Iterative approximations of periodic trajectories for nonlinear systems with discontinuous inputs

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Abstract: Nonlinear control-affine systems described by ordinary differential equations with bounded measurable input functions are considered. The problem of the existence of periodic trajectories to these systems is formulated in the sense of Carathéodory solutions. It is shown that, under the dominant linearization assumption, the periodic boundary value problem admits a unique solution for any admissible control. This solution can be obtained as the limit of the proposed simple iterative scheme and Newton-type method. Under additional technical assumptions, sufficient contraction conditions of the corresponding generating operators are derived analytically. The proposed iterative approach is applied for the computation of periodic solutions of a realistic chemical reaction model with discontinuous control inputs.

Keywords: nonlinear control system, periodic boundary value problem, discontinuous input function, Carathéodory solution, iterative scheme, nonlinear chemical reaction model

AMS subject classifications: 93C15, 34B15, 34A36, 47J25, 65L10, 92E20

Novelty statement:

- Existence and uniqueness of periodic solutions for nonlinear control systems with general measurable input functions
- Iterative schemes and numerical implementation of an algorithm for approximating the periodic solutions of discontinuous systems
- Analytical sufficient conditions for the convergence of the simple iteration and Newton-type methods
- Approximate periodic trajectories of a controlled chemical reaction model under arbitrary switching strategies

1. Introduction

Periodic optimal control problems have been attracting considerable interest in the mathematical literature [1–4] and play a significant role in a variety of emerging engineering applications (see, e.g., [5–9] and references therein). Our current paper is motivated by the previous analysis of nonlinear optimization problems with isoperimetric constraints [10, 11], where the main goal is to optimize the cost functional on the periodic trajectories under discontinuous control strategies. An important ingredient for achieving this goal relies on the description of the set of periodic solutions for a nonlinear control system with bang-bang inputs.

In the paper [11], the Chen–Fliess series have been exploited for constructing the τ-periodic solutions of nonlinear control-affine systems with switching. This approach allows representing the initial data of the considered system as a solution of nonlinear algebraic equations whose order depends on the remainder of the Chen–Fliess series for small $\tau > 0$. The construction of [11] has been also extended to a class of nonlinear chemical reaction models in non-affine form in [12]. Note that the convergence of the Chen–Fliess expansion is not guaranteed on an arbitrary time interval $[0, \tau]$ even for systems with analytic vector fields, and other approximation techniques should be developed to design periodic solutions for nonlinear control systems with large periods.

For different classes of nonlinear ordinary differential equations with regular right-hand sides, the problems of existence and approximation of periodic solutions have been studied by the method of generalized quasilinearization [13], comparison techniques [14], fixed point theory [15], reproducing kernel method [16], and other techniques. The presented list of references does not pretend to be complete.

To the best of our knowledge, a complete characterization of periodic trajectories of a given nonlinear system of ordinary differential equations with discontinuous right-hand sides remains open up to now. The purpose of this paper is to provide such a characterization together with efficient computational methods for approximating the solutions of nonlinear systems of the form

$$\dot{x}(t) = Ax(t) + g(x(t)) + u(t),\ x(t) = (x_1(t), ..., x_n(t))^T \in D \subset \mathbb{R}^n,\ t \in [0, \tau], \tag{1.1}$$

under the periodic boundary condition

$$x(\tau) = x(0), \tag{1.2}$$

where D is a domain containing the origin $x = 0 \in \mathbb{R}^n$, A is a constant $n \times n$ matrix, $g \in C^1(D; \mathbb{R}^n)$ stands for nonlinearity, and $u \in L^\infty([0, \tau]; \mathbb{R}^n)$ is an arbitrary input function (control).

For further analysis, we denote by $X = C([0, \tau]; \mathbb{R}^n)$ the Banach space of all continuous functions $x(t)$ from $[0, \tau]$ to $\mathbb{R}^n$ equipped with the norm

$$\|x(\cdot)\|_X := \sup_{t \in [0,\tau]} \|x(t)\|,$$

and $\|\xi\|$ is the Euclidean norm of a column vector $\xi \in \mathbb{R}^n$. The latter induces the 2-norm $\|A\| := \sup_{\|\xi\| \le 1} \|A\xi\|$ of a matrix A that will be exploited throughout the text. We will also use the notations $X_D = C([0, \tau]; D)$ and $X_{D'} = C([0, \tau]; D')$ if $D' \subset D$ is a closed domain.

As the function $u(t)$ is allowed to be discontinuous, we treat the solutions of (1.1) in the sense of Carathéodory [17, Chap. 1] as solutions of the following integral equation

$$x(t) = x_0 + \int_0^t [Ax(s) + g(x(s)) + u(s)]\, ds,\ x_0 = x(0).$$

The existence and uniqueness results for solutions to differential equations, with the right-hand sides being continuous in x and discontinuous in t under the Carathéodory conditions, are summarized in [17, Chap. 1]. By using the variation of constants method and introducing the matrix exponential e^{tA}, we can rewrite the above equation in the form

$$x(t) = e^{tA} x_0 + \int_0^t e^{(t-s)A} [g(x(s)) + u(s)]\, ds. \tag{1.3}$$

Let $x(t)$ be a solution of (1.3) with some $u(t)$ on $[0, \tau]$ such that $x(\tau) = x(0)$, then $x(t)$ can be extended to the τ-periodic function $\tilde{x}(t)$, defined for all $t \in \mathbb{R}$. If, moreover, the function $\tilde{u}(t)$ is τ-periodic and $\tilde{u}(t) = u(t)$ for all $t \in [0, \tau)$, then $\tilde{x}(t)$ satisfies (1.3) with the input $\tilde{u}(t)$ for all $t \in \mathbb{R}$. Because of this simple fact, we will refer to the Carathéodory solutions of boundary value problem (1.1)–(1.2) as periodic solutions. If $x(t)$ is a solution of (1.1)–(1.2), then formula (1.3) allows to represent its initial value $x_0 = x(0) = x(\tau)$:

$$x_0 = (e^{-\tau A} - I)^{-1} \int_0^\tau e^{-sA} [g(x(s)) + u(s)]\, ds, \tag{1.4}$$

provided that

$$\det\left(e^{-\tau A} - I\right) \neq 0. \tag{A1}$$

An immediate consequence of equation (1.4) *for the case of linear systems on* $D = \mathbb{R}^n$ is that the system (1.1) with $g(x) \equiv 0$ and any $u \in L^\infty([0,\tau];\mathbb{R}^n)$ *has a unique periodic solution* $x(t)$ on $t \in [0,\tau]$, and its initial data $x_0 = x(0)$ is defined by (1.4) if assumption $(A1)$ holds. The existence of periodic solutions was studied *for weakly nonlinear* boundary value problems with piecewise-constant right-hand sides in [18]. The latter paper develops the perturbation analysis techniques for nonlinear differential equations with switchings under periodic boundary conditions. A modified iterative scheme is proposed there for constructing approximate periodic solutions. Note that the contribution of [18] is limited to the systems of ordinary differential equations with a small parameter, and future study of systems of the form (1.1) with general nonlinearities $g(x)$ and merely measurable inputs $u(t)$ is needed.

The rest of this paper is organized as follows. In Section 2, a simple iteration method is adapted for constructing the periodic solutions of (1.1) under rather mild assumptions on $f(t)$ and $g(x)$. The convergence of this scheme is analyzed in terms of the period τ, growth rate of the matrix exponential e^{tA}, and the Lipschitz constant of $g(x)$ (Theorem 2.1). The operator formulation of the periodic boundary value problem for equation (1.1) is exploited in Section 3 to derive Newton's method. Sufficient contraction conditions of the Newton-type operator are formulated explicitly in Theorem 3.4 (the proof of this result is given in Appendix A). The proposed iterative approach is applied to a nonlinear chemical reaction model in Section 4. The final conclusions and perspectives are outlined in Section 5.

