On the evaluation of the alternating multiple t value $t(\overline{1}, \ldots, \overline{1}, 1, \overline{1}, \ldots, \overline{1})$

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Abstract

We prove an evaluation for the stuffle-regularised alternating multiple t value $t^{*,V}(\overline{1}, \ldots, \overline{1}, 1, \overline{1}, \ldots, \overline{1})$ in terms of V, the regularisation parameter, $\log(2)$, $\zeta(k)$ and $\beta(k)$. This arises by evaluating the corresponding generating series using the Evans-Stanton/Ramanujan asymptotics of a zero-balanced hypergeometric function ${}_3F_2$, and an evaluation established by Li in an alternative approach to Zagier's evaluation of $\zeta(2, \ldots, 2, 3, 2, \ldots, 2)$. We end with some discussion and conjectures on possible motivic applications.

Keywords Multiple zeta values · Multiple t values · Alternating MZVs · Motivic MZVs · Special values · Hypergeometric functions

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1 Introduction and statement

The *multiple zeta values* (MZVs) and *multiple t values* (MtVs) with signs $\varepsilon_i \in S^1 = \{z \in \mathbb{C} : |z| = 1\}$, and arguments $k_i \in \mathbb{Z}_{>0}$, such that $(k_d, \varepsilon_d) \neq (1, 1)$ for convergence, are defined by

$$\zeta\begin{pmatrix} \varepsilon_1, \ldots, \varepsilon_d \\ k_1, \ldots, k_d \end{pmatrix} := \sum_{0 < n_1 < \cdots < n_d} \frac{\varepsilon_1^{n_1} \cdots \varepsilon_d^{n_d}}{n_1^{k_1} \cdots n_d^{k_d}},$$

$$t\begin{pmatrix} \varepsilon_1, \ldots, \varepsilon_d \\ k_1, \ldots, k_d \end{pmatrix} := \sum_{0 < n_1 < \cdots < n_d} \frac{\varepsilon_1^{n_1} \cdots \varepsilon_d^{n_d}}{(2n_1 - 1)^{k_1} \cdots (2n_d - 1)^{k_d}}.$$

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Here, d is called the *depth* and $k_1 + \cdots + k_d$ is called the *weight* of the MZV or MtV. If all signs ε_i are (independently) equal to ± 1, we obtain the so-called *alternating* MZVs and MtVs, respectively. It is then common to simplify the notation (see [1, §1]) by combining the signs and arguments into a single string; in position i of the new string, one writes k_i if $\varepsilon_i = 1$ and $\overline{k_i}$ if $\varepsilon_i = -1$. For example

$$\zeta(\overline{k_1}, \overline{k_2}, k_3, k_4, \overline{k_5}) := \zeta\begin{pmatrix} -1, -1, & 1, & 1, & -1 \\ k_1, & k_2, & k_3, & k_4, & k_5 \end{pmatrix}.$$

The related Dirichlet beta function is defined by

$$\beta(s) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}$$

so that $t(\overline{k}) = -\beta(k)$, for $k \in \mathbb{Z}_{>0}$.

MZVs and by extension MtVs and other related objects are of significant interest in number theory (see [7, 12] for the foundational results around MZVs although Euler already studied the case $d \leq 2$, and see [8] for the recent (re-)introduction of MtVs following Nielsen's study of the case $d = 1$). They are also of interest for their applications to high energy physics (see [2] as a starting point). Typically one is interested in understanding identities and relations between MZVs and MtVs, either particular cases or the general structures thereof.

The main theorem of this note is an explicit evaluation and generating series for the stuffle-regularised $t^{*,V}(\{\overline{1}\}^a, 1, \{\overline{1}\}^b)$, where $\{k\}^n$ denotes the string $k, \ldots, k$ with n repetitions of k. Therefore, we briefly recall the idea of stuffle regularisation of MZVs (see [9] for more details) and correspondingly MtVs. The truncated $\zeta_M(1)$ is well known to satisfy

$$\zeta_M(1) := \sum_{n=1}^{M} \frac{1}{n} = \log(M) + \gamma + O\left(\frac{1}{M}\right),$$

where $\gamma = 0.577\ldots$ is the Euler–Mascheroni constant. So by application of the stuffle product (for example

$$\zeta(k_1)\zeta(\ell_1) = \sum_{0<n_1} \sum_{0<m_1} \frac{1}{n_1^{k_1}} \cdot \frac{1}{m_1^{\ell_1}}$$

$$= \left(\sum_{0<n_1<m_1} + \sum_{0<m_1<n_1} + \sum_{0<n_1=m_1} \right) \frac{1}{n_1^{k_1} m_1^{\ell_1}}$$

$$= \zeta(k_1, \ell_1) + \zeta(\ell_1, k_1) + \zeta(\ell_1 + k_1),$$

in the case of two single-zeta values; when signs are present, they will be multiplied in the $n_1 = m_1$ term), one has by induction that any truncated MZV satisfies

$$\zeta_M(k_1, \ldots, k_d) := \sum_{0 < n_1 < \cdots < n_d < M} \frac{1}{n_1^{k_1} \cdots n_d^{k_d}}$$

$$= Z^*(k_1, \ldots, k_d; \log(M) + \gamma) + O\left(\frac{\log^J M}{M}\right),$$

for some polynomial $Z^*(k_1, \ldots, k_d; U)$ with convergent MZV coefficients, and some J. This polynomial defines the regularised version of $\zeta^{*,U}(k_1, \ldots, k_d)$, with parameter $\zeta^{*,U}(1) = U$; in particular $\zeta^{*,U}(k_1, \ldots, k_d) = \zeta(k_1, \ldots, k_d)$, for $k_d \neq 1$. Essentially, one can formally extend the stuffle product to allow trailing 1's, then by considering, for $k_d \neq 1$, the difference

$$\zeta^{*,U}(k_1, \ldots, k_d, \{1\}^\alpha) - \frac{1}{\alpha}\zeta^{*,U}(k_1, \ldots, k_d, \{1\}^{\alpha-1})\zeta^{*,U}(1),$$

one obtains an expression with strictly fewer trailing 1's. This writes every (alternating) MZV as a polynomial in $\zeta^{*,U}(1) := U$ with convergent (alternating) MZV coefficients. The same process works for (alternating) MtVs, in particular, we have

$$t^{*,V}(\{\overline{1}\}^m, 1) = t^{*,V}(\{\overline{1}\}^m)t^{*,V}(1) - \sum_{i=0}^{m-1} t^{*,V}(\{\overline{1}\}^i, 1, \{\overline{1}\}^{m-i})$$

$$- \sum_{i=0}^{m-1} t^{*,V}(\{\overline{1}\}^i, \overline{2}, \{\overline{1}\}^{m-1-i})$$

$$= t(\{\overline{1}\}^m)V - \sum_{i=0}^{m-1} t(\{\overline{1}\}^i, 1, \{\overline{1}\}^{m-i}) - \sum_{i=0}^{m-1} t(\{\overline{1}\}^i, \overline{2}, \{\overline{1}\}^{m-1-i}). \quad (1)$$

The main theorem of this note is now as follows.

