac{\hat{n}\hat{\mathcal{A}}(x_-)}{2p_-} \right] \Bigg\} \Bigg\}.
\end{aligned}
\tag{6.2.9}
$$

At this point we have to solve for each term the corresponding traces.

6.2.1. Computation of the traces

For each of the four corrections to the probability we have one trace, which we will split into two parts in the following. One part contains the quantity $\tilde{M}_I$ or $\tilde{M}_{II}$ (this part gets the index M) and the other part contains the remaining term with the electron mass m (which will have the index m). This makes in total 8 traces to be solved. Fortunately, it turns out, that the four traces coming from $\tilde{P}_2^{(\delta)}$ and $\tilde{P}_2^{(\theta)}$ can be calculated on a similar way to, and at the end can be directly obtained from, the corresponding traces of $\tilde{P}_1^{(\delta)}$ and $\tilde{P}_1^{(\theta)}$ by changing $p_- \leftrightarrow p'_-$, $\Pi \leftrightarrow \Pi'$, and $\phi_{r,wr} \rightarrow \tilde{\phi}_{r,wr}$. Therefore we present only the results for the former four traces later. To perform the computation of the traces, Appendix A holds useful relations for γ-matrices. Further we introduce the short notation $\Lambda_i^\mu = \Lambda_i^\mu(q)$ in the following.

1st Trace: $R_1^{(\delta m)}$

We start with the first part of the trace from $\tilde{P}_1^{(\delta)}$, i.e.

$$
\begin{aligned}
R_1^{(\delta m)} =& \sum_{i=1}^{2} \frac{1}{4} \mathrm{Tr} \Bigg\{ (\hat{p} + m) \left[1 - \frac{\hat{n}\hat{\mathcal{A}}(x'_-)}{2p_-} \right] \hat{\Lambda}_i \left[1 + \frac{\hat{n}\hat{\mathcal{A}}(x'_-)}{2p'_-} \right] (\hat{p}' + m) \\
& \times \left[1 - \frac{\hat{n}\hat{\mathcal{A}}(x_-)}{2p'_-} \right] \hat{\Lambda}_i \left[1 + \frac{\hat{n}\hat{\mathcal{A}}(x_-)}{2p_-} \right] \frac{\hat{n}}{2p_-} \Bigg\}.
\end{aligned}
\tag{6.2.10}
$$

We notice that only an even number of γ-matrices is present if there is either $\hat{p}$ or $\hat{p}'$ in the trace. Instead, in the case of both or none of them being present, the number of γ-matrices is odd and the trace vanishes accordingly. Since $\{\hat{n}, \hat{\mathcal{A}}\} = 0$, $\{\hat{n}, \hat{\Lambda}_i\} = 0$, and $\hat{n}^2 = n^2 = 0$ the trace reduces to

$$
R_1^{(\delta m)} = \sum_{i=1}^{2} \frac{m}{4} \left\{ \mathrm{Tr} \left[\hat{p}\hat{\Lambda}_i\hat{\Lambda}_i \frac{\hat{n}}{2p_-} \right] + \mathrm{Tr} \left[\hat{\Lambda}_i \hat{p}' \hat{\Lambda}_i \frac{\hat{n}}{2p_-} \right] \right\}. \tag{6.2.11}
$$

With $(\Lambda_i \Lambda_j) = -\delta_{ij}$ we obtain

$$R_1^{(\delta m)} = m\left(\frac{p'_-}{p_-} - 1\right). \tag{6.2.12}$$

2nd Trace: $R_1^{(\delta M)}$

The second part of the trace from $\tilde{P}_1^{(\delta)}$ is

$$\begin{aligned} R_1^{(\delta M)} =& \sum_{i=1}^{2} \frac{1}{4} \mathrm{Tr}\Bigg\{ (\hat{p}+m)\left[1 - \frac{\hat{n}\hat{\mathcal{A}}(x'_-)}{2p_-}\right] \hat{\Lambda}_i \left[1 + \frac{\hat{n}\hat{\mathcal{A}}(x'_-)}{2p'_-}\right] (\hat{p}'+m) \\ & \times \left[1 - \frac{\hat{n}\hat{\mathcal{A}}(x_-)}{2p'_-}\right] \hat{\Lambda}_i \left[1 + \frac{\hat{n}\hat{\mathcal{A}}(x_-)}{2p_-}\right] \frac{\hat{n}}{2p_-} \tilde{M}_I^{x-} \Bigg\}. \end{aligned} \tag{6.2.13}$$

Due to the term $\hat{n}/2p_-$ the trace, including the quantity $\tilde{M}_I^{x-}$, simplifies to

$$\begin{aligned} R_1^{(\delta M)} =& \sum_{i=1}^{2} \frac{1}{4} \mathrm{Tr}\Bigg\{ (\hat{p}+m)\left[1 - \frac{\hat{n}\hat{\mathcal{A}}(x'_-)}{2p_-}\right] \hat{\Lambda}_i \left[1 + \frac{\hat{n}\hat{\mathcal{A}}(x'_-)}{2p'_-}\right] (\hat{p}'+m) \\ & \times \hat{\Lambda}_i \frac{\hat{n}}{2p_-} \left[-\frac{r}{u+r} \int_0^1 dw \widehat{\Delta\mathcal{A}}(\phi_{wr}^{x-}) - \frac{u}{u+r} \widehat{\Delta\mathcal{A}}(\phi_r^{x-}) \right] \Bigg\}. \end{aligned} \tag{6.2.14}$$

We rewrite now $(\hat{p}+m)$ and $(\hat{p}'+m)$ in the above expression using the relation $(\hat{p}+m) = \sum_s u_s(p)\bar{u}_s(p)$. It is easy to verify the following identities:

$$\bar{u}_s(p)\left[1 - \frac{\hat{n}\hat{\mathcal{A}}(x'_-)}{2p_-}\right] = \bar{u}_s(p)\frac{\hat{n}}{2p_-}\left[m + \hat{\Pi}(x'_-)\right] \tag{6.2.15}$$

and

$$\left[1 + \frac{\hat{n}\hat{\mathcal{A}}(x'_-)}{2p'_-}\right] u_{s'}(p') = \left[m + \hat{\Pi}'(x'_-)\right] \frac{\hat{n}}{2p'_-} u_{s'}(p'), \tag{6.2.16}$$

where we introduced the short notation $\hat{\Pi}(x'_-) = \hat{\Pi}_p(x'_-)$ and $\hat{\Pi}'(x'_-) = \hat{\Pi}_{p'}(x'_-)$ with the four-vector $\Pi_p^\mu(x'_-)$ defined in Eq. (2.9.4). Note that the above equalities stay valid under the simultaneous exchange of $p \leftrightarrow p'$ and $\Pi \leftrightarrow \Pi'$. Transforming the spinors back to $(\hat{p}+m)$ and $(\hat{p}'+m)$ we observe that due to the new appearance of $\hat{n}$ in the trace they reduce to $\hat{p}$ and $\hat{p}'$, respectively. The dependence on the electron mass m resulting from the relations in Eqs. (6.2.15) and (6.2.16) turns out to not contribute to the trace, as traces which scale linear with m have an odd number of γ-matrices and are therefore zero, and the trace scaling with m^2 turns out to vanish due to the contraction of $\hat{n}^2 = 0$. With this the trace reduces to

$$\begin{aligned} R_1^{(\delta M)} =& \sum_{i=1}^{2} \frac{1}{4} \mathrm{Tr}\Bigg\{ \hat{\Pi}(x'_-)\hat{\Lambda}_i\hat{\Pi}'(x'_-)\hat{\Lambda}_i \frac{\hat{n}}{2p_-} \\ & \times \left[-\frac{r}{u+r} \int_0^1 dw \widehat{\Delta\mathcal{A}}(\phi_{wr}^{x-}) - \frac{u}{u+r} \widehat{\Delta\mathcal{A}}(\phi_r^{x-}) \right] \Bigg\}. \end{aligned} \tag{6.2.17}$$

After anti-commutation of $\hat{\Lambda}_i$ and $\hat{\Pi}'(x'_-)$ the trace can be solved,

$$\begin{aligned}
R_1^{(\delta M)} =& \sum_{i=1}^{2}\Bigg\{(\Lambda_i \Pi'(x'_-))\left[\frac{r}{u+r}\left(\Lambda_i \int_0^1 dw \Delta\mathcal{A}(\phi_{wr}^{x_-})\right)+\frac{u}{u+r}\left(\Lambda_i \Delta\mathcal{A}(\phi_r^{x_-})\right)\right] \\
&-\frac{1}{2}\frac{p'_-}{p_-}\left[\frac{r}{u+r}\left(\Pi(x'_-)\int_0^1 dw\Delta\mathcal{A}(\phi_{wr}^{x_-})\right)+\frac{u}{u+r}\left(\Pi(x'_-)\Delta\mathcal{A}(\phi_r^{x_-})\right)\right] \\
&+\frac{1}{2}\left[\frac{r}{u+r}\left(\Pi'(x'_-)\int_0^1 dw\Delta\mathcal{A}(\phi_{wr}^{x_-})\right)+\frac{u}{u+r}\left(\Pi'(x'_-)\Delta\mathcal{A}(\phi_r^{x_-})\right)\right]\Bigg\}.
\end{aligned} \tag{6.2.18}$$

The expression can be simplified further using again the representation of the metric shown in Eq. (2.5.3). Thereafter the four-product of two arbitrary four-vectors B^μ and C^μ can be expressed as

$$(BC) = [(qB)C_- + (qC)B_-]/q_- - \sum_{i=1}^{2}(\Lambda_i B)(\Lambda_i C). \tag{6.2.19}$$

With this identity we can perform the remaining sum and obtain the final expression

$$\begin{aligned}
R_1^{(\delta M)} =& \frac{r}{u+r}\left[\frac{p'_-}{q_-}\left(q\int_0^1 dw\Delta\mathcal{A}(\phi_{wr}^{x_-})\right)-\frac{p'_-}{p_-}\left(\Pi(x'_-)\int_0^1 dw\Delta\mathcal{A}(\phi_{wr}^{x_-})\right)\right] \\
&+\frac{u}{u+r}\left[\frac{p'_-}{q_-}\left(q\Delta\mathcal{A}(\phi_r^{x_-})\right)-\frac{p'_-}{p_-}\left(\Pi(x'_-)\Delta\mathcal{A}(\phi_r^{x_-})\right)\right].
\end{aligned} \tag{6.2.20}$$

3rd Trace: $R_1^{(\theta m)}$

We continue with the first part of the trace from $\tilde{P}_1^{(\theta)}$ given by

$$\begin{aligned}
R_1^{(\theta m)} =& \sum_{i=1}^{2}\frac{1}{4}\mathrm{Tr}\Bigg\{(\hat{p}+m)\left[1-\frac{\hat{n}\hat{\mathcal{A}}(x'_-)}{2p_-}\right]\hat{\Lambda}_i\left[1+\frac{\hat{n}\hat{\mathcal{A}}(x'_-)}{2p'_-}\right](\hat{p}'+m) \\
&\times\left[1-\frac{\hat{n}\hat{\mathcal{A}}(x_-)}{2p'_-}\right]\hat{\Lambda}_i\left[1+\frac{\hat{n}\hat{\mathcal{A}}(x_-)}{2p_-}\right](\hat{p}+m)\Bigg\}.
\end{aligned} \tag{6.2.21}$$

First we observe that $(\hat{p}+m)^2 = 2m(\hat{p}+m)$ and the trace is therefore identical to $2m$ times the trace of the leading order nonlinear Compton scattering diagram. However, we computed this trace so far only within the LCFA, such that we have to perform the calculation here again. For this we rewrite all four squared brackets according to the relations in Eqs.(6.2.15) and (6.2.16). Again, we observe that the quantities $(\hat{p}+m)$ and $(\hat{p}'+m)$ reduce to $\hat{p}$ and $\hat{p}'$, respectively, and with $\hat{n}\hat{p}\hat{n} = 2p_-\hat{n}$ we obtain

$$\begin{aligned}
R_1^{(\theta m)} =& 2m\sum_{i=1}^{2}\frac{1}{4}\mathrm{Tr}\Bigg\{\frac{\hat{n}}{2p_-}\left[m+\hat{\Pi}(x'_-)\right]\hat{\Lambda}_i\left[m+\hat{\Pi}'(x'_-)\right]\frac{\hat{n}}{2p'_-} \\
&\times\left[m+\hat{\Pi}'(x_-)\right]\hat{\Lambda}_i\left[m+\hat{\Pi}(x_-)\right]\Bigg\}.
\end{aligned} \tag{6.2.22}$$