2. Simple iteration method

Let us introduce the operator $\mathcal{F} : X_D = C([0,\tau];D) \to X$ such that

$$\mathcal{F} : x(\cdot) \mapsto (\mathcal{F}x)(t) = e^{tA}c(x(\cdot)) + \int_0^t e^{(t-s)A}\left[g(x(s)) + u(s)\right] ds, \tag{2.1}$$

where the vector functional $c(x(\cdot))$ is defined by

$$c(x(\cdot)) = (e^{-\tau A} - I)^{-1} \int_0^\tau e^{-sA}\left[g(x(s)) + u(s)\right] ds. \tag{2.2}$$

If an initial function $x^{(0)} \in X_D$ is given, we generate the sequence of functions $x^{(k)} = \mathcal{F}x^{(k-1)}$ for $k = 1, 2, \ldots$. This sequence is well-defined, in particular, if $X_D = X$ (i.e., $D = \mathbb{R}^n$). If $D \neq \mathbb{R}^n$, we consequently assume that each value $(\mathcal{F}x^{(k)})(t)$ is in D for all $t \in [0,\tau]$, $k = 1, 2, \ldots$. Below, we propose sufficient conditions for the convergence of $x^{(k)}(t)$ to a τ-periodic solution of system (1.1).

Theorem 2.1. *Assume that* $(A1)$ *holds and there exist constants* $L \geq 0$, $M \geq 1$, $\omega > 0$, *and a closed convex domain* $D' \subset D$ *such that:*

$$\left\| \frac{\partial g(x)}{\partial x} \right\| \leq L,\ \|e^{tA}\| \leq Me^{\omega|t|} \quad \textit{for all}\ \ x \in D,\ t \in [-\tau,\tau]; \tag{A2}$$

$$(\mathcal{F}x)(\cdot) \in X_{D'} = C([0,\tau];D')\ \textit{for each}\ x(\cdot) \in X_{D'}; \tag{A3}$$

$$\frac{ML(1 + MR_\tau)(e^{\omega\tau} - 1)}{\omega} < 1, \tag{A4}$$

where

$$R_\tau = \left\|(e^{-\tau A} - I)^{-1}\right\|. \tag{2.3}$$

Then, for any $x^{(0)}(\cdot) \in X_{D'}$, *the sequence*

$$x^{(k)}(\cdot) = \mathcal{F}x^{(k-1)}(\cdot), \quad k = 1, 2, \ldots$$

converges to the limit $x^*(\cdot) \in X_{D'}$ *as* $k \to \infty$. *This limit function* $x^*(t)$, $t \in [0,\tau]$ *is the unique solution of system* (1.1) *such that* $x^*(0) = x^*(\tau)$.

Proof. Under assumptions $(A1)$–$(A2)$, the operator $\mathcal{F} : X_D \to X$ defined by (2.1) is Fréchet differentiable. Indeed, the Fréchet derivative $\mathcal{F}'_x : X \to X$ at $x(\cdot) \in X_D$ is a bounded linear operator such that its action on $\delta x(\cdot) \in X$ is:

$$(\mathcal{F}'_x(\delta x))(t) = e^{tA} dc_x(\delta x) + \int_0^t e^{(t-s)A} \left.\frac{\partial g(x)}{\partial x}\right|_{x=x(s)} \delta x(s)\, ds, \tag{2.4}$$

where

$$dc_x(\delta x) = (e^{-\tau A} - I)^{-1} \int_0^\tau e^{-sA} \left.\frac{\partial g(x)}{\partial x}\right|_{x=x(s)} \delta x(s)\, ds. \tag{2.5}$$

By taking into account inequalities $(A2)$, we conclude that

$$\begin{aligned}\|\mathcal{F}'_x(\delta x)\|_X \le & \left\{ \sup_{t\in[0,\tau]} \|e^{tA}\| \cdot \|(e^{-\tau A} - I)^{-1}\| \int_0^\tau \|e^{-sA}\| \cdot \left\|\frac{\partial g}{\partial x}\right\| ds \right. \\ & \left. + \sup_{t\in[0,\tau]} \int_0^t \|e^{(t-s)A}\| \cdot \left\|\frac{\partial g}{\partial x}\right\| ds \right\} \|\delta x\|_X \\ \le & \, ML \int_0^\tau e^{\omega s} ds \left\{1 + M \left\|(e^{-\tau A} - I)^{-1}\right\|\right\} \|\delta x\|_X.\end{aligned}$$

We see that the operator $\mathcal{F}$ is contractive on $X_{D'}$ if condition $(A4)$ holds, $\mathcal{F}(X_{D'}) \subseteq X_{D'}$ because of assumption $(A3)$, and the metric space $(X_{D'}, d)$ equipped with the distance $d(x, y) = \|x - y\|_X$ is complete. Thus, the assertion of Theorem 2.1 follows from the Banach fixed point theorem [19, Chap. XVI, §1, Thm. 1 and Chap. XVII, §1, Thm. 1]. ☐

Remark 2.1. *The initial function $x^{(0)}(t)$ can be taken, in particular, as $x^{(0)}(t) \equiv 0$. In this case, it is easy to see that the first approximation $x^{(1)}(t)$ is the solution of the linear system $\dot{x}^{(1)} = Ax^{(1)} + u(t) + g(0)$, $t \in [0, \tau]$, such that $x^{(1)}(0) = x^{(1)}(\tau)$.*

3. Newton's method

In this section, we propose a modification of Newton's method for computing the fixed points of $\mathcal{F}x$. For this purpose we introduce the nonlinear operator $\Psi : X_D \to X$ such that

$$\Psi x = x - \alpha(\mathcal{F}'_x - I)^{-1}(\mathcal{F}x - x), \tag{3.1}$$

where $\mathcal{F} : X_D \to X$ is introduced in (2.1)–(2.2), $\mathcal{F}'_x$ is the Fréchet derivative of the operator $\mathcal{F}$ at $x \in X_D$ defined by (2.4)–(2.5), I is the identity operator on X, and $\alpha > 0$ is a parameter. It is clear that, if the operator $(\mathcal{F}'_x - I)^{-1}$ is nonsingular, then the sets of solutions to the equations $\mathcal{F}x = x$ and $\Psi x = x$ are equivalent. The case $\alpha = 1$ corresponds to "the classical" Newton's method. For a given $x^{(0)} \in X_D$ and $\alpha = 1$, the convergence of the sequence $x^{(k)} = \Psi\, x^{(k-1)}$, $k = 1, 2, ...,$ generated by (3.1), can be (in principle) analyzed by the Newton–Kantorovich theorem [20] under a suitable assumption on $x^{(0)}$. In what follows, we will estimate the Fréchet derivative of Ψ at any $x \in X_D$ analytically. Then the convergence conditions will be obtained in a straightforward way by the Banach fixed point theorem with an arbitrary $\alpha \in (0, 1]$.

