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by

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JACOBI-EISENSTEIN SERIES OVER NUMBER FIELDS

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ABSTRACT. For any given totally real number field K, we compute the Fourier developments of the Jacobi Eisenstein series over K at the cusp at infinity. As a main application we prove, for any K with class number 1, that the L-series of the Jacobi Eisenstein series of weight $k \geq 3$ for indices with rank and modified level 1 coincide with the L-series of the Eisenstein series of weight $2k-2$ on the full Hilbert modular group of K. Moreover, under this correspondence the Fourier coefficients of the Jacobi Eisenstein series are related to the twisted L-series of the Hilbert Eisenstein series at the critical point by a Waldspurger type identity.

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1. Introduction and statement of results

In [Boy15] we developed a theory of Jacobi forms over (totally real) number fields and determined explicitly all Jacobi forms of singular weight over any given field. However, an essential part is still missing in this theory, namely, a Hecke theory and the precise relation to usual Hilbert modular forms. We expect a lifting of Jacobi forms over a given field K of weight k and index of rank 1 and modified level $\mathfrak{m}$ to Hilbert modular forms of weight $2k-2$ on the subgroup $\Gamma_0(\mathfrak{m})$ of the Hilbert modular group of K. This is justified by examples and by the intimate relation between Jacobi forms and half integral weight modular forms and the work of Shimura, Ikeda et al. on half integral weight Hilbert modular forms and their relation to integral weight forms. Moreover, we expect Waldspurger type identities between the Fourier coefficients of Jacobi Hecke eigenforms and the values of the twisted L-series of its associated Hilbert modular forms at the critical point. In short, we hope to extend the lifting theory of Skoruppa and Zagier [SZ88] for Jacobi forms over the rational numbers to Jacobi forms over arbitrary totally real number fields including the Waldspurger type identities as developed in [GKZ87].

In this article we do a step towards such a theory by considering the case of Jacobi Eisenstein series. The general (and much deeper case) including cusp forms will very likely afford a trace formula. First steps towards such a trace formula are done by Strömberg and Skoruppa [SS16]. For Eisenstein series the proposed extension of Skoruppa and Zagier's lifting can be done more explicitly once one has closed formulas for their Fourier coefficients. In this article we shall derive such formulas.

We explain our main result. Let K be a totally real number field with ring of integers $\mathfrak{o}$ and different $\mathfrak{d}$. Moreover, let $\mathfrak{c}$ be an integral $\mathfrak{o}$-ideal and ω be a totally positive element of K such that $\mathfrak{m} := \frac{1}{2}\omega\mathfrak{c}^2\mathfrak{d}$ is integral. Then, $[\mathfrak{c},\omega] := (\mathfrak{c},\beta)$, where $\beta : \mathfrak{c}\times\mathfrak{c} \to \mathfrak{d}^{-1}$, $\beta(x,y) = \omega xy$, defines an even totally positive definite $\mathfrak{o}$-lattice of modified level $\mathfrak{m}$. For any integer k, let $J_{k,[\mathfrak{c},\omega]}$ denote the space of Jacobi forms of weight k and index $[\mathfrak{c},\omega]$. Assume that the modified level $\mathfrak{m}$ of $[\mathfrak{c},\omega]$ equals $\mathfrak{o}$ and that $k \geq 3$. Assume furthermore that k is even if K has a unit of norm -1. Then there are exactly h_K-many Eisenstein series in $J_{k,[\mathfrak{c},\omega]}$ [Boy15, §4.5], where h_K denotes the class number of K. (If k is odd and K possesses a unit of norm -1 then every Jacobi form in $J_{k,[\mathfrak{c},\omega]}$ is a cusp form; see Prop. 2.4). In this note we are mainly interested in calculating the Fourier expansion of the Eisenstein series

$$E_{k,[\mathfrak{c},\omega]} = \sum_{A\in G_{4\mathfrak{m}}\backslash \mathrm{SL}(2,\mathfrak{o})} \sum_{t\in\mathfrak{c}} q^{\frac{1}{2}\omega t^2}\zeta^{\omega t}|_{k,\underline{L}}A,$$

where $G_{\mathfrak{l}}$, for a given integral ideal $\mathfrak{l}$, denotes the subgroup of all upper triangular matrices $\left[\begin{smallmatrix} d^{-1} & b \\ 0 & d\end{smallmatrix}\right]$ in $\mathrm{SL}(2,\mathfrak{o})$ with d in the subgroup $\mathfrak{o}_{\mathfrak{l}}$ of units d in $\mathfrak{o}$ with $d \equiv 1 \bmod \mathfrak{l}$ (for the other notations see Section 2).

Since we assume that the index $[\mathfrak{c},\omega]$ has modified level $\mathfrak{m} = \mathfrak{o}$ the Fourier expansion of $E_{k,[\mathfrak{c},\omega]}$ is of the form (see §2.4 and Theorem 3.1).

$$E_{k,[\mathfrak{c},\omega]} = [\mathfrak{o}^\times : \mathfrak{o}_4]\sum_{r\in\mathfrak{c}} q^{\frac{1}{2}\omega r^2}\zeta^{\omega r} + \sum_{\substack{\Delta\in\mathfrak{c}^2, r\in\mathfrak{c} \\ \Delta\ll 0,\ \Delta\equiv r^2 \bmod 4\mathfrak{c}^2}} e_{k,[\mathfrak{c},\omega]}(\Delta)\, q^{\frac{\omega}{8}(r^2-\Delta)}\zeta^{\frac{1}{2}\omega r}.$$

Main Theorem. *For any totally negative Δ in $\mathfrak{c}^2$ which is a square mod $4\mathfrak{c}^2$, we have*

$$e_{k,[\mathfrak{c},\omega]}(\Delta) = C\frac{\mathrm{N}(\Delta/4\mathfrak{d}\mathfrak{f}_\Delta^2)^{k-3/2}}{h_K}\sum_{\psi}\frac{L\left(\psi\left(\frac{\Delta}{*}\right)_0, k-1\right)}{L\left(\psi^2, 2k-2\right)}\gamma^\psi(\mathfrak{f}_\Delta/\mathfrak{c}),$$

where for any integral ideal $\mathfrak{f}$,

$$\gamma^{\psi}(\mathfrak{f}) = \sum_{\mathfrak{t}|\mathfrak{f}} \mu(\mathfrak{t})\psi(\mathfrak{t}) \left(\tfrac{\Delta}{\mathfrak{t}}\right)_0 \mathrm{N}(\mathfrak{t})^{k-2}\, \sigma_{2k-3}^{\psi^2}(\mathfrak{f}/\mathfrak{t}) .$$

Here $\left(\frac{\Delta}{*}\right)_0$ *is the Größencharacter associated to the extension* $K(\sqrt{\Delta})$ *of* K*, and* $\mathfrak{f}_\Delta$ *denotes the largest integral ideal in* K *whose square divides* Δ *such that* Δ *is a square modulo* $4\mathfrak{f}_\Delta^2$*. The sum is over all characters* ψ *of the class group of* K*, and, for any Größencharakter* χ *of* K*, we use* $L(\chi, s)$ *for its L-series.*

Moreover, $\mathrm{N}(\mathfrak{t})$ *denotes the norm of the ideal* $\mathfrak{t}$*,* h_K *denotes the class number of* K*,* μ *is the Möbius* μ*-function*[1] *of* K*, and* $\sigma_k^{\psi^2}(\mathfrak{f}) = \sum_{\mathfrak{t}|\mathfrak{f}} \mathrm{N}(\mathfrak{t})^k \psi^2(\mathfrak{f}/\mathfrak{t})$.

Finally, C *is a constant whose precise value is given in* (11).

In the formula for $\gamma^{\psi}(\mathfrak{f})$ we use $\left(\frac{\Delta}{\mathfrak{t}}\right)_0 = 0$ if $\mathfrak{t}$ is not relatively prime to $\Delta/\mathfrak{f}_\Delta^2$. The Größencharakter $\left(\frac{\Delta}{*}\right)_0$ was studied in detail in [BS]. In particular, it is shown loc.cit. that its conductor equals $\Delta/\mathfrak{f}_\Delta^2$.

As already explained we expect that there should be a Hecke equivariant lift of $J_{k,[\mathfrak{c},\omega]}$ to the space $M_{2k-2}(\mathrm{SL}(2,\mathfrak{o}))$ of Hilbert modular forms on $\mathrm{SL}(2,\mathfrak{o})$ of weight $2k-2$ with respect to a suitable Hecke theory for Jacobi forms over arbitrary totally real number fields, which has still to be developed. (Here we assume still that the modified level of $[\mathfrak{c},\omega]$ is 1.) In terms of L-series this lifting should work as follows. Call a number Δ in K a discriminant if it is integral and a square modulo 4. We say that two disriminants fall into the same class if their square roots generate the same quadratic extension over K. For any class $\mathfrak{D}$ of totally negative discriminants and any Jacobi form ϕ in $J_{k,[\mathfrak{c},\omega]}$ with Fourier coefficients $C_\phi(\Delta)$ we set

$$L_{\mathfrak{D}}(\phi, s) := L\left(\left(\tfrac{\Delta}{*}\right)_0, s-k+2\right) \sum_{\substack{\Delta\in\mathfrak{D}/\mathfrak{o}^{\times 2}\\ \mathfrak{c}^2|\Delta}} \frac{C_\phi(\Delta)}{\mathrm{N}(\mathfrak{f}_\Delta)^s}.$$

(Note that $C_\phi(\Delta)$ depends only on $\Delta\mathfrak{o}^{\times 2}$, see (4), and note that $J_{k,[\mathfrak{c},\omega]} = 0$ if k is odd and K possesses a unit of norm -1.) If the class number h_K of K is 1 and ϕ is a Hecke eigenform we expect that these Dirichlet series are all proportional, and in fact are proportional to the L-series of a Hecke eigenform in $M_{2k-2}(\mathrm{SL}(2,\mathfrak{o}))$.

Using the formula of the main theorem this can be easily checked for $E_{k,[\mathfrak{c},\omega]}$. For $h_K = 1$ the form $E_{k,[\mathfrak{c},\omega]}$ is the only Eisenstein series in $J_{k,[\mathfrak{c},\omega]}$ and should therefore be Hecke eigenform. For $h_K = 1$ we can also assume that $\mathfrak{c} = \mathfrak{o}$. The formula of the main theorem simplifies then to

$$e_{k,[\mathfrak{o},\omega]}(\Delta) = C\, \mathrm{N}(\omega\Delta/8\mathfrak{f}_\Delta^2)^{k-3/2} \frac{L\left(\left(\frac{\Delta}{*}\right)_0, k-1\right)}{\zeta_K(2k-2)} \gamma(\mathfrak{f}_\Delta),$$

where $\gamma(\mathfrak{f}) = \gamma^1(\mathfrak{f}) = \sum_{\mathfrak{t}|\mathfrak{f}} \mu(\mathfrak{t}) \left(\frac{\Delta}{\mathfrak{t}}\right)_0 \mathrm{N}(\mathfrak{t})^{k-2}\, \sigma_{2k-3}(\mathfrak{f}/\mathfrak{t})$. Moreover, for $h_K = 1$, every class of discriminants coincides with the nonzero multiples of a fundamental discriminant Δ_0, i.e. a discriminant Δ_0 with $\mathfrak{f}_{\Delta_0} = 1$. Accordingly we find

$$L_{\mathfrak{D}}(E_{k,[\mathfrak{o},\omega]}, s) = \text{const.} \sum_{\mathfrak{a}} \frac{\sigma_{2k-3}(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^s},$$

where the sum is over all integral ideals $\mathfrak{a}$. Note that the Dirichlet series is indeed the L-function of the unique Eisenstein series E_{2k-2} in $M_{2k-2}(\mathrm{SL}(2,\mathfrak{o}))$.