At this point we use Eq. (6.2.19) to expand every square bracket into a form similar to

$$\left[m+\hat{\Pi}(x_-)\right] = \left[m+\hat{n}\frac{(q\Pi(x_-))}{q_-}+\hat{q}\frac{(n\Pi(x_-))}{q_-}-\sum_{k=1}^{2}\hat{\Lambda}_k(\Lambda_k\Pi(x_-))\right], \tag{6.2.23}$$

with $(n\Pi(x_-)) = p_-$. Note that the term on the right hand side depending on $\hat{n}$ does not contribute to the trace and can be neglected. The number of possible combinations in Eq. (6.2.22) is reduced by the following observation: Considering the expression in between two $\hat{n}$ the two square brackets can depend each on m, $\hat{q}$, and $\hat{\Lambda}_k$. Now contributions where both square brackets only depend on m and $\hat{\Lambda}_k$ vanish as $n^2 = 0$ and if both square brackets only depend on $\hat{q}$ they vanish as $q^2 = 0$, too. Hence only contributions where one square bracket depends on $\hat{q}$ and the other one on m or $\hat{\Lambda}_k$ are non-zero. Further, considering only contributions with an even number of γ-matrices we end up with eight possible combinations

$$\begin{aligned}
R_1^{(\theta m)} =& \frac{m}{2}\sum_{i=1}^{2}\frac{1}{4p_-p'_-q_-^2}\Bigg\{m^2\Bigg\{\mathrm{Tr}\left[\hat{n}\hat{q}\hat{\Lambda}_i\hat{n}\hat{q}\hat{\Lambda}_i\right]p_-p'_- + \mathrm{Tr}\left[\hat{n}\hat{q}\hat{\Lambda}_i\hat{n}\hat{\Lambda}_i\hat{q}\right]p_-^2 \\
&+ \mathrm{Tr}\left[\hat{n}\hat{\Lambda}_i\hat{q}\hat{n}\hat{q}\hat{\Lambda}_i\right]p'^2_- + \mathrm{Tr}\left[\hat{n}\hat{\Lambda}_i\hat{q}\hat{n}\hat{\Lambda}_i\hat{q}\right]p_-p'_-\Bigg\} \\
&+ \sum_{k=1}^{2}\sum_{j=1}^{2}\Bigg\{\mathrm{Tr}\left[\hat{n}\hat{q}\hat{\Lambda}_i\hat{\Lambda}_j\hat{n}\hat{q}\hat{\Lambda}_i\hat{\Lambda}_k\right]p_-p'_-(\Lambda_j\Pi'(x'_-))(\Lambda_k\Pi(x_-)) \\
&+ \mathrm{Tr}\left[\hat{n}\hat{q}\hat{\Lambda}_i\hat{\Lambda}_j\hat{n}\hat{\Lambda}_k\hat{\Lambda}_i\hat{q}\right]p_-^2(\Lambda_j\Pi'(x'_-))(\Lambda_k\Pi'(x_-)) \\
&+ \mathrm{Tr}\left[\hat{n}\hat{\Lambda}_j\hat{\Lambda}_i\hat{q}\hat{n}\hat{q}\hat{\Lambda}_i\hat{\Lambda}_k\right]p'^2_-(\Lambda_j\Pi(x'_-))(\Lambda_k\Pi(x_-)) \\
&+ \mathrm{Tr}\left[\hat{n}\hat{\Lambda}_j\hat{\Lambda}_i\hat{q}\hat{n}\hat{\Lambda}_k\hat{\Lambda}_i\hat{q}\right]p_-p'_-(\Lambda_j\Pi(x'_-))(\Lambda_k\Pi'(x_-))\Bigg\}\Bigg\}.
\end{aligned} \tag{6.2.24}$$

These traces can be solved using anti-commutation relations. Again, we rewrite with Eq. (6.2.19) the remaining sum and we identify $(\Pi(x_-)\Pi(x'_-)) = m^2 - [\mathcal{A}(x'_-) - \mathcal{A}(x_-)]^2/2 = (\Pi'(x_-)\Pi'(x'_-))$. With these transformations we obtain for the expression

$$\begin{aligned}
R_1^{(\theta m)} =& -4m^3 + m\left(\frac{p_-}{p'_-}+\frac{p'_-}{p_-}\right)[\mathcal{A}(x'_-) - \mathcal{A}(x_-)]^2 \\
&+ 2m\frac{p_-}{q_-}\big[(q\Pi'(x'_-)) + (q\Pi'(x_-))\big] + 2m\frac{p'_-}{q_-}\big[(q\Pi(x'_-)) + (q\Pi(x_-))\big].
\end{aligned} \tag{6.2.25}$$

Now considering that the momenta are on-shell and exploiting their conservation law for the minus and perpendicular light-cone component, one can show that

$$\begin{aligned}
-2(q\Pi(x_-)) &= \Pi^2(x_-) - \Pi'^2(x_-) + q^2 - 2(q\Pi(x_-)) \\
&= (\Pi^\mu(x_-) - \Pi'^\mu(x_-) - q^\mu)(\Pi^\mu(x_-) + \Pi'^\mu(x_-) - q^\mu) \\
&= (\Pi_+(x_-) - \Pi'_+(x_-) - q_+)(\Pi_-(x_-) + \Pi'_-(x_-) - q_-) \\
&= 2p'_-(\Pi_+(x_-) - \Pi'_+(x_-) - q_+)
\end{aligned} \tag{6.2.26}$$

and analogously $(q\Pi'(x_-)) = -p_-(\Pi_+(x_-) - \Pi'_+(x_-) - q_+)$. Further one can rewrite $(\Pi_+(x_-) - \Pi'_+(x_-) - q_+) = -(m^2 q_-/(2p_- p'_-))[1 + \boldsymbol{\pi}^2_{\perp,e}(x_-)]$, where $\boldsymbol{\pi}_{\perp,e}(x_-)$ is defined in Eq. (4.1.9). With these observations we finally end up with the expression

$$R_1^{(\theta m)} = -4m^3 + m\left(\frac{p_-}{p'_-} + \frac{p'_-}{p_-}\right)\left\{\left[\mathcal{A}(x'_-) - \mathcal{A}(x_-)\right]^2 + m^2\left[2 + \boldsymbol{\pi}^2_{\perp,e}(x'_-) + \boldsymbol{\pi}^2_{\perp,e}(x_-)\right]\right\}. \tag{6.2.27}$$

4th Trace: $R_1^{(\theta M)}$

Now we solve the second part of the trace from $\tilde{P}_1^{(\theta)}$, i.e.

$$\begin{aligned} R_1^{(\theta M)} =& \sum_{i=1}^{2} \frac{1}{4}\mathrm{Tr}\Bigg\{(\hat{p}+m)\left[1 - \frac{\hat{n}\hat{\mathcal{A}}(x'_-)}{2p_-}\right]\hat{\Lambda}_i\left[1 + \frac{\hat{n}\hat{\mathcal{A}}(x'_-)}{2p'_-}\right](\hat{p}'+m) \\ &\times \left[1 - \frac{\hat{n}\hat{\mathcal{A}}(x_-)}{2p'_-}\right]\hat{\Lambda}_i\left[1 + \frac{\hat{n}\hat{\mathcal{A}}(x_-)}{2p_-}\right](\hat{p}+m)\tilde{M}_I\Bigg\}. \end{aligned} \tag{6.2.28}$$

First we rewrite all squared brackets into the form given in Eqs.(6.2.15) and (6.2.16). Further, by the transformation

$$\begin{aligned} (\hat{p}+m)\tilde{M}_I(\hat{p}+m) =& \left[\tilde{M}_I(m-\hat{p}) + \{\hat{p},\tilde{M}_I\}\right](\hat{p}+m) \\ =& \{\hat{p},\tilde{M}_I\}(\hat{p}+m). \end{aligned} \tag{6.2.29}$$

the trace depends on the anti-commutator between $\hat{p}$ and $\tilde{M}_I$. This anti-commutator can be calculated and is given by

$$\{\hat{p},\tilde{M}_I\} = K_0 + \frac{r}{u+r}\widehat{\Delta\mathcal{A}}(\phi_r)\hat{p} + \frac{r}{u}\,\frac{2u+r}{u+r}\int_0^1 dw\widehat{\Delta\mathcal{A}}(\phi_{wr})\widehat{\Delta\mathcal{A}}(\phi_r) + \frac{\hat{n}}{2p_-}\hat{K}_n, \tag{6.2.30}$$

where we introduced the quantities

$$\begin{aligned} K_0 =& -\frac{u+2r}{u+r}\left(\Delta\mathcal{A}(\phi_r)\right)^2 + 2\frac{r}{u}\,\frac{r}{u+r}\left(\int_0^1 dw\Delta\mathcal{A}(\phi_{wr})\right)^2 \\ &- \frac{r}{u}\int_0^1 dw\left(\Delta\mathcal{A}(\phi_{wr})\right)^2 - 2\frac{r}{u}\,\frac{r}{u+r}\left(\Delta\mathcal{A}(\phi_r)\int_0^1 dw\Delta\mathcal{A}(\phi_{wr})\right) \end{aligned} \tag{6.2.31}$$

and

$$\begin{aligned} \hat{K}_n =& 2m^2\frac{r}{u+r}\widehat{\Delta\mathcal{A}}(\phi_r) - 2(p\Delta\mathcal{A}(\phi_r))\frac{r}{u+r}\hat{p} \\ &- 2\left(p\int_0^1 dw\Delta\mathcal{A}(\phi_{wr})\right)\frac{r}{u}\,\frac{2u+r}{u+r}\widehat{\Delta\mathcal{A}}(\phi_r) \\ &+ 2(p\Delta\mathcal{A}(\phi_r))\frac{r}{u}\,\frac{2u+r}{u+r}\int_0^1 dw\widehat{\Delta\mathcal{A}}(\phi_{wr}). \end{aligned} \tag{6.2.32}$$

Note that the first term of the anti-commutator, i.e. K_0, can be moved out of the trace and the remaining expression in the trace is equal to $R_1^{(\theta m)}/(2m)$. For the second term in Eq. (6.2.30) we only have to keep the electron mass m from $(\hat{p}+m)$ in Eq. (6.2.29), since otherwise the $\hat{p}$ combines with the other $\hat{p}$ in the anti-commutator to $\hat{p}^2 = m^2$ and the $\hat{n}$ of Eqs.(6.2.15) and (6.2.16) can anti-commute yielding $\hat{n}^2 = 0$. According to a similar reason we have to keep for the third term only the $\hat{p}$ from $(\hat{p}+m)$. Finally, the last term in the anti-commutator, which is proportional to $\hat{n}$, vanishes due to an other $\hat{n}$ from the last squared bracket. We obtain

$$
\begin{aligned}
R_1^{(\theta M)} =& \frac{K_0}{2m} R_1^{(\theta m)} + \sum_{i=1}^{2} \frac{1}{4} \mathrm{Tr} \Bigg\{ \left[m + \hat{\Pi}(x'_-) \right] \hat{\Lambda}_i \left[m + \hat{\Pi}'(x'_-) \right] \frac{\hat{n}}{2p'_-} \\
& \times \left[m + \hat{\Pi}'(x_-) \right] \hat{\Lambda}_i \left[m + \hat{\Pi}(x_-) \right] \frac{\hat{n}}{2p_-} \\
& \times \left[\frac{r}{u+r} \widehat{\Delta\mathcal{A}}(\phi_r) + \frac{r}{u} \frac{2u+r}{u+r} \int_0^1 dw \widehat{\Delta\mathcal{A}}(\phi_{wr}) \widehat{\Delta\mathcal{A}}(\phi_r) \right] \Bigg\}.
\end{aligned}
\tag{6.2.33}
$$

Again Eq.(6.2.23) is used to rewrite all four square brackets. Similar considerations as discussed below Eq. (6.2.23) can be made to reduce the number of combinations. At the end 16 of them are non-zero and the corresponding traces can be calculated using anti-commutation relations. The result can be further simplified with Eq.(6.2.19) and we finally obtain

$$
\begin{aligned}
R_1^{(\theta M)} =& \frac{R_1^{(\theta m)}}{2m} \left[K_0 + \frac{r}{u} \frac{2u+r}{u+r} \left(\Delta\mathcal{A}(\phi_r) \int_0^1 dw \Delta\mathcal{A}(\phi_{wr}) \right) \right] \\
& + m^2 \frac{r}{u+r} \left(1 - \frac{p'_-}{p_-} \right) \left[(\Pi(x_-)\Delta\mathcal{A}(\phi_r)) - (\Pi(x'_-)\Delta\mathcal{A}(\phi_r)) \right] \\
& + \frac{r}{u} \frac{2u+r}{u+r} \sum_{k=1}^{2} \sum_{j=1}^{2} \left[\frac{p_-}{p'_-} (\Lambda_j \Pi'(x'_-))(\Lambda_k \Pi'(x_-)) + \frac{p'_-}{p_-} (\Lambda_j \Pi(x_-))(\Lambda_k \Pi(x'_-)) \right] \\
& \times \left[(\Lambda_j \Delta\mathcal{A}(\phi_r)) \left(\Lambda_k \int_0^1 dw \Delta\mathcal{A}(\phi_{wr}) \right) - (\Lambda_k \Delta\mathcal{A}(\phi_r)) \left(\Lambda_j \int_0^1 dw \Delta\mathcal{A}(\phi_{wr}) \right) \right].
\end{aligned}
\tag{6.2.34}
$$