Theorem 1.1 *Consider the following generating series of the multiple t values $t^{*,V}(\{\overline{1}\}^a, 1, \{\overline{1}\}^b)$, with stuffle regularisation $t^{*,V}(1) = V$ if necessary,*

$$F^V(x, y) = \sum_{a,b \geq 0} (-1)^{a+b} t^{*,V}(\{\overline{1}\}^a, 1, \{\overline{1}\}^b) x^a y^b.$$

Then the generating series has the following closed form expression

$$F^V(x, y) = \frac{1}{2}\left(\cos\left(\frac{\pi x}{4}\right) + \sin\left(\frac{\pi x}{4}\right)\right)$$

$$\cdot \left(A\left(\frac{x - y}{4}\right) - A\left(\frac{x + y}{4}\right) + 2A\left(\frac{x + y}{2}\right) - \log(2) + 2V\right)$$

$$+ \frac{1}{2}\left(\cos\left(\frac{\pi y}{4}\right) + \sin\left(\frac{\pi y}{4}\right)\right)$$

$$\cdot \left(-A\left(\frac{x - y}{8}\right) + A\left(\frac{x - y}{4}\right) - 2C\left(\frac{x + y}{2}\right) + \log(2)\right),$$

where

$$A(z) = \psi(1) - \tfrac{1}{2}(\psi(1+z) + \psi(1-z)) = \sum_{r=1}^{\infty} \zeta(2r+1)z^{2r},$$

$$C(z) = \tfrac{1}{8}\left(\psi\left(\tfrac{1}{4}+\tfrac{z}{4}\right) - \psi\left(\tfrac{1}{4}-\tfrac{z}{4}\right) - \psi\left(\tfrac{3}{4}+\tfrac{z}{4}\right) + \psi\left(\tfrac{3}{4}-\tfrac{z}{4}\right)\right) = \sum_{r=1}^{\infty} \beta(2r)z^{2r-1},$$

with $\psi(x) = \frac{d}{dx}\log\Gamma(x)$ the digamma function. In particular, each $t^{*,V}(\{\overline{1}\}^a, 1, \{\overline{1}\}^b)$ is a polynomial in V, Riemann zeta values $\zeta(k)$, $\log(2)$ and Dirichlet beta values $\beta(k)$.

Recall, $\{k\}^n$ denotes the string $k, \ldots, k$ with n repetitions of k. Then following evaluation follows directly from the generating series; using $\beta(1) = \frac{\pi}{4}$ this verifies the claimed polynomial form of the reduction. (Note that $F^V(x, y)$ is the generating series for $(-1)^{a+b}t^{*,V}(\{\overline{1}\}^a, 1, \{\overline{1}\}^b)$, so after extracting the relevant coefficient, we must still multiply by $(-1)^{a+b}$ to remove the extraneous signs from the MtV.)

$$t^{*,V}(\{\overline{1}\}^a, 1, \{\overline{1}\}^b)$$

$$= \sum_{\substack{r=2 \\ \text{even}}}^{a+b+1} \frac{(-1)^{\lfloor(a+b-r-1)/2\rfloor}}{2\,(a+b-r)!}\left(\frac{\pi}{4}\right)^{a+b-r}\binom{r}{b}\left(\frac{1}{4^r} - \frac{(-1)^b}{4^r} - \frac{2}{2^r}\right)\zeta(r+1)$$

$$+ \sum_{\substack{r=2 \\ \text{even}}}^{a+b+1} \frac{(-1)^{\lfloor(a+b-r-1)/2\rfloor}}{2\,(a+b-r)!}\left(\frac{\pi}{4}\right)^{a+b-r}\binom{r}{a}\left(\frac{(-1)^a}{8^r} - \frac{(-1)^a}{4^r}\right)\zeta(r+1)$$

$$- \sum_{\substack{r=1 \\ \text{odd}}}^{a+b+1} \frac{(-1)^{\lfloor(a+b-r-1)/2\rfloor}}{2\,(a+b-r)!}\left(\frac{\pi}{4}\right)^{a+b-r}\binom{r}{a}\left(\frac{2}{2^r}\right)\beta(r+1)$$

$$+ \delta_{a=0}\frac{(-1)^{\lfloor -b/2\rfloor}}{2\cdot b!}\left(\frac{\pi}{4}\right)^b \log(2) + \delta_{b=0}\frac{(-1)^{\lfloor -a/2\rfloor}}{2\cdot a!}\left(\frac{\pi}{4}\right)^a (2V - \log(2)).$$

2 Proof of Theorem 1.1

Firstly, recall that the $_pF_{p-1}$ hypergeometric function is defined as follows:

$$_pF_{p-1}\left[\begin{matrix} a_1, & \ldots, & a_p \\ b_1, & \ldots, & b_{p-1} \end{matrix}; x\right] := \sum_{m=0}^{\infty} \frac{(a_1)_m \cdots (a_p)_m}{(b_1)_m \cdots (b_{p-1})_m} \frac{x^m}{m!},$$

where $(a)_m = a(a+1)\cdots(a+m-1)$ is the ascending Pochhammer symbol. Asymptotic and transformation properties of the $_3F_2$ function will play a key role in the proof of our generating series evaluation.

