An equivariant generalisation of McDuff–Segal's group completion theorem

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In this short note, we prove a G–equivariant generalisation of McDuff–Segal's group completion theorem for finite groups G.

Contents

1 Introduction

Let G be a finite group. In order to state the theorem, we will need a few pieces of terminology:

Notation 1.1. In this note, two kinds of ring localisations will feature and we define and relate them here. Let $R \in \mathrm{CAlg}(\mathrm{Sp}_G)$ and $\underline{S} = \{S_H\}_{H\leq G}$ be a *G–subset* of the zeroth equivariant homotopy Mackey functor $\underline{\pi}_0 R$ of R. That is, for any $H \leq G$, $\underline{S}$ satisfies $\mathrm{Res}^G_H S_G \subseteq S_H \subseteq \pi_0^H R$. Now for any $A \in \mathrm{CAlg}(\mathrm{Sp}_G)$, we define

$$\mathrm{Map}^{\underline{S}^{-1}}_{\mathrm{CAlg}(\mathrm{Sp}_G)}(R, A) \qquad \text{and} \qquad \mathrm{Map}^{S_G^{-1}}_{\mathrm{CAlg}(\mathrm{Sp}_G)}(R, A)$$

to be subcomponents of $\mathrm{Map}_{\mathrm{CAlg}(\mathrm{Sp}_G)}(R, A)$ of commutative algebra maps $R \to A$ which send elements in $\underline{S}$ to units in $\underline{\pi}_0 A$ and send elements in S_G to units in $\pi_0^G A$, respectively. By general theory (cf. [Nik17, Appen. A] for example), we know that the latter mapping space is corepresented by a telescopic localisation $S_G^{-1}R$ of R against elements in $S_G \subseteq \pi_0^G R$. In particular, we have that $\underline{\pi}_* S_G^{-1} R \cong S_G^{-1} \underline{\pi}_* R$.

On the other hand, since the former mapping space is corepresentable by presentability, we write the corepresenting objects as $L_{\underline{S}^{-1}} R$. In general, this need not be given by a nice formula in terms of a telescopic localisation since we need to invert different sets of elements at different subgroups $H \leq G$ that do not all come from restricting elements from S_G (ie. the inclusion $\mathrm{Res}^G_H S_G \subseteq S_H$ might be proper), and so $\underline{\pi}_* L_{\underline{S}^{-1}} R$ need not admit a nice description as a Mackey functor with elements in $\underline{S}$ inverted. However, since maps $R \to A$ which invert $\underline{S}$ must necessarily invert S_G, we do have an inclusion

$$\mathrm{Map}^{\underline{S}^{-1}}_{\mathrm{CAlg}(\mathrm{Sp}_G)}(R, -) \hookrightarrow \mathrm{Map}^{S_G^{-1}}_{\mathrm{CAlg}(\mathrm{Sp}_G)}(R, -)$$

Thus, this inclusion is induced by a canonical comparison map in $\mathrm{CAlg}(\mathrm{Sp}_G)$

$$S_G^{-1} R \longrightarrow L_{\underline{S}^{-1}} R. \tag{1}$$

Notation 1.2. We write $\mathrm{CMon}(\mathcal{S}_G)$ for the category of commutative monoid objects in G–spaces. An object $M \in \mathrm{CMon}(\mathcal{S}_G)$ consists of $\mathbf{E}_\infty$–monoids M^H for every subgroup $H \leq G$ and the restriction map $M^H \to M^K$ associated to a subconjugation $K \leq H$ is a map of $\mathbf{E}_\infty$–monoids. This implies that $\mathbb{S}_G[M] := \Sigma^\infty_{G,+} M \in \mathrm{CAlg}(\mathrm{Sp}_G)$ (see for instance Observation 2.1). We write $\underline{\pi}_M \subseteq \underline{\pi}_0 \mathbb{S}_G[M]$ for the image of the Hurewicz map on the equivariant homotopy groups $\underline{\pi}_0 M \to \pi_0 \Omega^\infty_G \mathbb{S}_G[M] = \underline{\pi}_0 \mathbb{S}_G[M]$. This is clearly a G–subset in the sense defined above.

For a more highly structured input, let $\mathrm{CMon}_G(\mathcal{S}_G)$ be the category of G–commutative monoid G–spaces. An object $M \in \mathrm{CMon}_G(\mathcal{S}_G)$ consists of the data above together with "equivariant addition" maps $\oplus_{H/K} \colon M^K \to M^H$ for every $K \leq H$ satisfying double–coset formulas. Put differently, $\mathrm{CMon}_G(\mathcal{S}_G)$ are G–*semi*–Mackey functors valued in spaces, to emphasise the semiadditivity - but not additivity - of the situation, following [CMN+20]. Furthermore, we will also write $\mathrm{CAlg}_G(\mathrm{Sp}_G)$ for the G–commutative ring objects in G–spectra, ie. those $\mathbf{E}_\infty$–rings equipped with multiplicative norms. More details on all these in §2.

