Optimal Power Extraction from Active Particles with Hidden States

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We identify generic protocols achieving optimal power extraction from a single active particle subject to continuous feedback control under the assumption that its spatial trajectory, but not its instantaneous self-propulsion force, is accessible to direct observation. Our Bayesian approach draws on the Onsager-Machlup path integral formalism and is exemplified in the cases of free run-and-tumble and active Ornstein-Uhlenbeck dynamics in one dimension. Such optimal protocols extract positive work even in models characterized by time-symmetric positional trajectories and thus vanishing informational entropy production rates. We argue that the theoretical bounds derived in this work are those against which the performance of realistic active matter engines should be compared.

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Macroscopic living creatures such as horses and oxen have been utilized by humans for millennia to do useful work. A question of current theoretical and practical interest is the extent to which energy can be efficiently harvested from *microscopic* active systems [1–6], whose motion is subject to non-negligible noise. The efficiency of existing many-particle microscopic active matter engines, such as turbines driven by the persistent motion of *E. coli* bacteria in solution [7–9], is heavily limited by the difficulty of rectifying the incoherent motion of collections of individual swimmers with weak alignment interactions in the bulk. Even under idealized conditions, where individual active particles can be manipulated independently, strict upper bounds on extractable power are not well understood, particularly when only a subset of the observables characterizing active motion are accessible to direct observation [10–12]. Here, we present a generic framework for the identification of protocols achieving optimal power extraction from a single active particle under continuous feedback control with the assumption that the instantaneous net velocity, $\dot{x}(t)$, but *not* the fluctuating contribution originating from the self-propulsion, $w(t)$, is observable. This is typically the case for realistic active matter engines [1,7]. Our Bayesian approach, which draws on the Onsager-Machlup path integral formalism [13], applies to a generic stochastic self-propulsion process and is illustrated in the cases of free run-and-tumble (RnT) [14] and active Ornstein-Uhlenbeck (AOU) [15] dynamics in one dimension.

Both models are characterized by time-symmetric positional trajectories (Supplemental Material, Sec. SI [16]) and thus vanishing informational entropy production rates (iEPR) [24,25], defined as the Kullback-Leibler divergence [26] per unit time of the ensemble of forward paths and their time-reversed counterparts [27,28].

In the Markovian case, where all degrees of freedom are observable, the iEPR is proportional to the thermodynamic dissipation and thus provides a (loose) upper bound to the extractable power. This relation fails to apply in the presence of hidden states [12,29,30]. Indeed, we show that positive average power extraction remains possible even for vanishing iEPR upon Bayesian inference of the hidden state (cf. [31], where it is argued that vanishing local iEPR implies zero extractable work). Measurement-driven protocols of the type we discuss in the following incur a thermodynamic maintenance cost [32,33], but are not constrained by Landauer's principle in the same way as equilibrium information engines [6,34].

Definition of the optimal protocol.—Consider the overdamped Langevin equation for a generic active particle $\dot{x}(t) = w(t) + \gamma^{-1} F_{\mathrm{ext}}(t) + \sqrt{2D_x}\xi(t)$, where $\xi(t)$ is a white noise of unit covariance with associated diffusivity D_x and γ denotes the viscosity. We henceforth work in units whereby $\gamma = 1$. Here, $w(t)$ is a stochastic self-propulsion velocity, which for the time being we take to be measurable by an external observer tasked with controlling the applied force $F_{\mathrm{ext}}(t)$. In practice, $F_{\mathrm{ext}}(t)$ could be implemented using an optical trap [34] or, for a charged active colloid [35], through an external electric field of time-varying magnitude and direction. Positive average work is readily extracted by applying an $F_{\mathrm{ext}}(t)$ smaller than and opposite to the particle's self-propulsion [1,7,36]. Over a duration T this generates a noise-averaged total work by the particle against the known external force

$$\mathbb{E}_\xi[W_{\rm tot}[F_{\rm ext}]] = -\int_0^T dt\, F_{\rm ext}(t)\mathbb{E}_\xi[\dot{x}(t)] = -\int_0^T dt\, F_{\rm ext}(t)[w(t)+F_{\rm ext}(t)], \quad (1)$$