Introduce the multivariable Ti-function

$$\mathrm{Ti}_{k_1,\ldots,k_d}(z_1,\ldots,z_d) := \sum_{0<n_1<\cdots<n_d} \frac{z_1^{n_1} \cdots z_d^{n_d}}{(2n_1-1)^{k_1} \cdots (2n_d-1)^{k_d}}.$$

(Note this is slightly different to the version in [4]; here the arguments in the numerator have exponent n_i, which is more suited to the case of alternating t values.) Then for $(k_d, \varepsilon_d) \neq (1,1)$, we find $\mathrm{Ti}_{k_1,\ldots,k_d}(\varepsilon_1,\ldots,\varepsilon_d)$ evaluates to the alternating MtV with arguments $k_1,\ldots,k_d$ and signs $\varepsilon_1,\ldots,\varepsilon_d$. We consider the generating series of $\mathrm{Ti}_{\{1\}^{n+m+1}}(\{-1\}^m, z, \{-1\}^n)$ and seek to carefully evaluate the limit thereof as $z \to 1^-$, somehow dealing with the divergence caused by $\mathrm{Ti}_{1,\ldots,1}(-1,\ldots,-1,z) \xrightarrow{z\to 1^-} t(\{\bar{1}\}^n, 1)$.

Consider

$$G(x, y; z) := \sum_{a,b\geq 0} (-1)^{a+b} \mathrm{Ti}_{\{1\}^{a+b+1}}(\{-1\}^a, z, \{-1\}^b) x^a y^b$$

$$= \sum_{r=1}^{\infty} \prod_{0<k<r} \left(1 - \frac{(-1)^k x}{2k-1}\right) \frac{z^r}{2r-1} \prod_{\ell>r} \left(1 - \frac{(-1)^\ell y}{2\ell-1}\right).$$

Using standard evaluations, one can show

$$\prod_{k=1}^{\infty} \left(1 - \frac{(-1)^k y}{2k-1}\right) = \cos\left(\frac{\pi y}{4}\right) + \sin\left(\frac{\pi y}{4}\right),$$

and so we can rewrite $G(x, y; z)$ as follows.

$$G(x, y; z) = \left(\cos\left(\frac{\pi y}{4}\right) + \sin\left(\frac{\pi y}{4}\right)\right)$$

$$\cdot \sum_{r=1}^{\infty} \prod_{0<k<r} \left(1 - \frac{(-1)^k x}{2k-1}\right) \frac{z^r}{2r-1} \prod_{0<\ell\leq r} \left(1 - \frac{(-1)^\ell y}{2\ell-1}\right)^{-1}.$$

We split the summation in $G(x, y; z)$ into odd- and even- indexed terms, and sum each separately. We can check the following (where $(x)_k = x(x+1)\cdots(x+k-1)$ is, as already mentioned, the ascending Pochhammer symbol):

$$(r = 2m+2) \quad \prod_{0<k<r} \left(1 - \frac{(-1)^k x}{2k-1}\right) \frac{z^r}{2r-1} \prod_{0<\ell\leq r} \left(1 - \frac{(-1)^\ell y}{2\ell-1}\right)^{-1}$$

$$= \frac{(1+x)z^2}{(3-y)(1+y)} \cdot \frac{(1)_m \left(\frac{3}{4} - \frac{x}{4}\right)_m \left(\frac{5}{4} + \frac{x}{4}\right)_m}{\left(\frac{7}{4} - \frac{y}{4}\right)_m \left(\frac{5}{4} + \frac{y}{4}\right)_m} \frac{(z^2)^m}{m!},$$

$$(r = 2m + 1) \quad \prod_{0 < k < r} \left(1 - \frac{(-1)^k x}{2k - 1}\right) \frac{z^r}{2r - 1} \prod_{0 < \ell \le r} \left(1 - \frac{(-1)^\ell y}{2\ell - 1}\right)^{-1}$$

$$= \frac{z}{1 + y} \cdot \frac{(1)_m \left(\frac{3}{4} - \frac{x}{4}\right)_m \left(\frac{1}{4} + \frac{x}{4}\right)_m}{\left(\frac{3}{4} - \frac{y}{4}\right)_m \left(\frac{5}{4} + \frac{y}{4}\right)_m} \frac{(z^2)^m}{m!}.$$

Both of these are in exactly the right form to sum to a $_3F_2$-hypergeometric function (the summation index in both cases starts at $m = 0$), and so we obtain

$$G(x, y; z) = \left(\cos\left(\frac{\pi y}{4}\right) + \sin\left(\frac{\pi y}{4}\right)\right)$$

$$\cdot \left\{ \frac{(1 + x)z^2}{(3 - y)(1 + y)} \cdot {}_3F_2\left[\begin{matrix} 1, & \frac{3}{4} - \frac{x}{4}, & \frac{5}{4} + \frac{x}{4} \\ & \frac{7}{4} - \frac{y}{4}, & \frac{5}{4} + \frac{y}{4} \end{matrix}; z^2\right] \right.$$

$$\left. + \frac{z}{1 + y} \cdot {}_3F_2\left[\begin{matrix} 1, & \frac{3}{4} - \frac{x}{4}, & \frac{1}{4} + \frac{x}{4} \\ & \frac{3}{4} - \frac{y}{4}, & \frac{5}{4} + \frac{y}{4} \end{matrix}; z^2\right] \right\}.$$

We now consider how the divergence in $\mathrm{Ti}_{\{1\}^{m+1}}(\{-1\}^m, z)$ arises, so we can compensate for it in the generating series. We have by the stuffle multiplication of Ti, that

$$\mathrm{Ti}_{\{1\}^m, 1}(\{-1\}^m, z) = \mathrm{Ti}_{\{1\}^m}(\{-1\}^m)\mathrm{Ti}_1(z)$$

$$- \sum_{j=0}^{m-1} \mathrm{Ti}_{\{1\}^{m+1}}(\{-1\}^j, z, \{-1\}^{m-j})$$

$$- \sum_{j=0}^{m-1} \mathrm{Ti}_{\{1\}^j, 2, \{1\}^{m-1-j}}(\{-1\}^j, -z, \{-1\}^{m-1-j}).$$