We are now ready to state the theorem of this note:

Theorem 1.3. *Let $M \in \mathrm{CMon}(\mathcal{S}_G)$ be a commutative monoid G–space.*

(i) The group completion map $M \to \Omega BM$ induces an equivalence in $\mathrm{CAlg}(\mathrm{Sp}_G)$

$$L_{(\underline{\pi}_M)^{-1}}\mathbb{S}_G[M] \xrightarrow{\simeq} \mathbb{S}_G[\Omega BM]$$

(ii) Moreover, if M even had the structure of a G–commutative monoid G–space - ie. $M \in \mathrm{CMon}_G(\mathcal{S}_G)$ - then $\mathbb{S}_G[\Omega BM] \simeq L_{(\underline{\pi}_M)^{-1}}\mathbb{S}_G[M]$ refines to a G–commutative ring object. In other words, it lifts to an object in $\mathrm{CAlg}_G(\mathrm{Sp}_G)$. Furthermore, in this case, the canonical map from (1)

$$(\pi_M^G)^{-1}\mathbb{S}_G[M] \longrightarrow L_{(\underline{\pi}_M)^{-1}}\mathbb{S}_G[M] \simeq \mathbb{S}_G[\Omega BM]$$

is an equivalence so that we have the expected localisation effect on homotopy groups, ie. $\underline{\pi}_*\mathbb{S}_G[\Omega BM] \cong (\pi_M^G)^{-1}\underline{\pi}_*\mathbb{S}_G[M]$.

We emphasise again the point from Notation 1.1 that part (i) says nothing about how the homotopy Mackey functor of $\mathbb{S}_G[\Omega BM]$ looks like. It only says that $\mathbb{S}_G[\Omega BM]$ satisfies a universal property defined in terms of $\underline{\pi}_M \subseteq \underline{\pi}_0\mathbb{S}_G[M]$. We feel that the main force of the result lies in the highly structured setting of part (ii) where this result gives a formula for the homology of ΩBM in terms of that of M with respect to any Mackey functor coefficients, a commonly considered instance being Bredon homology. In this case for example, we obtain

$$H_*^G(\Omega BM; H\underline{\mathbb{Z}}) \cong H_*^G(M; H\underline{\mathbb{Z}})[(\pi_0^G M)^{-1}]$$

where $\underline{\mathbb{Z}}$ is the constant Mackey functor associated to the trivial G–action on $\mathbb{Z}$. In the body of the paper, however, we isolate a condition on $\underline{\pi}_M$ we call *torsion extension* (cf. Condition 3.1) which ensures that the different localisations from Notation 1.1 agree even in the absence of the norms. This might be usable and useful in specific cases of M. We expect this kind of result to be helpful in making equivariant computational analyses of group completions in the same way that the nonequivariant version is a computational staple.

Example 1.4. G–commutative monoid G–spaces, for which the localisation formula of Theorem 1.3 (ii) holds, are in abundant supply. One fertile source is small semiadditive ∞–categories equipped with G–actions, ie. objects in

$\mathrm{Fun}(BG, \mathrm{Cat}_\infty^\oplus)$. If $\mathcal{C}$ were one such instance, then $\{\mathcal{C}^{hH}\}_{H\leq G}$ assembles to a G–commutative monoid G–category. In other words, it is an object in $\mathrm{Mack}_G(\mathrm{Cat}_\infty^\oplus)$ (cf. [BGS20, §8] for an explanation of this). Then taking the groupoid core yields a G–commutative monoid G–space $\{(\mathcal{C}^{hH})^\simeq\}_{H\leq G} \in \mathrm{CMon}_G(\mathcal{S}_G)$. Concrete examples belonging to this template include equipping the trivial G–action on categories like finitely generated projective R–modules Proj_R for $R \in \mathrm{CRing}$ or perfect A–modules Perf_A for $A \in \mathrm{CAlg}(\mathrm{Sp})$. These yield the objects in $\mathrm{CMon}_G(\mathcal{S}_G)$

$$\{\mathrm{Map}(BH, \mathrm{Proj}_R^\simeq)\}_{H\leq G} \qquad \{\mathrm{Map}(BH, \mathrm{Perf}_A^\simeq)\}_{H\leq G}$$

the group completions of which give the so–called *Swan* equivariant K–theories. Another interesting source of semiadditive categories equipped with G–actions come from finite Galois extensions of fields $K \subseteq L$. In this case, the G–Galois action on $\mathrm{Vect}_L^{\mathrm{fd}}$ yields the G–commutative monoid G–space $\{(\mathrm{Vect}_{L^H}^{\mathrm{fd}})^\simeq\}_{H\leq G}$.

As far as we know, the theorem cannot be directly deduced from the classical group completion theorem. However, the method of proof will be a direct adaptation of that of Nikolaus via ring localisations in his [Nik17, Thm. 1]. The first part of the theorem will require only standard ∞–category theory, whereas in the more highly structured second part of Theorem 1.3 we will need the language of G–categories in order to discuss G–commutative rings succintly. While we were not able to find this equivariant group completion theorem written in the literature, it is probably known or at least expected among experts. We very much welcome a reference to where this result might have previously appeared and give the appropriate credits.