which constitutes the key observable of a hypothetical experiment. Above and henceforth, $\mathbb{E}_\phi[\bullet]$ is used to denote an average with respect to the steady-state distribution of the random variable ϕ. We will subsequently refer to $F_{\rm ext}(t)$ as "the protocol." The integrand of Eq. (1), corresponding to the instantaneous power output, can be maximized at each time t by applying the protocol $F^*_{\rm ext}(t) = -w(t)/2$. The corresponding steady-state average power output is

$$\lim_{T\to\infty} \frac{\mathbb{E}_\xi[W_{\rm tot}[F_{\rm ext}]]}{T} = \frac{\bar{w}^2}{4} + \frac{\mathbb{E}_w[(w(t)-\bar{w})^2]}{4}, \quad (2)$$

where $\bar{w} \equiv \mathbb{E}_w[w(t)]$ and we have invoked ergodicity to convert time averages to ensemble averages. The average power is smaller than the thermodynamic dissipation at $F_{\rm ext} = 0$, given by $D_x\dot{S}_i = \mathbb{E}_w[w^2(t)]$ [37,38], demonstrating that the entropy production rate $\dot{S}_i$ provides only a *loose* upper bound to the extractable power at low Reynolds number, due to the unavoidability of viscous effects when $\dot{x}(t) \neq 0$. We will henceforth refer to protocols $F^*_{\rm ext}(t)$ achieving the maximum average power output allowed under a particular set of constraints as *optimal*.

Consider now the case where the underlying dynamics of the active particle (in the form of the full set of governing equations) are known but the instantaneous self-propulsion velocity $w(t)$ is not accessible to direct observation, i.e., it is a hidden variable. Naïvely, this suggests positive work extraction is unattainable since there is no immediate indication of which direction and magnitude should be chosen for $F_{\rm ext}(t)$. However, since $w(t)$ can still be partially inferred from the history of $x(t)$, positive work can be extracted during transient periods of persistent motion. To see this, let $\mathbb{P}(w(T) = v|\{x\}_0^T)$ denote the posterior probability density that the instantaneous self-propulsion velocity of the active particle at current time T equals v *given* a particular spatial trajectory $\{x\}_0^T$ has been observed. The expected work extracted during a time window of duration T can be expressed as the following functional of the generic protocol $F_{\rm ext}(t)$,

$$\mathbb{E}_{\xi,w}[W_{\rm tot}[F_{\rm ext}]] = -\int_0^T dt \int_{-\infty}^{\infty} dv\, \mathbb{P}(v|\{x\}_0^t)F_{\rm ext}(t)[v+F_{\rm ext}(t)]. \quad (3)$$

The optimal protocol $F^*_{\rm ext}(t)$ is obtained from $\delta\mathbb{E}_{\xi,w}[W_{\rm tot}[F_{\rm ext}]]/\delta F_{\rm ext}|_{F^*_{\rm ext}} = 0$, whence

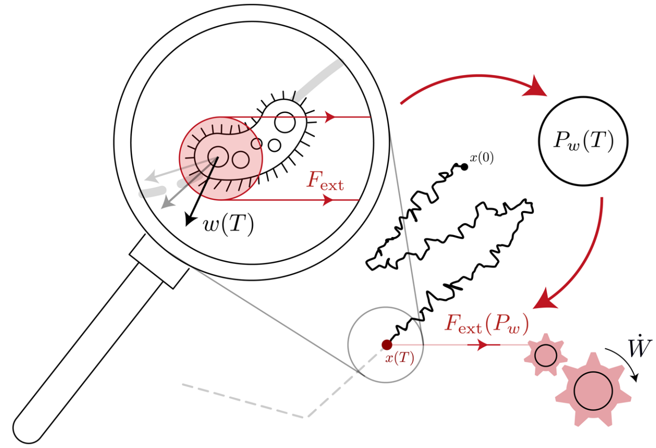

FIG. 1. Optimal power extraction from an active particle (here visualized as a bacterium) with hidden self-propulsion velocity is achieved by subjecting the latter to continuous feedback control, whereby the magnitude and direction of the protocol $F_{\rm ext}(t)$ are modulated according to the inferred self-propulsion velocity.