On rearranging and taking the generating series of both sides, we find that

$$\sum_{m \ge 0} (-1)^m \mathrm{Ti}_{\{1\}^m, 1}(\{-1\}^m, z)x^m - \mathrm{Ti}_1(z) \sum_{m \ge 0} (-1)^m \mathrm{Ti}_{\{1\}^m}(\{-1\}^m)x^m$$

$$= -\sum_{m \ge 0}(-1)^m \left(\sum_{j=0}^{m-1} \mathrm{Ti}_{\{1\}^{m+1}}(\{-1\}^j, z, \{1\}^{m-j})\right.$$

$$+ \sum_{j=0}^{m-1} \mathrm{Ti}_{\{1\}^j, 2, \{1\}^{m-1-j}}(\{-1\}^j, -z, \{-1\}^{m-1-j})\right)x^m$$

$$\xrightarrow{z \to 1^-} -\sum_{m \ge 0}(-1)^m \left(\sum_{j=0}^{m-1} t(\{\overline{1}\}^j, 1, \{\overline{1}\}^{m-j}) + \sum_{j=0}^{m-1} t(\{\overline{1}\}^j, \overline{2}, \{\overline{1}\}^{m-1-j})\right)x^m$$

$$= \sum_{m \ge 0}(-1)^m t^{*, V=0}(\{\overline{1}\}^m, 1)x^m,$$

as $z \to 1^-$. Here, we have recognised the limiting generating series as that of $t^{*,V=0}(\{\overline{1}\}^m, 1)$, where $t^{*,V=0}$ represents the stuffle regularisation polynomial of $t(\{\overline{1}\}^m, 1)$ evaluated at $V = 0$. (This is the regularisation where we take $t^{*,V=0}(1) = 0$, cf. (1).) We also notice that

$$\mathrm{Ti}_1(z) = \sqrt{z} \tanh^{-1}(\sqrt{z}),$$

and from [8, Corollary 6.1], the generating series of $t(\{\overline{1}\}^m)$ is given by

$$\sum_{m \geq 0} (-1)^m t(\{\overline{1}\}^m) x^m = \cos\left(\frac{\pi x}{4}\right) + \sin\left(\frac{\pi x}{4}\right). \tag{2}$$

We, hence, find the limit of the following combination gives the generating series of $t^{*,V=0}(\{\overline{1}\}^a, 1, \{\overline{1}\}^b)$, with a suitably stuffle-regularised variant $t^{*,V=0}(\{\overline{1}\}^a, 1)$ in the case $b = 0$.

$$\lim_{z \to 1^-} G(x, y; z) - \sqrt{z} \tanh^{-1}(\sqrt{z}) \cdot \left(\cos\left(\frac{\pi x}{4}\right) + \sin\left(\frac{\pi x}{4}\right)\right)$$
$$= \sum_{a \geq 0, b > 0} (-1)^{a+b} t(\{\overline{1}\}^a, 1, \{\overline{1}\}^b) x^a y^b + \sum_{a \geq 0} (-1)^a t^{*,V=0}(\{\overline{1}\}^a, 1) x^a =: F(x, y).$$

We now want to take the limit $\lim_{z \to 1^-} G(x, y; z) - \sqrt{z} \tanh^{-1}(\sqrt{z}) \cdot \left(\cos\left(\frac{\pi x}{4}\right) + \sin\left(\frac{\pi x}{4}\right)\right)$. For this, we recall the following result [5, Theorem 3] (proving a claim of Ramanujan), which treats the asymptotics of the 0-balanced hypergeometric function ${}_3F_2$, which both of the hypergeometric series in $G(x, y; z)$ are.

Theorem 2.1 (Evans-Stanton 1984 [5], Ramanujan) *If $a+b+c = d+e$, and $\mathrm{Re}(c) > 0$, then as $u \to 1^-$,*

$$\frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(d)\Gamma(e)} \cdot {}_3F_2\left[\begin{matrix} a, \ b, \ c \\ d, \ e \end{matrix}; u\right] = -\log(1 - u) + L + O((1 - u)\log(1 - u)),$$

where

$$L = -2\gamma - \psi(a) - \psi(b) + \sum_{k=1}^{\infty} \frac{(d - c)_k (e - c)_k}{(a)_k (b)_k k}.$$

Here, $\gamma = 0.577\ldots$ is the Euler–Mascheroni constant, $\psi(x) = \frac{d}{dx} \log \Gamma(x)$ is the digamma function, and $(x)_k = x(x+1)\cdots(x+k-1)$ is the ascending Pochhammer symbol.

If we apply it to both of the above ${}_3F_2$ functions, with $c = 1$ after appropriately permuting the arguments, then we find (after some simplification of the prefactor and the resulting gamma function combination) that

$$\left(\cos\left(\frac{\pi y}{4}\right) + \sin\left(\frac{\pi y}{4}\right) \right) \cdot \frac{(1+x)z^2}{(3-y)(1+y)} \cdot {}_3F_2\left[\begin{matrix} 1, \ \frac{3}{4} - \frac{x}{4}, \ \frac{5}{4} + \frac{x}{4} \\ \frac{7}{4} - \frac{y}{4}, \ \frac{5}{4} + \frac{y}{4} \end{matrix}; z^2\right]$$

$$= \frac{1}{4}z^2\left(\cos\left(\frac{\pi x}{4}\right) + \sin\left(\frac{\pi x}{4}\right)\right)\left(-2\gamma - \psi\left(\frac{3}{4} - \frac{x}{4}\right) - \psi\left(\frac{5}{4} + \frac{x}{4}\right) - \log(1 - z^2)\right.$$

$$\left. + \sum_{k=1}^{\infty} \frac{\left(\frac{3}{4} - \frac{y}{4}\right)_k\left(\frac{1}{4} + \frac{y}{4}\right)_k}{k\left(\frac{3}{4} - \frac{x}{4}\right)_k\left(\frac{5}{4} + \frac{x}{4}\right)_k} \right) + O((1-z)\log(1-z))$$

and

$$\left(\cos\left(\frac{\pi y}{4}\right) + \sin\left(\frac{\pi y}{4}\right) \right) \cdot \frac{z}{1+y} \cdot {}_3F_2\left[\begin{matrix} 1, \ \frac{3}{4} - \frac{x}{4}, \ \frac{1}{4} + \frac{x}{4} \\ \frac{3}{4} - \frac{y}{4}, \ \frac{5}{4} + \frac{y}{4} \end{matrix}; z^2\right]$$

$$= \frac{1}{4}z\left(\cos\left(\frac{\pi x}{4}\right) + \sin\left(\frac{\pi x}{4}\right)\right)\left(-2\gamma - \psi\left(\frac{3}{4} - \frac{x}{4}\right) - \psi\left(\frac{1}{4} + \frac{x}{4}\right) - \log(1 - z^2)\right.$$

$$\left. + \sum_{k=1}^{\infty} \frac{\left(-\frac{1}{4} - \frac{y}{4}\right)_k\left(\frac{1}{4} + \frac{y}{4}\right)_k}{k\left(\frac{3}{4} - \frac{x}{4}\right)_k\left(\frac{1}{4} + \frac{x}{4}\right)_k} \right) + O((1-z)\log(1-z)).$$