Lastly, a few words on organisation: we will briefly record some foundational materials in §2 to orient the reader who might not be familiar with the formalism of G–categories; in §3, we give a proof of Theorem 1.3. We will end with some remarks on how norms and localisations managed to interplay well in our situation and how this result fits in with the nonequivariant group completion theorem. Along the way, we will explain how geometric fixed points turn the mysterious localisation $L_{\underline{S}^{-1}}R$ into something familiar.

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2 Some preliminaries

We begin with the following observation, which requires no theory of G–categories:

Observation 2.1. The left adjoint in the adjunction

$$\mathcal{S}_G \underset{\Omega^\infty_G}{\overset{\mathbb{S}_G[-]}{\rightleftarrows}} \mathrm{Sp}_G$$

refines to a symmetric monoidal functor with the cartesian symmetric monoidal structure on $\mathcal{S}_G$ and the tensor product of G–spectra on Sp_G. Hence, applying the functor $\mathrm{CAlg}(-)$ we obtain an adjunction

$$\mathrm{CMon}(\mathcal{S}_G) \simeq \mathrm{CAlg}(\mathcal{S}_G^\times) \underset{\Omega^\infty_G}{\overset{\mathbb{S}_G[-]}{\rightleftarrows}} \mathrm{CAlg}(\mathrm{Sp}_G^\otimes) \tag{2}$$

Now, to set the stage for our discussions about the multiplicative norms, we collect here some basics on G–categories. The reader uninsterested in this refinement can skip right away to the proof of the first part of Theorem 1.3 in the next section.

We have chosen to travel light in this document and so we will refrain from giving a self–contained exposition of the required theory on G–categories. For the original sources of these materials, we refer the reader to [BDG+16; Nar17; Sha22], and a one–stop survey of G–categories can be found for example in [Hil22, Chap. 1]. In short, a G–category is an object in $\mathrm{Cat}_{\infty,G} := \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathrm{Cat}_\infty)$ and we will use the underline notation $\underline{\mathcal{D}}$ to denote a G–category and $\mathcal{D}_{G/H}$ for its value at $G/H \in \mathcal{O}_G^{\mathrm{op}}$. Important examples include the G–category of G–spaces $\{\underline{\mathcal{S}}_G \colon G/H \mapsto \mathcal{S}_H\}$ and of genuine G–spectra $\{\underline{\mathrm{Sp}}_G \colon G/H \mapsto \mathrm{Sp}_H\}$.

Lemma 2.2. *The G–adjunction $\mathbb{S}_G[-]\colon \underline{\mathcal{S}}_G \rightleftharpoons \underline{\mathrm{Sp}}_G : \Omega^\infty_G$ induces a G–adjunction*

$$\mathbb{S}_G[-]\colon \mathrm{CMon}_G(\underline{\mathcal{S}}_G) \rightleftarrows \mathrm{CAlg}_G(\underline{\mathrm{Sp}}_G) : \Omega^\infty_G$$

Proof. We know by [Nar17, §3] that the map $\mathbb{S}_G[-] := \Sigma^\infty_+$ refines to a G–symmetric monoidal functor $\mathbb{S}_G[-]\colon \underline{\mathcal{S}}^\times_G \longrightarrow \underline{\mathrm{Sp}}^\otimes_G$. This means that Ω^∞_G canonically refines to a G–lax symmetric monoidal structure. Hence applying CAlg_G on both sides and using that $\mathrm{CAlg}_G(\underline{\mathcal{S}}^\times_G) \simeq \mathrm{CMon}_G(\underline{\mathcal{S}}_G)$ we get the desired adjunction. □

Construction 2.3 (Equivariant group completions)**.** If we write B for the suspension in the category $\mathrm{CMon}(\mathcal{S})$ (which is very different from the suspension on the underlying space!), then we know that we have the adjunction

$$\mathrm{CMon}(\mathcal{S}) \underset{\longleftarrow}{\xrightarrow{\Omega B}} \mathrm{CGrp}(\mathcal{S})$$

so that ΩB implements the group completion functor on $\mathrm{CMon}(\mathcal{S})$. In fact, as slickly explained in [CDH+20, Prop. 3.3.5], ΩB *always* implements group completions in *any* semiadditive category with pullbacks and pushouts. We can then cofreely make this into a G–adjunction of G–categories

$$\underline{\mathrm{Cofree}}_G\big(\mathrm{CMon}(\mathcal{S})\big) \underset{\longleftarrow}{\xrightarrow{\Omega B}} \underline{\mathrm{Cofree}}_G\big(\mathrm{CGrp}(\mathcal{S})\big) \tag{3}$$

Here for an ordinary ∞–category $\mathcal{C}$, $\underline{\mathrm{Cofree}}_G(\mathcal{C}) \in \mathrm{Cat}_{\infty,G}$ is the G–category given by $\{G/H \mapsto \mathrm{Fun}(\mathcal{O}^{\mathrm{op}}_H, \mathcal{C})\}_{H\leq G}$ (cf. [Nar17, Def. 1.10] for instance, where the notation was just an underline instead of the word "cofree"). In this notation, the G–category of G–spaces is therefore given by $\underline{\mathcal{S}}_G = \underline{\mathrm{Cofree}}_G(\mathcal{S})$. Both G–categories in the adjunction are G–semiadditive, and so in particular ΩB preserves G–biproducts. Now by [Nar17, Thm. 2.32], if $\mathcal{C}$ admits finite limits, then

$$\mathrm{CMon}_G(\underline{\mathrm{Cofree}}_G(\mathcal{C})) \simeq \mathrm{Mack}_G(\mathcal{C}) \simeq \mathrm{Mack}_G(\mathrm{CMon}(\mathcal{C}))$$

Using this, we can then apply $\mathrm{CMon}_G(-)$ to the adjunction (3) to get a G–adjunction

$$\mathrm{CMon}_G(\underline{\mathcal{S}}_G) \underset{\longleftarrow}{\xrightarrow{\Omega B}} \mathrm{CGrp}_G(\underline{\mathcal{S}}_G)$$

Concretely, this implements group completion pointwise, and this is what we mean by the *equivariant group completion*.