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[1]This means $\mu(\mathfrak{a}) = (-1)^{\nu}$ if $\mathfrak{a}$ is squarefree, where ν equals the number of prime divisors of $\mathfrak{a}$, and $\mu(\mathfrak{a}) = 0$ otherwise.

Moreover, if we twist the L-series of the Eisenstein series in $M_{2k-2}\left(\mathrm{SL}(2,\mathfrak{o})\right)$ by $\left(\frac{\Delta}{*}\right)_0$, with a fundamental Δ (i.e. $\mathfrak{f}_\Delta = 1$), we obtain the series

$$\begin{aligned} L\left(E_{2k-2} \otimes \left(\tfrac{\Delta}{*}\right)_0, s\right) &= L\left(\left(\tfrac{\Delta}{*}\right)_0, s\right) L\left(\left(\tfrac{\Delta}{*}\right)_0, s-2k+3\right) \\ &= C_1 L\left(\left(\tfrac{\Delta}{*}\right)_0, s\right) L\left(\left(\tfrac{\Delta}{*}\right)_0, 2k-s-2\right) \end{aligned}$$

(where C_1 is a constant, and where the second equality, we applied the functional equation for $L\left(\left(\frac{\Delta}{*}\right)_0, s\right)$ under $s \to 1-s$). In particular,

$$L\left(E_{2k-2} \otimes \left(\tfrac{\Delta}{*}\right)_0, k-1\right) = C_1 L\left(\left(\tfrac{\Delta}{*}\right)_0, k-1\right)^2.$$

Therefore,

$$\mathrm{N}(\Delta)^{2k-3} L\left(E_2 \otimes \left(\tfrac{\Delta}{*}\right)_0, k-1\right) = C_2\, e_{k,[\mathfrak{o},\omega]}(\Delta)^2,$$

(where $C_2 = C_1 \zeta_K(2k-2)^2 / C^2\, \mathrm{N}(\omega/8)^{k-3/2}$), which is the desired Waldspurger type identity.

The plan of the article is as follows: in Section 2 we provide the notations that we use throughout the article and also some basic facts which will be needed in the article. In Section 3 we study, first of all, the basic properties of the general Jacobi Eisenstein series at the cusp at infinity, which is

$$E_{k,\underline{L},s} = \sum_{A \in G_{\mathfrak{l}} \backslash \mathrm{SL}(2,\mathfrak{o})} \sum_{\substack{r \in L^\sharp \\ r \equiv s \bmod L}} q^{\beta(r)} \zeta_\beta^r |_{k,\underline{L}} A,$$

where $\underline{L} = (L, \beta)$ is an arbitrary even totally positive definite $\mathfrak{o}$-lattice, s an isotropic element of $\underline{L}$ and $\mathfrak{l}$ the level of $\underline{L}$. In Section 4 we calculate the Fourier coefficients of $E_{k,\underline{L},s}$, which are given in Theorem 4.1 and, with some further simplifications for $s = 0$, in Theorem 4.3. These calculations are straight-forward and depend essentially on the generalized Lipschitz formula of Lemma 4.2. The essential part in the formula for $s = 0$ is identified as a Dirichlet series whose coefficients are representation numbers of the quadratic form induced by β on the finite abelian group $L/\mathfrak{a}L$ (see (15)). Calculating these numbers for arbitrary $\underline{L}$ would lead us to far away from the main goal of this article and we do not pursue this. In Sections 5 and 6 we restrict ourselves to the case of lattices $\underline{L}$ of index of rank and modified level one. In Proposition 5.2 we calculate the mentioned representation numbers. This calculation is not straight-forward and depends on Theorem 5.1, whose non-trivial proof is given in a separate note [BS]. In Section 6 we conclude with the proof of the main theorem. In Section 7 we calculate some integrals whose exact values are needed for the proof of the generalized Lipschitz formula Lemma 4.2.

2. Notations and basic facts

2.1. **The algebras $\mathcal{C}$ and $\mathcal{R}$.** Throughout this article K denotes a totally real number field over $\mathbb{Q}$ with discriminant D_K, ring of integers $\mathfrak{o}$, group of units $\mathfrak{o}^\times$ and different $\mathfrak{d}$. We use tr and N for the trace and the norm from K to $\mathbb{Q}$. Set $\mathcal{C} = \mathbb{C} \otimes_{\mathbb{Q}} K$ and $\mathcal{R} = \mathbb{R} \otimes_{\mathbb{Q}} K$. Note that $\mathcal{C}$ carries the structure of an algebra over $\mathbb{C}$ and of an algebra over K. We identify K and $\mathbb{C}$ with its canonical images in $\mathcal{C}$ (under the maps $a \mapsto 1 \otimes a$ and $z \mapsto z \otimes 1$), respectively. In this way the product of e.g. a complex number with an element of K is meaningfully defined as an element of $\mathcal{C}$. In particular, $\mathrm{SL}(2,K)$ acts on $\mathcal{C}$ via $(A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right], z) \mapsto Az = \frac{az+b}{cz+d}$. Note that $\mathcal{R}$ is a K-subalgebra of $\mathcal{C}$. We extend the $\mathbb{Q}$-linear map $\mathrm{tr}\left(.\right)$ and every embedding $\sigma : K \mapsto \mathbb{C}$ to a $\mathbb{C}$-linear map $\mathcal{C} \mapsto \mathbb{C}$. We let $\mathcal{H}$ to be the *Poincaré upper half plane* attached to K, i.e. the subset of all z in $\mathcal{C}$ such that $\Im(\sigma(z)) > 0$ for all embeddings σ of K. Finally N denotes the norm of the $\mathbb{C}$-algebra $\mathcal{C}$, i.e., for $c \in \mathcal{C}$, we have $\mathrm{N}(c) = \prod_\sigma \sigma(c)$. We use $e\{*\} := e^{2\pi i\, \mathrm{tr}(*)}$. Moreover, for n and r

in K, we use q^n and ζ^r for the functions $q^n(\tau) = e\{n\tau\}$ ($\tau \in \mathcal{H}$) and $\zeta^r(z) = e\{rz\}$ ($z \in \mathbb{C}$).

2.2. **Jacobi forms of lattice index over K.** From now on unless otherwise stated $\underline{L} = (L, \beta)$ will denote a totally positive definite even $\mathfrak{o}$-lattice with level $\mathfrak{l}$, rank $r_{\underline{L}}$ and $\det L = [L^\sharp : L]$ (see [Boy15, §3.1] for definition). We extend the $\mathfrak{o}$-bilinear map $\beta : L \times L \mapsto \mathfrak{d}^{-1}$ to a $\mathcal{C}$-bilinear map on $L_{\mathcal{C}} := \mathcal{C} \otimes_{\mathfrak{o}} L$ taking values in $\mathcal{C}$. We use $H(\underline{L})$ for the Heisenberg group associated to $\underline{L}$ [Boy15, Def. 3.14]. Recall that $H(\underline{L}) = \{(x, y, \xi) : x, y \in L \times L, \xi \in \mathbb{C}^*\}$ with multiplication $(x, y, \xi) \cdot (x', y', \xi') = (x + x', y + y', \xi\xi' e\{(\beta(x, y') - \beta(x', y))/2\})$. The groups $\mathrm{SL}(2, \mathfrak{o})$ and $H(\underline{L})$ act on the space of holomorphic functions on $\mathcal{H} \times L_{\mathcal{C}}$ via

$$\begin{aligned} \big(\phi|_{k,\underline{L}} A\big)(\tau, z) &= \mathrm{N}(c\tau + d)^{-k} e\left\{-\tfrac{c\beta(z)}{c\tau+d}\right\} \phi(A\tau, \tfrac{z}{c\tau+d}), \\ \big(\phi|_{k,\underline{L}}(x, y, \xi)\big)(\tau, z) &= \xi\, e\left\{\tau\beta(x) + \beta(x, z) + \tfrac{1}{2}\beta(x, y)\right\} \phi(\tau, z + x\tau + y). \end{aligned} \tag{1}$$

Here and in the following we use

$$\beta(x) = \tfrac{1}{2}\beta(x, x).$$

Occasionally we need "$|_{k,\underline{L}} A$" also for half integral k (which defines then no longer a proper action). The factor $\mathrm{N}(c\tau + d)^{-k}$ is then evaluated according to the following convention which we shall use throughout the article. For complex numbers $w \neq 0$ and r we let

$$w^r = \exp(r \operatorname{Log} w),$$

where Log is the main branch of the logarithm, i.e. that branch such that $\operatorname{Log} w = it$ with $-\pi < t \leq +\pi$ for $|w| = 1$.

Recall from [Boy15, Def. 3.45] that the space $J_{k,\underline{L}}$ of Jacobi forms over K of integral weight k and index $\underline{L}$ consists of all holomorphic functions on $\mathcal{H} \times L_{\mathcal{C}}$ which satisfy $\phi|_{k,\underline{L}} g = \phi$ for all g in $\mathrm{SL}(2, \mathfrak{o})$ and all g in $H(\underline{L})$. (If K equals the field of rational numbers we also have to impose the usual regularity condition at the cusps, which is automatically fulfilled by a Köcher type principle for $K \neq \mathbb{Q}$ [Boy15, Thm. 3.2].)

2.3. **Fourier expansions of Jacobi forms.** Every element ϕ in $J_{k,\underline{L}}$ possesses a Fourier development of the form

$$\phi = \sum_{\substack{n \in \mathfrak{d}^{-1}, r \in L^\sharp \\ n - \beta(r) \gg 0}} c_\phi(n, r)\, q^n \zeta_\beta^r, \tag{2}$$

where, for z in $L_{\mathcal{C}}$, we use $\zeta_\beta^r(z) = e\{\beta(r, z)\}$. For D in K and r in $L^\sharp$ such that $D \equiv -\beta(r) \bmod \mathfrak{d}^{-1}$, we set $C_\phi(D, r) = c_\phi(D + \beta(r), r)$. Then

$$C_\phi(D, r) = C_\phi(D, s) \quad \text{for} \quad r \equiv s \bmod L \tag{3}$$

(see [Boy15, remark after Them. 3.3]). Moreover, let u be a unit of K. Applying the matrix $\left[\begin{smallmatrix} u & 0 \\ 0 & u^{-1} \end{smallmatrix}\right]$ to $\phi \in J_{k,\underline{L}}$ we see that

$$C_\phi(u^2 D, ur) = \mathrm{N}(u)^k C_\phi(D, r) \tag{4}$$

holds true. Here we use the identity in (2).

2.4. **Jacobi forms with rank one index.** Every positive definite even $\mathfrak{o}$-lattice of rank one is isomorphic to a lattice of the form

$$[\mathfrak{c}, \omega] := (\mathfrak{c}, (x, y) \mapsto \omega xy),$$

where $\mathfrak{c}$ is a nonzero fractional $\mathfrak{o}$-ideal, ω is a totally positive element of K such that the *modified level*

$$\mathfrak{m} := \tfrac{1}{2}\omega\mathfrak{d}\mathfrak{c}^2$$

of $[\mathfrak{c}, \omega]$ is an integral ideal ([Boy15, Prop. 3.10]). Note that $\mathfrak{m}$ depends only on the isomorphism class of the lattice $[\mathfrak{c}, \omega]$. Indeed, if a runs through $K^\times$, then $[a\mathfrak{c}, a^{-2}\omega]$ runs through the isomorphism class of $[\mathfrak{c}, \omega]$ (see [Boy15, §3.1, Prop. 3.9]), and all members of this isomorphism class have obviously the same modified level. A simple calculation shows

$$[\mathfrak{c}, \omega]^\sharp = \mathfrak{c}/2\mathfrak{m}.$$

The Fourier expansion (2) of any ϕ in $J_{k,[\mathfrak{c},\omega]}$ can therefore be written in the form

$$\phi = \sum_{\substack{n\in\mathfrak{d}^{-1}, r\in\mathfrak{c}/2\mathfrak{m} \\ n-\frac{1}{2}\omega r^2 \gg 0}} c_\phi\left(n, r\right)\, q^n \zeta^{\omega r} = \sum_{\substack{\Delta\in\mathfrak{c}^2/\mathfrak{m}^2,\ r\in\mathfrak{c}/\mathfrak{m} \\ \Delta \ll 0,\ \Delta\equiv r^2 \bmod 4\mathfrak{c}^2/\mathfrak{m}}} C_\phi(\Delta, r)\, q^{\frac{\omega}{8}(r^2-\Delta)} \zeta^{\frac{\omega}{2}r}.$$

The coefficient $C_\phi(\Delta, r) = c_\phi\left(\frac{\omega}{8}(r^2 - \Delta), \frac{r}{2}\right)$ depends only on the coset $r + 2\mathfrak{c}$ as we saw in the previous section. We can assume if convenient that $\mathfrak{c}$ is integral and divisible by $\mathfrak{m}$ (after replacing $\mathfrak{c}$ by $a\mathfrak{c}$ with a suitable a in $K^\times$).

