Comments on Trace Anomaly Matching

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Abstract

The structure of type A and B trace anomalies is reanalyzed in terms of the universal behaviour of dimension -2 invariant amplitudes. Based on it a general argument for trace anomaly matching between the unbroken and broken phases of a CFT is given. The structure of moduli trace anomalies and their transformations under source reparametrizations is discussed in detail.

Contents

1 Introduction

Trace anomalies [1,2] have rather special properties compared with the other QFT anomalies. While chiral anomalies can be described generally in a topological framework, which allows their understanding independent of the group (continuous or discrete) or the order of the symmetry (zero form or higher form), trace anomalies do not have such a topological description. This difference is related to trace anomalies being "real", i.e. appearing as a real term in the Euclidean generating functional in counterdistinction to the chiral anomalies which appear as a phase (of course in Minkowski metric all terms being phases the distinction is not there). As a consequence, while the 't Hooft matching for chiral anomalies, i.e. the constancy of the anomaly along the RG flow, follows from the topological invariants being rigid, such an argument for matching is not available for trace anomalies. Nevertheless it is believed that trace anomalies are matched between the unbroken and spontaneously broken phases of a given CFT [3]. For this matching one should rely on the detailed analytic structure of the anomalous correlators.

The diffeomorphism and Weyl symmetry Ward identities obeyed by connected correlators of primary operators have the same form in the unbroken and broken phases. This follows from the fact that they are derived from operatorial relations which are the same in the two phases, evaluated on a Poincaré invariant vacuum, while the transformation of the vacuum under dilations and special conformal transformations is not used. Moreover in both phases the general analytic structure of the invariant amplitudes is the same. As a consequence the cohomological structures of the generating functional are the same in the two phases. Therefore the same local functionals of the sources can appear as anomalies. We mean by "matching" simply that the normalizations of the anomalies are the same in the two phases of a given theory. Since the functional dependence on the momentum invariants of the correlators is completely different in the two phases, "anomaly matching", if valid, gives non-trivial constraints on e.g. the structure and normalization of the amplitudes in the broken phase involving the dilaton.

Generically the spectrum of the broken phase is massive. The mass scale is provided by the non-zero vacuum expectation value of a scalar primary operator of positive dimension which causes the spontaneous breaking of conformal symmetry to Poincaré symmetry. There could be a decoupled massless subsector which still preserves conformal invariance. If present it should be factored out in the anomaly matching. In the rest the only generic massless field present is the dilaton, the Goldstone boson corresponding to the broken Weyl symmetry. Since trace anomalies require the contributions of massless fields as intermediate states in correlators, the anomaly matching fixes certain couplings of the dilaton. These couplings are normalized by the difference between the anomalies in the unbroken phase and the conformal sector of the broken phase, if present.

Proving anomaly matching for CFTs is not trivial [3]. The distinction between the two types of trace anomalies [4] ("type A" and "type B") played an important role. Type A anomalies have an analytic structure very similar to zero form, continuous group chiral anomalies. One could identify a dimension -2 invariant amplitude which, for special kinematic configurations where there is only one independent invariant q^2, has a $\frac{a}{q^2}$ dependence. The coefficient of this power gives the normalization of the type A anomaly. The existence of this is again a consequence of the conformal Ward identities which are also valid in the spontaneously broken phase. In addition one needs again the usual requirements of analyticity which are believed to be valid also in the broken phase and therefore the $\frac{1}{q^2}$ behaviour is also there. Using the general property that in the limit where we rescale the momentum to infinity the amplitudes in the broken phase should match those in the unbroken phase in the same limit, the coefficients of $\frac{1}{q^2}$ singularities should match since they originate from amplitudes which match. Moreover the dilaton contributes exactly to this amplitude and therefore the dilaton couplings are constrained by the type A anomaly coefficient calculated in the unbroken phase.

For type B anomalies the situation is considerably more involved. We start by reviewing the procedure for finding the normalization of type B anomalies [1]. Type B anomalies appear generically in correlators of integer dimensional primary operators with the energy-momentum tensor. One particular case involves correlators of just energy-momentum tensors in even dimensions. In the cohomological analysis type B anomalies are characterized by an anomaly density which is Weyl invariant and the anomaly does not vanish for x-independent Weyl parameter σ. For the standard example let us consider the Weyl anomaly in $d = 4$ for a CFT coupled to a background metric $g_{\mu\nu}$. Then the type

B anomaly is

$$\delta_\sigma W = c \int d^4x\, \sigma\, \sqrt{g}\, C^2 \tag{1.1}$$

where W is the generating functional for connected correlation functions of the energy-momentum tensor, C^2 is the square of the Weyl tensor and c the anomaly coefficient. For x-independent σ, (1.1) is also the variation of the correlators under dilations and therefore the anomaly is directly related to the only possible UV counterterm

$$\bar{c}\, \log \Lambda^2 \int d^4x\, \sqrt{g}\, C^2 \tag{1.2}$$

corresponding to a logarithmic UV divergence which is possible for integral dimension primaries in a CFT. The correlators are no longer invariant under dilations since after the subtraction of the counterterm (1.2) the finite correlator contains terms with logarithmic dependence on the invariants. Therefore in the unbroken phase the anomaly coefficient c can be identified by looking at the variation under dilations of a logarithmic term in the appropriate correlator, e.g. the two-point function, whose Λ dependence follows from the second term in the expansion of (1.2) around flat space.

$$\int d^4x \langle T_{\mu\nu}(x)\, T_{\rho\sigma}(0)\rangle e^{i\,p\cdot x} = \frac{4}{3}\bar{c}\, \log p^2/\Lambda^2 \Pi_{\mu\nu,\rho\sigma}(p) \equiv \Gamma^{(2)}_{\mu\nu,\rho\sigma}(p) \tag{1.3}$$

where $\Pi_{\mu\nu,\rho\sigma}$ is the unique tensor structure which is both conserved and traceless and satisfies the symmetry conditions which follow from Bose symmetry of the two-point function. Its explicit form is given in (4.15). Calculating the variation under dilations of (1.3) and comparing with the second variation of (1.1) around flat space gives $c = 2\,\bar{c}$.

In the broken phase, since the broken vacuum is not dilation invariant, the relation between Weyl transformations and dilations breaks down and the previous argument cannot be used. The high momentum behaviour of the correlator of two energy-momentum tensors is still given by the same $\bar{c}$ as in the unbroken phase, but we cannot relate it directly to the normalization of the possible Weyl anomaly.

In order to match type B anomalies we re-examine the above set-up and we arrive at a different way to extract the anomaly normalizations from universal features of the correlators. This new way is more general and can be applied uniformly for all trace anomalies and also gives an alternative and more rigorous way for proving the matching of type A anomalies. The general procedure will be to analyze the Ward identities following from diffeomorphism and Weyl invariance after the correlators are decomposed in invariant amplitudes. We will treat from the beginning the diffeomorphism Ward identities as non-anomalous and in the Weyl Ward identity we will introduce the anomalous terms with the structure prescribed by the cohomological analysis with a free normalization. The combined identities relate the anomaly to relations between dimension -2 amplitudes. Instead of trying to isolate power-like behaviour in one invariant when the other invariants are sent to potentially singular points, we link the anomaly normalization to the special, universal behaviour of certain amplitudes when one invariant is taken to infinity the others being generic. Then the anomaly matching between the two phases follows as a consequence of the equality of the respective amplitudes in the deep Euclidean limit. Equivalently the high invariant behaviour is equivalent to the validity of sum rules normalized to the anomalies. The sum rule is generically the integral over a discontinuity of an amplitude, one invariant being integrated while the other two are kept at generic values. In the broken phase another parameter which is kept fixed is the spontaneous breaking scale v and the sum rule is valid for the whole range from $v = 0$, the unbroken phase, to $v = \infty$, the IR limit of the broken phase.

The steps in this analysis are:

a) In the unbroken phase the logarithmically divergent amplitudes give the normalization of the anomaly through their relation to dilations as outlined above, but by themselves they are not anomalous. In the above example the logarithmically divergent amplitudes in the two- and three-point functions obey non-anomalous relations as evidenced by the counterterm (1.2), which is invariant both under diffeomorphisms and Weyl transformations. Therefore the amplitudes having UV divergences can be eliminated from the anomaly analysis.

b) Using the non-anomalous diffeomorphism Ward identities in the Weyl Ward identities, one obtains identities which involve only dimension -2 amplitudes. These identities relate the behaviour of the amplitudes when a particular kinematical invariant on which it depends goes to infinity to the anomaly normalizations in both phases.

c) In addition one obtains non-anomalous Ward identities which relate the dimension -2 amplitudes to cut-off independent expressions which are derived from the two-point function. In the unbroken phase this determines the high invariant behaviour of the respective amplitudes in terms of the two-point function and when used in b) relate the anomaly to the normalization of the two-point function, replacing the usual argument.

d) The dimension -2 amplitudes have the same deep Euclidean limit in the unbroken and broken phases. Using this fact for the combinations of amplitudes appearing in b), one establishes the equality of the anomalies in the two phases, i.e. "anomaly matching".

e) Once the existence and normalization of the anomalies in the broken phase are known, the constraints on the dilaton couplings follow from the known Weyl transformation of the dilaton.

f) The anomaly equation obeyed by the dimension -2 amplitudes and their known high momentum behaviour implies sum rules for their discontinuities, normalized by the anomaly. In the IR limit of the broken phase the sum rules are dominated by the dilaton contribution and the couplings of the dilaton can be determined.

After these steps we arrive at a characteristic Ward identity summarising the anomaly structure for both type A and B trace anomalies (and also the perturbative chiral anomalies). For the three-point function relevant for anomalies in $d = 4$ one has

$$s_1\, E_1(s_1, s_2, s_3) + s_2\, E_2(s_2, s_3, s_1) + s_3\, E_3(s_3, s_1, s_2) = ct \tag{1.4}$$

where $s_i \equiv p_i^2$ are the kinematical invariants (p_i are the three external momenta), E_i are dimension -2 amplitudes and ct is a constant which characterizes the strength of the anomaly, i.e. a or c. The basic Ward identity (1.4) can be translated into two equivalent, universal characterizations of the anomaly:

$$E_i \xrightarrow{s_i \to \infty} \frac{ct}{s_i} + \mathcal{O}\left(\frac{s_j, s_k}{s_i^2}[\log s_i]^p\right) \tag{1.5}$$

and

$$-\frac{1}{\pi}\int ds_i\, \mathrm{Im}_i E_i(s_i, s_j, s_k) = ct \tag{1.6}$$

where the imaginary part is obtained from the discontinuity with respect to the s_i invariant while the other two invariants s_j, s_k are kept fixed.

From comparing (1.4) in the deep Euclidean limit in the unbroken and broken phases, one reaches the conclusion that (1.4) and therefore (1.5) and (1.6) are valid with the same value of the anomaly ct also in the broken phase. This gives the most general statement about "anomaly matching". In the broken phase the functional dependence is completely different and the various amplitudes depend on the breaking scale v, but the anomaly equations are independent of v. In particular the relations are valid also for $v = \infty$, the IR regime of the broken phase, where they impose constraints on the dilaton couplings. In addition for type B anomalies the normalization of the anomaly obtained form the three-point function as outlined above is related to the two-point correlator in a universal fashion involving again only dimension -2 amplitudes.

In Section 2 we study in detail the above scenario for the simplest type B anomaly in $d = 4$, which involves scalar primary operators of dimension $+2$. We will refer to this as the $\Delta = 2$ model. The relatively simple kinematics allows us to follow in detail the steps outlined above. Whenever the explicit Ward identities realize a step described above we give the general, abstract form of the equations /arguments which are valid for all anomalies. This section contains therefore our general results with the simplest explicit realization.

In Section 3 the special features related to anomalies of higher dimensional primaries are studied, in particular for conformal moduli in $d = 4$. We show how the general arguments can be applied also in these cases by mapping the high dimension amplitudes to combinations of dimension -2 amplitudes.

In Section 4 we study in detail the analytic structure of the type B anomaly in the correlators of just energy-momentum tensors in $d = 4$ and apply the general procedure for the matching of both type A and type B anomalies.

We verify different aspects of the anomaly structure discussed in the main text by a Feynman diagram calculation in a free model in Appendix A. The calculations have general validity for the $\Delta = 2$ model, since different CFT with dimension +2 primaries have the same analytic structure for the relevant two and three-point correlators differing possibly just by their normalizations.

A simple explicit model for the spontaneous breaking of conformal symmetry is discussed in Appendix B. The various general features of the anomaly structure in the broken phase are verified and the role of the dilaton as an effective description of the anomaly difference for massive flows is also exemplified.

2 Detailed Analysis of the $\Delta = 2$ Model

Consider in $d = 4$ a CFT which has a dimension two primary scalar operator $\mathcal{O}$. An explicit realization of such a model is a free massless scalar ϕ for which $\mathcal{O} = \phi^2$. In Appendices A and B we present several explicit checks of our general results for this simple model, but our arguments will be independent of the actual realization.

We couple the operator to a source J which transforms under a Weyl transformation as

$$\delta_\sigma J = -2\,\sigma\,J \tag{2.1}$$

while the metric transforms as

$$\delta_\sigma g_{\mu\nu} = 2\,\sigma\,g_{\mu\nu} \tag{2.2}$$

The cohomological analysis gives a type B anomaly in the Weyl transformation of the generating functional of connected correlation functions of the energy-momentum tensor $T_{\mu\nu}$ and $\mathcal{O}$,

$$\delta_\sigma W = c \int d^4x\, \sigma \sqrt{g}\, J^2 \tag{2.3}$$

while diffeomorphisms are not anomalous.

Even though the theory is conformal, there are logarithmic UV divergences in momentum space correlators of integer dimensional operators, which require counterterms. In particular for correlators of two $\Delta = 2$ operators with any number of energy-momentum tensors the unique counterterm is

$$\bar{c}\,\log\Lambda^2 \int d^4x\, \sqrt{g}\, J^2 \tag{2.4}$$

The standard argument relates the anomaly coefficient c to the normalization $\bar{c}$ by considering an x-independent Weyl transformation which also represents dilations. Then the explicit breaking of dilations due to the presence of the cut-off Λ in the counterterm leads to a nonvanishing Weyl variation, i.e. to an anomaly (2.3), as discussed in the Introduction. This fixes to $c = 2\,\bar{c}$, as will be confirmed below.