5th Trace: $R_2^{(\delta m)}$

We move over to the first part of the trace from $\tilde{P}_2^{(\delta)}$ which is

$$
\begin{aligned}
R_2^{(\delta m)} =& \sum_{i=1}^{2} \frac{1}{4} \mathrm{Tr} \Bigg\{ (\hat{p}+m) \left[1 - \frac{\hat{n}\hat{\mathcal{A}}(x'_-)}{2p_-} \right] \hat{\Lambda}_i \left[1 + \frac{\hat{n}\hat{\mathcal{A}}(x'_-)}{2p'_-} \right] (\hat{p}'+m) \\
& \times \frac{\hat{n}}{2p'_-} \left[1 - \frac{\hat{n}\hat{\mathcal{A}}(x_-)}{2p'_-} \right] \hat{\Lambda}_i \left[1 + \frac{\hat{n}\hat{\mathcal{A}}(x_-)}{2p_-} \right] \Bigg\}.
\end{aligned}
\tag{6.2.35}
$$

The trace can be solved analogous to $R_1^{(\delta m)}$ or can be obtained directly from it by changing $p_- \leftrightarrow p'_-$. We therefore only present the result which is

$$
R_2^{(\delta m)} = m \left(\frac{p_-}{p'_-} - 1 \right). \tag{6.2.36}
$$

6th Trace: $R_2^{(\delta M)}$

The second part of the trace from $\tilde{P}_2^{(\delta)}$ is

$$\begin{aligned}R_2^{(\delta M)} =&\sum_{i=1}^{2}\frac{1}{4}\mathrm{Tr}\Bigg\{(\hat{p}+m)\left[1-\frac{\hat{n}\hat{\mathcal{A}}(x'_-)}{2p_-}\right]\hat{\Lambda}_i\left[1+\frac{\hat{n}\hat{\mathcal{A}}(x'_-)}{2p'_-}\right](\hat{p}'+m)\\&\times\tilde{M}_{II}^{x_-}\frac{\hat{n}}{2p'_-}\left[1-\frac{\hat{n}\hat{\mathcal{A}}(x_-)}{2p'_-}\right]\hat{\Lambda}_i\left[1+\frac{\hat{n}\hat{\mathcal{A}}(x_-)}{2p_-}\right]\Bigg\}.\end{aligned} \tag{6.2.37}$$

It can be solved analogous to $R_1^{(\delta M)}$ or can be obtained directly from it with the substitutions $p_-\leftrightarrow p'_-$, $\Pi\leftrightarrow\Pi'$, and $\phi_{r,wr}\rightarrow\tilde{\phi}_{r,wr}$. The final result is

$$\begin{aligned}R_2^{(\delta M)} =&\frac{r}{u+r}\left[\frac{p_-}{q_-}\left(q\int_0^1 dw\Delta\mathcal{A}(\tilde{\phi}_{wr}^{x_-})\right)-\frac{p_-}{p'_-}\left(\Pi'(x'_-)\int_0^1 dw\Delta\mathcal{A}(\tilde{\phi}_{wr}^{x_-})\right)\right]\\&+\frac{u}{u+r}\left[\frac{p_-}{q_-}\left(q\Delta\mathcal{A}(\tilde{\phi}_r^{x_-})\right)-\frac{p_-}{p'_-}\left(\Pi'(x'_-)\Delta\mathcal{A}(\tilde{\phi}_r^{x_-})\right)\right].\end{aligned} \tag{6.2.38}$$

7th Trace: $R_2^{(\theta m)}$

The first part of the trace from $\tilde{P}_2^{(\theta)}$ is

$$\begin{aligned}R_2^{(\theta m)} =&\sum_{i=1}^{2}\frac{1}{4}\mathrm{Tr}\Bigg\{(\hat{p}+m)\left[1-\frac{\hat{n}\hat{\mathcal{A}}(x'_-)}{2p_-}\right]\hat{\Lambda}_i\left[1+\frac{\hat{n}\hat{\mathcal{A}}(x'_-)}{2p'_-}\right](\hat{p}'+m)\\&\times(\hat{p}'+m)\left[1-\frac{\hat{n}\hat{\mathcal{A}}(x_-)}{2p'_-}\right]\hat{\Lambda}_i\left[1+\frac{\hat{n}\hat{\mathcal{A}}(x_-)}{2p_-}\right]\Bigg\}.\end{aligned} \tag{6.2.39}$$

This trace turns out to be equal to $R_1^{(\theta m)}$ as $(\hat{p}'+m)^2=2m(\hat{p}'+m)$, thus

$$R_2^{(\theta m)} = R_1^{(\theta m)}. \tag{6.2.40}$$

8th Trace: $R_2^{(\theta M)}$

Finally the second part of the trace from $\tilde{P}_2^{(\theta)}$ is

$$\begin{aligned}R_2^{(\theta M)} =&\sum_{i=1}^{2}\frac{1}{4}\mathrm{Tr}\Bigg\{(\hat{p}+m)\left[1-\frac{\hat{n}\hat{\mathcal{A}}(x'_-)}{2p_-}\right]\hat{\Lambda}_i\left[1+\frac{\hat{n}\hat{\mathcal{A}}(x'_-)}{2p'_-}\right](\hat{p}'+m)\\&\times\tilde{M}_{II}(\hat{p}'+m)\left[1-\frac{\hat{n}\hat{\mathcal{A}}(x_-)}{2p'_-}\right]\hat{\Lambda}_i\left[1+\frac{\hat{n}\hat{\mathcal{A}}(x_-)}{2p_-}\right]\Bigg\}.\end{aligned} \tag{6.2.41}$$

Also this trace can be solved analogous to $R_1^{(\theta M)}$ or can be obtained directly from it by changing $p_- \leftrightarrow p'_-$, $\Pi \leftrightarrow \Pi'$, and $\phi_{r,wr} \to \tilde{\phi}_{r,wr}$. The final result is

$$\begin{aligned}
R_2^{(\theta M)} =&\frac{R_1^{(\theta m)}}{2m}\left[\tilde{K}_0 + \frac{r}{u}\,\frac{2u+r}{u+r}\left(\Delta\mathcal{A}(\tilde{\phi}_r)\int_0^1 dw\Delta\mathcal{A}(\tilde{\phi}_{wr})\right)\right]\\
&+m^2\frac{r}{u+r}\left(1-\frac{p_-}{p'_-}\right)\left[(\Pi'(x_-)\Delta\mathcal{A}(\tilde{\phi}_r)) - (\Pi'(x'_-)\Delta\mathcal{A}(\tilde{\phi}_r))\right]\\
&-\frac{r}{u}\,\frac{2u+r}{u+r}\sum_{k=1}^{2}\sum_{j=1}^{2}\left[\frac{p_-}{p'_-}(\Lambda_j\Pi'(x'_-))(\Lambda_k\Pi'(x_-)) + \frac{p'_-}{p_-}(\Lambda_j\Pi(x_-))(\Lambda_k\Pi(x'_-))\right]\\
&\times\left[\left(\Lambda_j\Delta\mathcal{A}(\tilde{\phi}_r)\right)\left(\Lambda_k\int_0^1 dw\Delta\mathcal{A}(\tilde{\phi}_{wr})\right) - \left(\Lambda_k\Delta\mathcal{A}(\tilde{\phi}_r)\right)\left(\Lambda_j\int_0^1 dw\Delta\mathcal{A}(\tilde{\phi}_{wr})\right)\right],
\end{aligned} \tag{6.2.42}$$

where we introduced the quantity

$$\begin{aligned}
\tilde{K}_0 =& -\frac{u+2r}{u+r}\left(\Delta\mathcal{A}(\tilde{\phi}_r)\right)^2 + 2\frac{r}{u}\,\frac{r}{u+r}\left(\int_0^1 dw\Delta\mathcal{A}(\tilde{\phi}_{wr})\right)^2\\
&-\frac{r}{u}\int_0^1 dw\left(\Delta\mathcal{A}(\tilde{\phi}_{wr})\right)^2 - 2\frac{r}{u}\,\frac{r}{u+r}\left(\Delta\mathcal{A}(\tilde{\phi}_r)\int_0^1 dw\Delta\mathcal{A}(\tilde{\phi}_{wr})\right).
\end{aligned} \tag{6.2.43}$$

6.2.2. The final expression of the correction to the probability

With the expressions of the traces we have everything together to present the total correction to the probability from the mass operator. After adding the four probabilities in Eqs.(6.2.6)-(6.2.9) to $\tilde{P} = \tilde{P}_1 + \tilde{P}_2 = \tilde{P}_1^\delta + \tilde{P}_1^\theta + \tilde{P}_2^\delta + \tilde{P}_2^\theta$ and inserting the corresponding results for the traces, we obtain for the correction proportional to α^2 stemming from the one-loop mass operator to the probability of nonlinear Compton scattering

$$
\begin{aligned}
\tilde{P} =& \mathrm{Re}\Bigg\{ \int \frac{d^3q}{(2\pi)^3} \frac{\alpha^2}{p_- p'_- \omega} \int dx_- \int dx'_- \int_0^\infty \frac{dudr}{(u+r)^2} e^{-i\frac{r^2}{u+r}m^2} \\
& \times e^{i(p_+ - p'_+ - q_+)(x'_- - x_-) + i\int_{x_-}^{x'_-} d\varphi \left[\frac{(p\mathcal{A}(\varphi))}{p_-} - \frac{(p'\mathcal{A}(\varphi))}{p'_-} - \frac{\mathcal{A}^2(\varphi)}{2p_-} + \frac{\mathcal{A}^2(\varphi)}{2p'_-} \right]} \\
& \times \Bigg\{ \frac{u+2r}{u+r} m^2 q_- \left[\frac{1}{p'_-} \left(e^{T_2^{x_-}} - 1 \right) - \frac{1}{p_-} \left(e^{T_1^{x_-}} - 1 \right) \right] \\
& + e^{T_1^{x_-}} \frac{u}{u+r} \left[\frac{p'_-}{q_-} \left(q \Delta\mathcal{A}(\phi_r^{x_-}) \right) - \frac{p'_-}{p_-} \left(\Pi(x'_-) \Delta\mathcal{A}(\phi_r^{x_-}) \right) \right] \\
& + e^{T_2^{x_-}} \frac{u}{u+r} \left[\frac{p_-}{q_-} \left(q \Delta\mathcal{A}(\tilde{\phi}_r^{x_-}) \right) - \frac{p_-}{p'_-} \left(\Pi'(x'_-) \Delta\mathcal{A}(\tilde{\phi}_r^{x_-}) \right) \right] \\
& + e^{T_1^{x_-}} \frac{r}{u+r} \left[\frac{p'_-}{q_-} \left(q \int_0^1 dw \Delta\mathcal{A}(\phi_{wr}^{x_-}) \right) - \frac{p'_-}{p_-} \left(\Pi(x'_-) \int_0^1 dw \Delta\mathcal{A}(\phi_{wr}^{x_-}) \right) \right] \\
& + e^{T_2^{x_-}} \frac{r}{u+r} \left[\frac{p_-}{q_-} \left(q \int_0^1 dw \Delta\mathcal{A}(\tilde{\phi}_{wr}^{x_-}) \right) - \frac{p_-}{p'_-} \left(\Pi'(x'_-) \int_0^1 dw \Delta\mathcal{A}(\tilde{\phi}_{wr}^{x_-}) \right) \right] \\
& - \frac{i}{2} \frac{R_1^{(\theta m)}}{2m} \Bigg\{ 2m^2 \frac{u+2r}{u+r} \left[\int_{-\infty}^{x_-} d\phi \frac{1}{p_-} \left(e^{T_1} - 1 \right) + \int_{x_-}^{\infty} d\phi \frac{1}{p'_-} \left(e^{T_2} - 1 \right) \right] \\
& \quad + \int_{-\infty}^{x_-} d\phi \frac{1}{p_-} e^{T_1} K_1 + \int_{x_-}^{\infty} d\phi \frac{1}{p'_-} e^{T_2} K_2 \Bigg\} \\
& - \frac{i}{2} m^2 \frac{r}{u+r} \Bigg[\int_{-\infty}^{x_-} d\phi \frac{q_-}{p_-^2} e^{T_1} \left[\left(\mathcal{A}(x'_-) \Delta\mathcal{A}(\phi_r) \right) - \left(\mathcal{A}(x_-) \Delta\mathcal{A}(\phi_r) \right) \right] \\
& \quad - \int_{x_-}^{\infty} d\phi \frac{q_-}{p'^2_-} e^{T_2} \left[\left(\mathcal{A}(x'_-) \Delta\mathcal{A}(\tilde{\phi}_r) \right) - \left(\mathcal{A}(x_-) \Delta\mathcal{A}(\tilde{\phi}_r) \right) \right] \Bigg] \\
& - \frac{i}{2} \frac{r}{u} \frac{2u+r}{u+r} \sum_{k=1}^{2} \sum_{j=1}^{2} \left[\frac{p_-}{p'_-} (\Lambda_j \Pi'(x'_-))(\Lambda_k \Pi'(x_-)) + \frac{p'_-}{p_-} (\Lambda_j \Pi(x_-))(\Lambda_k \Pi(x'_-)) \right] \\
& \times \Bigg\{ \int_{-\infty}^{x_-} d\phi \frac{1}{p_-} e^{T_1} \Bigg[(\Lambda_j \Delta\mathcal{A}(\phi_r)) \left(\Lambda_k \int_0^1 dw \Delta\mathcal{A}(\phi_{wr}) \right) \\
& \quad - (\Lambda_k \Delta\mathcal{A}(\phi_r)) \left(\Lambda_j \int_0^1 dw \Delta\mathcal{A}(\phi_{wr}) \right) \Bigg] \\
& \quad - \int_{x_-}^{\infty} d\phi \frac{1}{p'_-} e^{T_2} \Bigg[\left(\Lambda_j \Delta\mathcal{A}(\tilde{\phi}_r) \right) \left(\Lambda_k \int_0^1 dw \Delta\mathcal{A}(\tilde{\phi}_{wr}) \right) \\
& \quad - \left(\Lambda_k \Delta\mathcal{A}(\tilde{\phi}_r) \right) \left(\Lambda_j \int_0^1 dw \Delta\mathcal{A}(\tilde{\phi}_{wr}) \right) \Bigg] \Bigg\} \Bigg\} \Bigg\},
\end{aligned}
\tag{6.2.44}
$$

