# EXOTIC PICARD GROUPS AND CHROMATIC VANISHING VIA THE GROSS-HOPKINS DUALITY

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ABSTRACT. In this paper, we study the exotic $K(h)$-local Picard groups $\kappa_h$ when $2p-1=h^2$ and the homological Chromatic Vanishing Conjecture when $p-1$ does not divide $h$. The main idea is to use the Gross-Hopkins duality to relate both questions to certain Greek letter element computations in chromatic homotopy theory. Classical results of Miller-Ravenel-Wilson then imply that an exotic element at height 3 and prime 5 is not detected by the type-2 complex $V(1)$. For the homological Vanishing Conjecture, we prove it holds modulo the invariant prime ideal $I_{h-1}$. We further show that this special case of the Vanishing Conjecture implies the exotic Picard group $\kappa_h$ is zero at height 3 and prime 5. Both results can be thought of as a first step towards proving the vanishing of $\kappa_3$ at prime 5.

**Keywords.** exotic Picard groups, Chromatic Vanishing Conjecture, Gross-Hopkins duality, Greek letter elements

## 0. INTRODUCTION

0.1. **Statement of main results.** The study of Picard groups in chromatic homotopy theory was initiated by Hopkins in [17, 33]. By analyzing the homotopy fixed point spectral sequence for the $K(h)$-local sphere, Hopkins-Mahowald-Sadofsky proved the following:

**Theorem** ([17, Proposition 7.5])**.** *The exotic $K(h)$-local Picard group $\kappa_h$ (see Definition 1.11) is zero when $p-1$ does not divide $h$ and $2p-1>h^2$.*

In this paper, we study $\kappa_h$ when $2p-1=h^2$. The smallest of such pairs is $h=3$ and $p=5$. Notice that this assumption already implies $(p-1)\nmid h$.

**Remark.** It is an open question in number theory whether there are infinitely primes $p$ such that $2p-1$ is a perfect square ([21, page 171]). Using SageMath [36], the authors are able to find $35,528,083$ positive integers $h$ less than $10^9$ such that $\frac{h^2+1}{2}$ is a prime number.

Our first main result is:

**Theorem** (A, Theorem 3.27, Corollary 3.28)**.** *Let $2p-1=h^2$. Suppose the type-$(h-1)$ Smith-Toda complex $V(h-2)=S^0/(p,v_1,\cdots,v_{h-2})$ exists at prime $p$. Then an exotic element $X\in\kappa_h$ cannot be detected by $V(h-2)$, i.e.*

$$L_{K(h)}\left(X\wedge V(h-2)\right)\simeq L_{K(h)}V(h-2).$$

*In particular,*

*(1) At height 3 and prime 5, an exotic element $X$ in $\mathrm{Pic}_{K(3)}$ cannot be detected by $V(1)=S^0/(5,v_1)$.*
*(2) At height 5 and prime 13, an exotic element $X$ in $\mathrm{Pic}_{K(5)}$ cannot be detected by $V(3)=S^0/(13,v_1,v_2,v_3)$.*

When $4p-3=h^2$, we prove a similar statement in Theorem 3.31 for a subgroup $\kappa_h^{(1)}$ of the exotic Picard group $\kappa_h$ defined in Section 1.3. In particular at $(h,p)=(3,3)$ and $(5,7)$, we show that $V(h-2)$ cannot detect elements in this subgroup of $\kappa_h$.

Our method is also used to study the following special case of the Chromatic Vanishing Conjecture (2.29), first proposed in [4, 5].

**Conjecture** (Reduced Homological Vanishing Conjecture, (RHVC))**.**

$$\mathbf{F}_p \cong H_0(\mathbf{G}_h;\mathbf{F}_{p^h}) \xrightarrow{\sim} H_0(\mathbf{G}_h;\pi_0(E_h)/p).$$

**Remark.** The Vanishing Conjecture was stated in terms of group *co*homology in [5, Conjecture 1.1.4]. This is equivalent to the homological versions when $(p-1)\nmid h$ by Poincaré duality. See Remark 2.30.

**Theorem** (B, Theorem 3.26)**.** *When $(p-1)\nmid h$, the RHVC holds modulo the ideal $I_{h-1}=(p,u_1,\cdots,u_{h-2})$, i.e. there are isomorphisms:*

$$\mathbf{F}_p \cong H_0(\mathbf{G}_h;\mathbf{F}_{p^h}) \xrightarrow{\sim} H_0(\mathbf{G}_h;\pi_0(E_h)/I_{h-1}).$$

Exotic Picard groups and the Vanishing Conjecture are related by:

**Theorem** (C, Theorem 3.32)**.** *If the RHVC holds at height 3, then $\kappa_3=0$ at $p=5$ and $\kappa_3^{(1)}=0$ at $p=3$, where $\kappa_3^{(1)}$ is a subgroup of $\kappa_3$ defined in Section 1.3*

For general heights and primes, we give some bounds on the divisibility of Greek letter elements that would imply the RHVC (when $(p-1)\nmid h$) and $\kappa_h=0$ (when $2p-1=h^2$) in Proposition 3.15.

0.2. **General strategy.** A summary of our strategy to study exotic Picard groups when $2p-1=h^2$ is as follows. We will show successively each claim below is implied by the following one.

I. $\kappa_h=0$.

II. $H_c^{h^2}(\mathbf{S}_h;\pi_{2p-2}(E_h))=H_c^{2p-1}(\mathbf{S}_h;\pi_{2p-2}(E_h))=0$.

III. $H_c^{h^2}(\mathbf{S}_h;\pi_{2p-2}(E_h)/p)=0$.

IV. $H_c^0(\mathbf{S}_h;\pi_{2h-2p+2}(E_h)\langle\det\rangle/(p,u_1^\infty,\cdots,u_{h-1}^\infty))=0$, where the determinant twist $\langle\det\rangle$ is defined in Definition 2.18 and the quotient mod $(p,u_1^\infty,\cdots,u_{h-1}^\infty)$ is explained in Definition 2.19.

V. $H_c^0\left(\mathbf{S}_h;\pi_{2h-2p+2-\frac{p^N|v_h|}{p-1}}(E_h)/J\right)=0$ for any open invariant ideal $J\trianglelefteq\pi_0(E_h)$ containing $p$ such that $v_h^{p^N}$ is invariant mod $J$.

VI. $\operatorname{Ext}_{BP_*BP}^{0,2h-2p+2-\frac{p^N|v_h|}{p-1}}(BP_*,v_h^{-1}BP_*/J)=0$ for any invariant ideal $J\trianglelefteq v_h^{-1}BP_*$ containing $p$ such that $v_h^{p^N}$ is invariant mod $J$.

VII. $H^{0,t}(M_1^{h-1})=0$ for any $t\equiv 2h-2p+2-\frac{p^N|v_h|}{p-1} \mod p^N|v_h|$ and all integers $N\geq 0$, where $M_1^{h-1}:=v_h^{-1}BP_*/(p,v_1^\infty,\cdots,v_{h-1}^\infty)$.

**II $\Longrightarrow$ I**: In [11], Goerss-Henn-Mahowald-Rezk defined a map that detects the exotic Picard group $\kappa_h$:

$$\mathrm{ev}_2\colon \kappa_h\to H_c^{2p-1}(\mathbf{G}_h;\pi_{2p-2}(E_h)).$$

Using the same argument as in [17], we will show this map is injective when $(p-1)\nmid h$ and $4p-3>h^2$ in Proposition 1.20.[1] As a result, $\kappa_h$ vanishes if $H_c^{2p-1}(\mathbf{G}_h;\pi_{2p-2}(E_h))=0$ when $2p-1=h^2$. By [9, Lemma 1.32] and [12, page 12], we have

$$H_c^s(\mathbf{G}_h;\pi_t(E_h))\cong H_c^s(\mathbf{S}_h;\pi_t(E_h))^{\mathrm{Gal}} \text{ for any } s \text{ and } t,$$

[1]A descent spectral sequence for $K(h)$-local Picard groups in [13, Example 6.18] implies this map is an isomorphism under the assumptions. See Proposition 1.25.

where $\mathbf{S}_h \leq \mathbf{G}_h$ is the automorphism group of the height $h$-Honda formal group. This indicates we just need to show the relevant group cohomology of $\mathbf{S}_h$ is zero.

**III $\Longrightarrow$ II**: Now suppose $2p-1=h^2$. By Theorem 2.8 of Lazard and the fact $\mathbf{S}_h$ has no finite $p$-group, $\mathrm{cd}_p(\mathbf{S}_h)=h^2$. When $(p-1)\nmid h$, the cohomology we are computing $H_c^{2p-1}(\mathbf{G}_h;\pi_{2p-2}(E_h)) = H_c^{h^2}(\mathbf{G}_h;\pi_{2p-2}(E_h))$ is a top degree cohomology. Using a Hochschild-Lyndon-Serre spectral sequence and the explicit formula of the action by the center $\mathbf{Z}_p^\times$ of $\mathbf{S}_h$, we show in Proposition 2.3 that

$$H_c^{h^2}(\mathbf{G}_h;\pi_{2p-2}(E_h)) \xrightarrow{\sim} H_c^{h^2}(\mathbf{G}_h;\pi_{2p-2}(E_h)/p).$$

Alternatively, the above isomorphism can be proved using the Poincaré duality between top degree cohomology and zero degree homology.

**IV $\Longrightarrow$ III**: There is another Poincaré duality between top and zero degree cohomology groups for any $p$-complete $\mathbf{G}_h$-module $M$:

$$H_c^{h^2}(\mathbf{S}_h;M)\cong H_c^0(\mathbf{S}_h;M^\vee)^\vee,$$

where $(-)^\vee := \hom_c(-,\mathbf{Q}_p/\mathbf{Z}_p)$ is the continuous equivariant Pontryagin dual (Definition 2.11). For $M=\pi_t(E_h)$, the dual $M^\vee$ is identified by Gross-Hopkins duality Corollary 2.22:

$$\pi_t(E_h)^\vee \cong \pi_{2h-t}(E_h)\langle\det\rangle/\mathfrak{m}^\infty,$$

where $\mathfrak{m}=(p,u_1,\cdots,u_{h-1})\trianglelefteq\pi_0(E_h)$ is the maximal ideal, mod $\mathfrak{m}^\infty$ is defined in Definition 2.19, and $\langle\det\rangle$ is the determinant twist defined in Definition 2.18). In the case when $t=2p-2$, we further have:

$$\begin{aligned} H_c^{h^2}(\mathbf{S}_h;\pi_{2p-2}(E_h)) &\cong H_c^{h^2}(\mathbf{S}_h;\pi_{2p-2}(E_h)/p)\\ &\cong H_c^0(\mathbf{S}_h;\pi_{2h-2p+2}(E_h)\langle\det\rangle/(p,u_1^\infty,\cdots,u_{h-1}^\infty))^\vee. \end{aligned}$$

**V $\Longrightarrow$ IV**: In [16], Gross-Hopkins identified the determinant twist mod $p>2$ with a limit of finite suspensions:

$$\pi_0(E_h)\langle\det\rangle/p\cong\Sigma^{\lim_{N\to\infty}\frac{p^N|v_h|}{p-1}}\pi_0(E_h)/p.$$

This is a limit in the *algebraic* $K(h)$-local Picard group. More precisely, let $J\trianglelefteq\pi_0(E_h)$ be an open invariant ideal containing $p$, such that $v_h^{p^N}$ is invariant modulo $J$. Then

$$\pi_0(E_h)\langle\det\rangle/J\cong\Sigma^{\frac{p^N|v_h|}{p-1}}\pi_0(E_h)/J.$$

By Proposition 2.27, we now have

$$\begin{aligned} &H_c^0(\mathbf{S}_h;\pi_{2h-2p+2}(E_h)\langle\det\rangle/(p,u_1^\infty,\cdots,u_{h-1}^\infty))\\ \cong &\operatorname*{colim}_{p\in J\trianglelefteq\pi_0(E_h)} H_c^0\left(\mathbf{S}_h;\pi_{2h-2p+2-\frac{p^N|v_h|}{p-1}}(E_h)\Big/ J\right). \end{aligned}$$

As a result, to show the left hand side is zero, it suffices to show every single term in the colimit system on right hand side is zero.

**VI $\Longrightarrow$ V** Using a Change of Rings theorem, Theorem 3.1, we relate the group cohomology of $\mathbf{G}_h$ with Ext-groups of $BP_*BP$-comodules:

$$H^s(\mathbf{G}_h;\pi_t(E_h)/J)\cong\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*,v_h^{-1}BP_*/J')$$

for some invariant ideal $J'\trianglelefteq v_h^{-1}BP_*$. When $J=(p,u_1^{j_1},\cdots,u_{h-1}^{j_{h-1}})$, we can take $J'=(p,v_1^{j_1},\cdots,v_{h-1}^{j_{h-1}})$. As a result, we need to compute $\mathrm{Ext}^{0,t}_{BP_*BP}(BP_*,v_h^{-1}BP_*/J')$ for certain values of $t$.

**VII $\Longrightarrow$ VI** For a $BP_*BP$-comodule $M$, we denote $\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*,M)$ by $H^{s,t}(M)$. The colimit of the cohomology groups $H^{0,t}(v_h^{-1}BP_*/J)$ over all invariant ideals $J\trianglelefteq v_h^{-1}BP_*$ containing $p$ is $H^{0,t}(M_1^{h-1})$, where

$M_1^{h-1} = v_h^{-1}BP_*/(p, v_1^\infty, \cdots, v_{h-1}^\infty)$. This is the group of mod-$p$ Greek letter elements at height $h$. Keeping track of the degree $t$, we have reduced our computation to the following:

**Proposition.** *Suppose* $2p-1 = h^2$*. If* $H^{0,t}(M_1^{h-1}) = 0$ *whenever* $t \equiv 2h - 2p + 2 - \frac{p^N|v_h|}{p-1} \mod p^N|v_h|$ *for some integer* $N \geq 0$*, then* $\kappa_h = 0$*.*

The argument above can also be used to study the Chromatic Vanishing Conjecture (2.29) in degree 0 homology groups when $(p-1) \nmid h$. This conjecture has been verified at all primes at heights 1 and 2 by explicit computations. It plays an essential role in Beaudry-Goerss-Henn's works in [5] to disprove and completely understand the Chromatic Splitting Conjecture at $h = p = 2$. The Vanishing Conjecture is wide open at $h \geq 3$. Using Gross-Hopkins duality and Change of Rings theorem, we can translate the Reduced Homological Vanishing Conjecture (RHVC) to Greek letter element computations:

**Proposition.** *Suppose* $p-1$ *does not divide* $h$*. If* $H^{0,t}(M_1^{h-1}) = \mathbf{F}_p$ *whenever* $t \equiv 2h - \frac{p^N|v_h|}{p-1} \mod p^N|v_h|$ *for some integer* $N \geq 0$*, then* $H_0(\mathbf{G}_h; \pi_0(E_h)/p) = \mathbf{F}_p$ *and the RHVC holds.*

0.3. **Greek letter element computations.** Next, we need to compute the Greek letter elements in $H^{0,t}(M_1^{h-1})$. Elements in this group are classified into three families in Proposition 3.3.