Lemma 2.1. *If* $\mathfrak{m} = \mathfrak{o}$*, the class of* r *in* $\mathfrak{c}/2\mathfrak{c}$ *is already uniquely determined by the congruence* $\Delta \equiv r^2 \bmod 4\mathfrak{c}^2$.

Proof. Namely, if r and s in $\mathfrak{c}$, the congruence $r^2 \equiv s^2 \bmod 4\mathfrak{c}^2$ implies $(2a\mathfrak{c})^2 \mid (ar + as)(ar - as)$, where a is any element of $K^\times$ such that $a\mathfrak{c}$ is integral; since $2a\mathfrak{c} \mid 2as$, so that $(ar + as) \equiv (ar - as) \bmod 2a\mathfrak{c}$, we conclude $2a\mathfrak{c} \mid ar - as, ar + as$, and hence in any case $r - s \in 2\mathfrak{c}$. □

We write therefore in the following simply $C_\phi(\Delta)$ for $C_\phi(\Delta, r)$.

2.5. **The theta expansion of Jacobi forms.** We shall also need the Jacobi-theta function from [Boy15, Def. 3.32]. For τ in $\mathcal{H}$ and z in $L_\mathcal{C}$ it is defined as

$$\vartheta_{\underline{L},s} = \sum_{\substack{t\in L^\sharp \\ t\equiv s \bmod L}} q^{\beta(t)} \zeta_\beta^t.$$

Here s is an element of the dual $L^\sharp$ of L, i.e. an element of $K \otimes_\mathfrak{o} L$ such that $\beta(s, x) \in \mathfrak{d}^{-1}$ for all x in L.

If ϕ is a Jacobi form in $J_{k,\underline{L}}$ then, on using (3) we can write its Fourier expansion in the form

$$\phi = \sum_{x\in L^\sharp/L} h_x\, \vartheta_{\underline{L},x}, \quad \text{where} \quad h_x = \sum_{\substack{D\gg 0 \\ D\equiv -\beta(x) \bmod \mathfrak{d}^{-1}}} C_\phi(D, x)\, q^D.$$

One can show that $\vartheta_{\underline{L},s}$ is a Jacobi form of weight $r_{\underline{L}}/2$ and index $\underline{L}$ on some subgroup of $\mathrm{SL}(2, \mathfrak{o})$. More generally, $\Theta(\underline{L}) = \langle \vartheta_{\underline{L},x} : x \in L^\sharp\rangle$ is invariant under the (projective) action of $\mathrm{SL}(2, \mathfrak{o})$ defined by $(A, \vartheta) \mapsto \vartheta|_{r_{\underline{L}}/2,\underline{L}} A^{-1}$ (see [Boy15, Thm. 3.1]), in other words, for any y in $L^\sharp/L$,

$$\vartheta_{\underline{L},y}|_{r_{\underline{L}}/2,\underline{L}} A^{-1} = \sum_{x\in L^\sharp/L} \vartheta_{\underline{L},x}\, \omega(A)_{x,y}, \tag{5}$$

where $A \mapsto \omega(A) := \left(\omega(A)_{x,y}\right)_{x,y\in L^\sharp/L}$ defines a projective representation of $\mathrm{SL}(2, \mathfrak{o})$. From [Boy15, Thm 3.4 (ii)] we have

Lemma 2.2. *The image of ω is finite.*

Later we shall also need:

Lemma 2.3. *Let $s \in L^\sharp$, k be integral and $A = \left[\begin{smallmatrix} d^{-1} & b \\ 0 & d \end{smallmatrix}\right] \in \mathrm{SL}(2, \mathfrak{o})$. Then,*

$$\vartheta_{\underline{L},s}|_{k,\underline{L}}A = \mathrm{N}(d)^{-k} e\left\{\tfrac{b}{d}\beta(s)\right\} \vartheta_{\underline{L},s/d}.$$

Remark. If $s \in \mathrm{Iso}(\underline{L})$, then clearly, $e\left\{\frac{b}{d}\beta(s)\right\} = 1$ (since d is a unit).

Proof. Using the definition of the slash action $|_{k,\underline{L}}$ in (1), we obtain

$$\begin{aligned}\vartheta_{\underline{L},s}|_{k,\underline{L}}A &= \mathrm{N}(d)^{-k} \sum_{\substack{t \in L^\sharp \\ t \equiv s \bmod L}} e\left\{(\tfrac{1}{d^2}\tau + \tfrac{b}{d})\beta(t)\right\} e\left\{\beta(t, \tfrac{z}{d})\right\} \\ &= \mathrm{N}(d)^{-k} \sum_{\substack{t \in L^\sharp \\ t \equiv s \bmod L}} q^{\beta(t/d)} \zeta_\beta^{t/d} e\left\{\tfrac{b}{d}\beta(t)\right\} = \mathrm{N}(d)^{-k} e\left\{\tfrac{b}{d}\beta(s)\right\} \vartheta_{\underline{L},s/d}.\end{aligned}$$

For the last identity we used that $\frac{b}{d}\beta(t) \equiv \frac{b}{d}\beta(s) \bmod L$ whenever $t \equiv s \bmod L$. Indeed, write $t = s + l$ ($l \in L$). Then we have $\beta(t) = \beta(s+l) = \beta(s) + \beta(l) + \beta(s, l)$. But $\beta(s, l) \in \mathfrak{d}^{-1}$ (since $t \in L^\sharp$ and $l \in L$). Also $\beta(l) \in \mathfrak{d}^{-1}$, since $\underline{L}$ is an even integral $\mathfrak{o}$-lattice. The claimed congruence follows now from the fact that d is a unit. □

2.6. **The subspace $J_{k,\underline{L}}^\infty$.** For a Jacobi form ϕ, we define the *singular part of* ϕ by

$$\phi_{\mathrm{sing}} = \sum_{s \in \mathrm{Iso}(\underline{L})} C_\phi(0, s) q^{\beta(r)} \zeta_\beta^r,$$

where

$$\mathrm{Iso}(\underline{L}) = \{s \in L^\sharp : \beta(s) \in \mathfrak{d}^{-1}\}.$$

Denote by $J_{k,\underline{L}}^\infty$ the subspace of ϕ in $J_{k,\underline{L}}$ such that $\phi_{\mathrm{sing}} = 0$.

The group of units $\mathfrak{o}^\times$ of $\mathfrak{o}$ acts on $\mathrm{Iso}(L)$ and on the set $\mathrm{Iso}(L)/L$ of L-orbits $s + L$. If ϕ is a Jacobi form in $J_{k,\underline{L}}$, then $C_\phi(0, s)$ depends only on $s + L$ in $\mathrm{Iso}(L)/L$, and by (4) we have $C_\phi(0, us) = \mathrm{N}(u)^k C_\phi(0, s)$ for any u in $\mathfrak{o}^\times$ and s in $\mathrm{Iso}(L)$. In particular, $C_\phi(0, s) = 0$ if there is a unit u with $us \equiv s \bmod L$ and $\mathrm{N}(u)^k = -1$. Set

(6)
$$\mathrm{Iso}_k(\underline{L}) = \left\{s \in \mathrm{Iso}(\underline{L}) : \text{for all } u \in \mathfrak{o}^\times \text{ with } us \equiv s \bmod L \text{ one has } \mathrm{N}(u)^k = +1\right\}$$

From the preceding discussion it is clear that

$$\phi_{\mathrm{sing}} = \sum_{s \in \mathrm{Iso}_k(\underline{L})/L} C_\phi(0, s)\, \vartheta_{\underline{L},s},$$

and that ϕ_{sing} is contained in the subspace $\Theta^{k-\mathrm{sing}}(\underline{L})$ of all ϑ in $\Theta(\underline{L})$ such that $\vartheta|_{k,\underline{L}}A = \vartheta$ for all upper triangular matrices A in $\mathrm{SL}(2, \mathfrak{o})$. We have then the exact sequence

$$0 \longrightarrow J_{k,\underline{L}}^\infty \xrightarrow{\subseteq} J_{k\underline{L}} \xrightarrow{\mathrm{sing}} \Theta^{k-\mathrm{sing}}(\underline{L}), \tag{7}$$

where sing maps a Jacobi form ϕ to its singular part ϕ_{sing}. As an immediate consequence of Theorem 3.1 below we have

Proposition 2.4. *For $k > 2 + \frac{n}{2}$, the map* sing *in* (7) *is surjective.*

In particular,

$$\dim J_{k,\underline{L}}^\infty = \dim \Theta^{k-\mathrm{sing}}(\underline{L}) = \mathrm{card}\left((\mathrm{Iso}_k(\underline{L})/L) / \mathfrak{o}^\times\right).$$

If the class number of K is one, then $J_{k,\underline{L}}^\infty$ is a complement of the subspace of cusp forms in $J_{k,\underline{L}}$ [Boy15, remark after Def. 3.47].

3. Jacobi-Eisenstein series

Throughout this section $\underline{L} = (L, \beta)$ denotes a totally positive definite even integral $\mathfrak{o}$-lattice of level $\mathfrak{l}$ and rank $r_{\underline{L}}$. We shall use $G = \mathrm{SL}(2, \mathfrak{o})$.

Definition. For integral k and $s \in \mathrm{Iso}_k(L)$ (see (6)), we define

$$E_{k,\underline{L},s} = \sum_{A \in G_{\mathfrak{l}} \backslash G} \vartheta_{\underline{L},s}|_{k,\underline{L}} A, \tag{8}$$

where $G_{\mathfrak{l}}$ denotes the subgroup of all $\left[\begin{smallmatrix} d^{-1} & b \\ 0 & d \end{smallmatrix}\right]$ in G such that $d \equiv 1 \bmod \mathfrak{l}$.

The sum in (8) is well-defined, i.e. the Ath term $\vartheta_{\underline{L},s}|_{k,\underline{L}} A$ in the sum defining $E_{k,\underline{L},s}$ depends indeed only on $G_{\mathfrak{l}} A$, as is easily deduced from Lemma 2.3 (for the deduction note that $ds \equiv s \bmod L$ for all s in $L^\sharp$ and $d \equiv 1 \bmod \mathfrak{l}$ since the level $\mathfrak{l}$ lies in the annihilator of $\underline{L}$ [Boy15, Prop. 1.5]).

