arXiv:2401.10047v1 [math.OC] 18 Jan 2024

Interpolatory Necessary Optimality Conditions for Reduced-order Modeling of Parametric Linear Time-invariant Systems

Preprint

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January 19, 2024

Abstract

Interpolatory necessary optimality conditions for $\mathcal{H}_2$-optimal reduced-order modeling of non-parametric linear time-invariant (LTI) systems are known and well-investigated. In this work, using the general framework of $\mathcal{L}_2$-optimal reduced-order modeling of parametric stationary problems, we derive interpolatory $\mathcal{H}_2 \otimes \mathcal{L}_2$-optimality conditions for parametric LTI systems with a general pole-residue form. We then specialize this result to recover known conditions for systems with parameter-independent poles and develop new conditions for a certain class of systems with parameter-dependent poles.

Keywords reduced-order modeling, parametric systems, optimization, interpolation, linear systems

1 Introduction

Consider a parametric linear time-invariant (LTI) system (full-order model (FOM))

$$\mathcal{E}(\mathsf{q})\dot{x}(t, \mathsf{q}) = \mathcal{A}(\mathsf{q})x(t, \mathsf{q}) + \mathcal{B}(\mathsf{q})u(t), \tag{1.1a}$$

$$y(t, \mathsf{q}) = \mathcal{C}(\mathsf{q})x(t, \mathsf{q}), \tag{1.1b}$$

where $\mathsf{q} \in \mathcal{Q} \subseteq \mathbb{R}^{n_{\mathsf{q}}}$ is the parameter vector; $u(t) \in \mathbb{R}^{n_{\mathrm{i}}}$ is the input; $x(t, \mathsf{q}) \in \mathbb{R}^{n}$ is the state; $y(t, \mathsf{q}) \in \mathbb{R}^{n_{\mathrm{o}}}$ is the output; and $\mathcal{E}(\mathsf{q}), \mathcal{A}(\mathsf{q}) \in \mathbb{R}^{n \times n}$, $\mathcal{B}(\mathsf{q}) \in \mathbb{R}^{n \times n_{\mathrm{i}}}$, and $\mathcal{C}(\mathsf{q}) \in \mathbb{R}^{n_{\mathrm{o}} \times n}$ are parametric matrices. Given the FOM in (1.1), the goal of parametric reduced-order modeling is to find a reduced parametric LTI system (reduced-order model (ROM))

$$\widehat{\mathcal{E}}(\mathsf{q})\dot{\widehat{x}}(t, \mathsf{q}) = \widehat{\mathcal{A}}(\mathsf{q})\widehat{x}(t, \mathsf{q}) + \widehat{\mathcal{B}}(\mathsf{q})u(t), \tag{1.2a}$$

$$\widehat{y}(t, \mathsf{q}) = \widehat{\mathcal{C}}(\mathsf{q})\widehat{x}(t, \mathsf{q}), \tag{1.2b}$$

where $\widehat{x}(t, \mathsf{q}) \in \mathbb{R}^{r}$ is the reduced state with $r \ll n$; $\widehat{y}(t, \mathsf{q}) \in \mathbb{R}^{n_{\mathrm{o}}}$ is the approximate output; and $\widehat{\mathcal{E}}(\mathsf{q}), \widehat{\mathcal{A}}(\mathsf{q}) \in \mathbb{R}^{r \times r}$, $\widehat{\mathcal{B}}(\mathsf{q}) \in \mathbb{R}^{r \times n_{\mathrm{i}}}$, and $\widehat{\mathcal{C}}(\mathsf{q}) \in \mathbb{R}^{n_{\mathrm{o}} \times r}$ are the reduced parametric matrices, such that $\widehat{y}$ approximates y for a wide range of inputs u and a set of parameters $\mathsf{q} \in \mathcal{Q} \subseteq \mathbb{R}^{n_{\mathsf{q}}}$. Parametric dynamical systems are ubiquitous in applications ranging from inverse problems to uncertainty quantification to optimization and model reduction of parametric systems has been a major research topic; we refer the reader to, e.g., [4, 3] for more details.

Both the FOM and ROM can be fully described by their (parametric) transfer functions, given by, respectively,

$$H(s, \mathsf{q}) = \mathcal{C}(\mathsf{q})(s\mathcal{E}(\mathsf{q}) - \mathcal{A}(\mathsf{q}))^{-1}\mathcal{B}(\mathsf{q}) \quad \text{and}$$

$$\widehat{H}(s, \mathsf{q}) = \widehat{\mathcal{C}}(\mathsf{q})\Big(s\widehat{\mathcal{E}}(\mathsf{q}) - \widehat{\mathcal{A}}(\mathsf{q})\Big)^{-1}\widehat{\mathcal{B}}(\mathsf{q}). \tag{1.3}$$

As in any approximation problem, one needs a metric to judge the quality of the approximation. For *non-parametric* LTI systems, i.e., when $\mathcal{E}, \mathcal{A}, \mathcal{B}, \mathcal{C}$ are constant matrices, the $\mathcal{H}_2$-norm has been one of the most commonly used metrics in (optimal) reduced-order modeling [1, 8, 12, 18]. For parametric LTI systems we consider here, the $\mathcal{H}_2 \otimes \mathcal{L}_2$-norm introduced in [2] provides a natural extension. The goal of $\mathcal{H}_2 \otimes \mathcal{L}_2$-optimal reduced-order modeling is to find a ROM that (locally) minimizes the $\mathcal{H}_2 \otimes \mathcal{L}_2$ error

$$\left\lVert H - \widehat{H} \right\rVert_{\mathcal{H}_2 \otimes \mathcal{L}_2} = \left(\int_{\mathcal{Q}} \left\lVert H(\cdot, \mathsf{q}) - \widehat{H}(\cdot, \mathsf{q}) \right\rVert_{\mathcal{H}_2}^2 \mathrm{d}\nu(\mathsf{q}) \right)^{1/2}, \tag{1.4}$$

where ν is a measure over $\mathcal{Q}$ and the $\mathcal{H}_2$ norm is given as

$$\|H(\cdot, \mathsf{q})\|_{\mathcal{H}_2} = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \|H(\imath\omega, \mathsf{q})\|_{\mathrm{F}}^2 \,\mathrm{d}\omega \right)^{1/2}.$$

The $\mathcal{H}_2 \otimes \mathcal{L}_2$ error gives an upper bound for the output error

$$\|y - \widehat{y}\|_{\mathcal{L}_\infty \otimes \mathcal{L}_2} \leqslant \left\| H - \widehat{H} \right\|_{\mathcal{H}_2 \otimes \mathcal{L}_2} \|u\|_{\mathcal{L}_2},$$

where

$$\|y\|_{\mathcal{L}_\infty \otimes \mathcal{L}_2} = \left(\int_{\mathcal{Q}} \|y(\cdot, \mathsf{q})\|_{\mathcal{L}_\infty}^2 \,\mathrm{d}\nu(\mathsf{q}) \right)^{1/2}$$

is the $\mathcal{L}_\infty \otimes \mathcal{L}_2$ norm of the output, further justifying the use of $\mathcal{H}_2 \otimes \mathcal{L}_2$ measure in parametric reduced-order modeling.

The $\mathcal{H}_2 \otimes \mathcal{L}_2$ norm with ν as the Lebesgue measure was introduced in Baur et al. [2]. There, for the special case where $\widehat{\mathcal{E}}$ and $\widehat{\mathcal{A}}$ are *parameter-independent*, the $\mathcal{H}_2 \otimes \mathcal{L}_2$-optimal reduced-order modeling problem was converted to a non-parametric $\mathcal{H}_2$-optimal reduced-order modeling problem and interpolatory optimality conditions could be established. For another simplified problem where the poles of $\widehat{H}$ do *not* vary with the parameter $\mathsf{q} \in \{|\mathsf{q}| = 1\} \subset \mathbb{C}$, Grimm [7] used an $\mathcal{H}_2 \otimes \mathcal{L}_2$ norm and derived interpolatory conditions and proposed an optimization algorithm.

A common assumption in parametric reduced-order modeling methods is parameter-separability. For the ROM (1.2), this would mean that the reduced quantities can be written as

$$\widehat{\mathcal{E}}(\mathsf{q}) = \sum_{\ell=1}^{n_{\widehat{\mathcal{E}}}} \widehat{\varepsilon}_\ell(\mathsf{q}) \widehat{E}_\ell, \quad \widehat{\mathcal{A}}(\mathsf{q}) = \sum_{i=1}^{n_{\widehat{\mathcal{A}}}} \widehat{\alpha}_i(\mathsf{q}) \widehat{A}_i, \tag{1.5a}$$

$$\widehat{\mathcal{B}}(\mathsf{q}) = \sum_{j=1}^{n_{\widehat{\mathcal{B}}}} \widehat{\beta}_j(\mathsf{q}) \widehat{B}_j, \quad \widehat{\mathcal{C}}(\mathsf{q}) = \sum_{k=1}^{n_{\widehat{\mathcal{C}}}} \widehat{\gamma}_k(\mathsf{q}) \widehat{C}_k, \tag{1.5b}$$

for some functions $\widehat{\varepsilon}_\ell, \widehat{\alpha}_i, \widehat{\beta}_j, \widehat{\gamma}_k \colon \mathcal{Q} \to \mathbb{R}$, constant matrices $\widehat{E}_\ell$, $\widehat{A}_i$, $\widehat{B}_j$, $\widehat{C}_k$, and positive integers $n_{\widehat{\mathcal{E}}}, n_{\widehat{\mathcal{A}}}, n_{\widehat{\mathcal{B}}}, n_{\widehat{\mathcal{C}}}$. We call a ROM of this form a structured ROM (StROM). This form has also been considered in $\mathcal{H}_2 \otimes \mathcal{L}_2$-optimal reduced-order modeling methods. In particular, Petersson [16] considered the case of a discretized $\mathcal{H}_2 \otimes \mathcal{L}_2$ norm, i.e., where ν is a sum of Dirac measures, proposing an optimization algorithm to find a locally $\mathcal{H}_2 \otimes \mathcal{L}_2$-optimal ROM. Additionally, Hund et al. [9] proposed an optimization algorithm for $\mathcal{H}_2 \otimes \mathcal{L}_2$-optimal reduced-order modeling using quadrature for the case of Lebesgue measure. Both of these works used matrix equation-based, Wilson-type conditions [18] and not interpolation.