In the following we will discuss an alternative argument which avoids amplitudes with UV divergences by using the high momentum behaviour of finite dimension -2 invariant amplitudes. The invariant amplitudes which contain UV divergences can be identified by expanding the metric dependence in the counterterm (2.4) in perturbations h around flat space η

$$g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu} \tag{2.5}$$

In the unbroken phase the two-point function which one obtains by expanding the generating functional to order J^2 is completely determined by the dimension of $\mathcal{O}$ and in momentum space has the expression

$$\Gamma^{(2)}(p^2) \equiv \langle \mathcal{O}(-p)\, \mathcal{O}(p) \rangle = -2\, \bar{c}\, \log p^2/\Lambda^2 \tag{2.6}$$

In the renormalized correlator the cut-off Λ is replaced by a finite scale but we will continue using the cut-off as a scale. Expanding (2.4) a logarithmically divergent term with the same normalization will appear also in the correlator of two operators $\mathcal{O}$ and one energy-momentum tensor. Expanding $\sqrt{g} = 1 - \frac{1}{2}\eta_{\mu\nu}h^{\mu\nu}$ (using (2.5)) one finds that the divergence will be in a structure proportional to $\eta_{\mu\nu}$.

We will now study this correlator by decomposing it into invariant amplitudes in momentum space as

$$\begin{aligned} \Gamma^{(3)}(q, k_1, k_2) &\equiv \langle T_{\mu\nu}(-q)\, \mathcal{O}(k_1)\, \mathcal{O}(k_2) \rangle \\ &= A\, \eta_{\mu\nu} + B\, q_\mu q_\nu + C\, (q_\mu\, r_\nu + q_\nu\, r_\mu) + D\, r_\mu r_\nu \end{aligned} \tag{2.7}$$

where A, B, C, D depend on the three Lorentz invariants q^2, k_1^2, k_2^2, with q_μ and $k_{1\mu}, k_{2\mu}$ the four momenta carried by the energy-momentum tensor and the two scalar operators, respectively. In (2.7) we have defined

$$r_\mu = k_{1\mu} - k_{2\mu} \tag{2.8a}$$

and by momentum conservation one has

$$q_\mu = k_{1\mu} + k_{2\mu} \tag{2.8b}$$

We remark that the amplitudes A, B, D are symmetric and the amplitude C is antisymmetric under the interchange of the momenta k_1, k_2. The amplitude A has dimension 0 while the amplitudes B, C, D have dimension -2 and are therefore finite, i.e. independent of the cut-off.

We now study the Ward identities which relate $\Gamma^{(3)}$ to $\Gamma^{(2)}$. Invariance under infinitesimal diffeomorphisms $x^\mu \to x^\mu - \xi^\mu(x)$ under which $g_{\mu\nu}$ and J transform as

$$\begin{aligned} \delta_\xi g_{\mu\nu} &= \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \\ \delta_\xi J &= \xi^\mu \partial_\mu J \end{aligned} \tag{2.9}$$

applied to the expansions of the generating functional leads to[1]

$$q^\mu \Gamma^{(3)}_{\mu\nu}(q,k_1,k_2) = k_{1\nu}\,\Gamma^{(2)}(k_2^2) + k_{2\nu}\Gamma^{(2)}(k_1^2) \tag{2.10}$$

Weyl invariance, defined as the variations with parameter $\sigma(x)$ given in eqs.(2.1) and (2.2), leads to the relation

$$\eta^{\mu\nu}\Gamma^{(3)}_{\mu\nu} = \Gamma^{(2)}(k_1^2) + \Gamma^{(2)}(k_2^2) + 2\,c \tag{2.11}$$

Relations (2.10) and (2.11), when rewritten in terms of the invariant amplitudes, give

$$\begin{aligned}
&A + q^2 B + q\cdot r\, C = \frac{1}{2}\left[\Gamma^{(2)}(k_1^2) + \Gamma^{(2)}(k_2^2)\right] \\
&q^2\, C + q\cdot r\, D = \frac{1}{2}\left[\Gamma^{(2)}(k_2^2) - \Gamma^{(2)}(k_1^2)\right] \\
&4\,A + q^2\, B + 2\,q\cdot r\, C + r^2\, D = 2\left[\Gamma^{(2)}(k_1^2) + \Gamma^{(2)}(k_2^2)\right] + 2\,c
\end{aligned} \tag{2.12}$$

where we used that the Ward identities which follow from diffeomorphism invariance are not anomalous and in the identity resulting from Weyl transformations we included the contribution of the anomaly obtained from the expansion of (2.3).

From (2.12) we could replace A by

$$\bar{A} \equiv A - \frac{1}{2}\left[\Gamma^{(2)}(k_1^2) + \Gamma^{(2)}(k_2^2)\right] \tag{2.13}$$

and all cut-off dependent terms disappear from the Ward identities. This is a consequence of the structure of the counterterm (2.4), which confirms that these terms obey the Ward identities. Generically there remains a difference between the two logarithms which does not contain the cut-off and therefore the possible anomalies are produced by finite amplitudes. More generally we can simply solve the first equation of (2.12) for A, replace it in the third and obtain

$$q^2\, C + q\cdot r\, D = \frac{1}{2}\left[\Gamma^{(2)}(k_2^2) - \Gamma^{(2)}(k_1^2)\right] \tag{2.14a}$$

$$-3\,q^2\, B - 2\,q{\cdot}r\, C + r^2\, D = 2\,c \tag{2.14b}$$

We stress that all amplitudes present in (2.14) have dimension -2 and the contribution from the two-point function is also finite, keeping the information about its overall normalization. The appearance of the Ward identities with the structure of (2.14) is generic and we will now discuss their properties and role for the matching in the general setting.

Equation (2.14b) is a particular instance of the general type of equations (1.4)

$$s_1\, E_1(s_1,s_2,s_3) + s_2\, E_2(s_2,s_3,s_1) + s_3\, E_3(s_3,s_1,s_2) = 2\,c \tag{2.15}$$

where $s_1 = q^2$, $s_2 = k_1^2$ and $s_3 = k_2^2$ are the three kinematical invariants and E_i are the dimension -2 amplitudes

$$E_1 = -3\,B - D\,, \qquad E_2 = 2(D-C)\,, \qquad E_3 = 2(D+C) \tag{2.16}$$

[1]More details about the derivation of the Ward identities will be given in Section 4.

The amplitudes with dimension -2 obey unsubtracted dispersion relations in any of the invariants when the other two invariants are kept fixed at generic values. We choose for each amplitude the invariant with the same index since this is the dependence constrained by the anomaly equation (2.15), i.e.

$$E_i(s_i, s_j, s_k) = \frac{1}{\pi} \int dx_i \frac{\mathrm{Im}_i E_i(x_i, s_j, s_k)}{x_i - s_i} \tag{2.17}$$

where Im_i indicates $\frac{1}{2i}\times$ the discontinuity in the variable s_i, while the other two invariants are kept fixed. We remark that (2.17) also contains the information about the analytic structure in the variables which are kept fixed, after doing appropriate analytic continuations. In a CFT the support of the integral is between 0 and ∞ for the x_i variable. We choose s_j, s_k to be real negative in order to have a nonsingular discontinuity.

The large s behaviour of a dimension -2 invariant amplitude in a CFT is generically $\frac{1}{s}[\log(s)]^p$ for any of the invariants, where the scale of s in the log is given by the invariants which are kept fixed. If the amplitude however satisfies (2.17), the behaviour is more restricted: taking a discontinuity in s_i of (2.15) we obtain

$$s_i \, \mathrm{Im}_i E_i + s_j \, \mathrm{Im}_i E_j + s_k \, \mathrm{Im}_i E_k = 0 \tag{2.18}$$

which implies

$$\mathrm{Im}_i E_i \xrightarrow[s_i \to \infty]{} \frac{1}{s_i^2} [\log s_i]^p + \ldots \tag{2.19a}$$

$$E_i \xrightarrow[s_i \to \infty]{} \frac{2\,c}{s_i} + \mathcal{O}\left(\frac{s_j, s_k}{s_i^2} [\log s_i]^p \right) \tag{2.19b}$$

Then s_i can be taken outside the dispersion relation, and comparing with (2.19) we obtain the sum rules for each of the invariant amplitudes:

$$-\frac{1}{\pi} \int ds_i \, \mathrm{Im}_i E_i(s_i, s_j, s_k) = 2\,c \tag{2.20}$$

Therefore if an invariant amplitude E_i which appears in an anomaly equation of the form (2.15) obeys any of the equivalent universal relations (2.19) or (2.20), the parameter c gives directly the anomaly coefficient. This special structure (2.15) of the Ward identity for dimension -2 amplitudes, relating it to the anomaly, is generic and common also to the type A trace anomaly and even to chiral anomalies. Once it is obeyed the high invariant behaviour of the amplitudes (2.19) or, equivalently, the sum rules (2.20) follow.

What makes type B anomalies special is the relation of the anomaly coefficient c to the two-point function. For type B anomalies typically there is a diffeomorphism Ward identity with a cut-off independent contribution of the two-point function which fixes the special high invariant contribution of the form (2.19) recovering this way the relation between the anomaly normalization and the two-point function, as we now show explicitly for the $\Delta = 2$ model.

For this model we analyze eq.(2.14) at

$$x \equiv k_1^2 - k_2^2 = 0 \tag{2.21}$$

which is not a singular point. The amplitudes depend on q^2 and on

$$k^2 \equiv k_1^2 = k_2^2 \tag{2.22}$$

Taking a derivative with respect to x at $x = 0$ of (2.14a) and evaluating (2.14b) at $x = 0$, we obtain

$$q^2\, \bar{C} + D = \frac{\bar{c}}{k^2} \tag{2.23a}$$

$$q^2\big(- 3\, B - D\big) + 4\, k^2\, D = 2\, c \tag{2.23b}$$

where we used that the amplitude C is odd in x and we defined a dimension -4 amplitude $\bar{C}$ by

$$\bar{C}(q^2, k^2) \equiv \frac{\partial C}{\partial x}(q^2, k^2, x = 0) \tag{2.24}$$

Now we can use the high k^2 behaviour for D extracted from (2.23a):

$$D \xrightarrow[k^2\to\infty]{} \frac{\bar{c}}{k^2} + \mathcal{O}\left(\frac{q^2}{k^4}[\log k^2]^p\right) \tag{2.25a}$$

and compare it with the relevant equation following from (2.19):

$$D \xrightarrow[k^2\to\infty]{} \frac{c}{2\, k^2} + \mathcal{O}\left(\frac{q^2}{k^4}[\log k^2]^p\right) \tag{2.25b}$$

leading to the equality $c = 2\,\bar{c}$. This argument, which relates the normalizations of the type B anomaly and of the two-point function in the unbroken phase is general for all the type B anomalies: besides the equation (2.14b) there is always an equation generalizing (2.14a) which relates the high invariant behaviour of the amplitude to the normalization of the two-point function. Comparing the two we get the desired relation between the anomaly and the two-point function normalizations without using UV divergent amplitudes.

Another special feature of type B anomalies is the appearance of effective IR poles, reflecting the role of the two-point correlator in the Ward identity. We will demonstrate this in the concrete setting for the $\Delta = 2$ model. We proved in the unbroken phase the relation between the anomaly normalization and the two-point function using the special kinematic configurations $k_1^2 = k_2^2 \equiv k^2$ and q^2. If we assume in addition the validity of dispersion relations in the "diagonal variable" k^2, by the argument following (2.15), we obtain the sum rule

$$-\frac{1}{\pi}\int dk^2\, \mathrm{Im}_{k^2} D(k^2, q^2) = \frac{c}{2} \tag{2.26}$$

If this sum rule is valid also for $q^2 = 0$ then, since there is no scale left for $\mathrm{Im}_{k^2} D$, we conclude

$$\mathrm{Im}_{k^2} D(k^2, 0) = -\frac{\pi\, c}{2}\,\delta(k^2) \qquad \text{and} \qquad D(k^2, 0) = \frac{c}{2\, k^2} \tag{2.27}$$

This functional dependence shows the presence of an effective zero-mass pole. The mechanism for its appearance is simple: for $q^2 = 0$, from eq.(2.14a) we obtain

$$\mathrm{Im}_{k^2} D(k^2, 0) = \lim_{k_1^2 \to k_2^2} \frac{\mathrm{Im}_{k_2^2}\Gamma^{(2)}(k_2^2) - \mathrm{Im}_{k_1^2}\Gamma^{(2)}(k_1^2)}{2(k_1^2 - k_2^2)} \tag{2.28}$$

Since[2] $\mathrm{Im}_{k^2} \log k^2 = -\pi\, \theta(k^2)$, the result is as in Figure 1, which is a regularized δ-function

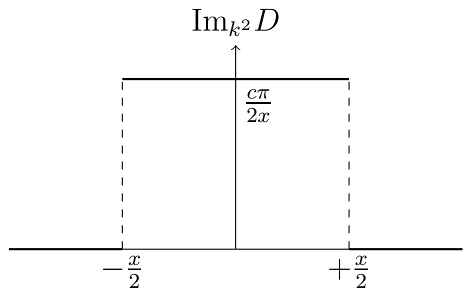

Figure 1

and therefore, using also (2.6) the limit is

$$\mathrm{Im}_{k^2} D(k^2, 0) \to -\bar{c}\, \pi\, \delta(k^2) \tag{2.29}$$

and comparing with (2.27) gives again $c = 2\,\bar{c}$. We remark that the appearance of the δ-function is the result of a "collision" between the branch points in k_1^2 and k_2^2. This effective pole is specific to type B anomalies and is different from the generic presence of poles following from the anomaly sum rules which represent the collapse of ordinary branch cuts in certain limits. In particular it is not matched in the broken phase where the analytic structure of the two-point correlator is completely different.

Once the high invariant behaviour and sum rules for dimension -2 amplitudes (2.19) and (2.20) are valid for type B, we could discuss in general the matching for all trace anomalies. We start our discussion of the anomaly matching with a summary of the structure of the spontaneously broken phase. Let us assume that there is another Poincaré invariant vacuum on which a nonzero dimensional scalar primary operator gets a vacuum expectation value. In such a situation the conformal symmetry is spontaneously broken and a mass scale v, introduced through the vacuum expectation value, the order parameter of the broken phase, is introduce in the theory. There are several general characteristics of the broken phase which we will use:

a) Following from Goldstone's theorem a zero mass scalar, the dilaton exists. The dilaton $\mathcal{D}$ has a linear coupling to the energy-momentum tensor with a dimensional strength f related to v

$$\langle 0|T_{\mu\nu}|\Sigma(q)\rangle = f\, q_\mu q_\nu \tag{2.30}$$

[2] We use the definition of the logarithm as a real analytic function on the first Riemann sheet with the branch cut on the positive real axis.

