Supplemental material for:
Topological superconductivity in quantum wires mediated by helical magnons

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I. DIAGONALIZING THE MAGNON HAMILTONIAN

We here discuss the procedure to derive and diagonalize the magnon Hamiltonian of a general non-colinear spin structure.

Defining the local spin axes

To write the spin operators in terms of bosonic operators, we first need to define the local spin axes. This is achieved by starting from a ferromagnetic configuration, where all spins are directed along the positive z-azis, and rotating the local spin axes at site i to the local spin direction $\mathbf{S}_i$. Following Refs. [1, 2] we can write the relation between the Cartesian frame and local spin axes as a rotation $\mathbf{r}_\alpha = R^i_{\alpha\beta}\mathbf{f}^i_\beta$, where $\mathbf{f}^i_\alpha$ are the local spin axes at site i with $\mathbf{f}^i_3 = \mathbf{S}_i$. We assume that an orthonormal coordinate system $\mathbf{e}_\alpha$ defining the spin spiral state can be written as

$$
\begin{aligned}
\mathbf{e}_1 &= (\cos\alpha\cos\beta, \cos\alpha\sin\beta, -\sin\alpha) \\
\mathbf{e}_2 &= (-\sin\beta, \cos\beta, 0) \\
\mathbf{e}_3 &= (\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\alpha)
\end{aligned} \tag{1}
$$

in the Cartesian basis. The the spin at site i can then be written as

$$
\begin{aligned}
\mathbf{S}_i &= \cos(\mathbf{q}\cdot\mathbf{r}_i)\mathbf{e}_1 + \sin(\mathbf{q}\cdot\mathbf{r}_i)\mathbf{e}_2 \\
&= \cos(\mathbf{q}\cdot\mathbf{r}_i)\begin{pmatrix}\cos\alpha\cos\beta \\ \cos\alpha\sin\beta \\ -\sin\alpha\end{pmatrix} + \sin(\mathbf{q}\cdot\mathbf{r}_i)\begin{pmatrix}-\sin\beta \\ \cos\beta \\ 0\end{pmatrix}.
\end{aligned} \tag{2}
$$

This equation implicitly defines the local spin axis via the relation

$$
\mathbf{f}^i_3 = \mathbf{S}_i = \begin{pmatrix}\sin\theta_i\cos\phi_i \\ \sin\theta_i\sin\phi_i \\ \cos\theta_i\end{pmatrix}, \tag{3}
$$

which allows to solve for θ_i and ϕ_i in terms of α, β and $(\mathbf{q}\cdot\mathbf{r}_i)$. The rotation matrix relating the Cartesian and

local spin axes is then

$$R^i_{\alpha\beta} = \begin{pmatrix} \cos\theta_i\cos\phi_i & -\sin\phi_i & \sin\theta_i\cos\phi_i \\ \cos\theta_i\sin\phi_i & \cos\phi_i & \sin\theta_i\sin\phi_i \\ -\sin\theta_i & 0 & \cos\theta_i \end{pmatrix}. \quad (4)$$

In the particular case discussed in the main text, where the propagation vector of the spiral lies along the wire axis ($\mathbf{q} = \hat{\mathbf{z}}$), we have $\alpha = \beta = 0$ if the spiral is in the xy-plane, $\alpha = \beta = -\pi/2$ if the spiral is in the xz-plane, and $\alpha = \pi/2$ and $\beta = 0$ if the spiral is in the yz-plane. This in turn gives the rotation angles $\theta_i = \pi/2$ and $\phi_i = \mathbf{q}\cdot\mathbf{r}_i$, $\theta_i = \mathbf{q}\cdot\mathbf{r}_i$ and $\phi_i = 0$, or $\theta_i = \mathbf{q}\cdot\mathbf{r}_i$ and $\phi_i = \pi/2$ in the respective cases.

The rotation matrix in each of the above cases simplifies to

$$R^i = \begin{pmatrix} 0 & -\sin\mathbf{q}\cdot\mathbf{r}_i & \cos\mathbf{q}\cdot\mathbf{r}_i \\ 0 & \cos\mathbf{q}\cdot\mathbf{r}_i & \sin\mathbf{q}\cdot\mathbf{r}_i \\ -1 & 0 & 0 \end{pmatrix} \quad (xy\text{-plane}) \quad (5)$$

$$R^i = \begin{pmatrix} \cos\mathbf{q}\cdot\mathbf{r}_i & 0 & \sin\mathbf{q}\cdot\mathbf{r}_i \\ 0 & 1 & 0 \\ -\sin\mathbf{q}\cdot\mathbf{r}_i & 0 & \cos\mathbf{q}\cdot\mathbf{r}_i \end{pmatrix} \quad (xz\text{-plane}) \quad (6)$$

$$R^i = \begin{pmatrix} 0 & -1 & 0 \\ \cos\mathbf{q}\cdot\mathbf{r}_i & 0 & \sin\mathbf{q}\cdot\mathbf{r}_i \\ -\sin\mathbf{q}\cdot\mathbf{r}_i & 0 & \cos\mathbf{q}\cdot\mathbf{r}_i \end{pmatrix}. \quad (yz\text{-plane}) \quad (7)$$

Expanding the Hamiltonian in bosonic operators

Having defined the local spin axes, our goal is now to obtain the low-energy excitations of the magnetic system. This is achieved by performing a Holstein-Primakoff expansion of the spin operators around the local spin axis $\mathbf{S}_i$. In terms of the rotation matrices $R^i_{\alpha\beta}$, the spin Hamiltonian is given by

$$H_s = -J\sum_{\langle ij\rangle\alpha}(R^i_{\alpha\beta}\hat{S}_{i\beta})(R^j_{\alpha\gamma}\hat{S}_{j\gamma}) \quad (8)$$
$$-\sum_{\langle ij\rangle}\epsilon_{\alpha\beta\gamma}D_{ij\alpha}(R^i_{\beta\delta}\hat{S}_{i\delta})(R^j_{\gamma\epsilon}\hat{S}_{j\epsilon}),$$

where $S_{i\alpha} = \mathbf{S}_i\cdot\mathbf{f}^i_\alpha$ is the component of the spin along the local α-axis.