$$F^*_{\rm ext}(T) = -\frac{1}{2}\int_{-\infty}^{\infty} dv\, \mathbb{P}(v|\{x\}_0^T)v = -\frac{\mathbb{E}_w[w(T)|\{x\}_0^T]}{2}, \quad (4)$$

where $\mathbb{E}_w[w(T)|\{x\}_0^T]$ denotes the posterior expectation of the self-propulsion velocity with respect to $\mathbb{P}(v|\{x\}_0^T)$. This is not to be confused with the expectation of $w(T)$ taken with respect to the corresponding prior probability $\mathbb{P}(v) = \int \mathcal{D}x\, \mathbb{P}(v|\{x\}_0^T)\mathbb{P}(\{x\}_0^T)$, which we denoted $\bar{w}$ and assume to be independent of T. Substituting the optimal force into the expression for the instantaneous power output, the integrand in Eq. (3) gives

$$\mathbb{E}_{\xi,w}[\dot{W}[F^*_{\rm ext}(t)]] = \frac{\bar{w}^2}{4} + \frac{\mathbb{E}_w[(w(T)-\bar{w})|\{x\}_0^T]^2}{4}; \quad (5)$$

cf. Eq. (2). In the following, we take $\bar{w} = 0$ to focus on the nontrivial term appearing on the right-hand side of Eq. (5). Figure 1 schematizes the feedback control described above.

Warm-up: The run-and-tumble particle.—We have reduced the problem of identifying the optimal protocol to the evaluation of the posterior expectation $\mathbb{E}_w[w(T)|\{x\}_0^T]$, Eq. (4). Now we proceed to show how this can be done for the case of RnT motion in one dimension, $\dot{x}(t) = \nu w(t) + F_{\rm ext}(t) + \sqrt{2D_x}\xi(t)$, whose binary internal self-propulsion mode $w(t)$ constitutes the simplest example of a state-space amenable to nontrivial coarse graining.

In particular, let $w(t) \in \{-1, 1\}$ be a dimensionless dichotomous noise with symmetric transition rate α. We seek the posterior probability that the particle is a right self-propeller, $w(T) = +1$, given its positional trajectory up to the current time T, which we denote $P_+(T) = \mathbb{P}[w(T) = +1|\{x\}_0^T]$ for compactness. The complementary probability is denoted $P_-(T) = \mathbb{P}[w(T) = -1|\{x\}_0^T]$. Defining the *confidence parameter* $Q[\{x\}_0^T] = \log(P_+(T)/P_-(T))$ and using $P_+(T) + P_-(T) = 1$, we can write

$$P_+(T) = \frac{e^Q}{1+e^Q} = \frac{1}{2} + \frac{e^Q - 1}{2(1+e^Q)}. \tag{6}$$

Equation (6) reduces to the prior probability $\mathbb{P}(w = \pm 1) = 1/2$ when $Q = 0$. To calculate $P_+(T)$ via Q we thus need to find an expression for the ratio of the conditional path probabilities. To do so, we first invoke Bayes' theorem,

$$\mathbb{P}[w(T) = \pm 1 | \{x\}_0^T] = \frac{\mathbb{P}[\{x\}_0^T | w(T) = \pm 1]}{2\mathbb{P}[\{x\}_0^T]}, \tag{7}$$

where we have used $\mathbb{P}[w(T) = \pm 1] = 1/2$. We can equivalently write

$$Q[\{x\}_0^T] = \log \frac{\mathbb{P}[\{x\}_0^T | w(T) = +1]}{\mathbb{P}[\{x\}_0^T | w(T) = -1]}, \tag{8}$$

reminiscent of a stochastic entropy [28]. We now introduce the notation for the average with respect to the distribution of $w(t)$ path probabilities conditioned on a particular final value $w(T)$,