We shall show in a moment that the sum defining $E_{k,\underline{L},s}$ is absolutely convergent for $k > 2 + r_{\underline{L}}/2$. Note that $E_{k,\underline{L},s} = E_{k,\underline{L},s'}$ for $s \equiv s' \bmod L$ (since then $\vartheta_{\underline{L},s} = \vartheta_{\underline{L},s'}$). Moreover, one has $E_{k,\underline{L},s} = N(u)^k E_{k,\underline{L},us}$ for all u in $\mathfrak{o}^\times$ (as one sees by replacing A by $\left[\begin{smallmatrix} u & 0 \\ 0 & u^{-1} \end{smallmatrix}\right] A$, which is allowed since $\left[\begin{smallmatrix} u & 0 \\ 0 & u^{-1} \end{smallmatrix}\right]$ normalizes $G_{\mathfrak{l}}$, and applying Lemma 2.3).

Theorem 3.1. *Suppose $k - r_{\underline{L}}/2 > 2$. The series* (8) *is absolutely and uniformly convergent on compact subsets of $\mathcal{H}$, and converges towards an element of $J_{k,\underline{L}}$. Its singular part equals*

$$\left(E_{k,\underline{L},s}\right)_{\mathrm{sing}} = \sum_{u \in \mathfrak{o}^\times / \mathfrak{o}_{\mathfrak{l}}} N(u)^k \vartheta_{\underline{L},us},$$

where $\mathfrak{o}_{\mathfrak{l}}$ denotes the group of units u in $\mathfrak{o}^\times$ such that $u \equiv 1 \bmod \mathfrak{l}$.

Proof. Write (8) in the form

$$E_{k,\underline{L},s} = \sum_{A \in G_{\mathfrak{l}} \backslash G} \mathrm{N}(c\tau + d)^{-(k - r_{\underline{L}}/2)} \vartheta_{\underline{L},s}|_{\frac{r_{\underline{L}}}{2},\underline{L}} A.$$

Using (5) we have

$$E_{k,\underline{L},s} = \sum_{A \in G_{\mathfrak{l}} \backslash G} \mathrm{N}(c\tau + d)^{-(k - r_{\underline{L}}/2)} \sum_{x \in L^\sharp / L} \vartheta_{\underline{L},x}\, \omega_{x,s}(A^{-1}).$$

Let $S = V \times C$, where $V \subset \mathcal{H}$ is a cusp sector in the sense of [Fre90, Ch. 1, §2, p. 29] and C is a compact subset of $L_{\mathcal{C}}$. We shall show in a moment that the sum in (8) is normally convergent on S. Since every compact subset of $\mathcal{H} \times L_{\mathcal{C}}$ is contained in such an S, this implies the first part of the theorem.

For verifying the normal convergence on S we note that each $\vartheta_{\underline{L},x}$ is bounded to above on S as follows from its Fourier expansion. Since the image of ω is finite (see Lemma 2.2) we have then that the absolute value of the inner sum in the last expression for $E_{k,\underline{L},s}$ is bounded to above by a constant γ independent of A (but dependent on S). Therefore, on S the sum of $\left|\vartheta_{\underline{L},s}|_{k,\underline{L}} A\right|$ over a set of representatives A for $G_{\mathfrak{l}} \backslash G$ is bounded to above by

$$\gamma \sum_{A \in G_{\mathfrak{l}} \backslash G} \sup_{(\tau,z) \in S} |\, \mathrm{N}(c\tau + d)^{-(k - r_{\underline{L}}/2)}|.$$

But this sum is convergent [Fre90, Chap. 1, Lemma 5.7].

The first part implies that $E_{k,\underline{L},s}$ is holomorphic. From the definition it is immediate that $E_{k,\underline{L},s}$ satisfies $E_{k,\underline{L},s}|_{k,\underline{L}} g = E_{k,\underline{L},s}$ for all g in G and all g in $H(\underline{L})$. Hence $E_{k,\underline{L},s}$ defines an element of $J_{k,\underline{L}}$ (for $K = \mathbb{Q}$ we still need to check that $E_{k,\underline{L},s}$ is holomorphic at infinity, which will follow from the considerations below).

For verifying the formula for the singular part of $E_{k,\underline{L},s}$ we consider the theta expansion of $E_{k,\underline{L},s}$, i.e. we write

$$E_{k,\underline{L},s} = \sum_{x \in L^\sharp/L} h_x \, \vartheta_{\underline{L},x}, \quad \text{where} \quad h_x = \sum_{A \in G_{\mathfrak{l}} \backslash G} \omega(A^{-1})_{x,s} \, \mathrm{N}(c\tau + d)^{-(k - r_{\underline{L}}/2)}.$$

For r in $\mathrm{Iso}(\underline{L})$ the coefficient $C_{E_{k,\underline{L},s}}(0,r)$ equals the constant term of h_r, which can be computed by setting $\tau = it$ for real t and let t tend to infinity. Since the sum defining h_x is uniformly convergent on cusp sectors we can interchange the limit and the sum and obtain

$$C_{E_{k,\underline{L},s}}(0,r) = \sum_{A = \left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right] \in G_{\mathfrak{l}} \backslash G_{\mathfrak{o}}} \omega(A^{-1})_{r,s} \, \mathrm{N}(d)^{-(k - r_{\underline{L}}/2)}$$

(where $G_{\mathfrak{o}}$ is the subgroup of G of all matrices of the form $\left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right]$). A set of representatives of $G_{\mathfrak{l}} \backslash G_{\mathfrak{o}}$ is given by the matrices $\left[\begin{smallmatrix} 1/d & 0 \\ 0 & d \end{smallmatrix}\right]$, where d runs through a set of representatives for $\mathfrak{o}^\times/\mathfrak{o}_{\mathfrak{l}}$. By Lemma 2.3 $\omega\left(\left[\begin{smallmatrix} d & 0 \\ 0 & 1/d \end{smallmatrix}\right]\right)_{r,s}$ equals $\mathrm{N}(d)^{-r_{\underline{L}}/2}$ if $r \equiv s/d \bmod L$, and 0 otherwise. The claimed formula is now obvious. □

For calculating the Fourier coefficients of Jacobi-Eisenstein series we write their definition (8) in a slightly different way. For this we need, first of all, a description of $G_{\mathfrak{l}} \backslash G$.

Lemma 3.2. *The application $A \mapsto (0,1)A$ indices a bijection*

$$G_{\mathfrak{l}} \backslash G \to \{(c,d) \in \mathfrak{o}^2 : \gcd(c,d) = \mathfrak{o}\}/\mathfrak{o}_{\mathfrak{l}}. \tag{9}$$

Remark. Note that representatives for the orbits of the right hand side of (9) are given by $(0,d)$, where d is a representative in $\mathfrak{o}^\times/\mathfrak{o}_{\mathfrak{l}}$, and by (c,d) $(c \neq 0)$, where c is a representative in $\mathfrak{o}/\mathfrak{o}_{\mathfrak{l}}$ and $d \in \mathfrak{o}$ such that $(c,d) = 1$.

Proof of Lemma 3.2. First we prove the well-definedness. For that suppose $A = \left[\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} * & * \\ c' & d' \end{smallmatrix}\right]$ lie in the same coset, i.e. we have $A = \left[\begin{smallmatrix} * & * \\ 0 & \gamma \end{smallmatrix}\right] B$ for some $\gamma \in \mathfrak{o}^\times$ with $d \equiv 1 \bmod \mathfrak{l}$. Hence, $(c,d) = \gamma(c',d')$, which shows that $(0,1)A$ and $(0,1)B$ lie in the same orbit under the action of $\mathfrak{o}_{\mathfrak{l}}$.

Now we prove the injectivity. For that suppose $(0,1)A = (c,d)$ and $(0,1)B = (c',d')$ $(A := \left[\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right],\ B := \left[\begin{smallmatrix} * & * \\ c' & d' \end{smallmatrix}\right] \in G)$ lie in the same orbit under the action of $\mathfrak{o}_{\mathfrak{l}}$. Hence, $c = c'u$ and $d = d'u$ for some $u \in \mathfrak{o}_{\mathfrak{l}}$. Therefore, one can form a matrix $U = \left[\begin{smallmatrix} * & * \\ 0 & u \end{smallmatrix}\right] \in G_{\mathfrak{l}}$ such that $A = UB$. Surjectivity follows from Bezout's theorem. □

Lemma 3.3. *Let $s \in L^\sharp$ and k be integral. Then, we have*

$$\vartheta_{\underline{L},s} = \sum_{x \in L} \left(q^{\beta(s)} \zeta_\beta^s\right)|_{k,\underline{L}}(x,0,1). \tag{10}$$

Proof. We shall calculate the right hand side of the claimed identity, and show that it equals $\vartheta_{\underline{L},s}$. Using the slash actions $|_{k,\underline{L}}$ as in (1), we have

$$\begin{aligned}\left(q^{\beta(s)} \zeta_\beta^s\right)|_{k,\underline{L}}(x,0,1)(\tau,z) &= e\{\tau\beta(x) + \beta(x,z)\}\, q^{\beta(s)} e\{\beta(s, z + x\tau)\} \\ &= q^{\beta(x+s)}(\tau) \zeta_\beta^{x+s}(z).\end{aligned}$$

Now inserting $q^{\beta(x+s)} \zeta_\beta^{x+s}$ into the right hand side of (10) and doing the substitution $x \mapsto x - s$ yields the result. □

Proposition 3.4. *We have*

$$E_{k,\underline{L},s}(\tau,z) = \sum_{u\in\mathfrak{o}^\times/\mathfrak{o}_\mathfrak{l}} \mathrm{N}(u)^k\,\vartheta_{\underline{L},us}(\tau,z)$$
$$+\sum_{\substack{c\in\mathfrak{o}/\mathfrak{o}_\mathfrak{l}\\ c\neq 0}}\ \sum_{\substack{d\in\mathfrak{o}\\ (c,d)=1}} \mathrm{N}(c\tau+d)^{-k}\sum_{x\in L} e\left\{\tfrac{a\tau+b}{c\tau+d}\beta(x+s)+\tfrac{1}{c\tau+d}\beta(x+s,z)-\tfrac{c}{c\tau+d}\beta(z)\right\}.$$

Here for a given (c,d), the pair (a,b) denotes any solution of $ad-bc=1$.

Proof. Using the expression for $\vartheta_{\underline{L},s}$ from Lemma 3.3 the defining formula of the Jacobi-Eisenstein series (8) can be written in the form

$$E_{k,\underline{L},s} = \sum_{A=\left[\begin{smallmatrix} a & b\\ 0 & d\end{smallmatrix}\right]\in G_\mathfrak{l}\backslash G} \vartheta_{\underline{L},s}|_{k,\underline{L}}A + \sum_{\substack{A=\left[\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right]\in G_\mathfrak{l}\backslash G\\ c\neq 0}}\ \sum_{x\in L}\left(q^{\beta(s)}\zeta_\beta^s\right)|_{k,\underline{L}}(x,0,1)A.$$

We have

$$\left\{\left(q^{\beta(s)}\zeta_\beta^s\right)|_{k,\underline{L}}(x,0,1)A\right\}(\tau,z) =$$
$$\mathrm{N}(c\tau+d)^{-k}e\left\{\tfrac{-c\beta(z)}{c\tau+d}\right\}e\left\{\beta(x)A\tau+\beta\big(x,\frac{z}{c\tau+d}\big)\right\}\times$$
$$e\left\{\beta(s)A\tau\right\}e\left\{\beta\left(s,\tfrac{z}{c\tau+d}+xA\tau\right)\right\}.$$

Inserting this into the second sum of the last formula for $E_{k,\underline{L},s}$, and using the representatives for $G_\mathfrak{l}\backslash G$ according to the remark after Lemma 3.2 proves the proposition. □

4. Fourier Coefficients of Jacobi-Eisenstein Series

In this section we shall calculate the Fourier coefficients of Jacobi-Eisenstein series $E_{k,\underline{L},s}$ at the cusp at infinity. A first formula is given by the following theorem.