The Holstein-Primakoff transformation is given to leading order in S^{-1} by

$$S_{i1} = \sqrt{S/2}(a^\dagger_i + a_i) \quad (9)$$
$$S_{i2} = i\sqrt{S/2}(a^\dagger_i - a_i)$$
$$S_{i3} = S - a^\dagger_i a_i.$$

Inserting this expansion into the spin Hamiltonian, and using that $\mathbf{f}^i_\alpha$ are columns of R_i, we find

$$H_s = \sum_{\langle ij\rangle}\left(b^d_{ij}a^\dagger_i a_i - \frac{1}{2}b^a_{ij}a^\dagger_i a^\dagger_j - \frac{1}{2}b^f_{ij}a^\dagger_i a_j + H.c.\right)$$
$$b^d_{ij} = JS(\mathbf{f}^i_3\cdot\mathbf{f}^j_3) + S\mathbf{D}_{ij}\cdot(\mathbf{f}^i_3\times\mathbf{f}^j_3) \quad (10)$$
$$b^a_{ij} = JS(\mathbf{f}^i_+\cdot\mathbf{f}^j_+) + S\mathbf{D}_{ij}\cdot(\mathbf{f}^i_+\times\mathbf{f}^j_+)$$
$$b^f_{ij} = JS(\mathbf{f}^i_+\cdot\mathbf{f}^j_-) + S\mathbf{D}_{ij}\cdot(\mathbf{f}^i_+\times\mathbf{f}^j_-)$$

Here we have defined the vectors $\mathbf{f}^i_\pm = \mathbf{f}^i_1 \pm i\mathbf{f}^i_2$ to simplify the notation.

Transforming to momentum space

To write the Hamiltonian in momentum space, we note that H_s is periodic in the magnetic unit cell (mUC) of length La, where a is the lattice constant of the electronic system. This implies that we can write $r_i = r_l + R_n$ and break the sum over lattice sites i into a sum over sites l within the mUC, and a sum over vectors R_n connecting different mUC's. Similarly, we let the momentum p run over the magnetic Brillouin zone (mBZ) $[-\pi/La, \pi/La]$. We can now write the Fourier transformed operators as

$$a_i = \frac{1}{\sqrt{N_m}}\sum_p e^{ipr_i}a_p, \quad (11)$$

and the first term of the Hamiltonian becomes

$$\sum_{\langle ij\rangle}b^d_{ij}a^\dagger_i a_i = \sum_{pp'n}\sum_{l\langle l'\rangle}b^d_{ll'}e^{i(p'-p)(r_l+R_n)}a^\dagger_{pl}a_{p'l}$$
$$= \sum_{pl}\left(\sum_{\langle l'\rangle}b^d_{ll'}\right)a^\dagger_{pl}a_{pl}. \quad (12)$$

Similarly, the second and the third terms of the Hamiltonian become

$$\sum_{\langle ij\rangle}b^a_{ij}a^\dagger_i a^\dagger_j = \sum_{pp'n}\sum_{l\langle l'\rangle}b^a_{ll'}e^{-i(p'+p)(r_l+R_n)}e^{-ip'\delta_{ll'}}a^\dagger_{pl}a^\dagger_{p'l'}$$
$$= \sum_{pl}\sum_{\langle l'\rangle}b^a_{ll'}e^{ip\delta_{ll'}}a^\dagger_{pl}a^\dagger_{-pl'} \quad (13)$$

$$\sum_{\langle ij\rangle}b^f_{ij}a^\dagger_i a_j = \sum_{pp'n}\sum_{l\langle l'\rangle}b^f_{ll'}e^{i(p'-p)(r_l+R_n)}e^{ip'\delta_{ll'}}a^\dagger_{pl}a_{p'l'}$$
$$= \sum_{pl}\sum_{\langle l'\rangle}b^f_{ll'}e^{ip\delta_{ll'}}a^\dagger_{pl}a_{pl'} \quad (14)$$

Putting the pieces together we have

$$H_s = \sum_{pl}\sum_{\langle l'\rangle}\left(b^d_{ll'}a^\dagger_{pl}a_{pl} - \frac{1}{2}b^a_{ll'}e^{ip\delta_{ll'}}a^\dagger_{pl}a^\dagger_{-pl'} \quad (15)\right.$$
$$\left. - \frac{1}{2}b^f_{ll'}e^{ip\delta_{ll'}}a^\dagger_{pl}a_{pl'} + H.c.\right)$$

Thus, for a spin spiral of wavelength La, we have a Bogoliubov tight-binding Hamiltonian of size $2L \times 2L$ to

solve for each momentum p. Due to the degeneracy of the Bogoliubov formalism, this gives rise to L physical magnon bands with positive energies.

Diagonalizing the magnon Hamiltonian

To diagonalize the magnon Hamiltonian, we write it in block form as

$$H_s = \sum_p \Phi_p^\dagger \mathcal{H}_{sp} \Phi_p, \tag{16}$$

where $\Phi_p^\dagger = (a_{p1}^\dagger \, a_{p2}^\dagger \, \ldots \, a_{pL}^\dagger \, a_{-p1} \, a_{-p2} \, \ldots \, a_{-pL})$ and the Hamiltonian matrix at momentum p is

$$\mathcal{H}_{sp} = \begin{pmatrix} \mathcal{H}_{p1} & \mathcal{H}_{-p2}^* \\ \mathcal{H}_{p2} & \mathcal{H}_{-p1}^* \end{pmatrix}. \tag{17}$$

Compared to the fermionic analog, additional complications arise when diagonalizing quadratic boson Hamiltonians of the Bogoliubov form [3], stemming from the fact that care must be taken to preserve the bosonic commutation relations. This in particular means that the magnon Hamiltonian cannot be diagonalized by a standard unitary transformation, but instead by a para-unitary matrix satisfying $T_p^{-1} = \tau_z T_p^\dagger \tau_z$ with τ_z the third Pauli matrix in Nambu space. A general procedure to diagonalize such Hamiltonians is described in detail in Ref. [3].

In brief, the procedure is as follows: (1) Write the Hamiltonian $\mathcal{H}_{sp}$ on the form $\mathcal{H}_{sp} = K_p^\dagger K_p$, where K_p is an Hermitian upper diagonal matrix. Numerically this step can be achieved by a Cholesky factorization. (2) Diagonalize the Hermitian matrix $K_p \tau_z K_p^\dagger$ by some standard numerical method. (3) Order the $2L$ energy eigenvalues from largest to smallest in a vector $E_p = (\epsilon_{Lp}, \epsilon_{L-1,p}, \ldots, \epsilon_{1p}, -\epsilon_{1p}, -\epsilon_{2p}, \ldots, -\epsilon_{Lp})$, and store the corresponding $2L$ eigenvectors $\mathbf{w}_{ip}$ as columns of a matrix W_p. (4) Construct the diagonal matrix $D_p = \sqrt{|E_p|}$. (5) Find the inverse transformation diagonalizing the Hamiltonian from $T_p^{-1} = K_p^{-1} W_p D_p$.