For the next construction, we use the notation $\underline{\mathrm{Fin}}_{*G}$ for the G–category of finite pointed G–sets. That is, it is the G–category $\{G/H \mapsto \mathrm{Fin}_{*H} := \mathrm{Fun}(BH, \mathrm{Fin}_*)\}$. Nardin used this to give a definition of G–symmetric monoidal categories in [Nar17, §3] much like the nonequivariant situation from [Lur17]. See also [NS22] for a more recent treatment. Suffice to say, in this setting, a G–symmetric monoidal category is a G–category $\underline{\mathcal{D}}^{\otimes}$ equipped with a map to $\underline{\mathrm{Fin}}_{*G}$ satisfying appropriate cocartesianness and G–operadic conditions, and G–commutative ring objects are then G–inert sections to this map.

Construction 2.4 (Forgetting norms)**.** First note that we have an adjunction

$$i \colon * \rightleftharpoons \mathcal{O}_G^{\mathrm{op}} : p$$

where i is the inclusion of G/G, which is an initial object in $\mathcal{O}_G^{\mathrm{op}}$. Hence, applying $\mathrm{Fun}(-, \mathrm{Cat}_\infty)$ we obtain an adjunction of $(\infty, 2)$–categories

$$p^* \colon \mathrm{Cat}_\infty \rightleftharpoons \mathrm{Cat}_{\infty,G} : i^*$$

Since $p^*(\mathcal{C}) = \underline{\mathrm{const}}_G(\mathcal{C}) := \mathcal{C} \times \mathcal{O}_G^{\mathrm{op}}$ and $i^*(\underline{\mathcal{D}}) = \mathcal{D}_{G/G} := \mathrm{ev}_{G/G}\underline{\mathcal{D}}$, we see that

$$\mathrm{Fun}_G\left(\underline{\mathrm{const}}_G(\mathcal{C}), \underline{\mathcal{D}}\right) \simeq \mathrm{Fun}\left(\mathcal{C}, \mathcal{D}_{G/G}\right) \tag{4}$$

In particular, the fully faithful functor of 1–categories $\mathrm{Fin}_* \to \mathrm{Fin}_{*G} := \mathrm{Fun}(BG, \mathrm{Fin}_*)$ given by $n \mapsto \coprod^n G/G$ induces a G–functor

$$q \colon \underline{\mathrm{const}}_G(\mathrm{Fin}_*) \longrightarrow \underline{\mathrm{Fin}}_{*G} \quad :: \quad (n, G/H) \mapsto \coprod^n H/H$$

Therefore, for a G–symmetric monoidal category $\underline{\mathcal{D}}^{\otimes} \in \mathrm{CMon}_G(\underline{\mathrm{Cat}}_{\infty,G})$, using the adjunction (4) again, we obtain the map

$$\mathrm{CAlg}_G(\underline{\mathcal{D}}^{\otimes}) := \mathrm{Fun}_G^{\underline{\mathrm{inert}}}(\underline{\mathrm{Fin}}_{*G}, \underline{\mathcal{D}}^{\otimes}) \xrightarrow{q^*} \mathrm{Fun}^{\mathrm{inert}}(\mathrm{Fin}_*, \mathcal{D}^{\otimes}_{G/G}) =: \mathrm{CAlg}(\mathcal{D}^{\otimes}_G)$$

Here $\mathrm{Fun}_G^{\underline{\mathrm{inert}}}$ denotes the G–inert sections of $\underline{\mathcal{D}}^{\otimes} \to \underline{\mathrm{Fin}}_{*G}$ and similarly for $\mathrm{Fun}^{\mathrm{inert}}$. We have used that $\mathrm{Fun}_G(\underline{\mathrm{const}}_G(\mathrm{Fin}_*), \underline{\mathcal{D}}^{\otimes}) \times_{\mathrm{Fun}_G(\underline{\mathrm{const}}_G(\mathrm{Fin}_*), \underline{\mathrm{Fin}}_{*G})} \{q\}$ is equivalent to $\mathrm{Fun}(\mathrm{Fin}_*, \mathcal{D}^{\otimes}_{G/G}) \times_{\mathrm{Fun}(\mathrm{Fin}_*, \mathrm{Fin}_{*G})} \{q\}$ by (4) to analyse the target. Intuitively, the functor q^* forgets the norm structures on a G–commutative ring object in $\underline{\mathcal{D}}$ and so we will also denote it by fgt in the sequel.