where we introduced the quantities

$$\begin{aligned} K_1 = &-\frac{u+2r}{u+r}\left(\Delta\mathcal{A}(\phi_r)\right)^2 + 2\frac{r}{u}\,\frac{r}{u+r}\left(\int_0^1 dw \Delta\mathcal{A}(\phi_{wr})\right)^2 \\ &-\frac{r}{u}\int_0^1 dw \left(\Delta\mathcal{A}(\phi_{wr})\right)^2 + \frac{r}{u}\,\frac{2u-r}{u+r}\left(\Delta\mathcal{A}(\phi_r)\int_0^1 dw \Delta\mathcal{A}(\phi_{wr})\right) \end{aligned} \tag{6.2.45}$$

and

$$\begin{aligned} K_2 = &-\frac{u+2r}{u+r}\left(\Delta\mathcal{A}(\tilde{\phi}_r)\right)^2 + 2\frac{r}{u}\,\frac{r}{u+r}\left(\int_0^1 dw \Delta\mathcal{A}(\tilde{\phi}_{wr})\right)^2 \\ &-\frac{r}{u}\int_0^1 dw \left(\Delta\mathcal{A}(\tilde{\phi}_{wr})\right)^2 + \frac{r}{u}\,\frac{2u-r}{u+r}\left(\Delta\mathcal{A}(\tilde{\phi}_r)\int_0^1 dw \Delta\mathcal{A}(\tilde{\phi}_{wr})\right). \end{aligned} \tag{6.2.46}$$

Note that this probability only contains the corrections contributing to the physical amplitude whereas terms canceling with contributions from the vertex correction were neglected. For the total correction in leading order α also contributions from the polarization operator and the vertex correction have to be considered.

6.3. Comparison with the probability including the decay of particle states

In Chapter 4 the probability of nonlinear Compton scattering including the decay of the particle states was derived, which corresponds to a resummation of an infinite series of Feynman diagrams. This series also includes the two Feynman diagrams of the corrections calculated in this chapter. Hence, within some further approximations, the probability with damping in Eq. (4.1.6) should implicitly include the new result in Eq. (6.2.44) and we want to compare both probabilities in the following. As the steps of comparing the probability $\tilde{P}$ with Eq. (4.1.6) is similar for its two parts $\tilde{P}_1$ and $\tilde{P}_2$, we will first restrict us to the comparison of $\tilde{P}_1$, where the incoming electron is corrected by the mass operator (see Fig. 6.2), and will briefly present the case of $\tilde{P}_2$ at the end.

In order to do so we first have to extract in Eq. (4.1.6) the contribution coming from the considered Feynman diagram and apply for both probabilities, the one in Eq. (4.1.6) and $\tilde{P}_1$, the same approximations.

6.3.1. Rewriting the probability of nonlinear Compton scattering with damping

For the comparison we start from the probability including the particle states decay given in Eq. (4.1.6) on page 41 which is

$$\begin{aligned} P^{(e^-\to e^-\gamma)}_{j,s,s'} = &\int \frac{d^3q}{16\pi^2}\frac{\alpha}{p_-p'_-\omega}\int d\phi_+ e^{2\mathrm{Im}\left\{\frac{m}{p_-}\int_{-\infty}^{\phi_+} d\varphi M_s(p,\varphi) + \int_{\phi_+}^{\infty} d\varphi\left[\frac{m}{p'_-}M_{s'}(p',\varphi)+\frac{m}{q_-}P_j(q,\varphi)\right]\right\}} \\ &\times \int d\phi_- e^{i\frac{m^2}{2p_-}\frac{q_-}{p'_-}\left\{[1+\boldsymbol{\pi}^2_{\perp,e}(\phi_+)]\phi_- + \frac{\boldsymbol{\varepsilon}^2(\phi_+)}{m^2}\frac{\phi_-^3}{12}\right\}} T_{j,s,s'}. \end{aligned} \tag{6.3.1}$$

As mentioned already we first have to extract the part of the expression corresponding to the correction of the incoming electron by one mass operator. This is achieved by neglecting the damping of the electron and photon out-state and expressing the remaining exponential damping of the electron in-state as a series, i.e.

$$e^{2\mathrm{Im}\left\{\frac{m}{p_-}\int_{-\infty}^{\phi_+} d\varphi M_s(p,\varphi)\right\}} \to 1 + 2\mathrm{Im}\left[\frac{m}{p_-}\int_{-\infty}^{\phi_+} d\varphi M_s(p,\varphi)\right] + \cdots . \tag{6.3.2}$$

Now the first summand, i.e. 1, corresponds to the leading order diagram without corrections (see Fig. 6.1). The second summand, i.e. $2\mathrm{Im}\left[\frac{m}{p_-}\int_{-\infty}^{\phi_+} d\varphi M_s(p,\varphi)\right]$, corresponds to the Feynman diagram we are looking for, where the incoming electron is corrected by one mass operator (see Fig. 6.2). Higher order summands instead correspond to diagrams with corrections by multiple mass operators on the incoming electron state. Hence we only keep the second summand for our considerations. Further, we have to sum (average) over the final (initial) quantum numbers in order to compare the probability with the new result, i.e. we have to consider the probability

$$\begin{aligned}\frac{1}{2}\sum_{s,s',j} P_{j,s,s'}^{(e^-\to e^-\gamma),M} =& \sum_{s,s',j}\int \frac{d^3q}{16\pi^2}\frac{\alpha}{p_-p'_-\omega}\int d\phi_+ \mathrm{Im}\left[\frac{m}{p_-}\int_{-\infty}^{\phi_+} d\varphi M_s(p,\varphi)\right] \\ &\times \int d\phi_- e^{i\frac{m^2}{2p_-}\frac{q_-}{p'_-}\left\{[1+\boldsymbol{\pi}^2_{\perp,e}(\phi_+)]\phi_- + \frac{\boldsymbol{\varepsilon}^2(\phi_+)}{m^2}\frac{\phi_-^3}{12}\right\}} T_{j,s,s'}.\end{aligned} \tag{6.3.3}$$

The sum over the final spin and polarization can be performed straight forward, where for the trace the expressions in Eqs. (4.3.1) and (4.3.2) on page 47 are used. We obtain

$$\begin{aligned}\frac{1}{2}\sum_{s',j} T_{j,s,s'} =& -2m^2 - 2\left(2 + \frac{q_-}{p_-}\frac{q_-}{p_- - q_-}\right)\mathcal{A}_0^2\psi'^2(\phi_+)\frac{\phi_-^2}{4} \\ &- ism\mathcal{A}_0\psi'(\phi_+)\phi_-\frac{q_-}{p_-}.\end{aligned} \tag{6.3.4}$$

The sum over the initial spin includes the trace and the imaginary part of the mass operator. With the one-loop mass operator presented in Eq. (3.2.19) on page 33 this sum yields

$$\begin{aligned}\sum_s \mathrm{Im}\left[\frac{m}{p_-}\int_{-\infty}^{\phi_+} d\varphi M_s(p,\varphi)\right]\sum_{s',j} T_{j,s,s'} =& \frac{2\alpha m^2}{\pi p_-}\int_{-\infty}^{\phi_+} d\varphi \int_0^\infty du \int_0^\infty \frac{dv}{(1+v)^3} \\ \times \Bigg\{ \mathrm{Im}\left[e^{-iu\left(1+\frac{u^2\chi_p^2(\varphi)}{3v^2}\right)}\right] & \frac{5+7v+5v^2}{3}\frac{\chi_p^2(\varphi)}{v^2} u \Bigg[-2m^2 \\ - 2\left(2 + \frac{q_-}{p_-}\frac{q_-}{p_- - q_-}\right)&\mathcal{A}_0^2\psi'^2(\phi_+)\frac{\phi_-^2}{4}\Bigg] \\ + \mathrm{Re}\left[e^{-iu\left(1+\frac{u^2\chi_p^2(\varphi)}{3v^2}\right)}\right] & \chi_p(\varphi) im\mathcal{A}_0\psi'(\phi_+)\phi_-\frac{q_-}{p_-}\Bigg\}.\end{aligned} \tag{6.3.5}$$

Now we replace this expression into Eq. (6.3.3) and observe the following: The only dependence on ϕ_- is in the exponential phase term, where it depends on an odd power of ϕ_-, and in the trace. In the trace in Eq. (6.3.5) we see that there are two terms, the first one (in the second and third line) depending on an even power of ϕ_- (more explicit on $\phi_-^0 = 1$ and ϕ_-^2) and the second one (in the fourth line) depending on an odd power of ϕ_- (linearly on ϕ_-). Since the overall integrand of the integral in $d\phi_-$ in Eq. (6.3.3) has to be even, only the real part of the phase contributes to the first part of the trace and only the imaginary part of the phase contributes to the second part of the trace. With this observation we conclude that the final expression of the probability is

$$\begin{aligned}\frac{1}{2}\sum_{s,s',j} P_{j,s,s'}^{(e^-\to e^-\gamma),M} &= \int \frac{d^3q}{(2\pi)^3}\frac{\alpha^2 m^2}{p_-^2 p'_-\omega}\int d\phi_+ \int d\phi_- \int_0^\infty du \int_0^\infty \frac{dv}{(1+v)^3}\int_{-\infty}^{\phi_+} d\varphi \\ &\times \Bigg\{ \mathrm{Re}\left[e^{i\frac{m^2}{2p_-}\frac{q_-}{p'_-}\left\{[1+\boldsymbol{\pi}_{\perp,e}^2(\phi_+)]\phi_- + \frac{\boldsymbol{\varepsilon}^2(\phi_+)}{m^2}\frac{\phi_-^3}{12}\right\}}\right] \mathrm{Im}\left[e^{-iu\left(1+\frac{u^2\chi_p^2(\varphi)}{3v^2}\right)}\right] \\ &\quad \times \frac{5+7v+5v^2}{3}\frac{\chi_p^2(\varphi)}{v^2}u\left[-2m^2 - 2\left(2+\frac{q_-}{p_-}\frac{q_-}{p_- - q_-}\right)\mathcal{A}_0^2\psi'^2(\phi_+)\frac{\phi_-^2}{4}\right] \\ &- \mathrm{Im}\left[e^{i\frac{m^2}{2p_-}\frac{q_-}{p'_-}\left\{[1+\boldsymbol{\pi}_{\perp,e}^2(\phi_+)]\phi_- + \frac{\boldsymbol{\varepsilon}^2(\phi_+)}{m^2}\frac{\phi_-^3}{12}\right\}}\right] \mathrm{Re}\left[e^{-iu\left(1+\frac{u^2\chi_p^2(\varphi)}{3v^2}\right)}\right] \\ &\quad \times \chi_p(\varphi) m \mathcal{A}_0 \psi'(\phi_+)\phi_- \frac{q_-}{p_-}\Bigg\}.\end{aligned} \tag{6.3.6}$$