The outlined scheme results in the diagonal magnon operators $\Psi_p^\dagger = (\alpha_{p1}^\dagger \, \alpha_{p2}^\dagger \, \ldots \, \alpha_{pL}^\dagger \, \alpha_{-p1} \, \alpha_{-p2} \, \ldots \, \alpha_{-pL})$, which are related to the original bosonic operators by $\Psi_p = T_p \Phi_p$ via the matrices

$$T_p = \begin{pmatrix} U_p & V_{-p}^* \\ V_p & U_{-p}^* \end{pmatrix} \tag{18}$$

$$T_p^{-1} = \begin{pmatrix} U_p^\dagger & -V_p^\dagger \\ -V_{-p}^T & U_{-p}^T \end{pmatrix}. \tag{19}$$

This finally gives the diagonal magnon Hamiltonian

$$H_s = \sum_{nk} \Omega_{nk} \alpha_{nk}^\dagger \alpha_{nk}, \tag{20}$$

where $\Omega_{nk} = |\epsilon_{nk}|$.

II. DIAGONALIZING THE ELECTRONIC HAMILTONIAN

Electronic structure in presence of a spin spiral

We now consider the effect of the static spin spiral $\mathbf{S}_i$ on the itinerant electrons. In the local coordinate frame, the spin-electron coupling is given by

$$H_{s-e} = -g \sum_{i\alpha} \left[R^i_{x\alpha} \hat{s}_{ix} \hat{S}_{i\alpha} + R^i_{y\alpha} \hat{s}_{iy} \hat{S}_{i\alpha} + R^i_{z\alpha} \hat{s}_{iz} \hat{S}_{i\alpha} \right]. \tag{21}$$

We initially focus on the terms proportional to S_{i3} (the local z-axis), which lie along the local moments. Expanding the spin operators as $S_{i3} = S - a^\dagger_i a_i$, and assuming that the quadratic terms can be neglected (i.e., we assume a low average magnon occupation), these terms are independent of the magnon operators and determine the coupling to the static spin spiral. Explicitly this term is given by

$$H_{s-e} = -2gS \sum_{i\sigma\sigma'} c^\dagger_{i\sigma} \begin{pmatrix} \cos\theta_i & \sin\theta_i e^{-i\phi_i} \\ \sin\theta_i e^{i\phi_l} & -\cos\theta_i \end{pmatrix} c_{i\sigma'} \tag{22}$$

where θ_i and ϕ_i are the polar and azimuthal angles of the spin at site i in the global Cartesian frame. We denote the matrix in this equation by $M_{i\sigma\sigma'}$, and note that it carries both site and spin indexes.

To write the Hamiltonian in momentum space, we use that the matrix M is periodic over the mUC of length La. This motivates us to write $r_i = r_l + R_n$ and break the sum over lattice sites i into a sum over sites l within the mUC, and a sum over vectors R_n connecting different mUC's. Similarly, the sum over momenta in the electronic Brillouin zone (eBZ) can be split according to $p = k + \nu q$, where k runs over the mBZ and νq connects the different mBZ's within the eBZ. For this sum to cover the original eBZ we should take q as the momentum of the magnetic spiral and $\nu \in \{0, 1, \ldots, L-1\}$.

We can now write the coupling Hamiltonian in k-space by utilizing the Fourier transform

$$c_{i\sigma} = \frac{1}{\sqrt{N}} \sum_p e^{ipr_i} c_{k\sigma} = \frac{1}{\sqrt{N}} \sum_{k\nu} e^{i(k+\nu q)r_i} c_{k+\nu q,\sigma}. \tag{23}$$

The resulting expression is

$$H_{s-e} = -2gS \sum_{kk'\nu\nu'} \sum_{ln\sigma\sigma'} e^{i(k+\nu q - k' - \nu' q)(r_l + R_n)} \tag{24}$$
$$\times c^\dagger_{k+\nu q,\sigma} M_{l\sigma\sigma'} c_{k'+\nu' q,\sigma'}$$
$$= -2gS \sum_{k\nu\nu'} \sum_{l\sigma\sigma'} e^{i(\nu-\nu')qr_l} c^\dagger_{k+\nu q,\sigma} M_{l\sigma\sigma'} c_{k+\nu' q,\sigma'},$$

where in the second line we have used that $qR_n \in 2\pi\mathbb{Z}$ to perform the sum over R_n. We note that $\sum_l e^{i(\nu-\nu')qr_l} = \delta_{\nu\nu'}$, so that if M is periodic in the electronic unit cell

(eUC), such that we can perform the sum over l, this expression reduces to the expected diagonal form.

We can now use that $(\nu - \nu')q = \nu'' q + G$, where G is a reciprocal lattice vector of the electronic system, to bring the sum over k back to the eBZ. Shifting $k \to k - \nu q/2$, relabeling $\nu'' = \nu$ and defining $\zeta_{\nu l} = -2ge^{-i\nu q r_l} S/\sqrt{N}$, we find

$$H_{s-e} = \sum_{k\sigma\sigma'\nu} c^\dagger_{k+\nu q/2,\sigma} f^{\sigma\sigma'}_\nu c_{k-\nu q/2,\sigma'} \tag{25}$$
$$f^{\sigma\sigma'}_\nu = \sum_l \zeta_{\nu l} \begin{pmatrix} \cos\theta_l & \sin\theta_l e^{-i\phi_l} \\ \sin\theta_l e^{i\phi_l} & -\cos\theta_l \end{pmatrix}.$$

We note that this Hamiltonian is Hermitian since $\zeta^*_{\nu l} = \zeta_{-\nu l}$ and $f^\dagger_\nu = f_{-\nu}$.

General diagonalization scheme

In general, to diagonalize the Hamiltonian of Eq. 25, it is necessary to write it in terms of the four-component spinor

$$\Phi_{k\nu} = \begin{pmatrix} c_{k+\frac{\nu q}{2},\uparrow} & c_{k+\frac{\nu q}{2},\downarrow} & c_{k-\frac{\nu q}{2},\uparrow} & c_{k-\frac{\nu q}{2},\downarrow} \end{pmatrix}. \tag{26}$$

Also including the kinetic energy term H_e, the electronic Hamiltonian in presence of a static spin spiral becomes

$$H_{s-e} = \frac{1}{2} \sum_{k\nu} \Phi^\dagger_{k\nu} F_\nu \Phi_{k\nu} \tag{27}$$
$$F_\nu = \begin{pmatrix} \epsilon_{k+\nu q/2} & f_\nu \\ f_{-\nu} & \epsilon_{k-\nu q/2} \end{pmatrix}.$$

Here $\epsilon_{k\pm\nu q/2}$ is a diagonal 2×2 matrix, and f_ν is the coupling matrix of Eq. 25. We note that expanding the two-component spinor notation to a four-component formalism introduces an artificial redundancy, such that the eigenvalues of Eq. 27 are doubly degenerate.