The $\mathcal{H}_2 \otimes \mathcal{L}_2$-norm was also used by Brunsch [6] to derive error bounds within a reduced-order modeling framework for parametric LTI systems with symmetric positive definite $\mathcal{E}(\mathsf{q})$ and $-\mathcal{A}(\mathsf{q})$. The method is based on sparse-grid interpolation in the parameter domain. It satisfies (Hermite) interpolation conditions and preserves stability, but has no proven optimality properties. See also [10, 5, 17] for some data-driven approaches.

In our recent work on $\mathcal{L}_2$-optimal reduced-order modeling [14], we covered both LTI systems and parametric stationary problems. We developed interpolatory necessary optimality conditions in [15] for certain types of StROMs, including *non-parametric* LTI systems and parametric stationary problems. We also showed that the interpolatory conditions of [7] can be derived from our generalized $\mathcal{L}_2$-optimality conditions. However, as stated before, [7] assumes the poles are fixed.

Therefore, unlike for the non-parametric LTI problems for which interpolatory optimality conditions for $\mathcal{H}_2$ model reduction have been well-established [12, 8, 1], there is a significant gap in the development of interpolatory optimality conditions for $\mathcal{H}_2 \otimes \mathcal{L}_2$-optimal parametric ROM construction, except for the special cases mentioned above. Our goal in this paper is to close this gap and to develop interpolatory optimality conditions for the more general setting of parametric LTI systems. Additionally, we show that our analysis contains the earlier conditions from [2] as a special case.

We provide background in Section 2. While Section 3 covers the general parametric diagonal StROMs (D-StROMs) case, Sections 4 and 5 focus on simplified cases, leading to optimality conditions that can be directly linked to the bitangential Hermite interpolation framework. We conclude with Section 6.

2 Background

Here we recall one of the main results of [13], specifically, the necessary $\mathcal{L}_2$-optimality conditions for D-StROMs, which will form the foundation of our analysis.

Given a parameter-to-output mapping

$$H \colon \mathcal{P} \to \mathbb{C}^{n_{\mathrm{o}} \times n_{\mathrm{i}}},$$

the goal in [13] is to construct a StROM

$$\widehat{\mathcal{K}}(\mathsf{p}) \widehat{X}(\mathsf{p}) = \widehat{\mathcal{F}}(\mathsf{p}), \tag{2.1a}$$

$$\widehat{H}(\mathsf{p}) = \widehat{\mathcal{G}}(\mathsf{p}) \widehat{X}(\mathsf{p}), \tag{2.1b}$$

with a parameter-separable form

$$\widehat{\mathcal{K}}(\mathsf{p}) = \sum_{i=1}^{n_{\widehat{\mathcal{K}}}} \widehat{\kappa}_i(\mathsf{p}) \widehat{K}_i, \quad \widehat{\mathcal{F}}(\mathsf{p}) = \sum_{j=1}^{n_{\widehat{\mathcal{F}}}} \widehat{\zeta}_j(\mathsf{p}) \widehat{F}_j, \quad \widehat{\mathcal{G}}(\mathsf{p}) = \sum_{k=1}^{n_{\widehat{\mathcal{G}}}} \widehat{\eta}_k(\mathsf{p}) \widehat{G}_k, \tag{2.2}$$

where $\widehat{X}(\mathsf{p}) \in \mathbb{C}^{r \times n_{\mathrm{i}}}$ is the reduced state, $\widehat{H}(\mathsf{p}) \in \mathbb{C}^{n_{\mathrm{o}} \times n_{\mathrm{i}}}$ is the approximate output, $\widehat{\mathcal{K}}(\mathsf{p}) \in \mathbb{C}^{r \times r}$, $\widehat{\mathcal{F}}(\mathsf{p}) \in \mathbb{C}^{r \times n_{\mathrm{i}}}$, $\widehat{\mathcal{G}}(\mathsf{p}) \in \mathbb{C}^{n_{\mathrm{o}} \times r}$, $\widehat{\kappa}_i, \widehat{\zeta}_j, \widehat{\eta}_k \colon \mathcal{P} \to \mathbb{C}$, $\widehat{K}_i \in \mathbb{C}^{r \times r}$, $\widehat{F}_j \in \mathbb{C}^{r \times n_{\mathrm{i}}}$, and $\widehat{G}_k \in \mathbb{C}^{n_{\mathrm{o}} \times r}$. The goal is to construct $\widehat{\mathcal{K}}(\mathsf{p})$, $\widehat{\mathcal{F}}(\mathsf{p})$, and $\widehat{\mathcal{G}}(\mathsf{p})$ such that $\widehat{H}(\mathsf{p}) = \widehat{\mathcal{G}}(\mathsf{p}) \widehat{\mathcal{K}}(\mathsf{p})^{-1} \widehat{\mathcal{F}}(\mathsf{p})$ is an optimal $\mathcal{L}_2$-approximation to the original mapping $H(\mathsf{p})$, i.e.,

$$\left\| H - \widehat{H} \right\|_{\mathcal{L}_2} = \left(\int_{\mathcal{P}} \left\| H(\mathsf{p}) - \widehat{H}(\mathsf{p}) \right\|_{\mathrm{F}}^2 \mathrm{d}\mu(\mathsf{p}) \right)^{1/2} \tag{2.3}$$

is minimized where μ is a measure over $\mathcal{P}$. The $\mathcal{H}_2 \otimes \mathcal{L}_2$ norm (1.4) is a special case of the $\mathcal{L}_2$-norm (2.3) for appropriately defined p and μ, a fact we exploit in Sections 3 to 5. We will use the notation $(\widehat{K}_i, \widehat{F}_j, \widehat{G}_k)$ to denote the StROM specified by (2.1) and (2.2).

We assume all $\widehat{K}_i$'s are diagonal (and in return so is $\widehat{\mathcal{K}}(\mathsf{p})$ in (2.1)); thus $\widehat{H}$ for a D-StROM has a "pole-residue" form, i.e.,

$$\widehat{H}(\mathsf{p}) = \widehat{\mathcal{G}}(\mathsf{p})\widehat{\mathcal{K}}(\mathsf{p})^{-1}\widehat{\mathcal{F}}(\mathsf{p}) = \sum_{\ell=1}^{r} \frac{g_\ell(\mathsf{p}) f_\ell(\mathsf{p})^*}{k_\ell(\mathsf{p})}, \tag{2.4}$$

where $k_\ell(\mathsf{p})$ is the ℓth diagonal entry of $\widehat{\mathcal{K}}(\mathsf{p})$, $f_\ell(\mathsf{p}) = \widehat{\mathcal{F}}(\mathsf{p})^* e_\ell$, and $g_\ell(\mathsf{p}) = \widehat{\mathcal{G}}(\mathsf{p}) e_\ell$. With this pole-residue form in hand, we have the optimality conditions for D-StROMs (Corollary 2.4 in [13]).