3 Proof of the theorem

Armed with the preliminaries, we can now give the proof of the theorem. We break it up into parts (i) and (ii). We emphasise again that the theory of G–categories is not required in the first part.

Proof of Theorem 1.3 (i). The proof is exactly the same as that of [Nik17, Thm. 1]. To wit, we first claim that $\Omega BM \in \mathrm{CMon}(\mathcal{S}_G)$ satisfies the following universal property: for every $X \in \mathrm{CMon}(\mathcal{S}_G)$, the map

$$\mathrm{Map}_{\mathrm{CMon}(\mathcal{S}_G)}(\Omega BM, X) \to \mathrm{Map}^{(\underline{\pi}_0 M)^{-1}}_{\mathrm{CMon}(\mathcal{S}_G)}(M, X)$$

induced by $\eta\colon M \to \Omega BM$ is an equivalence, where $\mathrm{Map}^{(\underline{\pi}_0 M)^{-1}}_{\mathrm{CMon}(\mathcal{S}_G)} \subseteq \mathrm{Map}_{\mathrm{CMon}(\mathcal{S}_G)}$ means the subcomponents of maps where $\underline{\pi}_0 M$ is sent to elements that admit additive inverses in $\underline{\pi}_0 X$. The map lands in this subcomponent since ΩBM is group complete. To prove the claim, define $X^\times$ as the pullback in $\mathrm{CMon}(\mathcal{S}_G)$

$$\begin{array}{ccc} X^\times & \xrightarrow{i} & X \\ \downarrow & \lrcorner & \downarrow \\ (\underline{\pi}_0 X)^\times & \hookrightarrow & \underline{\pi}_0 X \end{array}$$

Now consider the commuting diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{CMon}(\mathcal{S}_G)}(\Omega BM, X) & \xrightarrow{\eta^*} & \mathrm{Map}^{(\underline{\pi}_0 M)^{-1}}_{\mathrm{CMon}(\mathcal{S}_G)}(M, X) \\ {\scriptstyle i_*}\uparrow & & \uparrow{\scriptstyle i_*} \\ \mathrm{Map}_{\mathrm{CMon}(\mathcal{S}_G)}(\Omega BM, X^\times) & \xrightarrow[\eta^*]{} & \mathrm{Map}_{\mathrm{CMon}(\mathcal{S}_G)}(M, X^\times) \end{array}$$

The left vertical i_* is an equivalence since ΩBM is group complete and $(-)^\times$ is the right adjoint to the inclusion $\mathrm{CGrp}(\mathcal{S}_G) \subseteq \mathrm{CMon}(\mathcal{S}_G)$; the bottom η^* is an equivalence since $X^\times$ is group complete and ΩBM is the group completion of M by Construction 2.3; the right vertical i_* is an equivalence because maps in $\mathrm{Map}^{(\underline{\pi}_0 M)^{-1}}$ are precisely those that land in $X^\times$ by definition. Therefore, all in all, the top horizontal η^* is also an equivalence, as claimed.

Now by the adjunction (2), for any $A \in \mathrm{CAlg}(\mathrm{Sp}_G)$, we have

$$\begin{aligned} \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Sp}_G)}(\mathbb{S}_G[\Omega BM], A) &\simeq \mathrm{Map}_{\mathrm{CMon}(\mathcal{S}_G)}(\Omega BM, \Omega^\infty_G A) \\ &\simeq \mathrm{Map}^{(\underline{\pi}_0 M)^{-1}}_{\mathrm{CMon}(\mathcal{S}_G)}(M, \Omega^\infty_G A) \\ &\simeq \mathrm{Map}^{(\underline{\pi}_M)^{-1}}_{\mathrm{CAlg}(\mathrm{Sp}_G)}(\mathbb{S}_G[M], A) \end{aligned} \tag{5}$$

where the second equivalence is by the claim above. By Notation 1.1, $\mathbb{S}_G[\Omega BM]$ computes $L_{(\underline{\pi}_M)^{-1}}\mathbb{S}_G[M]$, as desired. □

We now turn to the task of refining to normed structures when the input is more highly structured, ie. when $M \in \mathrm{CMon}_G(\underline{\mathcal{S}}_G)$. Before that, it would be useful to formulate the following intermediate notion together with a couple of easy consequences which would help us identify the homotopy groups of the abstract localisation we have so far.

Condition 3.1 (Torsion extensions)**.** Let $R \in \mathrm{CAlg}(\mathrm{Sp}_G)$ and $\underline{S} \subseteq \underline{\pi}_0 R$ be a G–subset of the zeroth equivariant homotopy groups of R. We say that $\underline{S}$ satisfies the *torsion extension condition* if for any $H \leq G$, the inclusion $\mathrm{Res}^G_H S_G \subseteq S_H$ is a torsion extension, ie. for any $a \in S_H$, there exists a $r \in \pi_0^H R$ such that $r \cdot a \in \mathrm{Res}^G_H S_G$.

Remark 3.2. Since this is just an intermediate notion, we have not invested too much time in choosing a satisfying name. The reason for this choice was an analogy in the case of modules: if $I \subseteq J \subseteq R$ are R–submodules satisfying the analogous condition, then J/I is a torsion R–module. In any case, the next three lemmas should clarify our interest in this condition.