6.3.2. Rewriting the new probability $\tilde{P}_1$

Now we modify the probability in Eq. (6.2.44) in order to make it comparable with the one in Eq. (6.3.6). As mentioned before, we first consider only the part of the probability corresponding to $\tilde{P}_1$. We assume to have a linearly polarized background field with four-potential (times electron charge) $\mathcal{A}^\mu(\phi) = (0, \boldsymbol{\mathcal{A}}(\phi))$ and $\boldsymbol{\mathcal{A}}(\phi) = \mathcal{A}_0\psi(\phi)\boldsymbol{a}_1$, such that the last term containing the double sum can be neglected. Therefore the probability under consideration is

$$\begin{aligned}
\tilde{P}_1^{\text{linear}} =&\text{Re}\Bigg\{ \int \frac{d^3q}{(2\pi)^3}\frac{\alpha^2}{p_-p'_-\omega}\int dx_-\int dx'_-\int_0^\infty \frac{dudr}{(u+r)^2}e^{-i\frac{r^2}{u+r}m^2} \\
&\times e^{i(p_+-p'_+-q_+)(x'_--x_-)+i\int_{x_-}^{x'_-}d\varphi\left[\frac{(p\mathcal{A}(\varphi))}{p_-}-\frac{(p'\mathcal{A}(\varphi))}{p'_-}-\frac{\mathcal{A}^2(\varphi)}{2p_-}+\frac{\mathcal{A}^2(\varphi)}{2p'_-}\right]} \\
&\times\Bigg\{-\frac{u+2r}{u+r}m^2\frac{q_-}{p_-}\left(e^{T_1^{x_-}}-1\right) \\
&+e^{T_1^{x_-}}\frac{u}{u+r}\left[\frac{p'_-}{q_-}\left(q\Delta\mathcal{A}(\phi_r^{x_-})\right)-\frac{p'_-}{p_-}\left(\Pi(x'_-)\Delta\mathcal{A}(\phi_r^{x_-})\right)\right] \\
&+e^{T_1^{x_-}}\frac{r}{u+r}\left[\frac{p'_-}{q_-}\left(q\int_0^1 dw\Delta\mathcal{A}(\phi_{wr}^{x_-})\right)-\frac{p'_-}{p_-}\left(\Pi(x'_-)\int_0^1 dw\Delta\mathcal{A}(\phi_{wr}^{x_-})\right)\right] \\
&-\frac{i}{2}\frac{R_1^{(\theta m)}}{2m}\Bigg\{2m^2\frac{u+2r}{u+r}\int_{-\infty}^{x_-}d\phi\frac{1}{p_-}\left(e^{T_1}-1\right)+\int_{-\infty}^{x_-}d\phi\frac{1}{p_-}e^{T_1}K_1\Bigg\} \\
&-\frac{i}{2}m^2\frac{r}{u+r}\int_{-\infty}^{x_-}d\phi\frac{q_-}{p_-^2}e^{T_1}\left[\left(\mathcal{A}(x'_-)\Delta\mathcal{A}(\phi_r)\right)-\left(\mathcal{A}(x_-)\Delta\mathcal{A}(\phi_r)\right)\right]\Bigg\}\Bigg\},
\end{aligned} \tag{6.3.7}$$

where $R_1^{(\theta m)}$ is given in Eq. (6.2.27) and K_1 in Eq. (6.2.45). The initial and final particles are on-shell and their momenta fulfill the relations $p_- = p'_- + q_-$ and $\boldsymbol{p}_\perp = \boldsymbol{p}'_\perp + \boldsymbol{q}_\perp$, such that we can rewrite the second exponential function as presented in Eq. (4.1.3). Further, we have to apply the LCFA in order to be able to compare the result with Eq. (6.3.6). As the LCFA is only valid in the case of $\xi_0 \gg 1$ [9,39] we assume this in the following and also expect that $\chi_0 \sim 1$. Similar as in the case of the probability with damping, we perform the LCFA here in two steps: First we transform the terms corresponding to contributions from the mass operator, and after that we transform the remaining part of the expression. The first part, the transformation of the mass operator contributions, is similar to Section 2.9.3 and we use Eqs. (2.9.19)-(2.9.26) to perform it [9,38]. With that we obtain the expression

$$\begin{aligned}
\tilde{P}_1^{L} =&\text{Re}\Bigg\{ \int \frac{d^3q}{(2\pi)^3}\frac{\alpha^2m^2}{p_-^2p'_-\omega}\int dx_-\int dx'_-\int_0^\infty \frac{dudr}{(u+r)^2}e^{-i\frac{r^2}{u+r}m^2} \\
&\times e^{i\frac{m^2q_-}{2p_-p'_-}\int_{x'_-}^{x_-}d\varphi\left[1+\left(\frac{\boldsymbol{p}_\perp}{m}-\frac{p_-}{q_-}\frac{\boldsymbol{q}_\perp}{m}-\frac{\boldsymbol{\mathcal{A}}_\perp(\varphi)}{m}\right)^2\right]} \\
&\times\Bigg\{-\frac{u+2r}{u+r}q_-\left(e^{T_{1,L}^{x_-}}-1\right) \\
&-e^{T_{1,L}^{x_-}}\frac{urp'_-(2u+r)}{(u+r)^2}m\chi_p(x_-)\left[\frac{p_-}{q_-}\left(qa_1\right)-\left(\Pi(x'_-)a_1\right)\right] \\
&-\frac{i}{2}\frac{R_1^{(\theta m)}}{2m^3}\int_{-\infty}^{x_-}d\phi\Bigg\{2m^2\frac{u+2r}{u+r}\left(e^{T_{1,L}}-1\right)+e^{T_{1,L}}K_1^L\Bigg\} \\
&+\frac{i}{2}m^3\frac{2ur^2}{(u+r)^2}\int_{-\infty}^{x_-}d\phi\frac{q_-}{p_-}e^{T_{1,L}}\chi_p(\phi)\left[\left(\mathcal{A}(x'_-)a_1\right)-\left(\mathcal{A}(x_-)a_1\right)\right]\Bigg\}\Bigg\},
\end{aligned} \tag{6.3.8}$$

where now the variable K_1 reduced to

$$K_1^L = \left[12 + 16\frac{r}{u} + 4\frac{r^2}{u^2}\right]\frac{u^3r^2m^6}{3(u+r)^3}\chi_p^2(\phi), \tag{6.3.9}$$

e^{T_1} to

$$e^{T_{1,L}} = \exp\left[-i\frac{u^2r^4m^6}{3(u+r)^3}\chi_p^2(\phi)\right], \tag{6.3.10}$$

and analogously $\exp(T_{1,L}^{x_-}) = \exp[-iu^2r^4m^6\chi_p^2(x_-)/(3(u+r)^3)]$. Before we continue with the second step of the transformation according to the LCFA, we perform some term transformations to simplify the intermediate expression. Similar as in Section 2.9.3 we can perform a series of substitutions, first $v = r/u$ and $\tilde{u} = um^2/(1+v)$, and then $u = \tilde{u}v^2$ [49]. After this substitutions we make use of Eq. (B.0.18). Additionally we note that $\left[\frac{p_-}{q_-}(qa_1) - \big(\Pi(x'_-)a_1\big)\right] = \left(\boldsymbol{p}_\perp - \frac{p_-}{q_-}\boldsymbol{q}_\perp - \boldsymbol{\mathcal{A}}_\perp(x'_-)\right)\cdot\boldsymbol{a}_1$ and end up with the expression

$$\begin{aligned}\tilde{P}_1^L =&\mathrm{Re}\Bigg\{\int\frac{d^3q}{(2\pi)^3}\frac{\alpha^2m^2}{p_-^2p'_-\omega}\int dx_-\int dx'_-\int_0^\infty du\int_0^\infty\frac{dv}{(1+v)^3}\\
&\times e^{i\frac{m^2q_-}{2p_-p'_-}\int_{x'_-}^{x_-}d\varphi\left[1+\left(\frac{\boldsymbol{p}_\perp}{m}-\frac{p_-}{q_-}\frac{\boldsymbol{q}_\perp}{m}-\frac{\boldsymbol{\mathcal{A}}_\perp(\varphi)}{m}\right)^2\right]}\\
&\times\Bigg\{e^{-i\left(u+\frac{u^3\chi_p^2(x_-)}{3v^2}\right)}\Bigg[(1+v-3v^2)\frac{u\chi_p^2(x_-)}{3v^2}q_-\\
&-\frac{p'_-}{m}\left(1+\frac{2}{v}\right)\chi_p(x_-)\left(\boldsymbol{p}_\perp-\frac{p_-}{q_-}\boldsymbol{q}_\perp-\boldsymbol{\mathcal{A}}_\perp(x'_-)\right)\cdot\boldsymbol{a}_1\Bigg]\\
&-i\frac{R_1^{(\theta m)}}{2m}\int_{-\infty}^{x_-}d\phi e^{-i\left(u+\frac{u^3\chi_p^2(\phi)}{3v^2}\right)}(5+7v+5v^2)\frac{u\chi_p^2(\phi)}{3v^2}\\
&+im\int_{-\infty}^{x_-}d\phi\frac{q_-}{p_-}e^{-i\left(u+\frac{u^3\chi_p^2(\phi)}{3v^2}\right)}\chi_p(\phi)\left[\big(\mathcal{A}(x'_-)a_1\big)-\big(\mathcal{A}(x_-)a_1\big)\right]\Bigg\}\Bigg\}.\end{aligned} \tag{6.3.11}$$

At this stage we apply the LCFA to the remaining part of the equation, similar to Chapter 4 below Eq. (4.1.4). We perform the substitution $\phi_+ = (x_- + x'_-)/2$ and $\phi_- = x_- - x'_-$, such that $x_- = \phi_+ + \phi_-/2$, $x'_- = \phi_+ - \phi_-/2$, and $dx_-dx'_- = d\phi_+d\phi_-$ [39]. According to the LCFA we expand now all the terms around $\phi_- \to 0$ in the following way:

Since the mass operator is already in the LCFA we keep only the zero order for its contributions, i.e. $x_- \to \phi_+$. This applies in Eq. (6.3.11) to the function $\chi_p(x_-)$ and to the integration boundary of the integral in $d\phi$. The remaining functions depending on x_- and x'_- in the pre-exponential have to be expanded up to first order (notice that the quantity $R_1^{(\theta m)}$ depends on ϕ_-, too). Finally we split the exponential phase in the second line of Eq. (6.3.11) into two parts like in Eq. (4.1.5) and expand one part linearly in ϕ_- and the other up to cubic order in ϕ_- as discusses below Eq. (4.1.5) [38, 39].

We obtain for the expression of $\tilde{P}_1$ in the LCFA

$$
\begin{aligned}
\tilde{P}_1^{\text{LCFA}} =&\text{Re}\Bigg\{\int \frac{d^3q}{(2\pi)^3}\frac{\alpha^2 m^2}{p_-^2 p'_- \omega}\int d\phi_+ \int d\phi_- \int_0^\infty du \int_0^\infty \frac{dv}{(1+v)^3} \\
&\times e^{i\frac{m^2 q_-}{2p_- p'_-}\left\{\left[1+\boldsymbol{\pi}^2_{\perp,e}(\phi_+)\right]\phi_- + \frac{\boldsymbol{\mathcal{A}}'^2_\perp(\phi_+)}{m^2}\frac{\phi_-^3}{12}\right\}} \\
&\times \Bigg\{ e^{-i\left(u+\frac{u^3\chi_p^2(\phi_+)}{3v^2}\right)}\Bigg[(1+v-3v^2)\frac{u\chi_p^2(\phi_+)}{3v^2}q_- \\
&- p'_-\left(1+\frac{2}{v}\right)\chi_p(\phi_+)\left(\boldsymbol{\pi}_{\perp,e}(\phi_+) + \frac{\phi_-}{2}\frac{\boldsymbol{\mathcal{A}}'_\perp(\phi_+)}{m}\right)\cdot \boldsymbol{a}_1\Bigg] \\
&- i\frac{R^{(\theta m)}_{1,\text{LCFA}}}{2m}\int_{-\infty}^{\phi_+} d\phi e^{-i\left(u+\frac{u^3\chi_p^2(\phi)}{3v^2}\right)}(5+7v+5v^2)\frac{u\chi_p^2(\phi)}{3v^2} \\
&+ im\int_{-\infty}^{\phi_+} d\phi \frac{q_-}{p_-}e^{-i\left(u+\frac{u^3\chi_p^2(\phi)}{3v^2}\right)}\chi_p(\phi)\phi_-\boldsymbol{\mathcal{A}}'_\perp(\phi_+)\cdot \boldsymbol{a}_1\Bigg\}\Bigg\},
\end{aligned}
\tag{6.3.12}
$$

where we have the trace

$$
\begin{aligned}
R^{(\theta m)}_{1,\text{LCFA}} =& -4m^3 - m\left(\frac{p_-}{p'_-}+\frac{p'_-}{p_-}\right)\phi_-^2\boldsymbol{\mathcal{A}}'^2_\perp(\phi_+) \\
&+ 2m^3\left(\frac{p_-}{p'_-}+\frac{p'_-}{p_-}\right)\left[1+\boldsymbol{\pi}^2_{\perp,e}(\phi_+) + \frac{\boldsymbol{\mathcal{A}}'^2_\perp(\phi_+)}{m^2}\frac{\phi_-^2}{4}\right],
\end{aligned}
\tag{6.3.13}
$$

and where $\boldsymbol{\pi}_{\perp,e}(\phi_+)$ is defined in Eq. (4.1.9). Now the second line in Eq. (6.3.13) is proportional to the derivative in ϕ_- of the exponent in the second line of Eq. (6.3.12), such that it vanishes when integrating over ϕ_- according to a property of the Airy function presented in Eq. (B.0.6). Also the term depending on $\boldsymbol{\pi}_{\perp,e}(\phi_+)$ in the fourth line of Eq. (6.3.12) does not contribute as the corresponding integrand is odd in the integral over $d^2\boldsymbol{q}_\perp$. Neglecting both terms the expression of the probability in the LCFA reduces to

$$
\begin{aligned}
\tilde{P}_1^{\text{LCFA}} =&\text{Re}\Bigg\{\int \frac{d^3q}{(2\pi)^3}\frac{\alpha^2 m^2}{p_-^2 p'_- \omega}\int d\phi_+ \int d\phi_- \int_0^\infty du \int_0^\infty \frac{dv}{(1+v)^3} \\
&\times e^{i\frac{m^2 q_-}{2p_- p'_-}\left\{\left[1+\boldsymbol{\pi}^2_{\perp,e}(\phi_+)\right]\phi_- + \frac{\boldsymbol{\mathcal{A}}'^2_\perp(\phi_+)}{m^2}\frac{\phi_-^3}{12}\right\}}\Bigg\{ e^{-i\left(u+\frac{u^3\chi_p^2(\phi_+)}{3v^2}\right)} \\
&\times \left[(1+v-3v^2)\frac{u\chi_p^2(\phi_+)}{3v^2}q_- - \frac{p'_-}{2}\left(1+\frac{2}{v}\right)\phi_-\frac{\mathcal{A}_0\psi'(\phi_+)}{m}\chi_p(\phi_+)\right] \\
&+ i\int_{-\infty}^{\phi_+} d\phi\, e^{-i\left(u+\frac{u^3\chi_p^2(\phi)}{3v^2}\right)}\Bigg\{m\frac{q_-}{p_-}\phi_-\mathcal{A}_0\psi'(\phi_+)\chi_p(\phi) \\
&+ \left[2m^2+\frac{1}{2}\left(\frac{p_-}{p'_-}+\frac{p'_-}{p_-}\right)\phi_-^2\mathcal{A}_0^2\psi'^2(\phi_+)\right](5+7v+5v^2)\frac{u\chi_p^2(\phi)}{3v^2}\Bigg\}\Bigg\}\Bigg\}.
\end{aligned}
\tag{6.3.14}
$$

The above equation only depends on the derivative of the background field and is therefore manifestly gauge invariant.