Theorem 2.1. *Suppose that $\mathcal{P} \subseteq \mathbb{C}^{n_\mathsf{p}}$, μ is a measure over $\mathcal{P}$, the function H is in $\mathcal{L}_2(\mathcal{P}, \mu; \mathbb{C}^{n_\mathrm{o} \times n_\mathrm{i}})$, functions $\widehat{\kappa}_i, \widehat{\zeta}_j, \widehat{\eta}_k \colon \mathcal{P} \to \mathbb{C}$ are measurable and satisfy*

$$\int_{\mathcal{P}} \left(\frac{\sum_{j=1}^{n_{\widehat{\mathcal{F}}}} \left|\widehat{\zeta}_j(\mathsf{p})\right| \sum_{k=1}^{n_{\widehat{\mathcal{G}}}} |\widehat{\eta}_k(\mathsf{p})|}{\sum_{i=1}^{n_{\widehat{\mathcal{K}}}} |\widehat{\kappa}_i(\mathsf{p})|} \right)^2 \mathrm{d}\mu(\mathsf{p}) < \infty, \tag{2.5}$$

$\widehat{K}_i \in \mathbb{C}^{r \times r}$, $\widehat{F}_j \in \mathbb{C}^{r \times n_\mathrm{i}}$, $\widehat{G}_k \in \mathbb{C}^{n_\mathrm{o} \times r}$, and

$$\operatorname*{ess\,sup}_{\mathsf{p} \in \mathcal{P}} \left\| \widehat{\kappa}_i(\mathsf{p}) \widehat{\mathcal{K}}(\mathsf{p})^{-1} \right\|_{\mathrm{F}} < \infty, \quad i = 1, 2, \ldots, n_{\widehat{\mathcal{K}}}, \tag{2.6}$$

where $\widehat{\mathcal{K}}$ is as in (2.2). Furthermore, let $(\widehat{K}_i, \widehat{F}_j, \widehat{G}_k)$ be an $\mathcal{L}_2$-optimal D-StROM of H with $\widehat{H}$ as in (2.4). Then

$$\int_{\mathcal{P}} \frac{\overline{\widehat{\eta}_k(\mathsf{p})} H(\mathsf{p}) f_\ell(\mathsf{p})}{\overline{k_\ell(\mathsf{p})}} \,\mathrm{d}\mu(\mathsf{p}) = \int_{\mathcal{P}} \frac{\overline{\widehat{\eta}_k(\mathsf{p})} \widehat{H}(\mathsf{p}) f_\ell(\mathsf{p})}{\overline{k_\ell(\mathsf{p})}} \,\mathrm{d}\mu(\mathsf{p}), \tag{2.7a}$$

$$\int_{\mathcal{P}} \frac{\overline{\widehat{\zeta}_j(\mathsf{p})} g_\ell(\mathsf{p})^* H(\mathsf{p})}{\overline{k_\ell(\mathsf{p})}} \,\mathrm{d}\mu(\mathsf{p}) = \int_{\mathcal{P}} \frac{\overline{\widehat{\zeta}_j(\mathsf{p})} g_\ell(\mathsf{p})^* \widehat{H}(\mathsf{p})}{\overline{k_\ell(\mathsf{p})}} \,\mathrm{d}\mu(\mathsf{p}), \tag{2.7b}$$

$$\begin{aligned} &\int_{\mathcal{P}} \frac{\overline{\widehat{\kappa}_i(\mathsf{p})} g_\ell(\mathsf{p})^* H(\mathsf{p}) f_\ell(\mathsf{p})}{\overline{k_\ell(\mathsf{p})}^2} \,\mathrm{d}\mu(\mathsf{p}) \\ &= \int_{\mathcal{P}} \frac{\overline{\widehat{\kappa}_i(\mathsf{p})} g_\ell(\mathsf{p})^* \widehat{H}(\mathsf{p}) f_\ell(\mathsf{p})}{\overline{k_\ell(\mathsf{p})}^2} \,\mathrm{d}\mu(\mathsf{p}), \end{aligned} \tag{2.7c}$$

for $i = 1, 2, \ldots, n_{\widehat{\mathcal{K}}}$, $j = 1, 2, \ldots, n_{\widehat{\mathcal{F}}}$, $k = 1, 2, \ldots, n_{\widehat{\mathcal{G}}}$, and $\ell = 1, 2, \ldots, r$.

Theorem 2.1 establishes the interpolatory optimality conditions (2.7) for $\mathcal{L}_2$-optimal approximation. We showed in [13] that various structured reduced-order modeling problems appear as a special case of Theorem 2.1 and derived interpolatory optimality conditions for important classes of *non-parametric* structured LTI systems. In this paper, we extend this analysis to parametric LTI systems.

3 $\mathcal{H}_2 \otimes \mathcal{L}_2$-optimal Parametric Interpolation

Here we use Theorem 2.1 to derive interpolatory conditions for $\mathcal{H}_2 \otimes \mathcal{L}_2$-optimal reduced-order approximation of the FOM (1.1) using D-StROMs. But first we need to establish what the assumptions (2.5) and (2.6) appearing in $\mathcal{L}_2$-optimal approximation for the structure (2.2) correspond to in the case of $\mathcal{H}_2 \otimes \mathcal{L}_2$ approximation with the structures of StROMs in (1.5).

Lemma 3.1. *Let $\mathsf{p} = (s, \mathsf{q})$, $\mathcal{P} = \imath\mathbb{R} \times \mathcal{Q}$, and $\mu = \frac{1}{2\pi}\lambda_{\imath\mathbb{R}} \times \nu$ where $\lambda_{\imath\mathbb{R}}$ is the Lebesgue measure over $\imath\mathbb{R}$ and ν is a measure over $\mathcal{Q} \subseteq \mathbb{C}^{n_\mathsf{q}}$. Furthermore, for StROM in (1.5), let the functions $\widehat{\varepsilon}_\ell, \widehat{\alpha}_i, \widehat{\beta}_j, \widehat{\gamma}_k \colon \mathcal{Q} \to \mathbb{C}$ be measurable. Then the condition*

$$\int_{\mathcal{Q}} \frac{\left(\sum_{j=1}^{n_{\widehat{\mathcal{B}}}} \left|\widehat{\beta}_j(\mathsf{q})\right| \sum_{k=1}^{n_{\widehat{\mathcal{C}}}} |\widehat{\gamma}_k(\mathsf{q})|\right)^2}{\sum_{\ell=1}^{n_{\widehat{\mathcal{E}}}} |\widehat{\varepsilon}_\ell(\mathsf{q})| \sum_{i=1}^{n_{\widehat{\mathcal{A}}}} |\widehat{\alpha}_i(\mathsf{q})|} \,\mathrm{d}\nu(\mathsf{q}) < \infty, \tag{3.1}$$

is equivalent to (2.5) and the conditions

$$\operatorname*{ess\,sup}_{\mathsf{q} \in \mathcal{Q}} |\widehat{\varepsilon}_\ell(\mathsf{q})| \left\| s\left(s\widehat{\mathcal{E}}(\mathsf{q}) - \widehat{\mathcal{A}}(\mathsf{q})\right)^{-1} \right\|_{\mathcal{L}_\infty} < \infty, \tag{3.2a}$$

$$\operatorname*{ess\,sup}_{\mathsf{q} \in \mathcal{Q}} |\widehat{\alpha}_i(\mathsf{q})| \left\| \left(s\widehat{\mathcal{E}}(\mathsf{q}) - \widehat{\mathcal{A}}(\mathsf{q})\right)^{-1} \right\|_{\mathcal{L}_\infty} < \infty, \tag{3.2b}$$

for $\ell = 1, \ldots, n_{\widehat{\mathcal{E}}}$ and $i = 1, \ldots, n_{\widehat{\mathcal{A}}}$, are equivalent to (2.6).

Proof. First note that with the choices of $\mathsf{p} = (s, \mathsf{q})$, $\mathcal{P} = \imath\mathbb{R} \times \mathcal{Q}$, and $\mu = \frac{1}{2\pi}\lambda_{\imath\mathbb{R}} \times \nu$, the $\mathcal{L}_2$-norm in (2.3) recovers the $\mathcal{H}_2 \otimes \mathcal{L}_2$norm in (1.4). Now note that the integral in (2.5), for the StROM $(s\widehat{\mathcal{E}}(\mathsf{q}) - \widehat{\mathcal{A}}(\mathsf{q}), \widehat{\mathcal{B}}(\mathsf{q}), \widehat{\mathcal{C}}(\mathsf{q}))$ as in (1.5), takes the form

$$\int_{\mathcal{Q}} \int_{-\infty}^{\infty} \left(\frac{\sum_{j=1}^{n_{\widehat{\mathcal{B}}}} \left|\widehat{\beta}_j(\mathsf{q})\right| \sum_{k=1}^{n_{\widehat{\mathcal{C}}}} |\widehat{\gamma}_k(\mathsf{q})|}{|\omega| \sum_{\ell=1}^{n_{\widehat{\mathcal{E}}}} |\widehat{\varepsilon}_\ell(\mathsf{q})| + \sum_{i=1}^{n_{\widehat{\mathcal{A}}}} |\widehat{\alpha}_i(\mathsf{q})|} \right)^2 \mathrm{d}\omega \,\mathrm{d}\nu(\mathsf{q}).$$

Using that $\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(a|x|+b)^2} = \frac{2}{ab}$ for positive a and b, the above integral becomes equal to the one in (3.1), up to scaling by 2.