Explicit solution for a perpendicular spiral

Specializing to a spin spiral in the xy-plane, propagating along wire axis in the z-direction, we have $\theta_i = \pi/2$ and $\phi_i = qr_i$. The spin-electron coupling in this case simplifies considerably, and we have $f^{\sigma\sigma'}_{\nu l} = \zeta_{\nu l}(e^{-iqr_l}\delta_{\sigma\uparrow}\delta_{\sigma'\downarrow} + e^{iqr_l}\delta_{\sigma\downarrow}\delta_{\sigma'\uparrow})$. Performing the sum over l, we find that the Hamiltonian can be cast on the form

$$H_e = \frac{1}{\sqrt{N}} \sum_k \Phi^\dagger_k \begin{pmatrix} \epsilon_{k-\frac{q}{2}} & -gS \\ -gS & \epsilon_{k+\frac{q}{2}} \end{pmatrix} \Phi_k. \tag{28}$$

Here $\Phi_k = (c_{k-\frac{q}{2},\uparrow} \ c_{k+\frac{q}{2},\downarrow})^T$ is a two-component spinor, and the bare energies are $\epsilon_k = -2t\cos k$. Since any matrix on the form $h_k = a_k I + \mathbf{b}_k \cdot \boldsymbol{\tau}$ has eigenvalues $\epsilon_{k\pm} = a_k \pm (\mathbf{b}_k \cdot \mathbf{b}_k)^{1/2}$, the energies of the electronic

system are given by

$$\epsilon_{k\tau} = \frac{\epsilon_{k-\frac{q}{2}} + \epsilon_{k+\frac{q}{2}}}{2} + \frac{\tau}{2}\sqrt{(\epsilon_{k+\frac{q}{2}} - \epsilon_{k-\frac{q}{2}})^2 + 4g^2S^2}.$$

$$(29)$$

These energy bands are illustrated in Fig. 2 of the main text. The operators for the corresponding eigenstates are found by a rotation of the original basis,

$$\begin{pmatrix} d_{kd} \\ d_{ku} \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta_k}{2} & \sin\frac{\theta_k}{2} \\ -\sin\frac{\theta_k}{2} & \cos\frac{\theta_k}{2} \end{pmatrix} \begin{pmatrix} c_{k-q/2,\uparrow} \\ c_{k+q/2,\downarrow} \end{pmatrix}, \qquad (30)$$

where the rotation angle θ_k is implicitly defined via the relations

$$\sin\theta_k = \frac{2gS}{\sqrt{(\epsilon_{k+q/2} - \epsilon_{k-q/2})^2 + 4g^2S^2}} \qquad (31\text{a})$$

$$\cos\theta_k = \frac{\epsilon_{k+q/2} - \epsilon_{k-q/2}}{\sqrt{(\epsilon_{k+q/2} - \epsilon_{k-q/2})^2 + 4g^2S^2}}. \qquad (31\text{b})$$

III. ELECTRON-MAGNON COUPLING

We now consider the parts of the spin-electron coupling proportional to S_{i1} and S_{i2}, giving rise to the electron-magnon interaction. These terms can be written as [1, 2]

$$H_{s-e} = -2g\sqrt{\frac{S}{2}} \sum_{i\sigma} \left[(\cos\theta_i - \sigma)e^{-i\sigma\phi_i} c_{i\sigma}^\dagger c_{i,-\sigma} a_i + H.c. \right]$$
$$+ 2g\sqrt{\frac{S}{2}} \sum_{i\sigma} \left[\sigma\sin\theta_i c_{i\sigma}^\dagger c_{i\sigma} a_i + H.c. \right], \quad (32)$$

where θ_i and ϕ_i are the polar and azimuthal angles of the spin spiral in the global Cartesian frame. Decomposing the lattice vector r_i and the electronic Fourier transform as discussed above, the operator part of this expression is

$$\sum_i c_{i\sigma}^\dagger c_{i\sigma'} a_i \quad (33)$$
$$= \frac{1}{\sqrt{NL}} \sum_{pk} \sum_{\nu\nu'l} e^{-i(\nu-\nu')qr_l} a_{pl} c_{p+k+\nu q,\sigma}^\dagger c_{k+\nu'q,\sigma'}.$$

Here the sum over l runs over the L electronic sites within a mUC. Noting that we can always write $(\nu - \nu')q = \nu''q + G$, where G is a reciprocal lattice vector of the electronic system, the above expression becomes

$$\sum_i c_{i\sigma}^\dagger c_{i\sigma'} a_i = \frac{1}{\sqrt{NL}} \sum_{pk} \sum_{\nu l} e^{-i\nu qr_l} a_{pl} c_{k+p+\nu q,\sigma}^\dagger c_{k\sigma'}. \quad (34)$$

Here it is implied that the electronic momentum k runs over the full eBZ, while the magnon momentum p runs over the mBZ. Collecting all the terms of the spin-electron coupling, we have the Hamiltonian

$$H_{s-e} = \sum_{pk\sigma\sigma'} \sum_{\nu l} \left[g_{\nu l}^{\sigma\sigma'} a_{pl} c_{k+p+\nu q,\sigma}^\dagger c_{k\sigma'} + H.c. \right] \quad (35)$$

$$g_{\nu l}^{\sigma\sigma'} = 2g e^{-i\nu qr_l} \sqrt{\frac{S}{2NL}} \quad (36)$$
$$\times \begin{pmatrix} \sin\theta_l & -[\cos\theta_l - 1]e^{-i\phi_l} \\ -[\cos\theta_l + 1]e^{i\phi_l} & -\sin\theta_l \end{pmatrix}.$$

This expression agrees with those derived in Refs. [1, 2].

We can now expand the operators a_{lp} in terms of the magnon operators α_{np}. Due to the non-collinear magnetic structure, these operators are related by a Bogoliubov transform, and we can write

$$a_{lp} = \sum_n (U_{lnp}^\dagger \alpha_{np} - V_{lnp}^\dagger \alpha_{n,-p}^\dagger) \quad (37)$$
$$a_{lp}^\dagger = \sum_n (U_{lnp}^T \alpha_{np}^\dagger - V_{lnp}^T \alpha_{n,-p}).$$

Here U and V are $L \times L$ blocks of the para-unitary matrix that diagonalizes the magnon Hamiltonian [3]. The new form of the electron-magnon coupling is then

$$H_{s-e} = \sum_{pk\sigma\sigma'} \sum_{n\nu} \left[(\eta_{n\nu p}^{\sigma\sigma'} \alpha_{np} + \bar{\eta}_{n\nu p}^{\sigma\sigma'} \alpha_{n,-p}^\dagger) c_{k+p+\nu q,\sigma}^\dagger c_{k\sigma'} \right]$$

$$\eta_{n\nu p}^{\sigma\sigma'} = \sum_l \left[g_{\nu l}^{\sigma\sigma'} U_{lnp}^\dagger - (g_{-\nu l}^{\sigma'\sigma})^* V_{ln,-p}^T \right]$$

$$\bar{\eta}_{n\nu p}^{\sigma\sigma'} = \sum_l \left[-g_{\nu l}^{\sigma\sigma'} V_{lnp}^\dagger + (g_{-\nu l}^{\sigma'\sigma})^* U_{ln,-p}^T \right]$$