Lemma 3.3. *If $R \in \mathrm{CAlg}(\mathrm{Sp}_G)$ and $\underline{S} \subseteq \underline{\pi}_0 R$ is a multiplicatively closed G–subset satisfying Condition 3.1, then the canonical map $S_G^{-1}R \longrightarrow L_{\underline{S}^{-1}}R$ from (1) is an equivalence.*

Proof. Fix $H \leq G$ and $A \in \mathrm{CAlg}(\mathrm{Sp}_H)$. As explained in Notation 1.1, the canonical map induces an inclusion of subcomponents

$$\mathrm{Map}^{\underline{S}^{-1}}_{\mathrm{CAlg}(\mathrm{Sp}_H)}(\mathrm{Res}^G_H R, A) \hookrightarrow \mathrm{Map}^{(\mathrm{Res}^G_H S_G)^{-1}}_{\mathrm{CAlg}(\mathrm{Sp}_H)}(\mathrm{Res}^G_H R, A)$$

Hence all we have to do is to show that all components in the target are hit. So suppose $\varphi\colon \mathrm{Res}^G_H R \to A$ inverts elements in $\mathrm{Res}^G_H S_G$ and let $a \in S_H$. By hypothesis, there exists a $r \in \pi_0^H R$ such that $r \cdot a \in \operatorname{Res}^G_H S_G$. Since φ inverts $r \cdot a$, it must have inverted a too. Therefore, since a was arbitrary, we see that φ must have inverted all of S_H as required. ☐

Lemma 3.4. *Let $R \in \mathrm{CAlg}_{\underline{G}}(\underline{\mathrm{Sp}}_G)$ be a G–commutative ring object and $\underline{S} \subseteq \underline{\pi}_0 R$ be a G–subset that is closed under the norms. Then $\underline{S}$ satisfies Condition 3.1.*

Proof. Fix $H \leq G$ and let $a \in S_H$. We want to show that there is an $r \in \pi_0^H R$ such that $r \cdot a \in \operatorname{Res}^G_H S_G$. For this, consider $\mathrm{N}^G_H a \in \pi_0^G R$ which is in fact in $S_G \subseteq \pi_0^G R$ by hypothesis. Then by the norm double coset formula, we get

$$\operatorname{Res}^G_H \mathrm{N}^G_H a = \prod_{g \in H \backslash G / H} \mathrm{N}^H_{H^g \cap H} g_* \operatorname{Res}^H_{H \cap H^g} a \in \operatorname{Res}^G_H S_G$$

where a is a factor on the right (ie. when $g = e$), whence the claim. ☐

Lemma 3.5. *If $M \in \mathrm{CMon}_{\underline{G}}(\underline{\mathcal{S}}_G)$ is a G–commutative monoid G–space, then $\underline{\pi}_M \subseteq \underline{\pi}_0 \mathbb{S}_G[M]$ is closed under the norms.*

Proof. First of all, by Lemma 2.2 we know $\mathbb{S}_G[M]$ refines to a G–commutative ring object. Now fix $H \leq G$ and suppose we have $n \in \pi_0^H M$ with associated element $\overline{n} \in \pi_0^H \mathbb{S}_G[M]$. Thus by definition the normed element $\mathrm{N}^G_H \overline{n} \in \pi_0^G \mathbb{S}_G[M]$ is given by

$$\mathbb{S}_G = \mathrm{N}^G_H \mathbb{S}_H \xrightarrow{\mathrm{N}^G_H \overline{n}} \mathrm{N}^G_H \operatorname{Res}^G_H \mathbb{S}_G[M] \simeq \mathbb{S}_G[\prod_{G/H} \operatorname{Res}^G_H M] \xrightarrow{\mathbb{S}[\oplus_{G/H}]} \mathbb{S}_G[M]$$

Here $\oplus_{G/H} \colon \prod_{G/H} \operatorname{Res}^G_H M \to M$ is the G–semiadditivity adjunction counit of an object $M \in \mathrm{CMon}_{\underline{G}}(\underline{\mathcal{S}}_G)$. The middle equivalence is since $\mathbb{S}_G[\prod^G_H -] \simeq \mathrm{N}^G_H \mathbb{S}_H[-]$ from the G–symmetric monoidality of the functor $\mathbb{S}_G[-]$ from Lemma 2.2

Now, the natural transformation $(-) \Rightarrow \Omega^\infty_G \mathbb{S}_G(-)$ from Lemma 2.2 together with the adjunction counit $\prod^G_H \operatorname{Res}^G_H M \xrightarrow{\oplus_{G/H}} M$ yield the commuting diagram

$$\begin{array}{ccc}
* & \longrightarrow & \Omega^\infty_G \mathbb{S}_G \\
{\scriptstyle n}\downarrow & & \downarrow{\scriptstyle \Omega^\infty_G \mathrm{N}^G_H \overline{n}} \\
\prod^G_H \operatorname{Res}^G_H M & \longrightarrow & \Omega^\infty_G \mathbb{S}_G[\prod^G_H \operatorname{Res}^G_H M] \simeq \Omega^\infty_G \mathrm{N}^G_H \operatorname{Res}^G_H \mathbb{S}_G[M] \\
{\scriptstyle \oplus_{G/H}}\downarrow & & \downarrow{\scriptstyle \Omega^\infty_G \mathbb{S}_G[\oplus_{G/H}]} \\
M & \longrightarrow & \Omega^\infty_G \mathbb{S}_G[M]
\end{array}$$

This implies that the normed up element $\mathrm{N}_H^G\overline{n} \in \pi_0^G\mathbb{S}_G[M]$ already came from the element $\oplus_{G/H}n \in \pi_0^G M$ and so $\underline{\pi}_M \subseteq \underline{\pi}_0\mathbb{S}_G[M]$ is closed under norms. □

We now cash in all the work we have done to complete the proof of the theorem.