$$\overline{\bullet}^{(v)} \equiv \int \mathcal{D}w \bullet \mathbb{P}[\{w(t)\}_0^T | w(T) = v], \tag{9}$$

which allows us to express the path probabilities in Eq. (8) as

$$\mathbb{P}[\{x\}_0^T | w(T) = +1] = \overline{\mathbb{P}[\{x\}_0^T | \{w\}_0^T]}^{(+1)}, \tag{10a}$$

$$\mathbb{P}[\{x\}_0^T | w(T) = -1] = \overline{\mathbb{P}[\{x\}_0^T | \{w\}_0^T]}^{(-1)}. \tag{10b}$$

Finally, we invoke the Onsager-Machlup path integral form [13] of the conditional path probability in the Stratonovich discretization

$$\mathbb{P}[\{x\}_0^T | \{w\}_0^T] \propto \exp\left(-\frac{1}{4D_x}\int_0^T dt\,(\dot{x}_c(t) - \nu w(t))^2\right), \tag{11}$$

where $\dot{x}_c = \dot{x} - F_{\text{ext}}$ denotes the velocity in the reference frame where the externally imposed drift is subtracted away. Substituting Eq. (11) into Eq. (10), combining the resulting expressions with Eq. (8), and canceling common $w(t)$-independent factors appearing in the numerator and denominator, we eventually arrive at

$$Q[\{x\}_0^T] = \log\left(\overline{\exp\left(\frac{\nu}{2D_x}\int_0^T dt\,\dot{x}_c(t)w(t)\right)}^{(+1)}\right) - \log\left(\overline{\exp\left(\frac{\nu}{2D_x}\int_0^T dt\,\dot{x}_c(t)w(t)\right)}^{(-1)}\right), \tag{12}$$

where we have also used $w^2(t) = 1$ for all $t \in [0, T]$. To make further progress we exploit the identity between the logarithm of a moment-generating function and its cumulant-generating function [39,40], as well as the parity of the cumulants (see Supplemental Material, Sec. SII [16]). This leads to

$$Q[\{x\}_0^T] = \sum_{n\,\text{odd}}^{\infty} \frac{\text{Pe}^n}{2^{n-1}n!}\overline{Y^n[\{x\}_0^T]}^{(+1),c}, \tag{13}$$

with Péclet number $\text{Pe} = \nu^2/(D_x\alpha)$ and

$$Y^n[\{x\}_0^T] = \int_0^T dt_1 \ldots dt_n \prod_{i=1}^n \left(\frac{\dot{x}_c(t_i)\alpha}{\nu}\right) w(t_i), \tag{14}$$

where the superscript c in expectations, e.g., $\overline{\bullet}^{(v),c}$, denotes the corresponding cumulant. Substituting Eq. (13) into Eq. (6), combined with Eq. (4), returns the optimal protocol.

Computing the right-hand side of Eq. (13) is unfeasible in general. However, $Q[\{x\}_0^T]$ can be computed analytically in the low-Pe asymptotic regime. To leading order in $\text{Pe} \ll 1$, only the first cumulant $\overline{Y[\{x\}_0^T]}^{(+1),c}$ is required, which in turn draws on $\overline{w(t)}^{(+1),c} = \overline{w(t)}^{(+1)} = \exp[-2\alpha(T-t)]$, Supplemental Material, Sec. SII [16], whence we find

$$Q[\{x\}_0^T] = \text{Pe}\int_0^T dt \left(\frac{\alpha \dot{x}_c(t)}{\nu}\right) e^{-2\alpha(T-t)} + \mathcal{O}(\text{Pe}^3). \tag{15}$$

In order to conveniently apply the optimal protocol under continuous feedback control, we can differentiate Eq. (15) with respect to T and use the Leibniz integration rule [assuming $\dot{x}_c(t) = 0$ for $t < 0$] to obtain a differential equation for the time evolution of Q, i.e., $\dot{Q}(T) = \nu\dot{x}_c(T)/D_x - 2\alpha Q(T)$. Remarkably, upon substituting for $\dot{x}_c$ and rescaling time by the switching rate, $T' = \alpha T$, the Langevin equation for $Q(T')$ reads like that of a RnT particle in a harmonic potential with self-propulsion speed and diffusivity both equal to the Péclet number, i.e.,

$$\frac{dQ(T')}{dT'} = \text{Pe}\,w(T') - 2Q(T') + \sqrt{2\text{Pe}}\,\xi(T'). \tag{16}$$