In order to represent the action of the operator $(\mathcal{F}'_x - I)^{-1}$ on a vector function $\delta y(\cdot) \in X$, we solve the following functional equation with respect to $\delta x(\cdot) \in X$:

$$(\mathcal{F}'_x - I)\delta x = \delta y,$$

or, equivalently,

$$\mathcal{F}'_x(\delta x) - \delta x = \delta y.$$

The above equation can be rewritten because of (2.4) in the form

$$dc_x(\delta x) + \int_0^t e^{-sA} \left.\frac{\partial g(x)}{\partial x}\right|_{x=x(s)} \delta x(s) ds - e^{-tA}\delta x(t) = e^{-tA}\delta y(t),\; t \in [0, \tau]. \tag{3.2}$$

By differentiating this formula with respect to t and introducing the function $\delta z(t) = \delta x(t) + \delta y(t)$, we obtain

$$\dot{\delta} z(t) = \left(A + \left.\frac{\partial g(x)}{\partial x}\right|_{x=x(t)} \right) \delta z(t) - \left.\frac{\partial g(x)}{\partial x}\right|_{x=x(t)} \delta y(t). \tag{3.3}$$

Let $\Phi_x(t) \in \mathrm{M}at(n \times n)$ be the fundamental matrix of the corresponding homogeneous system, i.e.

$$\Phi_x(0) = I, \;\; \dot{\Phi}_x(t) = \left(A + \left.\frac{\partial g(x)}{\partial x}\right|_{x=x(t)} \right) \Phi_x(t) \;\; \text{for } t \in [0,\tau]. \tag{3.4}$$

Note that the matrix $\Phi_x^{-1}(s)$ is well-defined for all $s \in [0,\tau]$ due to the uniqueness of solutions to the Cauchy problem (3.4). Then the variation of constants method yields the general solution of (3.3):

$$\delta z(t) = \Phi_x(t)\delta z(0) - \Phi_x(t) \int_0^t \Phi_x^{-1}(s) \left.\frac{\partial g(x)}{\partial x}\right|_{x=x(s)} \delta y(s) ds.$$

We rewrite the above formula with respect to $\delta x(t) = \delta z(t) - \delta y(t)$ as

$$\delta x(t) = \Phi_x(t) C - \Phi_x(t) \int_0^t \Phi_x^{-1}(s) \left.\frac{\partial g(x)}{\partial x}\right|_{x=x(s)} \delta y(s) ds - \delta y(t). \tag{3.5}$$

The integration constant $C = \delta x(0) + \delta y(0)$ is defined from (3.2). Indeed, the relation (3.2) at $t=0$ implies that $C = \delta x(0) + \delta y(0) = \delta x_c(\delta x)$, then the integration constant C can be eliminated by substituting (3.5) into (2.5). Thus we have:

$$C = M_x^{-1} \left(e^{-\tau A} - I\right)^{-1}$$
$$\times \int_0^\tau e^{-tA} \left.\frac{\partial g(x)}{\partial x}\right|_{x(t)} \left\{ \delta y(t) + \Phi_x(t) \int_0^t \Phi_x^{-1}(s) \left.\frac{\partial g(x)}{\partial x}\right|_{x(s)} \delta y(s) ds \right\} dt, \tag{3.6}$$

where

$$M_x = \left(e^{-\tau A} - I\right)^{-1} \int_0^\tau e^{-tA} \left.\frac{\partial g(x)}{\partial x}\right|_{x(t)} \Phi_x(t) dt - I. \tag{3.7}$$

The above procedure describes the evaluation of $\delta x = (\mathcal{F}'_x - I)^{-1}\delta y$ for any $\delta y(\cdot) \in X$, provided that the matrix M_x in (3.7) is nonsingular. Hence, for a fixed parameter $\alpha > 0$ in (3.1), the action of Ψ on a given $x \in X_D$ is defined by the following rule:

$$\Psi : x \mapsto \delta y = \mathcal{F}x - x \mapsto \delta x = (\mathcal{F}'_x - I)^{-1}\delta y \mapsto \Psi x = x - \alpha \delta x. \tag{3.8}$$

In the sequel, we assume that $g \in C^2(D;\mathbb{R}^n)$ and that the Hessian matrices of the components of $g(x) = (g_1(x), ..., g_n(x))^T$ are bounded, i.e.

$$\left\| \frac{\partial^2 g_k(x)}{\partial x_i \partial x_j} \right\| \le \bar{H} \quad \text{for all } \; x \in D, \; k = 1, 2, ..., n, \tag{A5}$$

with some constant $\bar{H} \ge 0$. For the analysis of Newton's scheme, we present auxiliary lemmas concerning the matrix-valued map $\Phi_x(t)$ and its differential.

Lemma 3.1. *Let assumption (A2) hold, and let $x(t) \in D$ be a continuous function on $t \in [0,\tau]$. Then the fundamental matrix $\Phi_x(t)$ in (3.4) satisfies the following norm estimates:*

$$\|\Phi_x(t)\| \le \phi_L(t), \;\; \|\Phi_x^{-1}(t)\| \le \phi_L(t), \;\; \phi_L(t) = \sqrt{n} e^{(\|A\|+L)t}, \quad t \in [0,\tau]. \tag{3.9}$$

Proof. Estimates (3.9) easily follow from Grönwall's inequality. □

Lemma 3.2. *For a given $t \in [0, \tau]$, consider the map $\Phi_x(t) : x(\cdot) \in X_D \mapsto \Phi_x(t) \in Mat(n \times n)$ defined by (3.4). Then the differential $d\Phi_x(t)(dx)$ of $\Phi_x(t)$ along $dx(\cdot) \in X$ is equal to*

$$d\Phi_x(t)(dx) = \Phi_x(t) \int_0^t \Phi_x^{-1}(s)\Gamma(x(s), dx(s))\Phi_x(s)ds, \tag{3.10}$$

where the entries of the $n \times n$ matrix $\Gamma(x(s), dx(s))$ are

$$\Gamma_{ki}(\xi, d\xi) = \sum_{j=1}^{n} \frac{\partial^2 g_k(\xi)}{\partial \xi_i \partial \xi_j} d\xi_j. \tag{3.11}$$

Moreover,

$$d\left(\Phi_x^{-1}(t)\right)(dx) = -\Phi_x^{-1}(t)\Big(d\Phi_x(t)(dx)\Big)\Phi_x^{-1}(t). \tag{3.12}$$

The proof of Lemma 3.2 is presented in A.

To establish convergence conditions of Newton's method with the operator Ψ defined by (3.1), (3.8), we evaluate the Fréchet derivative $\Psi'_x : X \to X$ of Ψ at a given $x(\cdot) \in X_D$. Namely, the action of Ψ'_x on a $dx(\cdot) \in X$ can be computed by expanding the linear part of $\Psi(x + dx) - \Psi(x)$ with respect to dx and using (3.8):

$$\Psi'_x(dx) = dx - \alpha d(\mathcal{F}'_x - I)^{-1}(d\delta y), \tag{3.13}$$

where $d\delta y$ is the differential of $\delta y = \mathcal{F}(x) - x$ at x along dx, which can be computed from (2.4)–(2.5):

$$d\delta y(t) = -dx(t) + e^{tA}dc_x(dx) + \int_0^t e^{(t-s)A} \left.\frac{\partial g(x)}{\partial x}\right|_{x(s)} dx(s)ds. \tag{3.14}$$

Then the main technical task is to evaluate $d(\mathcal{F}'_x - I)^{-1}(d\delta y)$ by computing the differential of the right-hand side of (3.5) along dx. We have:

$$\begin{aligned} d(\mathcal{F}'_x - I)^{-1}(d\delta y)(t) &= \Phi_x(t)dC(dx) - \Phi_x(t)\int_0^t \Phi_x^{-1}(s) \left.\frac{\partial g}{\partial x}\right|_{x(s)} d\delta y(s)ds - d\delta y(t) \\ &+ d\Phi_x(t)C - d\Phi_x(t)\int_0^t \Phi_x^{-1}(s) \left.\frac{\partial g}{\partial x}\right|_{x(s)} \delta y(s)ds \\ &+ \Phi_x(t)\int_0^t \Phi_x^{-1}(s)\left\{\Phi_x^{-1}(s)d\Phi_x(s)\left.\frac{\partial g}{\partial x}\right|_{x(s)} - \left.\frac{\partial^2 g}{\partial x^2}\right|_{x(s)} dx(s)\right\}\delta y(s)ds. \end{aligned} \tag{3.15}$$