(1) Family I elements are of the form $\frac{v_h^s}{pv_1 \cdots v_{h-1}}$, where $(s,p) = 1$. In Proposition 3.6, we prove Family I elements contribute to a copy $\mathbf{F}_p$ in $H_c^{h^2}(\mathbf{G}_h; \pi_0(E_h)/p)$ via Gross-Hopkins duality, which is predicted in the RHVC. This family does not contribute to $H_c^{h^2}(\mathbf{G}_h; \pi_{2p-2}(E_h)/p)$.
(2) Family II elements are of the form $\frac{1}{pv_1^{d_1} \cdots v_{h-1}^{d_{h-1}}}$, where $(p, v_1^{d_1}, \cdots, v_{h-1}^{d_{h-1}})$ is an invariant ideal. In Corollary 3.11, we show this family does not contribute to either $H_c^{h^2}(\mathbf{G}_h; \pi_0(E_h)/p)$ or $H_c^{h^2}(\mathbf{G}_h; \pi_{2p-2}(E_h)/p)$.
(3) Family III elements are of the form $\frac{y_{h,N}^s}{pv_1^{d_1} \cdots v_{h-1}^{d_{h-1}}}$, where $y_{h,N}$ is some replacement of $v_h^{p^N}$, $(s,p) = 1$ and $(p, v_1^{d_1}, \cdots, v_{h-1}^{d_{h-1}}, y_{h,N}^s)$ is an invariant regular ideal. While the precise conditions on the $d_i$'s are out of reach in the general situation, we established some bounds in Proposition 3.12 which would imply this family does not contribute to either $H_c^{h^2}(\mathbf{G}_h; \pi_0(E_h)/p)$ or $H_c^{h^2}(\mathbf{G}_h; \pi_{2p-2}(E_h)/p)$.

Combining the three cases above, we obtain the bounds on divisibility of Greek letter elements that would imply the RHVC (when $(p-1) \nmid h$) and vanishing of $\kappa_h$ (when $2p-1 = h^2$) in Proposition 3.15.

In [26], Miller-Ravenel-Wilson computed $H^{0,*}(M_{h-1}^1)$, where $M_{h-1}^1 := v_h^{-1}BP_*/(p, v_1, \cdots, v_{h-2}, v_{h-1}^\infty)$. Using Gross-Hopkins duality and Morava's Change of Rings Theorem, the Miller-Ravenel-Wilson computation yields when $(p-1) \nmid h$,

$$H_c^{h^2}(\mathbf{G}_h; \pi_0(E_h)/I_{h-1}) = \mathbf{F}_p,$$
$$H_c^{h^2}(\mathbf{G}_h; \pi_{2p-2}(E_h)/I_{h-1}) = 0.$$

It follows from first isomorphism that the RHVC holds modulo the ideal $I_{h-2} = (p, u_1, \cdots, u_{h-2}) \trianglelefteq \pi_0(E_h)$. This is the statement of Main Theorem B 3.26. The second group cohomology measures if there is an exotic element in $\mathrm{Pic}_{K(h)}$ detected by the type-$(h-1)$ Smith-Toda complex $V(h-2) := S^0/(p, v_1, \cdots, v_{h-2})$, provided the latter exists. Consequently, its vanishing yields Theorem A (3.27). At height 3 and prime 5, we further show in Theorem C (3.32) that the RHVC implies $\kappa_3 = 0$. This proof relies on the Miller-Ravenel-Wilson results.

**Remark** (3.29 and 3.30)**.** We learned from a referee that it is an open question whether $V(h)$ exists when $h \geq 4$ at *any* prime. By [20, Corollary 7.11], if $X \wedge_{K(h)} V \simeq V$ for all $X \in \kappa_h$ and finite complexes $V$ of type $n$, then $\kappa_h = 0$. Main Theorem A (3.27) can therefore be thought of as a first step towards showing $\kappa_h = 0$

when $2p-1=h^2$, since it implies $X \wedge_{K(h)} V$ for any cofibers $V$ of $v_h$-self maps of $V(h-2)$. Our choices of finite complexes are restricted to cofibers of the Smith-Toda complex $V(h-2)$, because we do not have better Greek letter element computations beyond $H^0(M^1_{h-1})$ in [26] when $h \geq 3$.

0.4. **Notations and Conventions.** Throughout, we will let $E_h$ denote a fixed Morava $E$-theory based on a height $h$ formal group, typically the height $h$ Honda formal group $\Gamma_h$. For a $K(h)$-local spectrum $X$, we will write $(E_h)_*X$ for the completed $E_h$-homology of $X$. That is, we write

$$(E_h)_*X := \pi_*(L_{K(h)}(E_h \wedge X)).$$

We will also write $X \wedge_{K(h)} Y$ for the $K(h)$-local smash product $L_{K(h)}(X \wedge Y)$.

Denote by $\mathbf{W} := \mathbf{WF}_{p^h}$ the ring of Witt vectors over $\mathbf{F}_{p^h}$. We will write $\mathbf{S}_h$ for the Morava stabilizer group, i.e. the automorphisms of a $\Gamma_h$, and we will write $\mathbf{G}_h$ for the extended Morava stabilizer group.

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## 1. The $K(h)$-local Picard group

1.1. **Definitions.** In chromatic homotopy theory, we study the stable homotopy category of spectra $\mathsf{Sp}$ via the height filtration of the moduli stack of formal groups at each prime $p$. One such layer in this filtration is the category of $K(h)$-local spectra $\mathsf{Sp}_{K(h)}$, where $K(h)$ is the Morava $K$-theory at $h$ and prime $p$. Like $\mathsf{Sp}$, the category $\mathsf{Sp}_{K(h)}$ also has a symmetric monoidal structure

$$X \wedge_{K(h)} Y := L_{K(h)}(X \wedge Y).$$

For $\mathsf{Sp}$, its Picard group is given by

**Theorem 1.1** ([17, page 90])**.** *The map* $\mathbf{Z} \to \mathrm{Pic}(\mathsf{Sp}), n \mapsto S^n$ *is an isomorphism of groups.*

The Picard group $\mathrm{Pic}_{K(h)}$ for $\mathsf{Sp}_{K(h)}$, however, is still not fully understood. Here we give a filtration on $\mathrm{Pic}_{K(h)}$ via a sequence of algebraic detection maps $\mathrm{ev}_i$. The first fact is:

**Theorem 1.2** ([17, Theorem 1.3])**.** *The followings are equivalent:*
- $X \in \mathsf{Sp}_{K(h)}$ *is invertible.*
- $(E_h)_*(X)$ *is an invertible graded* $(E_h)_*$*-module.*

As $E_h$ is even periodic, an invertible graded $(E_h)_*$-module is either itself or its suspension. This yields the zeroth detection map:

$$\mathrm{ev}_0\colon \mathrm{Pic}_{K(h)} \xrightarrow{X \mapsto (E_h)_*(X)} \mathrm{Pic}(\text{graded } (E_h)_*\text{-modules}) = \mathbf{Z}/2.$$

**Proposition 1.3.** $\mathrm{ev}_0$ *is a surjective group homomorphism.*

*Proof.* We can check $\mathrm{ev}_0$ is a group homomorphism using the Künneth theorem. It is surjective since $\mathrm{ev}_0(S^1) = \pi_*(\Sigma E_h)$ is concentrated in odd degrees. □

Denote the kernel of $\mathrm{ev}_0$ by $\mathrm{Pic}^0_{K(h)}$. This is the group of invertible $K(h)$-local spectra whose $E_h$-homology is concentrated in even degrees. For any spectrum $X$, its $E_h$-homology is not only a graded $(E_h)_*$-module, but also a *graded* $\pi_*(E_h \wedge_{K(h)} E_h)$*-comodule*. In the case when $X \in \mathrm{Pic}^0_{K(h)}$, this *graded* comodule structure is determined by $(E_h)_0(X)$ as an *ungraded* $\pi_0(E_h \wedge_{K(h)} E_h)$-comodule. This gives rise to the first detection map:

$$\mathrm{ev}_1\colon \mathrm{Pic}^0_{K(h)} \xrightarrow{X \mapsto (E_h)_0(X)} \mathrm{Pic}((\pi_0(E_h), \pi_0(E_h \wedge_{K(h)} E_h))\text{-comodules}).$$

To identify the target of $\mathrm{ev}_1$, we use the following lemma.

**Lemma 1.4** ([19])**.** *There is an isomorphism of Hopf algebroids:*

$$(\pi_0(E_h), \pi_0(E_h \wedge_{K(h)} E_h)) \cong (\pi_0(E_h), \mathrm{Map}_c(\mathbf{G}_h; \pi_0(E_h))),$$

*where* $\mathbf{G}_h = \mathbf{S}_h \rtimes \mathrm{Gal}(\mathbf{F}_{p^h}/\mathbf{F}_p)$ *and* $\mathbf{S}_h$ *is the automorphism group of the height-$h$ Honda formal group.*

It follows that a $\pi_0(E_h \wedge_{K(h)} E_h)$-comodule $M$ is equivalent to a $\pi_0(E_h)$-module together with a *continuous* $\mathbf{G}_h$-action such that the following diagram commutes for all $g \in \mathbf{G}_h$: ([17, page 118])

$$\begin{array}{ccc} \pi_0(E_h) \otimes M & \xrightarrow{g \otimes g} & \pi_0(E_h) \otimes M \\ \downarrow & & \downarrow \\ M & \xrightarrow{\quad g \quad} & M \end{array}$$

The Picard group of such $\mathbf{G}_h$-$\pi_0(E_h)$-modules is computed by a continuous group cohomology of $\mathbf{G}_h$:

**Proposition 1.5** ([17, Proposition 8.4])**.**

$$\mathrm{Pic}(\textit{continuous } \mathbf{G}_h\text{-}\pi_0(E_h)\textit{-modules}) \cong H^1_c(\mathbf{G}_h; \pi_0(E_h)^\times).$$

As a result, the first detection map is a group homomorphism:

$$\mathrm{ev}_1\colon \mathrm{Pic}^0_{K(h)} \to H^1_c(\mathbf{G}_h; \pi_0(E_h)^\times). \tag{1.6}$$

**Definition 1.7.** The Picard group of *graded* $\mathbf{G}_h$-$(E_h)_*$-modules is called the **algebraic $K(h)$-local Picard group**, denoted by $\mathrm{Pic}^{alg}_{K(h)}$. The Picard group of *ungraded* $\mathbf{G}_h$-$\pi_0(E_h)$-modules is denoted by $\mathrm{Pic}^{alg,0}_{K(h)}$.

Thus, by Proposition 1.5, we have

$$\mathrm{Pic}^{alg,0}_{K(h)} = H^1_c(\mathbf{G}_h; \pi_0(E_h)^\times).$$

The first detection map $\mathrm{ev}_1$ then extends to the full Picard group $\mathrm{Pic}_{K(h)}$, which we will also denote by $\mathrm{ev}_1$.

**Proposition 1.8.** *The $K(h)$-local Picard groups we have introduced so far are related by a map of short exact sequences:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Pic}^0_{K(h)} & \longrightarrow & \mathrm{Pic}_{K(h)} & \longrightarrow & \mathbf{Z}/2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{ev}_1} & & \downarrow{\scriptstyle \mathrm{ev}_1} & & \| & & \\ 0 & \longrightarrow & \mathrm{Pic}^{alg,0}_{K(h)} & \longrightarrow & \mathrm{Pic}^{alg}_{K(h)} & \longrightarrow & \mathbf{Z}/2 & \longrightarrow & 0 \end{array}$$

**Remark 1.9.** *It is known that the short exact sequences do not split at height $h = 1$ for all primes [17], and at height 2 for $p \geq 3$ [11].*

**Corollary 1.10.** *The two $\mathrm{ev}_1$ maps in the diagram above have isomorphic kernels and cokernels.*

This corollary justifies the usage of $\mathrm{ev}_1$ for both detection maps.

1.2. **Exotic Picard groups.** Now the question turns to whether $\mathrm{ev}_1$ is injective or surjective. The surjectivity problem is hard and involves obstruction theory. In certain cases, we can show $\mathrm{ev}_1$ is injective.

**Definition 1.11.** The **exotic $K(h)$-local Picard group** $\kappa_h$ is the kernel of $\mathrm{ev}_1$ in (1.6).

**Theorem 1.12** ([17, Proposition 7.5])**.** *The exotic Picard group $\kappa_h$ vanishes when $(p-1)\nmid h$ and $2p-1>h^2$.*

The detection of elements in $\kappa_h$ lies in the **homotopy fixed point spectral sequence** (HFPSS) to compute the $\pi_*(X)$ for $X\in \mathsf{Sp}_{K(h)}$:

$$E_2^{s,t}=H_c^s(\mathbf{G}_h;(E_h)_t(X))\Longrightarrow \pi_{t-s}(X)\,. \tag{1.13}$$

For any $X\in\kappa_h$, the $E_2$-page of the HFPSS to compute its homotopy groups is isomorphic to as that for $S^0_{K(h)}$. The potential differences between the two spectral sequences are the higher differentials. We will show that the higher differentials are necessarily zero under the assumption $2p-1>h^2$ and $(p-1)\nmid h$. To see this, we need the following basic facts about the HFPSS:

**Lemma 1.14** ([9, Lemma 1.32],[12, Page 12])**.** *For any $\mathbf{G}_h$-$\pi_0(E_h)$-module $M$, we have an isomorphism $H_c^s(\mathbf{G}_h;M)\cong H_c^s(\mathbf{S}_h;M)^{\mathrm{Gal}}$.*

**Lemma 1.15** (Sparseness, [12, Remark 1.4])**.** *The continuous group cohomology $H_c^s(\mathbf{S}_h;\pi_t(E_h))$ is zero unless $2(p-1)$ divides $t$.*

**Lemma 1.16** (Horizontal vanishing line, [12, Proposition 1.6])**.** *The $p$-adic Lie group $\mathbf{S}_h$ has cohomological dimension $h^2$ if $(p-1)\nmid h$.*

It follows that the HFPSS (1.13) has a horizontal vanishing line at $s=h^2$ when $(p-1)\nmid h$.

**Lemma 1.17** (0-line, [9, Lemma 1.33])**.** $H_c^0(\mathbf{G}_h;\pi_t(E_h))=\begin{cases}\mathbf{Z}_p, & t=0;\\ 0, & \text{otherwise.}\end{cases}$

*Proof of Theorem 1.12.* We need to show that when $(p-1)\nmid h$ and $h^2<2p-1$, a $K(h)$-local spectrum $X$ is weakly equivalent to $S^0_{K(h)}$ if there is a $\mathbf{G}_h$-equivariant isomorphism $(E_h)_*(X)\cong(E_h)_*$.

Under this assumption, HFPSS for $X$ collapses at $E_2$-page by sparseness (Lemma 1.15). As a result, any unit $[\iota_X]\in E_2^{0,0}(X)=\mathbf{Z}_p$ is a permanent cycle and induces a map $S^0\to X$. This map factors as $S^0\to S^0_{K(h)}\xrightarrow{\iota_X}X$ since $X$ is $K(h)$-local. As $\iota_X\colon S^0_{K(h)}\to X$ induces an isomorphism on the $E_2$-page of the HFPSS, it is a weak equivalence by [8, Theorem 5.3]. □

In the general case, the first possible non-trivial differential in (1.13) for $X\in\kappa_h$ is $d_{2p-1}$. Let's consider the possible $d_{2p-1}$-differentials supported by $E_{2p-1}^{0,0}(X)=E_2^{0,0}(X)=\mathbf{Z}_p$.