We now proceed to make the connection with the rate of work extraction. First of all, we have by combining Eqs. (4) and (6) that the optimal protocol is given to leading order in $Q \sim \text{Pe}$ by $F^*_{\text{ext}}(T) = -(\nu/4)Q + \mathcal{O}(Q^2)$. When the optimal protocol is applied at all times, the resulting noise-averaged power output, Eq. (5), is given by $\mathbb{E}_\xi[\dot{W}_{\text{RnT}}[F^*_{\text{ext}}(t)]] = \nu^2 Q^2(T)/16 + \mathcal{O}(\text{Pe}^2)$. Taking a further expectation with respect to the dichotomous noise $w(t)$ and exploiting the mapping of the Q dynamics onto those of a RnT particle in a harmonic potential, Eq. (16), whence $\mathbb{E}_{\xi,w}[Q^2] = (1 + \text{Pe}/4)\text{Pe}/2$ [14], we eventually arrive at

$$\mathbb{E}_{\xi,w}[\dot{W}_{\mathrm{RnT}}(F^*_{\mathrm{ext}})] = \frac{\nu^2}{4}\frac{\mathrm{Pe}}{8} + \mathcal{O}(\mathrm{Pe}^2), \qquad (17)$$

which constitutes a tight upper bound to the average extractable power from a RnT particle with hidden self-propulsion velocity in the low-Pe regime. Higher moments of the fluctuating power output under F^*_{ext} can be computed similarly, see Supplemental Material, Sec. SIII [16].

A boundary-update protocol.—We further introduce an independent approach to computing the posterior probability $P_+(T)$ in real time. This novel "boundary-update" protocol, described in full detail in Supplemental Material Sec. SVI [16], both saturates the bound (17) and is conjectured to achieve optimality for *all* Pe. It draws on the *conditional* splitting probabilities of the RnT process, which, to the best of our knowledge, we compute here for the first time. These are the probabilities that a particle initialized at $x_0 \in [-L/2, L/2]$ in a given statistical superposition of internal states exits said interval through either the left or right boundary in either a left or right self-propulsion state. Knowledge of the splitting statistics is used in combination with Bayes' theorem to update the posterior distribution of the internal state $w(t)$ each time the particle is observed to undergo a net displacement larger than $L/2$ in the reference frame where the deterministic drift is subtracted away, $\dot{x}_c = \dot{x} - F_{\mathrm{ext}}$. In the limit $L \to 0$, the posterior updating frequency diverges and we conjecture that optimal inference is achieved. Figure 2 shows application of the boundary-update approach indeed produces an average power output matching the bounds Eqs. (17) and (2) in the low- and high-Pe limits, respectively.

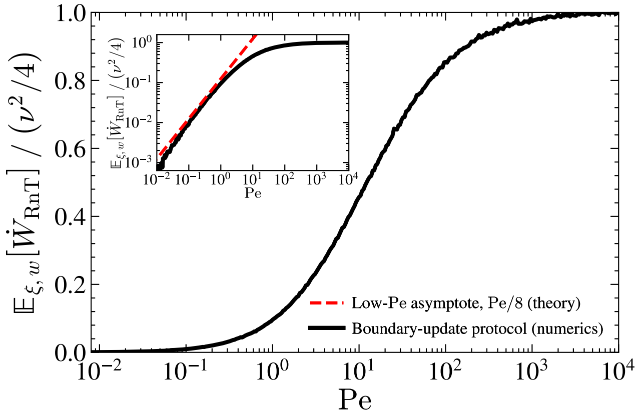

FIG. 2. Average power extracted from a RnT particle with hidden self-propulsion velocity upon application of the boundary-update protocol, the numerical implementation of which is discussed in detail in Supplemental Material, Sec. SVI [16]. The extractable power, which is positive for all Pe, asymptotically approaches that of a situation where the internal state is known, Eq. (2), as $\mathrm{Pe} \to \infty$ and is in excellent agreement with the theoretical bound in the low-Pe limit, Eq. (17).