Here, the differential of C is computed from (3.6):

$$\begin{aligned} dC(dx) &= M_x^{-1}(e^{-\tau A} - I)^{-1}\int_0^\tau e^{-sA}\left.\frac{\partial g}{\partial x}\right|_{x(s)} \\ &\times\left\{d\delta y(s) + \Phi_x(s)\int_0^s \Phi_x^{-1}(v)\left.\frac{\partial g}{\partial x}\right|_{x(v)} d\delta y(v)dv + d\Phi_x(s)\int_0^s \Phi_x^{-1}(v)\left.\frac{\partial g}{\partial x}\right|_{x(v)}\delta y(v)dv\right. \\ &\left.-\Phi_x(s)\int_0^s \Phi_x^{-1}(v)\left(\Phi_x^{-1}(v)d\Phi_x(v)\left.\frac{\partial g}{\partial x}\right|_{x(v)} - \left.\frac{\partial^2 g}{\partial x^2}\right|_{x(v)} dx(v)\right)\delta y(v)dv\right\}ds \\ &+M_x^{-1}(e^{-\tau A} - I)^{-1}\int_0^\tau e^{-sA}\left(\left.\frac{\partial^2 g}{\partial x^2}\right|_{x(s)} dx(s)\right) \\ &\times\left\{\delta y(s) + \Phi_x(s)\int_0^s \Phi_x^{-1}(v)\left.\frac{\partial g}{\partial x}\right|_{x(v)}\delta y(v)dv\right\}ds \\ &+d(M_x^{-1})(dx)(e^{-\tau A} - I)^{-1}\int_0^\tau e^{-sA}\left.\frac{\partial g}{\partial x}\right|_{x(s)} \\ &\times\left\{\delta y(s) + \Phi_x(s)\int_0^s \Phi_x^{-1}(v)\left.\frac{\partial g}{\partial x}\right|_{x(v)}\delta y(v)dv\right\}ds, \end{aligned} \tag{3.16}$$

where

$$d(M_x^{-1})(dx) = -M_x^{-2}(e^{-\tau A} - I)^{-1} \times \int_0^\tau e^{-sA} \left\{ \left(\frac{\partial^2 g}{\partial x^2}\bigg|_{x(s)} dx(s) \right) \Phi_x(s) + \frac{\partial g}{\partial x}\bigg|_{x(s)} d\Phi_x(s) \right\} ds. \tag{3.17}$$

The matrix norm of M_x^{-1} can be estimated by the following lemma.

Lemma 3.3. *Let assumptions (A1)–(A2) be satisfied, and let*

$$S = \frac{\sqrt{n} M L R_\tau (e^{(\|A\|+L+\omega)\tau} - 1)}{\|A\| + L + \omega} < 1. \tag{A6}$$

Then $\|M_x^{-1}\| \leq \frac{1}{1-S}$.

This lemma is proved in A. The presented auxiliary results are needed to establish the following contraction theorem for the operator Ψ.

Theorem 3.4. *Let the operator* $\Psi : X_D \to X$ *be defined by (3.1), (3.8) with some* $\alpha \in (0, 1]$*, and let* $x(t) \in D$ *be a continuous vector function on* $t \in [0, \tau]$ *such that*

$$\|g(x(t))\| \leq \gamma_x \quad \text{for all} \quad t \in [0, \tau]. \tag{3.18}$$

If, in addition, assumptions (A1)–(A2), (A5)–(A6) are satisfied, then the Fréchet derivative of Ψ *at* $x(\cdot)$ *exists and admits the estimate*

$$\|\Psi'_x(dx)\|_X \leq \rho \|dx\|_X \quad \text{for all} \quad dx \in X \tag{3.19}$$

with

$$\rho = 1 - \alpha(1 - \rho^*), \ \rho^* = \rho_1 + \rho_2 + \rho_3 + \rho_4 + \rho_5 \geq 0, \tag{3.20}$$

$$\begin{aligned}
\rho_1 &= \frac{ML(1 + MR_\tau)(e^{\omega\tau} - 1)}{\omega}, \\
\rho_2 &= L(1 + \rho_1)\bar{\rho}, \ \bar{\rho} = \frac{\phi_L(\tau)(\phi_L(\tau) - \sqrt{n})}{\|A\| + L}, \\
\rho_3 &= \frac{\sqrt{n}\bar{H}L\mu_x\bar{\rho}(\phi_L(\tau) + \sqrt{n})}{2} \left\{ \frac{MR_T(e^{\omega\tau} - 1)}{(1 - S)\omega} \right. \\
&\quad \left. + \frac{\phi_L(\tau) - \sqrt{n}}{\|A\| + L} \left(1 + \phi_L(\tau) \left(1 + \frac{\phi_L(\tau) + \sqrt{n}}{2} \right) \right) \right\}, \\
\rho_4 &= \sqrt{n}\bar{H}\mu_x\bar{\rho}, \\
\rho_5 &= \frac{MR_\tau(e^{\omega\tau} - 1)\phi_L(\tau)}{(1 - S)\omega} \Big\{ L(1 + \rho_1)(1 + \phi_L^2(\tau)) \\
&\quad + \frac{\sqrt{n}\bar{H}L\mu_x\bar{\rho}\phi_L(\tau)(\phi_L(\tau) + \sqrt{n})(1 + L\phi_L^2(\tau))}{2} + \sqrt{n}\bar{H}\mu_x(1 + L + L\phi_L^2(\tau)) \\
&\quad + \frac{\sqrt{n}\bar{H}(2\phi_L(\tau) + L\bar{\rho}(\phi_L(\tau) + \sqrt{n}))}{2(1 - S)} \Big\}.
\end{aligned} \tag{3.21}$$

In the above expressions,

$$\mu_x = \|x(\cdot)\|_X + \frac{M(1 + MR_\tau e^{\omega\tau})(e^{\omega\tau} - 1)(\|u(\cdot)\|_X + \gamma_x)}{\omega}, \tag{3.22}$$

and the expressions for $\bar{H}$, L, M, R_τ, S, γ_x, ω, ϕ_L *are given in (A2), (A5), (A6), (2.3), (3.9), (3.18).*

The proof of this theorem is given in A. By applying the Banach fixed point theorem [19] to the operator Ψ in the case $\rho < 1$ (i.e. $\rho^* < 1$), we deduce the following corollary.

Corollary of Theorem 3.4. *Let assumptions (A1)–(A2), (A5)–(A6) be satisfied, and let there exist a bounded closed convex domain* $D' \subset D$ *and a constant* $\alpha \in (0, 1]$ *such that the operator* Ψ *defined by (3.1), (3.8) satisfies the following property:*

$$(\Psi x)(\cdot) \in X_{D'} = C([0, \tau]; D') \text{ for each } x(\cdot) \in X_{D'}. \tag{A7}$$

Assume, moreover, that $\rho^* = \rho_1 + \rho_2 + \rho_3 + \rho_4 + \rho_5 < 1$*, where the constants* ρ_1*, ...,* ρ_5 *are defined by* (3.21) *with*

$$\gamma_x = \sup_{x \in D'} \|g(x)\|, \; \mu_x = \sup_{x \in D'} \|x\| + \frac{M(1 + MR_\tau e^{\omega\tau})(e^{\omega\tau} - 1)(\|u(\cdot)\|_X + \gamma_x)}{\omega},$$

and the values of $\bar{H}$*,* L*,* M*,* R_τ*,* S*,* ω*,* $\phi_L(\tau)$ *are given by (A2), (A5), (A6),* (2.3)*,* (3.9)*.*