**Construction 1.18** ([11, Construction 3.2])**.** Fix an $\mathbf{G}_h$-equivariant isomorphism $f^X\colon(E_h)_*\xrightarrow{\sim}(E_h)_*(X)$ and let $\iota_X=f^X(1)\in(E_h)_0(X)$. The differential

$$d_{2p-1}^X\colon E_{2p-1}^{0,0}(X)\longrightarrow E_{2p-1}^{2p-1,2p-2}(X)$$

is determined by the image of $\iota_X$. Define a homomorphism $\phi^X$ via the following commutative diagram:

$$\begin{array}{ccc} H_c^0(\mathbf{G}_h;\pi_0(E_h)) & \overset{\phi^X}{\dashrightarrow} & H_c^{2p-1}(\mathbf{G}_h;\pi_{2p-2}(E_h))\\ {\scriptstyle (f^X)_*}\downarrow\cong & & \cong\downarrow{\scriptstyle (f^X)_*}\\ H_c^0(\mathbf{G}_h;(E_h)_0(X)) & \xrightarrow{d_{2p-1}^X} & H_c^{2p-1}(\mathbf{G}_h;(E_h)_{2p-2}(X)) \end{array}$$

One can check that $\phi^X(1)$ is independent of the choice of $f^X$. We define the next detection map $\mathrm{ev}_2\colon\kappa_h\to H_c^{2p-1}(\mathbf{G}_h;\pi_{2p-2}(E_h))$ by setting $\mathrm{ev}_2(X):=\phi^X(1)$.

**Proposition 1.19.** *The map* $\mathrm{ev}_2\colon \kappa_h \to H_c^{2p-1}(\mathbf{G}_h;\pi_{2p-2}(E_h))$ *is a group homomorphism.*

*Proof.* It suffices to check $\mathrm{ev}_2(X\wedge_{K(h)}Y)=\mathrm{ev}_2(X)+\mathrm{ev}_2(Y)$. This follows from the Künneth isomorphism which is compatible with the $\mathbf{G}_h$-actions:

$$(E_h)_*(X\wedge_{K(h)}Y)\cong (E_h)_*X\otimes_{(E_h)_*}(E_h)_*Y.$$

This implies

$$\begin{aligned}E_{2p-1}^{s,t}(X\wedge_{K(h)}Y)&=E_2^{s,t}(X\wedge_{K(h)}Y)\\&\cong E_2^{s,t}(X)\otimes_{E_2^{0,0}(S^0)}E_2^{s,t}(Y)\\&=E_{2p-1}^{s,t}(X)\otimes_{E_{2p-1}^{0,0}(S^0)}E_{2p-1}^{s,t}(Y).\end{aligned}$$

Now by the multiplicative structure of the spectral sequence and the Leibniz rule, we have

$$\begin{aligned}&d_{2p-1}^{X\wedge_{K(h)}Y}(\iota_X\wedge\iota_Y)=d_{2p-1}^X(\iota_X)\otimes\iota_Y+\iota_X\otimes d_{2p-1}^Y(\iota_Y)\\\Longrightarrow\ &\mathrm{ev}_2(X\wedge_{K(h)}Y)=\phi^{X\wedge_{K(h)}Y}(1)=\phi^X(1)+\phi^Y(1)=\mathrm{ev}_2(X)+\mathrm{ev}_2(Y).\end{aligned}$$

□

**Proposition 1.20.** *The map* $\mathrm{ev}_2\colon\kappa_h\to H_c^{2p-1}(\mathbf{G}_h;\pi_{2p-2}(E_h))$ *is injective when* $4p-3>h^2$ *and* $(p-1)\nmid h$. *In particular, it is injective when* $2p-1=h^2$.

*Proof.* For any $X\in\ker\mathrm{ev}_2$, a unit $[\iota_X]$ in $E_2^{0,0}(X)$ does not support a $d_{2p-1}$-differential. By Sparseness (Lemma 1.15), the next possible non-trivial differential is $d_{4p-3}^X\colon E_{4p-3}^{0,0}(X)\to E_{4p-3}^{4p-3,4p-2}(X)$. The target of this differential is zero, since it is above the horizontal vanishing line at $s=h^2$ under our assumption. The same argument shows $[\iota_X]$ does not support any higher differentials and is thus a permanent cycle. The rest of the proof is identical to that of Theorem 1.12. □

This finishes the first implication II $\Longrightarrow$ I in Section 0.2. The goal of this paper is to answer the following question:

**Question 1.21.** Is $\kappa_h=0$ when $2p-1=h^2$?

Proposition 1.20 implies this would be true if

$$H_c^{2p-1}(\mathbf{G}_h;\pi_{2p-2}(E_h))=H_c^{h^2}(\mathbf{G}_h;\pi_{2p-2}(E_h))=0.$$

1.3. **A filtration on $K(h)$-local Picard groups.** The main results of this paper do not depend on this subsection. Following the construction above, one can define $\kappa_h^{(1)}:=\ker\mathrm{ev}_2$ and construct the next algebraic detection map using the $d_{4p-3}$-differential:

$$\mathrm{ev}_3\colon\kappa_h^{(1)}\longrightarrow E_{2p}^{4p-3,4p-4}(S^0)=E_{4p-3}^{4p-3,4p-4}(S^0).$$

Eventually, we get a descent filtration on $\mathrm{Pic}_{K(h)}$ (see [3, §3.3]):

$$
\begin{array}{ccc}
\cdots & & \cdots \\
\cup\!\shortmid & & \\
\kappa_h^{(m)} & \xrightarrow{\mathrm{ev}_{m+2}} & E_{2m(p-1)+2}^{2(m+1)(p-1)+1,2(m+1)(p-1)} \\
\cup\!\shortmid & & \\
\cdots & & \cdots \\
\cup\!\shortmid & & \\
\kappa_h^{(1)} & \xrightarrow{\mathrm{ev}_3} & E_{2p}^{4p-3,4p-4} \\
\cup\!\shortmid & & \\
\kappa_h & \xrightarrow{\mathrm{ev}_2} & E_2^{2p-1,2p-2} = H_c^{2p-1}(\mathbf{G}_h;\pi_{2p-2}(E_h)) \\
\cup\!\shortmid & & \\
\mathrm{Pic}_{K(h)}^0 & \xrightarrow{\mathrm{ev}_1} & \mathrm{Pic}(\mathbf{G}_h\text{-}\pi_0(E_h)\text{-modules}) \cong H_c^1(\mathbf{G}_h;\pi_0(E_h)^\times) \\
\cup\!\shortmid & & \\
\mathrm{Pic}_{K(h)} & \overset{\mathrm{ev}_0}{\twoheadrightarrow} & \mathrm{Pic}(\text{graded } (E_h)_*\text{-modules}) \cong \mathbf{Z}/2.
\end{array}
\tag{1.22}
$$

Each term in this tower is the kernel of the horizontal detection map right below it.

**Remark 1.23.** For each fixed $p$ and $h$, (1.22) is a finite (hence Hausdorff) filtration on $\kappa_h$. This is because the HFPSS (1.13) for $S^0_{K(h)}$ has a horizontal vanishing line on the $E_r$-page when $r$ is large enough by [5, Theorem 2.3.9]. As a result, the target of $\mathrm{ev}_m$ will eventually be zero and $\kappa_h^{(m)} = \kappa_h^{(m+1)} = \cdots = 0$ when $m \gg 0$.

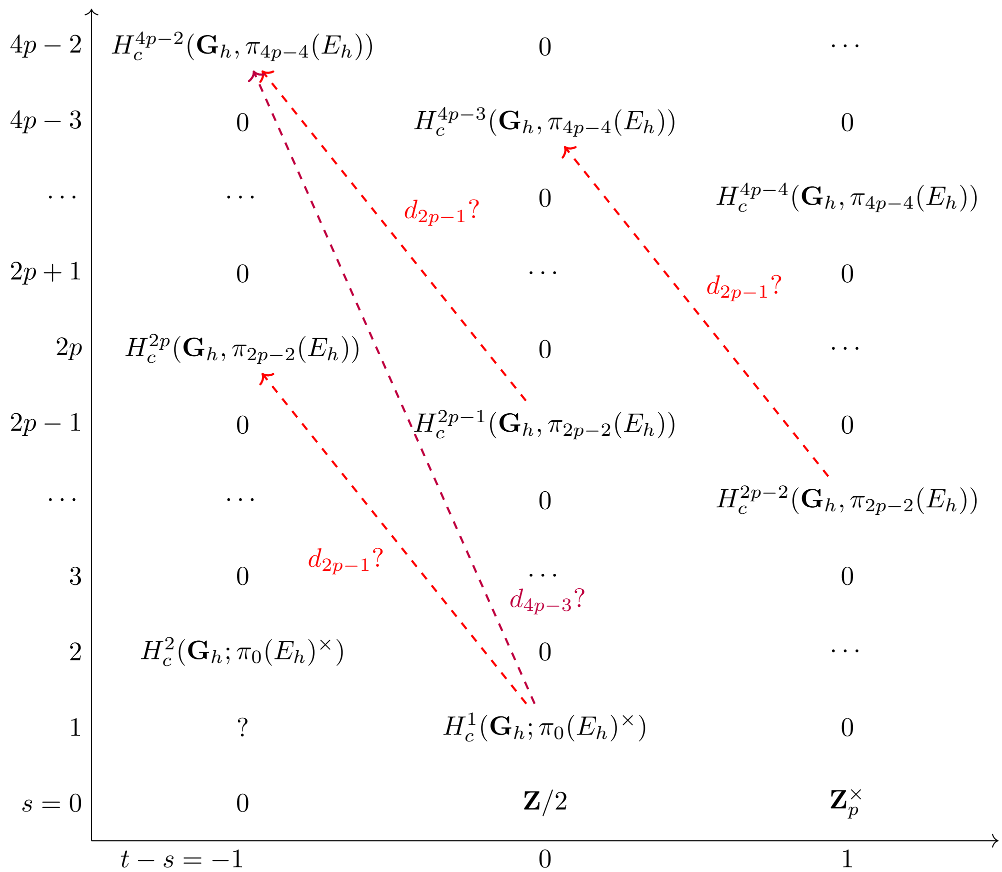
The right column in (1.22) is the 0-stem of a spectral sequence (similar to the one found in [25, Theorem 3.2.1]) to compute the homotopy groups of the Picard *spectrum* $\mathfrak{pic}_{K(h)}$ for $\mathsf{Sp}_{K(h)}$. Indeed, $\pi_0\left(\mathfrak{pic}_{K(h)}\right) = \mathrm{Pic}_{K(h)}$. In a recent paper [13], Heard has proved the following:

**Theorem 1.24** ([13, Example 6.18])**.** *There is a descent spectral sequence (DSS) for $\mathfrak{pic}_{K(h)}$ that converges when $t-s\geq 0$, whose $E_2$-page is:*

$$E_2^{s,t} = \begin{cases} 0, & t<0; \\ \mathbf{Z}/2, & s=t=0; \\ H_c^s(\mathbf{G}_h;\pi_0(E_h)^\times), & t=1; \\ H_c^s(\mathbf{G}_h;\pi_{t-1}(E_h)), & t\geq 2, \end{cases} \Longrightarrow \pi_{t-s}\left(\mathfrak{pic}_{K(h)}\right).$$

Let's analyze the $-1,0,1$-columns on the $E_2$-page of the descent spectral sequence Theorem 1.24, illustrated below in Adams grading. On this page of the spectral sequence:

- $E_2^{0,0} = H_c^0(\mathbf{G}_h;\mathbf{Z}/2) = \mathbf{Z}/2$. The non-zero element is a permanent cycle, since it represents $S^1$ in $\mathrm{Pic}_{K(h)}$. So $E_\infty^{0,0} = E_2^{0,0} = \mathbf{Z}/2$.
- $E_2^{0,1} = H_c^0(\mathbf{G}_h;\pi_0(E_h)^\times) = \mathbf{Z}_p^\times$. This term does not support any higher differential, because they represent permanent cycles $\mathbf{Z}_p^\times \subseteq \pi_0\left(S^0_{K(h)}\right)^\times \cong \pi_1\left(\mathfrak{pic}_{K(h)}\right)$.
- $E_2^{1,1} = H_c^1(\mathbf{G}_h;\pi_0(E_h)^\times) = \mathrm{Pic}^{alg,0}_{K(h)}$. For degree reasons, this term cannot be hit by a differential. But it may support one. As a result, $E_\infty^{1,1}$ is a subgroup of $H_c^1(\mathbf{G}_h;\pi_0(E_h)^\times)$.
- By Lemma 1.15, the next possibly nonzero terms in the $-1,0,1$-stems are when $t=2p-1$. In the 0-stem, it is $E_2^{2p-1,2p-1} = H_c^{2p-1}(\mathbf{G}_h;\pi_{2p-2}(E_h))$. The only possible differential that could hit this term is $d_{2p-1}\colon E_2^{0,1}\to E_2^{2p-1,2p-1}$. But since elements in $E_2^{0,1} = \mathbf{Z}_p^\times$ are all permanent cycles, this differential is zero. On the other hand, there is room for $E_2^{2p-1,2p-1}$ to support a differential. As a result, $E_\infty^{2p-1,2p-1}$ is a subgroup of $E_2^{2p-1,2p-1} = H_c^{2p-1}(\mathbf{G}_h;\pi_{2p-2}(E_h))$.




Now we can compare the $E_\infty$-page of the descent spectral spectral sequence for Picard spaces in Theorem 1.24 and the filtration in (1.22). Notice when $t \geq 2$, the $E_2^{s,t}$-term in Theorem 1.24 is the the same as $E_2^{s,t-1}$ in HFPSS (1.13) for $X = S^0_{K(h)}$. The Picard group $\mathrm{Pic}_{K(h)} = \pi_0\left(\mathfrak{pic}_{K(h)}\right)$ is an extension of the terms $E_\infty^{s,s}$ in Theorem 1.24. More precisely, we have a descending filtration $\mathrm{Pic}_{K(h)} = F^0 \supseteq F^1 \supseteq F^2 \supseteq F^3 \supseteq \cdots$, where the layers are related by short exact sequences:

$$0 \longrightarrow F^{s+1} \longrightarrow F^s \longrightarrow E_\infty^{s,s} \longrightarrow 0\ , \qquad s \geq 0.$$

As is mentioned in Remark 1.23, this is essentially a finite filtration since $E_\infty^{s,s} = 0$ when $s \gg 0$. In this filtration, we have $F^1 = \mathrm{Pic}^0_{K(h)}$ and $F^2 = F^3 = \cdots = F^{2p-1} = \kappa_h$ is the exotic $K(h)$-local Picard group. The ev-maps can then be defined as composite maps:

$$\begin{array}{ccc} & & E_\infty^{0,0} \\ & \nearrow & \| \\ F^0 = \mathrm{Pic}_{K(h)} & \xrightarrow{\mathrm{ev}_0} & E_2^{0,0} \end{array} \qquad\qquad \begin{array}{ccc} & & E_\infty^{1,1} \\ & \nearrow & \downarrow \\ F^1 = \mathrm{Pic}^0_{K(h)} & \xrightarrow{\mathrm{ev}_1} & E_2^{1,1} \end{array}$$

$$\begin{array}{ccc} & \nearrow\!\!\!\!\nearrow & E_\infty^{2p-1,2p-1} \\ & & \downarrow \\ F^{2p-1} = \kappa_h & \xrightarrow{\mathrm{ev}_2} & E_2^{2p-1,2p-1} \end{array} \qquad \begin{array}{ccc} & \nearrow\!\!\!\!\nearrow & E_\infty^{4p-3,4p-3} \\ & & \downarrow \\ F^{4p-3} = \kappa_h^{(1)} & \xrightarrow{\mathrm{ev}_3} & E_{2p}^{4p-3,4p-3} \end{array}$$

For $\mathrm{ev}_3$, the only differential that can hit $E_2^{4p-3,4p-3}$ is $d_{2p-1}$. So $E_{2p}^{4p-3,4p-3}$ cannot be hit by a differential, but it may support one. As a result, $E_\infty^{4p-3,4p-3}$ is a subgroup of $E_{2p}^{4p-3,4p-3}$.

From the factorizations above, we can see $\mathrm{ev}_1$ and $\mathrm{ev}_2$ are surjective precisely when $E_2^{1,1} = E_\infty^{1,1}$ and $E_2^{2p-1,2p-1} = E_\infty^{2p-1,2p-1}$. This will be the case if the targets of the potential differentials supported at $E_2^{1,1}$ and $E_2^{2p-1,2p-1}$ are above the horizontal vanishing line on the $E_2$-page.