Proof of Theorem 1.3 (ii). The main point is that we have commuting squares

$$\begin{array}{ccc} \mathrm{CMon}_G(\underline{\mathcal{S}}_G) & \underset{\Omega^\infty_G}{\overset{\mathbb{S}_G[-]}{\rightleftarrows}} & \mathrm{CAlg}_G(\underline{\mathrm{Sp}}^{\otimes}_G) \\ \downarrow{\scriptstyle \mathrm{fgt}} & & \downarrow{\scriptstyle \mathrm{fgt}} \\ \mathrm{CMon}(\mathcal{S}_G) & \underset{\Omega^\infty_G}{\overset{\mathbb{S}_G[-]}{\rightleftarrows}} & \mathrm{CAlg}(\mathrm{Sp}^{\otimes}_G) \end{array}$$

gotten by using the G–lax symmetric monoidality of the adjunction $\mathbb{S}_G[-] \dashv \Omega^\infty_G$ from Lemma 2.2 together with the fact that the forgetful functor is implemented by precomposition by Construction 2.4. In fact, for this proof we only need that the $(\mathbb{S}_G[-], \mathrm{fgt})$ square commutes. Thus, if $M \in \mathrm{CMon}_G(\underline{\mathcal{S}}_G)$ so that ΩBM is again in $\mathrm{CMon}_G(\underline{\mathcal{S}}_G)$ by Construction 2.3, then $\mathbb{S}_G[\Omega BM]$ - which is equivalent to $L_{(\underline{\pi}_M)^{-1}}\mathbb{S}_G[M]$ by part (i) of the theorem - naturally refines to the structure of an object in $\mathrm{CAlg}_G(\underline{\mathrm{Sp}}^{\otimes}_G)$, ie. it canonically attains the multiplicative norms. The final statement of part (ii) is then a direct combination of Lemmas 3.3 to 3.5. □

Remark 3.6. The norm closure of the subset $\underline{\pi}_M \subseteq \underline{\pi}_0\mathbb{S}_G[M]$ from Lemma 3.5 should have indicated why the localisation $(\underline{\pi}_M)^{-1}\mathbb{S}_G[M]$ even had a chance of attaining the multiplicative norms. In general, a localisation on a G–commutative ring need not refine again to a G–commutative ring, as is well documented for instance in [HH13]. Nonetheless, the norm closure of a multiplicative subset is a necessary and sufficient condition for the localisation to refine to the structure of a G–commutative ring. This can be deduced for example from [QS22, Lem. 5.27].

As we have remarked in Notation 1.1, the abstract localisation $L_{\underline{S}^{-1}}R$ has no reason to have a nice description in general. Notwithstanding, it does interact well with the geometric fixed points, and we end this note with some explanations regarding this as well as how this fits Theorem 1.3 with the classical group completion theorem.

Observation 3.7. Let $\underline{S} \subseteq \underline{\pi}_0 R$ be a multiplicative G–subset for some $R \in \mathrm{CAlg}(\mathrm{Sp}_G)$. Recall that we have a lax symmetric monoidal Bousfield localisation $\Phi^G \colon \mathrm{Sp}_G \rightleftharpoons \mathrm{Sp} : \Xi^G$ which then induces a Bousfield localisation $\Phi^G \colon \mathrm{CAlg}(\mathrm{Sp}_G) \rightleftharpoons \mathrm{CAlg}(\mathrm{Sp}) : \Xi^G$. Here for $X \in \mathrm{Sp}$, $\Xi^G X$ is the G–spectrum such that $(\Xi^G X)^G \simeq X$ and $(\Xi^G X)^H \simeq 0$ for $H \lneq G$. Classically, this is also written as $\widetilde{E\mathcal{P}} \otimes X$ where $\mathcal{P}$ is the proper family of subgroups of G. We claim that the resulting equivalence $\mathrm{Map}_{\mathrm{CAlg}(\mathrm{Sp}_G)}(R, \Xi^G A) \simeq \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Sp})}(\Phi^G R, A)$ restricts to an equivalence

$$\mathrm{Map}^{\underline{S}^{-1}}_{\mathrm{CAlg}(\mathrm{Sp}_G)}(R, \Xi^G A) \simeq \mathrm{Map}^{(\Phi^G S_G)^{-1}}_{\mathrm{CAlg}(\mathrm{Sp})}(\Phi^G R, A)$$