Then, for any $x^{(0)}(\cdot) \in X_{D'}$*, the sequence*

$$x^{(k)}(\cdot) = \Psi x^{(k-1)}(\cdot), \quad k = 1, 2, ...$$

converges to the limit $x^*(\cdot) \in X_{D'}$ *as* $k \to \infty$*. This limit function* $x^*(t)$*,* $t \in [0, \tau]$ *is the unique solution of system* (1.1) *such that* $x^*(0) = x^*(\tau)$*.*

4. Application to a controlled chemical reaction model

In this section, we apply the proposed iterative scheme for studying periodic trajectories of the chemical reaction model considered in [10, 11]:

$$\dot{x}(t) = Ax(t) + g(x(t)) + u(t), \;\; x(t) \in D \subset \mathbb{R}^2, \; u(t) \in U \subset \mathbb{R}^2, \; t \in [0, \tau], \tag{4.1}$$

where

$$\begin{aligned} D &= \{(x_1, x_2)^T \,|\, x_1 > -1, \, x_2 > -1\}, \\ U &= \{(u_1, u_2)^T \,|\, u_1 \in [u_1^{min}, u_1^{max}], \, u_2 \in [u_2^{min}, u_2^{max}]\}, \\ A &= \begin{pmatrix} -\phi_1 - k_1 e^{-\varkappa} & -k_1 \varkappa e^{-\varkappa} \\ -k_2 e^{-\varkappa} & -\phi_2 - k_2 \varkappa e^{-\varkappa} \end{pmatrix}, \\ g(x) &= \begin{pmatrix} k_1 e^{-\varkappa} - \phi_1 x_1 - k_1(x_1 + 1) e^{-\varkappa/(x_2+1)} \\ k_2 e^{-\varkappa} - \phi_2 x_2 - k_2(x_1 + 1) e^{-\varkappa/(x_2+1)} \end{pmatrix} - Ax. \end{aligned}$$

System (4.1) describes deviations of the dimensionless concentration $x_1(t)$ and temperature $x_2(t)$ from their reference values in a reactor under controlling the inlet concentration and temperature by $u(t)$. This system admits the trivial equilibrium $x_1 = x_2 = 0$ for $u(t) \equiv 0$, which corresponds to the steady-state reactor operation. In the general case, the function $u(t)$ can encode complicated control scenarios and is assumed to be of class $L^\infty\big([0, \tau]; U\big)$. The constraints $x_1 < -1$ and $x_2 < -1$ in D postulate that the corresponding physical concentration and the absolute temperature in Kelvin should be positive. The physical meaning of the parameters of system (4.1) is discussed in [10], and we take the following values for numerical simulations [11]:

$$\begin{aligned} &\phi_1 = \phi_2 = 1, \; \varkappa = 17.77, \; k_1 = 5.819 \cdot 10^7, k_2 = -8.99 \cdot 10^5, \\ &u_1^{max} = -u_1^{min} = 1.798, \; u_2^{max} = -u_2^{min} = 0.06663. \end{aligned}$$

To test the simple iteration method described in Section 2, we fix a grid size $n_G \in \mathbb{N}$ and consider the uniform partition of $[0, \tau]$ with the step size $\Delta t = \tau / n_G$: $t_j = j \Delta t$ for $j = 0, 1, ..., n_G$. Let $x_j^{(k)}$ denote the value of an approximate solution of the operator equation $\mathcal{F}x(\cdot) = x(\cdot)$ at $t = t_j$ corresponding to the iteration number k from Theorem 2.1. We start from the trivial initial approximation $x_j^{(0)} = 0$ for $j = \overline{0, n_G}$. Then the iteration with index k is obtained from $\{x_j^{(k-1)}\}_{j=0}^{n_G}$ by applying the rectangle quadrature rule to approximate the integrals in (2.1) and (2.2). The resulting algorithm is summarized below for a given number of iterations n_I and arbitrary dimension n of the state vector of (1.1).

The above algorithm has been implemented in Maple 2020 with the use of the `MatrixExponential` function to evaluate $e^{\pm t_j A}$. In the subsequent simulations, we define the function $u(t)$ to be constant on each subinterval of a partition

$$0 = \tau_0 < \tau_1 < ... < \tau_N = \tau,$$

i.e.

$$u(t) = \sum_{i=1}^N u^{(i)} \chi_{[\tau_{i-1}, \tau_i)}(t), \; u(\tau) = u^{(N)}, \; \chi_{[\tau_{i-1}, \tau_i)}(t) = \begin{cases} 1, & t \in [\tau_{i-1}, \tau_i), \\ 0, & t \notin [\tau_{i-1}, \tau_i). \end{cases} \tag{4.2}$$

Algorithm 1 Simple iteration method

Require: A, $g(x)$, $u(t)$, τ, n_G, n_I

Ensure: $x_j^{(n_I)} \approx x(t_j)$ is a discrete-time approximation of the τ-periodic solution $x(t)$ of system (1.1) after n_I iterations

```
Δt ← τ/n_G                                        ▷ Time step size
for j = 0 to n_G do
    t_j ← j * Δt
    x_j^(0) ← 0_n                                 ▷ 0_n is the n-dimensional column vector of zeros
    E_j ← e^(t_j A)                               ▷ Matrix exponentials
    E_{-j} ← e^(-t_j A)
end for
M_0 ← (E_{-n_G} − I)^(-1)                         ▷ The inverse of e^(-τA) − I
for k = 1 to n_I do                               ▷ k is the iteration number
    S_0 ← 0_n
    for j = 1 to n_G do                           ▷ S_j ≈ ∫_0^{t_j} e^(-sA)(u(s) + g(x^(k-1)(s)))ds
        S_j ← S_{j-1} + Δt * E_{1-j} * (u(t_{j-1}) + g(x_{j-1}^(k-1)))
    end for
    c ← M_0 * S_{n_G}                             ▷ The approximation of c in (2.2)
    x_0^(k) ← c
    for j = 1 to n_G do                           ▷ The approximation of x^(k) = F x^(k-1) at t = t_j
        x_j^(k) ← E_j * (c + S_j)
    end for
end for
```

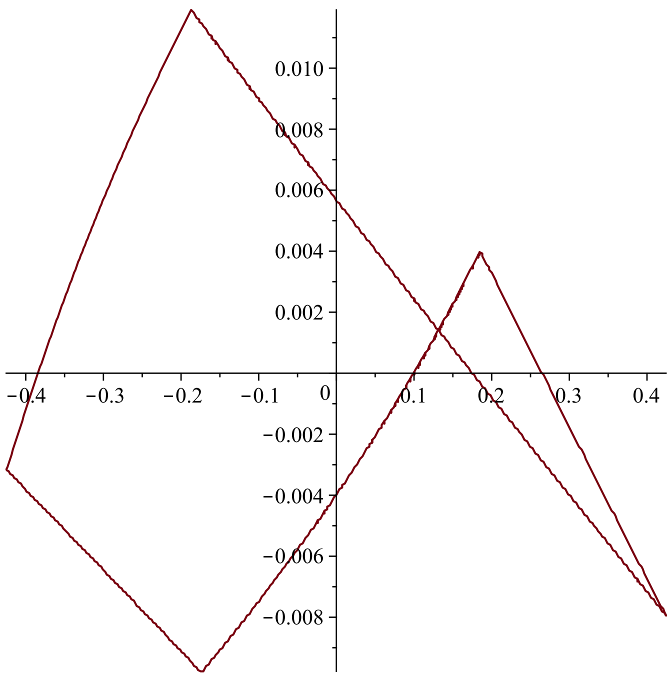

Figure 1: Periodic trajectory of (4.1) with $u(t)$ of the form (4.2)–(4.4): $\tau = 1$, $x(0) \approx (-0.42603, -0.00314)^T$.