Formally one has Weyl invariance also in the presence of the coupling (2.30) if we attribute to the dilaton the Weyl transformation

$$\Sigma \to \Sigma + \sigma \tag{2.31}$$

where we used the normalized dimensionless dilaton field $\Sigma \equiv \mathcal{D}/f$. In the broken phase Weyl invariance is limited to transformations where $\sigma(x)$ falls off to zero at large x. As a consequence e.g. the relation between dilations and Weyl invariance with x-independent σ is not anymore valid.

b) Since the operatorial relations of the CFT are not changed and the derivation of the diffeomorphism and Weyl Ward identities used only the Poincaré invariance of the vacuum, all the Ward identities we used in the unbroken phase remain valid. Also the analyticity properties of the invariant amplitudes remain valid. Anomalies can appear only as real parts in Ward identities corresponding to the same type of anomaly functionals as in the unbroken phase. Therefore the basic Ward identity for the dimension -2 amplitudes E_i^B in the broken phase, characterized by the mass scale v, is valid but with an a priori different normalization of the anomaly c^B:

$$s_1\, E_1^B(s_1, s_2, s_3, v^2) + s_2\, E_2^B(s_1, s_2, s_3, v^2) + s_3\, E_3^B(s_1, s_2, s_3, v^2) = 2\, c^B \tag{2.32}$$

The anomalies match if $c^B = c$. From its high momentum analysis we conclude, as in the unbroken phase,

$$E_i^B \xrightarrow[s_i \to \infty]{} \frac{2\, c^B}{s_i} + \mathcal{O}\left(\frac{s_j, s_k, v^2}{s_i^2}[\log s_i]^p\right) \tag{2.33}$$

and

$$-\frac{1}{\pi}\int ds_i\, \mathrm{Im}_i E_i^B(s_i, s_j, s_k, v) = 2\, c^B \tag{2.34}$$

where we made explicit the possible dependence of the discontinuities on the breaking scale v.

c) For a given correlator the deep Euclidean limit of the amplitude in the broken phase coincides with the limit in the unbroken phase. Denoting invariant amplitudes in the two phases as $\mathcal{A}^B$ and $\mathcal{A}$, respectively we have

$$\lim_{\lambda\to\infty} \frac{\mathcal{A}(\lambda\, q_1^2, \lambda\, q_2^2, \dots)}{\mathcal{A}_B(\lambda\, q_1^2, \lambda\, q_2^2, \dots)} = 1 \tag{2.35}$$

the deviations being of order $1/\lambda$ or v^2/q_i^2 when the invariants are taken to ∞. This is simply a consequence of the fact that the OPE of the operators are not changed in the broken phase and therefore the UV structure of the correlators remains the same even though for finite values of the invariants the structure of the amplitudes changes in the broken phase, the spectrum of the theory being generically massive, etc.

Anomaly matching is now an immediate consequence. Consider the combination of dimension -2 invariant amplitudes which appear in the "anomaly equations" (2.15) and

(2.32), respectively, for a configuration in the deep Euclidean limit for generic (i.e. avoiding special points like $s_j = 0$) configurations. We have

$$\lim_{\lambda\to\infty} \frac{\lambda s_1\, E_1(\lambda s_1, \lambda s_2, \lambda s_3) + \lambda s_2\, E_2(\lambda s_1, \lambda s_2, \lambda s_3) + \lambda s_3\, E_3(\lambda s_1, \lambda s_2, \lambda s_3)}{\lambda s_1\, E_1^B(\lambda s_1, \lambda s_2, \lambda s_3) + \lambda s_2\, E_2^B(\lambda s_1, \lambda s_2, \lambda s_3) + \lambda s_3\, E_3^B(\lambda s_1, \lambda s_2, \lambda s_3)} = \frac{c}{c^B} = 1 \tag{2.36}$$

where we used that c, c^B do not depend on the invariants or on the breaking scale v.

Once the equality of the anomalies is established, it follows that asymptotic values of E_i for taking s_i to ∞ and the sum rules (2.20) and (2.34) also match. The sum rules (2.34), now normalized to, c are valid for the whole range of v including $v = 0$, the unbroken phase.

We can relate the sum rules in special limits to particular contributions in the two phases. Consider the sum rule for the discontinuity in q^2 of the amplitude $-3B - D$, which we denoted by E_1 in the unbroken phase:

$$-\frac{1}{\pi} \int dq^2\, \mathrm{Im}_{q^2} E_1(q^2, k_1^2, k_2^2) = 2\, c \tag{2.37}$$

At $k_1^2 = 0$ or $k_2^2 = 0$ the amplitude is singular since one has a branch point. One can, however, approach the configuration $k_1^2 = k_2^2 \equiv k^2 = 0$ as a limit in k^2 approaching 0. Since (2.37) holds also in the limit and the integrand has dimension -2 this implies

$$\mathrm{Im}_{q^2} E_1(q^2, 0, 0) = -2\, c\, \pi\, \delta(q^2) \tag{2.38}$$

This pole-like discontinuity, which is reached in a very special way due to the singularity of the limit, gives a universal characterization of the anomaly. As shown above the characterizations through the high invariant limit of the amplitude (2.19) or equivalently the sum rules (2.20) are much more general and mathematically unambiguous.

In the broken phase generically the sum rule is saturated with massive states. If however we go to the deep IR limit, i.e. $v \to \infty$ when all the masses are sent to ∞ then also in the broken phase the saturation will be due only to massless states and (2.38) must be valid. Generically in the broken phase there is a sector which preserves conformal invariance which therefore could contribute to c. Outside this sector the only generic massless state is the dilaton whose coupling in the q^2-channel can produce the $\delta(q^2)$ dependence normalized to the coupling of the dilaton to the rest of the diagram. Therefore the anomaly matching will constrain the dilaton couplings requiring their proportionality to the difference between the anomaly in the unbroken phase and the anomaly of the conformal sector in the broken phase.

In the deep IR limit of the broken phase the dilaton reproduces completely the anomaly. We recall the implementation of this general relation. Assume that in the presence of the external sources, the metric $g_{\mu\nu}$ and the sources J coupled to the additional primaries, one has an anomaly:

$$\delta_\sigma W(g_{\mu\nu}, J) = \int d^4x\, \sigma \mathcal{A}(g_{\mu\nu}, J) \tag{2.39}$$

where W is the generating functional and $\mathcal{A}$ is the local anomaly functional containing the normalization mentioned above. Then the dilaton effective action $S(g_{\mu\nu}, J, \Sigma)$, whose variation reproduces the anomaly (2.39), is

$$S(g_{\mu\nu}, J, \Sigma) = -\int_0^1 dt \int d^4x\, \Sigma\, \mathcal{A}(g_{\mu\nu}^{-t\,\Sigma}, J^{-t\,\Sigma}) + \Psi(g_{\mu\nu}^{-\Sigma}, J^{-\Sigma}) \tag{2.40}$$

where $g_{\mu\nu}^{-t\,\Sigma}, J^{-t\,\Sigma}$ are the sources transformed by a Weyl parameter $\sigma = -t\,\Sigma$ and Ψ an arbitrary diffeomorphism invariant functional, contributing a Weyl invariant term. For the $\Delta = 2$ model the dilaton effective action is

$$S(g_{\mu\nu}, J, \Sigma) = -c \int d^4x \sqrt{g}\, \Sigma\, J^2 + \Psi\big(g_{\mu\nu} \exp(-2\Sigma), J \exp(2\,\Sigma)\big) \tag{2.41}$$

The second term is invariant under diffeomorphism and Weyl transformations. The first term represents the "dilaton coupling" to the two operators $\mathcal{O}\,\mathcal{O}$. In principle using its analytic and covariance properties it can be separated from the general off-shell $\langle \Sigma\, \mathcal{O}\, \mathcal{O} \rangle$ correlator. In the IR limit of the broken phase when the effective action is expanded in powers of all the momenta it is singled out by being the only "ultralocal" term which contains J^2 and survives when all momenta are zero. More generally, in the deep IR the dilaton effective action is given by a polynomial expansion in momenta and the dilaton coupling will always be defined such that it corresponds to the lowest independent terms in the momentum expansion around zero.

3 Analysis of the Moduli Problem

Consider, again in $d = 4$, a dimension four scalar primary which has the special property that it does not have a β-function, i.e. in particular its structure constant vanishes. Such a primary, called "modulus" in the following, will have nevertheless a Type B anomaly induced by its two-point function.[3] In the first part of this section we will study the anomaly structure of a CFT with one modulus. This will be generalized in the second part to the case of several moduli.

3.1 The Anomaly Structure

The high dimension of the modulus compared with the $\Delta = 2$ model of the previous section, produces new features which we will analyze. Coupling the operator to a source J, which is Weyl invariant, the anomaly is

$$\delta_\sigma W = c \int d^4x\, \sigma\, \sqrt{g}\, J\, \Delta_4 J \tag{3.1}$$

[3] Various aspects of this type of anomalies were studied in refs. [5–8].

where Δ_4 is the Fradkin-Tseytlin-Paneitz-Riegert (FTPR) operator [9–11] with the special property that it transforms homogeneously under Weyl transformations with weight -4, i.e. $\Delta_4 \to e^{-4\sigma}\Delta_4$. Its explicit form will be given in (3.17).

The anomaly reflects the logarithmic divergence in correlators of two moduli with any number of energy-momentum tensors. The corresponding diffeomorphism and Weyl invariant counterterm is

$$\bar{c}\,\log\Lambda^2 \int d^4x\,\sqrt{g}\,J\,\Delta_4 J \tag{3.2}$$

The two-point function is

$$\langle\mathcal{O}(p)\,\mathcal{O}(-p)\rangle \equiv \Gamma^{(2}(p) = -2\,\bar{c}\,(p^2)^2\,\log p^2/\Lambda^2 \tag{3.3}$$

As for the previous case, we want to understand the cut-off independent characterization of the anomaly as it appears in the correlator of two moduli and one energy-momentum tensor. Since the modulus operator $\mathcal{O}$ is a Lorentz scalar, the decomposition in invariant amplitudes is formally identical to the one in the previous section, (2.7), but now the invariant amplitudes have positive dimensions: $+4$ for the A amplitude and $+2$ for the B, C, D amplitudes. The derivation of the Ward identities is similar; the diffeo identities are identical to the first two equations of (2.12), while the Weyl equation, the third equation in (2.12), does not have a right hand side besides the anomaly since the source J is invariant under a Weyl transformation.

The amplitudes have UV divergences. Since we want to preserve conformal invariance, the normalization conditions corresponding to power divergences are put to zero. Therefore the A and B, C, D amplitudes obey triply, respectively doubly subtracted dispersion relations. In order to deal with finite amplitudes we use the fact that the logarithmically divergent contributions obey the Ward identities, since the counterterm is invariant under both diffeomorphisms and Weyl transformations. We can then shift the amplitudes by terms containing logarithms and these shifts will remove the two-point contributions which contain the cut-off since the counterterm fixes completely the form of the logarithmic divergence. We list the shifts which follow from the structure of the counterterm (3.2):

$$\begin{aligned}
A &\to A + \frac{\bar{c}}{6}\big(q^4 - q^2(k_1^2+k_2^2) - 6\,k_1^2\,k_2^2\big)\,\big(\log k_1^2/\Lambda^2 + \log k_2^2/\Lambda^2\big)\\
B &\to B - \frac{\bar{c}}{6}\big(q^2 - k_1^2 - k_2^2\big)\,\big(\log k_1^2/\Lambda^2 + \log k_2^2/\Lambda^2\big)\\
C &\to C - \frac{\bar{c}}{2}\big(k_1^2 - k_2^2\big)\,\big(\log k_1^2/\Lambda^2 + \log k_2^2/\Lambda^2\big)\\
D &\to D + \frac{\bar{c}}{2}\big(q^2 + k_1^2 + k_2^2\big)\,\big(\log k_1^2/\Lambda^2 + \log k_2^2/\Lambda^2\big)
\end{aligned} \tag{3.4}$$

After these shifts the Ward Identities for the finite shifted amplitudes have the form

$$A + q^2\,B + q\cdot r\,C = \frac{\bar{c}}{2}\big(k_1^4 - k_2^4\big)\big(\log k_2^2 - \log k_1^2\big) \tag{3.5a}$$

$$q^2\,C + q\cdot r\,D = \frac{\bar{c}}{2}\big(k_1^4 + k_2^4\big)\big(\log k_1^2 - \log k_2^2\big) \tag{3.5b}$$

$$4\,A + q^2\,B + 2\,q\cdot r\,C + r^2\,D = c\left((k_1^2)^2 + (k_2^2)^2\right) \tag{3.5c}$$

where we assumed that the Weyl Ward identity can be anomalous and we used the form of the anomaly (3.1) expanded around the flat metric $\eta_{\mu\nu}$.

Using (3.5a) we re-express (3.5c) in terms of the dimension 2 amplitudes

$$-q^2\big(3\,B{+}D\big){-}2\,q{\cdot}r\;C{+}2\big(k_1^2{+}k_2^2\big)D{+}2\,\bar{c}\big(k_1^4{-}k_2^4\big)\big(\log k_1^2{-}\log k_2^2\big) = \frac{c}{4}\big((k_1^2)^2{+}(k_2^2)^2\big) \tag{3.6}$$

We can absorb the additional k dependent term in the l.h.s. in a redefinition of C:

$$C \equiv \bar{C} - \bar{c}\,\big(k_1^2 + k_2^2\big)\big(\log k_1^2 - \log k_2^2\big) \tag{3.7}$$

Now (3.7) has the form

$$s_1\,E_1(s_1,s_2,s_3) + s_2\,E_2(s_1,s_2,s_3) + s_3\,E_3(s_1,s_2,s_3) = Q(s_1,s_2,s_3) \tag{3.8}$$

which generalizes (2.15). We pause again to discuss in general the properties of this positive dimensional anomaly structure. In (3.7) the amplitudes E_i have dimension N, while Q, which contains the normalization of the anomaly, is a homogenous polynomial in s_1, s_2, s_3 of dimension $N+2$. To keep the discussion general, we take N to be a positive even integer or 0. The amplitudes E_i have the special feature that they do not contain the cut-off scale, i.e. they are UV convergent in spite of their non-negative dimension. This means that they have convergent dispersion relations in any of the s variables. This is possible only if the amplitude as an analytic function has all its singularities in a dimension -2 function and the overall dimension is made up by integer powers of the s variables. Therefore the amplitudes E_i have the special form

$$E_i(s_1,s_2,s_3) = \Sigma_k P_k(s_1,s_2,s_3)\tilde{E}_i^{(k)}(s_1,s_2,s_3) \tag{3.9}$$

where P_k are monomials formed from the s-variables of total dimension $N+2$ and $E_i^{(k)}$ are dimension -2 analytic functions. The summation is over all monomials which are compatible with the total dimension N. The dispersion relations for E_i will be convergent: the discontinuity is coming from $\tilde{E}_i^{(k)}$ multiplied by the monomial and in the dispersion relation itself the monomial is simply taken outside the integral if it does not involve the integration variable s_i; if s_i is part of the monomial it provides "free" subtractions at $s_i = 0$, getting also outside the integral.