Since

$$\lim_{z\to 1^-} \left(\cos\left(\frac{\pi x}{4}\right) + \sin\left(\frac{\pi x}{4}\right) \right)$$

$$\cdot \left(\frac{1}{4}z^2\log(1-z^2) + \frac{1}{4}z\log(1-z^2) + \sqrt{z}\tanh^{-1}(\sqrt{z})\right)$$

$$= \frac{3}{2}\log(2)\left(\cos\left(\frac{\pi x}{4}\right) + \sin\left(\frac{\pi x}{4}\right)\right),$$

we find these asymptotic formulae lead to following limit for $G(x, y; z)$ that we seek

$$F(x, y) = \lim_{z\to 1^-} G(x, y; z) - \sqrt{z}\tanh^{-1}(\sqrt{z}) \cdot \left(\cos\left(\frac{\pi x}{4}\right) + \sin\left(\frac{\pi x}{4}\right)\right)$$

$$= \frac{1}{4}\left(\cos\left(\frac{\pi x}{4}\right) + \sin\left(\frac{\pi x}{4}\right)\right)$$

$$\cdot \left(-4\gamma - 6\log(2) - \psi\left(\frac{1}{4} + \frac{x}{4}\right) - 2\psi\left(\frac{3}{4} - \frac{x}{4}\right) - \psi\left(\frac{5}{4} + \frac{x}{4}\right)\right.$$

$$\left. + \sum_{k=1}^{\infty} \frac{\left(\frac{3}{4} - \frac{y}{4}\right)_k\left(\frac{1}{4} + \frac{y}{4}\right)_k}{k\left(\frac{3}{4} - \frac{x}{4}\right)_k\left(\frac{5}{4} + \frac{x}{4}\right)_k} + \sum_{k=1}^{\infty} \frac{\left(-\frac{1}{4} - \frac{y}{4}\right)_k\left(\frac{1}{4} + \frac{y}{4}\right)_k}{k\left(\frac{3}{4} - \frac{x}{4}\right)_k\left(\frac{1}{4} + \frac{x}{4}\right)_k} \right).$$

Using the same observation as in Proposition 1 of [13], we may rewrite these Pochhammer sums in a more useful way. Specifically we have

$$\sum_{k=1}^{\infty} \frac{(a)_k(b)_k}{k(c)_k(d)_k} = \frac{d}{dZ}\bigg|_{Z=0} {}_3F_2\left[\begin{matrix} a, \ b, \ Z \\ c, \ d \end{matrix}; 1\right],$$

so it follows that

$$F(x, y) = \frac{1}{4}\left(\cos\left(\frac{\pi x}{4}\right) + \sin\left(\frac{\pi x}{4}\right)\right)$$

$$\cdot \left\{-4\gamma - 6\log(2) - \psi\left(\frac{1}{4} + \frac{x}{4}\right) - 2\psi\left(\frac{3}{4} - \frac{x}{4}\right) - \psi\left(\frac{5}{4} + \frac{x}{4}\right)\right.$$

$$\left. + \frac{d}{dZ}\Big|_{Z=0}\left({}_3F_2\left[\begin{matrix}\frac{3}{4} - \frac{y}{4}, \ \frac{1}{4} + \frac{y}{4}, \ Z \\ \frac{3}{4} - \frac{x}{4}, \ \frac{5}{4} + \frac{x}{4}\end{matrix}; 1\right] + {}_3F_2\left[\begin{matrix}-\frac{1}{4} - \frac{y}{4}, \ \frac{1}{4} + \frac{y}{4}, \ Z \\ \frac{3}{4} - \frac{x}{4}, \ \frac{1}{4} + \frac{x}{4}\end{matrix}; 1\right]\right)\right\}.$$

Apply to the second hypergeometric term, the following contiguous function relation (with arguments in this order)

$$(a - b)p \cdot {}_3F_2\left[\begin{matrix}a, \ b, \ c \\ p, \ q\end{matrix}; z\right]$$

$$= b(a - p) \cdot {}_3F_2\left[\begin{matrix}a, \ b+1, \ c \\ p+1, \ q\end{matrix}; z\right] - a(b - p) \cdot {}_3F_2\left[\begin{matrix}a+1, \ b, \ c \\ p+1, \ q\end{matrix}; z\right],$$

and we find

$$F(x, y) = \frac{1}{4}\left(\cos\left(\frac{\pi x}{4}\right) + \sin\left(\frac{\pi x}{4}\right)\right)$$

$$\cdot \left\{-4\gamma - 6\log(2) - \psi\left(\frac{1}{4} + \frac{x}{4}\right) - 2\psi\left(\frac{3}{4} - \frac{x}{4}\right) - \psi\left(\frac{5}{4} + \frac{x}{4}\right)\right.$$

$$+ \frac{d}{dZ}\Big|_{Z=0}\left({}_3F_2\left[\begin{matrix}\frac{3}{4} - \frac{y}{4}, \ \frac{1}{4} + \frac{y}{4}, \ Z \\ \frac{3}{4} - \frac{x}{4}, \ \frac{5}{4} + \frac{x}{4}\end{matrix}; 1\right]\right.$$

$$+ \frac{x - y - 4}{2(x - 3)} \cdot {}_3F_2\left[\begin{matrix}-\frac{1}{4} - \frac{y}{4}, \ \frac{5}{4} + \frac{y}{4}, \ Z \\ \frac{7}{4} - \frac{x}{4}, \ \frac{1}{4} + \frac{x}{4}\end{matrix}; 1\right]$$

$$\left.\left. + \frac{x + y - 2}{2(x - 3)} \cdot {}_3F_2\left[\begin{matrix}\frac{3}{4} - \frac{y}{4}, \ \frac{1}{4} + \frac{y}{4}, \ Z \\ \frac{7}{4} - \frac{x}{4}, \ \frac{1}{4} + \frac{x}{4}\end{matrix}; 1\right]\right)\right\}. \tag{3}$$