From here the remaining step is to expand the electronic operators in the eigenstate basis. To do this, we first need to shift the momentum k to bring the electronic Hamiltonian into the form of Eq. 28, in a manner that depends on the spin projections of the operators. More specifically, for $\sigma = \sigma'$ we can just shift k to write

$$\sum_{k\nu} (\eta_{n\nu p}^{\sigma\sigma'} \alpha_{np} + \bar{\eta}_{n\nu p}^{\sigma\sigma'} \alpha_{n,-p}^\dagger) c_{k+p+\nu q,\uparrow}^\dagger c_{k\uparrow} \quad (38)$$
$$= \sum_{k\nu} (\eta_{n\nu p}^{\sigma\sigma'} \alpha_{np} + \bar{\eta}_{n\nu p}^{\sigma\sigma'} \alpha_{n,-p}^\dagger) c_{k+p+\nu q-\frac{q}{2},\uparrow}^\dagger c_{k-\frac{q}{2},\uparrow}$$

$$\sum_{k\nu} (\eta_{n\nu p}^{\sigma\sigma'} \alpha_{np} + \bar{\eta}_{n\nu p}^{\sigma\sigma'} \alpha_{n,-p}^\dagger) c_{k+p+\nu q,\downarrow}^\dagger c_{k\downarrow} \quad (39)$$
$$= \sum_{k\nu} (\eta_{n\nu p}^{\sigma\sigma'} \alpha_{np} + \bar{\eta}_{n\nu p}^{\sigma\sigma'} \alpha_{n,-p}^\dagger) c_{k+p+\nu q+\frac{q}{2},\downarrow}^\dagger c_{k+\frac{q}{2},\downarrow}$$

For $\sigma = -\sigma'$, we can bring the Hamiltonian into the correct form by shifting both k and ν, such that

$$\sum_{k\nu} (\eta_{n\nu p}^{\sigma\sigma'} \alpha_{np} + \bar{\eta}_{n\nu p}^{\sigma\sigma'} \alpha_{n,-p}^\dagger) c_{k+p+\nu q,\uparrow}^\dagger c_{k\downarrow} \quad (40)$$
$$= \sum_{k\nu} (\eta_{n\nu p}^{\sigma\sigma'} \alpha_{np} + \bar{\eta}_{n\nu p}^{\sigma\sigma'} \alpha_{n,-p}^\dagger) c_{k+p+(\nu+1)q-\frac{q}{2},\uparrow}^\dagger c_{k+\frac{q}{2},\downarrow}$$

$$\sum_{k\nu} (\eta_{n\nu p}^{\sigma\sigma'} \alpha_{np} + \bar{\eta}_{n\nu p}^{\sigma\sigma'} \alpha_{n,-p}^\dagger) c_{k+p+\nu q,\downarrow}^\dagger c_{k\uparrow} \quad (41)$$
$$= \sum_{k\nu} (\eta_{n\nu p}^{\sigma\sigma'} \alpha_{np} + \bar{\eta}_{n\nu p}^{\sigma\sigma'} \alpha_{n,-p}^\dagger) c_{k+p+(\nu-1)q+\frac{q}{2},\downarrow}^\dagger c_{k-\frac{q}{2},\uparrow}$$

This results in the coupling Hamiltonian

$$H_{s-e} = \sum_{kp\tau\tau'} \sum_{n\nu} \left[(\eta_{kpn\nu}^{\tau\tau'} \alpha_{np} + \bar{\eta}_{kpn\nu}^{\tau\tau'} \alpha_{n,-p}^\dagger) d_{k+p+\nu q,\tau}^\dagger d_{k\tau'} \right] \quad (42)$$

where

$$\eta_{kpn\nu}^{\tau\tau'} = \sum_{\sigma\sigma'} S_{\tau\sigma}(k+p+\nu q) \eta_{pn\nu'}^{\sigma\sigma'} N_{\sigma\sigma'} S_{\sigma'\tau'}(k) \quad (43)$$

$$N_{\sigma\sigma'} = \begin{pmatrix} \delta_{\nu\nu'} & \delta_{\nu-1,\nu'} \\ \delta_{\nu+1,\nu'} & \delta_{\nu\nu'} \end{pmatrix}.$$

Here S is the matrix relating the original electronic basis with the eigenbases, and N is a matrix taking into account the shift of the Umklapp index ν for the off-diagonal components. We note that for a spiral in the xy-plane, where $\phi_l = qr_l$, the matrix N can be absorbed

by redefining the spin-electron coupling matrix as

$$g^{\sigma\sigma'}_{\nu l} = 2ge^{-i\nu qr_l}\sqrt{\frac{S}{2NL}}$$

$$\times \begin{pmatrix} \sin\theta_l & -[\cos\theta_l - 1] \\ -[\cos\theta_l + 1] & -\sin\theta_l \end{pmatrix}. \tag{44}$$

Ferromagnetic limit

For a collinear ferromagnet, we have only one magnetic sub-lattice and so $L \to \infty$ and $q = 0$. This implies that the magnon operators are given by the Fourier transform of the spin-flip operators, $\alpha_p = a_p$, such that $U = 1$ and $V = 0$. Further, assuming $\theta = \pi/2$ as above, the spin-electron coupling matrix becomes

$$g_{\sigma\sigma'} = 2g\sqrt{\frac{S}{2N}}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}. \tag{45}$$

Consequently we have $\eta_{\sigma\sigma'} = g_{\sigma\sigma'}$, and since the electronic eigenstates are given by the trivial rotation $c_{k\sigma} = 2^{-1/2}(d_{k\uparrow} + \sigma d_{k\downarrow})$, we find

$$H_{s-e} = \sum_{pk\sigma\sigma'} \left[(g_{\sigma\sigma'}\alpha_p + g^\dagger_{\sigma\sigma'}\alpha^\dagger_{-p})c^\dagger_{k+p,\sigma}c_{k\sigma'} \right] \tag{46}$$

$$= 2g_0 \sum_{pk\tau\tau'} (\alpha_p + \alpha^\dagger_{-p})d^\dagger_{k+p,\tau}d_{k,-\tau}.$$

Therefore, in the ferromagnetic limit, magnon scattering only couples states within different electronic bands, which generically favors s-wave pairing.