Next, the conditions in (2.6) become

$$\operatorname*{ess\,sup}_{\mathsf{q} \in \mathcal{Q}} \operatorname*{ess\,sup}_{\omega \in \mathbb{R}} \left\| \imath\omega\widehat{\varepsilon}_\ell(\mathsf{q}) \left(\imath\omega\widehat{\mathcal{E}}(\mathsf{q}) - \widehat{\mathcal{A}}(\mathsf{q})\right)^{-1} \right\|_{\mathrm{F}} < \infty,$$

$$\operatorname*{ess\,sup}_{\mathsf{q} \in \mathcal{Q}} \operatorname*{ess\,sup}_{\omega \in \mathbb{R}} \left\| \widehat{\alpha}_i(\mathsf{q}) \left(\imath\omega\widehat{\mathcal{E}}(\mathsf{q}) - \widehat{\mathcal{A}}(\mathsf{q})\right)^{-1} \right\|_{\mathrm{F}} < \infty,$$

which simplify to

$$\operatorname*{ess\,sup}_{\mathsf{q} \in \mathcal{Q}} |\widehat{\varepsilon}_\ell(\mathsf{q})| \operatorname*{ess\,sup}_{s \in \imath\mathbb{R}} \left\| s\left(s\widehat{\mathcal{E}}(\mathsf{q}) - \widehat{\mathcal{A}}(\mathsf{q})\right)^{-1} \right\|_{\mathrm{F}} < \infty,$$

$$\operatorname*{ess\,sup}_{\mathsf{q} \in \mathcal{Q}} |\widehat{\alpha}_i(\mathsf{q})| \operatorname*{ess\,sup}_{s \in \imath\mathbb{R}} \left\| \left(s\widehat{\mathcal{E}}(\mathsf{q}) - \widehat{\mathcal{A}}(\mathsf{q})\right)^{-1} \right\|_{\mathrm{F}} < \infty.$$

Since $\|\cdot\|_{\mathrm{F}}$ and $\|\cdot\|_2$ are equivalent norms, the above conditions are equivalent to (3.2). ☐

The work [9] used the assumptions that $\mathcal{Q} \subset \mathbb{R}^{n_\mathsf{q}}$ is compact, ν is a finite Borel measure over $\mathcal{Q}$, $\widehat{\varepsilon}_\ell, \widehat{\alpha}_i, \widehat{\beta}_j, \widehat{\gamma}_k \colon \mathcal{Q} \to \mathbb{R}$ are continuous, $\widehat{\mathcal{E}}(\mathsf{q})$ is invertible and $\widehat{\mathcal{E}}(\mathsf{q})^{-1}\widehat{\mathcal{A}}(\mathsf{q})$ has all eigenvalues in the open left half-plane for all $\mathsf{q} \in \mathcal{Q}$. Therefore, we see that the assumptions of the earlier work [9] on $\mathcal{H}_2 \otimes \mathcal{L}_2$ approximation are indeed a special case of the ones we derived in Lemma 3.1.

Now that we established the assumptions of Theorem 2.1 for parametric LTI systems, we are ready to derive the corresponding interpolatory $\mathcal{H}_2 \otimes \mathcal{L}_2$-optimality conditions based on the conditions (2.7). We use $\widehat{\mathcal{E}}(\mathsf{q}) = I$ and all $\widehat{A}_i$ being diagonal. (The result can be extended to parametric diagonal $\widehat{\mathcal{E}}$; only the expressions become more involved.)

Theorem 3.2. *Given the full-order parametric transfer function H, let $\widehat{H}$ in (1.3) be an $\mathcal{H}_2 \otimes \mathcal{L}_2$-optimal D-StROM for H with $\widehat{\mathcal{E}}(\mathsf{q}) = I$ and all $\widehat{A}_i$ being diagonal in (1.5a). Let $\lambda_\ell(\mathsf{q})$ denote the ℓth diagonal entry of $\widehat{\mathcal{A}}(\mathsf{q})$. Moreover, define $c_\ell(\mathsf{q}) = \widehat{\mathcal{C}}(\mathsf{q})e_\ell$ and $b_\ell(\mathsf{q}) = \widehat{\mathcal{B}}(\mathsf{q})^* e_\ell$, where $\widehat{\mathcal{B}}(\mathsf{q})$ and $\widehat{\mathcal{C}}(\mathsf{q})$ are as defined in (1.5b). Then*

$$\widehat{H}(s,\mathsf{q}) = \sum_{\ell=1}^{r} \frac{c_\ell(\mathsf{q}) b_\ell(\mathsf{q})^*}{s - \lambda_\ell(\mathsf{q})} \tag{3.3}$$

and

$$\begin{aligned}&\int_{\mathcal{Q}} \widehat{\gamma}_k(\mathsf{q}) H\left(-\overline{\lambda_\ell(\mathsf{q})},\mathsf{q}\right) b_\ell(\mathsf{q})\,\mathrm{d}\nu(\mathsf{q}) \\ &= \int_{\mathcal{Q}} \widehat{\gamma}_k(\mathsf{q}) \widehat{H}\left(-\overline{\lambda_\ell(\mathsf{q})},\mathsf{q}\right) b_\ell(\mathsf{q})\,\mathrm{d}\nu(\mathsf{q}),\end{aligned} \tag{3.4a}$$

$$\begin{aligned}&\int_{\mathcal{Q}} \widehat{\beta}_j(\mathsf{q}) c_\ell(\mathsf{q})^* H\left(-\overline{\lambda_\ell(\mathsf{q})},\mathsf{q}\right) \mathrm{d}\nu(\mathsf{q}) \\ &= \int_{\mathcal{Q}} \widehat{\beta}_j(\mathsf{q}) c_\ell(\mathsf{q})^* \widehat{H}\left(-\overline{\lambda_\ell(\mathsf{q})},\mathsf{q}\right) \mathrm{d}\nu(\mathsf{q}),\end{aligned} \tag{3.4b}$$

$$\begin{aligned}&\int_{\mathcal{Q}} \widehat{\alpha}_i(\mathsf{q}) c_\ell(\mathsf{q})^* \frac{\partial H}{\partial s}\left(-\overline{\lambda_\ell(\mathsf{q})},\mathsf{q}\right) b_\ell(\mathsf{q})\,\mathrm{d}\nu(\mathsf{q}) \\ &= \int_{\mathcal{Q}} \widehat{\alpha}_i(\mathsf{q}) c_\ell(\mathsf{q})^* \frac{\partial \widehat{H}}{\partial s}\left(-\overline{\lambda_\ell(\mathsf{q})},\mathsf{q}\right) b_\ell(\mathsf{q})\,\mathrm{d}\nu(\mathsf{q}),\end{aligned} \tag{3.4c}$$

for $\ell = 1,2,\ldots,r$, $k = 1,2,\ldots,n_{\widehat{\mathcal{C}}}$, $j = 1,2,\ldots,n_{\widehat{\mathcal{B}}}$, $i = 1,2,\ldots n_{\widehat{\mathcal{A}}}$, where $\widehat{\alpha}_i$, $\widehat{\beta}_j$, and $\widehat{\gamma}_k$ are as defined in (1.5).

Proof. The pole-residue form (3.3) follows from the general diagonal pole-residue form (2.4), with $k_\ell(s,\mathsf{q}) = s - \lambda_\ell(\mathsf{q})$, $f_\ell(s,\mathsf{q}) = b_\ell(\mathsf{q})$, and $g_\ell(s,\mathsf{q}) = c_\ell(\mathsf{q})$. Then, with this structure, the optimality conditions (3.4) follow from diagonal conditions after applying the Cauchy integral formula. For instance, the left-hand side of the right tangential Lagrange condition (2.7a) becomes

$$\begin{aligned}&\int_{\mathcal{P}} \frac{\overline{\gamma_k(\mathsf{p})} H(\mathsf{p}) b_\ell(\mathsf{p})}{\overline{a_\ell(\mathsf{p})}}\,\mathrm{d}\mu(\mathsf{p}) \\ &= \int_{\mathcal{Q}} \int_{-\infty}^{\infty} \frac{\widehat{\gamma}_k(\mathsf{q}) H(\imath\omega,\mathsf{q}) b_\ell(\mathsf{q})}{\overline{\imath\omega - \lambda_\ell(\mathsf{q})}}\,\mathrm{d}\omega\,\mathrm{d}\nu(\mathsf{q}) \\ &= \int_{\mathcal{Q}} \int_{-\infty}^{\infty} \frac{\widehat{\gamma}_k(\mathsf{q}) H(\imath\omega,\mathsf{q}) b_\ell(\mathsf{q})}{-\imath\omega - \overline{\lambda_\ell(\mathsf{q})}}\,\mathrm{d}\omega\,\mathrm{d}\nu(\mathsf{q}) \\ &= \frac{1}{\imath} \int_{\mathcal{Q}} \oint_{\imath\mathbb{R}} \frac{\widehat{\gamma}_k(\mathsf{q}) H(s,\mathsf{q}) b_\ell(\mathsf{q})}{-s - \overline{\lambda_\ell(\mathsf{q})}}\,\mathrm{d}s\,\mathrm{d}\nu(\mathsf{q}) \\ &= \frac{2\pi}{\imath} \int_{\mathcal{Q}} \widehat{\gamma}_k(\mathsf{q}) H\left(-\overline{\lambda_\ell(\mathsf{q})},\mathsf{q}\right) b_\ell(\mathsf{q})\,\mathrm{d}\nu(\mathsf{q}),\end{aligned}$$

which yields (3.4a). The remaining two conditions (3.4b)–(3.4c) follow similarly from (2.7b) and (2.7c). ☐

Recall that $\mathcal{H}_2$-optimal approximation of *non-parametric* LTI systems requires bitangential Hermite interpolation of the FOM transfer function H at the mirror images of the reduced-order poles [8, 1]. We showed in our earlier works [14, 15, 13] that bitangential Hermite interpolation as necessary conditions for optimality extend to many other $\mathcal{H}_2/\mathcal{L}_2$ approximation settings as well. Even though the $\mathcal{H}_2 \otimes \mathcal{L}_2$ optimality conditions (3.4) derived here have an integral form, they still have the similar bitangential Hermite interpolation structure as before. To arrive at these more familiar form of bitangential Hermite interpolations, we need to have explicit expressions for the functions $\widehat{\alpha}_i, \widehat{\beta}_j, \widehat{\gamma}_k, \lambda_\ell, b_\ell, c_\ell$. In the next two sections we focus on such cases.