Using (3.9) we can repeat our discussion following (2.15) to determine the special features of the dimension -2 amplitudes $\tilde{E}_i^{(k)}(s_1,s_2,s_3)$ related to the presence of the anomaly polynomial Q in the r.h.s. Taking an asymptotic expansion in each of the variables s_i and equating the corresponding terms on the two sides of the equation one obtains the equivalent relations

$$\tilde{E}_i^{(k)} \xrightarrow[s_i\to\infty]{} \frac{f_i^k(c)}{s_i} + \mathcal{O}\left(\frac{s_j, s_k}{s_i^2}[\log s_i]^p\right) \tag{3.10}$$

and

$$-\frac{1}{\pi}\int dx_i \mathrm{Im}_i \tilde{E}_i^{(k)}(x_i, s_j, s_k) = f_i^k(c) \tag{3.11}$$

where $f_i^k(c)$ are pure numbers depending on the anomaly polynomial Q. In particular some of these coefficients could be zero if the appropriate term does not appear in Q. The leading terms in the asymptotic expansion give relations as specified by (3.11) while non-leading ones have generically sums of terms. Saturating (3.11) with δ-function type discontinuities at special configurations when only one invariant is left is generically problematic also in this case. For type B anomalies at least one of the expressions appearing in (3.10) is given by a diffeomorphism Ward identity involving the two-point function and then comparing it with (3.10) we get the desired relation between the anomaly in the unbroken phase and the normalization of the two-point function.

We return now to the moduli anomaly. To simplify again the argument we choose the nonsingular kinematical configuration $k_1^2 = k_2^2 \equiv k^2$. For the D amplitude the relevant expansion is

$$D(q^2, k^2) = (k^2)^2 \tilde{D}(q^2, k^2) + ... \tag{3.12}$$

and then clearly the behaviour of $\tilde{D}$ is similar to the behaviour of D for the $\Delta = 2$ model of the previous section. From the asymptotic expansion of (3.5b) in k^2 we obtain

$$\tilde{D} \to \frac{\bar{c}}{k^2} \tag{3.13}$$

and using the expansion in (3.6) we find $c = 2\,\bar{c}$ from the matching with the $(k^2)^2$ in the anomaly polynomial.

Clearly after the expansion of the positive dimension amplitudes in terms of dimension -2 amplitudes the moduli problem (and a similar one for the anomalies of dimension 3 scalar operators) are mapped to the $\Delta = 2$ case. The anomaly matching follows from an argument similar to the one used in Section 2 for the $\Delta = 2$ model: one considers again the ratio of (3.9) in the unbroken and broken phases in the deep Euclidean limit. Equating the ratio of the anomaly polynomials in the same limit the equality of the normalizations follows. In particular for a single modulus the dilaton effective action is:

$$S(g_{\mu\nu}, J, \Sigma) = -c\int d^4x\, \Sigma\, \sqrt{g}\, J\, \Delta_4 J + \Psi(g_{\mu\nu} \exp -2\Sigma, J) \tag{3.14}$$

where Ψ is an arbitrary diffeomorphism invariant functional. The dilaton coupling is defined by the normalization of the unique local term with four derivatives in the expansion of the effective action.

We comment that a similar analysis can be performed for the type B anomaly generated by a dimension $+3$ scalar primary coupled to a dimension -1 source J where

$$\delta_\sigma W = c\int d^4x\, \sigma\, \sqrt{g}\, J\, \Delta_2 J \qquad\qquad \Delta_2 = \Box - \tfrac{1}{6}R \tag{3.15}$$

3.2 The role of source reparametrizations

We consider now moduli in $d = 4$, i.e. dimension 4 scalar primaries $\mathcal{O}_i, i = 1, ..., N$, which have the special property that the structure constants of any three moduli (including the same modulus) vanish. This prevents the appearance of higher than linear terms of $\log \Lambda^2$ in the three-point correlator and therefore a vanishing of the β function in lowest order. We assume that the higher order constraints leading to the exact vanishing of the β function are also fulfilled. Then the moduli can be added to the action with finite coefficients and conformality is preserved. We diagonalize the two-point correlators of the moduli which is always possible in a unitary CFT and we normalize the operators such that after diagonalization the two-point function is proportional to the unit matrix.

We couple the moduli to sources J^i via[4]

$$\int d^4x \sqrt{\gamma} \sum_{i=1}^{N} J^i \mathcal{O}_i \tag{3.16}$$

The sources are inert under Weyl transformations and as a consequence one can redefine them through local functions without interfering with their Weyl transformation properties. We will discuss the meaning and role of these transformations for type B anomalies. Since different type B anomalies can mix under the transformations we need a complete list of these anomalies.

In order to simplify the analysis, we will use the one-to-one relation between type B anomalies and logarithmic counterterms which is valid in the unbroken phase and we will classify the logarithmic counterterms. Since counterterms preserve the diffeomorphism and Weyl symmetries, they are constructed from local scalar integrands which transform homogeneously with weight -4 under Weyl transformations. We list these local expressions for a single source J when J is acted upon by derivative operators.

$$I_1(J) \equiv \Delta_4 J \equiv \Big(\Box^2 + \frac{1}{3}(\nabla^\mu R)\,\nabla_\mu + 2\,R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}R\,\Box\Big)J \tag{3.17}$$

where Δ_4 is the FTPR operator [9–11].

$$I_2(J) \equiv \Box J\,\Box J + 2\,\nabla^\mu\nabla^\nu J\,\nabla_\mu\nabla_\nu J + 4\,\nabla^\mu J\,\nabla_\mu\Box J + 4\left(R^{\mu\nu} - \frac{1}{6}g^{\mu\nu}R\right)\nabla_\mu J\,\nabla_\nu J \tag{3.18}$$

$$I_3(J) \equiv \Box J\nabla^\mu J\nabla_\mu J + 2\nabla^\mu\nabla^\nu J\nabla_\mu J\nabla_\nu J \tag{3.19}$$

$$I_4(J) \equiv \nabla^\mu J\nabla_\mu J\nabla^\nu J\nabla_\nu J \tag{3.20}$$

The counterterms contain the integrated expressions

$$C_k \equiv \int d^4x\sqrt{\gamma}\,I_k(J) \tag{3.21}$$

[4]In this section we use γ for the space-time metric and reserve g for the Zamolodchikov metric.

Since I_1, I_3 are total derivatives the corresponding C_1, C_3 vanish. Of course we expect these vanishings since in the unbroken phase there is no expectation value for $\mathcal{O}$ and the correlator of three $\mathcal{O}$ also vanishes. Therefore we will get just the counterterms C_2, C_4. We can get form the densities I_1, I_3 non-vanishing counterterms if we multiply them with additional powers of J. These counterterms are, however, not independent. For example

$$\int d^4x \sqrt{\gamma}\, J\, I_1(J) = -C_2 \tag{3.22}$$

and

$$\int d^4x \sqrt{\gamma}\, J\, I_3(J) = -C_4 \tag{3.23}$$

The counterterms produce type B anomalies by using the same integrands in the variation of the effective action[5]

$$\delta_\sigma W = c_k \int d^4x\, \sigma(x) \sqrt{\gamma}\, I_k(J) \qquad\qquad k = 2, 4 \tag{3.24}$$

where c_k gives the normalization of the anomaly.

The equivalence of counterterms of the type discussed above, which was based on integration by parts, can produce additional terms when derivatives act on σ. These expressions, being Weyl invariant and vanishing for x-independent σ, represent therefore, if cohomologically nontrivial, possible type A anomalies. It is an interesting question if type A anomalies for moduli can be realized physically, but we limit our discussion just to type B and therefore we ignore the possible type A anomalies which may appear in the equivalence relations.

When we have more than one source, any combination of sources in the expressions above would produce a priori independent anomalies. The counterterms /anomalies (3.18),(3.20) represent just the simplest terms in infinite families of moduli anomalies. Consider, as an example starting from (3.18), the family of anomalies

$$\begin{aligned} c_{\{k\}ij} \int d^4x\, \sigma \sqrt{\gamma}\, J^{k_1} \dots J^{k_n} \Big(&\Box J^i \,\Box J^j + 2\, \nabla^\mu \nabla^\nu J^i\, \nabla_\mu \nabla_\nu J^j \\ &+ 4\, \nabla^\mu J^i\, \nabla_\mu \Box J^j + 4\big(R^{\mu\nu} - \tfrac{1}{6} g^{\mu\nu} R\big) \nabla_\mu J^i\, \nabla_\nu J^j \Big) \end{aligned} \tag{3.25}$$

with a priori independent universal, i.e. scheme independent normalizations $c_{\{k\}ij}$. Obviously these expressions are still Weyl invariant and they represent new possible anomalies.

They correspond to single logarithmic divergences in the correlators of $k+2$ moduli with theory dependent normalizations. In expressions (3.25) summation over all $(k+2)!$ permutations of the $k+2$ indices of $c_{\{k\}ij}$ are understood, irrespective of whether they are all different or not. This ensures Bose symmetry of the moduli correlators which are derived from them. This also applies to all the following expressions.

[5] This relation to type B anomalies which, by definition, have a Weyl invariant anomaly density, is the reason why above we listed only Weyl invariant densities.

The reduction of counterterms built from the I_1, I_3 structures to the counterterms of type C_2, C_4 is valid in the general situation when we have an arbitrary number of sources and the structures can be multiplied by arbitrary products of sources. This is a consequence of the complete symmetrization on the sources which is valid for all our expressions. Indeed let us consider a typical identity for a single source used in the integration by parts when proving the equivalence of the counterterms:

$$\Box J^p = pJ^{p-1}\Box J + p(p-1)J^{p-2}\,\nabla_\mu J\nabla^\mu J \tag{3.26}$$

The same relation is valid if we have complete symmetrization of p sources, i.e.

$$\Box(J_{i_1}...J_{i_p})_S = p(\Box J_{i_1}...J_{i_p})_S + p(p-1)(\nabla_\mu J_{i_1}\nabla^\mu J_{i_2}...J_{i_p})_S \tag{3.27}$$

where we put an index S to remind us that the expressions are completely symmetrized. It follows that we can repeat the integration by parts manipulations for many sources by simply doing the calculation for a single source and in the final result replace the appropriate expressions for many sources, completely symmetrized. We obtain that generically an expression based on I_1 gives us linear combinations of expressions based on I_2 and I_4 while an expression based on I_3 gives just expressions of the I_4 type.

A convenient way to treat all the independent anomalies corresponding to I_2 and I_4 is to sum over the sources and then the two families of anomalies will be characterized by two families of functions $h_{ij}(J)$ and $t_{ijkl}(J)$, respectively:

$$\begin{aligned}\int d^4x\,\sigma\,\sqrt{\gamma}\,h_{ij}(J)\Big(&\Box J^i\,\Box J^j + 2\,\nabla^\mu\nabla^\nu J^i\,\nabla_\mu\nabla_\nu J^j \\ &+ 4\,\nabla^\mu J^i\,\nabla_\mu\Box J^j + 4\big(R^{\mu\nu} - \tfrac{1}{6}g^{\mu\nu}R\big)\nabla_\mu J^i\,\nabla_\nu J^j\Big)\end{aligned} \tag{3.28}$$

$$\int d^4x\,\sigma\,\sqrt{\gamma}\,t_{ijkl}(J)\nabla^\mu J^i\nabla_\mu J^j\nabla^\nu J^k\nabla_\nu J^l \tag{3.29}$$

The coefficients $c_{\{k\}ij}$ are recovered from $h_{ij}(J)$ and $t_{ijkl}(J)$ by a Taylor expansion around $J_i = 0$ for $i = 1, ..., N$.

These two sets of functions contain all the information about the Type B anomalies of moduli in $d = 4$. All the expressions are completely symmetrized in the sources. A partial constraint following from this is that h_{ij} and t_{ijkl} are symmetric under interchange of i with j and of k with l.

Consider now possible reparametrizations of the sources

$$J^k = f^k(J') \tag{3.30}$$

where f^k are Taylor expandable around $J'^i = 0$, invertible as a power series and start with a normalized linear term, i.e.

$$f^k(J') = J'^k + \mathcal{O}((J')^2) \tag{3.31}$$

Obviously the sources J'^i are also inert under Weyl transformations. The change of variables induces reparametrizations in the generating functional, which now will depend on J', and therefore a reparametrization of the coefficients of the terms which contain single logarithms which are at the origin of the anomalies. To follow the same convention after the change of variables we do a complete summation over permutations of the J' variables. Since the possible anomalies with J'^i as sources are characterized by the same basis of space-time integrands, now written in terms of J'^i, the new coefficient functions can be read off the explicit form of the anomaly. As the simplest example we consider the anomaly (3.29) which was first discussed in [12].

Doing the reparametrization of the anomaly the transformation of the coefficient function is

$$t'_{ijkl}(J') = t_{mnpr}(J(J'))\frac{\partial J^m}{\partial J'^i}\frac{\partial J^n}{\partial J'^j}\frac{\partial J^p}{\partial J'^k}\frac{\partial J^r}{\partial J'^l} \tag{3.32}$$

A similar procedure will give the transformation rules for $h'_{ij}(J')$, which will involve a linear combination of $h_{ij}(J)$ and $t_{ijkl}(J)$. Clearly the functional dependence of the anomaly functions is not fully physical since changing the sources through reparametrization does not change the physical meaning while changing the functional form. We stress that there is no a priori constraint on the transformation properties of the anomaly functions: they follow from the explicit form of the anomalies we choose as a basis by simply doing the change of variables on the anomaly formulae.

We need to understand the relation between the reparametrizations and the correlators of the moduli. The original sources J^i were coupled to the moduli operators $\mathcal{O}_j$ through the coupling (3.16). The correlators of the moduli were defined by the Taylor expansion of the generating functional, i.e. the coefficient of $J^{k_1}(x_1)...J^{k_n}(x_n)$ in its expansion gives the correlator

$$\langle \mathcal{O}_{k_1}(x_1)\,\mathcal{O}_{k_2}(x_2)\cdots\mathcal{O}_{k_n}(x_n)\rangle \tag{3.33}$$

After the change of variables, expanding now in powers of J', their coefficients will be given by linear combinations of the general form

$$\begin{aligned}\langle \mathcal{O}_{k_1}(x_1)\,\mathcal{O}_{k_2}(x_2)\cdots\mathcal{O}_{k_n}(x_n)\rangle &+ \sum_{i\neq j} a^r_{ij}\delta(x_i-x_j)\langle \mathcal{O}_{k_1}(x_1)\cdots\mathcal{O}_{k_r}(x_j)\cdots\mathcal{O}_{k_n}(x_n)\rangle\\ &+\sum_{i\neq j\neq k\neq i} b^r_{ijk}\delta(x_i-x_j)\,\delta(x_i-x_k)\langle \mathcal{O}_{k_1}(x_1)\cdots\mathcal{O}_{k_r}(x_k)\cdots\mathcal{O}_{k_n}(x_n)\rangle+\ldots\end{aligned} \tag{3.34}$$

The second term of the first line is a sum over $(n-1)$-point functions where the two operators $\mathcal{O}_{k_i}(x_i)$ and $\mathcal{O}_{k_j}(x_j)$ have been replaced by $\delta(x_i-x_j)a^r_{ij}\mathcal{O}_{k_r}(x_j)$, the third line is a sum of $(n-2)$-point functions where three operators have been replaced by one, etc. The coefficients a^r_{ij}, b^r_{ijk} etc. are determined by the (derivatives of the) functions f^r in the change of basis (3.30). Therefore, compared with the original correlators, the new ones have "semilocal" contributions when expressed in terms of the old correlators, i.e. contributions of lower order correlators multiplied by δ-functions. The only correlators which are not changed by the reparametrization invariance are the two-point functions since the one-point functions, which could have given a semilocal contribution, are zero in the unbroken phase. So depending on the sources, semilocal contributions could be present. We do not, however, have an intrinsic definition for such contributions and we can identify only their relative appearance between two sets of correlators which correspond to two choices of sources, which are related by a reparametrization. We therefore conclude that all choices of sources which are related by reparametrizations are equivalent and contain the same universal information. Therefore, all dependence on the sources, including the functional form of the anomaly functions, should be taken modulo reparametrizations in order to obtain the regularization independent universal information about the respective CFT.