**Proposition 1.25.** *Suppose $(p-1) \nmid h$. Theorem 1.24 implies:*

*(1) [28, Remark 2.6] The map $\mathrm{ev}_1\colon \mathrm{Pic}^0_{K(h)} \to \mathrm{Pic}^{alg,0}_{K(h)} := H^1_c(\mathbf{G}_h; \pi_0(E_h)^\times)$ is an isomorphism when $2p-1 > h^2$ and is a surjection when $2p-1 = h^2$.*

*(2) The map $\mathrm{ev}_2\colon \kappa_h \to H_c^{2p-1}(\mathbf{G}_h; \pi_{2p-2}(E_h))$ is an isomorphism when $4p-3 > h^2$ and is a surjection when $4p-3 = h^2$.*

*Proof.* The injectivity parts are from Theorem 1.12 and Proposition 1.20, respectively.

By sparseness (Lemma 1.15), the first possible non-trivial differentials supported at the two terms are

$$\begin{aligned} d_{2p-1}\colon E_2^{1,1} &\longrightarrow E_2^{2p,2p-1} = H_c^{2p}(\mathbf{G}_h; \pi_{2p-2}(E_h)), \\ d_{2p-1}\colon E_2^{2p-1,2p-1} &\longrightarrow E_2^{4p-2,4p-3} = H_c^{4p-2}(\mathbf{G}_h; \pi_{4p-4}(E_h)). \end{aligned}$$

Under the assumptions, the targets of the two $d_{2p-1}$-differentials are above the horizontal vanishing line at $s = h^2$ in the respective cases. As a result, their targets vanish and $E_2^{1,1} = E_\infty^{1,1}$, $E_2^{2p-1,2p-1} = E_\infty^{2p-1,2p-1}$. This proves the surjectivity part. □

**Remark 1.26.** *While the proof of Proposition 1.25 depends on Theorem 1.24, the statements have been verified independent of the descent spectral sequence in many cases, sometimes even without the assumption that $(p-1) \nmid h$:*

*(1) The map $\mathrm{ev}_1$ is known to be surjective when*
- *$h = 1$ [17, Corollary 2.6 for $p > 2$, Lemma 3.4 for $p = 2$].*
- *$h = 2, p > 2$ [11, Theorem 2.9].*
- *$2(p-1) > h^2 + h$ for general $h$ and $p$ [28, Theorem 2.5].*

*It is an open question whether the map $\mathrm{ev}_1$ is surjective or not in the $h = p = 2$ case.*

*(2) The map $\mathrm{ev}_2$ is known to be an isomorphism when*
- *$h = 1, p = 2$ [11, Remark 3.3].*
- *$h = 2, p = 3$ [11, Theorem 3.4].*

**Remark 1.27.** *The filtration (1.22) for $\kappa_2$ at prime 2 has been completed studied in [3]. In particular, they showed that the detection maps*

$$\mathrm{ev}_3\colon \kappa_2^{(1)} \to E_4^{5,4} \text{ is not surjective;}$$

$$\mathrm{ev}_4\colon \kappa_2^{(2)} \to E_6^{7,6} \text{ is injective.}$$

*See [3, Theorem 12.30] for the full details.*

We conclude this subsection by noting Theorem 1.24 implies the following:

**Corollary 1.28.** *When $2p-1 = h^2$, then the followings are equivalent:*

*(1)* $\mathrm{ev}_1\colon \mathrm{Pic}_{K(h)} \xrightarrow{\sim} \mathrm{Pic}^{alg}_{K(h)}$ *is an isomorphism.*
*(2)* $\mathrm{ev}_1\colon \mathrm{Pic}^0_{K(h)} \xrightarrow{\sim} \mathrm{Pic}^{alg,0}_{K(h)}$ *is an isomorphism.*
*(3)* $\kappa_h := \ker \mathrm{ev}_1 = 0$.
*(4)* $H^{2p-1}_c(\mathbf{G}_h;\pi_{2p-2}(E_h)) = H^{h^2}_c(\mathbf{G}_h;\pi_{2p-2}(E_h)) = 0$.

*Proof.* (1) $\iff$ (2) follows from Corollary 1.10. By Proposition 1.25, $\mathrm{ev}_1$ is surjective and $\mathrm{ev}_2$ is an isomorphism when $2p-1=h^2$. This implies (2) $\iff$ (3) and (3) $\iff$ (4), respectively. □

## 2. Duality

In Proposition 1.20, we have established that there is an isomorphism

$$\mathrm{ev}_2\colon \kappa_h \xrightarrow{\sim} H^{2p-1}_c(\mathbf{G}_h;\pi_{2p-2}(E_h))$$

under the conditions that $4p-3>h^2$ and $h$ is not divisible by $p-1$. In particular, this is true when $2p-1=h^2$. In light of this injection, we are thus interested in determining the group $H^{h^2}_c(\mathbf{G}_h;\pi_{2p-2}(E_h))$. The purpose of this section is reduce this computation using duality argument. We will prove the successive implications II $\Longleftarrow$ III $\Longleftarrow$ IV $\Longleftarrow$ V mentioned in Section 0.2:

**Proposition 2.1.** *Suppose $(p-1)\nmid h$.*
*(1) (Proposition 2.3)* $H^{h^2}_c(\mathbf{G}_h;\pi_{2p-2}(E_h)) \cong H^{h^2}_c(\mathbf{G}_h;\pi_{2p-2}(E_h)/p)$.
*(2) (Proposition 2.27) For a general $t\in\mathbf{Z}$, we have*

$$H^{h^2}_c(\mathbf{G}_h;\pi_t(E_h)/p) \cong \left[\operatorname*{colim}_{p\in J\trianglelefteq\pi_0(E_h)} H^0_c\left(\mathbf{G}_h;\pi_{2h-t-\frac{p^N|v_h|}{p-1}}(E_h)\Big/J\right)\right]^\vee,$$

*where $J\trianglelefteq\pi_0(E_h)$ ranges through all open invariant ideals containing $p$ and $N$ is the smallest integer such that $v_h^{p^N}$ is invariant mod $J$. The colimit system is described in Definition 2.19.*

### 2.1. Reduction to mod-$p$ coefficients.

The purpose of this subsection is to prove (1) in Proposition 2.1. This is the second step III $\Longrightarrow$ II in Section 0.2.

**Lemma 2.2** (Bounded torsion, [12, page 8])**.** *The cohomology group $H^*_c(\mathbf{G}_h;\pi_{2p-2}(E_h))$ is $p$-torsion.*

**Proposition 2.3.** *If $(p-1)\nmid h$, then we have an isomorphism:*

$$H^{h^2}_c(\mathbf{G}_h;\pi_{2p-2}(E_h)) \xrightarrow{\sim} H^{h^2}_c(\mathbf{G}_h;\pi_{2p-2}(E_h)/p).$$

*Proof.* Let $M=\pi_{2p-2}(E_h)$. There is a short exact sequence of $\mathbf{G}_h$-$\pi_0(E_h)$-modules

$$0 \longrightarrow M \xrightarrow{p} M \longrightarrow M/p \longrightarrow 0. \tag{2.4}$$

This short exact sequence induces a long exact sequence in cohomology

$$\cdots \to H^k_c(\mathbf{G}_h;M) \xrightarrow{p} H^k_c(\mathbf{G}_h;M) \to H^k(\mathbf{G}_h;M/p) \xrightarrow{\delta} H^{k+1}(\mathbf{G}_h;M) \to \cdots \tag{2.5}$$

By Lemma 2.2, all the multiplication-by-$p$ maps in (2.5) are zero. Since $p-1$ does not divide $h$, $\mathrm{cd}_p(\mathbf{G})=h^2$ by Lemma 1.16. As a result, the cohomology groups $H^s_c(\mathbf{G}_h;-)=0$ when $s>h^2$. This means the long exact sequence (2.5) ends with

$$0\to H^{h^2}_c(\mathbf{G}_h;M)\to H^{h^2}_c(\mathbf{G}_h;M/p)\to 0$$

and we get the desired isomorphism. □

**Remark 2.6.** *Let $M = \pi_{2p-2}(E_h)$ as above. When $s = 0$, we have $\delta\colon H^0_c(\mathbf{G}_h; M/p) \xrightarrow{\sim} H^1_c(\mathbf{G}_h; M)$. When $1 \le s \le h^2 - 1$, there is a short exact sequence instead:*

$$0 \to H^s_c(\mathbf{G}_h; M) \to H^s_c(\mathbf{G}_h; M/p) \xrightarrow{\delta} H^{s+1}_c(\mathbf{G}_h; M) \to 0.$$

*Since all three groups above are $\mathbf{F}_p$-vector spaces, the short exact sequence splits (non-canonically). As a result, we have $H^s_c(\mathbf{G}_h; M/p) \cong H^s_c(\mathbf{G}_h; M) \oplus H^{s+1}_c(\mathbf{G}_h; M)$ for $1 \le s \le h^2 - 1$.*

**Remark 2.7.** *The claims above hold for any $M = \pi_t(E_h)$, where $t = 2m(p-1)$ and $p \nmid m$.*

2.2. **Poincaré duality.** The Morava stabilizer group $\mathbf{G}_h$ is not just a profinite group, but is also a compact $p$-adic Lie group of dimension $h^2$. This imposes a great deal of more structures on its (co-)homology. In this section, we review the theory of Poincaré duality for $p$-adic analytic groups following [35]. Recall that for a property $P$, a profinite group $G$ is said to be virtually $P$ if there is an open normal subgroup of $G$ which is $P$. A profinite group $G$ has Poincaré duality of dimension $d$ if

$$H^d_c(G, \mathbf{Z}_p[\![G]\!]) \cong \mathbf{Z}_p$$

as abelian groups ([35, (4.4.1)]).

**Theorem 2.8** (Lazard, [35, Theorem 5.1.9])**.** *Let $G$ be a compact $p$-adic analytic group. Then $G$ is a virtual Poincaré duality group of dimension $d = \dim G$.*

In the case of the Morava stabilizer group, $\mathbf{S}_h$ is a virtual Poincaré duality group of dimension $h^2$. When $(p-1) \nmid h$, then $\mathbf{S}_h$ contains no $p$-torsion subgroups. In fact, its maximal finite subgroup is cyclic of order $p^h - 1$ [1, Table 5.3.1]. Under this assumption, $\mathbf{S}_h$ is a Poincaré duality group of dimension $h^2$ (as opposed to a *virtual* one).

Now $G$ being a profinite group having Poincaré duality of dimension $n$ implies that there is a *dualizing module* $D(G)$ such that there are natural isomorphisms [35, Theorem 4.4.3] for continuous $G$-modules $M$ that are inverse limits of discrete $G$-modules:

$$H^{n-k}_c(G; M) \longrightarrow H^c_k(G; D(G)\widehat{\otimes}_{\mathbf{Z}_p} M),$$

and for discrete $p$-torsion $G$-modules

$$H^c_{n-k}(G; M) \longrightarrow H^k_c(G; \hom_{\mathbf{Z}_p}(D_p(G), M)).$$

The dualizing module $D(G)$ is given by

$$D(G) = H^n_c(G; \mathbf{Z}_p[\![G]\!]).$$

Note that, as the coefficients $\mathbf{Z}_p[\![G]\!]$ has a left $G$-action, the dualizing module $D(G)$ has a corresponding right $G$-action. See [6, §4.5] for further details.

In the case when $G$ is the Morava Stabilizer group $\mathbf{G}_h$, Strickland has calculated the dualizing module $D(\mathbf{G}_h)$ along with its $\mathbf{G}_h$-action.

**Theorem 2.9** (Strickland, [34])**.** *As a $\mathbf{G}_h$-module, $H^{h^2}_c(\mathbf{G}_h; \mathbf{Z}_p[\![\mathbf{G}_h]\!]) \cong \mathbf{Z}_p$ has the trivial $\mathbf{G}_h$-action.*

**Corollary 2.10.** *Assume $(p-1) \nmid h$. The dualizing module $I_p$ for $\mathbf{G}_h$ is $\mathbf{Z}_p^\vee \cong \mathbf{Q}_p/\mathbf{Z}_p$ with the trivial $\mathbf{G}_h$-action. Hence, we have a duality*

$$H^{h^2-k}_c(\mathbf{G}_h; M) \cong H^c_k(\mathbf{G}_h; M)$$

*that is natural in $p$-profinite continuous $\mathbf{G}_h$-modules $M$.*

**Definition 2.11.** Write $(-)^\vee$ for $\hom_c(M, I_p(G))$. If $M$ has a continuous $G$-action, we endow $M^\vee$ with a left $G$-action via

$$(g\cdot f)(x) := f(g^{-1}x).$$

In the case of $G = \mathbf{G}_h$, Corollary 2.10 implies $M^\vee$ is the continuous Pontryagin dual $M^\vee \cong \hom_c(M, \mathbf{Z}/p^\infty)$.

As usual, this also induces a version of Poincaré duality for $p$-profinite $\mathbf{G}_h$-modules $M$ in purely cohomological terms when $(p-1)\nmid h$: ([6, Theorem 4.26])

$$H_c^k(\mathbf{G}_h; M) \cong H_c^{h^2-k}(\mathbf{G}_h; M^\vee)^\vee. \tag{2.12}$$

**Corollary 2.13.** *Assume $(p-1)\nmid h$. We have the following duality:*

$$H_c^{h^2}(\mathbf{S}_h; \pi_t(E_h)) \cong H_0(\mathbf{S}_h; \pi_t(E_h)), \qquad H_c^{h^2}(\mathbf{S}_h; \pi_t(E_h)/p) \cong H_0(\mathbf{S}_h; \pi_t(E_h)/p); \tag{2.14}$$

$$H_c^{h^2}(\mathbf{S}_h; \pi_t(E_h)) \cong H_c^0(\mathbf{S}_h; \pi_t(E_h)^\vee)^\vee, \qquad H_c^{h^2}(\mathbf{S}_h; \pi_t(E_h)/p) \cong H_c^0(\mathbf{S}_h; (\pi_t(E_h)/p)^\vee)^\vee. \tag{2.15}$$

**Remark 2.16.** *Using the duality* (2.14)*, we can give another proof of Proposition 2.3 by showing:*

*(1) The group homology $H_*(\mathbf{G}_h; \pi_{2p-2}(E_h))$ is $p$-torsion. This is because the orbit of the action by $\mathbf{Z}_p^\times \subseteq \mathbf{S}_h$ is already $p$-torsion.*

*(2) Apply $H_*$ to the short exact sequence* (2.4) *to get the a long exact sequence like* (2.5)*. Equivalently, we are essentially applying* (2.14) *to every term in* (2.5)*.*

2.3. **Gross-Hopkins duality.** Now we want to use (2.15) to compute $H_c^{h^2}(\mathbf{G}_h; M/p)$ where $M = E_t$. To do so, we have to identify the $\mathbf{G}_h$-equivariant Pontryagin dual of $M$. This is realized by Gross-Hopkins duality.

**Remark 2.17.** *For the purpose of Question 1.21, we only need to study the case when $t = 2p-2$. Later for the Vanishing Conjecture, we also need the $t = 0$ case. So we will give a uniform treatment for all $t \in \mathbf{Z}$ in the remainder of this section.*

We remind the reader the definition of the determinant twist. The group $\mathbf{S}_h$ can be realized as a subgroup of $\mathrm{GL}_h(\mathbf{W})$. Thus, taking the determinant, we have a map

$$\det\colon \mathbf{S}_h \to \mathbf{W}^\times.$$

It turns out that this map actually factors through $\mathbf{Z}_p^\times$. We extend this to the extended Morava stabilizer group via the composite

$$\det\colon \mathbf{G}_h \cong \mathbf{S}_h \rtimes \mathrm{Gal} \longrightarrow \mathbf{Z}_p^\times \times \mathrm{Gal} \xrightarrow{proj} \mathbf{Z}_p^\times.$$

This results in a $\mathbf{G}_h$-action on $\mathbf{Z}_p$.