4 Parameters in Inputs and Outputs

The work [2] considered parametric LTI systems with parameter-dependence only in $\mathcal{B}$ and $\mathcal{C}$; specifically, the ROM of the form

$$\begin{aligned}\widehat{\mathcal{E}}(\mathsf{q}) &= I, & \widehat{\mathcal{A}}(\mathsf{q}) &= \widehat{A}, \\ \widehat{\mathcal{B}}(\mathsf{q}) &= \widehat{B}_1 + \mathsf{q}_1\widehat{B}_2, & \widehat{\mathcal{C}}(\mathsf{q}) &= \widehat{C}_1 + \mathsf{q}_2\widehat{C}_2,\end{aligned} \tag{4.1}$$

with $\widehat{A} = \operatorname{diag}(\lambda_1,\lambda_2,\ldots,\lambda_r)$ and $\mathcal{Q} = [0,1]^2$. Therefore,

$$\begin{gathered}\mathsf{q} = (\mathsf{q}_1,\mathsf{q}_2),\ n_{\widehat{\mathcal{A}}} = 1,\ \widehat{\alpha}_1(\mathsf{q}) = 1, \\ n_{\widehat{\mathcal{B}}} = 2,\ \widehat{\beta}_1(\mathsf{q}) = 1,\ \widehat{\beta}_2(\mathsf{q}) = \mathsf{q}_1, \\ n_{\widehat{\mathcal{C}}} = 2,\ \widehat{\gamma}_1(\mathsf{q}) = 1,\ \widehat{\gamma}_2(\mathsf{q}) = \mathsf{q}_2, \\ \lambda_\ell(\mathsf{q}) = \lambda_\ell,\ b_\ell(\mathsf{q}) = b_{\ell,1} + \mathsf{q}_1 b_{\ell,2},\ c_\ell(\mathsf{q}) = c_{\ell,1} + \mathsf{q}_2 c_{\ell,2},\end{gathered}$$

where $b_{\ell,i} = \widehat{B}_i^* e_\ell$ and $c_{\ell,i} = \widehat{C}_i e_\ell$ for $i = 1,2$. In [2], only single-input single-output (SISO) systems were considered. Here we consider multiple-input multiple-output (MIMO) systems. (Further extensions are possible, see, e.g., [11].)

Note that the reduced transfer function is bilinear in terms of the parameters q_1 and q_2:

$$\begin{aligned}\widehat{H}(s,\mathsf{q}) &= \left(\widehat{C}_1 + \mathsf{q}_2\widehat{C}_2\right)\left(sI - \widehat{A}\right)^{-1}\left(\widehat{B}_1 + \mathsf{q}_1\widehat{B}_2\right) \\ &= \widehat{H}_{11}(s) + \mathsf{q}_1\widehat{H}_{12}(s) + \mathsf{q}_2\widehat{H}_{21}(s) + \mathsf{q}_1\mathsf{q}_2\widehat{H}_{22}(s)\end{aligned} \tag{4.2}$$

where

$$\widehat{H}_{ij}(s) = \widehat{C}_i\left(sI - \widehat{A}\right)^{-1}\widehat{B}_j, \quad i,j \in \{1,2\}.$$

We assume the same form for the full transfer function

$$H(s,\mathsf{q}) = H_{11}(s) + \mathsf{q}_1 H_{12}(s) + \mathsf{q}_2 H_{21}(s) + \mathsf{q}_1\mathsf{q}_2 H_{22}(s), \tag{4.3}$$

where $H_{ij} \in \mathcal{H}_2$ for $i, j \in \{1, 2\}$. However, contrary to [2], we do not need to assume that H_{ij} has a finite-dimensional state space, i.e., they can contain non-rational terms. We only require the ROM to have a finite-dimensional state space. Thus, the theory we develop applies not only to MIMO systems but also to non-rational transfer functions.

Following [2], we define the auxiliary transfer functions

$$\mathcal{H}(s) = \begin{bmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \end{bmatrix}, \tag{4.4a}$$

$$\begin{aligned} \widehat{\mathcal{H}}(s) &= \begin{bmatrix} \widehat{H}_{11}(s) & \widehat{H}_{12}(s) \\ \widehat{H}_{21}(s) & \widehat{H}_{22}(s) \end{bmatrix} \\ &= \begin{bmatrix} \widehat{C}_1 \\ \widehat{C}_2 \end{bmatrix} \left(sI - \widehat{A}\right)^{-1} \begin{bmatrix} \widehat{B}_1 & \widehat{B}_2 \end{bmatrix}. \end{aligned} \tag{4.4b}$$

Note that $\mathcal{H}$ and $\widehat{\mathcal{H}}$ satisfy

$$H(s, \mathsf{q}) = \begin{bmatrix} I_{n_{\mathrm{o}}} & \mathsf{q}_2 I_{n_{\mathrm{o}}} \end{bmatrix} \mathcal{H}(s) \begin{bmatrix} I_{n_{\mathrm{i}}} \\ \mathsf{q}_1 I_{n_{\mathrm{i}}} \end{bmatrix}, \tag{4.5a}$$

$$\widehat{H}(s, \mathsf{q}) = \begin{bmatrix} I_{n_{\mathrm{o}}} & \mathsf{q}_2 I_{n_{\mathrm{o}}} \end{bmatrix} \widehat{\mathcal{H}}(s) \begin{bmatrix} I_{n_{\mathrm{i}}} \\ \mathsf{q}_1 I_{n_{\mathrm{i}}} \end{bmatrix}. \tag{4.5b}$$

We obtain the following result, were we take ν to be the Lebesgue measure over $\mathcal{Q}$ (as in [2]).

Theorem 4.1. *Let $H, \widehat{H}$ be as in (4.3) and (4.1) and $\mathcal{H}, \widehat{\mathcal{H}}$ as in (4.4a) and (4.4b). Furthermore, let $\widehat{H}$ be an $\mathcal{H}_2 \otimes \mathcal{L}_2$-optimal ROM for H. Define*

$$\mathfrak{b}_\ell = \begin{bmatrix} I_{n_{\mathrm{i}}} & \frac{1}{2} I_{n_{\mathrm{i}}} \\ \frac{1}{2} I_{n_{\mathrm{i}}} & \frac{1}{3} I_{n_{\mathrm{i}}} \end{bmatrix} \begin{bmatrix} b_{\ell,1} \\ b_{\ell,2} \end{bmatrix}, \quad \mathfrak{c}_\ell = \begin{bmatrix} I_{n_{\mathrm{o}}} & \frac{1}{2} I_{n_{\mathrm{o}}} \\ \frac{1}{2} I_{n_{\mathrm{o}}} & \frac{1}{3} I_{n_{\mathrm{o}}} \end{bmatrix} \begin{bmatrix} c_{\ell,1} \\ c_{\ell,2} \end{bmatrix}. \tag{4.6}$$

Then for $\ell = 1, 2, \ldots, r$, we have

$$\mathcal{H}(-\overline{\lambda_\ell}) \mathfrak{b}_\ell = \widehat{\mathcal{H}}(-\overline{\lambda_\ell}) \mathfrak{b}_\ell, \tag{4.7a}$$

$$\mathfrak{c}_\ell^* \mathcal{H}(-\overline{\lambda_\ell}) = \mathfrak{c}_\ell^* \widehat{\mathcal{H}}(-\overline{\lambda_\ell}), \tag{4.7b}$$

$$\mathfrak{c}_\ell^* \mathcal{H}'(-\overline{\lambda_\ell}) \mathfrak{b}_\ell = \mathfrak{c}_\ell^* \widehat{\mathcal{H}}'(-\overline{\lambda_\ell}) \mathfrak{b}_\ell. \tag{4.7c}$$