Now, comparing the third line of the expression with the fourth and fifth line, we observe that the first term in the third line is comparable to the one in the fifth line and the second term in the third line is comparable to the one in the fourth line. An important difference is, however, that taking into account an overall factor of $1/p_-$, the terms in the third line are proportional to q_-/p_- and p'_-/p_-, whereas the fourth and fifth line scale with $2(m^2/p_-)(q_-/p_-)\int_{-\infty}^{\phi_+} d\phi$ and $2(m^2/p_-)\int_{-\infty}^{\phi_+} d\phi$, respectively. Employing the substitution $\tilde{\phi} = \omega_0\phi$ in order to make the integral in ϕ dimensionless, the last two lines receive an additional factor $1/\omega_0$ and, due to the relation $m^2/(\omega_0 p_-) = \xi_0/\chi_0$, we conclude that they scale with the factor ξ_0/χ_0. This factor is large as within the LCFA $\xi_0 \gg 1$ and we further approximate $\chi_0 \sim 1$. On the other hand the factors q_-/p_- and p'_-/p_- are bound to positive values between zero and unity. Therefore, the contribution of the third line in the above equation is negligible within the LCFA and we can approximate the probability by

$$
\begin{aligned}
\tilde{P}_1^{\mathrm{LCFA}} \approx -\operatorname{Im}\Bigg\{ &\int \frac{d^3q}{(2\pi)^3}\frac{\alpha^2 m^2}{p_-^2 p'_-\omega}\int d\phi_+ \int d\phi_- \int_0^\infty du \int_0^\infty \frac{dv}{(1+v)^3} \\
&\times e^{i\frac{m^2 q_-}{2p_-p'_-}\left\{\left[1+\boldsymbol{\pi}_{\perp,e}^2(\phi_+)\right]\phi_- + \frac{\boldsymbol{\mathcal{A}}_\perp'^2(\phi_+)}{m^2}\frac{\phi_-^3}{12}\right\}} \int_{-\infty}^{\phi_+} d\phi\, e^{-i\left(u+\frac{u^3\chi_p^2(\phi)}{3v^2}\right)} \\
&\times \Bigg\{ \left[2m^2 + \frac{1}{2}\left(\frac{p_-}{p'_-}+\frac{p'_-}{p_-}\right)\phi_-^2\mathcal{A}_0^2\psi'^2(\phi_+)\right](5+7v+5v^2)\frac{u\chi_p^2(\phi)}{3v^2} \\
&\quad + m\frac{q_-}{p_-}\phi_-\mathcal{A}_0\psi'(\phi_+)\chi_p(\phi)\Bigg\}\Bigg\}.
\end{aligned}
\tag{6.3.15}
$$

With the remaining expression we perform analogous transformations as in the discussion below Eq. (6.3.5) and obtain the final equation

$$
\begin{aligned}
\tilde{P}_1^{\mathrm{LCFA}} = &\int \frac{d^3q}{(2\pi)^3}\frac{\alpha^2 m^2}{p_-^2 p'_-\omega}\int d\phi_+ \int d\phi_- \int_0^\infty du \int_0^\infty \frac{dv}{(1+v)^3}\int_{-\infty}^{\phi_+} d\phi \\
&\times \Bigg\{ \operatorname{Re}\left[e^{i\frac{m^2 q_-}{2p_-p'_-}\left\{\left[1+\boldsymbol{\pi}_{\perp,e}^2(\phi_+)\right]\phi_- + \frac{\boldsymbol{\mathcal{A}}_\perp'^2(\phi_+)}{m^2}\frac{\phi_-^3}{12}\right\}}\right] \operatorname{Im}\left[e^{-i\left(u+\frac{u^3\chi_p^2(\phi)}{3v^2}\right)}\right] \\
&\quad \times \left[-2m^2 - \frac{1}{2}\left(\frac{p_-}{p'_-}+\frac{p'_-}{p_-}\right)\mathcal{A}_0^2\psi'^2(\phi_+)\phi_-^2\right](5+7v+5v^2)\frac{u\chi_p^2(\phi)}{3v^2} \\
&\quad - \operatorname{Im}\left[e^{i\frac{m^2 q_-}{2p_-p'_-}\left\{\left[1+\boldsymbol{\pi}_{\perp,e}^2(\phi_+)\right]\phi_- + \frac{\boldsymbol{\mathcal{A}}_\perp'^2(\phi_+)}{m^2}\frac{\phi_-^3}{12}\right\}}\right] \operatorname{Re}\left[e^{-i\left(u+\frac{u^3\chi_p^2(\phi)}{3v^2}\right)}\right] \\
&\quad \times m\frac{q_-}{p_-}\chi_p(\phi)\mathcal{A}_0\psi'(\phi_+)\phi_-\Bigg\},
\end{aligned}
\tag{6.3.16}
$$

which precisely coincide with Eq. (6.3.6). We thus proved that the new probability $\tilde{P}_1$, considering a linearly polarized plane wave and the LCFA, is included in the probability of nonlinear Compton scattering with damping.

6.3.3. Comparison of $\tilde{P}_2$ with the probability including the damping

So far we only compared half of $\tilde{P}$ in Eq. (6.2.44), namely the part corresponding to $\tilde{P}_1$. Now we briefly present the main steps of the comparison of the other half, $\tilde{P}_2$, with the probability of nonlinear Compton scattering including the particles states decay, which is performed analogously to the previous one.

We start with rewriting the probability including the damping. This time the exponential damping term is expanded as

$$e^{D^{\mathrm{NC}}_{j,s,s'}} \to 1 + 2\mathrm{Im}\left[\frac{m}{p'_-}\int_{\phi_+}^{\infty} d\varphi M_s(p',\varphi)\right] + \dots. \tag{6.3.17}$$

Keeping only the first order of the expansion and summing (averaging) over the final (initial) quantum numbers gives us finally the expression of the probability

$$\begin{aligned}
\frac{1}{2}\sum_{s,s',j} P^{(e^-\to e^-\gamma),M'}_{j,s,s'} = &\int \frac{d^3q}{(2\pi)^3}\frac{\alpha^2 m^2}{p_- p'^2_- \omega}\int d\phi_+ \int d\phi_- \int_0^{\infty} du \int_0^{\infty}\frac{dv}{(1+v)^3}\int_{\phi_+}^{\infty} d\varphi \\
&\times \Bigg\{ \mathrm{Re}\left[e^{i\frac{m^2}{2p_-}\frac{q_-}{p'_-}\left\{[1+\boldsymbol{\pi}^2_{\perp,e}(\phi_+)]\phi_- + \frac{\boldsymbol{\varepsilon}^2(\phi_+)}{m^2}\frac{\phi_-^3}{12}\right\}}\right]\mathrm{Im}\left[e^{-iu\left(1+\frac{u^2\chi^2_{p'}(\varphi)}{3v^2}\right)}\right] \\
&\quad\times \frac{5+7v+5v^2}{3}\frac{\chi^2_{p'}(\varphi)}{v^2}u\left[-2m^2 - 2\left(2+\frac{q_-}{p_-}\frac{q_-}{p_- - q_-}\right)\mathcal{A}_0^2\psi'^2(\phi_+)\frac{\phi_-^2}{4}\right] \\
&- \mathrm{Im}\left[e^{i\frac{m^2}{2p_-}\frac{q_-}{p'_-}\left\{[1+\boldsymbol{\pi}^2_{\perp,e}(\phi_+)]\phi_- + \frac{\boldsymbol{\varepsilon}^2(\phi_+)}{m^2}\frac{\phi_-^3}{12}\right\}}\right]\mathrm{Re}\left[e^{-iu\left(1+\frac{u^2\chi^2_{p'}(\varphi)}{3v^2}\right)}\right] \\
&\quad\times \chi_{p'}(\varphi) m\mathcal{A}_0\psi'(\phi_+)\phi_-\frac{q_-}{p'_-}\Bigg\}.
\end{aligned} \tag{6.3.18}$$

Next, we have to adapt the expression of $\tilde{P}_2$ following an analog procedure and using the same assumptions as in the previous Subsection. For this, first we derive the expression of $\tilde{P}_2$ in the LCFA in a linearly polarized plane wave. Instead of recalculating all steps, we can alternatively obtain the expression $\tilde{P}_2^{\mathrm{LCFA}}$ directly from $\tilde{P}_1^{\mathrm{LCFA}}$ in Eq. (6.3.12) by symmetry considerations. As we can observe in Eq. (6.2.44) apart from some different signs and different integration boundaries in the integral in $d\phi$, both $\tilde{P}_1$ and $\tilde{P}_2$ are similar under the exchange of $p \leftrightarrow p'$, $\Pi(x'_-) \leftrightarrow \Pi'(x'_-)$, $\phi_r \leftrightarrow \tilde{\phi}_r$, and $\phi_{wr} \leftrightarrow \tilde{\phi}_{wr}$. Now, when performing the LCFA for the mass operator an important difference is however that in ϕ_r and $\tilde{\phi}_r$, and similarly in ϕ_{wr} and $\tilde{\phi}_{wr}$, the correction to the phase ϕ has a different relative sign. This results into the replacement $\chi_p(\phi) \to -\chi_{p'}(\phi)$ in Eq. (6.3.12) for the LCFA expression of $\tilde{P}_2$. The two terms that vanish with the integrals in ϕ_- and $d^2\boldsymbol{q}_\perp$ vanish in this case, too (note that $\left(\boldsymbol{p}'_\perp - \frac{p'_-}{q_-}\boldsymbol{q}_\perp - \boldsymbol{\mathcal{A}}_\perp(x'_-)\right) = \left(\boldsymbol{p}_\perp - \frac{p_-}{q_-}\boldsymbol{q}_\perp - \boldsymbol{\mathcal{A}}_\perp(x'_-)\right)$). Hence we obtain for the probability $\tilde{P}_2$

in a linearly polarized plane wave and in the LCFA

$$
\begin{aligned}
\tilde{P}_2^{\rm LCFA} =&\mathrm{Re}\Bigg\{\int\frac{d^3q}{(2\pi)^3}\frac{\alpha^2m^2}{p_-p_-'^2\omega}\int d\phi_+\int d\phi_-\int_0^\infty du\int_0^\infty\frac{dv}{(1+v)^3}\\
&\times e^{i\frac{m^2q_-}{2p_-p_-'}\left\{\left[1+\boldsymbol{\pi}_{\perp,e}^2(\phi_+)\right]\phi_-+\frac{\boldsymbol{\mathcal{A}}_\perp'^2(\phi_+)}{m^2}\frac{\phi_-^3}{12}\right\}}\Bigg\{e^{-i\left(u+\frac{u^3\chi_{p'}^2(\phi_+)}{3v^2}\right)}\\
&\times\left[-(1+v-3v^2)\frac{u\chi_{p'}^2(\phi_+)}{3v^2}q_-+\frac{p_-}{2}\left(1+\frac{2}{v}\right)\phi_-\frac{\mathcal{A}_0\psi'(\phi_+)}{m}\chi_{p'}(\phi_+)\right]\\
&+i\int_{\phi_+}^\infty d\phi\,e^{-i\left(u+\frac{u^3\chi_{p'}^2(\phi)}{3v^2}\right)}\Bigg\{m\frac{q_-}{p_-'}\phi_-\mathcal{A}_0\psi'(\phi_+)\chi_{p'}(\phi)\\
&+\left[2m^2+\frac{1}{2}\left(\frac{p_-}{p_-'}+\frac{p_-'}{p_-}\right)\phi_-^2\mathcal{A}_0^2\psi'^2(\phi_+)\right](5+7v+5v^2)\frac{u\chi_{p'}^2(\phi)}{3v^2}\Bigg\}\Bigg\}\Bigg\}.
\end{aligned}
\tag{6.3.19}
$$

With the same argumentation as below Eq. (6.3.14) the third line of the above equation can be neglected within the LCFA since its contributions are smaller by an factor of $1/\xi_0 \ll 1$ in comparison to the remaining expression. Therefore, we obtain for the probability $\tilde{P}_2$ within the LCFA the expression

$$
\begin{aligned}
\tilde{P}_2^{\rm LCFA} \approx &-\mathrm{Im}\Bigg\{\int\frac{d^3q}{(2\pi)^3}\frac{\alpha^2m^2}{p_-p_-'^2\omega}\int d\phi_+\int d\phi_-\int_0^\infty du\int_0^\infty\frac{dv}{(1+v)^3}\\
&\times e^{i\frac{m^2q_-}{2p_-p_-'}\left\{\left[1+\boldsymbol{\pi}_{\perp,e}^2(\phi_+)\right]\phi_-+\frac{\boldsymbol{\mathcal{A}}_\perp'^2(\phi_+)}{m^2}\frac{\phi_-^3}{12}\right\}}\int_{\phi_+}^\infty d\phi\,e^{-i\left(u+\frac{u^3\chi_{p'}^2(\phi)}{3v^2}\right)}\\
&\times\Bigg\{\left[2m^2+\frac{1}{2}\left(\frac{p_-}{p_-'}+\frac{p_-'}{p_-}\right)\phi_-^2\mathcal{A}_0^2\psi'^2(\phi_+)\right](5+7v+5v^2)\frac{u\chi_{p'}^2(\phi)}{3v^2}\\
&+m\frac{q_-}{p_-'}\phi_-\mathcal{A}_0\psi'(\phi_+)\chi_{p'}(\phi)\Bigg\}\Bigg\}.
\end{aligned}
\tag{6.3.20}
$$

After rewriting the expression as discussed below Eq. (6.3.5), it precisely coincides with Eq. (6.3.18). Therefore we conclude that the complete probability $\tilde{P}$ is, for a linear polarized plane wave and within the LCFA, included in the probability of nonlinear Compton scattering with damping.