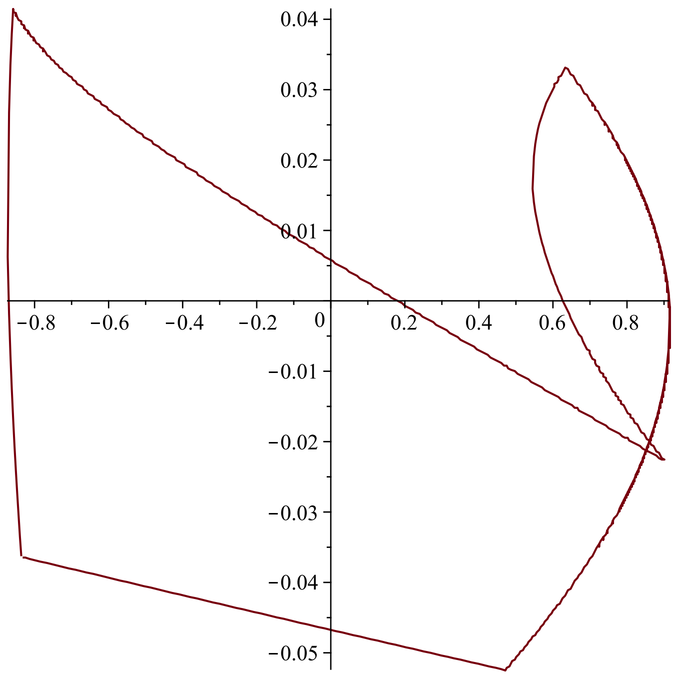

Figure 2: Periodic trajectory of (4.1) with $u(t)$ of the form (4.2)–(4.4): $\tau = 5$, $x(0) \approx (-0.83567, -0.03637)^T$.

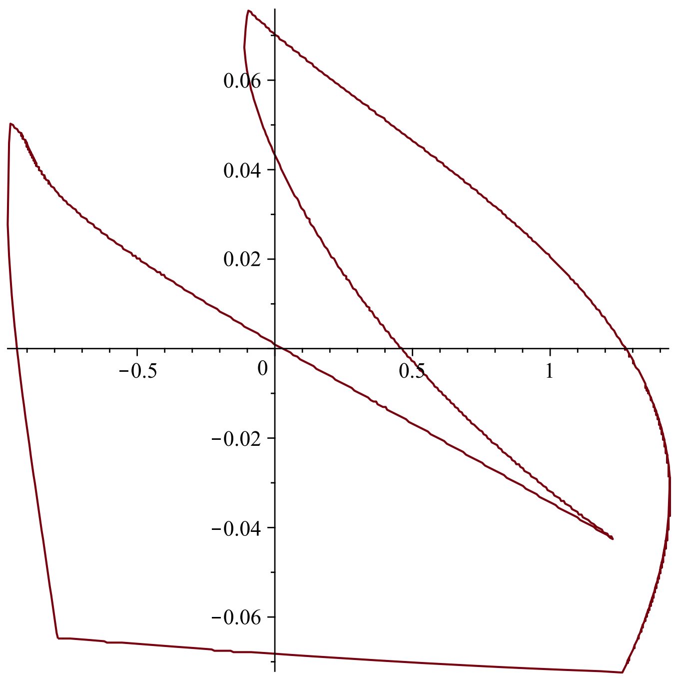

Figure 3: Periodic trajectory of (4.1) with $u(t)$ of the form (4.2)–(4.4): $\tau = 10$, $x(0) \approx (-0.78953, -0.06456)^T$.

The function $u(t)$ in (4.2) corresponds to a family of bang-bang controls with the values $u^{(i)} \in \partial U$ taken at the boundary of U.

To illustrate the behavior of periodic solutions of system (4.1) with discontinuous $u(t)$ of the form (4.2), we fix $N = 5$ and consider the switching sequence

$$u^{(1)} = u^{(3)} = -u^{(4)} = \begin{pmatrix} u_1^{max} \\ u_2^{min} \end{pmatrix}, \; u^{(2)} = -u^{(5)} = \begin{pmatrix} u_1^{max} \\ u_2^{max} \end{pmatrix} \tag{4.3}$$

together with the following switching time parameterization:

$$\tau_0 = 0, \; \tau_1 = 0.1\tau, \; \tau_2 = 0.3\tau, \; \tau_3 = 0.5\tau, \; \tau_4 = 0.8\tau, \; \tau_5 = \tau. \tag{4.4}$$

Algorithm 1 has been executed for system (4.1) with the above choice of $u(t)$ on the grid of size $n_G = 10^5$ and the number of iterations $n_I = 5$. The resulting plots of $x_j^{(n_I)} = (x_{j1}^{(n_I)}, x_{j2}^{(n_I)})^T$, $j = \overline{0, n_G}$, are shown in Fig. 1–Fig. 3 for different values of time horizon τ.

We also check the accuracy of Algorithm 1 by computing the discrepancies

$$d^{(k)}(n_G) = \frac{1}{|J_A|} \sum_{j \in J_A} \left\| \frac{1}{\Delta t}(x_j^{(k)} - x_{j-1}^{(k)}) - Ax_j^{(k)} - u(t_j) - g(x_j^{(k)}) \right\| \tag{4.5}$$

depending on the iteration number k for different grid sizes n_G, where

$$J_A = \{j = \overline{0, n_G} \,|\, t_j \notin \{\tau_0, \tau_1, ..., \tau_N\}\}.$$

The above $d^{(k)}(n_G)$ measures the difference between the finite-difference approximation of $\dot{x}(t)$ and the right-hand side of system (4.1) evaluated at the grid points. The simulation results are summarized in Table 1 up to the maximum iterations number $n_I = 5$.

Iteration number (k)	Discrepancy $(n_G = 10^5)$	Discrepancy $(n_G = 10^6)$
1	0.0604597	0.018644978405663
2	0.0026356	0.000311006947477
3	0.0000429	0.000014457699874
4	0.0000257	$4.49025854660252 \cdot 10^{-6}$
5	0.0000255	$4.39804889269695 \cdot 10^{-6}$

Table 1: Discrepancies $d^{(k)}(n_G)$ for $n_G = 10^5$ and $n_G = 10^6$, $\tau = 1$.

As we can see in Table 1, the resulting discrepancies (as approximation error measures) are monotonically decreasing with the iteration number increase. A significant error reduction ($d^{(3)}(10^5) \approx 4.29 \cdot 10^{-5}$) is already achieved at the third iteration with the time step size $\Delta t = 10^{-5}$, and a more refined time step $\Delta t = 10^{-6}$ results in the discrepancy $d^{(4)}(10^6) \approx 4.49 \cdot 10^{-6}$ at the fourth iteration. As the rectangle quadrature rule is used to approximate the integrals with *arbitrary measurable* input functions $u \in L^\infty[0, \tau]$ in Algorithm 1 and simple finite differences are taken into account in (4.5), it is natural to expect no significant improvement of the decay of approximation errors at subsequent iterations, when the order of $d^{(k)}(n_G)$ becomes comparable with the time step n_G^{-1}.

5. Conclusion and future work

The key contribution of this paper provides general existence and uniqueness conditions for the τ-periodic solutions of nonlinear control systems of the form (1.1) with discontinuous input functions (of class L^∞) and general nonlinearities (of class C^1) in Theorem 2.1. Our crucial assumption is the dominant linearization condition $\det \left(e^{-\tau A} - I\right) \neq 0$, which relates the period τ with spectral properties of the matrix exponent of A from (1.1).