Proof. Based on the conditions in (3.4), we need to compute the integrals

$$\int_{\mathcal{Q}} H(-\overline{\lambda_\ell}, \mathsf{q}) b_\ell(\mathsf{q}) \,\mathrm{d}\mathsf{q}, \ \int_{\mathcal{Q}} \mathsf{q}_2 H(-\overline{\lambda_\ell}, \mathsf{q}) b_\ell(\mathsf{q}) \,\mathrm{d}\mathsf{q},$$
$$\int_{\mathcal{Q}} c_\ell(\mathsf{q})^* H(-\overline{\lambda_\ell}, \mathsf{q}) \,\mathrm{d}\mathsf{q}, \ \int_{\mathcal{Q}} \mathsf{q}_1 c_\ell(\mathsf{q})^* H(-\overline{\lambda_\ell}, \mathsf{q}) \,\mathrm{d}\mathsf{q},$$
$$\int_{\mathcal{Q}} c_\ell(\mathsf{q})^* \frac{\partial H}{\partial s}(-\overline{\lambda_\ell}, \mathsf{q}) b_\ell(\mathsf{q}) \,\mathrm{d}\mathsf{q},$$

and similarly with $\widehat{H}$. Starting with the first, we find that

$$\begin{aligned} &\int_{\mathcal{Q}} H(-\overline{\lambda_\ell}, \mathsf{q}) b_\ell(\mathsf{q}) \,\mathrm{d}\mathsf{q} \\ &= \int_{\mathcal{Q}} \big(H_{11}(-\overline{\lambda_\ell}) + \mathsf{q}_1 H_{12}(-\overline{\lambda_\ell}) + \mathsf{q}_2 H_{21}(-\overline{\lambda_\ell}) \\ &\qquad + \mathsf{q}_1 \mathsf{q}_2 H_{22}(-\overline{\lambda_\ell})\big)(b_{\ell,1} + \mathsf{q}_1 b_{\ell,2}) \,\mathrm{d}\mathsf{q} \\ &= H_{11}(-\overline{\lambda_\ell}) b_{\ell,1} + \frac{1}{2} H_{12}(-\overline{\lambda_\ell}) b_{\ell,1} \\ &\quad + \frac{1}{2} H_{21}(-\overline{\lambda_\ell}) b_{\ell,1} + \frac{1}{4} H_{22}(-\overline{\lambda_\ell}) b_{\ell,1} \\ &\quad + \frac{1}{2} H_{11}(-\overline{\lambda_\ell}) b_{\ell,2} + \frac{1}{3} H_{12}(-\overline{\lambda_\ell}) b_{\ell,2} \\ &\quad + \frac{1}{4} H_{21}(-\overline{\lambda_\ell}) b_{\ell,2} + \frac{1}{6} H_{22}(-\overline{\lambda_\ell}) b_{\ell,2} \\ &= \begin{bmatrix} I_{n_{\mathrm{o}}} & \frac{1}{2} I_{n_{\mathrm{o}}} \end{bmatrix} \mathcal{H}(-\overline{\lambda_\ell}) \begin{bmatrix} I_{n_{\mathrm{i}}} & \frac{1}{2} I_{n_{\mathrm{i}}} \\ \frac{1}{2} I_{n_{\mathrm{i}}} & \frac{1}{3} I_{n_{\mathrm{i}}} \end{bmatrix} \begin{bmatrix} b_{\ell,1} \\ b_{\ell,2} \end{bmatrix} \end{aligned}$$

where we used (4.3) and (4.5a). The second integral becomes

$$\begin{aligned} &\int_{\mathcal{Q}} \mathsf{q}_2 H(-\overline{\lambda_\ell}, \mathsf{q}) b_\ell(\mathsf{q}) \,\mathrm{d}\mathsf{q} \\ &= \begin{bmatrix} \frac{1}{2} I_{n_{\mathrm{o}}} & \frac{1}{3} I_{n_{\mathrm{o}}} \end{bmatrix} \mathcal{H}(-\overline{\lambda_\ell}) \begin{bmatrix} I_{n_{\mathrm{i}}} & \frac{1}{2} I_{n_{\mathrm{i}}} \\ \frac{1}{2} I_{n_{\mathrm{i}}} & \frac{1}{3} I_{n_{\mathrm{i}}} \end{bmatrix} \begin{bmatrix} b_{\ell,1} \\ b_{\ell,2} \end{bmatrix}. \end{aligned}$$

Stacking these two vertically (and using (4.6)) gives us the left-hand side in the right Lagrange tangential condition (4.7a). The other conditions follow similarly. □

Therefore, for this special case of (4.1), we obtain more familiar bitangential Hermite interpolation. More specifically, $\mathcal{H}_2 \otimes \mathcal{L}_2$-optimal reduced-order modeling of H with $\widehat{H}$ in (4.1) is equivalent to a *weighted* $\mathcal{H}_2$-optimal reduced-order modeling for $\mathcal{H}$ with $\widehat{\mathcal{H}}$. Thus, our general framework in Theorem 3.2 not only recovers the results from [2] but also extends it to MIMO systems and eliminates the need for the full-model to have a rational transfer function.

Interpolatory conditions of Theorem 4.1 are in terms of the parametric function $\widehat{\mathcal{H}}$, not the original parametric transfer function H. The next result gives explicit interpolatory conditions in terms of H for SISO systems.

Corollary 4.2. *Let the assumptions in Theorem 4.1 hold. Furthermore, let $n_{\mathrm{i}} = n_{\mathrm{o}} = 1$. If $\mathfrak{b}_{\ell,1} \neq 0$ and $\mathfrak{c}_{\ell,1} \neq 0$, then*

$$\begin{aligned} H\left(-\overline{\lambda_\ell}, \frac{\mathfrak{b}_{\ell,2}}{\mathfrak{b}_{\ell,1}}, \mathsf{q}_2\right) &= \widehat{H}\left(-\overline{\lambda_\ell}, \frac{\mathfrak{b}_{\ell,2}}{\mathfrak{b}_{\ell,1}}, \mathsf{q}_2\right), \\ \partial_{\mathsf{q}_2} H\left(-\overline{\lambda_\ell}, \frac{\mathfrak{b}_{\ell,2}}{\mathfrak{b}_{\ell,1}}, \mathsf{q}_2\right) &= \partial_{\mathsf{q}_2} \widehat{H}\left(-\overline{\lambda_\ell}, \frac{\mathfrak{b}_{\ell,2}}{\mathfrak{b}_{\ell,1}}, \mathsf{q}_2\right), \\ H\left(-\overline{\lambda_\ell}, \mathsf{q}_1, \frac{\overline{\mathfrak{c}_{\ell,2}}}{\overline{\mathfrak{c}_{\ell,1}}}\right) &= \widehat{H}\left(-\overline{\lambda_\ell}, \mathsf{q}_1, \frac{\overline{\mathfrak{c}_{\ell,2}}}{\overline{\mathfrak{c}_{\ell,1}}}\right), \\ \partial_{\mathsf{q}_1} H\left(-\overline{\lambda_\ell}, \mathsf{q}_1, \frac{\overline{\mathfrak{c}_{\ell,2}}}{\overline{\mathfrak{c}_{\ell,1}}}\right) &= \partial_{\mathsf{q}_1} \widehat{H}\left(-\overline{\lambda_\ell}, \mathsf{q}_1, \frac{\overline{\mathfrak{c}_{\ell,2}}}{\overline{\mathfrak{c}_{\ell,1}}}\right), \\ H'\left(-\overline{\lambda_\ell}, \frac{\mathfrak{b}_{\ell,2}}{\mathfrak{b}_{\ell,1}}, \frac{\overline{\mathfrak{c}_{\ell,2}}}{\overline{\mathfrak{c}_{\ell,1}}}\right) &= \widehat{H}'\left(-\overline{\lambda_\ell}, \frac{\mathfrak{b}_{\ell,2}}{\mathfrak{b}_{\ell,1}}, \frac{\overline{\mathfrak{c}_{\ell,2}}}{\overline{\mathfrak{c}_{\ell,1}}}\right), \end{aligned}$$

for all $\mathsf{q}_1, \mathsf{q}_2 \in \mathbb{C}$ and $\ell = 1, 2, \ldots, r$.

Proof. The proof follows directly from (4.7) using (4.5) and

$$\partial_{\mathsf{q}_2} H(s, \mathsf{q}) = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathcal{H}(s) \begin{bmatrix} 1 \\ \mathsf{q}_1 \end{bmatrix}$$

and similar expressions for $\partial_{\mathsf{q}_2} \widehat{H}$, $\partial_{\mathsf{q}_1} H$, and $\partial_{\mathsf{q}_1} \widehat{H}$. □

This states that $\mathcal{H}_2 \otimes \mathcal{L}_2$-optimality for the full-order and reduced-order structure in (4.3) and (4.2) requires that, in the parameter space, interpolation is enforced over lines instead of just a finite number of points (as generically done for parametric systems).

5 Parameter in Dynamics

In the previous section, we considered a special case where the parametric dependence was only in $\widehat{\mathcal{B}}$ and $\widehat{\mathcal{C}}$. Now, we consider the case where $\mathcal{Q} = [a, b] \subset \mathbb{R}$, $a < b$, and the FOM and ROM have the form, respectively,

$$\begin{aligned} \mathcal{E}(\mathsf{q}) &= I, \quad \mathcal{A}(\mathsf{q}) = A_1 + \mathsf{q}A_2, \\ \mathcal{B}(\mathsf{q}) &= B, \quad \mathcal{C}(\mathsf{q}) = C \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} \widehat{\mathcal{E}}(\mathsf{q}) &= I, \quad \widehat{\mathcal{A}}(\mathsf{q}) = \widehat{A}_1 + \mathsf{q}\widehat{A}_2, \\ \widehat{\mathcal{B}}(\mathsf{q}) &= \widehat{B}, \quad \widehat{\mathcal{C}}(\mathsf{q}) = \widehat{C} \end{aligned} \tag{5.2}$$

with

$$\begin{aligned} A_k &= \operatorname{diag}(\nu_{k,1}, \nu_{k,2}, \ldots, \nu_{k,n}) \text{ and} \\ \widehat{A}_k &= \operatorname{diag}(\lambda_{k,1}, \lambda_{k,2}, \ldots, \lambda_{k,r}), \end{aligned}$$

for $k = 1, 2$. In other words, we are assuming the parametric dependencies appear only in the dynamics matrices $\mathcal{A}$ and $\widehat{\mathcal{A}}$ and they are both composed of only two terms which are simultaneously diagonalizable. With these parametric forms, the full-order and reduced-order transfer functions have the pole-residue forms

$$H(s, \mathsf{q}) = \sum_{i=1}^{n} \frac{\Phi_i}{s - \nu_i(\mathsf{q})}, \quad \widehat{H}(s, \mathsf{q}) = \sum_{i=1}^{r} \frac{c_i b_i^*}{s - \lambda_i(\mathsf{q})}, \tag{5.3}$$

where

$$\nu_i(\mathsf{q}) = \nu_{1,i} + \mathsf{q}\nu_{2,i} \quad \text{and} \quad \lambda_i(\mathsf{q}) = \lambda_{1,i} + \mathsf{q}\lambda_{2,i}. \tag{5.4}$$