All three terms are of the following form, with $X = \frac{1}{4} + \frac{y}{4}, Y = \frac{1}{4} + \frac{x}{4}$, then $X = -\frac{1}{4} - \frac{y}{4}, Y = \frac{3}{4} - \frac{x}{4}$ and then $X = \frac{1}{4} + \frac{y}{4}, Y = \frac{3}{4} - \frac{x}{4}$, respectively:

$$\frac{d}{dZ}\Big|_{Z=0} {}_3F_2\left[\begin{matrix}X, \ 1-X, \ Z \\ 1-Y, \ 1+Y\end{matrix}; 1\right]$$

$$= \psi(1 + Y) + \psi(1 - Y) - \psi(1 - X + Y) - \psi(1 - X - Y)$$

$$- \frac{\sin(\pi X)}{\sin(\pi Y)}\left[\psi(1 - X + Y) - \psi(1 - X - Y) - \psi\left(1 - \frac{X-Y}{2}\right) + \psi\left(1 - \frac{X+Y}{2}\right)\right]. \tag{4}$$

This evaluation is given in Eq. 9 of [10], and hence, reduces $F(x, y)$ to a combination of digamma functions.

After substituting in the indicated X, Y values, and some amount of simplification (see Remark 2.2 below), we find

$$
\begin{aligned}
F(x, y) = &\frac{1}{2}\left(\cos\left(\frac{\pi x}{4}\right) + \sin\left(\frac{\pi x}{4}\right)\right) \\
&\cdot \left(A\left(\frac{x-y}{4}\right) - A\left(\frac{x+y}{4}\right) + 2A\left(\frac{x+y}{2}\right) - \log(2)\right) \\
&+ \frac{1}{2}\left(\cos\left(\frac{\pi y}{4}\right) + \sin\left(\frac{\pi y}{4}\right)\right) \\
&\cdot \left(-A\left(\frac{x-y}{8}\right) + A\left(\frac{x-y}{4}\right) - 2C\left(\frac{x+y}{2}\right) + \log(2)\right),
\end{aligned}
\tag{5}
$$

where A and C are certain generating series of ζ- and β-values as defined below. The series A is defined as in [13], namely

$$
A(z) := \psi(1) - \tfrac{1}{2}(\psi(1+z) + \psi(1-z)) = \sum_{r=1}^{\infty} \zeta(2r+1)z^{2r}.
$$

The generating C is defined as follows, to give an analogous Dirichlet-β generating series:

$$
C(z) := \tfrac{1}{8}\left(\psi\left(\tfrac{1}{4} + \tfrac{z}{4}\right) - \psi\left(\tfrac{1}{4} - \tfrac{z}{4}\right) - \psi\left(\tfrac{3}{4} + \tfrac{z}{4}\right) + \psi\left(\tfrac{3}{4} - \tfrac{z}{4}\right)\right) = \sum_{r=1}^{\infty} \beta(2r)z^{2r-1}.
$$

This formula for C follows by interchanging the summation, and applying partial fractions to the result, in

$$
\begin{aligned}
\sum_{r=1}^{\infty} \beta(2r)z^{2r-1} &= \sum_{r=1}^{\infty}\sum_{k=1}^{\infty} \frac{-(-1)^k}{(2k-1)^{2r}} z^{2r-1} \\
&= \sum_{m=1}^{\infty} \frac{(-1)^m}{2(1-2m+z)} - \frac{(-1)^m}{2(1-2m-z)}.
\end{aligned}
$$

Remark 2.2 Since the steps of simplification of $F(x, y)$ are rather involved, we indicate some strategy to follow in the verification process. The goal is to check $I = $(3)–(5) $= 0$, where (3) is evaluated via the formula for $\frac{\mathrm{d}}{\mathrm{d}Z}$ of a certain ${}_3F_2$ hypergeometric series given in (4).

Via the functional equation,

$$
\psi(z+1) - \psi(z) = \frac{1}{z},
$$

every argument of $(x-3)I$ (we multiply by $(x-3)$ to eliminate rational function coefficients at this point) can be reduced to one of the form $\alpha + \beta x + \delta y$, where $\alpha = -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}$. Using the functional equation

$$\psi(-z) = \psi(z) + \frac{1}{z} + \pi \cot(\pi z),\tag{6}$$

we can also reduce the $\alpha = -\frac{1}{4}$ arguments to $\frac{1}{4}$. In particular, we are left with only the following arguments:

$$\left\{\psi\left(-\tfrac{x}{2}-\tfrac{y}{2}\right), \psi\left(-\tfrac{x}{4}-\tfrac{y}{4}\right), \psi\left(\tfrac{x}{4}-\tfrac{y}{4}\right), \psi\left(\tfrac{1}{4}-\tfrac{x}{8}-\tfrac{y}{8}\right),\right.$$
$$\psi\left(\tfrac{x}{8}-\tfrac{y}{8}\right), \psi\left(-\tfrac{x}{8}+\tfrac{y}{8}\right), \psi\left(\tfrac{1}{2}-\tfrac{x}{8}+\tfrac{y}{8}\right), \psi\left(\tfrac{1}{4}+\tfrac{x}{8}+\tfrac{y}{8}\right),$$
$$\left.\psi\left(-\tfrac{x}{4}+\tfrac{y}{4}\right), \psi\left(\tfrac{x}{4}+\tfrac{y}{4}\right), \psi\left(\tfrac{1}{2}+\tfrac{x}{4}+\tfrac{y}{4}\right), \psi\left(\tfrac{x}{2}+\tfrac{y}{2}\right)\right\}.$$

Applying the symmetry in (6) further times, reduces $-\frac{x}{2}-\frac{y}{2}$, $-\frac{x}{4}-\frac{y}{4}$, $-\frac{x}{4}+\frac{y}{4}$, $-\frac{x}{8}+\frac{y}{8}$ to their positive counterparts. Using (6) (again, undoing the previous step in one case!) and the functional equation

$$\psi(2z) = \tfrac{1}{2}\left(\psi\left(z+\tfrac{1}{2}\right) + \psi(z)\right) + \log(2),\tag{7}$$

with $z = z' - \frac{1}{4}$, we can express $\psi\left(\frac{1}{4}-\frac{x}{8}-\frac{y}{8}\right)$ in terms of $\psi\left(-\frac{1}{4}+\frac{x}{8}+\frac{y}{8}\right)$ first, then in terms of $\psi\left(\frac{1}{4}+\frac{x}{8}+\frac{y}{8}\right)$ and $\psi\left(-\frac{1}{2}-\frac{x}{4}-\frac{y}{4}\right)$.