Theorem 4.1. *Let k be integral, let $\underline{L}$ be a totally positive definite even integral $\mathfrak{o}$-lattice of $\mathfrak{o}$-rank $r_{\underline{L}}$, and let s in $Iso_k(\underline{L})$. Assume that $k-\frac{1}{2}r_{\underline{L}}>2$. Then, we have*

$$E_{k,\underline{L},s} = \sum_{u\in\mathfrak{o}^\times/\mathfrak{o}_\mathfrak{l}}\mathrm{N}(u)^k\,\vartheta_{\underline{L},us} + C\sum_{\substack{(n,r)\in\mathfrak{d}^{-1}\times L^\sharp\\ D:=n-\beta(r)\gg 0}}\left[\mathrm{N}(D)^{k-1-\frac{1}{2}r_{\underline{L}}}\sum_{\substack{c\in\mathfrak{o}/\mathfrak{o}_\mathfrak{l}\\ c\neq 0}}\frac{\gamma_{n,r}(c,s)}{\mathrm{N}(c)^k}\right]q^n\zeta_\beta^r,$$

where

$$\gamma_{n,r}(c,s) := \sum_{\substack{d \bmod c\\ (c,d)=1}}\ \sum_{x\in L/cL} e\left\{\tfrac{a}{c}\beta(x+s)-\tfrac{1}{c}\beta(r,x+s)+\tfrac{d}{c}n\right\},$$

where the a in the inner sum denotes any element in $\mathfrak{o}$ with $ad\equiv 1 \bmod c$, and where C is given by (11) below.

For the proof of the theorem we use the following formula.

Lemma 4.2. *For $k-\frac{r_L}{2}>1$, one has*

$$\sum_{(t,p)\in\mathfrak{o}\times L}\mathrm{N}(\tau+t)^{-k}e\left\{-\tfrac{\beta(z+p)}{\tau+t}\right\} = C\sum_{\substack{(n,r)\in\mathfrak{d}^{-1}\times L^\sharp\\ D:=n-\beta(r)\gg 0}} N(D)^{k-1-\frac{1}{2}r_{\underline{L}}}\,q^n\zeta_\beta^r,$$

where

$$C = \frac{i^{-kn_K}(2\pi)^{k+1-\frac{1}{2}r_{\underline{L}}}}{\Gamma\left(k-\frac{1}{2}r_{\underline{L}}\right)^{n_K}}D_K^{-\frac{1}{2}}\det(\mathrm{tr}\,(\underline{L}))^{-\frac{1}{2}}. \tag{11}$$

Proof. Let $F(\tau, z)$ denote the left hand side of the claimed formula. Since F is periodic and holomorphic in (τ, z) with respect to the period lattice $\mathfrak{o} \times L$, we can write it in the form

$$F(\tau, z) = \sum_{(n,r)\in\mathfrak{d}^{-1}\times L^\sharp} \gamma(n, r)\, q^n \zeta_\beta^r \tag{12}$$

for suitable coefficients $\gamma(n, r)$. For finding the $\gamma(n, r)$ we apply the Poisson summation formula (see e.g. [Ser79, Part II, §6, Prop. 15]), which in this case implies that, for all τ and z, one has

$$\gamma(n, r)\, q^n \zeta_\beta^r = C_1 \int_{\mathcal{R}\times(\mathcal{R}\otimes_\mathfrak{o} L)} f(t, p)\, e\left\{-nt - \beta(r, p)\right\}\, dt\, dp. \tag{13}$$

Here, $f(t, p)$ denotes the (t, p)-th term of the sum defining of $F(\tau, z)$, and dt and dp are Haar measures on $\mathcal{R}$ and $\mathcal{R} \otimes_\mathfrak{o} L$, respectively. Moreover,

$$C_1 = \mathrm{vol}_{dt}(\mathcal{R}/\mathfrak{o})^{-1}\, \mathrm{vol}_{dp}((\mathcal{R} \otimes_\mathfrak{o} L)/L)^{-1}.$$

Inserting the formula for $f(t, p)$ we obtain

$$\gamma(n, r) = C_1 \int_{\mathcal{R}} \mathrm{N}(\tau + t)^{-k} e\left\{-n(\tau + t)\right\} \times \\ \int_{\mathcal{R}\otimes_\mathfrak{o} L} e\left\{-\tfrac{\beta(z+p)}{\tau+t}\right\} e\left\{-\beta(r, z + p)\right\}\, dt\, dp.$$

Note that $\gamma(n, r)$, being a coefficient in the Fourier development (12), does neither depend on z nor on τ. In other words, for evaluating the integral defining $\gamma(n, r)$ we are free to choose z and τ. Denote the inner integral by J. After completing the square J becomes

$$J = e\left\{(\tau + t)\beta(r)\right\} \int_{\mathcal{R}\otimes_\mathfrak{o} L} e\left\{-\tfrac{\beta(z+p+(\tau+t)r)}{\tau+t}\right\} dp.$$

Choosing $z = -(\tau + t)r$, we have $J = e\left\{(\tau + t)\beta(r)\right\} \Lambda(\tau + t)$, where, for any τ in $\mathcal{H}$, we set

$$\Lambda(\tau) := \int_{\mathcal{R}\otimes_\mathfrak{o} L} e\left\{-\tfrac{\beta(p)}{\tau}\right\} dp.$$

This is a standard Gaussian integral with respect to the (complex-valued) quadratic form $p \mapsto \mathrm{tr}\left(-\beta(p)/\tau\right)$ on the real vector space $\mathcal{R}\otimes_\mathfrak{o} L$. Its value equals $C_2\, \mathrm{N}(\tau)^{r_{\underline{L}}/2}$ with

$$C_2 = \Lambda(i)\, \mathrm{N}(i)^{-r_{\underline{L}}/2}.$$

(For this note that the integral and $\mathrm{N}(\tau)^{r_{\underline{L}}/2}$ are holomorphic functions of τ, and it suffices therefore to prove the claimed identity for $\tau = iv$ with v in $\mathcal{R}$, which is quickly checked by substituting $p\sqrt{v}$ for p.) We therefore obtain

$$\gamma(n, r) = C_1 C_2 \int_{\mathcal{R}} \mathrm{N}(\tau + t)^{-k+r_{\underline{L}}/2} e\left\{(\tau + t)\left(\beta(r) - n\right)\right\}\, dt.$$

Recall that $\gamma(n, r)$ does not depend on τ.

Let $D := n - \beta(r)$. If $\sigma(D) \le 0$ for at least one embedding σ we can choose τ such that $\sigma(\tau) = iv$ with a positive real v and let v tend to infinity. But then the integral on the right tends to 0, and we have $\gamma(n, r) = 0$. If $D \gg 0$, we replace τ by τ/D and substitute t/D for t, which yields

$$\gamma(n, r) = C_1 C_2 C_3 N(D)^{k-1-r_{\underline{L}}/2},$$

where

$$C_3 = \int_{\mathcal{R}} \mathrm{N}(\tau + t)^{-k+r_{\underline{L}}/2} e\left\{-(\tau + t)\right\}\, dt.$$

Using the formulas for C_2 and C_3 from Corollary 7.2 and Proposition 7.3, we obtain the claimed formula. $\square$

Proof of Theorem 4.1. From Proposition 3.4 we have

$$E_{k,\underline{L},s} = \sum_{u\in\mathfrak{o}^\times/\mathfrak{o}_\mathfrak{l}} \mathrm{N}(u)^k \vartheta_{\underline{L},us} + I_s,$$

where

$$I_s = \sum_{\substack{c\in\mathfrak{o}/\mathfrak{o}_\mathfrak{l}\\ c\neq 0}} \mathrm{N}(c)^{-k} \sum_{\substack{d\in\mathfrak{o}\\ (c,d)=1}} \mathrm{N}(\tau+\tfrac{d}{c})^{-k} \sum_{x\in L} e\left\{-\tfrac{c}{c\tau+d}\beta(z-\tfrac{x+s}{c}) + \tfrac{a}{c}\beta(x+s)\right\}$$

(and where we used $\frac{a}{c} - \frac{1}{c(c\tau+d)} = \frac{a\tau+b}{c\tau+d}$). Replacing d by $d+ct$, where d runs through a set of representatives for $d \bmod c$ and t through $\mathfrak{o}$, and replacing x by $x-cp$ with x running through a set of representatives for L/cL and p through L, the sum I_s can be rewritten in the form

$$I_s = \sum_{\substack{c\in\mathfrak{o}/\mathfrak{o}_\mathfrak{l}\\ c\neq 0}} \mathrm{N}(c)^{-k} \sum_{\substack{d \bmod c\\ (c,d)=1}} \sum_{x\in L/cL} e\left\{\tfrac{a}{c}\beta(x+s)\right\} F(\tau+\tfrac{d}{c}, z-\tfrac{x+s}{c}), \tag{14}$$

where

$$F(\tau,z) = \sum_{(t,p)\in\mathfrak{o}\times L} \mathrm{N}(\tau+t)^{-k} e\left\{-\tfrac{\beta(z+p)}{\tau+t}\right\}.$$

Inserting the Fourier expansion of F from Lemma 4.2 into (14) gives

$$I_s = C \sum_{\substack{c\in\mathfrak{o}/\mathfrak{o}_\mathfrak{l}\\ c\neq 0}} \mathrm{N}(c)^{-k} \sum_{\substack{d \bmod c\\ (c,d)=1}} \sum_{x\in L/cL}$$
$$\sum_{\substack{(n,r)\in\mathfrak{d}^{-1}\times L^\sharp\\ D:=n-\beta(r)\gg 0}} \mathrm{N}(D)^{k-1-\frac{1}{2}r_{\underline{L}}} e\left\{\tfrac{a}{c}\beta(x+s) + \tfrac{nd}{c} - \beta(r,\tfrac{x+s}{c})\right\} q^n \zeta_\beta^r.$$

Changing the order of summation and collecting the exponential together yields the formula given in the theorem. $\square$

If we take $s=0$ then $E_{k,\underline{L},s}$ can be further simplified. Namely, we have

Theorem 4.3. *Under the same assumptions and notations as in Theorem 4.1 we have for $s=0$*

$$\tfrac{1}{[\mathfrak{o}^\times:\mathfrak{o}_\mathfrak{l}]} E_{k,\underline{L},0} = \vartheta_{\underline{L},0} + \tfrac{C}{[\mathfrak{o}^\times:\mathfrak{o}_\mathfrak{l}]} \sum_{\substack{(n,r)\in\mathfrak{d}^{-1}\times L^\sharp\\ D:=n-\beta(r)\gg 0}} \mathrm{N}(D)^{k-1-\frac{1}{2}r_{\underline{L}}} L(D,r;k-1)\, q^n \zeta_\beta^r.$$

Here, for $D\in K^\times$ and r in $L^\sharp$ such that $D+\beta(r)\equiv 0 \bmod \mathfrak{d}^{-1}$, we use $L(D,r;s)$ for the Dirichlet series

$$L(D,r;s) = \sum_{C\in \mathrm{Cl}(K)} \Big(\sum_{\mathfrak{b}\in C^{-1}} \frac{\mu(\mathfrak{b})}{\mathrm{N}(\mathfrak{b})^{s-r_{\underline{L}}+1}}\Big)\Big(\sum_{\mathfrak{a}\in C} \frac{N_\mathfrak{a}(D,r)}{\mathrm{N}(\mathfrak{a})^s}\Big),$$

where the first sum is over all ideal classes C of K, and the inner sums are over all integral ideals $\mathfrak{b}$ in C^{-1} and $\mathfrak{a}$ in C, respectively. Moreover, μ is the Möbius function on the semigroup of nonzero integral ideals of K, and

$$N_\mathfrak{a}(D,r) = \mathrm{card}\left(\{x\in L/\mathfrak{a}L : \beta(x-r)\equiv -D \bmod \mathfrak{a}\mathfrak{d}^{-1}\}\right). \tag{15}$$

Remark. Note that the assumption $s = 0 \in \mathrm{Iso}_k(\underline{L})$ implies k is even if K contains a unit of norm -1, and vice versa.