**Definition 2.18.** The $\mathbf{G}_h$-action on $\mathbf{Z}_p$ above is denoted by $\mathbf{Z}_p\langle\det\rangle$. Given a Morava module $M$ we write $M\langle\det\rangle$ for the Morava module

$$M\langle\det\rangle \cong M \otimes_{\mathbf{Z}_p} \mathbf{Z}_p\langle\det\rangle$$

with the diagonal $\mathbf{G}_h$-action. We refer to $M\langle\det\rangle$ as the *determinant twist* of $M$.

**Definition 2.19.** We now describe the quotient mod $\mathfrak{m}^\infty$. Let $M$ be a $\mathbf{G}_h$-$\pi_0(E_h)$-module, we define

$$M/\mathfrak{m}^\infty := \operatorname*{colim}_{J\trianglelefteq\pi_0(E_h)} M/J, \tag{2.20}$$

where $J$ ranges over all open invariant ideals of $\pi_0(E_h)$. Suppose $J \subseteq J'$ is an inclusion of open invariant ideals of $\pi_0(E_h)$. Then we have a $\mathbf{G}_h$-equivariant isomorphism:

$$M/J' \cong \{[m] \in M/J \mid x\cdot[m] = 0, \forall x \in J'\}.$$

This gives the structure map $M/J' \to M/J$ in the colimit system. Similarly, in the mod-$p$ case, we have

$$M/(p, u_1^\infty, \cdots, u_{h-1}^\infty) := \operatorname*{colim}_{p\in J \trianglelefteq \pi_0(E_h)} M/J,$$

where $J$ ranges over all invariant ideals of $\pi_0(E_h)$ containing $p$.

**Theorem 2.21** (Gross-Hopkins). *Let $\mathfrak{m} \trianglelefteq \pi_0(E_h)$ be the maximal ideal.*

*(1)* [34] *There is a $\mathbf{G}_h$-equivariant perfect pairing of $\mathbf{G}_h$-$\pi_0(E_h)$-modules:*

$$\rho\colon \pi_0(E_h)/\mathfrak{m}^\infty \otimes_{\pi_0(E_h)} \Omega^{h-1} \longrightarrow \mathbf{Q}_p/\mathbf{Z}_p,$$

*where $\Omega^{h-1}$ is the top exterior power of the module of continuous Kähler differentials for $\pi_0(E_h)$ relative to $\mathbf{W}$.*

*(2)* [15] *The module $\Omega^{h-1}$ is $\mathbf{G}_h$-equivariantly equivalent to the bundle $\omega^{\otimes h}\langle\det\rangle$ over the Lubin-Tate deformation space, where $\omega = \pi_2(E_h)$ is the sheaf of invariant of differentials and $\langle\det\rangle$ is the determinant twist.*

**Corollary 2.22** (See [34, Proposition 19]). *The $\mathbf{G}_h$-equivariant Pontryagin dual of $\pi_t(E_h)$ is*

$$(\pi_t(E_h))^\vee \cong (\pi_{2h-t}(E_h))\langle\det\rangle/\mathfrak{m}^\infty.$$

*Proof.* The $\mathbf{G}_h$-equivariant perfect pairing $\rho$ in Theorem 2.21 can be rewritten as:

$$\rho\colon \pi_0(E_h)/\mathfrak{m}^\infty \otimes_{\pi_0(E_h)} \Omega^{h-1} \cong \pi_t(E_h) \otimes_{\pi_0(E_h)} \pi_{-t}(E_h)/\mathfrak{m}^\infty \otimes_{\pi_0(E_h)} \Omega^{h-1} \longrightarrow \mathbf{Q}_p/\mathbf{Z}_p.$$

This implies the $\mathbf{G}_h$-equivariant Pontryagin dual of $\pi_t(E_h)$ is $\pi_{-t}(E_h)/\mathfrak{m}^\infty \otimes_{\pi_0(E_h)} \Omega^{h-1}$, which is $\mathbf{G}_h$-equivariantly isomorphic to $(\pi_{2h-t}(E_h))\langle\det\rangle/\mathfrak{m}^\infty$ by part (2) of Theorem 2.21. ☐

Applying (2.12), we have proved:

$$H_c^{h^2}(\mathbf{G}_h; \pi_t(E_h)) \cong H_c^0\left(\mathbf{G}_h; (\pi_{2h-t}(E_h))\langle\det\rangle/\mathfrak{m}^\infty\right)^\vee. \tag{2.23}$$

The formula holds with $\pi_t(E_h)$ replaced by $\pi_t(E_h)/p$. This yields the third implication IV $\Longrightarrow$ III in Section 0.2 when $t = 2p-2$. Notice (2.20) is a filtered colimit, and the group $\mathbf{G}_h$ is topologically finitely generated (since it is a finite dimensional $p$-adic Lie group), we have

**Proposition 2.24.** *There are isomorphisms:*

$$\operatorname*{colim}_{J\trianglelefteq E_h} H_c^0(\mathbf{G}_h; M/J) \xrightarrow{\sim} H_c^0(\mathbf{G}_h; M/\mathfrak{m}^\infty),$$

$$\operatorname*{colim}_{p\in J\trianglelefteq E_h} H_c^0(\mathbf{G}_h; M/J) \xrightarrow{\sim} H_c^0(\mathbf{G}_h; M/(p, u_1^\infty, \cdots, u_{h-1}^\infty)).$$

Now set $M = E_{2h-2p+2}\langle\det\rangle$. In order to prove

$$H_c^0(\mathbf{G}_h; M/(p, u_1^\infty, \cdots, u_{h-1}^\infty))^\vee = 0,$$

it suffices to show $H_c^0(\mathbf{G}_h; M/J) = 0$ for a cofinal system of invariant ideals $J \trianglelefteq \pi_0(E_h)$ containing $p$. To do that, we need to identify the determinant twist $\pi_0(E_h)\langle\det\rangle$ mod $p$. The following theorem was originally stated in [16, Corollary 7] and a nice proof appears in [12, Theorem 1.32]:

**Theorem 2.25** (Gross-Hopkins). *When $p > 2$, there is an isomorphism of $\mathbf{G}_h$-$\pi_0(E_h)$-modules:*

$$\pi_0(E_h)\langle\det\rangle/p \cong \pi_0\left(\Sigma^{\lim_{N\to\infty} \frac{p^N|v_h|}{p-1}} E_h\right)/p.$$

*More precisely, let $J \trianglelefteq \pi_0(E_h)$ be an open invariant ideal containing $p$, such that $v_h^{p^N}$ is invariant modulo $J$, then*

$$\pi_0(E_h)\langle\det\rangle/J \cong \pi_0\left(\Sigma^{\frac{p^N|v_h|}{p-1}} E_h\right)/J.$$

**Remark 2.26.** *Suppose $v_h^{p^{N'}}$ is also invariant mod $J$ for some $N' < N$. Then*

$$\pi_0(E_h)\langle\det\rangle/J \cong \pi_0\left(\Sigma^{\frac{p^{N'}|v_h|}{p-1}} E_h\right)/J.$$

*This is compatible with the statement in Theorem 2.25. This is because*

$$\begin{aligned}\frac{p^{N'}|v_h|}{p-1} &\equiv \frac{p^N|v_h|}{p-1} \mod p^{N'}|v_h| \\ \Longrightarrow \pi_0\left(\Sigma^{\frac{p^{N'}|v_h|}{p-1}} E_h\right)/J &\cong \pi_0\left(\Sigma^{\frac{p^N|v_h|}{p-1}} E_h\right)/J.\end{aligned}$$

For each open invariant ideal $J$, there is a smallest $N$ such that $v_h^{p^N}$ is invariant mod $J$. It follows from this proposition that

$$M/J = \pi_{2h-2p+2}(E_h)\langle\det\rangle/J \cong \pi_{2h-2p+2-\frac{p^N|v_h|}{p-1}}(E_h)\Big/ J.$$

Combining all the duality arguments in Corollary 2.13 and Corollary 2.22 with the identification of the determinant twist $\pi_0(E_h)\langle\det\rangle$ mod $p$ in Theorem 2.25, we have proved part (2) in Proposition 2.1.

**Proposition 2.27.** *Suppose $(p-1)\nmid h$. Then there is an isomorphism:*

$$H_c^{h^2}(\mathbf{G}_h;\pi_t(E_h)/p) \cong \left[\operatorname*{colim}_{p\in J\trianglelefteq\pi_0(E_h)} H_c^0\left(\mathbf{S}_h;\pi_{2h-t-\frac{p^N|v_h|}{p-1}}(E_h)\Big/ J\right)^{\mathrm{Gal}}\right]^\vee,$$

*where $J \trianglelefteq \pi_0(E_h)$ ranges through all opening invariant ideals containing $p$ and $N$ is the smallest integer such that $v_h^{p^N}$ is invariant mod $J$.*

From this, we get the implication V $\Longrightarrow$ IV in Section 0.2. Consequently, Question 1.21 now reduces to checking

$$H_c^0\left(\mathbf{G}_h;\pi_{2h-2p+2-\frac{p^N|v_h|}{p-1}}(E_h)\Big/ J\right) = 0 \tag{2.28}$$

for a cofinal system of invariant ideals $J$ containing $p$, where $N$ is the smallest number such that $v_h^{p^N}$ is invariant mod $J$.

2.4. **The Chromatic Vanishing Conjecture.** A closely related computation is the Chromatic Vanishing Conjecture. Consider the natural inclusion $\iota:\mathbf{W}\hookrightarrow\pi_0(E_h)$, which is $\mathbf{G}_h$-equivariant. Explicit computations at height 2 in [2, 5, 11, 14, 23, 32] show that this inclusion induces isomorphisms in group cohomology of $\mathbf{G}_2$ for all primes and degrees. At $h=p=2$, this isomorphism plays an essential role in disproving and completely understanding the Chromatic Splitting Conjecture by Beaudry-Goerss-Henn in [5]. Observing this phenomenon, Hans-Werner Henn first raised the question if there is a conceptual reason for the isomorphisms. This leads to a more general conjecture:

**Conjecture 2.29** (Chromatic Vanishing Conjecture, [4, Conjecture 1.1], [5, Conjecture 1.1.4])**.** The followings are true for all heights $h$, primes $p$, and (co)-homological degrees $s$:

(1) (Integral) The continuous group cohomology and homology of $\operatorname{coker}(\iota)$ vanish so that

$$\iota_*\colon H_c^s(\mathbf{G}_h;\mathbf{W}) \xrightarrow{\sim} H_c^s(\mathbf{G}_h;\pi_0(E_h)), \qquad \iota_*\colon H_s(\mathbf{G}_h;\mathbf{W}) \xrightarrow{\sim} H_s(\mathbf{G}_h;\pi_0(E_h)).$$

(2) (Reduced) The continuous group cohomology and homology of $\operatorname{coker}(\iota\otimes \mathbf{W}/p)$ vanish so that

$$\iota_*\colon H_c^s(\mathbf{G}_h;\mathbf{F}_p) \xrightarrow{\sim} H_c^s(\mathbf{G}_h;\pi_0(E_h)/p), \qquad \iota_*\colon H_s(\mathbf{G}_h;\mathbf{F}_p) \xrightarrow{\sim} H_s(\mathbf{G}_h;\pi_0(E_h)/p).$$

**Remark 2.30** ([4, page 692])**.**

*(1) By Corollary 2.10 and* (2.12)*, the cohomological and homological versions of Conjecture 2.29 are equivalent when $(p-1)\nmid h$.*
*(2) The reduced version of conjecture implies the integral version by the Five Lemma and a* $\lim^1$ *exact sequence.*
*(3) The conjecture is a tautology when $h=1$, since $\mathbf{Z}_p^\times$ acts on $\pi_0(E_1)\cong \mathbf{Z}_p$ trivially.*
*(4) At $h=2$, the conjecture has been proved for all primes.*
*(5) The proof for $s=0$ at all heights can be found in [9, Lemma 1.33].*

**Remark 2.31** (Hopkins, [7, Theorem 8.1], [18, §5.3], [24] for $p\geq 5$; Karamanov [22] for $p=3$)**.** *When $h=2$ and $p\geq 3$, the* additive *Vanishing Conjecture in cohomological degree* 1 *can be used to show a* multiplicative *version of the conjecture:*

$$H_c^1(\mathbf{G}_h;\mathbf{W}^\times) \xrightarrow{\sim} H_c^1(\mathbf{G}_h;\pi_0(E_h)^\times).$$

*From there, we can compute the algebraic $K(2)$-local Picard groups when $p\geq 3$:*

$$\operatorname{Pic}_{K(2)}^{alg,0}\cong \mathbf{Z}_p\oplus\mathbf{Z}_p\oplus \mathbf{Z}/(p^2-1).$$

*Combined with Proposition 1.20 and Remark 1.26, we know $\operatorname{Pic}_{K(2)}^{alg}\cong \operatorname{Pic}_{K(2)}\cong \mathbf{Z}_p\oplus\mathbf{Z}_p\oplus\mathbf{Z}/|v_2|$ when $p\geq 5$. The group is topologically generated by $S^1_{K(2)}$ and $S^0_{K(2)}\langle\det\rangle$. Those two generators are related by Theorem 2.25 and the fact that $\operatorname{ev}_1\colon \operatorname{Pic}_{K(2)}\xrightarrow{\sim}\operatorname{Pic}_{K(2)}^{alg}$ is an isomorphism when $p\geq 5$:*

$$S^0\langle\det\rangle\wedge_{K(2)} V(1)\simeq S^{2(p+1)}\wedge_{K(2)} V(1).$$

The case of Conjecture 2.29 relevant to Question 1.21 is if the following holds when $(p-1)\nmid h$:

$$\begin{aligned}&\iota_*\colon \mathbf{F}_p = H_0(\mathbf{G}_h;\mathbf{F}_p)\xrightarrow{\sim} H_0(\mathbf{G}_h;\pi_0(E_h)/p)\\ \iff &\iota_*\colon \mathbf{F}_p = H_c^{h^2}(\mathbf{G}_h;\mathbf{F}_p)\xrightarrow{\sim} H_c^{h^2}(\mathbf{G}_h;\pi_0(E_h)/p).\end{aligned}$$

As this is the reduced version of Conjecture 2.29 in homological degree 0, we will call it the **Reduced Homological Vanishing Conjecture** (RHVC). It follows immediately that

$$\text{(RHVC)}\qquad H_0(\mathbf{G}_h;\pi_0(E_h)/p)\cong H_0(\mathbf{G}_h;\mathbf{F}_{p^h})\cong\mathbf{F}_p.$$

This is the formula we want to prove. Setting $t=0$ in Proposition 2.27, we get an isomorphism when $(p-1)\nmid h$:

$$H_c^{h^2}(\mathbf{G}_h;\pi_0(E_h)/p)\cong\left[\operatorname*{colim}_{p\in J\trianglelefteq\pi_0(E_h)} H_c^0\left(\mathbf{G}_h;\pi_{2h-\frac{p^N|v_h|}{p-1}}(E_h)\Big/ J\right)\right]^\vee.$$

As a result, to prove (RHVC), it suffices to show that

$$H_c^0\left(\mathbf{G}_h;\pi_{2h-\frac{p^N|v_h|}{p-1}}(E_h)\Big/ J\right)=\mathbf{F}_p \tag{2.32}$$

for a cofinal system of invariant ideals $J$ containing $p$, where $N$ is the smallest number such that $v_h^{p^N}$ is invariant mod $J$, and that the structure maps in the colimit are non-zero.