Let ν be the Lebesgue measure over $[a, b]$. Furthermore, for any $\sigma_a, \sigma_b \in \mathbb{C}_-$, define $f_{\sigma_a,\sigma_b} \colon \mathbb{C}_+^2 \to \mathbb{C}$ as

$$f_{\sigma_a,\sigma_b}(s_a, s_b) = \frac{b - a}{(s_b - \sigma_b) - (s_a - \sigma_a)} \ln\left(\frac{s_b - \sigma_b}{s_a - \sigma_a}\right). \tag{5.5}$$

Additionally, define the functions $G, \widehat{G} \colon \mathbb{C}_+^2 \to \mathbb{C}$ by

$$G(s_a, s_b) = \sum_{i=1}^{n} f_{\nu_i(a),\nu_i(b)}(s_a, s_b)\Phi_i \quad \text{and} \tag{5.6a}$$

$$\widehat{G}(s_a, s_b) = \sum_{i=1}^{r} f_{\lambda_i(a),\lambda_i(b)}(s_a, s_b) c_i b_i^*. \tag{5.6b}$$

Note that G and $\widehat{G}$ depend on the pole-residue forms (5.3) of H and $\widehat{H}$, respectively. Thus, one can consider G as the full-order *modified* function and $\widehat{G}$ the reduced-order one. Based on this setup, we are ready to state the interpolatory optimality conditions in this setting.

Theorem 5.1. *Let H and $\widehat{H}$ be as given in (5.3) and let G and $\widehat{G}$ be as defined in (5.6). If $\widehat{H}$ is an $\mathcal{H}_2 \otimes \mathcal{L}_2$-optimal D-StROM for H, then*

$$G\left(-\overline{\lambda_i(a)}, -\overline{\lambda_i(b)}\right) b_i = \widehat{G}\left(-\overline{\lambda_i(a)}, -\overline{\lambda_i(b)}\right) b_i, \tag{5.7a}$$

$$c_i^* G\left(-\overline{\lambda_i(a)}, -\overline{\lambda_i(b)}\right) = c_i^* \widehat{G}\left(-\overline{\lambda_i(a)}, -\overline{\lambda_i(b)}\right), \tag{5.7b}$$

$$c_i^* \frac{\partial G}{\partial s_a}\left(-\overline{\lambda_i(a)}, -\overline{\lambda_i(b)}\right) b_i = c_i^* \frac{\partial \widehat{G}}{\partial s_a}\left(-\overline{\lambda_i(a)}, -\overline{\lambda_i(b)}\right) b_i, \tag{5.7c}$$

$$c_i^* \frac{\partial G}{\partial s_b}\left(-\overline{\lambda_i(a)}, -\overline{\lambda_i(b)}\right) b_i = c_i^* \frac{\partial \widehat{G}}{\partial s_b}\left(-\overline{\lambda_i(a)}, -\overline{\lambda_i(b)}\right) b_i, \tag{5.7d}$$

for $i = 1, 2, \ldots, r$.

Proof. Due to the special form of the ROM in (5.2) and its transfer function $\widehat{H}$ in (5.3), the quantities $b_\ell, c_\ell, \widehat{\gamma}_k, \widehat{\beta}_j$ in (3.3) and (3.4) in Theorem 3.2 are parameter-independent. Thus, to analyze the first Lagrange conditions (3.4a) and (3.4b) for the special case (5.1) and (5.2), it is enough to focus on the integrals

$$\int_{\mathcal{Q}} H\left(-\overline{\lambda_i(\mathsf{q})}, \mathsf{q}\right) \mathrm{d}\nu(\mathsf{q}) \quad \text{and} \quad \int_{\mathcal{Q}} \widehat{H}\left(-\overline{\lambda_i(\mathsf{q})}, \mathsf{q}\right) \mathrm{d}\nu(\mathsf{q}).$$

We start with the first integral involving H. Using the pole-residue form of H from (5.3) and the expression for $\lambda_i(\mathsf{q})$ from (5.4), we obtain

$$\begin{aligned} \int_{\mathcal{Q}} H\left(-\overline{\lambda_i(\mathsf{q})}, \mathsf{q}\right) \mathrm{d}\nu(\mathsf{q}) &= \int_a^b H\left(-\overline{\lambda_i(\mathsf{q})}, \mathsf{q}\right) \mathrm{d}\mathsf{q} \\ &= \int_a^b \sum_{j=1}^{n} \frac{\Phi_j}{-\overline{\lambda_i(\mathsf{q})} - \nu_{1,j} - \mathsf{q}\nu_{2,j}} \mathrm{d}\mathsf{q} \\ &= \sum_{j=1}^{n} \int_a^b \frac{\Phi_j}{-\overline{\lambda_{1,i}} - \mathsf{q}\overline{\lambda_{2,i}} - \nu_{1,j} - \mathsf{q}\nu_{2,j}} \mathrm{d}\mathsf{q} \\ &= \sum_{j=1}^{n} \int_a^b \frac{\Phi_j}{-\overline{\lambda_{1,i}} - \nu_{1,j} + \mathsf{q}\left(-\overline{\lambda_{2,i}} - \nu_{2,j}\right)} \mathrm{d}\mathsf{q}. \end{aligned}$$

Then integrating the last equality gives

$$\begin{aligned} &\int_{\mathcal{Q}} H\left(-\overline{\lambda_i(\mathsf{q})}, \mathsf{q}\right) \mathrm{d}\nu(\mathsf{q}) \\ &= \sum_{j=1}^{n} \frac{\Phi_j}{-\overline{\lambda_{2,i}} - \nu_{2,j}} \ln\left(\frac{-\overline{\lambda_{1,i}} - \nu_{1,j} + b\left(-\overline{\lambda_{2,i}} - \nu_{2,j}\right)}{-\overline{\lambda_{1,i}} - \nu_{1,j} + a\left(-\overline{\lambda_{2,i}} - \nu_{2,j}\right)}\right) \\ &= \sum_{j=1}^{n} \frac{\Phi_j(b - a)}{\left(-\overline{\lambda_i(b)} - \nu_j(b)\right) - \left(-\overline{\lambda_i(a)} - \nu_j(a)\right)} \\ &\qquad \times \ln\left(\frac{-\overline{\lambda_i(b)} - \nu_j(b)}{-\overline{\lambda_i(a)} - \nu_j(a)}\right) \end{aligned}$$

$$= G\Big(-\overline{\lambda_i(a)}, -\overline{\lambda_i(b)}\Big).$$

Similarly, one can show that

$$\int_{\mathcal{Q}} \widehat{H}\Big(-\overline{\lambda_i(\mathsf{q})}, \mathsf{q}\Big)\, \mathrm{d}\nu(\mathsf{q}) = \widehat{G}\Big(-\overline{\lambda_i(a)}, -\overline{\lambda_i(b)}\Big).$$

Therefore, the first two optimality conditions (3.4a) and (3.4b) in Theorem 3.2 lead to the interpolatory conditions (5.7a) and (5.7b).

To derive the remaining two conditions (5.7c) and (5.7d) from (3.4c), we now consider the integrals

$$\int_{\mathcal{Q}} \mathsf{q}^k \frac{\partial H}{\partial s}\Big(-\overline{\lambda_i(\mathsf{q})}, \mathsf{q}\Big)\, \mathrm{d}\nu(\mathsf{q}), \quad k = 0, 1. \tag{5.8}$$

We will need the expressions for the partial derivatives of G. It directly follows from (5.6a) that

$$\frac{\partial G}{\partial s_a}(s_a, s_b) = \sum_{i=1}^{n} \frac{\partial f_{\nu_i(a),\nu_i(b)}}{\partial s_a}(s_a, s_b)\Phi_i. \tag{5.9}$$

Similar expressions hold for $\frac{\partial G}{\partial s_b}(s_a, s_b)$, $\frac{\partial \widehat{G}}{\partial s_a}(s_a, s_b)$, and $\frac{\partial \widehat{G}}{\partial s_b}(s_a, s_b)$ as well. Thus, to compute these partial derivatives, we simply focus on f_{σ_a,σ_b} and obtain, via direct differentiation of (5.5), that

$$\begin{aligned}
&\frac{\partial f_{\sigma_a,\sigma_b}}{\partial s_a}(s_a, s_b) \\
&= \frac{b-a}{\big((s_b-\sigma_b)-(s_a-\sigma_a)\big)^2} \ln\left(\frac{s_b-\sigma_b}{s_a-\sigma_a}\right) \\
&\quad - \frac{b-a}{(s_b-\sigma_b)-(s_a-\sigma_a)} \cdot \frac{1}{s_a-\sigma_a}, \quad \text{and} \\
&\frac{\partial f_{\sigma_a,\sigma_b}}{\partial s_b}(s_a, s_b) \\
&= -\frac{b-a}{\big((s_b-\sigma_b)-(s_a-\sigma_a)\big)^2} \ln\left(\frac{s_b-\sigma_b}{s_a-\sigma_a}\right) \\
&\quad + \frac{b-a}{(s_b-\sigma_b)-(s_a-\sigma_a)} \cdot \frac{1}{s_b-\sigma_b}.
\end{aligned}$$