Proof of Theorem 4.3. It is easy to see that the first summand in Theorem 4.1 for $s = 0$ reads $[\mathfrak{o}^\times : \mathfrak{o}_\mathfrak{l}]\,\vartheta_{\underline{L},0}$. We now simplify the second summand. For that we have to calculate the sum

$$(16) \qquad S := \sum_{\mathfrak{c}} \mathrm{N}(c)^{-k} \sum_{\substack{d \bmod c\\(c,d)=1}} \sum_{x \in L/cL} e\left\{\tfrac{a}{c}\beta(x) - \beta(r, \tfrac{x}{c}) + \tfrac{d}{c}n\right\}.$$

Here we replaced the sum of c in $\mathfrak{o}/\mathfrak{o}_\mathfrak{l}$ by $[\mathfrak{o}^\times : \mathfrak{o}_\mathfrak{l}]$ times the sum over $c \in \mathfrak{o}/\mathfrak{o}^\times$, which becomes then a sum over all integral principal ideals $\mathfrak{c} = \mathfrak{o}c$.

We replace x by dx in the inner sum and write

$$\tfrac{a}{c}\beta(dx) - \beta(r, \tfrac{dx}{c}) + \tfrac{d}{c}n \equiv \tfrac{d}{c}(\beta(x) - \beta(r,x) + n) \bmod \mathfrak{d}^{-1},$$

where we use that $ad \equiv 1 \bmod c$. Note that $N := \beta(x) - \beta(r,x) + n$ is in $\mathfrak{d}^{-1}$. We change the order of the last two sums in (16), and calculate the sum

$$\tilde{S} := \sum_{\substack{d \bmod c\\(c,d)=1}} e\left\{\tfrac{d}{c}N\right\}.$$

For this we write

$$\tilde{S} = \sum_{d \bmod c} \left(\sum_{\mathfrak{a}|(c,d)} \mu(\mathfrak{a})\right) e\left\{\tfrac{d}{c}N\right\} = \sum_{\mathfrak{a}|c} \mu(\mathfrak{a}) \sum_{d \in \mathfrak{a}/\mathfrak{c}} e\left\{\tfrac{d}{c}N\right\}.$$

Since $\chi : d \mapsto e\left\{\tfrac{d}{c}N\right\}$ defines a linear character, the sum $\sum_{d\in\mathfrak{a}/\mathfrak{c}} e\left\{\tfrac{d}{c}N\right\}$ equals zero unless χ is trivial, when it equals card $(\mathfrak{a}/\mathfrak{c})$. But the character χ is trivial if and only if $\tfrac{d}{c}N \in \mathfrak{d}^{-1}$ for all $d \in \mathfrak{a}$, i.e. if $\tfrac{\mathfrak{a}}{c}N \in \mathfrak{d}^{-1}$, or, equivalently if $N \in \mathfrak{c}\mathfrak{a}^{-1}\mathfrak{d}^{-1}$. Moreover, the order of the group $\mathfrak{a}/\mathfrak{c}$ equals $\mathrm{N}(\mathfrak{c}\mathfrak{a}^{-1})$ (as follows from the exact sequence $0 \to \mathfrak{a}/c\mathfrak{o} \to \mathfrak{o}/c\mathfrak{o} \to \mathfrak{o}/\mathfrak{a} \to 0$).

Therefore (16) becomes

$$S = \sum_{\mathfrak{c}} \sum_{\mathfrak{a}|\mathfrak{c}} \tfrac{\mu(\mathfrak{c}/\mathfrak{a})\,\mathrm{N}(\mathfrak{a})}{\mathrm{N}(\mathfrak{c})^k} \,\mathrm{card}\left(\{x \in L/\mathfrak{c}L : \beta(x-r) + D \equiv 0 \bmod \mathfrak{a}\mathfrak{d}^{-1}\}\right),$$

where we replaced $\mathfrak{a}$ by $\mathfrak{c}\mathfrak{a}^{-1}$ and wrote $\beta(x) - \beta(r,x) + n = \beta(x-r) + D$.

Interchanging the summation and writing $\mathfrak{c} = \mathfrak{a}\mathfrak{b}$, we obtain (recall that $\mathfrak{c}$ runs through principle ideals, so that for fixed $\mathfrak{a}$ the ideal $\mathfrak{b}$ runs through the integral ideals in the ideal class $\mathrm{cl}(\mathfrak{a}^{-1})$ of $\mathfrak{a}^{-1}$)

$$S = \sum_{\mathfrak{a}} \sum_{\mathfrak{b}\in\mathrm{cl}(\mathfrak{a})^{-1}} \tfrac{\mu(\mathfrak{b})\,\mathrm{N}(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^k\,\mathrm{N}(\mathfrak{b})^k} \,\mathrm{card}\left(\{x \in L/\mathfrak{a}\mathfrak{b}L : \beta(x-r) + D \equiv 0 \bmod \mathfrak{a}\mathfrak{d}^{-1}\}\right).$$

Since the defining congruence of S depends only on $x \bmod \mathfrak{a}L$, we can suppress the $\mathfrak{b}$ when counting S and multiply the result by the cardinality of the kernel of the natural map $L/\mathfrak{a}\mathfrak{b}L \to L/\mathfrak{a}L$, which equals $\mathrm{N}(\mathfrak{b})^{r_{\underline{L}}}$ (since L is isomorphic as $\mathfrak{o}$-module to $\mathfrak{s} \times \mathfrak{o}^{r_{\underline{L}}-1}$ for some integral ideal $\mathfrak{s}$, so that the kernel $\mathfrak{a}L/\mathfrak{a}\mathfrak{b}L$ is isomorphic as $\mathfrak{o}$-module to $\mathfrak{a}\mathfrak{s}/\mathfrak{a}\mathfrak{b}\mathfrak{s} \times (\mathfrak{a}/\mathfrak{a}\mathfrak{b})^{r_{\underline{L}}-1}$, which in turn is isomorphic as $\mathfrak{o}$-module to $(\mathfrak{o}/\mathfrak{b})^{r_{\underline{L}}}$). Thus,

$$S = \sum_{\mathfrak{a}} \sum_{\mathfrak{b}\in\mathrm{cl}(\mathfrak{a})^{-1}} \frac{\mu(\mathfrak{b})}{\mathrm{N}(\mathfrak{b})^{k-r_{\underline{L}}}} \frac{N_{\mathfrak{a}}(D,r)}{\mathrm{N}(\mathfrak{a})^{k-1}}.$$

The formula of the theorem becomes now obvious. ∎

5. Explicit formulas for the $N_{\mathfrak{a}}(D,r)$ in the rank one case

For rank one $\mathfrak{o}$-lattices of modified level $\mathfrak{o}$ the numbers $N_{\mathfrak{a}}(D,r)$ defined in (15) can be calculated explicitly using a result in [BS]. For explaining this result and our formulas for $N_{\mathfrak{a}}(D,r)$ we need to introduce some notations.

Let K denotes an arbitrary (not necessarily totally real) number field. Let Δ be a non-zero integer in K which is a square modulo 4. For a prime ideal $\mathfrak{p} \nmid \Delta$, we set $\left(\frac{\Delta}{\mathfrak{p}}\right) = +1$ or $= -1$ accordingly as Δ is a square modulo $4\mathfrak{p}$ or not. Of course, for $\mathfrak{p} \nmid 2$ the number Δ is a square modulo $4\mathfrak{p}$ if and only if it is square modulo $\mathfrak{p}$ as follows from the Chinese remainder theorem. We continue $\left(\frac{\Delta}{*}\right)$ to a homomorphism of the group $\mathfrak{I}_\Delta$ of fractional ideals relatively prime[2] to Δ onto the group $\{\pm 1\}$.

As is shown in [BS, Thm. 4] the homomorphism $\left(\frac{\Delta}{*}\right)$ defines a Größencharakter modulo Δ. Its conductor equals $\Delta/\mathfrak{f}_\Delta{}^2$, where $\mathfrak{f}_\Delta$ is the maximal integral ideal dividing Δ such that Δ is a square modulo $4\mathfrak{f}_\Delta^2$ (and where *maximal* refers to the partial ordering defined by *division of ideals*). We use $\left(\frac{\Delta}{*}\right)_0$ for the primitive Größencharakter modulo $\Delta/\mathfrak{f}_\Delta^2$ induced by $\left(\frac{\Delta}{*}\right)$. Using the Größencharakter $\left(\frac{\Delta}{*}\right)_0$ we define a function χ_Δ on the semigroup of all integral ideals $\mathfrak{a}$ by setting

$$\chi_\Delta(\mathfrak{a}) := \begin{cases} \mathrm{N}(\mathfrak{g}) \left(\frac{\Delta}{\mathfrak{a}/\mathfrak{g}^2}\right)_0 & \text{if } (\mathfrak{a},\Delta) = \mathfrak{g}^2 \text{ and } \Delta \text{ is a square mod } 4\mathfrak{g}^2 \\ 0 & \text{otherwise.} \end{cases}$$

Note that, for an integral square $\mathfrak{g}^2 \mid \Delta$, the condition that Δ is a square mod $4\mathfrak{g}^2$ is equivalent to $\mathfrak{g} \mid \mathfrak{f}_\Delta$. Of course, χ_Δ is no longer a homomorphism, but it remains multiplicative in the sense that $\chi_\Delta(\mathfrak{a}\mathfrak{b}) = \chi_\Delta(\mathfrak{a})\chi_\Delta(\mathfrak{b})$ whenever $\mathfrak{a}$ and $\mathfrak{b}$ are relatively prime.

Theorem 5.1. [BS, Thm. 6] *For any integral ideal* $\mathfrak{a}$*, one has*

$$\operatorname{card}\left(\{x \in \mathfrak{o}/2\mathfrak{a} : x^2 \equiv \Delta \bmod 4\mathfrak{a}\}\right) = \sum_{\substack{\mathfrak{b}\mid\mathfrak{a} \\ \mathfrak{a}/\mathfrak{b} \text{ squarefree}}} \chi_\Delta(\mathfrak{b}). \tag{17}$$

(The sum is over all integral ideals diving $\mathfrak{a}$ *and such that* $\mathfrak{a}/\mathfrak{b}$ *is squarefree.)*

Remark. In terms of Dirichlet series the formula of the theorem can be rewritten as

$$\sum_{\mathfrak{a}} \frac{\operatorname{card}\left(\{x \in \mathfrak{o}/2\mathfrak{a} : x^2 \equiv \Delta \bmod 4\mathfrak{a}\}\right)}{\mathrm{N}(\mathfrak{a})^s} = \frac{\zeta_K(s)}{\zeta_K(2s)} L(\chi_\Delta, s).$$

The L-series $L(\chi_\Delta, s)$ coincides up to a finite number of Euler factors with the L-series

$$L\left(\left(\tfrac{\Delta}{*}\right)_0, s\right) = \sum_{(\mathfrak{a},\Delta/\mathfrak{f}_\Delta^2)=1} \left(\tfrac{\Delta}{\mathfrak{a}}\right)_0 \mathrm{N}(\mathfrak{a})^{-s}$$

of the Größencharakter $\left(\frac{\Delta}{*}\right)_0$ (the sum being over all integral ideals relatively prime to $\Delta/\mathfrak{f}_\Delta^2$). More precisely, one has (see Lemma 6.1)

$$L(\chi_\Delta, s) = L\left(\left(\tfrac{\Delta}{*}\right)_0, s\right) \sum_{\mathfrak{t}\mid\mathfrak{f}_\Delta} \frac{\mu(\mathfrak{t}) \left(\frac{\Delta}{\mathfrak{t}}\right)_0}{\mathrm{N}(\mathfrak{t})^s} \sigma_{1-2s}(\mathfrak{f}_\Delta/\mathfrak{t}),$$

where we use $\left(\frac{\Delta}{\mathfrak{t}}\right)_0 = 0$ if $\mathfrak{t}$ is not relatively prime to $\Delta/\mathfrak{f}_\Delta^2$, and where $\mu(\mathfrak{a})$ is the Möbius μ-function of K.