## 3. Greek letter elements

3.1. **The change of rings theorem.** In this section we will prove the main theorems. The first step is to translate (2.28) and (2.32) to **Greek letter element** computations in chromatic homotopy theory. We refer readers to [26, §1 and §3] and [31, §5.1] for an introduction. The transition from $\mathbf{G}_h$-$\pi_0(E_h)$-modules to $BP_*BP$-comodules is achieved by the following theorem:

**Theorem 3.1** (Morava's Change of Rings Theorem, [10, Theorem 6.5])**.** *Let $M$ be a $BP_*BP$-comodule such that $I_h^n M = 0$ for some $n$, where $I_h = (p, u_1, \cdots, u_{h-1})$. Then there is a natural isomorphism:*

$$r_*\colon \operatorname{Ext}^{s,t}_{BP_*BP}(BP_*, v_h^{-1}M) \xrightarrow{\sim} H^s_c(\mathbf{G}_h; \pi_t(E_h)\otimes_{BP_*} M),$$

*where $r_*$ is induced by a ring homomorphism $r\colon BP_* \to \pi_*(E_h)$ defined below:*

$$r(v_i) = \begin{cases} u_i u^{1-p^i}, & i<h;\\ u^{1-p^h}, & i=h;\\ 0, & i>h.\end{cases}$$

Let $p \in J \trianglelefteq \pi_0(E_h)$ be an open invariant ideal containing $p$. For our computation, $M$ is a $BP_*BP$-comodule such that

$$\pi_0(E_h)\otimes_{BP_*} M \cong \pi_0(E_h)/J.$$

**Lemma 3.2.** *When $J = (p, u_1^{j_1}, \cdots, u_{h-1}^{j_{h-1}})$, we can take $M := BP_*/J'$, where $J' = (p, v_1^{j_1}, \cdots, v_{h-1}^{j_{h-1}})$.*

The implication VI $\Longrightarrow$ V in Section 0.2 then follows from Theorem 3.1. We now need to compute $\operatorname{Ext}^{0,t}_{BP_*BP}(BP_*, v_h^{-1}BP_*/J')$ for a family of invariant ideals $J'$ and certain values of $t$.

3.2. **Families of Greek letter elements.** From now on, for a graded $BP_*BP$-comodule $M$, we will write

$$H^{0,t}(M) := \operatorname{Ext}^{0,t}_{BP_*BP}(BP_*, M).$$

Suppose $J' = (p, v_1^{j_1}, \cdots, v_{h-1}^{j_{h-1}})$ for some $j_i \geq 0$. The right hand term can be more explicitly identified as the submodule of primitive elements $x$ of degree $t$ in the comodule $M_1^{h-1} := v_h^{-1}BP_*/(p, v_1^\infty, \ldots, v_{h-1}^\infty)$, such that $v_i^{j_i}x = 0$ for all $1 \leq i \leq h-1$. This establishes the final implication VII $\Longrightarrow$ VI in Section 0.2.

As a result, we need to compute $H^{0,t}(M_1^{h-1})$. The computation of this Ext-group in general heights are beyond our reach, but we can at least place elements within three distinct families.

**Proposition 3.3.** *Let $M^m_{h-m} = v_h^{-1}BP_*/(p, v_1, \cdots, v_{h-m-1}, v_{h-m}^\infty, \cdots, v_{h-1}^\infty)$. Then for $0 \leq m < h$, the cohomology group $H^{0,*}(M^m_{h-m})$ is generated as an $\mathbf{F}_p$-vector space by elements of the following families:*

I. $\frac{v_h^s}{pv_1\cdots v_{h-1}}$, *where $(s,p)=1$.*

II. $\frac{1}{pv_1^{d_1}\cdots v_{h-1}^{d_{h-1}}}$, *where $(p, v_1^{d_1}, \cdots, v_{h-1}^{d_{h-1}})$ is an invariant ideal and $d_1 = \cdots = d_{h-m-1} = 1$.*

III. $\frac{y^s_{m,N}}{pv_1^{d_1}\cdots v_{h-1}^{d_{h-1}}}$, *where $(p, v_1^{d_1}, \cdots, v_{h-1}^{d_{h-1}}, y^s_{m,N})$ is an invariant ideal with $d_1 = \cdots = d_{h-m-1} = 1$, $y_{m,N} \equiv y_{m-1,N} \mod (p, v_1, \cdots, v_{h-m})$, $N \geq 1$ and $(s,p)=1$.*

*Here, the degrees of elements are given by:*

$$\left|\frac{y^s_{m,N}}{pv_1^{d_1}\cdots v_{h-1}^{d_{h-1}}}\right| = sp^N|v_h| - \sum_{i=1}^{h-1} d_i|v_i|.$$

*Proof.* We prove this by induction on $m$. By [31, Proposition 5.1.12], the zeroth cohomology of $M_h^0 = v_h^{-1}BP_*/I_h$ is $\mathbf{F}_p[v_h^{\pm 1}]$. Identifying the $M_h^0 \subseteq M_1^{h-1}$ as a subcomodule consisting of elements that are $v_i$-torsion for all $1 \le i \le h-1$, we have proved the $m=0$ case where $y_{0,N} = v_h^{p^N}$.

The $m=1$ case was proved by Miller-Ravenel-Wilson in [26, Theorem 5.10] (see full statements in Theorem 3.17 and Theorem 3.22). Their inductive step from $m=0$ to $m=1$ also applies to the $m>1$ case, as summarized below. Recall that there are short exact sequences of $BP_*BP$-comodules

$$0 \to M_{h-m}^m \longrightarrow M_{h-m-1}^{m+1} \xrightarrow{\cdot v_{h-m-1}} M_{h-m-1}^{m+1} \to 0,$$

which leads to the $v_{h-m-1}$-Bockstein spectral sequence

$$H^{s,t}(M_{h-m}^m) \otimes \mathbf{F}_p[v_{h-m-1}]/(v_{h-m-1}^\infty) \Longrightarrow H^{s,t}(M_{h-m-1}^{m+1}).$$

Alternatively, we can consider the long exact sequence of cohomology groups

$$0 \to H^0\left(M_{h-m}^m\right) \longrightarrow H^0\left(M_{h-m-1}^{m+1}\right) \xrightarrow{\cdot v_{h-m-1}} H^0\left(M_{h-m-1}^{m+1}\right) \xrightarrow{\delta} H^1\left(M_{h-m}^m\right) \to \cdots$$

As a result, $H^0\left(M_{h-m}^m\right)$ is the subgroup of $v_{h-m+1}$-torsion elements in $H^0\left(M_{h-m-1}^{m+1}\right)$. On the other hand, the Bockstein spectral sequence implies for any element $x \in H^0\left(M_{h-m-1}^{m+1}\right)$, there is a $k$ such that $v_{h-m+1}^k x \in H^0\left(M_{h-m}^m\right)$. We can therefore obtain an additive basis for $H^0\left(M_{h-m-1}^{m+1}\right)$ from that for $H^0\left(M_{h-m}^m\right)$ by taking their quotients of powers of $v_{h-m+1}$.

Let $[x] \in H^0\left(M_{h-m-1}^{m+1}\right)$. It is can be divided by $v_{h-m+1}$ in $H^0\left(M_{h-m-1}^{m+1}\right)$ iff $\delta([x]) = [0]$ in the long exact sequence above. Pick a representative cocycle $x$ for $[x]$. From the definition of the connecting homomorphism in long exact sequence, we know $\delta([x])$ is represented by the cocycle $d(\frac{x}{v_{h-m-1}})$, where $d$ is the cobar differential. This cocycle being zero in $H^1\left(M_{h-m}^m\right)$ means that $d(\frac{x}{v_{h-m-1}}) = d(\varepsilon)$ for some correcting term $\varepsilon \in M_{h-m}^m$. Now set $x' = x - v_{h-m-1}\cdot\varepsilon$. Then $x' \equiv x \mod v_{h-m-1}$ and $x'$ can be divided by $v_{h-m-1}$ in $H^0\left(M_{h-m-1}^{m+1}\right)$.

Then the inductive hypothesis says $H^0\left(M_{h-m}^m\right)$ is generated by the three family of elements $\left\{\frac{v_h^s}{pv_1\cdots v_{h-1}}\right\} \cup \left\{\frac{1}{pv_1^{d_1}\cdots v_{h-1}^{d_{h-1}}}\right\} \cup \left\{\frac{y_{h,N}^s}{pv_1^{d_1}\cdots v_{h-1}^{d_{h-1}}}\right\}$. Apply the procedure above to those generators $[x]$ until $\delta([x]/v_{h-m-1}^k) \neq [0] \in H^1\left(M_{h-m}^m\right)$, we obtain an additive basis for $H^0\left(M_{h-m-1}^{m+1}\right)$. It remains to check the new basis obtained from Families I and II generators in $H^0\left(M_{h-m}^m\right)$ have the desired forms. For Family II, the claim follows from the cobar differential $d(1) = 0$.

For Family I, we can compute the cobar differential using [31, (6.1.13)]

$$\delta\left(\frac{v_h^s}{pv_1\cdots v_{h-m-1}v_{h-m}\cdots v_{h-1}}\right) = d\left(\frac{v_h^s}{pv_1\cdots v_{h-m-1}^2 v_{h-m}\cdots v_{h-1}}\right) = \frac{sv_h^{s-1}t_{m+1}^{p^{h-m-1}}}{pv_1\cdots v_{h-1}}.$$

This is a non-zero cocycle in $H^1(M_{h-m}^m)$ by [31, Theorem 6.5.12].[2] As a result, the zero cocycle $\left[\frac{v_h^s}{pv_1\cdots v_{h-1}}\right]$ is not $v_{h-m-1}$-divisible in $H^0\left(M_{h-m-1}^{m+1}\right)$. This proves the form of Family I elements. □

**Remark 3.4.** To get a full account of $H^0(M_1^{h-1})$ using the method above, we will need to have knowledge of $H^0(M_2^{h-2})$ and $H^1(M_2^{h-2})$. This in terms requires the knowledge of of $H^0(M_3^{h-3})$, $H^2(M_3^{h-3})$, and $H^3(M_3^{h-3})$. In the end, we will need to know $H^*(M_h^0)$ for $0 \le * \le h-1$ to compute $H^0(M_1^{h-1})$. These groups are only the inputs of the Bockstein spectral sequences. We still need to compute the cobar differentials

[2]Note that the $h_{i,j}$ in the cited theorem is represented by the cocycle $t_i^{p^j}$.

to determine the additive bases at each step. This is why getting an additive basis for $H^0(M_1^{h-1})$ is out of reach using the current technology.

One particular technical point in this computation is to find the correcting terms $\varepsilon$ in the proof above. Without them, Baird's Lemma 3.8 would have given us the full basis. For a particular computation where one has to add correcting terms, a classic example arises from the $v_1$-Bockstein spectral sequence

$$H^*(M_2^0)\otimes \mathbf{F}_p[v_1]/(v_1^\infty) \implies H^*(M_1^1)$$

for primes $p\geq 5$. For example, as shown in [31] and [26] (cf. [7] for another account) the class $\frac{v_2^{p^2}}{pv_1^{p^2+1}}$ in the $E_1$-page of the $v_1$-BSS is a permanent cycle and so detects a class in $H^0(M_1^1)$. However, the element it detects is

$$\frac{v_2^{p^2}}{pv_1^{p^2+1}}-\frac{v_2^{p^2-p+1}}{pv_1^2}-\frac{v_2^{-p}v_3^p}{pv_1}\in M_1^1.$$

We now analyze degrees of elements in the three families in $H^0(M_1^{h-1})$ and study the degrees of corresponding elements in $H^{h^2}(\mathbf{G}_h;\pi_*(E_h))$ under duality. In **Family I**, the degrees of elements are given by:

$$\left|\frac{v_h^s}{pv_1\cdots v_{h-1}}\right| = s|v_h|-\sum_{i=1}^{h-1}|v_i| = s|v_h|+2h-\frac{|v_h|}{p-1}. \tag{3.5}$$

**Proposition 3.6.** *Let $J\trianglelefteq \pi_0(E_h)$ be an open invariant ideal containing $p$, such that $v_h^{p^N}$ is invariant modulo $J$. Then the Family I element $\frac{v_h^s}{pv_1\cdots v_{h-1}}$ determines a copy of $\mathbf{F}_p$ in $H_c^{h^2}(\mathbf{G}_h;\pi_t(E_h)/J)$ via Gross-Hopkins duality Proposition 2.27 and the change-of-rings Theorem 3.1, where*

$$t\equiv -\left(s+\frac{p^N-1}{p-1}\right)|v_h| \mod p^N|v_h|. \tag{3.7}$$

*In particular,*

- *Elements in Family I contribute to $H_c^{h^2}(\mathbf{G}_h;\pi_t(E_h)/p)$ only when $|v_h|$ divides $t$.*
- *Family I elements determine a copy of $\mathbf{F}_p$ in $H_c^{h^2}(\mathbf{G}_h;\pi_0(E_h)/p)$.*

*Proof.* By Proposition 2.27 and Theorem 3.1, we have isomorphisms

$$\begin{aligned} H_c^{h^2}(\mathbf{G}_h;\pi_t(E_h)/J) &\cong \left(H_c^0\left(\mathbf{G}_h;E_{2h-t-\frac{p^N|v_h|}{p-1}}\Big/ J\right)\right)^\vee \\ &\cong \left(H^{0,2h-t-\frac{p^N|v_h|}{p-1}}(M_1^{h-1}/J')\right)^\vee, \end{aligned}$$

where $J'\trianglelefteq BP_*$ is an invariant ideal corresponding to $J$ as in Lemma 3.2. By construction, elements in Family I are in $H^{0,*}(M_1^{h-1}/J')$ for all $J'$. To prove the claim, we need to compare the degrees of Family I elements (3.5) and the target degree $2h-t-\frac{p^N|v_h|}{p-1}$ above. Notice the $BP_*BP$-comodule $M_1^{h-1}/J'$ is $p^N|v_h|$-periodic by assumption. Solving for $t$ in the residue equation:

$$2h-t-\frac{p^N|v_h|}{p-1}\equiv s|v_h|+2h-\frac{|v_h|}{p-1} \mod p^N|v_h|,$$

we obtain the congruence relation for $t$ in (3.7). In particular, the number $t$ is necessarily divisible by $|v_h|$. Solving for $s$ when $t=0$, we obtain the Famliy I element

$$\frac{v_h^{mp^N}\cdot v_h^{-\frac{p^N-1}{p-1}}}{pv_1\cdots v_{h-1}}\in H^{0,2h-\frac{p^N|v_h|}{p-1}}(M_1^{h-1})$$

that contributes to a copy of $\mathbf{F}_p\subseteq H_c^{h^2}(\mathbf{G}_h;\pi_0(E_h)/J)$ for some $m$. The claims about $H_c^{h^2}(\mathbf{G}_h;\pi_t(E_h)/p)$ then follows by passing to the colimit. □

It follows that we can prove (2.28) and (2.32) by showing elements in Families II and III do not contribute to $H_c^{h^2}(\mathbf{G}_h;\pi_0(E_h)/J)$ and $H_0(\mathbf{G}_h;\pi_{2p-2}(E_h)/J)$ for any open invariant ideal $J$ containing $p$.