A generic active particle.—Having explored the particular case of RnT motion in some detail, we now expand our scope to a one-dimensional active particle with self-propulsion velocity $w(t)$ evolving according to a generic (discrete- or continuous-state) stochastic process [41]. Following Eq. (4), the identification of the optimal protocol requires us to compute the posterior expectation of the self-propulsion velocity, which can be conveniently expressed as

$$\mathbb{E}_w[w(T)|\{x\}_0^T] = \mathrm{Tr}_v[v\mathbb{P}[w(T)=v|\{x\}_0^T]] = \frac{\mathrm{Tr}_v\left[v\cdot\overline{\exp\left(-\frac{\mathrm{Pe}}{4}\int_0^T dt\,\frac{\mu}{\sigma_w^2}(\dot{x}_c(t)-w(t))^2\right)}^{(v)}\mathbb{P}(v)\right]}{\mathrm{Tr}_v\left[\overline{\exp\left(-\frac{\mathrm{Pe}}{4}\int_0^T dt\,\frac{\mu}{\sigma_w^2}(\dot{x}_c(t)-w(t))^2\right)}^{(v)}\mathbb{P}(v)\right]}, \qquad (18)$$

with $\mathrm{Pe} = \sigma_w^2/(\mu D_x)$, $\sigma_w^2 = \mathbb{E}_w[w^2]$, and μ a characteristic inverse timescale associated with the self-propulsion dynamics. Here, Tr_v denotes an integral (sum) over the continuous (discrete) state space. We have also invoked Bayes' theorem to write

$$\mathbb{P}[w(T)=v|\{x\}_0^T] = \frac{\mathbb{P}(v)\overline{\mathbb{P}[\{x\}_0^T|\{w(t)\}_0^T]}^{(v)}}{\mathbb{P}[\{x(t)\}_0^T]}, \qquad (19)$$

where $\overline{\bullet}^{(v)}$ is defined as in Eq. (9), and we have used the normalization condition $1 = \mathrm{Tr}_v\mathbb{P}[w(T)=v|\{x\}_0^T]$ to divide by a factor of unity throughout, producing the same type of cancellations of v-independent terms observed in the RnT case. We can rewrite Eq. (18) in a compact form as

$$\mathbb{E}_w[w(T)|\{x\}_0^T] = \frac{\mathrm{Tr}_v\left[v\cdot e^{\mathcal{L}[\{x\}_0^T,v]}\mathbb{P}(v)\right]}{\mathrm{Tr}_v[e^{\mathcal{L}[\{x\}_0^T,v]}\mathbb{P}(v)]} \qquad (20)$$

by introducing the cumulant-generating function

$$\mathcal{L}[\{x\}_0^T, v] = \sum_{n=1}^{\infty}\frac{(-\mathrm{Pe})^n}{2^{2n}n!}\overline{\left[\int_0^T dt\,\frac{\mu}{\sigma_w^2}(\dot{x}_c(t)-w(t))^2\right]^n}^{(v),c}. \qquad (21)$$

If no further assumptions can be made regarding the process $w(t)$, one can now truncate the sum and substitute the resulting expression into Eq. (20) to obtain, by invoking Eq. (4) and recalling $\mathbb{E}_w[w] = 0$, the optimal protocol in the asymptotic case $\mathrm{Pe} \ll 1$,

$$F_{\mathrm{ext}}^{*}(T)=\mathrm{Tr}_{v}\left[v\frac{\mathrm{Pe}}{8}\int_{0}^{T}dt\frac{\mu}{\sigma_{w}^{2}}\left(\overline{w^{2}(t)}^{(v),c}-2\dot{x}_{c}(t)\overline{w(t)}^{(v),c}\right)\mathbb{P}(v)\right]. \tag{22}$$

The form of Eq. (22) matches the RnT result, Eq. (15), except for the appearance of a term depending on the second-order cumulant $\overline{w^2(t)}^{(v),c}$, which was absent in the RnT case due to the norm of the self-propulsion velocity being constant. The correlation functions of the hidden state $w(t)$ in Eq. (22) can be reconstructed from observable trajectories (see Supplemental Material, Sec. SV [16]), allowing us to relax *a posteriori* the requirement that the equations governing the dynamics of $w(t)$ be known, and to extract work even in this case.