[2]A fractional ideal is called *relatively prime* to Δ, if it is of the form $\mathfrak{a}/\mathfrak{b}$ with integral ideals $\mathfrak{a}$ and $\mathfrak{b}$ both of which have no prime ideal common with Δ.

We are now ready to prove our explicit formulas for the numbers $N_{\mathfrak{a}}(D,r)$ defined in (15) in the case of rank one lattices. For this let

$$[\mathfrak{c},\omega] = (\mathfrak{c},(x,y)\mapsto \omega xy)$$

be an even rank one lattice. Recall that this means that $\mathfrak{c}$ is a nonzero fractional $\mathfrak{o}$-ideal, and ω is a totally positive element of K such that the modified level of $[\mathfrak{c},\omega]$

$$\mathfrak{m} = \tfrac{1}{2}\omega\mathfrak{c}^2\mathfrak{d}$$

is an integral ideal. The numbers $N_{\mathfrak{a}}(D,r)$ of (15) take here the form

$$N_{\mathfrak{a}}(D,r) = \mathrm{card}\left(\left\{x\in\mathfrak{c}/\mathfrak{a}\mathfrak{c} : \tfrac{1}{2}\omega(x-r)^2 \equiv -D \bmod \mathfrak{a}\mathfrak{d}^{-1}\right\}\right).$$

Proposition 5.2. *Assume $\mathfrak{m}=\mathfrak{o}$. For any integral ideal $\mathfrak{a}$, any $D\in K^\times$ and $r\in L^\sharp$ such that $D\equiv -\frac{1}{2}\omega r^2 \bmod \mathfrak{d}^{-1}$, one has*

$$N_{\mathfrak{a}}(D,r) = \mathrm{N}(\mathfrak{c})^{-1}\sum_{\substack{\mathfrak{b}\mid\mathfrak{a}\\ \mathfrak{a}/\mathfrak{b}\ \textit{squarefree}}} \chi_\Delta(\mathfrak{b}\mathfrak{c}^2),$$

where $\Delta = -8D/\omega$.

Proof. Multiplying the congruence defining $N_{\mathfrak{a}}(D,r)$ by $8/\omega$ and setting $y=2r-2x$ gives

$$N_{\mathfrak{a}}(D,r) = \mathrm{card}\left(\{y\in\mathfrak{c}/2\mathfrak{a}\mathfrak{c} : y\equiv 2r \bmod 2\mathfrak{c},\quad y^2\equiv\Delta \bmod 4\mathfrak{a}\mathfrak{c}^2\}\right),$$

(where we also used $\mathfrak{m}=\frac{1}{2}\omega\mathfrak{c}^2\mathfrak{d}=\mathfrak{o}$ and $L^\sharp=\frac{1}{2}\mathfrak{c}\mathfrak{m}^{-1}=\frac{1}{2}\mathfrak{c}$). Note that by assumption $(2r)^2\equiv\Delta \bmod 4\mathfrak{c}^2$, so that the second congruence $y^2\equiv\Delta \bmod 4\mathfrak{a}\mathfrak{c}^2$ implies $y^2\equiv(2r)^2 \bmod 4\mathfrak{c}^2$. But this implies the first congruence $y\equiv 2r \bmod 2\mathfrak{c}$ (for the short argument see the proof of Lemma 2.1). We conclude that

$$N_{\mathfrak{a}}(D,r) = \mathrm{card}\left(\left\{y\in\mathfrak{c}/2\mathfrak{a}\mathfrak{c} : y^2\equiv\Delta \bmod 4\mathfrak{a}\mathfrak{c}^2\right\}\right).$$

After replacing $[c,\omega]$ by $[a\mathfrak{c},a^{-2}\omega]$ with a suitable a in $K^\times$ we can assume that $\mathfrak{c}$ is integral. We then have available the natural reduction map $\mathfrak{c}/2\mathfrak{a}\mathfrak{c}^2\to\mathfrak{c}/2\mathfrak{a}\mathfrak{c}$, and hence we can count the solutions y in $\mathfrak{c}$ of $\Delta\equiv y^2 \bmod 4\mathfrak{a}\mathfrak{c}^2$ modulo $2\mathfrak{a}\mathfrak{c}^2$ instead of $2\mathfrak{a}\mathfrak{c}$, so that

$$N_{\mathfrak{a}}(D,r) = \mathrm{N}(\mathfrak{c})^{-1}\,\mathrm{card}\left(\left\{y\in\mathfrak{c}/2\mathfrak{a}\mathfrak{c}^2 : y^2\equiv\Delta \bmod 4\mathfrak{a}\mathfrak{c}^2\right\}\right).$$

Since Δ is in $(2r)^2+4\mathfrak{c}^2$ and $2r$ is in $\mathfrak{c}$, any integral solutions y of $\Delta\equiv y^2 \bmod 4\mathfrak{a}\mathfrak{c}^2$ is already in $\mathfrak{c}$. Hence

$$N_{\mathfrak{a}}(D,r) = \mathrm{N}(\mathfrak{c})^{-1}\,\mathrm{card}\left(\left\{y\in\mathfrak{o}/2\mathfrak{a}\mathfrak{c}^2 : y^2\equiv\Delta \bmod 4\mathfrak{a}\mathfrak{c}^2\right\}\right),$$

and we can apply Theorem 5.1 to conclude

$$N_{\mathfrak{a}}(D,r) = \mathrm{N}(\mathfrak{c})^{-1}\sum_{\substack{\mathfrak{b}\mid\mathfrak{a}\mathfrak{c}^2\\ \mathfrak{a}\mathfrak{c}^2/\mathfrak{b}\ \text{squarefree}}}\chi_\Delta(\mathfrak{b}) = \mathrm{N}(\mathfrak{c})^{-1}\sum_{\substack{\mathfrak{b}\mid\mathfrak{a}\\ \mathfrak{a}/\mathfrak{b}\ \text{squarefree}}}\chi_\Delta(\mathfrak{b}\mathfrak{c}^2).$$

For the second identity note that any $\mathfrak{b}\mid\mathfrak{a}\mathfrak{c}^2$ such that $\mathfrak{a}\mathfrak{c}^2/\mathfrak{b}$ is squarefree and $(\mathfrak{b},\Delta)$ is a square, is necessarily divisible by $\mathfrak{c}^2$ (Otherwise there would exist a prime ideal $\mathfrak{p}$ with $\beta<2\gamma$, where β and γ are the orders of $\mathfrak{b}$ and $\mathfrak{c}$ at $\mathfrak{p}$. But $\mathfrak{c}^2\mid\Delta$ and $(\mathfrak{b},\Delta)$ being a square implies then that β is even, whence $2\gamma-\beta\geq 2$, contradiction that $\mathfrak{a}\mathfrak{c}^2/\mathfrak{b}$ is squarefree.) ∎

6. Fourier coefficients in the rank one case

In this section we finally derive the formulas for the Fourier coefficients of Jacobi-Eisenstein series for lattices of rank and modified level one which we discussed in § 1 (see Main Theorem).

Proof of Main Theorem. Let $D = -\Delta\omega/8$ and write $D = n - \frac{1}{2}\omega r^2$ ($r \in \mathfrak{c}/2$ and $n \in \mathfrak{d}^{-1}$). From Theorem 4.3

$$e_{k,[\mathfrak{c},\omega]}(D) = C\,\mathrm{N}(D)^{k-3/2} L(D, r; k-1),$$

where

$$L(D, r; s) = \sum_{C \in \mathrm{Cl}(K)} \sum_{\mathfrak{b} \in C^{-1}} \frac{\mu(\mathfrak{b})}{\mathrm{N}(\mathfrak{b})^s} \sum_{\mathfrak{a} \in C} \frac{N_{\mathfrak{a}}(D, r)}{\mathrm{N}(\mathfrak{a})^s}$$

with

$$N_{\mathfrak{a}}(D, r) = \mathrm{card}\left(\{x \in \mathfrak{c}/\mathfrak{a}\mathfrak{c} : \tfrac{1}{2}\omega(x-r)^2 \equiv -D \bmod \mathfrak{a}\mathfrak{d}^{-1}\}\right).$$

(Note that from the formula of the numbers $N_{\mathfrak{a}}(D, r)$ in Proposition 5.2 we see that they don't depend actually on r. Therefore, from now on we use instead of $N_{\mathfrak{a}}(D, r)$ and $L(D, r; s)$, $N_{\mathfrak{a}}(D)$ and $L(D; s)$, respectively.)

In the series $L(D; s)$ we write the sum over $\mathfrak{a} \in C$ as $h_K^{-1} \sum_{\mathfrak{a}} \sum_{\psi} \psi(\mathfrak{a})\psi(C)^{-1}$, where $\mathfrak{a}$ runs through all integral ideals of K and ψ runs through all characters of the class group of K (we write $\psi(\mathfrak{a})$ for the value of ψ at the class of $\mathfrak{a}$), and where h_K is the class number of K. Exchanging sums and moving the sum over ψ to the front, $L(D; s)$ becomes

$$h_K L(D; s) = \sum_{\psi} \sum_{\mathfrak{b}} \frac{\mu(\mathfrak{b})\psi(\mathfrak{b})}{\mathrm{N}(\mathfrak{b})^s} \sum_{\mathfrak{a}} \psi(\mathfrak{a}) \frac{N_{\mathfrak{a}}(D)}{\mathrm{N}(\mathfrak{a})^s} = \sum_{\psi} \frac{1}{L(\psi, s)} \sum_{\mathfrak{a}} \psi(\mathfrak{a}) \frac{N_{\mathfrak{a}}(D)}{\mathrm{N}(\mathfrak{a})^s},$$

where $\mathfrak{b}$ is now also running through all integral ideals. We now insert our formula for $N_{\mathfrak{a}}(D)$, which gives

$$h_K L(D; s) = \sum_{\psi} \frac{1}{L(\psi, s)} \sum_{\mathfrak{a}\text{ squarefree}} \frac{\psi(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^s} \sum_{\mathfrak{a}} \frac{\psi(\mathfrak{a})\chi_\Delta(\mathfrak{a}\mathfrak{c}^2)/\,\mathrm{N}(\mathfrak{c})}{\mathrm{N}(\mathfrak{a})^s}.$$

Note that

$$\sum_{\mathfrak{a}\text{ squarefree}} \frac{\psi(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^s} = \frac{L(\psi, s)}{L(\psi^2, 2s)},$$

so that

$$h_K L(D; s) = \sum_{\psi} \frac{1}{L(\psi^2, 2s)} \sum_{\mathfrak{a}} \frac{\psi(\mathfrak{a})\chi_\Delta(\mathfrak{a}\mathfrak{c}^2)/\,\mathrm{N}(\mathfrak{c})}{\mathrm{N}(\mathfrak{a})^s}.$$

In other words, we have

$$h_K L(D; s) = \psi(\mathfrak{c})^{-2}\,\mathrm{N}(\mathfrak{c})^{2s-1} \sum_{\psi} \frac{1}{L(\psi^2, 2s)} L_{\mathfrak{c}^2}(\psi\chi_\Delta, s),$$

where $L_{\mathfrak{c}^2}(\psi\chi_\Delta, s)$ is defined in (18) below. The theorem follows now using the subsequent lemma. □