It should be emphasized that sufficient contraction conditions of the corresponding operator formulations are presented explicitly for the simple iteration (Theorem 2.1) and Newton-type

(Theorem 3.4) methods. Our case study justifies the applicability of the proposed iteration scheme to nonlinear control systems describing non-isothermal chemical reaction models. Up to our knowledge, the question of existence and uniqueness of periodic solutions for the model considered in Section 4 with arbitrary discontinuous inputs has not been analyzed so far. The presented Algorithm 1 allows the use of general discontinuous functions $u(t)$ without any regularity assumptions except the integrability on $[0,\tau]$. Because of this general purpose, no high-order quadrature formulas have been incorporated in Algorithm 1. As a direction for further improvement, adaptive quadrature rules can be applied for the approximation of integrals with $u(t)$, taking into account the regularity of input signals on the intervals (τ_i,τ_{i+1}) in the context of bang-bang parameterizations considered in Section 4. We consider the development of such adaptive schemes as well as numerical implementation of Newton's method as topics for future research.

Acknowledgements

The first author was supported by the German Research Foundation (DFG) under Grant ZU 359/2-1.

Availability of data and materials

The datasets used and analysed during the current study are available from the corresponding author on reasonable request.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed to the study conception and technical content. The first draft of the manuscript was written by Alexander Zuyev, and all authors commented on further versions of the manuscript. All authors have read and approved the final manuscript.

A. Proof of Theorem 3.4 and auxiliary results

Proof. of Lemma 3.2. For a given $x \in X_D$ and arbitrary $dx \in X$, we consider the fundamental matrix $\Phi_{x+dx}(t)$ defined by substituting $x(t)+dx(t)$ for $x(t)$ in (3.4):

$$\Phi_{x+dx}(0) = I,\;\; \dot{\Phi}_{x+dx}(t) = \left(A + \left.\frac{\partial g(\xi)}{\partial \xi}\right|_{\xi = x(t)+dx(t)} \right) \Phi_{x+dx}(t) \;\text{ for } t\in[0,\tau].$$

Then $\Delta\Phi_x(t) = \Phi_{x+dx}(t) - \Phi_x(t)$ satisfies the differential equation

$$\begin{aligned} \dot{\Delta\Phi}_x(t) &= \left(A + \left.\frac{\partial g(\xi)}{\partial \xi}\right|_{\xi = x(t)} \right) \Delta\Phi_x(t) \\ &\quad + \left(\left.\frac{\partial g(\xi)}{\partial \xi}\right|_{\xi = x(t)+dx(t)} - \left.\frac{\partial g(\xi)}{\partial \xi}\right|_{\xi = x(t)} \right) \Phi_{x+dx}(t). \end{aligned} \tag{A.1}$$

We observe that the (k,i) entry of the $n\times n$ matrix $\left.\frac{\partial g(\xi)}{\partial \xi}\right|_{\xi = x(t)+dx(t)} - \left.\frac{\partial g(\xi)}{\partial \xi}\right|_{\xi = x(t)}$ is $\Gamma_{ki}(x(t),dx(t)) + o(\|dx(t)\|)$ for small $dx(t)$, provided that g is of class C^2 and Γ_{ki} is defined by (3.11). Let $d\Phi_x(t)(dx)$

be the linear part of $\Delta\Phi_x(t)$ with respect to dx. Then we derive the following linear differential equation for $d\Phi_x(t)(dx)$ from (A.1):

$$\begin{aligned}\frac{d}{dt}d\Phi_x(t)(dx) = &\left(A + \left.\frac{\partial g(\xi)}{\partial \xi}\right|_{\xi=x(t)}\right) d\Phi_x(t)(dx)\\ &+ \Gamma(x(t), dx(t))\Phi_x(t),\ t \in [0, \tau].\end{aligned} \tag{A.2}$$

The variation of constants method yields formula (3.10) for the solution of (A.2) with the initial condition $d\Phi_x(0)(dx) = 0$. To prove the last assertion of Lemma 3.2, we note that

$$\begin{aligned}\Phi^{-1}_{x+dx}(t) &= (\Phi_x(t) + d\Phi_x(t)(dx) + o(\|dx\|_X))^{-1}\\ &= \left(\Phi_x(t)\left\{I + \Phi_x^{-1}(t)d\Phi_x(t)(dx) + o(\|dx\|_X)\right\}\right)^{-1}\\ &= \left\{I - \Phi_x^{-1}(t)d\Phi_x(t)(dx) + o(\|dx\|_X)\right\}\Phi_x^{-1}(t)\\ &= \Phi_x^{-1}(t) - \Phi_x^{-1}(t)\Big(d\Phi_x(t)(dx)\Big)\Phi_x^{-1}(t) + o(\|dx\|_X).\end{aligned}$$

The last formula implies (3.12), which completes the proof of Lemma 3.2. □

Proof. of Lemma 3.3. We denote the matrix

$$B = \left(e^{-\tau A} - I\right)^{-1}\int_0^\tau e^{-tA}\left.\frac{\partial g(x)}{\partial x}\right|_{x(t)}\Phi_x(t)dt,$$

and use the Neumann series for the inverse of M_x defined in (3.7):

$$M_x^{-1} = -(I - B)^{-1} = -\sum_{k=0}^{\infty} B^k. \tag{A.3}$$

The above series converges under assumption $(A6)$:

$$\|B\| \le R_\tau ML\int_0^\tau e^{\omega t}\phi_L(t)dt = S < 1,$$

where the constants M, L, ω, R_τ are defined in $(A2)$ and (2.3), and $\|\Phi_x(t)\| \le \phi_L(t) = \sqrt{n}e^{(\|A\|+L)t}$ is estimated by Lemma 3.1. Moreover, (A.3) together with the triangle inequality implies the assertion of Lemma 3.3:

$$\|M_x^{-1}\| \le \sum_{k=0}^{\infty}\|B^k\| \le \sum_{k=0}^{\infty} S^k = \frac{1}{1-S}.$$

□

Proof. of Theorem 3.4. Formulas (3.13), (3.14), (3.15) allow to represent the Fréchet derivative $\Psi'_x : X \to X$ at $x(\cdot) \in X_D$ in the following way:

$$\Psi'_x(dx) = (1-\alpha)dx - \alpha\Big(\Delta_1(dx) + \Delta_2(dx) + \Delta_3(dx) + \Delta_4(dx) + \Delta_5(dx)\Big), \tag{A.4}$$

where the operators $\Delta_x^{(i)} : X \to X$ are

$$\Delta_x^{(1)}(dx)(t) = -e^{tA}dc_x(dx) - \int_0^t e^{(t-s)A}\left.\frac{\partial g(x)}{\partial x}\right|_{x(s)} dx(s)ds, \tag{A.5}$$

$$\Delta_x^{(2)}(dx)(t) = -\Phi_x(t)\int_0^t \Phi_x^{-1}(s)\left.\frac{\partial g}{\partial x}\right|_{x(s)} d\delta y(s)ds, \tag{A.6}$$

$$\begin{aligned}\Delta_x^{(3)}(dx)(t) =& d\Phi_x(t)C - d\Phi_x(t)\int_0^t \Phi_x^{-1}(s)\left.\frac{\partial g}{\partial x}\right|_{x(s)}\delta y(s)ds\\ &+ \Phi_x(t)\int_0^t \Phi_x^{-2}(s)d\Phi_x(s)\left.\frac{\partial g}{\partial x}\right|_{x(s)}\delta y(s)ds,\end{aligned} \tag{A.7}$$

$$\Delta_x^{(4)}(dx)(t) = -\Phi_x(t)\int_0^t \Phi_x^{-1}(s)\left\{\left.\frac{\partial^2 g}{\partial x^2}\right|_{x(s)} dx(s)\right\}\delta y(s)ds, \tag{A.8}$$

$$\Delta_x^{(5)}(dx)(t) = \Phi_x(t)dC(dx), \tag{A.9}$$

where $d\delta y(t)$ and $dC(dx)$ are defined by (3.14) and (3.16), respectively. Then estimate (3.19) follows from (A.4) and the triangle inequality, provided that

$$\|\Delta_x^{(i)}(dx)\|_X \le \rho_i\|dx\|_X \quad \text{for all } dx \in X,\ \ i = \overline{1,5}. \tag{A.10}$$