To see this, since $\Phi^G \Xi^G \simeq \mathrm{id}$, we know Φ^G induces an inclusion

$$\mathrm{Map}^{\underline{S}^{-1}}_{\mathrm{CAlg}(\mathrm{Sp}_G)}(R, \Xi^G A) \hookrightarrow \mathrm{Map}^{(\Phi^G S_G)^{-1}}_{\mathrm{CAlg}(\mathrm{Sp})}(\Phi^G R, A)$$

To see that this is even an equivalence, suppose we have $\varphi \colon \Phi^G R \to A$ which inverts $\Phi^G S_G \subseteq \pi_0 \Phi^G R$. The adjoint $\overline{\varphi} \colon R \to \Xi^G A$ is given by the composite

$$\overline{\varphi} \colon R \xrightarrow{\eta} \Xi^G \Phi^G R \xrightarrow{\Xi^G \varphi} \Xi^G A$$

where the adjunction unit η is a map of $\mathbf{E}_\infty$–rings and sends elements in S_G to elements in $\Phi^G S_G$. Therefore, $\overline{\varphi}$ must invert all elements in S_G. Moreover, since for $H \lneq G$, $(\Xi^G A)^H$ are equivalent to the zero rings, the maps $\mathrm{Res}^G_H \overline{\varphi} \colon \mathrm{Res}^G_H R \to \mathrm{Res}^G_H \Xi^G A \simeq 0$ send everything to units for trivial reasons, and so in total $\overline{\varphi}$ indeed inverts elements in $\underline{S}$ as was to be shown.

Proposition 3.8. *Let $R \in \mathrm{CAlg}(\mathrm{Sp}_G)$ and let $\underline{S} \subseteq \underline{\pi}_0 R$ be a multiplicative subset. Then the canonical map $\Phi^G R \to \Phi^G L_{\underline{S}^{-1}} R$ induces an equivalence $(\Phi^G S_G)^{-1} \Phi^G R \simeq \Phi^G L_{\underline{S}^{-1}} R$.*

Proof. Let $A \in \mathrm{CAlg}(\mathrm{Sp})$. Then

$$\begin{aligned}
\mathrm{Map}_{\mathrm{CAlg}(\mathrm{Sp})}(\Phi^G L_{\underline{S}^{-1}} R, A) &\simeq \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Sp}_G)}(L_{\underline{S}^{-1}} R, \Xi^G A) \\
&\simeq \mathrm{Map}^{\underline{S}^{-1}}_{\mathrm{CAlg}(\mathrm{Sp}_G)}(R, \Xi^G A) \\
&\simeq \mathrm{Map}^{(\Phi^G S_G)^{-1}}_{\mathrm{CAlg}(\mathrm{Sp})}(\Phi^G R, A) \\
&\simeq \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Sp})}((\Phi^G S_G)^{-1} \Phi^G R, A)
\end{aligned}$$

where the third equivalence is by Observation 3.7. ☐

Remark 3.9. Let $M \in \mathrm{CMon}(\mathcal{S}_G)$. We claim that $\Phi^G \pi^G_M = \pi_{M^G} \subseteq \pi_0 \mathbb{S}[M^G]$ where π_{M^G} is the image of the nonequivariant Hurewicz map $\pi_0 M^G \to \pi_0 \mathbb{S}[M^G]$. Given this, we see by Proposition 3.8 that

$$\Phi^G(\underline{\pi}_M^{-1} \mathbb{S}_G[M]) \simeq (\pi_{M^G})^{-1} \mathbb{S}[M^G]$$

and so applying Φ^G reduces Theorem 1.3 (i) to the classical group completion theorem formulated for example in [Nik17, Thm. 1]. To prove the claim, we want to show that the inclusion $\Phi^G \pi^G_M \subseteq \pi_{M^G}$ is surjective. We know that there is a commuting square

$$\begin{array}{ccc} \mathcal{S}_G & \xrightarrow{\mathbb{S}_G[-]} & \mathrm{Sp}_G \\ {\scriptstyle (-)^G}\downarrow & & \downarrow{\scriptstyle \Phi^G} \\ \mathcal{S} & \xrightarrow{\mathbb{S}} & \mathrm{Sp} \end{array}$$

which yields a commuting square

$$\begin{array}{ccc} \pi_0^G M = \pi_0 \mathrm{Map}_{\mathcal{S}_G}(*, M) & \xrightarrow{\mathbb{S}_G[-]} & \pi_0^G \mathbb{S}_G[M] = \pi_0 \mathrm{Map}_{\mathrm{Mod}_{\mathrm{Sp}_G}(\mathbb{S}_G[M])}(\mathbb{S}_G[M], \mathbb{S}_G[M]) \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \Phi^G} \\ \pi_0(M^G) = \pi_0 \mathrm{Map}_{\mathcal{S}}(*, M^G) & \xrightarrow{\mathbb{S}[-]} & \pi_0 \mathbb{S}[M^G] = \pi_0 \mathrm{Map}_{\mathrm{Mod}_{\mathrm{Sp}}(\mathbb{S}[M^G])}(\mathbb{S}[M^G], \mathbb{S}[M^G]) \end{array}$$

This implies that $\Phi^G \pi^G_M \subseteq \pi_{M^G}$ is surjective, as desired.

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