Let Δ be an integer in K, let $\mathfrak{c}$ be an integral ideal of K such that $\mathfrak{c}^2 \mid \Delta$ and Δ is a square modulo $4\mathfrak{c}^2$. We set

$$L_{\mathfrak{c}^2}(\psi\chi_\Delta, s) := \sum_{\mathfrak{c}^2 \mid \mathfrak{a}} \frac{\psi(\mathfrak{a})\chi_\Delta(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^s}. \tag{18}$$

Lemma 6.1. *For the L-series* (18), *one has in the notations of Theorem 1*

$$\psi(\mathfrak{c})^{-2}\,\mathrm{N}(\mathfrak{c})^{2s-1}\,\frac{L_{\mathfrak{c}^2}(\psi\chi_\Delta,s)}{L\left(\psi\left(\frac{\Delta}{*}\right)_0,s\right)} = \mathrm{N}(\mathfrak{f}_\Delta/\mathfrak{c})^{1-2s}\sum_{\mathfrak{t}\mid\mathfrak{f}_\Delta/\mathfrak{c}}\mu(\mathfrak{t})\psi(\mathfrak{t})\left(\tfrac{\Delta}{\mathfrak{t}}\right)_0\mathrm{N}(\mathfrak{t})^{s-1}\,\sigma_{2s-1}^{\psi^2}\left(\mathfrak{f}_\Delta/\mathfrak{c}\mathfrak{t}\right).$$

Proof. Since $\psi\chi_\Delta$ is multiplicative $L_{\mathfrak{c}^2}(\psi\chi_\Delta,s)$ has an Euler product, say

$$\psi(\mathfrak{c})^{-2}\,\mathrm{N}(\mathfrak{c})^{2s-1}\,\frac{L_{\mathfrak{c}^2}(\psi\chi_\Delta,s)}{L(\psi\left(\frac{\Delta}{*}\right)_0,s)} = \prod_{\mathfrak{p}\mid\Delta}F_{\mathfrak{p}}(s)\,.$$

Note that the Euler product is only over the prime ideals dividing Δ, since for $\mathfrak{p}$ not dividing Δ, we have $\chi_\Delta(\mathfrak{p}^k) = \left(\frac{\Delta}{\mathfrak{p}}\right)^k$. For $\mathfrak{p}\mid\Delta$, we have

$$F_{\mathfrak{p}}(s) = \begin{cases} \sigma_{1-2s}^{\psi^2}\left(\mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{f}_\Delta)}\right) & \text{if } \mathfrak{p}\mid\Delta_0,\ \mathfrak{p}\nmid\mathfrak{c},\\ \sigma_{1-2s}^{\psi^2}\left(\mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{f}_\Delta/\mathfrak{c})}\right) & \text{if } \mathfrak{p}\mid\Delta_0,\ \mathfrak{p}\mid\mathfrak{c},\\ \sum_{\mathfrak{t}\mid\mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{f}_\Delta)}}\mu(\mathfrak{t})\psi(\mathfrak{t})\left(\frac{\Delta}{\mathfrak{t}}\right)_0\mathrm{N}(\mathfrak{t})^{-s}\,\sigma_{1-2s}^{*\psi^2}\left(\mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{f}_\Delta)}/\mathfrak{t}\right) & \text{if } \mathfrak{p}\nmid\Delta_0,\ \mathfrak{p}\nmid\mathfrak{c},\\ \sum_{\mathfrak{t}\mid\mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{f}_\Delta/\mathfrak{c})}}\mu(\mathfrak{t})\psi(\mathfrak{t})\left(\frac{\Delta}{\mathfrak{t}}\right)_0\mathrm{N}(\mathfrak{t})^{-s}\,\sigma_{1-2s}^{*\psi^2}\left(\mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{f}_\Delta/\mathfrak{c})}/\mathfrak{t}\right) & \text{if } \mathfrak{p}\nmid\Delta_0,\ \mathfrak{p}\mid\mathfrak{c}, \end{cases}$$

where $\Delta_0 = \Delta/\mathfrak{f}_\Delta^2$, and $\sigma_{1-2s}^{*\psi^2}(\mathfrak{f}) = \sum_{\mathfrak{t}\mid\mathfrak{f}}\mathrm{N}(\mathfrak{t})^{1-2s}\psi^2(\mathfrak{t}) = \mathrm{N}(\mathfrak{f})^{1-2s}\sigma_{2s-1}^{\psi^2}(\mathfrak{f})$.

Indeed, for verifying the first two rows it suffices to note that, for $\mathfrak{p}\mid\Delta_0$, the coefficient $\chi_\Delta(\mathfrak{p}^k)$ equals $\mathrm{N}(\mathfrak{p})^{k/2}$ if $k\le 2v_{\mathfrak{p}}(\mathfrak{f}_\Delta)$ and k is even, and it equals 0 otherwise.

For the last two lines, i.e. for $\mathfrak{p}\nmid\Delta_0$, we find $\chi_\Delta(\mathfrak{p}^k) = \mathrm{N}(\mathfrak{p})^{k/2}$ if $k\le 2v_{\mathfrak{p}}(\mathfrak{f}_\Delta)$ and k is even, and $\chi_\Delta(\mathfrak{p}^k) = \mathrm{N}(\mathfrak{p})^{v_{\mathfrak{p}}(\mathfrak{f}_\Delta)}\left(\frac{\Delta}{\mathfrak{p}}\right)_0^k$ for $k\ge 2v_{\mathfrak{p}}(\mathfrak{f}_\Delta)$. Hence, if $\mathfrak{p}\nmid\mathfrak{c}$, we obtain

$$F_{\mathfrak{p}}(s) = (1-\psi(\mathfrak{p})\left(\frac{\Delta}{\mathfrak{p}}\right)_0\mathrm{N}(\mathfrak{p})^{-s})\,\sigma_{1-2s}^{*\psi^2}\left(\mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{f}_\Delta)}\right) + \mathrm{N}(\mathfrak{p})^{v_{\mathfrak{p}}(f_\Delta)}\psi(\mathfrak{p})^{2v_{\mathfrak{p}}(\mathfrak{f}_\Delta)+1}\left(\frac{\Delta}{\mathfrak{p}}\right)_0\mathrm{N}(\mathfrak{p})^{-(2v_{\mathfrak{p}}(\mathfrak{f}_\Delta)+1)s},$$

which equals the claimed expression. Finally, for $\mathfrak{p}\mid\mathfrak{c}$, the calculation is similar. The lemma is now obvious. ☐

7. Appendix

In this appendix we calculate some integrals which we needed in the proof of Lemma 4.2.

Proposition 7.1. *Let $\underline{L} = (L,\beta)$ be a positive definite (not necessarily integral) $\mathbb{Z}$-lattice, and μ a Haar measure on $\mathbb{R}\otimes_{\mathbb{Z}}L$. Then*

$$\int_{\mathbb{R}\otimes_{\mathbb{Z}}L} e^{-2\pi\beta(p)}\,d\mu(p) = \frac{\mu(V/L)}{\sqrt{\det(\underline{L})}},$$

where $\det(\underline{L})$ *is the determinant of* $\underline{L}$ *(i.e. the determinant of any Gram matrix of the bilinear form β with respect to a given $\mathbb{Z}$-basis of L).*

Proof. Let a_j be a $\mathbb{Z}$-basis of L, let x_j be the dual basis, and let $x = (x_1,\dots,x_n)$ be the isomorphism of $V := \mathbb{R}\otimes_{\mathbb{Z}}L$ with $\mathbb{R}^n$ with coordinate functions x_j. Then $q(p) = \frac{1}{2}\beta(p,p)$ becomes $q = \frac{1}{2}xFx^t$ with the Gram matrix $F = (\beta(a_i,a_j))_{i,j}$, and there is a $\lambda > 0$ such that $\mu(A) = \lambda\int_{x(A)}dx$ for all measurable A in V, where dx is the usual Lebesgue measure on $\mathbb{R}^n$. In particular, x maps a fundamental mesh of L in V to a fundamental mesh of $\mathbb{Z}^n$ in $\mathbb{R}^n$ (since x maps L to $\mathbb{Z}^n$), and therefore $\lambda = \mu(V/L)$. In the coordinates x_j our integral becomes

$$\int_V e^{-2\pi\beta(p)}\, d\mu(p) = \lambda \int_{\mathbb{R}^n} e^{-\pi x F x^t}\, dx = \lambda \det(F)^{-1/2},$$

which is the claimed formula. $\square$

As immediate consequence we obtain

Corollary 7.2. *For any totally positive definite $\mathfrak{o}$-lattice $\underline{L} = (L, \beta)$ and any Haar measure ν on $\mathcal{R} \otimes_{\mathfrak{o}} L$, one has*

$$\int_{\mathcal{R}\otimes_{\mathfrak{o}} L} e\left\{i\beta(p)\right\} d\nu(p) = \frac{\nu\left((\mathcal{R}\otimes_{\mathfrak{o}} L)/L\right)}{\sqrt{\det(\operatorname{tr}\underline{L})}},$$

where $\operatorname{tr}\underline{L}$ is the $\mathbb{Z}$-lattice $(L, \operatorname{tr}\circ\beta)$.

Proposition 7.3. *For any real $r > 1$ and $\tau \in \mathcal{H}$, we have*

$$\int_{\mathcal{R}} \mathrm{N}(\tau + t)^{-r} e\left\{-(\tau+t)\right\}\, dt = \mathrm{N}(i)^{-r} \frac{(2\pi)^{(r+1)n_K}}{\Gamma(r)^{n_K}}, \tag{19}$$

where dt denotes the Haar measure on $\mathcal{R}$, which becomes the usual Lebesgue measure when we identify $\mathcal{R}$ with $\mathbb{R}^{n_K}$ using the $\mathbb{R}$-linear continuations of the embeddings σ of K to $\mathcal{R}$ as coordinate functions.

Proof. First of all note that $\gamma := \operatorname{Log}\mathrm{N}(\tau) - \sum_\sigma \operatorname{Log}\sigma(\tau)$ does not depend on τ (since γ is holomorphic in $\mathcal{H}$ with values in $2\pi i\mathbb{Z}$). Choosing $\tau = i$, we find $e^{-r\gamma} = e(rn_K/4)/\mathrm{N}(i)^r$, hence $\mathrm{N}(\tau)^{-r} = e(rn_K/4)\,\mathrm{N}(i)^{-r}\prod_\sigma \sigma(\tau)^{-r}$ Extending the embeddings σ of K to $\mathbb{R}$-linear maps on $\mathcal{R}$ they become coordinate functions of an isomorphism of $\mathcal{R}$ with $\mathbb{R}^{n_K}$. Applying the isomorphism whose coordinate functions are the $\mathbb{R}$-linear extensions of the embeddings σ, the integral in question becomes the product $e(rn_K/4)\,\mathrm{N}(i)^{-r}\prod_\sigma I_\sigma$, where

$$I_\sigma = \int_{-\infty}^{+\infty} \frac{e^{-2\pi i(\tau_\sigma + u)}}{(\tau_\sigma + u)^r}\, du = \int_{\Im(u)=c} \frac{e^{-2\pi i u}}{u^r}\, du = \frac{-1}{i^{r-1}} \int_{\Re(z)=c} z^{-r} e^{2\pi z}\, dz,$$

with $\tau_\sigma = \sigma(\tau)$ and $c = \Im(\tau_\sigma)$ (which is positive), and where, for the last identity we set $u = iz$. But $f(t) := \frac{1}{2\pi i}\int_{\Re(z)=c} z^{-r} e^{tz}\, dz$ is the inverse Laplace transform of z^{-r}, whence $z^{-r} = \int_0^\infty f(t)\, e^{-zt}\, dt$. On the other hand $z^{-r}\Gamma(r) = \int_0^\infty e^{-zt}\, t^r \frac{dt}{t}$, and hence $f(t) = t^r/\Gamma(r)$. $\square$

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