Once this is understood we could obtain additional information about the anomaly functions. Let us consider, as an example, the three J contributions in the anomalies. Since there is no intrinsic anomaly starting with three J, terms of this type can appear only in the h_{ij} anomaly. In light of the previous discussion this means that this must be true modulo reparametrizations, i.e. there should exist a choice of sources for which the three J terms vanish. But in an arbitrary parametrization one could have a three J term, reflecting a semilocal three-point function:

$$\delta(x-y)\langle\mathcal{O}(y)\,\mathcal{O}(z)\rangle \tag{3.35}$$

It follows that three J contributions in the h_{ij} anomaly are possible since by choosing the quadratic terms in the $f^k(J)$ functions we could put them to zero.

We do not have similar constraints for (3.29). Even though the structure constants vanish for three moduli, one can still have an unremovable logarithm in correlators with four and more moduli: in a block decomposition one has couplings between two moduli and other primaries which are not moduli and a logarithm may be produced.

Finally we want to use the previous discussion to produce a basis for the two remaining independent anomalies which have simple transformation rules for the anomaly functions characterizing them. We stress that this is not a logical necessity which imposes constraints on the theory but just a convenient choice. The functions t_{ijkl} for the (3.29) anomaly already have simple transformation rules (3.32), so we will concentrate on (3.28).

The new form should still be a functional only of h_{ij} which contains all the universal information. Therefore the new terms we add could depend only on h_{ij}. They should be Weyl invariant in order that the new form continues to be a type B anomaly. We have the option to add a term based on the kinematical structure $I_3(J)$ which we know to be reducible to $I_4(J)$. We use this option normalizing the contribution to a "connection" derived from h_{ij}. In addition we can add a term with the form of (3.29) but again with a normalization dependent on h_{ij}. Therefore at the end these two modifications amount to a redefinition of t_{ijkl} by an additive h_{ij} dependent term. We arrive therefore at the new form of (3.28)

$$\hat{\mathcal{A}} = \int \sqrt{\gamma}\,\sigma\, g_{ij}\Big\{\hat{\Box}J^i\,\hat{\Box}J^j + 2\,\hat{\nabla}^\mu\hat{\nabla}^\nu J^i\,\hat{\nabla}_\mu\hat{\nabla}_\nu J^j + 4\,\nabla^\mu J^i\,\hat{\nabla}_\mu\hat{\Box}J^j + 4\left(R^{\mu\nu} - \frac{1}{6}g^{\mu\nu}R\right)\nabla_\mu J^i\,\nabla_\nu J^j\Big\} \tag{3.36}$$

where

$$\begin{aligned}\hat{\nabla}_\mu\hat{\nabla}_\nu J^i &= \nabla_\mu\nabla_\nu J^i + \Gamma^i_{kl}\,\partial_\mu J^k\,\partial_\nu J^l \\ \hat{\nabla}_\mu\hat{\Box}J^i &= \nabla_\mu\hat{\Box}j^i + \Gamma^i_{kl}\,\nabla_\mu J^k\,\hat{\Box}J^l\end{aligned} \tag{3.37}$$

and

$$\Gamma^i_{jk} = \frac{1}{2}g^{im}\left(\partial_k g_{mj} + \partial_l g_{jm} - \partial_m g_{ij}\right) \tag{3.38}$$

and we replaced h_{ij} with g_{ij} to stress that they have different transformation properties under reparametrizations. Indeed if we work out the reparametrization of (3.36) we obtain that g_{ij} transforms as a symmetric tensor. We remark again that (3.28) and (3.36) are completely equivalent as far as the universal information they carry is concerned and are equally correct forms for the type B anomaly. Due to the simple transformation properties of g_{ij} it is easy to check the vanishing requirement for thee three J contribution in a given "frame": simply one can choose Riemann normal coordinates where $g_{ij}(0) = \delta_{ij}$ and $\Gamma^i_{jk}(0) = 0$, and then from the form of (3.36) it is evident.

The expression (3.36) is clearly Weyl invariant, being a linear combination of the previous anomalies with special choices of the anomaly functions.

We now discuss the meaning and transformation properties of the "Zamolodchikov metric" related to the above general discussion. The Zamolodchikov metric is defined [13] by first deforming the original CFT through the addition of a term

$$\sum_k \bar{J}^k \int d^4x\,\sqrt{\gamma}\,\mathcal{O}_k \tag{3.39}$$

where $\bar{J}^k$ are finite deformation parameters. Then the two-point functions of the moduli $\langle\mathcal{O}_j\mathcal{O}_l\rangle$ are studied in the deformed theory and their normalization, defined by a matrix $\bar{g}_{jl}(\bar{J})$, gives the "Zamolodchikov metric". Since the analytic structure of the two-point function is completely fixed by conformal invariance, the normalization is also the normalization of the logarithmic dependence and therefore the metric is closely related to the type B anomaly. We want to study the details of this relation, in particular the covariance of the metric under a reparametrization of the deformation parameters $\bar{J}^k$.

Though the generating functional's dependence on the sources is defined as an expansion in J^k around $J^k = 0$, it is believed to have a finite radius of convergence and therefore can be expanded also around $\bar{J}^k$, giving the correlators of the deformed theory.

To discuss this concretely, let us choose the "covariant scheme" (3.36) and expand $J^k(x)$ as:

$$J^k(x) = \bar{J}^k + \tilde{J}^k(x) \tag{3.40}$$

Then Taylor expanding in $\tilde{J}$ and keeping only quadratic terms, we obtain for the anomaly

$$\mathcal{A} = \int d^4x\, \sigma \sqrt{\gamma}\, g_{ij}(\bar{J}) \Big[\Box \tilde{J}^j \, \Box \tilde{J}^l + 2\, \nabla^\mu \nabla^\nu \tilde{J}^j \, \nabla_\mu \nabla_\nu \tilde{J}^l \\ + 4\, \nabla^\mu \tilde{J}^j \, \nabla_\mu \Box \tilde{J}^l + 4 \big(R^{\mu\nu} - \tfrac{1}{6} g^{\mu\nu} R \big) \nabla_\mu \tilde{J}^j \, \nabla_\nu \tilde{J}^l \Big] \tag{3.41}$$

This expression can be completed with the additional terms in $\tilde{J}^k$ to make it a type B anomaly in the deformed theory at generic J^k, but they are not needed for our argument. The lowest term above represents the metric in $\tilde{J}^k$ at $\bar{J}^k$, which is the normalization of the two-point function of the moduli without insertions of the energy-momentum tensor, i.e. the Zamolodchikov metric.

Therefore one can immediately identify $g_{ij}(\bar{J})$ with the Zamolodchikov metric. It contains all the universal information about the infinite class of type B anomalies defined in the undeformed theory, i.e. the coefficients of the single logarithms in correlators of any number of moduli operators. The covariance properties of the Zamolodchikov metric are simply inherited from those of the anomaly metric. Even though the metric reflects the two-point function at the deformed point it reflects the infinite summation of all higher order correlators. Different parametrizations of the deformed point contain the scheme dependence as semi-local contributions could contribute and produce the transformation of the metric. The semilocal contributions do not have an intrinsic (universal) meaning, unless some higher symmetry is introduced.

The above argument misses an important aspect of the Zamolodchikov metric as characterizing also the global features of the moduli space. It assumes that the "path" between the "perturbative expansion point $J = 0$" and the finite point $\bar{J}$ is unique. This is not generically true: one can have "holonomies" on the moduli space related to its global properties. The anomaly approach being intrinsically perturbative misses the information about this structure and it is a very interesting problem to try to find such a global information in the anomalies.

Finally the above identification gives us a simple argument for the matching of the $g_{ij(J)}$ anomaly normalizations. The Zamolodchikov metric gives the normalization of the two-point function for the deformed theory at $\bar{J}^k$. As we argued in Section 3 this normalization is matched through the relation to the three-point function to the anomaly in the broken phase. The matching occurs for a give scheme for J and it is covariant under a reparametrization.

Similar considerations can be made for the second anomaly (3.29) in the deformed theory. Its normalization for the lowest term, the logarithmic term in the correlator of four moduli, defines a Zamolodchikov-Osborn tensor which is given by $t_{ijlm}(\bar{J}^j)$. Its covariance properties under repametrizations of $\bar{J}^k$ are again given by (3.32).

4 Energy-Momentum Tensor Three-Point Function

We now turn to the case in which Weyl anomalies were originally discussed, the three-point function of the energy-momentum tensor in $d = 4$,

$$\Gamma^{(3)}_{\mu\nu,\rho\sigma,\alpha\beta}(k_1, k_2, k_3) \equiv \langle T_{\mu\nu}(k_1)\, T_{\rho\sigma}(k_2)\, T_{a\beta}(k_3)\rangle \qquad \text{with} \qquad k_1 + k_2 + k_3 = 0 \tag{4.1}$$

It is the $\mathcal{O}(h^3)$ term in the expansion of the effective action around Minkowski space. It exhibits both Type A and Type B anomalies [14–17]. Here we will be mainly concerned with the latter. Compared to the correlation functions which we discussed in the previous sections, the one discussed here is far more involved due to the more complicated tensor structure.

The correlation function (4.1) has dimension four and, in addition to the symmetries implied by the symmetry of $T_{\mu\nu}$, it has S_3 Bose symmetry under permutation of the three pairs of indices. As already discussed in detail in [14], there are 137 possible tensor structures, each of which is multiplied by an invariant amplitude which is a function of the three independent kinematical invariants k_1^2, k_2^2, k_3^2. The tensor indices can be carried by the three momenta and the Minkowski metric. As the total dimension $\Gamma^{(3)}$ is four, the amplitudes which multiply tensor structures where all six indices are carried by momenta, have dimension -2, those where four indices are carried by momenta have dimension 0, those where two indices are carried by momenta have dimension $+2$ and, finally, those where all indices are carried by the metric, have dimension $+4$. Their numbers are 27, 63, 42 and 5, respectively. Among those only the 27 dimension -2 amplitudes are scheme independent and therefore unambiguous and our analysis is based on them.

In general space-time dimension the tensor structures are independent, however in integer dimensions there are dimension-dependent special identities, so-called Schouten identities. For $d = 4$, which we are interested in, there is one identity among the dimension zero tensor structures which is the vanishing of the third metric variation of $\int \sqrt{g} E_4$ (c.f. (4.13)). It vanishes as the integrand is a total derivatives.

Our aim is to generalize the analysis of the $\Delta = 2$ model of Section 2. More specifically, by choosing appropriate linear combinations we look for diffeomorphism and Weyl Ward identities which involve only the 27 dimension -2 amplitudes. In a second step we use a diffeomorphism Ward identity to fix the normalization of the Type B anomaly in terms of the normalization of the two-point function.

Due to Bose symmetry the invariant amplitudes come in S_3 orbits, where S_3 acts on the arguments. There are orbits of length 6, 3, 2 and 1. In the first case the amplitudes have no symmetry under exchange of any two arguments while in the last case they are totally symmetric. To make the S_3 symmetry manifest we use a basis for the tensor structures where the independent momenta are chosen as follows: the indices (μ, ν) are carried by k_2, k_3, indices (ρ, σ) by k_3, k_1 and (α, β) by k_1, k_2. Of course the indices can also be carried by the metric. A similar discussion can be found in [17], which we closely followed.

As for the final analysis of the Ward identities only the dimension -2 amplitudes, where all six tensor indices are carried by momenta, enter, we will only enumerate those. We introduce the following notation for their tensor structures, e.g.

$$(23;13;12) = k^2_\mu k^3_\nu k^1_\rho k^3_\sigma k^1_\alpha k^2_\beta \tag{4.2}$$

and for the invariant amplitudes

$$A_I^{\{ijl\}} = A_I(k_i^2, k_j^2, k_l^2) \tag{4.3}$$

Then the seven S_3 orbits of the dimension -2 amplitudes are

$$\begin{aligned}
&(1) \quad A_1^{\{123\}}(22;11;11) + A_1^{\{213\}}(22;11;22) + A_1^{\{132\}}(33;11;11) \\
&\qquad\qquad + A_1^{\{231\}}(22;33;22) + A_1^{\{312\}}(33;33;11) + A_1^{\{321\}}(33;33;22) \\
&(2) \quad A_2^{\{123\}}(33;11;12) + A_2^{\{213\}}(22;33;12)k + A_2^{\{132\}}(22;13;11) \\
&\qquad\qquad + A_2^{\{231\}}(23;11;22) + A_2^{\{312\}}(33;13;22) + A_2^{\{321\}}(23;33;11) \\
&(3) \quad A_3^{\{123\}}(22;33;11) + A_3^{\{213\}}(33;11;22) \qquad A_3^{\{ijk\}} = A_3^{\{jki\}} = A_3^{\{kij\}} \\
&(4) \quad A_4^{\{123\}}(22;11;12) + A_4^{\{312\}}(33;13;11) + A_4^{\{321\}}(23;33;22) \qquad A_4^{\{ijk\}} = A_4^{\{jik\}} \\
&(5) \quad A_5^{\{123\}}(23;13;12) \qquad A_5^{\{123\}} = A_5^{\{231\}} = A_5^{\{312\}} = A_5^{\{213\}} = A_5^{\{321\}} = A_5^{\{132\}} \\
&(6) \quad A_6^{\{123\}}(23;11;12) + A_6^{\{213\}}(22;13;12) + A_6^{\{132\}}(23;13;11) \\
&\qquad\qquad + A_6^{\{231\}}(23;13;22) + A_6^{\{312\}}(33;13;12) + A_6^{\{321\}}(23;33;12) \\
&(7) \quad A_7^{\{123\}}(23;11;11) + A_7^{\{231\}}(22;13;22) + A_7^{\{312\}}(33;33;12) \qquad A_7^{\{ijk\}} = A_7^{\{ikj\}}
\end{aligned} \tag{4.4}$$

This defines the 27 dimension -2 invariant amplitudes, but there are only seven independent functions of three arguments. In any particular CFT they are fixed.