IV. EFFECTIVE ELECTRON-ELECTRON ATTRACTION AND GAP EQUATION

Using the electron-magnon coupling derived above, we can now write the total Hamiltonian as

$$H = \sum_{\tau\mathbf{k}} \epsilon_{\tau\mathbf{k}} d^\dagger_{\tau\mathbf{k}} d_{\tau\mathbf{k}} + \sum_{n\mathbf{p}} \Omega_{n\mathbf{p}} \alpha^\dagger_{n\mathbf{p}} \alpha_{n\mathbf{p}} \tag{47}$$
$$+ \sum_{\mathbf{k}\mathbf{p}\tau\tau'} \sum_{n\nu} \left[(\eta^{\tau\tau'}_{\mathbf{k}\mathbf{p}n\nu} \alpha_{n\mathbf{p}} + \bar{\eta}^{\tau\tau'}_{\mathbf{k}\mathbf{p}n\nu} \alpha^\dagger_{n,-\mathbf{p}}) d^\dagger_{\mathbf{k}+\mathbf{p}+\nu\mathbf{q},\tau} d_{\mathbf{k}\tau'} \right].$$

Here, to distinguish the momenta from the four-momenta to be introduced shortly, we use a bold font even though we are working with a one-dimensional system. We note that apart from the non-trivial spin structure of the electron-magnon coupling, the Hamiltonian above is analogous to the standard electron-phonon Hamiltonian employed within Bardeen-Cooper-Schrieffer (BCS) and Eliashberg theory, and can therefore be handled with similar methods. To integrate out the magnons we define the finite temperature functional integral [4, 5]

$$Z = \int \mathcal{D}[\bar{d}, d, \bar{\alpha}, \alpha] e^{S[\bar{d}, d, \bar{\alpha}, \alpha]}. \tag{48}$$

The action $S[\bar{d}, d, \bar{\alpha}, \alpha]$ is given in terms of the inverse propagators $G^{-1}_{0\tau}(k) = i\omega_m - \epsilon_{\tau\mathbf{k}} + \mu$ and $D^{-1}_{0n}(p) = i\Omega_m - \Omega_{n\mathbf{p}}$, where ω_m and Ω_m are fermionic and bosonic Matsubara frequencies, respectively, and $k = (i\omega_m, \mathbf{k})$ and $p = (i\Omega_m, \mathbf{p})$ are momentum four-vectors. Explicitly, we have

$$S = \sum_{\tau k} \bar{d}_{\tau k} G^{-1}_{0\tau}(k) d_{\tau k} + \sum_{np} \bar{\alpha}_{np} D^{-1}_{0n}(p) \alpha_{np} \tag{49}$$
$$- \sum_{\tau\tau'n} \sum_{\nu kp} \left[(\eta^{\tau\tau'}_{n\nu pk} \alpha_{np} + \bar{\eta}^{\tau\tau'}_{n\nu pk} \bar{\alpha}_{n,-p}) \bar{d}_{k+p+\nu q,\tau} d_{k\tau'} \right].$$

Defining the generalized electronic density operators

$$f_{np} = \sum_{k\nu\tau\tau'} \eta^{\tau\tau'}_{n\nu pk} \bar{d}_{k+p+\nu q,\tau} d_{k\tau'} \tag{50}$$
$$\bar{f}_{np} = \sum_{k\nu\tau\tau'} \bar{\eta}^{\tau\tau'}_{n\nu pk} \bar{d}_{k+p+\nu q,\tau} d_{k\tau'},$$

we can write the magnon-dependent part of the partition function as

$$Z_m[\bar{c}, c] = \int \mathcal{D}[\bar{\alpha}, \alpha] e^{S_{e-m}} \tag{51}$$
$$S_{e-m} = \sum_{np} \bar{\alpha}_{np} D^{-1}_{0n}(p) \alpha_{np} - \sum_{np} (\bar{f}_{np} \bar{\alpha}_{n,-p} + f_{np} \alpha_{np}).$$

Since this action is quadratic, we can diagonalize it by shifting the magnon variables as $\bar{\alpha}_{np} \to \bar{\alpha}_{np} + f_{np} D_{0n}(p)$ and $\alpha_{np} \to \alpha_{np} + \bar{f}_{n,-p} D_{0n}(p)$, resulting in

$$S_{e-m} = \sum_{np} \left[\bar{\alpha}_{np} D^{-1}_{0n}(p) \alpha_{np} - f_{np} \bar{f}_{n,-p} D_{0n}(p) \right].$$

Noting that $f_{np} \bar{f}_{n,-p}$ is an even function of p, we can replace $D_{0n}(p)$ with the symmetric combination

$$\mathcal{D}_{0n}(p) = \frac{D_{0n}(p) + D_{0n}(-p)}{2} = \frac{2\Omega_{np}}{(i\Omega_m)^2 - \Omega^2_{np}}. \tag{52}$$

Integrating over the magnon modes now results in the effective electronic action

$$S_e = \sum_{\tau k} \bar{d}_{\tau k} G^{-1}_{0\tau}(k) d_{\tau k} - \sum_{np} f_{np} \bar{f}_{n,-p} \mathcal{D}_{0n}(p) \tag{53}$$
$$= \sum_{\tau k} \bar{d}_{\tau k} G^{-1}_{0\tau}(k) d_{\tau k} - \sum_{kk'p} \sum_{\nu\nu'} \sum_{\tau_1\tau_2\tau_3\tau_4}$$
$$\times U^{\tau_1\tau_2\tau_3\tau_4}_{pkk'\nu\nu'} \bar{d}_{k+p+\nu q,\tau_1} d_{k\tau_2} \bar{d}_{k'-p+\nu'q,\tau_3} d_{k'\tau_4},$$

Working in the static limit $\omega_m \to 0$ and $\Omega_m \to 0$, the effective electron-electron interaction is

$$U^{\tau_1\tau_2\tau_3\tau_4}_{pkk'\nu\nu'} = -2 \sum_n \frac{\eta^{\tau_1\tau_2}_{n\nu pk} \bar{\eta}^{\tau_3\tau_4}_{n\nu',-p,k'}}{\Omega_{np}}. \tag{54}$$

Assuming that $k' = -k$ and $\nu' = -\nu$, as appropriate for zero-momentum Cooper pairs, and using that $G^{-1}_{0\tau}(k) = \mu - \epsilon_{\tau\mathbf{k}}$ in the static limit, we arrive at the effective BCS Hamiltonian

$$H_{\mathrm{BCS}} = \sum_{\tau\mathbf{k}} (\epsilon_{\tau\mathbf{k}} - \mu) d^\dagger_{\tau\mathbf{k}} d_{\tau\mathbf{k}} \tag{55}$$
$$- \sum_{\mathbf{k}\mathbf{k}'} \sum_{\tau_1\tau_2\tau_3\tau_4} U^{\tau_1\tau_2\tau_3\tau_4}_{\mathbf{k}\mathbf{k}'} d^\dagger_{\mathbf{k}',\tau_1} d_{\mathbf{k}\tau_2} d^\dagger_{-\mathbf{k}',\tau_3} d_{-\mathbf{k},\tau_4}.$$

Here we have defined the new momentum variable $k' = k + p + \nu q$, and the interaction is given by

$$U^{\tau_1\tau_2\tau_3\tau_4}_{\mathbf{k}\mathbf{k}'} = -2 \sum_n \frac{\eta^{\tau_1\tau_2}_{n\nu pk} \bar{\eta}^{\tau_3\tau_4}_{n,-\nu,-p,-k}}{\Omega_{np}}. \tag{56}$$

This is the effective interaction used to obtain the results in the main text.