To prove the first inequality in (A.10), we compare formula (A.5) with (2.4) and note that $-\Delta_x^{(1)}(dx) = \mathcal{F}_x'(dx)$. We also note that the ρ_1 given in (3.21) coincides with the left-hand side of $(A4)$, so that the inequality $\|\Delta_x^{(1)}(dx)\|_X \le \rho_1\|dx\|_X$ follows from the proof of Theorem 2.1. The latter inequality together with (3.14), (A.5) implies that

$$\|d\delta y\|_X \le (1+\rho_1)\|dx\|_X. \tag{A.11}$$

Because of Lemma 3.1, assumption $(A2)$, and inequality (A.11), the expression (A.6) is estimated as

$$\|\Delta_x^{(2)}(dx)\|_X \le L\phi_L(\tau)\left(\int_0^\tau \phi_L(s)ds\right)\|d\delta y\|_X \le \rho_2\|dx\|_X,$$

where $\phi_L(\cdot)$ and ρ_2 are defined in (3.9) and (3.21), respectively.

Under assumption $(A5)$, the norm of the matrix $\Gamma(x(s), dx(s))$ in (3.11) is estimated by the norm of $dx(s) \in \mathbb{R}^n$ as

$$\|\Gamma(x(s), dx(s))\| \le \sqrt{n}\bar{H}\|dx(s)\|. \tag{A.12}$$

We use the above inequality to estimate $d\Phi_x(t)(dx)$ in (3.10):

$$\begin{aligned}\|d\Phi_x(\cdot)(dx)\|_X &= \sup_{t\in[0,\tau]}\|d\Phi_x(t)(dx)\| \le \sqrt{n}\bar{H}\sup_{t\in[0,\tau]}\left(\phi_L(t)\int_0^t \phi_L^2(s)\|dx(s)\|\right)ds\\ &\le \frac{\sqrt{n}\bar{H}\phi_L(\tau)(\phi_L^2(\tau)-n)}{2(\|A\|+L)}\|dx\|_X.\end{aligned} \tag{A.13}$$

Furthermore, we use the Cauchy–Schwarz and triangle inequalities to derive the following estimate from (2.1) under assumption (3.18):

$$\|\delta y\|_X = \|\mathcal{F}x - x\|_X \le \mu_x, \tag{A.14}$$

where μ_x is defined in (3.22). By putting together (3.6), (A.13), (A.14) and Lemma 3.3, we estimate the norm of (A.7) with the use of the constant ρ_3 from (3.21):

$$\|\Delta_x^{(3)}(dx)\|_X \le \rho_3\|dx\|_X.$$

We also estimate the norm of $\Delta_x^{(4)}(dx)$ in (A.8) with the use of Lemma 3.1, (A.12), and (A.14):

$$\|\Delta_x^{(4)}(dx)\|_X \le \sqrt{n}\bar{H}\mu_x\phi_L(\tau)\left(\int_0^\tau \phi_L(s)ds\right)\|dx\|_X = \rho_4\|dx\|_X,$$

where ρ_4 is defined in (3.21).

It remains to estimate $\Delta_x^{(5)}(dx)$ in (A.9) by exploiting representation (3.16) together with Lemma 3.1:

$$\|\Delta_x^{(5)}(dx)\|_X \le \phi_L(\tau)\|dC(dx)\|,\ dC = dC_1(dx) + dC_2(dx), \tag{A.15}$$

where the two summands in $dC(dx)$ are

$$dC_1(dx) = M_x^{-1}(e^{-\tau A} - I)^{-1}\int_0^\tau e^{-sA}\zeta(s)ds,$$

$$
\begin{aligned}
dC_2(dx) = d(M_x^{-1})(dx)(e^{-\tau A} - I)^{-1} \int_0^\tau e^{-sA} \left.\frac{\partial g}{\partial x}\right|_{x(s)} \\
\times \left\{ \delta y(s) + \Phi_x(s) \int_0^s \Phi_x^{-1}(v) \left.\frac{\partial g}{\partial x}\right|_{x(v)} \delta y(v) dv \right\} ds,
\end{aligned}
$$

and

$$
\begin{aligned}
\zeta(s) = & \left.\frac{\partial g}{\partial x}\right|_{x(s)} \Bigg\{ d\delta y(s) + \Phi_x(s) \int_0^s \Phi_x^{-1}(v) \left.\frac{\partial g}{\partial x}\right|_{x(v)} d\delta y(v) dv \\
& + d\Phi_x(s) \int_0^s \Phi_x^{-1}(v) \left.\frac{\partial g}{\partial x}\right|_{x(v)} \delta y(v) dv \\
& -\Phi_x(s) \int_0^s \Phi_x^{-1}(v) \left(\Phi_x^{-1}(v) d\Phi_x(v) \left.\frac{\partial g}{\partial x}\right|_{x(v)} - \left.\frac{\partial^2 g}{\partial x^2}\right|_{x(v)} dx(v) \right) \delta y(v) dv \Bigg\} \\
& + \left(\left.\frac{\partial^2 g}{\partial x^2}\right|_{x(s)} dx(s) \right) \left\{ \delta y(s) + \Phi_x(s) \int_0^s \Phi_x^{-1}(v) \left.\frac{\partial g}{\partial x}\right|_{x(v)} \delta y(v) dv \right\}.
\end{aligned}
$$

As the assumptions of Lemma 3.3 are satisfied, then

$$
\|dC_1(dx)\| \leq \frac{MR_\tau(e^{\omega\tau} - 1)}{(1 - S)\omega} \|\zeta\|_X, \tag{A.16}
$$

and the norm of ζ is estimated with the use of Lemma 3.1 and (A.11), (A.12), (A.13), (A.14) as

$$
\begin{aligned}
\|\zeta\|_X \leq \Big(& L(1 + \rho_1)(1 + \phi_L^2(\tau)) + \sqrt{n}\bar{H}\mu_x(1 + L + L\phi_L^2(\tau)) \\
& + \frac{\sqrt{n}\bar{H}L\mu_x\bar{\rho}\phi_L(\tau)(\phi_L(\tau) + \sqrt{n})(1 + L\phi_L^2(\tau))}{2} \Big) \|dx\|_X.
\end{aligned} \tag{A.17}
$$

Furthermore, we derive the following estimate of $dC_2(dx)$ by applying Lemma 3.3 and (3.17), (A.12), (A.13):

$$
\|dC_2(dx)\| \leq \frac{\sqrt{n}\bar{H}MR_\tau(e^{\omega\tau} - 1)}{(1 - S)^2\omega} \left(\phi_L(\tau) + \frac{L\bar{\rho}}{2}(\phi_L(\tau) + \sqrt{n}) \right) \|dx\|_X. \tag{A.18}
$$

It is easy to see that inequalities (A.15), (A.16), (A.17), (A.18) imply

$$
\|\Delta_x^{(5)}(dx)\|_X \leq \rho_5 \|dx\|_X
$$

with ρ_5 defined in (3.21).

We have shown that estimates (A.10) hold with ρ_i defined by (3.21). Then assertion (3.19) follows from (A.4) and (3.20) by applying the triangle inequality. □

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