In Supplemental Material, Sec. SIV [16], we apply the general result obtained above to the specific case of a one-dimensional AOU process, the simplest canonical active particle model with a continuous self-propulsion state [15]. We find the average extractable power from an AOU particle with hidden self-propulsion velocity in the low-Pe asymptote is bound above by

$$\mathbb{E}_{\xi,w}[\dot{W}_{\mathrm{AOU}}(F_{\mathrm{ext}}^{*})]=\frac{\sigma_{w}^{2}}{4}\frac{\mathrm{Pe}}{16}+\mathcal{O}(\mathrm{Pe}^{2}), \tag{23}$$

and further compute the second moment of the power output distribution (Supplemental Material, Sec. SIII [16]).

Langevin dynamics: High-Pe asymptotics.—When the dynamics of $w(t)$ are described by a Langevin process, Eq. (18) also allows us to explore the high-Pe asymptote through a saddle-point expansion. For the particular case of the AOU process (as defined in Supplemental Material, Sec. SIV [16]), we can write, using the Onsager-Machlup form of $\mathbb{P}[\{w\}_0^T]$,

$$\overline{\exp\left(-\frac{\mathrm{Pe}}{4}\int_{0}^{T}dt\frac{\mu}{\sigma_{w}^{2}}(\dot{x}_{c}(t)-w(t))^{2}\right)}^{(v)}\propto\int\mathcal{D}w\,e^{-\mathcal{N}[w(t);\{x\}_{0}^{T}]}\delta[w(T)-v], \tag{24}$$

with the actionlike functional

$$\mathcal{N}[w(t);\{x\}_{0}^{T}]=\mu\int_{0}^{T}dt\left[\mathrm{Pe}\left(\frac{\dot{x}_{c}(t)-w(t)}{2\sigma_{w}}\right)^{2}+\left(\frac{\dot{w}(t)/\mu+w(t)}{2\sigma_{w}}\right)^{2}\right], \tag{25}$$

which combines a "potential" term (prefactor Pe), penalizing departures from $w(t)=\dot{x}_c$, and a "kinetic" term (unit prefactor) penalizing changes in $w(t)$ that are exceedingly fast or slow compared to the characteristic inverse timescale μ of the self-propulsion dynamics. Even at high Pe, the second term cannot be ignored since the boundary condition $w(T)=v$ in general prevents $w(t)=\dot{x}_c(t)$ from being an accessible trajectory for the functional integral. We define $w^*(t;v)$ as the path that minimizes Eq. (25), $\delta\mathcal{N}[w]/\delta w|_{w^*}=0$, whence

$$m\ddot{w}^{*}(t)=\mu^{2}(w^{*}(t)-\dot{x}_{c}(t))+\mu^{2}mw^{*}(t), \tag{26}$$

with $m=1/\mathrm{Pe}$ and boundary condition $w^*(T)=v$. Equation (26) is purposefully arranged to resemble the Newtonian dynamics of a particle of mass m in an unstable, time-dependent harmonic potential $V(w^*,t)=-[\mu^2(\dot{x}_c(t)-w^*)^2/2+m\mu^2w^{*2}/2]$. Remarkably, the high-Pe limit corresponds to the overdamped limit of Eq. (26), whereby $m\to 0$ and the potential term dominates. For $m\ll 1$, Eq. (26) is solved by combining an exponential ansatz with the particular solution $w^*(t;v)=\dot{x}_c(t)+\mathcal{O}(m)$, whence

$$w^{*}(t)=\dot{x}_{c}(t)+(v-\dot{x}_{c}(T))e^{\sqrt{\frac{1+m}{m}}\mu(t-T)}+\mathcal{O}(m). \tag{27}$$