The Ward identities are derived as follows. We expand the generating functional for connected correlation functions as

$$\begin{aligned}
W &= \log \int D\phi \, e^{-S[\phi,g]} \\
&= \frac{1}{2!} \int dx\, dy\, \tilde{\Gamma}^{(2)}_{\mu\nu,\rho\sigma}(x,y)\, h^{\mu\nu}(x)\, h^{\rho\sigma}(y) \\
&\quad + \frac{1}{3!} \int dx\, dy\, dz\, \tilde{\Gamma}^{(3)}_{\mu\nu,\rho\sigma,\alpha\beta}(x,y,z)\, h^{\mu\nu}(x)\, h^{\rho\sigma}(y)\, h^{\alpha\beta}(z) + \dots
\end{aligned} \tag{4.5}$$

where

$$h^{\mu\nu} = g^{\mu\nu} - \eta^{\mu\nu} \tag{4.6}$$

We note that $\tilde{\Gamma}^{(n)}$ differs from the n-point function of the energy-momentum tensor by a factor $(-1/2)^n$. This follows form the definition

$$\langle T_{\mu\nu} \rangle = -\frac{2}{\sqrt{g}} \frac{\delta}{\delta g^{\mu\nu}} W \tag{4.7}$$

We are interested in the variation of W under infinitesimal diffeomorphisms and Weyl transformations of the metric

$$\begin{aligned}\delta_\xi g_{\mu\nu} &= \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu && \text{(diffeo.)}\\ \delta_\sigma g_{\mu\nu} &= 2\,\sigma\, g_{\mu\nu} && \text{(Weyl)}\end{aligned} \tag{4.8}$$

Expanded to first order in $h^{\mu\nu}$ these translate to

$$\begin{aligned}\delta_\xi h^{\mu\nu} &= -\partial^\mu \xi^\nu - \partial^\nu \xi^\mu - h^{\mu\rho}\partial_\rho \xi^\nu - h^{\nu\rho}\partial_\rho \xi^\mu + \xi^\rho \partial_\rho h^{\mu\nu} && \text{(diffeo.)}\\ \delta_\sigma h^{\mu\nu} &= -2\,\sigma\,(\eta^{\mu\nu} + h^{\mu\nu}) && \text{(Weyl)}\end{aligned} \tag{4.9}$$

As before, we assume a regularization which preserves diffeomorphism invariance of W. Evaluating $\delta_\xi W = 0$ at $\mathcal{O}(h^2)$ results in the diffeomorphism Ward identity, which in momentum space reads

$$\begin{aligned}k_1^\mu \Gamma^{(3)}_{\mu\nu,\rho\sigma,\alpha\beta}(k_1,k_2,k_3) = {}& k_{1\rho}\Gamma^{(2)}_{\nu\sigma,\alpha\beta}(k_3) + k_{1\sigma}\Gamma^{(2)}_{\rho\nu,\alpha\beta}(k_3) + k_{1\alpha}\Gamma^{(2)}_{\rho\sigma,\nu\beta}(k_2)\\ &+ k_{1\beta}\Gamma^{(2)}_{\rho\sigma,\alpha,\nu}(k_2) - k_{2\nu}\Gamma^{(2)}_{\rho\sigma,\alpha\beta}(k_3) - k_{3\nu}\Gamma^{(2)}_{\rho\sigma,\alpha\beta}(k_2)\end{aligned} \tag{4.10}$$

Due to the anomaly, W is not invariant under Weyl transformations:

$$\delta_\sigma W = \int d^4x\,\sqrt{g}\,\sigma\,\mathcal{A}(x) \tag{4.11}$$

In $d = 4$ there are two cohomologically non-trivial CP-even solutions to the Wess-Zumino consistency condition, parametrized by theory dependent constants a and c, and a cohomologically trivial one, parametrized by b,

$$\mathcal{A} = c\,C^2 - a\,E_4 + b\,\Box R \tag{4.12}$$

where

$$E_4 = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4\,R^{\mu\nu}R_{\mu\nu} + R^2 \tag{4.13}$$

is the Euler density and $C^2 = C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}$ the square of the Weyl tensor. Only the cohomologically non-trivial solutions are true anomalies as they cannot be removed by addition of a local counterterm to the generating functional.

The Weyl Ward identity in momentum space is

$$\eta^{\mu\nu}\Gamma^{(3)}_{\mu\nu,\rho\sigma,\alpha\beta}(k_1,k_2,k_3) = 2\,\Gamma^{(2)}_{\rho\sigma,\alpha\beta}(k_2) + 2\,\Gamma^{(2)}_{\rho\sigma,\alpha\beta}(k_3) + \mathcal{A}_{\rho\sigma,\alpha\beta}(k_2,k_3) \tag{4.14}$$

where $\mathcal{A}_{\rho\sigma,\alpha\beta}$ is obtained from the expansion of (4.12) to $\mathcal{O}(h^2)$.

For each of the Ward identities (4.10) and (4.14) there are two more, where the divergence and trace are w.r.t. to the second and third energy-momentum tensor, respectively. They easily follow from those given by Bose symmetry. All these Ward identities hold both in the broken and in the unbroken phase, a priori with different anomaly coefficients.

The final ingredient which we need is the two-point function $\Gamma^{(2)}_{\mu\nu,\rho\sigma}(k)$. It is not anomalous and its tensor structure is fixed by conservation and tracelessness to be that of the $\mathcal{O}(h^2)$ term in the expansion of C^2, as discussed in the Introduction:

$$\Gamma^{(2)}_{\mu\nu.\rho\sigma}(k) = \Pi_{\mu\nu,\rho\sigma}(k)\, f(k^2) = \left(\pi_{\mu\nu}\,\pi_{\rho\sigma} - \frac{3}{2}\big(\pi_{\mu\rho}\,\pi_{\nu\sigma} + \pi_{\mu\sigma}\,\pi_{\nu\rho}\big)\right) f(k^2) \tag{4.15}$$

where

$$\pi_{\mu\nu} = k^2\,\eta_{\mu\nu} - k_\mu\,k_\nu \tag{4.16}$$

In the unbroken phase,

$$f(k^2) = \frac{4}{3}\bar{c}\,\log(k^2/\Lambda^2) \tag{4.17}$$

while in the broken phase $f(k^2)$ is more complicated and not known generally, except that for $k^2 \gg v^2$ it approaches (4.17).

The analysis of the Ward identities now proceeds in several steps, which are analogous to the ones which we followed in Section 2. Due to the large number of tensor structures and invariant amplitudes and the fact that their range of dimension is from $+4$ to -2, it is considerably more involved and we will skip most of the straightforward but tedious details.[6]

In the first step we insert the expansion of $\Gamma^{(3)}_{\mu\nu,\rho\sigma,\alpha\beta}$ in invariant amplitudes into the (non-anomalous) diffeomorphism Ward identities. Separating the resulting tensor structures[7] leads to a large number of homogeneous and inhomogeneous linear relations between the invariant amplitudes. The coefficients are homogeneous polynomials in the three kinematical invariants and the inhomogeneities, if present, are $f(k_i)$ multiplied by a non-negative power of k_i^2. By taking linear combinations we obtain relations which involve only dimension -2 amplitudes and $f(k_i^2)$. The simplest such relation which we find and which we will use later, is

$$\big(s_1 - s_2 - s_3\big)A_1^{\{213\}} + \big(s_3 + s_1 - s_2\big)A_1^{\{123\}} + \big(s_2 - s_1\big)A_4^{\{123\}} = 4\big(f(s_1) - f(s_2)\big) \tag{4.18}$$

plus two others which are related by Bose symmetry. Here, as before,

$$s_1 = k_1^2\,, \qquad s_2 = k_2^2\,, \qquad s_3 = k_3^2 \tag{4.19}$$

This identity is satisfied in the unbroken and in the broken phase, where in the former $f(k^2)$ is given by (4.17). Note that in these relations the UV cut-off Λ cancels, as required by the fact the all amplitudes in these relations are of dimension -2 and therefore finite, i.e. cut-off independent.

We now turn to the Weyl Ward identities. Inserting the expansion of $\Gamma^{(3)}$ in invariant amplitudes leads to new inhomogeneous linear relations between them, where the inhomogeneities now contain $f(k_i^2)$ and the anomaly coefficients a and c and b. Again all coefficients are simple homogeneous polynomials of the kinematical invariants.

[6] For which we used the xAct Mathematica package [18].

[7] Here it is advantageous to convert to a basis which involves only two of the momenta, e.g. k_1 and k_2.

As in the analysis of Section 2, adding to them appropriate linear combinations of the diffeomorphisms Ward identities, we obtain (anomalous) relations which involve only dimension -2 amplitudes. The simplest such relation which, furthermore, does not contain $f(k_i^2)$ and only the type B Weyl anomaly coefficient c, is of the general type of (1.4),

$$s_1\,E_1 + s_2\,E_2 + s_3\,E_3 = \frac{8}{3}c \tag{4.20a}$$

with

$$\begin{aligned}
E_1 &= A_1^{\{213\}} + A_2^{\{132\}} - A_2^{\{231\}} - \frac{1}{2}A_4^{\{123\}} + A_6^{\{123\}} - A_6^{\{213\}} - A_7^{\{123\}} + A_7^{\{231\}} \\
E_2 &= A_1^{\{123\}} - A_2^{\{132\}} + A_2^{\{231\}} - \frac{1}{2}A_4^{\{123\}} - A_6^{\{123\}} + A_6^{\{213\}} + A_7^{\{123\}} - A_7^{\{231\}} \\
E_3 &= -A_1^{\{123\}} - A_1^{\{213\}} + A_2^{\{132\}} + A_2^{\{231\}} - \frac{1}{2}A_4^{\{123\}} - A_7^{\{123\}} - A_7^{\{231\}}
\end{aligned} \tag{4.20b}$$

and again two more related by Bose symmetry. We will show below that the anomaly c is fixed by $\bar{c}$, the normalization of the two-point function.

We also find anomalous Weyl Ward identities between the dimension -2 amplitudes which involve only the type A anomaly coefficient a, e.g.

$$s_1\,E_1 + s_2\,E_2 + s_3\,E_3 = -16\,a \tag{4.21a}$$

with

$$\begin{aligned}
E_1 = {}& 2A_1^{\{123\}} + 4A_1^{\{213\}} + 2A_1^{\{132\}} + 10A_1^{\{231\}} + 2A_1^{\{312\}} + 2A_1^{\{321\}} + A_2^{\{123\}} - 9A_2^{\{213\}} \\
& - 2A_2^{\{132\}} - 2A_2^{\{312\}} - 2A_2^{\{321\}} + 2A_3^{\{123\}} - 4A_3^{\{213\}} - 3A_4^{\{123\}} - 2A_4^{\{312\}} + 3A_4^{\{321\}} \\
& - A_5^{\{123\}} + A_6^{\{123\}} + 3A_6^{\{213\}} + 2A_6^{\{132\}} - A_6^{\{312\}} + A_6^{\{123\}} - 2A_7^{\{123\}} + 2A_7^{\{231\}} + 7A_7^{\{312\}} \\
E_2 = {}& -2A_1^{\{213\}} + 6A_1^{\{231\}} + A_2^{\{123\}} - 9A_2^{\{213\}} + 2A_2^{\{231\}} + 4A_2^{\{312\}} - 2A_3^{\{213\}} + A_4^{\{123\}} - 3A_4^{\{321\}} \\
& - A_5^{\{123\}} - A_6^{\{123\}} + A_6^{\{213\}} - 4A_6^{\{231\}} + A_6^{\{312\}} + 9A_6^{\{231\}} + 4A_7^{\{231\}} - 9A_7^{\{312\}} \\
E_3 = {}& -4A_1^{\{213\}} - 10A_1^{\{231\}} - 4A_1^{\{321\}} - A_2^{\{123\}} + 5A_2^{\{213\}} + 4A_2^{\{231\}} - 2A_2^{\{312\}} - 4A_3^{\{213\}} - A_4^{\{123\}} \\
& + 7A_4^{\{321\}} - A_5^{\{123\}} + A_6^{\{123\}} + A_6^{\{213\}} + 2A_6^{\{231\}} + A_6^{\{312\}} - 5A_6^{\{231\}} - 2A_7^{\{231\}} + 5A_7^{\{312\}}
\end{aligned} \tag{4.21b}$$

We now analyze the Ward identities. We start with (4.18). At $s_2 = s_1$ it is satisfied identically and contains no information. A non-trivial relation is obtained if we first take the derivative w.r.t. s_1 before setting $s_2 = s_1$ and then taking the limit $s_1 \to \infty$ while keeping s_3 fixed. In doing so we recall that the amplitudes behave as $A \sim \frac{1}{s_i}\log^p s_i$ for $s_i \to \infty$. Therefore, in this limit, $\partial_{s_i}A$ is suppressed by one additional power. If we furthermore use $\partial_{s_1}f(s_1) = 4\,\bar{c}/(3s_1)$ as $s_1 \to \infty$, which is valid in both phases, we obtain from (4.18) the relation

$$2\,A_1^{\{113\}} - A_4^{\{113\}} = \frac{16\,\bar{c}}{3\,s_1} \tag{4.22}$$

We now take the same limit in eq. (4.20). This yields

$$2\,A_1^{\{113\}} - A_4^{\{113\}} = \frac{8\,c}{3\,s_1} \tag{4.23}$$

Comparison gives

$$c = 2\,\bar{c} \neq 0 \tag{4.24}$$

The normalization of the type A anomaly, (4.21) cannot be reduced to the two-point function. Any regularization respecting diffeomorphism invariance will produce the dimension -2 amplitudes corresponding to the three energy-momentum correlators which appear in (4.21) and the value of a can be simply read off. In dimensional regularization a is determined by the 0/0 contribution of a dimension zero amplitude. This amplitudes vanishes in $d = 4$ due to the Schouten identity [4].

We will not discuss the dilaton effective action for this case but refer instead to the literature, e.g. [3].

5 Conclusions

Our main result is a uniform description of type A and B trace anomalies in $d = 4$. As we show the information about the anomaly is carried by a Ward identity of the general form

$$s_1\,E_1(s_1, s_2, s_3) + s_2\,E_2(s_2, s_3, s_1) + s_3\,E_3(s_3, s_1, s_2) = ct \tag{5.1}$$

where $s_i \equiv p_i^2$ are the kinematical invariants (p_i are the three external momenta), E_i are dimension -2 amplitudes, selected depending of the anomaly type and ct is a constant which characterizes the strength of the anomaly being respectively related to a or c. The basic Ward identity (5.1) can be translated into two equivalent, universal characterizations of the anomaly:

$$E_i \xrightarrow{s_i \to \infty} \frac{ct}{s_i} + \mathcal{O}\left(\frac{s_j, s_k}{s_i^2}[\log s_i]^p\right) \tag{5.2}$$

and

$$-\frac{1}{\pi}\int ds_i\,\mathrm{Im}_i E_i(s_i, s_j, s_k) = ct \tag{5.3}$$

where the imaginary part is obtained from the discontinuity with respect to the s_i invariant while the other two invariants s_j, s_k are kept fixed.