Use (6) to reduce $\psi\left(-\frac{1}{2}+z\right)$ to $\psi\left(\frac{1}{2}+z\right)$, and apply (7) to reduce this to $\psi(z)$ and $\psi(2z)$, where necessary. Then using (6), we can finally reduce all arguments to the following set of 5 possibilities:

$$\left\{\psi\left(\tfrac{x}{4}-\tfrac{y}{4}\right), \psi\left(\tfrac{x}{8}-\tfrac{y}{8}\right), \psi\left(\tfrac{1}{4}+\tfrac{x}{8}+\tfrac{y}{8}\right), \psi\left(\tfrac{x}{4}+\tfrac{y}{4}\right), \psi\left(\tfrac{x}{2}+\tfrac{y}{2}\right)\right\}.$$

At this point, the coefficients of each of these 5 arguments are a trigonometric rational function, as is the constant coefficient of the ψ-polynomial. Each of these trig functions can be written as a rational function (with $\sqrt{2}$-coefficients) in

$$\cos\left(\frac{\pi x}{8}\right), \sin\left(\frac{\pi x}{8}\right), \cos\left(\frac{\pi y}{8}\right), \sin\left(\frac{\pi y}{8}\right),$$

using the addition formulae for sin, cos, and then reducing $\frac{\pi x}{2}$- and $\frac{\pi x}{4}$-arguments to the above via the double-angle formulae. (For cosine, $\cos(2x) = \cos(x)^2 - \sin(x)^2$ seems to be the more useful variant.) At this point, the coefficients of ψ's vanish identically (without relating powers of sin and cos), whereas the constant coefficient factors into an expression involving $\cos^2\left(\frac{\pi x}{8}\right) + \sin^2\left(\frac{\pi x}{8}\right) - \cos^2\left(\frac{\pi y}{8}\right) - \sin^2\left(\frac{\pi y}{8}\right)$, which is 0 via Pythagoras. This shows that (3)–(5) = 0 as claimed, hence verifying the expression for $F(x, y)$ given in (5).

At this point, we have the generating series involving (for $b = 0$) the regularised values $t^{*, V=0}$ at $V = 0$; however, for MtVs, the most natural stuffle regularisation would seem to be $V = \log(2)$. Fortunately, since $t(\{\bar{1}\}^m, 1)$ only involves a single argument 1, the regularisation polynomial is given (cf. (1)) by

$$t^{*, V}(\{\bar{1}\}^m, 1) = t(\{\bar{1}\}^m)V + t^{*, V=0}(\{\bar{1}\}^m, 1),$$

meaning that we can (re-)construct the entire polynomial from this constant term. In particular (see (2)),

$$\sum_{m \geq 0} (-1)^m t^{*,V}(\{\overline{1}\}^m, 1) x^m$$

$$= V\left(\cos\left(\frac{\pi x}{4}\right) + \sin\left(\frac{\pi x}{4}\right)\right) + \sum_{m \geq 0} (-1)^m t^{*,V=0}(\{\overline{1}\}^m, 1) x^m.$$

We can, therefore, give the arbitrary stuffle regularisation generating series as follows:

$$F^V(x, y) = \frac{1}{2}\left(\cos\left(\frac{\pi x}{4}\right) + \sin\left(\frac{\pi x}{4}\right)\right)$$
$$\cdot \left(A\left(\frac{x - y}{4}\right) - A\left(\frac{x + y}{4}\right) + 2A\left(\frac{x + y}{2}\right) - \log(2) + 2V\right)$$
$$+ \frac{1}{2}\left(\cos\left(\frac{\pi y}{4}\right) + \sin\left(\frac{\pi y}{4}\right)\right)$$
$$\cdot \left(-A\left(\frac{x - y}{8}\right) + A\left(\frac{x - y}{4}\right) - 2C\left(\frac{x + y}{2}\right) + \log(2)\right). \qquad (8)$$

The final step is to extract an explicit evaluation for $t(\{\overline{1}\}^a, 1, \{\overline{1}\}^b)$ from this generating series. One can check easily that

$$[x^a y^b] \sum_{i=0}^{\infty} f(i) x^i \cdot \sum_{j=0}^{\infty} g(j)(x + y)^j = \sum_{n=0}^{a+b} \binom{n}{b} g(a + b - n) f(n),$$

where $[x^a y^b]$ denotes the coefficient of $x^a y^a$ in the terms thereafter. So we can readily extract the following formulae from $F^V(x, y)$, where as before $\{k\}^n$ denotes the string $k, \ldots, k$ with n repetitions of k. (Remember that, $F^V(x, y)$ is the generating series for $(-1)^{a+b} t^{*,V}(\{\overline{1}\}^a, 1, \{\overline{1}\}^b)$, so one must take $(-1)^{a+b}[x^a y^b] F^V(x, y)$, in order to obtain the following formula.)

$$t^{*,V}(\{\overline{1}\}^a, 1, \{\overline{1}\}^b)$$

$$= \sum_{\substack{r=2 \\ \text{even}}}^{a+b+1} \frac{(-1)^{\lfloor (a+b-r-1)/2 \rfloor}}{2(a + b - r)!}\left(\frac{\pi}{4}\right)^{a+b-r} \binom{r}{b}\left(\frac{1}{4^r} - \frac{(-1)^b}{4^r} - \frac{2}{2^r}\right) \zeta(r + 1)$$

$$+ \sum_{\substack{r=2 \\ \text{even}}}^{a+b+1} \frac{(-1)^{\lfloor (a+b-r-1)/2 \rfloor}}{2(a + b - r)!}\left(\frac{\pi}{4}\right)^{a+b-r} \binom{r}{a}\left(\frac{(-1)^a}{8^r} - \frac{(-1)^a}{4^r}\right) \zeta(r + 1)$$

$$- \sum_{\substack{r=1 \\ \text{odd}}}^{a+b+1} \frac{(-1)^{\lfloor (a+b-r-1)/2 \rfloor}}{2(a + b - r)!}\left(\frac{\pi}{4}\right)^{a+b-r} \binom{r}{a}\left(\frac{2}{2^r}\right) \beta(r + 1)$$

$$+ \delta_{a=0} \frac{(-1)^{\lfloor -b/2 \rfloor}}{2 \cdot b!}\left(\frac{\pi}{4}\right)^b \log(2) + \delta_{b=0} \frac{(-1)^{\lfloor -a/2 \rfloor}}{2 \cdot a!}\left(\frac{\pi}{4}\right)^a (2V - \log(2)).$$