Noting the second functional derivative of $\mathcal{N}$ is independent of $w(t)$, we perform a change of variables $w(t)\to\delta w(t)+w^*(t;v)$ in the functional integral, Eq. (24), to rewrite Eq. (18) exactly as

$$\mathbb{E}_{w}[w(T)|\{x\}_{0}^{T}]=\frac{\int dv\,v\cdot e^{-\mathcal{N}[w^{*}(t;v);\{x\}_{0}^{T}]}\mathbb{P}(v)}{\int dv\,e^{-\mathcal{N}[w^{*}(t;v);\{x\}_{0}^{T}]}\mathbb{P}(v)}. \tag{28}$$

Substituting Eq. (27) into Eq. (25) we thus have, to leading order in large Pe,

$$\mathcal{N}[w^{*}(t)]=\frac{\sqrt{\mathrm{Pe}}}{8}\left[\left(\frac{v-\dot{x}_{c}(T)}{\sigma_{w}}\right)^{2}+\mathcal{O}\left(\mathrm{Pe}^{-\frac{1}{2}}\right)\right], \tag{29}$$

which draws only on the potential term. Further substituting Eq. (29) into Eq. (28) and performing all the resulting Gaussian integrals in closed form, we arrive at the following expression for the posterior expectation of the self-propulsion velocity at high Pe:

$$\mathbb{E}_{w}[w(T)|\{x\}_{0}^{T}]=\left(1-\frac{4}{\sqrt{\mathrm{Pe}}}\right)\dot{x}_{c}(T)+\mathcal{O}(\mathrm{Pe}^{-1}). \tag{30}$$

In other words, the prior distribution $\mathbb{P}[w]$ weakly biases our posterior estimation $\mathbb{E}_w[w(T)|\{x\}_0^T]$ away from $\dot{x}_c(T)$ and towards the prior expectation $\mathbb{E}_w[w(T)]=0$. Using Eq. (5), the high-Pe asymptotic average power output, having applied the optimal protocol, is thus given by

$$\mathbb{E}_{\xi,w}[\dot{W}_{\mathrm{AOU}}(F_{\mathrm{ext}}^{*})]=\frac{\sigma_{w}^{2}}{4}\left(1-\frac{8}{\sqrt{\mathrm{Pe}}}\right)+\mathcal{O}(\mathrm{Pe}^{-1}). \tag{31}$$

Conclusion.—We have identified generic continuous feedback protocols achieving maximum average power extraction from active particles with a (zero-mean) hidden self-propulsion state. These optimal protocols can be written in closed form in the asymptotes $\mathrm{Pe} \ll 1$ and $\mathrm{Pe} \gg 1$, and provide upper bounds to the average extractable work by *any* such protocol (cf. [6]), e.g., Eqs. (17), (23), and (31). These bounds are those against which the performance of autonomous active matter engines, which typically do not have access to the self-propulsion states of the individual constituent particles [1,2], should be compared. Furthermore, our "boundary-update" approach enables work extraction in experimental settings where real-time particle tracking is unfeasible, since only the detection of first-passage events is required for its implementation.

The optimal protocol is generally non-Markovian. However, this difficulty can be circumvented at $\mathrm{Pe} \ll 1$ by embedding the dynamics in a higher dimensional phase space [42], e.g., via the auxiliary dynamics in Eq. (16). Analogously to equilibrium information engines [6,34], the thermodynamic cost of operating the feedback control can be identified with the increase in the total entropy production rate upon expanding the phase space to include such auxiliary variables [17]. In an idealized situation where the operating temperature of the measurement device is arbitrary, and can thus be chosen to be arbitrarily small, the associated dissipation is negligible [34]. The unique utility of information engines operating on active particles arises from their nonvanishing efficiency even when the measurement device and the particle are coupled to the *same* heat bath [6]. Future work will characterize the efficiency of the optimal protocols in this case.

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