6.4. Probability corrections in the LCFA

We derived the expressions of $\tilde{P}_1$ and $\tilde{P}_2$ for a linearly polarized plane wave and within the LCFA in Eqs. (6.3.15) and (6.3.20), respectively, and here we will perform the remaining integrals in $d^2\boldsymbol{q}_\perp$ and $d\phi_-$ for both expressions. For this we follow the same procedure as in Section 4.3. First we transform the integral in the photon momentum to light-cone coordinates via $dq^3 = (\omega/q_-)dq_-d^2\boldsymbol{q}_\perp$ [39] and use Eq. (4.3.4) to perform the integral in the perpendicular photon momentum which is Gaussian. After that we employ the substitution $\tilde{\phi} = \left[q_- \mathcal{A}_0^2 \psi'^2(\phi_+)/(8p_- p'_-)\right]^{1/3} \phi_-$ and define $z = \left[q_-/(p'_- \chi_p(\phi_+))\right]^{2/3}$. With Eqs. (B.0.2)-(B.0.4) the integral in ϕ_- can be expressed in terms of Airy-functions and we finally obtain the expressions

$$\begin{aligned}\tilde{P}_1^{\mathrm{LCFA}} = &- \mathrm{Im}\Bigg\{ \int_0^{p_-} \frac{dq_-}{2\pi} \frac{2\alpha^2 m^4}{p_-^3} \int d\phi_+ \int_0^\infty du \int_0^\infty \frac{dv}{(1+v)^3} \\ &\times \int_{-\infty}^{\phi_+} d\phi \, e^{-i\left(u + \frac{u^3 \chi_p^2(\phi)}{3v^2}\right)} \Bigg\{ i \frac{q_-}{p_-} \chi_p(\phi) \frac{\mathrm{Ai}(z)}{\sqrt{z}} \mathrm{sgn}(\psi'(\phi_+)) \\ &+ \left[\mathrm{Ai}_1(z) + \left(\frac{p_-}{p'_-} + \frac{p'_-}{p_-}\right) \frac{\mathrm{Ai}'(z)}{z}\right] (5 + 7v + 5v^2) \frac{u \chi_p^2(\phi)}{3v^2} \Bigg\}\Bigg\}\end{aligned} \tag{6.4.1}$$

and

$$\begin{aligned}\tilde{P}_2^{\mathrm{LCFA}} = &- \mathrm{Im}\Bigg\{ \int_0^{p_-} \frac{dq_-}{2\pi} \frac{2\alpha^2 m^4}{p_-^2 p'_-} \int d\phi_+ \int_0^\infty du \int_0^\infty \frac{dv}{(1+v)^3} \\ &\times \int_{\phi_+}^{\infty} d\phi \, e^{-i\left(u + \frac{u^3 \chi_{p'}^2(\phi)}{3v^2}\right)} \Bigg\{ i \frac{q_-}{p'_-} \chi_{p'}(\phi) \frac{\mathrm{Ai}(z)}{\sqrt{z}} \mathrm{sgn}(\psi'(\phi_+)) \\ &+ \left[\mathrm{Ai}_1(z) + \left(\frac{p_-}{p'_-} + \frac{p'_-}{p_-}\right) \frac{\mathrm{Ai}'(z)}{z}\right] (5 + 7v + 5v^2) \frac{u \chi_{p'}^2(\phi)}{3v^2} \Bigg\}\Bigg\},\end{aligned} \tag{6.4.2}$$

where $\mathrm{sgn}(\psi'(\phi_+))$ denotes again the sign of the pulse shape function $\psi'(\phi_+)$. As discussed in the previous Section these probability corrections are included in the probability with damping in Eq. (4.3.6).

7. Conclusions

Throughout this thesis we investigated corrections to the first order strong field QED processes nonlinear Compton scattering and nonlinear Breit-Wheeler pair production. Our first goal was to derive probabilities for both, nonlinear Compton scattering and nonlinear Breit-Wheeler pair production, which stay valid in the limit of large phase length of a pulsed plane wave background field.

For this in Chapter 3 we investigated which Feynman diagrams of the corrections to nonlinear Compton scattering and nonlinear Breit-Wheeler pair production have to be considered in order to obtain useful results in the limit of a long pulse duration of the laser pulse. We observed that the summation of these corrections is precisely included when calculating the S-matrix with the exact states obtained from the Schwinger-Dyson equations for the electron, positron, and photon. As it was already shown in Refs. [48, 64, 65], this exact electron, positron and photon states are not stable inside a plane wave background field, but "decay", meaning that electrons and positrons emit photons and photons decay into electron-positron pairs. Their expressions were obtained within the LCFA and in leading order of α (see also Refs. [48, 64, 65]).

In Chapters 4 and 5 we used these exact electron, positron, and photon states to calculate the probabilities for nonlinear Compton scattering and nonlinear Breit-Wheeler pair production, respectively. This probabilities contain corrections due to the particles states decay in form of an exponential damping term and, in contrast to the expressions without corrections, they are valid for long pulse durations of the background field, too. The new exponential damping scales with the pulse duration and depends on the particles momenta and their spin/polarization quantum numbers. It becomes important when the product $\alpha\xi_0\Phi_L \gtrsim 1$, with Φ_L being the phase length of the laser pulse and considering that $\chi_0 \sim 1$ and $\kappa_0 \sim 1$. When neglecting the damping, the results reduce to the probabilities without radiative corrections. For all the calculations a particular spin and polarization basis was chosen. We proved however, that the results for the total probability are independent of the choice of this basis after summing over the corresponding spin/polarization quantum number. Further we verified that the result of nonlinear Compton scattering is in agreement with that of a Poissonian distribution in the classical limit, as it was found in Ref. [42].

Finally, we investigated in Chapter 6 the first order in α corrections to nonlinear Compton scattering coming from Feynman diagrams where the mass operator is acting either on the incoming or on the outgoing electron state. This calculation was performed in a more general case without employing the LCFA and for a plane wave background field with arbitrary transverse polarization. The corrections to the probability of nonlinear Compton scattering were summed (averaged) over the quantum numbers of the outgoing (incoming) particles and integrated over the outgoing particles momenta. At the end they were compared to the probability including the particles states decay computed in Chapter 4. Here we observed that within the approximations in Chapter 4 the contributions of the new probability corrections are included in the probability with damping. In this context also the expressions of the new probability corrections within the LCFA and for a linearly polarized plane wave were presented.

In order to obtain the complete first order in α correction to the probability of nonlinear Compton scattering also the correction by the polarization operator and the vertex correction have to be considered. The polarization operator correction was investigated in parallel by an other student, whereas the vertex correction has still to be computed in future. Especially the contribution of the vertex correction is of interest, as it is not included in the resummation corresponding to the probability with damping.

Appendix

A. Dirac matrices

The Dirac matrices are generated by the two-dimensional unity-matrix and the Pauli-matrices [11, 46]

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.0.1}$$

and they have the following form:

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_{2\times2} & 0 \\ 0 & -\mathbb{1}_{2\times2} \end{pmatrix}, \qquad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}, \tag{A.0.2}$$

with $j = 1, 2, 3$. They fulfill the conditions $(\gamma^0)^2 = \mathbb{1}_{4\times4}$, $(\gamma^j)^2 = -\mathbb{1}_{4\times4}$, $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$, and the anti-commutator is given by

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}. \tag{A.0.3}$$

A product of three γ-matrices can be rewritten by the following identity [49, 78]

$$\gamma^\mu\gamma^\nu\gamma^\rho = \eta^{\mu\nu}\gamma^\rho + \eta^{\nu\rho}\gamma^\mu - \eta^{\mu\rho}\gamma^\nu - i\epsilon^{\tau\mu\nu\rho}\gamma_\tau\gamma^5, \tag{A.0.4}$$

where $\epsilon^{0123} = +1$ and we will introduce γ^5 shortly in Eq. (A.0.8). Further the following relations can be helpful when calculating traces:

$$\mathrm{Tr}[\gamma^\mu] = 0 = \mathrm{Tr}[\gamma^{\mu_1}...\gamma^{\mu_j}] \text{ for } j \text{ being an odd number}, \tag{A.0.5}$$

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu] = 4\eta^{\mu\nu}, \tag{A.0.6}$$

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}). \tag{A.0.7}$$

The 5th Dirac matrix is defined as [11, 46]

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & 0 \end{pmatrix}. \tag{A.0.8}$$

Its square is equal to the identity matrix and it anti-commutes with the other Dirac matrices, i.e. $(\gamma^5)^2 = 1$ and $\{\gamma^5, \gamma^\mu\} = 0$. Useful for traces with γ^5 are the following identities:

$$\mathrm{Tr}[\gamma^5] = 0 = \mathrm{Tr}[\gamma^5\gamma^\mu] = \mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu] = \mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho], \tag{A.0.9}$$

$$\mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = -4i\epsilon^{\mu\nu\rho\sigma}, \tag{A.0.10}$$

$$\mathrm{Tr}[\gamma^5\gamma^{\mu_1}...\gamma^{\mu_j}] = 0 \text{ for } j \text{ being an odd number}, \tag{A.0.11}$$

$$\begin{aligned}\mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\alpha\gamma^\beta\gamma^\delta] = -4i(&\eta^{\mu\nu}\epsilon^{\rho\alpha\beta\delta} + \eta^{\nu\rho}\epsilon^{\mu\alpha\beta\delta} - \eta^{\mu\rho}\epsilon^{\nu\alpha\beta\delta} \\ &- \eta^{\beta\delta}\epsilon^{\alpha\mu\nu\rho} + \eta^{\alpha\delta}\epsilon^{\beta\mu\nu\rho} - \eta^{\alpha\beta}\epsilon^{\delta\mu\nu\rho}).\end{aligned} \tag{A.0.12}$$

Another useful matrix when dealing with γ-matrices is their commutator [11, 46]

$$\sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu). \tag{A.0.13}$$

This matrix is anti-symmetric ($\sigma^{\mu\nu} = -\sigma^{\nu\mu}$) and commutes with γ^5, i.e. $[\sigma^{\mu\nu}, \gamma^5] = 0$. For the product of both matrices there exists the relation

$$\gamma^5\sigma^{\mu\nu} = \frac{i}{2}\epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta}. \tag{A.0.14}$$