Now suppose an element $\frac{1}{pv_1^{d_1}\cdots v_{h-1}^{d_{h-1}}}$ in **Family II** determines a non-zero element in $H_c^{h^2}(\mathbf{G}_h;\pi_t(E_h)/J)$, where $v_h^{p^N}$ is invariant modulo $J$. Then we have

$$\begin{aligned}-\sum_{i=1}^{h-1}d_i|v_i|&\equiv 2h-\frac{p^N|v_h|}{p-1}-t && \mod p^N|v_h|\\ \Longrightarrow\ t&\equiv 2h+\sum_{i=1}^{h-1}d_i|v_i|-\frac{p^N|v_h|}{p-1} && \mod p^N|v_h|.\end{aligned}$$

To estimate the bounds for $t$, we use the following lemma.

**Lemma 3.8** (Baird, [26, Lemma 7.6])**.** *Let $s_1,\dots,s_h$ be a sequence of positive integers, and let $p^{e_i}$ be the largest power of $p$ dividing $s_i$. Then the sequence*

$$p,v_1^{s_1},\dots,v_n^{s_n}$$

*is an invariant ideal if and only if $s_i\le p^{e_{i+1}}$ for $1\le i<n$.*

In our case $s_h=p^N$, so the largest possible values of $d_i$ is when $d_1=d_2=\cdots=d_{h-1}=p^N$. The smallest possible value is when all the $d_i$'s are 1. From this we get:

$$-\frac{(p^N-1)|v_h|}{p-1}\le t\le 2h(1-p^N)\mod p^N|v_h|. \tag{3.9}$$

Thus we have proved the following result:

**Proposition 3.10.** *Elements in Family II contribute to $H_c^{h^2}(\mathbf{G}_h;\pi_t(E_h)/J)$ via Gross-Hopkins duality Proposition 2.27 and the change-of-rings Theorem 3.1 only when $t$ satisfies* (3.9)*, where $v_h^{p^N}$ is invariant modulo $J$.*

**Corollary 3.11.** *Elements in Family II do not contribute to $H_c^{h^2}(\mathbf{G}_h;\pi_0(E_h)/p)$ or $H_c^{h^2}(\mathbf{G}_h;\pi_{2p-2}(E_h)/p)$.*

*Proof.* This is because the residue class of $t=0$ or $2p-2$ never falls into the bounds in (3.9). □

Now it remains to analyze elements in **Family III**. When $h=2$, this was computed by Miller-Ravenel-Wilson in [26]. In the next subsection, we will study the implications of their computations. Nevertheless, we can get some general bounds for the $d_i$'s that would imply the RHVC and vanishing of $\kappa_h$ when $2p-1=h^2$.

**Proposition 3.12.**

*(1) Elements in Family III do not contribute through Gross-Hopkins duality and the change-of-rings theorem to $H_c^{h^2}(\mathbf{G}_h;\pi_0(E_h)/p)$ if for all invariant ideals of the form $J=(p,v_1^{d_1},\cdots,v_{h-1}^{d_{h-1}},y_{h,N}^s)$, we have*

$$\sum_{i=1}^{h-1} d_i|v_i| < \frac{p^N|v_h|}{p-1}-2h. \tag{3.13}$$

*(2) Similarly, these elements do not contribute through Gross-Hopkins duality and the change-of-rings theorem to $H_c^{h^2}(\mathbf{G}_h;\pi_{2p-2}(E_h)/p)$ if for all invariant ideals of the form $(p,v_1^{d_1},\cdots,v_{h-1}^{d_{h-1}},y_{h,N}^s)$, we have*

$$\sum_{i=1}^{h-1} d_i|v_i| < \frac{p^N|v_h|}{p-1}-2h+2p-2. \tag{3.14}$$

*Proof.* Similar to the Family II cases, suppose an element $\frac{y_{h,N}^s}{pv_1^{d_1}\cdots v_{h-1}^{d_{h-1}}}$ in Family III corresponds to non-zero element in $H_c^{h^2}(\mathbf{G}_h;\pi_t(E_h)/J)$, where $v_h^{p^N}$ is invariant modulo $J$. Then we have

$$\begin{aligned} s|y_{h,N}|-\sum_{i=1}^{h-1} d_i|v_i| &\equiv 2h-\frac{p^N|v_h|}{p-1}-t && \mod p^N|v_h| \\ \implies t &\equiv 2h+\sum_{i=1}^{h-1} d_i|v_i|-\frac{p^N|v_h|}{p-1} && \mod p^N|v_h|. \end{aligned}$$

We want to show $t$ cannot be congruent to 0 or $2p-2$ from this residue equation. Similar to the Family II case, we have $d_i\geq 1$. From this, we get the same lower bound for $t$ as in (3.9):

$$t\geq 2h+\sum_{i=1}^{h-1}|v_i|-\frac{p^N|v_h|}{p-1}=\frac{(1-p^N)|v_h|}{p-1}.$$

The right hand side of this inequality is greater than both $-p^N|v_h|$ and $-p^N|v_h|+2p-2$. The bounds (3.13) imply $t<0$ in the residue equation. The lower and upper bounds together show that $t\not\equiv 0$ in the residue equation. Similarly, we can show the other bound (3.14) implies $t\not\equiv 2p-2$ in the residue equation. ☐

The analysis above yields:

**Proposition 3.15.**

*(1) Suppose $p-1\nmid h$. If the bounds* (3.13) *hold, then the RHVC is true.*
*(2) Suppose $2p-1=h^2$. If the bounds* (3.14) *hold, then $\kappa_h=0$. In particular, the first bounds* (3.13) *imply both the RHVC and $\kappa_h=0$ in this case.*

*Proof.* In Proposition 2.27, we showed there is an isomorphism of groups using the duality theorems:

$$H_c^{h^2}(\mathbf{G}_h;\pi_t(E_h)/p)\cong \operatorname*{colim}_{p\in J\trianglelefteq\pi_0(E_h)} H_c^0\left(\mathbf{G}_h;\pi_{2h-t-\frac{p^N|v_h|}{p-1}}(E_h)\Big/ J\right)^\vee,$$

where $J\trianglelefteq\pi_0(E_h)$ ranges through all open invariant ideals containing $p$ and $v_h^{p^N}$ is invariant mod $J$. Recall:

(1) Combining the Poincaré duality between homology and cohomology (2.14) and the isomorphism above, we proved in (2.32) the RHVC reduces to the computation:

$$H_c^0\left(\mathbf{G}_h;\pi_{2h-\frac{p^N|v_h|}{p-1}}(E_h)\Big/ J\right)=\mathbf{F}_p.$$

(2) By Proposition 1.20, $\kappa_h$ injects into $H_c^{h^2}(\mathbf{G}_h;\pi_{2p-2}(E_h))$ when $2p-1=h^2$. The latter is isomorphic to $H_c^{h^2}(\mathbf{G}_h;\pi_{2p-2}(E_h)/p)$ by Proposition 2.3. In (2.32), we concluded the vanishing of $\kappa_h$ would follow from

$$H_c^0\left(\mathbf{G}_h;\pi_{2h-(2p-2)-\frac{p^N|v_h|}{p-1}}(E_h)\Big/ J\right)=0.$$

By the Change-of-Rings Theorem 3.1, the two degree-zero cohomology groups are identified with Ext-groups of $BP_*BP$-comodule $BP_*/J'$ in the corresponding internal degrees. They can be further viewed as a subgroups of $H^{0,*}(M_1^{h-1})$. So we need to show

$$H^{0,*}(M_1^{h-1})=\begin{cases}\mathbf{F}_p & *=2h-\frac{p^N|v_h|}{p-1}, \quad \text{for the RHVC;}\\ 0 & *=2h-(2p-2)-\frac{p^N|v_h|}{p-1}, \quad \text{for } \kappa_h=0.\end{cases}$$

By Proposition 3.3, elements in $H^{0,*}(M_1^{h-1})$ are classified into three families:

- Proposition 3.6 says elements in Family I contribute a copy of $\mathbf{F}_p$ to $H^{0,*}(M_1^{h-1})$ when $*=2h-\frac{p^N|v_h|}{p-1}$. They have no contribution when $*=2h-(2p-2)-\frac{p^N|v_h|}{p-1}$.
- Corollary 3.11 shows elements in Family II do not contribute to $H^{0,*}(M_1^{h-1})$ when $*=2h-\frac{p^N|v_h|}{p-1}$ or $2h-(2p-2)-\frac{p^N|v_h|}{p-1}$.
- The two bounds (3.13) and (3.14) in Proposition 3.12 would respectively imply Family III elements do not contribute to $H^{0,*}(M_1^{h-1})$ when $*=2h-\frac{p^N|v_h|}{p-1}$ or $2h-(2p-2)-\frac{p^N|v_h|}{p-1}$.

Combining the three families above, we conclude the two bounds (3.13) and (3.14) in Proposition 3.12 would respectively imply

$$H^{0,2h-\frac{p^N|v_h|}{p-1}}(M_1^{h-1})=\mathbf{F}_p \implies \text{RHVC},$$
$$H^{0,2h-(2p-2)-\frac{p^N|v_h|}{p-1}}(M_1^{h-1})=0 \implies \kappa_h=0.$$

As the first bound (3.13) is stronger than the second (3.14), it would imply both the RHVC and $\kappa_h=0$ when $2p-1=h^2$. $\square$

**Remark 3.16.** *Baird's Lemma 3.8 implies that elements in $H^{0,*}(M_1^{h-1})$ with numerator $v_h^{sp^N}$ for some $N\geq 1$ and $(s,p)=1$ must be of the form:*

$$\frac{v_h^{spN}}{pv_1^{s_1}\cdots v_{h-1}^{s_{h-1}}},$$

*such that the sequence $(s_1,\cdots,s_{h-1},sp^N)$ satisfies $s_i\leq p^{v_p(s_{i+1})}$. It follows that the largest values of the $s_i$'s are $s_1=s_2=\cdots=s_{h-1}=p^N$. One can then check that*

$$\sum_{i=1}^{h-1}s_i|v_i|=p^N\sum_{i=1}^{h-1}|v_i|=p^N\left(\frac{2(p^h-1)}{p-1}-2h\right)=\frac{p^N|v_h|}{p-1}-p^N\cdot 2h$$

*This is strictly smaller than both bounds* (3.13) *and* (3.14) *since $N\geq 1$. As is explained in Remark 3.4, we can add correcting terms in lower Bockstein filtrations to $v_h^{sp^N}$ to increase their $v_i$-divisibility for $1\leq i\leq h-1$. This is why we cannot deduce from Baird's Lemma 3.8 that the bounds* (3.13) *and* (3.14) *are always satisfied* .

3.3. **Consequences of the Miller-Ravenel-Wilson computation.** Recall that $M^1_{h-1}$ is defined to be $v_h^{-1}BP_*/(p, v_1, \cdots, v_{h-2}, v_{h-1}^\infty)$. In this subsection, we discuss some consequences of the computations of $H^0(M^1_{h-1})$ in [26] on the RHVC when $(p-1)\nmid h$ and the exotic Picard groups when $2p-1=h^2$. The computations at height 2 are given by:

**Theorem 3.17** (Miller-Ravenel-Wilson, [26, Theorem 5.3])**.**

$$H^{0,*}(M^1_1)\cong \mathbf{F}_p\left\{\frac{v_2^s}{pv_1}\,\middle|\, s\in\mathbf{Z}, p\nmid s\right\}\bigoplus \mathbf{F}_p\left\{\frac{1}{pv_1^j}\,\middle|\, j\geq 1\right\}$$
$$\bigoplus \mathbf{F}_p\left\{\frac{x_N^s}{pv_1^{e_1}}\,\middle|\, N\geq 1, s\in\mathbf{Z}, p\nmid s, 1\leq e_1\leq p^N+p^{N-1}-1\right\},$$

*where $x_N$ is defined inductively by*

$$\begin{aligned}
x_0 &= v_2,\\
x_1 &= x_0^p - v_1^p v_2^{-1} v_3,\\
x_2 &= x_1^p - v_1^{p^2-1} v_2^{(p-1)p+1} - v_1^{p^2+p-1} v_2^{p^2-2p} v_3,\\
x_N &= x_{N-1}^p - 2v_1^{(p+1)(p^{N-1}-1)} v_2^{(p-1)(p^{N-1}+1)}, \qquad N\geq 3.
\end{aligned}$$

*The internal degree of $x_N^s$ is $sp^N|v_2|-e_1|v_1|$.*

Using Gross-Hopkins duality Proposition 2.27, the results above imply the top degree cohomology groups of $\mathbf{G}_2$ with coefficients in $\pi_t(E_2)/p$ are:

**Proposition 3.18.** *Let $[\alpha]\in H^4_c(\mathbf{G}_2;\pi_t(E_2)/p)$ be a non-zero cohomology class. If $[\alpha]$ corresponds to an element $\frac{x_N^s}{pv_1^{e_1}}\in H^{0,*}(M^1_1)$ for some $N\geq 1$ via the Gross-Hopkins duality, then*

$$t\equiv -\frac{(p^N-1)|v_2|}{p-1}+(e_1-1)|v_1| \mod p^N|v_2|.$$

*Proof.* By assumption, the element $\frac{x_N^s}{pv_1^{e_1}}$ is in the image of $H^{0,sp^N|v_2|-e_1|v_1|}(M_1^1/J)$ for some $J$ containing $p$ where $BP_*/J$ has a $v_2^{p^N}$-self map. The Poincaré duality (2.14) gives an isomorphism:

$$H^4_c(\mathbf{G}_2;\pi_t(E_2)/p)\cong H^0_c(\mathbf{G}_2;\pi_{4-t}(E_2)\langle\det\rangle/(p,u_1^\infty))^\vee.$$

By Theorem 2.25, the determinant twist mod $J$ is identified with:

$$\pi_{4-t}(E_2)\langle\det\rangle/J=\pi_{4-t}\left(\Sigma^{\frac{p^N|v_2|}{p-1}}E_2\right)\Big/ J=\pi_{4-t-\frac{p^N|v_2|}{p-1}}(E_2)/J.$$

The claim now follows by solving for $t$ in the residue equation:

$$4-t-\frac{p^N|v_2|}{p-1}\equiv sp^N|v_2|-e_1|v_1| \mod p^N|v_2|. \qquad\square$$

In this way, we have recovered the patterns of the top-degree cohomology $H^4_c(\mathbf{G}_2,\pi_t(E_2)/p)$ in the computation by Behrens in [7, Figure 3.2] when $p\geq 5$.