Linearized gap equation

Following the procedure outlined in Refs. [2, 6], the gap equation can be obtained by minimizing the free energy F, which has the form

$$F = \sum_{k\tau} \epsilon_{k\tau} - \frac{1}{2} \sum_{k\eta} E_{k\eta} + \frac{1}{2} \sum_{\tau_1\tau_2} \Delta^{\tau_1\tau_2}_k b^\dagger_{k\tau_2\tau_1} \tag{57}$$
$$- \frac{1}{\beta} \sum_{k\eta} \ln\left(1 + e^{-\beta E_{k\eta}}\right).$$

Here $E_{k\tau}$ are the eigenvalues of the mean-field decoupled BCS Hamiltonian (the so-called Bogoliubov-de Gennes [BdG] Hamiltonian), and has the structure

$$E_{k\tau} = \sqrt{\epsilon^2_{kd} + \epsilon^2_{ku} + \frac{1}{2} \mathrm{Tr}\, \Delta_k \Delta^\dagger_k + \frac{\tau}{2} \sqrt{A_k}}. \tag{58}$$

The index η takes values $(\pm)$, and the function A_k is a complicated expression with the key feature that $A_k \to (\epsilon_d^2 - \epsilon_u^2)^2$ as $\Delta \to 0$. Therefore, in the limit $\Delta \to 0$ the BdG eigenvalues $E_{k+} \to \epsilon_{ku}$ and $E_{k-} \to \epsilon_{kd}$. The expression $b_{k\tau_2\tau_1}^{\dagger} = \langle d_{k\tau_2}^{\dagger} d_{-k,\tau_1}^{\dagger} \rangle$.

Differentiating the free energy with respect to the gap $\Delta_k^{\tau_1\tau_2}$ results in the gap equation

$$\Delta_k^{\tau_1\tau_2} = \sum_{k'\tau_3\tau_4} U_{kk'}^{\tau_1\tau_2\tau_3\tau_4} \sum_{\eta} \left(\Delta_{k'}^{\tau_3\tau_4} + \eta B_{k'}^{\tau_3\tau_4} \right) \qquad (59)$$

$$\times \frac{\tanh(\beta E_{k\eta})}{2E_{k\eta}}.$$

In the limit $\Delta \to 0$ the term proprtional to B_k is expected to be sub-dominant, and so the linearized gap equation reads

$$\Delta_k^{\tau_1\tau_2} = \sum_{k'\tau_3\tau_4} U_{kk'}^{\tau_1\tau_2\tau_3\tau_4} \Delta_{k'}^{\tau_3\tau_4} \sum_{\tau} \frac{\tanh(\beta\epsilon_{k\tau})}{2\epsilon_{k\tau}},$$

V.   ANALYTICAL SOLUTION FOR THE ZERO TEMPERATURE GAP

In the zero temperature limit, where $\beta \to \infty$, the single-band linearized gap equation simplifies to

$$\Delta_{kv} = \frac{1}{N} \sum_{k'} \frac{U_{kk'vv'}}{2E_{k'}} \Delta_{k'v'}. \qquad (60)$$

Here $v = \pm$ is an index denoting the two points on the Fermi surface. Replacing the sum with an integral, $N^{-1} \sum_k \to \int d\epsilon \rho(\epsilon)$, and assuming the density of states is constant over the energy range $[\mu - \Omega, \mu + \Omega]$, we find

$$\Delta_\tau = \sum_{\tau'} \frac{\rho_F U_{\tau\tau'}}{2} \Delta_{\tau'} \int d\epsilon \frac{1}{\sqrt{(\epsilon - \mu)^2 + \Delta_{\tau'}^2}}. \qquad (61)$$

Here Ω is the characteristic magnon energy, and all quantities are implicitly dependent on the chemical potential μ. The momentum dependence of the gap has been neglected since its variation over the Fermi surface is assumed to be negligible.

Restricting the integral to the region $[-\Omega, \Omega]$, we have

$$\frac{1}{2} \int_{-\Omega}^{\Omega} d\epsilon \frac{1}{\sqrt{(\epsilon - \mu)^2 + \Delta^2}} = \ln(\Omega/\Delta). \qquad (62)$$

Further assuming a gap with triplet symmetry $\Delta_+ = -\Delta_- = \Delta$, as is necessary for a single non-degenerate band, we have

$$1 = \rho_F \ln(\Omega/\Delta)(U_{++} - U_{+-}). \qquad (63)$$

Defining $U_0 = U_{++} - U_{+-}$ and rewriting this equation, the solution for the gap becomes

$$\Delta = \Omega e^{-1/\rho_F U_0}. \qquad (64)$$

This expression is of the standard BCS form, and shows that the magnitude of the gap scales exponentially with the density of states at the Fermi level and with the spin-electron coupling.

VI. TOPOLOGICAL PHASE DIAGRAM

Topology in the effective single-band limit

We here discuss the topological phases of the superconducting system in the effective single-band limit. In this limit, we can write the Hamiltonian as

$$H = \sum_k \Phi_k^\dagger \begin{pmatrix} d_{3k} & \Delta_k^* \\ \Delta_k & -d_{3k} \end{pmatrix} \Phi_k, \qquad (65)$$

where $\Phi_k^\dagger = (c_k^\dagger \ c_{-k})$ is a Nambu spinor field, $d_{3k} = (1/2)(\epsilon_{\tau k} - \mu)$, and $\Delta_k = d_{1k} + id_{2k}$. The Hamiltonian can be written more compactly on the form

$$\mathcal{H}_k = d_{1k}\sigma_x + d_{2k}\sigma_y + d_{3k}\sigma_z, \qquad (66)$$

from which its eigenvalues are obtained as $E_{k\pm} = \pm d_k$ with $d_k = \sqrt{d_{1k}^2 + d_{2k}^2 + d_{3k}^2}$. For later convenience, we perform a $\pi/2$ rotation around the x-axis, such that the resulting Hamiltonian is

$$\begin{aligned} \mathcal{H}'_k &= e^{-i\sigma_x \pi/4} \mathcal{H}_k e^{i\sigma_x \pi/4} \\ &= d'_{1k}\sigma_x + d'_{2k}\sigma_y + d'_{3k}\sigma_z, \end{aligned} \qquad (67)$$

where $d'_1 = d_1$, $d'_2 = -d_3$ and $d'_3 = d_2$. In the following, we neglect the primes since all calculations will be performed in the new rotated basis.