Using the pole-residue form of H from (5.3) in the first integral in (5.8) gives

$$\begin{aligned}
&\int_{\mathcal{Q}} \frac{\partial H}{\partial s}\Big(-\overline{\lambda_i(\mathsf{q})}, \mathsf{q}\Big)\, \mathrm{d}\nu(\mathsf{q}) = \int_a^b \frac{\partial H}{\partial s}\Big(-\overline{\lambda_i(\mathsf{q})}, \mathsf{q}\Big)\, \mathrm{d}\mathsf{q} \\
&= \int_a^b \sum_{j=1}^{n} \frac{-\Phi_j}{\Big(-\overline{\lambda_i(\mathsf{q})} - \nu_{1,j} - \mathsf{q}\nu_{2,j}\Big)^2}\, \mathrm{d}\mathsf{q} \\
&= \sum_{j=1}^{n} \int_a^b \frac{-\Phi_j}{\big(-\overline{\lambda_{1,i}} - \nu_{1,j} + \mathsf{q}(-\overline{\lambda_{2,i}} - \nu_{2,j})\big)^2}\, \mathrm{d}\mathsf{q} \\
&= \sum_{j=1}^{n} \frac{-\Phi_j}{-\overline{\lambda_{2,i}} - \nu_{2,j}} \left(\frac{1}{-\overline{\lambda_{1,i}} - \nu_{1,j} + b(-\overline{\lambda_{2,i}} - \nu_{2,j})} \right. \\
&\qquad \left. - \frac{1}{-\overline{\lambda_{1,i}} - \nu_{1,j} + a(-\overline{\lambda_{2,i}} - \nu_{2,j})}\right).
\end{aligned}$$

After various algebraic manipulations to replace $\lambda_{k,i}$ and $\nu_{k,j}$ by $\lambda_i(\cdot)$ and $\nu_j(\cdot)$ and using (5.9), we obtain

$$\begin{aligned}
&\int_{\mathcal{Q}} \frac{\partial H}{\partial s}\Big(-\overline{\lambda_i(\mathsf{q})}, \mathsf{q}\Big)\, \mathrm{d}\nu(\mathsf{q}) \\
&= \sum_{j=1}^{n} \frac{-\Phi_j(b-a)}{\Big(-\overline{\lambda_i(b)} - \nu_j(b)\Big) - \Big(-\overline{\lambda_i(a)} - \nu_j(a)\Big)} \\
&\qquad \times \left(\frac{1}{-\overline{\lambda_i(b)} - \nu_j(b)} - \frac{1}{-\overline{\lambda_i(a)} - \nu_j(a)}\right) \\
&= -(b-a)\bigg(\frac{\partial G}{\partial s_a}\Big(-\overline{\lambda_i(a)}, -\overline{\lambda_i(b)}\Big) \\
&\qquad + \frac{\partial G}{\partial s_b}\Big(-\overline{\lambda_i(a)}, -\overline{\lambda_i(b)}\Big)\bigg).
\end{aligned}$$

Following the same derivations, one obtains similar expressions involving $\widehat{H}$ and $\widehat{G}$, which shows that

$$\begin{aligned}
&c_i^* \left(\frac{\partial G}{\partial s_a}\Big(-\overline{\lambda_i(a)}, -\overline{\lambda_i(b)}\Big) + \frac{\partial G}{\partial s_b}\Big(-\overline{\lambda_i(a)}, -\overline{\lambda_i(b)}\Big)\right) b_i \\
&= c_i^* \left(\frac{\partial \widehat{G}}{\partial s_a}\Big(-\overline{\lambda_i(a)}, -\overline{\lambda_i(b)}\Big) + \frac{\partial \widehat{G}}{\partial s_b}\Big(-\overline{\lambda_i(a)}, -\overline{\lambda_i(b)}\Big)\right) b_i.
\end{aligned} \tag{5.10}$$

Focusing on the second integral in (5.8), we find

$$\begin{aligned}
&\int_{\mathcal{Q}} \mathsf{q}\frac{\partial H}{\partial s}\Big(-\overline{\lambda_i(\mathsf{q})}, \mathsf{q}\Big)\, \mathrm{d}\nu(\mathsf{q}) = \int_a^b \mathsf{q}\frac{\partial H}{\partial s}\Big(-\overline{\lambda_i(\mathsf{q})}, \mathsf{q}\Big)\, \mathrm{d}\mathsf{q} \\
&= \sum_{j=1}^{n} \int_a^b \frac{-\mathsf{q}\Phi_j}{\big(-\overline{\lambda_{1,i}} - \nu_{1,j} + \mathsf{q}(-\overline{\lambda_{2,i}} - \nu_{2,j})\big)^2}\, \mathrm{d}\mathsf{q} \\
&= \sum_{j=1}^{n} \frac{-\Phi_j}{(-\overline{\lambda_{2,i}} - \nu_{2,j})^2} \left(\frac{-\overline{\lambda_{1,i}} - \nu_{1,j}}{-\overline{\lambda_i(b)} - \nu_j(b)} \right. \\
&\qquad \left. - \frac{-\overline{\lambda_{1,i}} - \nu_{1,j}}{-\overline{\lambda_i(a)} - \nu_j(a)} + \ln\left(\frac{-\overline{\lambda_i(b)} - \nu_j(b)}{-\overline{\lambda_i(a)} - \nu_j(a)}\right)\right).
\end{aligned}$$

After more tedious algebraic manipulations, we obtain

$$\begin{aligned}
&\int_{\mathcal{Q}} \mathsf{q}\frac{\partial H}{\partial s}\Big(-\overline{\lambda_i(\mathsf{q})}, \mathsf{q}\Big)\, \mathrm{d}\nu(\mathsf{q}) \\
&= \sum_{j=1}^{n} \frac{\Phi_j(b-a)}{\Big(-\overline{\lambda_i(b)} - \nu_j(b)\Big) - \Big(-\overline{\lambda_i(a)} - \nu_j(a)\Big)} \\
&\qquad \times \left(\frac{b}{-\overline{\lambda_i(b)} - \nu_j(b)} - \frac{a}{-\overline{\lambda_i(a)} - \nu_j(a)}\right) \\
&\quad + \sum_{j=1}^{n} \frac{-\Phi_j(b-a)^2}{\Big(\Big(-\overline{\lambda_i(b)} - \nu_j(b)\Big) - \Big(-\overline{\lambda_i(a)} - \nu_j(a)\Big)\Big)^2} \\
&\qquad \times \ln\left(\frac{-\overline{\lambda_i(b)} - \nu_j(b)}{-\overline{\lambda_i(a)} - \nu_j(a)}\right) \\
&= a\frac{\partial G}{\partial s_a}\Big(-\overline{\lambda_i(a)}, -\overline{\lambda_i(b)}\Big) + b\frac{\partial G}{\partial s_b}\Big(-\overline{\lambda_i(a)}, -\overline{\lambda_i(b)}\Big).
\end{aligned}$$

As before, with similar expressions involving $\widehat{H}$ and $\widehat{G}$, we obtain

$$\begin{aligned} & c_i^* \left(a \frac{\partial G}{\partial s_a} \left(-\overline{\lambda_i(a)}, -\overline{\lambda_i(b)} \right) + b \frac{\partial G}{\partial s_b} \left(-\overline{\lambda_i(a)}, -\overline{\lambda_i(b)} \right) \right) b_i \\ &= c_i^* \left(a \frac{\partial \widehat{G}}{\partial s_a} \left(-\overline{\lambda_i(a)}, -\overline{\lambda_i(b)} \right) + b \frac{\partial \widehat{G}}{\partial s_b} \left(-\overline{\lambda_i(a)}, -\overline{\lambda_i(b)} \right) \right) b_i. \end{aligned} \tag{5.11}$$

Then (5.10) and (5.11) give the last two optimality conditions (5.7c) and (5.7d), thus concluding the proof. □

Theorem 5.1 proves that for this class of parametric LTI systems, bitangential Hermite interpolation, once again, forms the foundation of the $\mathcal{L}_2$-optimal approximation. The interpolation is based on a modified, two-variable transfer function G, and has to be enforced at the reflected boundary values of the poles. This is the first such result for parametric LTI systems where the system poles vary with the parameters. Therefore, we have extended the classical bitangential Hermite interpolation conditions from non-parametric $\mathcal{H}_2$-optimal approximation to parametric $\mathcal{H}_2 \otimes \mathcal{L}_2$-optimal approximation.

6 Conclusion

We derived interpolatory necessary optimality conditions for $\mathcal{H}_2 \otimes \mathcal{L}_2$-optimal reduced-order modeling of parametric LTI systems with general diagonal structure. Then we give conditions for special cases where only inputs and outputs or only the dynamics are parameterized.

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