**Corollary 3.19.** *$H^4_c(\mathbf{G}_2;\pi_t(E_2)/p)\neq 0$ iff either $|v_2|$ divides $t$, or $|v_1|$ divides $t$ and there is an $N\geq 1$ such that*

$$-\frac{(p^N-1)|v_2|}{p-1}\leq t\leq -\frac{(p^N-1)|v_2|}{p-1}+|v_1|(p^N+p^{N-1}-2) \qquad \bmod p^N|v_2|$$
$$= -2p^N-2p^{N-1}-2p+6 \qquad \bmod p^N|v_2|.$$

*Proof.* In degrees divisible by $|v_2|$, we have elements corresponding to $\frac{v_2^s}{pv_1}$. When $|v_2|\nmid t$, this follows from Proposition 3.18 and the bounds for $e_1$ in Theorem 3.17: $1\leq e_1\leq p^N+p^{N-1}-1$. □

We have therefore recovered the following result of Shimomura and Yabe in [32]:

**Corollary 3.20.** *The RHVC holds and $H^4_c(\mathbf{G}_2;\pi_{2p-2}(E_2))=0$ when $h=2$ and $p\geq 5$.*

**Remark 3.21.** *Shimomura and Yabe proved the cohomological version of Conjecture 2.29 at $h=2$ and $p\geq 5$, which is equivalent to the homological version by Poincaré duality Corollary 2.10.*

*Proof.* When $|v_2|\nmid t$, the upper bounds for $t$ above are always negative, which implies when $p\geq 5$

$$H_0(\mathbf{G}_2;\pi_0(E_2)/p)\cong H^4_c(\mathbf{G}_2;\pi_0(E_2)/p)=\mathbf{F}_p,$$
$$H_0(\mathbf{G}_2;\pi_{2p-2}(E_2))\cong H^4_c(\mathbf{G}_2;\pi_{2p-2}(E_2))\cong H^4_c(\mathbf{G}_2;\pi_{2p-2}(E_2)/p)=0.$$

We have therefore verified (RHVC) and the vanishing of the top degree cohomology group $H^4_c(\mathbf{G}_2;\pi_{2p-2}(E_2))$. □

At height $h\geq 3$, $H^0(M^1_{h-1})$ is described as follows:

**Theorem 3.22** (Miller-Ravenel-Wilson, [26, Theorem 5.10])**.** *Define $a_{h,N}$ by the recursive formula: $a_{h,0}=1$, $a_{h,1}=p$, and*

$$a_{h,N}=\begin{cases} pa_{h,N-1}, & 1<N\not\equiv 1 \mod (h-1);\\ pa_{h,N-1}+p-1, & 1<N\equiv 1 \mod (h-1).\end{cases}$$

*Recall $M^1_{h-1}=v_h^{-1}BP_*/(p,v_1,\cdots,v_{h-2},v_{h-1}^\infty)$. Then $H^0(M^1_{h-1})$ is an $\mathbf{F}_p$-vector space generated by*

I. $\frac{v_h^s}{pv_1\cdots v_{h-1}}$, *where $p\nmid s\in\mathbf{Z}$.*

II. $\frac{1}{pv_1\cdots v_{h-2}v_{h-1}^j}$, *where $j\geq 1$.*

III. $\frac{x_{h,N}^s}{pv_1\cdots v_{h-2}v_{h-1}^{e_{h-1}}}$, *where $p\nmid s\in\mathbf{Z}$, $1\leq e_{h-1}\leq a_{h,N}$, and $x_{h,N}$ is is defined inductively by*

$$x_{h,0}=v_p,$$
$$x_{h,1}=v_h^p-v_{h-1}^pv_h^{-1}v_{h+1},$$
$$x_{h,N}=x_{h,N-1}^p \qquad \text{for } 1<N\not\equiv 1 \mod (h-1),$$
$$x_{h,N}=x_{h,N-1}^p-v_{h-1}^{\frac{(p^{N-1}-1)(p^h-1)}{p^{h-1}-1}}v_h^{p^N-p^{N-1}+1} \qquad \text{for } 1<N\equiv 1 \mod (h-1).$$

**Lemma 3.23.** *The closed formula of $a_{h,N}$ is given by:*

$$a_{h,N}=p^N+\frac{(p-1)(p^{N-1}-p^{r-1})}{p^{h-1}-1},$$

*where $1\leq r\leq h-1$ is an integer such that $N\equiv r \mod (h-1)$.* [3]

[3] $r$ is *not* the usual residue of $N$ mod $h-1$ since $r=h-1$ when $(h-1)\mid N$.

Like Corollary 3.19, we now have:

**Proposition 3.24.** *Assume $(p-1)\nmid h$ and let $I_{h-1}=(p,u_1,\cdots,u_{h-2})\trianglelefteq\pi_0(E_h)$. Then the cohomology group $H_c^{h^2}(\mathbf{G}_h;\pi_t(E_h)/I_{h-1})$ is zero unless $|v_h|$ divides $t$, or there is an $N\geq 1$ such that*

$$t\equiv -\frac{(p^N-1)|v_h|}{p-1}+k\cdot|v_{h-1}|\mod p^N|v_h| \text{ for some } 0\leq k\leq a_{h,N}-1.$$

In particular, the closed formula for $a_{h,N}$ in Lemma 3.23 implies the upper bounds for $t$ above are always negative. Like the $h=2$ and $p\geq 5$ case in Corollary 3.19, this shows that when $(p-1)\nmid h$:

$$\begin{aligned} H_c^{h^2}(\mathbf{G}_h;\pi_0(E_h)/I_{h-1})&=\mathbf{F}_p,\\ H_c^{h^2}(\mathbf{G}_h;\pi_{2p-2}(E_h)/I_{h-1})&=0. \end{aligned} \tag{3.25}$$

**Theorem 3.26** (Main Theorem B)**.** *When $(p-1)\nmid h$, the Homological Vanishing Conjecture is true modulo the ideal $I_{h-1}=(p,u_1,\cdots,u_{h-2})$.*

3.4. **Conclusions at small heights and primes.** Recall that by Theorem 1.24, there is an isomorphism when $2p-1=h^2$:

$$\kappa_h \xrightarrow[(1.20)]{\sim} H_c^{2p-1}(\mathbf{G}_h;\pi_{2p-2}(E_h)) \xrightarrow[(2.3)]{\sim} H_c^{h^2}(\mathbf{G}_h;\pi_{2p-2}(E_h)/p).$$

At $p=5$ and $h=3$, to use our method to compute $H_c^9(\mathbf{G}_3;\pi_8(E_3)/5)$, we need to know $H^{0,*}(M_1^2)$ at prime $p=5$. It is also needed to verify the RHVC at height $h=3$ and $p>2$ (which implies $(p-1)\nmid h$). This computation also appears in Yexin Qu's thesis [29]. By Proposition 3.15, we need to check that for each $1\leq e_2\leq a_{3,N}$, if there is element $\frac{y_N}{pv_1^{e_1}v_2^{e_2}}\in H^0(M_1^2)$, then

$$e_1\cdot|v_1|+e_2\cdot|v_2|<\frac{p^N|v_3|}{p-1}-2\cdot 3.$$

When $e_2=1$, we have $e_1<\frac{p^N(p^2+p+1)-3}{p-1}-(p+1)$. When $e_2$ attains its maximum $a_{3,N}$ in Theorem 3.22, this translates to

$$e_1<\frac{p^{N-1}(p^2+p+1)-3}{p-1}+p^{r-1},\quad r=\begin{cases}1, & N\text{ is odd;}\\ 2, & N\text{ is even.}\end{cases}$$

We observe that both bounds are larger (looser) than the bounds $a_{3,N}$ for $v_2$-divisibility itself. However, it is not clear how to verify them without computing the Greek letter elements in $H^0(M_1^2)$. Nevertheless, the vanishing result in (3.25) does have concrete implications on exotic elements in $\mathrm{Pic}_{K(h)}$ when $2p-1=h^2$, provided the relevant Smith-Toda complexes exist.

**Theorem 3.27** (Main Theorem A)**.** *Let $2p-1=h^2$. Suppose the type-$(h-1)$ Smith-Toda complex $V(h-2)=S^0/(p,v_1,\cdots,v_{h-2})$ exists at prime $p$. Then an exotic element $X\in\kappa_h$ cannot be detected by $V(h-2)$; that is,*

$$X\wedge_{K(h)}V(h-2)\simeq L_{K(h)}V(h-2).$$

*Proof.* Using the topology of $\mathrm{Pic}_{K(h)}$ described in [20, Proposition 14.3.(d)], we know that if the image of $X\in\kappa_h$ under the composite

$$\kappa_3\xrightarrow{\mathrm{ev}_2} H_c^{h^2}(\mathbf{G}_h;\pi_{2p-2}(E_h))\twoheadrightarrow H_c^{h^2}(\mathbf{G}_3;\pi_{2p-2}(E_h)/I_{h-1})$$

is zero, then $X\wedge_{K(h)}V(h-2)=L_{K(h)}V(h-2)$, provided $V(h-2)=S^0/(p,v_1,\cdots,v_{h-2})$ exists. Since the target of this map is zero by (3.25), the equivalence above is true for any $X\in\kappa_h$ when $2p-1=h^2$. □

**Corollary 3.28.**

*(1) At height 3 and prime 5, an exotic element $X$ in $\mathrm{Pic}_{K(3)}$ cannot be detected by $V(1) = S^0/(5, v_1)$.*
*(2) At height 5 and prime 13, an exotic element $X$ in $\mathrm{Pic}_{K(5)}$ cannot be detected by $V(3) = S^0/(13, v_1, v_2, v_3)$.*

*Proof.* The Smith-Toda complexes $V(1)$ and $V(3)$ have been constructed for $p \geq 3$ and $p \geq 7$ by Adams-Toda and Smith-Toda, respectively [30, Example 2.4.1]. □

**Remark 3.29.** *A referee has pointed out to us that it is an open question whether $V(4)$ exists any any prime (see discussions at the end of [31, §5.6]). Recall that Smith-Toda complexes $V(n)$ are constructed as cofibers of $v_n$-self maps of $V(n-1)$ that induce multiplication by $v_n$ on $BP$-homology groups. This means that we do not know the existence of $V(n)$ for $n \geq 4$ at any prime $p$. As a result, it is unclear whether we have a similar statement at the next pair of height and prime $(h,p) = (9, 41)$ satisfying $2p - 1 = h^2$, which would require the existence of $V(7)$ at the prime $p = 41$.*

*In [27], Nave proved the non-existence of the Smith-Toda complex $V(h)$ when $2h = p + 1$. This does not overlap with our consideration of the potential Smith-Toda complexes $V(h-2)$ when $h^2 = 2p - 1$.*

**Remark 3.30.** *By [20, Corollary 7.11], a $K(h)$-local spectrum $X$ is equivalent to $L_{K(h)}S^0$ iff $X \wedge_{K(h)} V \simeq L_{K(h)}V$ for all finite complexes of type $h$. This means if $X \wedge_{K(h)} V \simeq L_{K(h)}V$ for all $X \in \kappa_h$ and finite complexes $V$ of type $n$, then $\kappa_h = 0$. Theorem 3.27 can be thought of as a first step towards showing $\kappa_h = 0$ when $2p-1 = h^2$, since it implies $X \wedge_{K(h)} V \simeq L_{K(h)}V$ for any cofibers $V$ of $v_h$-self maps of $V(h-2)$. Our choices of finite complexes are restricted to cofibers of the Smith-Toda complexes $V(h-2)$, because we do not have better Greek letter element computation results beyond Theorem 3.22 in [26] when $h \geq 3$.*

We can also use the same technique to study the subgroup $\kappa_h^{(1)}$ of $\kappa_h$ when $4p - 3 = h^2$. Recall from (1.22), $\kappa_h^{(1)}$ is the kernel of detection map

$$\mathrm{ev}_2\colon \kappa_h \longrightarrow H_c^{2p-1}(\mathbf{G}_h; \pi_{2p-2}(E_h)).$$

In terms of the homotopy fixed point spectral sequence, it consists of exotic $K(h)$-local spheres $X$, such that $E_2^{0,0}(X) \cong \mathbf{Z}_p$ does not support a $d_{2p-1}$-differential. Using similar argument as in Proposition 1.20, one can show that the detection map:

$$\mathrm{ev}_3\colon \kappa_h^{(1)} \longrightarrow E_{2p}^{4p-3,4p-4}$$

injective because the target of the next detection map is above the horizontal vanishing line at $s = h^2 = 4p-3$ of the $E_2$-page. The target of this detection map is a subquotient of

$$E_2^{4p-3,4p-4} = H_c^{4p-3}(\mathbf{G}_h; \pi_{4p-4}(E_h)) = H_c^{h^2}(\mathbf{G}_h; \pi_{4p-4}(E_h)).$$

By Proposition 3.24, we know $H_c^{h^2}(\mathbf{G}_h; \pi_{4p-4}(E_h)/I_{h-1}) = 0$ when $(p-1) \nmid h$. This implies:

**Theorem 3.31.** *Let $X$ be an exotic element in $\mathrm{Pic}_{K(h)}$ where $h$ and $p$ satisfies $4p - 3 = h^2$. Suppose the Smith-Toda complex $V(h-2)$ exists. If $X \in \ker \mathrm{ev}_2$, i.e. the $E_2^{0,0}(X)$-term in the HFPSS (1.13) does not support a $d_{2p-1}$-differential, then $X \wedge_{K(h)} V(h-2) \simeq L_{K(h)}V(h-2)$. In particular, this is true when $(h,p) = (3,3)$ and $(h,p) = (5,7)$.*

We end this paper with a discussion on the relation between the RHVC and exotic Picard groups.

**Theorem 3.32** (Main Theorem C)**.** *At height 3, the RHVC implies $\kappa_3 = 0$ when $p = 5$ and $\kappa_3^{(1)} = 0$ when $p = 3$.*

*Proof.* We will prove the contra-positive statement at $p = 5$ first. Suppose $\kappa_3 \neq 0$ at $p = 5$. By Proposition 1.20 and Proposition 2.3, we know $H_c^9(\mathbf{G}_3; \pi_8(E_3)/5) \neq 0$. Let $x$ be a nonzero element in this group. Under the isomorphism in Proposition 2.27, $x$ corresponds to a family of non-zero elements (2.28)

$$\xi_J \in H_c^0\left(\mathbf{G}_3; \pi_{2\cdot 3-(2\cdot 5-2)-\frac{5^N(2\cdot 5^3-2)}{5-1}}(E_3)\Big/ J\right)$$

for cofinal system of open invariant ideals $J$ in $\pi_0(E_3)$ that contains 5. By Proposition 3.24:

$$H_c^0\left(\mathbf{G}_3;\pi_{2\cdot 3-(2\cdot 5-2)-\frac{5^N(2\cdot 5^3-2)}{5-1}}(E_3)\Big/(5,v_1,v_2^\infty)\right)=0,$$

which implies the element $\xi_J$ cannot be $v_1$-torsion. By Proposition 3.6 and Corollary 3.11, the $\xi_J$'s are necessarily Family III Greek letter elements in Proposition 3.3. As result, we obtain a compatible family of non-zero Family-III elements

$$\xi_J'=v_1\alpha_J\in H_c^0\left(\mathbf{G}_3;\pi_{2\cdot 3-\frac{5^N(2\cdot 5^3-2)}{5-1}}(E_3)\Big/J\right).$$

Again by Proposition 2.27, $\xi_J'$ corresponds a non-zero element $x'\in H_c^9(\mathbf{G}_3;\pi_0(E_3)/5)$. Recall from Proposition 3.6, this group already has a copy of $\mathbf{F}_5$ coming from Family I elements through Gross-Hopkins duality. The new addition of $x'$ in this group from Family III elements shows that its dimension is at least 2, which contradicts the RHVC.

At $p=3$, we know $\kappa_3^{(1)}$ injects into the $E_{2p}^{4p-3,4p-4}$-term in the HFPSS for the $K(3)$-local sphere. If $\kappa_3^{(1)}\neq 0$, then neither is $E_{2p}^{4p-3,4p-4}=E_6^{9,8}$. This implies $E_2^{9,8}=H_c^9(\mathbf{G}_3;\pi_8(E_3))\neq 0$, since $E_6^{9,8}\neq 0$ is its subquotient. The rest of the argument is entirely the same as the $p=5$ case.

In this way, we conclude $\kappa_3\neq 0$ at $p=5$ and $\kappa_3^{(1)}=0$ at $p=3$ implies the RHVC is false at the respective primes. These are the contra-positive statements of the theorem. □

**Remark 3.33.** *This proof relies on Proposition 3.24, a consequence of the Miller-Ravenel-Wilson computation Theorem 3.22. In general, the implication would hold at height $h$ if we knew*

$$H^{0,2h-(2p-2)-\frac{p^N|v_h|}{p-1}}(M_2^{h-2})=0 \tag{3.34}$$

*for all $N$. Miller-Ravenel-Wilson have calculated $H^{0,*}(M_{h-1}^1)$ for all $h$. To prove (3.34) one would have to calculate $h-3$ many Bockstein spectral sequences, which seems dizzyingly beyond our reach with current technology.*

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