Assuming that the gap function is real, $\Delta_k = d_{1k}$, the eigenstates of the Hamiltonian are

$$|v_{\tau k}\rangle = \frac{1}{\sqrt{2d^2}} \begin{pmatrix} \tau d \\ \Delta_k - i\epsilon_{\tau k} \end{pmatrix} \qquad (68)$$

and the corresponding Berry connections are

$$\begin{aligned} A_\tau(k) &= i\langle v_{\tau k}|\partial_k|v_{\tau k}\rangle \\ &= \frac{1}{2d^2}(\epsilon_{\tau k}\partial_k\Delta_k - \Delta_k\partial_k\epsilon_{\tau k}). \end{aligned} \qquad (69)$$

To check the validity of this definition, we calculate the Berry phase $\phi = \int_{-\pi}^{\pi} A(k)dk$ for the Kitaev chain [7], where $\epsilon_k = 2t\cos k - \mu$ and $\Delta_k = \sin k$. This gives the Berry connection

$$A(k) = \frac{1}{2} \frac{2t\Delta - \Delta\mu\cos k}{(2t\cos k - \mu)^2 + \Delta^2\sin^2 k}, \qquad (70)$$

which integrates to π in the interval $\mu \in [-2t, 2t]$ and zero otherwise. Therefore, the Berry phase can be identified with the $\mathbb{Z}_2$ index expected for a time-reversal breaking superconductor, measuring the winding of the vector $\mathbf{d} = (d_1 \ d_2 \ d_3)$ on the unit sphere.

Since the effective single-band model discussed in the main text can be adiabatically connected to the Kitaev model by continuous deformation of the dispersion and gap function, its topological phase diagram can be obtained by counting the number of intersections N of a line at constant chemical potential with the dispersion ϵ_k. The topological index is then given by $\mathbb{Z}_2 = (N/2)$ mod 2.

Topology in the full two-band model

To obtain the topological phase diagram of the full two-band model, we first consider the case with $\Delta_s = \Delta_p = 0$. In this limit, the model corresponds to two independent copies of the Kiteav model, and a $\mathbb{Z}_2$ invariant can be defined separately for each band. Denoting these invariants by Z_d and Z_u, the $\mathbb{Z}_2$ index of the full system is given by the product $Z_d Z_u$.

Since the addition of inter-band pairings cannot close the superconducting gap, the full two-band model is therefore adiabatically connected to the $\Delta_s = \Delta_p = 0$ limit. The topological phase diagram of the two-band model is hence determined by the topology in this limit. The phase diagram then follows by noting that when the chemical potential is in the effective single-band regime, the system is in a topological phase, while outside this regime the system is trivial. Consequently, the phase diagram can be constructed by counting the number of times a line at constant chemical potential μ crosses the bands $\epsilon_{k\tau}$, and is shown in Fig. 3 of the main text. For $g = 0$ the system is always in a trivial regime, while for $g \neq 0$ two regions with non-trivial topology grow out of the upper and lower band edges. The size of these regions increase linearly with the magnitude of g.

Properties of the open chain

To study the qualitative properties of the spectrum and edge mode wavefunctions of a quantum wire with open boundary conditions, we construct an effective Kitaev chain mimicking the properties of the real superconducting wire. In fact, since the gap equation is solved in momentum space, and takes into account the full effects of the static spin structure with periodic boundary conditions, it is a highly non-trivial task to reconstruct an exact real-space model describing our system. To generate the results in Fig. 1, we have solved the Kitaev model for a chain with $L = 4000$ sites and with open boundary conditions, as described by the Hamiltonian [7]

$$H = -\mu \sum_{i=1}^{N} n_i - \sum_{i=1}^{N-1} \left[tc_i^\dagger c_{i+1} + \Delta c_i c_{i+1} + H.c. \right]. \quad (71)$$

The Kitaev model is a model of spinless fermions, as effectively realized by our model within the single-band regime. For the numerical results we assume an electronic hopping of $t = 1$ eV and a gap magnitude of $\Delta = 2$ meV, with the chemical potential chosen to reproduce the Fermi level density of states of our full superconducting model. Although we expect these results to give a faithful representation of the qualitative superconducting properties (due to the topological nature both of the spectrum and the edge modes), these results do not arise from the same exact model as our bulk results.

As expected, we find two degenerate Majorana zero modes inside the superconducting gap of size 2Δ. The
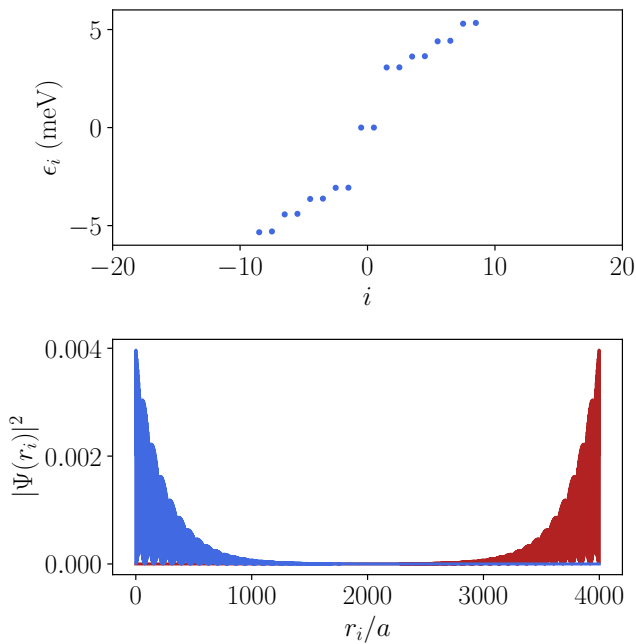
FIG. 1. Eigenvalue spectrum and edge mode wave functions of an effective Kitaev chain with $L = 4000$ sites, electronic hopping amplitude $t = 1$ eV and gap magnitude $\Delta = 2$ meV, mimicking the properties of the superconducting state.

wave functions corresponding to these modes are localized to the edges of the wire, whose spread decreases exponentially with the gap magnitude Δ. For a macroscopic wire with a typical length of 400 nm (corresponding to $a = 1$ Å), we therefore estimate a gap of ~ 1 meV is necessary to avoid mixing of the Majorana zero modes. However, we also note that it is possible to fabricate substantially longer quantum wires, such that edge mode mixing can be avoided also for smaller gaps.

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