Traces with $\sigma^{\mu\nu}$ can be solved with the following identities:

$$\mathrm{Tr}[\sigma^{\mu\nu}] = 0, \tag{A.0.15}$$

$$\mathrm{Tr}[\sigma^{\mu\nu}\gamma^{\mu_1}...\gamma^{\mu_j}] = 0 \text{ for } j \text{ being an odd number}, \tag{A.0.16}$$

$$\mathrm{Tr}[\sigma^{\mu\nu}\gamma^\rho\gamma^\tau] = 4i(\eta^{\mu\tau}\eta^{\nu\rho} - \eta^{\mu\rho}\eta^{\nu\tau}), \tag{A.0.17}$$

$$\mathrm{Tr}[\gamma^5\sigma^{\mu\nu}\gamma^\rho\gamma^\tau] = 4\epsilon^{\mu\nu\rho\tau}, \tag{A.0.18}$$

$$\begin{aligned}\mathrm{Tr}[\gamma^5\sigma^{\mu\nu}\gamma^\rho\gamma^\alpha\gamma^\beta\gamma^\delta] = 4(&\eta^{\nu\rho}\epsilon^{\mu\alpha\beta\delta} - \eta^{\mu\rho}\epsilon^{\nu\alpha\beta\delta} - \eta^{\beta\delta}\epsilon^{\alpha\mu\nu\rho}\\ &+ \eta^{\alpha\delta}\epsilon^{\beta\mu\nu\rho} - \eta^{\alpha\beta}\epsilon^{\delta\mu\nu\rho}),\end{aligned} \tag{A.0.19}$$

$$\begin{aligned}\mathrm{Tr}[\gamma^5\gamma^\mu\sigma^{\nu\rho}\gamma^\alpha\gamma^\beta\gamma^\delta] = 4(&\eta^{\mu\nu}\epsilon^{\rho\alpha\beta\delta} - \eta^{\mu\rho}\epsilon^{\nu\alpha\beta\delta} - \eta^{\beta\delta}\epsilon^{\alpha\mu\nu\rho}\\ &+ \eta^{\alpha\delta}\epsilon^{\beta\mu\nu\rho} - \eta^{\alpha\beta}\epsilon^{\delta\mu\nu\rho}).\end{aligned} \tag{A.0.20}$$

Together the matrices $1_{4\times4}$, γ^5, γ^μ, $i\gamma^\mu\gamma^5$, $\sigma^{\mu\nu}$ form a complete set in the sense that an arbitrary matrix M can be decomposed into a linear combination of those matrices [3, 48], i.e.

$$M = m_1 1_{4\times4} + m_5\gamma^5 + c_\mu\gamma^\mu + m_{5\mu}i\gamma^\mu\gamma^5 + m_{\mu\nu}\sigma^{\mu\nu}, \tag{A.0.21}$$

where the coefficients are obtained by solving the following traces

$$m_1 = \frac{1}{4}\mathrm{Tr}\left[1_{4\times4}M\right], \tag{A.0.22}$$

$$m_5 = \frac{1}{4}\mathrm{Tr}\left[\gamma^5 M\right], \tag{A.0.23}$$

$$m_\mu = \frac{1}{4}\mathrm{Tr}\left[\gamma_\mu M\right], \tag{A.0.24}$$

$$m_{5\mu} = \frac{1}{4}\mathrm{Tr}\left[i\gamma_\mu\gamma^5 M\right], \tag{A.0.25}$$

$$m_{\mu\nu} = \frac{1}{8}\mathrm{Tr}\left[\sigma_{\mu\nu}M\right]. \tag{A.0.26}$$

B. Airy functions

The so-called Airy-function solves the following differential equation [75]:

$$\mathrm{Ai}''(z) = z\mathrm{Ai}(z). \tag{B.0.1}$$

Its integral representation is given by [79]

$$\mathrm{Ai}(z) = \int_{-\infty}^{\infty} \frac{d\tilde{\phi}}{2\pi} e^{iz\tilde{\phi}+i\frac{\tilde{\phi}^3}{3}} \tag{B.0.2}$$

and for the derivative by

$$\mathrm{Ai}'(z) = i\int_{-\infty}^{\infty} \frac{d\tilde{\phi}}{2\pi} \tilde{\phi} e^{iz\tilde{\phi}+i\frac{\tilde{\phi}^3}{3}}. \tag{B.0.3}$$

Further, the integral over the Airy function can be expressed as

$$\mathrm{Ai}_1(z) = \int_z^{\infty} dx\mathrm{Ai}(x) = i\int_{-\infty}^{\infty} \frac{d\tilde{\phi}}{2\pi} \frac{1}{\tilde{\phi}+i0} e^{iz\tilde{\phi}+i\frac{\tilde{\phi}^3}{3}}. \tag{B.0.4}$$

Using partial integration one obtains for the integral

$$\int_{-\infty}^{\infty} \frac{d\tilde{\phi}}{2\pi} \frac{1}{(\tilde{\phi}+i0)^2} e^{iz\tilde{\phi}+i\frac{\tilde{\phi}^3}{3}} = z\mathrm{Ai}_1(z) + \mathrm{Ai}'(z). \tag{B.0.5}$$

Another useful identity can be derived from the differential equation (B.0.1) and the integral representation of the Airy function in Eq. (B.0.2). As can be seen in the following proof, the derivative of the exponential function with respect to the integration variable in the integral representation of the Airy function in Eq. (B.0.2) vanishes:

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{dt}{2\pi} \frac{\partial}{\partial t} e^{izt+i\frac{t^3}{3}} &= \int_{-\infty}^{\infty} \frac{dt}{2\pi} [iz + it^2] e^{izt+i\frac{t^3}{3}} = \int_{-\infty}^{\infty} \frac{dt}{2\pi} \left[iz - i\frac{\partial^2}{\partial z^2}\right] e^{izt+i\frac{t^3}{3}} \\ &= i[z\mathrm{Ai}(z) - \mathrm{Ai}''(z)] = 0. \end{aligned} \tag{B.0.6}$$

In the limit of small arguments ($z \ll 1$) the Airy function can be expanded to [80]

$$\frac{\mathrm{Ai}(z)}{\sqrt{z}} \overset{z\ll 1}{\approx} \frac{1}{3^{2/3}\Gamma\left(\frac{2}{3}\right)\sqrt{z}} + \mathcal{O}(\sqrt{z}), \tag{B.0.7}$$

$$\frac{\mathrm{Ai}'(z)}{z} \overset{z\ll 1}{\approx} -\frac{1}{3^{1/3}\Gamma\left(\frac{1}{3}\right)z} + \mathcal{O}(z), \tag{B.0.8}$$

$$\mathrm{Ai}_1(z) \overset{z\ll 1}{\approx} \frac{1}{3} + \mathcal{O}(z), \tag{B.0.9}$$

where $\Gamma(x)$ is the Gamma-function [75]. Contrary the expansion of the Airy functions for large arguments ($z \gg 1$) is [80]

$$\frac{\mathrm{Ai}(z)}{\sqrt{z}} \overset{z\gg 1}{\approx} e^{-\frac{2}{3}z^{3/2}} \left[\frac{1}{2\sqrt{\pi}} z^{-\frac{3}{4}} - \frac{5}{96\sqrt{\pi}} z^{-\frac{9}{4}} + \frac{385}{9216\sqrt{\pi}} z^{-\frac{15}{4}} + \mathcal{O}\left(z^{-\frac{21}{4}}\right)\right], \tag{B.0.10}$$

$$\frac{\mathrm{Ai}'(z)}{z} \overset{z\gg 1}{\approx} e^{-\frac{2}{3}z^{3/2}} \left[-\frac{1}{2\sqrt{\pi}} z^{-\frac{3}{4}} - \frac{7}{96\sqrt{\pi}} z^{-\frac{9}{4}} + \frac{455}{9216\sqrt{\pi}} z^{-\frac{15}{4}} + \mathcal{O}\left(z^{-\frac{21}{4}}\right)\right], \tag{B.0.11}$$

$$\mathrm{Ai}_1(z) \overset{z\gg 1}{\approx} e^{-\frac{2}{3}z^{3/2}} \left[\frac{1}{2\sqrt{\pi}} z^{-\frac{3}{4}} - \frac{41}{96\sqrt{\pi}} z^{-\frac{9}{4}} + \frac{9241}{9216\sqrt{\pi}} z^{-\frac{15}{4}} + \mathcal{O}\left(z^{-\frac{21}{4}}\right)\right]. \tag{B.0.12}$$

The Airy function and its derivative are related to the Bessel functions of second kind by [75]

$$\mathrm{Ai}(z) = \frac{\sqrt{z}}{\pi\sqrt{3}} \mathrm{K}_{\pm 1/3}\left(\frac{2}{3} z^{\frac{3}{2}}\right), \tag{B.0.13}$$

$$\mathrm{Ai}'(z) = -\frac{z}{\pi\sqrt{3}} \mathrm{K}_{\pm 2/3}\left(\frac{2}{3} z^{\frac{3}{2}}\right). \tag{B.0.14}$$

Other functions related to the Airy-function are (see Appendix C in [9] and Appendix F in [48])

$$f(z) = i \int_0^\infty du e^{-i\left(uz+\frac{u^3}{3}\right)}, \tag{B.0.15}$$

with its derivative

$$f'(z) = \int_0^\infty du\ u\ e^{-i\left(uz+\frac{u^3}{3}\right)}, \tag{B.0.16}$$

and the function

$$f_1(z) = \int_0^\infty \frac{du}{u} e^{-iuz}\left(e^{-i\frac{u^3}{3}} - 1\right). \tag{B.0.17}$$

Here the functions $f_1(z)$ and $f'(z)$ obey the following useful relation [48, 49]:

$$\int_0^\infty dv \frac{1+2v}{(1+v)^3} f_1\left((v/\chi_0)^{\frac{2}{3}}\right) = -\int_0^\infty dv \frac{1+v-3v^2}{3(1+v)^3}\left(\frac{\chi_0}{v}\right)^{\frac{2}{3}} f'\left((v/\chi_0)^{\frac{2}{3}}\right). \tag{B.0.18}$$

C. Summary Notation

All calculations we performed using units, where $\epsilon_0 = \hbar = c = 1$, with ϵ_0 being the vacuum permittivity, $\hbar$ the reduced Planck constant, and c the speed of light. Further, e is the electron charge with $e < 0$ and m is the electron mass.

We use the metric tensor $\eta^{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, where Greek letters in indices taking the values 0, 1, 2, 3. The Einstein notation is always assumed, meaning that indices with identical Greek letters are summed.

An arbitrary vector is presented as $\boldsymbol{v} = (v^1, v^2, v^3)$. Four-vectors have the components $v^\mu = (v^0, v^1, v^2, v^3) = (v^0, \boldsymbol{v})$ and in the covariant form $v_\mu = (v_0, v_1, v_2, v_3) = (v^0, -\boldsymbol{v})$. Both forms are related by the metric via $v^\mu = \eta^{\mu\nu} v_\nu$. The scalar product of two arbitrary vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is presented as $\boldsymbol{v} \cdot \boldsymbol{w} = v^1 w^1 + v^2 w^2 + v^3 w^3$. A four-product of two arbitrary four-vectors v^μ and w^μ is given by $(vw) = v^\mu w_\mu = v^0 w^0 - \boldsymbol{v} \cdot \boldsymbol{w}$. For the square of a vector we have $\boldsymbol{v}^2 = \boldsymbol{v} \cdot \boldsymbol{v}$ and for a four-vector $v^2 = v^\mu v_\mu$.

The space-time four-vector is defined as $x^\mu = (t, \boldsymbol{x})$, where t is the time and $\boldsymbol{x}$ the space vector. On-shell electrons and positrons with energy ε and momentum $p^\mu = (\varepsilon, \boldsymbol{p})$ fulfill the relation $p^2 = m^2$.

Some other definitions:	
$\alpha = e^2/(4\pi)$	fine structure constant
$\partial_\mu = \partial/(\partial x^\mu)$	derivative operator
$\epsilon^{\mu\nu\rho\lambda} = +1$	anti-symmetric four-tensor with $\epsilon^{0123} = +1$
$\delta(x)$	Dirac delta function
$[v^\mu, w^\nu] = v^\mu w^\nu - w^\nu v^\mu$	commutator
$\{v^\mu, w^\nu\} = v^\mu w^\nu + w^\nu v^\mu$	anti-commutator
$... \pm i0 = \lim\limits_{d \to +0} ... \pm id$	a small imaginary part to avoid divergences
γ^μ, γ_5, $\sigma^{\mu\nu}$	Dirac matrix and commutator (see Appendix A)
$\hat{v} = v_\mu \gamma^\mu$	contraction of four-vector with Dirac matrix
$\bar{u} = u^\dagger \gamma^0$	for a spinor u
$\bar{M} = \gamma^0 M^\dagger \gamma^0$	for a matrix M
$\xi_0 = \lvert e \rvert E_0 / m\omega_0$	classical nonlinearity parameter (see Section 2.1)
$\chi_p(\phi)$, $\kappa_q(\phi)$	quantum nonlinearity parameter (see Section 2.8)
$\mathcal{A}(\phi)$, $\mathcal{A}_0$, $\mathcal{F}^{\mu\nu}(\phi)$, $\psi(\phi)$	quantities of the background field (see Section 2.3)
$\Delta\mathcal{A}^\mu(\phi_r) = \mathcal{A}^\mu(\phi_r) - \mathcal{A}^\mu(\phi)$	
n^μ, $\tilde{n}^\mu$, a_j^μ	light-cone basis (see Section 2.4)
v_-, v_+, $\boldsymbol{v}_\perp$, ϕ	light-cone coordinates (see Section 2.4)
$\Lambda_j^\mu(q)$, ϵ_j^μ	polarization four-vector (see Section 2.5)
$u_s(p)$, $v_s(p)$, $E(p,x)$, ζ_p^μ	quantities of the fermion (see Section 2.6)
$\mathrm{Ai}(z)$	Airy function (see Appendix B)
$\Pi^\mu(\phi) = \Pi_p^\mu(\phi)$, $\Pi'^\mu(\phi) = \Pi_{p'}^\mu(\phi)$	short notation, $\Pi_p^\mu(\phi)$ defined in Eq. (2.9.4)

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