After the amplitudes entering the anomaly equation are identified any single one of the conditions (5.2), (5.3) implies all the others and also the validity of the basic equation (5.1) with the same normalization. This depends crucially on the invariant amplitudes having dimension -2 and obeying the standard analyticity of QFT. In particular the high invariant behaviour for one of the amplitudes can be related to the two-point function for type B and to the structure of an invariant amplitude in the three-point function in dimensional regularization which vanishes in $d = 4$ for type A. Once the basic equation

(5.1) is established trace anomaly matching is immediate: in the deep Euclidean limit the invariant amplitudes of the unbroken and broken phases match and since the anomaly is a constant this forces a and c to be the same in the two phases. The basic consequence is then that the anomaly is invariant along the "flow", i.e. a, c are independent of the breaking scale v for the whole range $v = 0$, corresponding to the UV unbroken phase, to $v = \infty$, the deep IR of the broken phase. This is happening while the individual invariant amplitudes have a nontrivial dynamical dependence on the breaking scale v along the "flow".

Interestingly the same type of equation (5.1) is obeyed by chiral anomalies in $d = 4$. This type of equation generalizes the anomaly information related to "Dolgov-Zakharov" poles [19–21]. If $s_j, s_k = 0$ in (5.2), the sum rule is necessarily saturated by a $ct\,\delta(s_i)$ singularity signaling a "pole". Since, however, the configuration chosen is singular and the amplitudes E_i having branch points at $s_j, s_k = 0$, the limit to the special configuration should be taken carefully along special lines. Moreover the "poles" are effectively representing a collapsed branch cut or a collision of two logarithmic branch points in the limit. The relation between the dimension -2 invariant amplitudes appearing in the different anomalies is puzzling. In particular the chiral anomaly amplitudes have opposite P and T parities compared with the trace anomaly ones and related to that they appear in a phase in the Euclidean configuration space. Moreover when the conformal group is extended to the superconformal one [22] they appear in the same supermultiplet. Understanding the similarities/differences of these structures as reflected in the equations (5.1) obeyed by all of them is an interesting question.

The three-point correlator of energy momentum tensors can be used to constrain the possible values of the anomalies in unitary CFT as discussed in [23]. In certain kinematical configurations the correlator reduces to a diagonal matrix element of one energy momentum tensor between two states obtained by acting on the vacuum with the other two, respectively. It would be interesting to understand if this interpretation carries over for the dimension -2 amplitudes constraining again their structure.

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Appendix A Explicit Realizations: Unbroken Phase

The general discussion presented in Sections 2 and 4 of the Ward identities and how the anomaly is captured by the properties of dimension -2 amplitudes, can be explicitly verified with the simplest CFT, namely a free massless scalar field. Here the correlators are one loop Feynman diagrams. As the analytic structure of two- and three-point functions is completely fixed by conformal symmetry, the results derived for this simple model are universal, the only free parameter being the normalization, i.e. the actual strength of the anomaly. Furthermore, while the three-point function of $T_{\mu\nu}$ computed via a one-loop Feynman diagram is not identical to $\Gamma^{(3)}_{\mu\nu,\rho\sigma,\alpha\beta}$, the dimensional -2 amplitudes can be unambiguously obtained as they are not contaminated by semilocal terms.

In this appendix we discuss the unbroken phase while in Appendix B we discuss a simple explicit calculable model of spontaneous breaking for which the results for the broken phase can be checked.

For the conformally coupled scalar with action

$$S = \frac{1}{2}\int d^d x\,\sqrt{g}\Big(\nabla^\mu\phi\,\nabla_\mu\phi + \xi\,R\,\phi^2\Big) \qquad\qquad \xi = \frac{d-2}{4(d-1)} \tag{A.1}$$

the on-shell traceless and conserved energy-momentum tensor is

$$T_{\mu\nu}(\phi) = \frac{2}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}}\Big|_{g_{\mu\nu}=\eta_{\mu\nu}} = \partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\,\partial^\rho\phi\,\partial_\rho\phi + \xi\big(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu\big)\phi^2 \tag{A.2}$$

We are interested in $d = 4$.

In the next section, when we discuss the spontaneously broken phase of this simple model, we need to consider a massive free scalar. In the Lagrangian the mass term is $-\frac{1}{2}M^2\,\phi^2$ which contributes to the energy-momentum tensor as

$$\Delta T_{\mu\nu} = \frac{1}{2}\eta_{\mu\nu}M^2\,\phi^2 \tag{A.3}$$

which leads to an explicit breaking of Weyl invariance, i.e. on-shell one how has $T^\mu_\mu = M^2\,\phi^2$. In this Appendix we will use these general expressions for $M = 0$.

We start with the discussion of the correlator

$$\langle T_{\mu\nu}(-q)\,\mathcal{O}(k_1)\,\mathcal{O}(k_2)\rangle \qquad \text{with} \qquad \mathcal{O} = \phi^2 \tag{A.4}$$

The only contributing Feynman diagrams are logarithmically divergent triangle graphs. There are two graphs with equal contributions to the amplitudes. From our discussion in Section 2 it follows that the divergent part only contributes to the amplitude A in the decomposition (2.7). This can be easily isolated and the finite amplitudes can be recognized by their tensor structures. We assumed that the anomaly appears in Weyl invariance; therefore a convenient regularization is dimensional regularization which respects diffeomorphism invariance. The finite, dimension -2 amplitudes are unambiguous, not being affected by the contributions of semi-local terms of the type discussed in Section 3.2.

There are different ways to obtain the finite amplitudes. It turns out that in order to explicitly check the features of the invariant amplitudes that we have discussed in Section 2, the most convenient way is the Passarino-Veltman [24] decomposition, which amounts to expressing all Feynman integrals with non-trivial tensor structure in terms of basic scalar integrals. This is most easily demonstrated on a simple example. Consider the one-loop integral[8],[9]

$$B_\mu(p) = \int \frac{d^dl}{\pi^{d/2}} \frac{l_\mu}{(l^2 - M^2)((l+p)^2 - M^2)} = p_\mu B_1(p) \tag{A.5}$$

where we have used that the index μ can only be carried by the external momentum p_μ. The following simple manipulation

$$\begin{aligned} p^\mu B_\mu(p) &= \int \frac{d^dl}{\pi^{d/2}} \frac{p\cdot l}{(l^2 - M^2)((l+p)^2 - M^2)} = \frac{1}{2}\int \frac{d^dl}{\pi^{d/2}} \frac{((l+p)^2 - M^2) - (l^2 - M^2) - p^2}{(l^2 - M^2)((l+p)^2 - M^2)} \\ &= -\frac{1}{2}p^2\, B_0(p) \end{aligned} \tag{A.6}$$

leads to

$$B_1(p) = -\frac{1}{2}B_0(p) \tag{A.7}$$

We have used the freedom to shift the loop momentum and we have defined the basic scalar two-point one-loop integral

$$\begin{aligned} B_0(p) &= \int \frac{d^dl}{\pi^{d/2}} \frac{1}{(l^2 - M^2)((l+p)^2 - M^2)} \\ &= -\frac{2}{d-4} + B_0^f + \text{const.} + \mathcal{O}(d-4) \end{aligned} \tag{A.8}$$

where

$$B_0^f(p) = -\int_0^1 dx \log \frac{\big(x(1-x)p^2 - M^2\big)}{\mu^2} \tag{A.9}$$

and μ is the arbitrary renormalization scale. Similarly, one can decompose

$$C_\mu(k_1,k_2) = \int \frac{d^dl}{\pi^{d/2}} \frac{l_\mu}{((l^2 - M^2)((l+k_1)^2 - M^2)((l-k_2)^2 - M^2)} \tag{A.10}$$

and

$$C_{\mu\nu}(k_1,k_2) = \int \frac{d^dl}{\pi^{d/2}} \frac{l_\mu\, l_\nu}{((l^2 - M^2)((l+k_1)^2 - M^2)((l-k_2)^2 - M^2)} \tag{A.11}$$

and express them in terms of B_0 and C_0 where

$$C_0(k_1,k_2) = \int \frac{d^dl}{\pi^{d/2}} \frac{1}{\big(l^2 - M^2\big)\big((k_1+l)^2 - M^2\big)\big((k_2-l)^2 - M^2\big)} \tag{A.12}$$

is the scalar triangle. The tensor indices are now carried by $k_{1\mu}, k_{2\nu}$ and $\eta_{\mu\nu}$.

[8]The measure for the loop integration has been chosen to avoid factors of 4π which can easily be inserted, if needed.

[9]With the discussion of Appendix B in mind, we treat the massive scalar field.

The final result for the invariant amplitudes B, C, D in $d = 4$, which one obtains using the PV decomposition of the three-point function $\langle T_{\mu\nu}\,\phi^2\,\phi^2\rangle$ is

$$\begin{aligned}
\lambda^4\,B &= 2\,r^2\,\lambda^2 + \frac{1}{12}\big[q^2\,r^2 + 2(q{\cdot}r)^2\big]\big[3\,q^4 - 4\,(q{\cdot}r)^2 - 2\,q^2\,r^2 + 3\,r^4\big]C_0 + 4\,r^2\,\lambda^2\,M^2\,C_0\\
&\quad - \frac{1}{2}\Big[\big(2(q{\cdot}r)^3 + (q^2 - r^2)(2(q{\cdot}r)^2 + q^2 r^2) + q{\cdot}r(q^2 r^2 - 3\,r^4)\big)B_0^f(k_1^2) + (k_1 \leftrightarrow k_2)\Big]\\
&\quad - (q^2 - r^2)\big[2(q{\cdot}r)^2 + q^2\,r^2\big]B_0^f(q^2)\\
\lambda^4\,C &= -2\,q{\cdot}r\,\lambda^2 - \frac{1}{4}q^2\,q{\cdot}r\big[3(q^2 - r^2)^2 - 4\,\lambda^2\big]C_0 - 3\,q^2\,q{\cdot}r\,(q^2 - r^2)B_0^f(q^2) - 4\,q{\cdot}r\,\lambda^2\,M^2\,C_0\\
&\quad + \frac{1}{2}\big[\big((q{\cdot}r)^2(3\,q^2 - r^2) + 3\,q{\cdot}r\,q^2(q^2 - r^2) - 2\,q^2\,r^4\big)B_0^f(k_1^2) - (k_1 \leftrightarrow k_2)\big]\\
\lambda^4\,D &= 2\,q^2\,\lambda^2 + \frac{1}{4}q^4\big[3(q^2 - r^2)^2 - 4\,\lambda^2\big]C_0 + 3\,q^4(q^2 - r^2)B_0^f(q^2) + 4\,q^2\,\lambda^2\,M^2\,C_0\\
&\quad + \frac{1}{2}\big[\big(3\,q^4(r^2 - q{\cdot}r - q^2) - 2\,(q{\cdot}r)^3 + 5\,q{\cdot}r\,q^2\,r^2\big)B_0^f(k_1^2) + (k_1 \leftrightarrow k_2)\big]
\end{aligned} \tag{A.13}$$

where

$$\lambda^2 = q^4 + k_1^4 + k_2^4 - 2\,q^2\,k_1^2 - 2\,q^2\,k_2^2 - 2\,k_1^2\,k_2^2 \tag{A.14}$$

is the triangle function.

The expressions for the dimension -2 amplitudes above are independent of the renormalization scale μ. They can be rewritten in terms of differences of logarithms. Given these explicit expressions for the invariant amplitudes, it is now straightforward to check that the Ward identities (2.14) are satisfied. In the normalization chosen here (2.14a) is satisfied with $c = 2$ and $\Gamma^{(2)}(k^2) = -2\,\log k^2/\mu^2$, consistent with the general discussion of the $\Delta = 2$ model in Section 2.

We remark that for the massive scalar the r.h.s. of (2.14b) evaluates to $4 + 8\,M^2\,C_0$. The additional term reflects the explicit violation of traceless of the energy-momentum tensor by the mass term. We will come back to this in Appendix B.

With the help of (A.13) we can also check the asymptotic behaviour of the amplitudes E_i. Both in the massless and massive cases one finds, as expected,

$$E_i \xrightarrow[s_i \to \infty]{} \frac{4}{s_i} \tag{A.15}$$

Given the expressions for B, C, D we can also check the sum rules. Using the Cutkosky rule one derives e.g., valid for $k_1^2, k_2^2 < 0$,

$$
\begin{aligned}
&\mathrm{Im}_{q^2} E_1 = \mathrm{Im}_{q^2}(-3\,B - D) \\
&= \pi \Bigg\{ -\frac{3(q^2 - r^2)(q^4 + 2\,(q\cdot r)^2 + q^2\,r^2)}{\lambda^4} \sqrt{1 - \frac{4\,M^2}{q^2}} \\
&+ \Big(8\,(q^2 + 3\,r^2)\lambda^2\,M^2 + \frac{1}{2}\big(q^4 + 2\,(q\cdot r)^2 + q^2\,r^2\big)\big(3\,q^4 - 4\,(q\cdot r)^2 - 2\,q^2\,r^2 + 3\,r^4\big)\Big) \\
&\qquad \times \frac{1}{\lambda^5}\tanh^{-1}\left(\frac{\sqrt{1-\frac{4M^2}{q^2}}}{q^2 - k_1^2 - k_2^2}\,\lambda\right)\Bigg\}\theta(q^2 - 4\,M^2)
\end{aligned}
\tag{A.16}
$$

From this one computes

$$
-\frac{1}{\pi}\int_{4\,M^2}^{\infty} dq^2\,\mathrm{Im}_{q^2} E_1(q^2, k_1^2, k_2^2, M^2) = 4 \tag{A.17}
$$

For later use we have again presented the results for a massive scalar but, of course the result being independent of M, it is also valid for the discontinuity evaluated at $M = 0$.

The computation of the correlation function of three energy-momentum tensors

$$
\langle T_{\mu\nu}(k_1)\,T_{\rho\sigma}(k_2)\,T_{\alpha\beta}(k_3)\rangle \qquad \text{with} \qquad k_1 + k_2 + k_3 = 0 \tag{A.18}
$$

takes more effort. Rather than doing a Passarino-Veltman decomposition, we have derived for the 27 dimension -2 amplitudes expressions involving integration of the two Feynman parameters (we work again in dimensional regularization). They all have the form

$$
\int_0^1 dx \int_0^{1-x} dy\; \frac{P(x,y)}{x\,y\,k_3^2 + x(1-x-y)k_1^2 + y(1-x-y)k_2^2 - M^2} \tag{A.19}
$$

where $P(x,y)$ are polynomials in the Feynman parameters. For the unbroken phase which we discuss here, $M^2 = 0$.

The calculation is straightforward, however the detailed results are too long to present here. But they were used to check all the Ward identities which we have written in Section 4, in particular that the combination of amplitudes in the Weyl Ward identities are constants, independent of the kinematical invariants. Also the diffeomorphism Ward identity (4.18) has been verified in this way. More precisely, the Ward identities are satisfied in this simple model for $(4\pi)^2\,c = \frac{1}{120}$ and $(4\pi)^2\,a = \frac{1}{360}$, which are known values for the free scalar; see e.g. [2].

Appendix B Explicit Realizations: Broken Phase

In this Appendix we check the general setup for the anomaly structure in the broken phase within a simple model proposed in [25]. Consider two massless scalar fields ϕ and

φ interacting through a marginal perturbation:

$$L = \frac{1}{2}\partial_\mu \phi \, \partial^\mu \phi + \frac{1}{2}\partial_\mu \varphi \, \partial^\mu \varphi - g \, \phi^2 \, \varphi^2 \tag{B.1}$$

The fields are coupled conformally to a background metric. Both in the unbroken and broken phases we will take a limit where g goes to 0 and therefore the beta function(s) vanish, thus not disturbing conformality. Therefore the unbroken phase is made simply from two decoupled massless scalar fields. Consider now the breaking: we give the field φ a vacuum expectation value v, i.e.

$$\langle \varphi \rangle = v \tag{B.2}$$

In order to calculate in the broken vacuum, we can alternatively shift in the Lagrangian the field φ

$$\varphi = v + \tilde{\varphi} \tag{B.3}$$

and calculate with the usual Feynman rules for the field $\tilde{\varphi}$ which has zero *vev*. The dimensionless dilaton Σ, which transforms linearly under Weyl transformations, is

$$\Sigma = \log \left(1 + \frac{\tilde{\varphi}}{v} \right) \simeq \frac{\tilde{\varphi}}{v} + \mathcal{O}(\varphi^2) \tag{B.4}$$

starting linearly in $\tilde{\varphi}$. Since the original energy-momentum tensor is

$$T_{\mu\nu}(\phi, \varphi) = T_{\mu\nu}(\phi) + T_{\mu\nu}(\varphi) + \frac{1}{2}\eta_{\mu\nu} \, g \, \phi^2 \, \varphi^2 \tag{B.5}$$

the shift produces a linear coupling of the dilaton in the energy-momentum tensor,

$$\frac{1}{3} v^2 \big(\eta_{\mu\nu} \Box - \partial_\mu \partial_\nu \big) \Sigma \tag{B.6}$$

which leads to

$$\langle 0 | T_{\mu\nu} | \Sigma(p) \rangle = \frac{1}{3} v^2 \, p_\mu \, p_\nu \tag{B.7}$$

Covariantly the above coupling is translated into a $v^2 \, \Sigma \, R$ term in the effective Lagrangian, where R is the curvature scalar. Also a mass term for the ϕ field with $M^2 = 2 \, g \, v^2$ and a cubic coupling $-2M^2 \, \Sigma \, \phi^2$ are produced.

We will take the limit

$$g \to 0 \, , \qquad v \to \infty \, , \qquad M^2 = 2 \, g \, v^2 = \text{fixed} \tag{B.8}$$

The dimension 2 operator will be

$$\mathcal{O}(x) = \phi^2(x) \tag{B.9}$$

The broken phase is defined by the Feynman diagrams which survive this limit. All the correlators of the ϕ^2 operators and energy-momentum tensors coupled directly or through the dilaton have a scale M. This is the consequence of the dilaton having the propagator proportional to $\frac{1}{v^2}$ which cancels v^2 in the dilaton coupling to the scalar curvature.

We recapitulate the content of the broken phase:

a) A massive scalar field ϕ with mass M with standard massive propagator and an energy-momentum tensor containing the conformal improvement and the mass term.

b) A massless dilaton field Σ with propagator normalized to $\frac{1}{v^2}$.

c) The dilaton is coupled to the massive field through a $M^2 \Sigma \phi^2$ coupling and to the energy-momentum tensor by a $v^2 p_\mu p_\nu$ coupling. All the diagrams involving correlators of the massive energy-momentum tensor and operators made of the massive field have the scale M and are well defined in the limit.

d) The dilaton has a Weyl invariant kinetic term inherited from the φ field. Its energy-momentum tensor is decoupled from the rest of the system and has the v independent trace anomalies expected for a free massless field. The kinetic term contains dilaton self-interactions with the scale v which goes to ∞ in the limit considered, but being decoupled we will ignore this sector.

We start with the discussion of the $\Delta = 2$ model in this particular broken phase. The two-point function is simply the mass term corrrelator for a massive field, i.e. it is logarithmically divergent. After renormalization it is given by

$$\Gamma^{(2)}(p^2) = \Gamma^{(2)}(\mu^2) + \frac{1}{(4\,\pi)^2}(p^2 - \mu^2) \int_{4M^2}^{\infty} dx \frac{\sqrt{1 - \frac{4M^2}{x}}}{(x - p^2)(x - \mu^2)} \tag{B.10}$$

Its exact form will not play a role in our calculation. As discussed in Section 2, after using diffeomorphism invariance the logarithmically divergent contributions of the two and three-point functions drop out and we are left with Ward identity (1.4) which involves only the dimension -2 amplitudes of the $\langle T_{\mu\nu}\,\mathcal{O}\,\mathcal{O}\rangle$ correlator in the broken phase. In the limit (B.8) there are two diagrams which survive (see Figure 2) corresponding to the

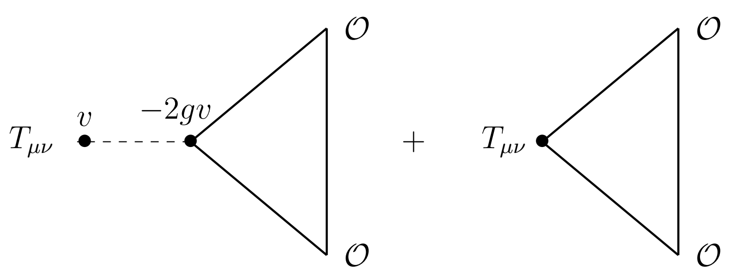

Figure 2

coupling of the energy-momentum tensor through the dilaton and directly. Taking the trace of the energy-momentum tensor gives the combination entering (1.4) whose right hand side is the anomaly, a constant independent on the kinematical invariants and the scale M. We remark that the dilaton contribution to the trace is, with opposite sign, equal to the contribution of the correlator of $M^2\,\phi^2$ with two $\mathcal{O}$ operators. Therefore an alternative interpretation of the anomaly equation in this very special broken phase is that it represents an anomaly in the Ward identity satisfied by the trace of a free massive scalar

$$T^{\mu}_{\mu} - M^2 \mathcal{O} \simeq 0 \tag{B.11}$$

which is valid for the free massive scalar with energy-momentum tensor

$$T_{\mu\nu} = \partial_\mu \phi \, \partial_\nu \phi - \frac{1}{2} \eta_{\mu\nu} \, \partial^\alpha \phi \, \partial_\alpha \phi + \frac{1}{6} \big(\eta_{\mu\nu} \Box - \partial_\mu \partial_\nu \big) \phi^2 + \frac{1}{2} \eta_{\mu\nu} \, M^2 \, \phi^2 \tag{B.12}$$

evaluated in a correlator with two $\mathcal{O}$ operators.

For our very simple model the limit (B.8) selected the diagrams which participate in the relevant Ward identities: the two diagrams of Figure 2 and the two-point function (B.10). Then using the regularization which respects diffeomorphism invariance one arrives at (2.14b) which defines the anomaly in the spontaneously broken phase in terms of the dimension -2 amplitudes contained in the two diagrams of Figure 2. Without such a complete analysis even the meaning of the anomaly in the spontaneously broken phase is not clear. In particular we intend to analyze in the future if the interesting models with potential control of the broken phase proposed in refs. [6, 7] can be brought to this level of meaningful analysis. We continue now the analysis of our simple model.

We can now check the Weyl Ward identity (2.14b) in the broken phase for this model. This requires the knowledge of the dimension -2 amplitudes contributed by the two diagrams.

The first diagram, using the linear dilaton coupling (B.6), gives a contribution ΔB to the amplitude B

$$\Delta B = -\frac{8}{3} \frac{M^2}{q^2} \, C_0 \tag{B.13}$$

with C_0 as in (A.12). C_0 and the amplitudes B, C, D corresponding to the second diagram are now those for the massive case. The amplitudes C and D are not modified. We gave in (A.16) the expression for the contribution to the discontinuity in q^2 of the E_1 amplitude from the second diagram. The contribution to the sum rule from the first diagram, i.e. the contribution from ΔB, is zero. The reason is simply that, at high q^2, ΔB behaves as $\frac{(\log q^2)^2}{(q^2)^2}$ with the power $\frac{1}{q^2}$, whose coefficient is the sum rule contribution, missing. Therefore at finite M the anomaly is controlled by the second diagram in Figure 2 which, as shown in the previous Appendix, saturates the sum-rule. This verifies anomaly matching explicitly.

We discuss now the anomaly in the IR limit of the broken phase, i.e. when M goes to ∞. Since one takes the limit of M first we cannot consider anymore the high momentum behaviour of the amplitudes or the sum rules derived from them. We should use instead directly (2.32). As the anomaly is independent of M we expect the matching to work also at $M = \infty$. The second diagram vanishes in this limit (we remind that we are discussing all the time the dimension -2 amplitudes). We have to evaluate the first, i.e. the dilaton diagram. It has a finite limit giving

$$E_1 = \frac{4}{q^2} \tag{B.14}$$

while $E_2 = E_3 = 0$. Therefore the anomaly equation (2.32) is satisfied with $c^B = c = 2$. Since the matching happened due to the specific value of the dilaton coupling to two $\mathcal{O}$ operators we see explicitly the connection between the anomaly matching and the constraints on dilaton couplings.

We comment on two additional features of this calculation:

a) The same limit appears in the calculation of the anomaly of the massless scalar field when Pauli-Villars regularization is used. Then the trace of the energy-momentum tensor in a correlator with two $\mathcal{O}$ operators is given by the explicit violation introduced by the Pauli-Villars regulator. Therefore the limit with opposite sign represents the anomaly.

b) As discussed above, in this simple model the anomaly in the broken phase for finite M is related to the anomaly of a massive scalar. One can relate therefore the spontaneous breaking in the conformal theory to a "massive flow" specifically of the ϕ scalar which starts massless in the UV and in the IR has an infinite mass. As we described above for finite M one had the anomaly in the correlator of the energy-momentum tensor of the massive scalar with two $\mathcal{O}$ operators. At $M = \infty$ this correlator vanishes and therefore the anomaly in the IR is zero. Hence, from the-point of view of the massive flow one has different anomalies in the UV and IR. In the broken CFT description one has anomaly matching and a physical dilaton degree of freedom in the IR. As a consequence the first, i.e. the dilaton diagram, makes up the difference as calculated above. Therefore generally for a massive flow it is natural to describe it in terms of a dilaton source (not a physical state) which contains the structure of the nonvanishing difference between the UV and IR anomalies on the massive flow [25–27].[10]

One can use the same model to verify the general results presented in Section 4 for the correlator of three energy-momentum tensors in the broken phase. This is much more involved and we discuss here only the analysis of the $M \to \infty$ limit. We start with the discussion of the general set up. In the broken phase we should consider for the three-point function all the contributions where the energy-momentum tensor couples directly to the massive loop or through up to three dilatons. Since we are interested in the anomalous part of the effective action, in principle we should isolate the contributions of the dimension -2 amplitudes entering the anomalous equations, i.e. (4.20) and (4.21). In the $M \to \infty$ limit the contribution of the diagram with direct couplings of all three energy-momentum tensors vanishes for dimension -2 amplitudes.

For the diagrams where the energy-momentum tensor couples through at least one dilaton we use a short cut. Locality of the anomaly implies that in order that the diagram contributes to the anomalous Ward identity its expression should have exactly one dilaton propagator. Therefore in the $M \to \infty$ limit, after factoring out the propagator, the rest of the diagram should give a dimensionless coefficient multiplying four momenta. The interpretation of the coefficient is that of the normalization of a two-dilaton – one-metric perturbation or three-dilaton terms in the anomalous Wess-Zumino action. In order to

[10]For a bootstrap approach see [28].

generate these terms in the limit one has to expand the respective triangle Feynman diagram in the momenta carried by the additional dilatons. The momenta cancel the additional propagators and the $M \to \infty$ limit is finite.

In addition the diagrams with no external momenta have also positive powers of M in the expansion. These terms are non-anomalous since after taking the trace of the energy-momentum tensor they have dilaton propagators and of course the anomaly cannot have such an analytic structure. These effective non-anomalous tree diagrams which involve dilaton propagators arise from the non-anomalous kinetic term of the dilaton. Therefore the contributions with positive powers of M in the limit $M \to \infty$ can be included as a "renormalization" of the kinetic term.

To summarize, the anomalous contribution of the diagrams with couplings through the dilaton is the finite contribution (through expansion in momenta) in the $M \to \infty$ limit. These anomalous contributions for $2, 3, 4$ dilatons were calculated in [25]. Here we complete the calculation for the single dilaton which gives the linear coupling of the dilaton to the anomaly curvature polynomials. By our discussion above, this is captured by the left diagram in Figure 2 where the two operators $\mathcal{O}$ are replaced by energy-momentum tensors. This amounts to computing the correlator

$$\langle M^2\, \phi^2(q)\, T_{\mu\nu}(k_1)\, T_{\rho\sigma}(k_2)\rangle \tag{B.15}$$

More specifically we computed the unambiguous finite part of this dimension $+2$ correlator, where all four tensor indices are carried by the two momenta k_1 and k_2. The invariant amplitudes have the general form (A.19), but in the limit $M^2 \to \infty$ we can replace the denominator by $-M^2$. In this case the integral over the two Feynman parameters becomes trivial. This (finite) part of (B.15) should be compared with the $\mathcal{O}(h^2)$ expansion of the anomaly (4.12). In fact, we keep from it only the piece where all four tensor indices are carried by the momenta of the two gravitons $h^{\mu\nu}$. This leads to an over-determined system of linear equations which is solved by $(4\pi)^2(a, c, b) = \frac{1}{360}(1, 3, 2)$ as expected [2].

In addition to the contribution to the anomaly we have discussed so far, there is also the dilaton loop which contributes with equal coefficients as ϕ such that the total anomaly is that of two free scalars. This dilaton contribution is generic and not special to this simple model.

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