This completes the proof of Theorem 1.1, and the explicit evaluation thereafter.   □

3 Potential motivic applications

In [3] and [13] with the case $\zeta(\{2\}^a, 3, \{2\}^b)$, in [11] with $t(\{2\}^a, 3, \{2\}^b)$ and in [4] with the case $t^{*,V=0}(\{2\}^a, 1, \{2\}^b)$, the various identities (and the arithmetic of their coefficients) were applied to show linear independence and/or basis results on the motivic level. It should, therefore, be possible to lift Theorem 1.1 to a motivic version and show on the motivic level some independence and/or basis results for alternating MtVs.

Equivalently (after extending coefficients to $\mathbb{Q}(i)$), one should also obtain results about coloured MZVs of level $N = 4$ (i.e. $\varepsilon_i \in \{\pm 1, \pm i\}$, roots of unity of order $N = 4$). In general, we have the following expression for MtVs of level N in terms of coloured MZVs of level $2N$ (after fixing some choice of square roots, which we symmetrise over anyway)

$$
t\binom{\varepsilon_1, \ldots, \varepsilon_d}{k_1, \ldots, k_d} = \sum_{0 < n_1 < \cdots < n_d} \frac{(1 - (-1)^{n_1})\varepsilon_1^{\frac{n_1+1}{2}}}{2\,n_1^{k_1}} \cdots \frac{(1 - (-1)^{n_d})\varepsilon_d^{\frac{n_d+1}{2}}}{2\,n_d^{k_d}}
$$
$$
= \frac{1}{2^d} \sum_{\substack{\eta_1^2 = \varepsilon_1 \\ \vdots \\ \eta_d^2 = \varepsilon_d}} \eta_1 \cdots \eta_d \zeta\binom{\eta_1, \ldots, \eta_d}{k_1, \ldots, k_d}.
$$

So alternating MtVs are expressed in terms of coloured MZVs of level $N = 4$; note, however that $\eta_1 \cdots \eta_d = \pm 1$ if an even number of $\varepsilon_i = -1$, and $\eta_1 \cdots \eta_d = \pm i$ if an odd number of $\varepsilon_i = -1$. In particular, the MtVs correspond to the real part of purely real, respectively, the imaginary part of purely imaginary, combinations in each case.

Optimistically one expects some result of the following form, where $\{1, \overline{1}\}^\times$ denotes the set of all words with entries 1 or $\overline{1}$.

Idea 3.1 *The stuffle-regularised alternating MtVs*

$$
\{t^{*,V}(w) \mid w \in \{1, \overline{1}\}^\times\}
$$

are linearly independent and form a basis for the space of alternating MtVs.

Some issues do arise in attempting to investigate this, both from the motivic viewpoint and from the classical viewpoint. The issues should not be insurmountable, but we do postpone the motivic investigation for the moment.

As just noted, for $k_i \in \mathbb{Z}_{>0} \cup \overline{\mathbb{Z}_{>0}}$ (with $\overline{k_i}$ denoting the argument k_i has associated sign $\varepsilon_i = -1$), one only has that

$$
i^{\#\{k_i \in \overline{\mathbb{Z}_{>0}}\}} t(k_1, \ldots, k_d)
$$

is a $\mathbb{Q}$-linear combination of coloured MZVs of level $N = 4$. So that one must extend the coefficients to $\mathbb{Q}(i)$, in order to say anything about level $N = 4$ coloured MZVs from an MtV result, and vice versa.

Second, one must redevelop the background in [11], and [4] to extend the formulae for the motivic derivations D_{2r+1} to the case of alternating t values. An deeper issue here is that there are more primitive elements at level $N = 4$, so that $\ker D_{<N} = \zeta^{\mathrm{m}}(N)\mathbb{Q} \oplus \zeta^{\mathrm{m}}\left(\begin{smallmatrix} i \\ N \end{smallmatrix}\right)\mathbb{Q}$ is no longer one dimensional (see [6, Corollary 5.1.3]). Lifting identities by application of the period map requires identifying 2 unknown rational coefficients. Working over $\mathbb{Q}$, one can appeal to the real and imaginary parts for this, but after tensoring by $\mathbb{Q}(i)$, this is not necessarily so straightforward.

Finally, the natural regularisation $V = \log(2)$, for $t^{*,V}(1)$ immediately fails to give a basis in Idea 3.1. One has that

$$t^{*,V}(1, 1) = \frac{1}{2}t^{*,V}(1)^2 - \frac{1}{2}t(2) = \frac{1}{2}V^2 - \frac{\pi^2}{16}$$

$$t(1, \overline{1}) = \frac{1}{2}G - \frac{\pi}{8}\log(2)$$

$$t^{*,V}(\overline{1}, 1) = t(\overline{1})t^{*,V}(1) - t(1, \overline{1}) - t(\overline{2}) = -\frac{\pi}{4}V + \frac{1}{2}G + \frac{\pi}{8}\log(2)$$

$$t(\overline{1}, \overline{1}) = \frac{1}{2}t(\overline{1})^2 - \frac{1}{2}t(2) = -\frac{\pi^2}{32}$$

Here, $G = \beta(2) = 0.9159\ldots$ is the Catalan constant. We refer to [8, Sect. 6] for these evaluations. In the case $V = \log(2)$, we have

$$t^{*,V=\log(2)}(\overline{1}, 1) = -\frac{\pi}{8}\log(2) + \frac{1}{2}G = t(1, \overline{1}).$$

So the stuffle-regularised $t^{*,V}(w)$, $w \in \{1, \overline{1}\}^{\times}$ cannot be a basis for $V = \log(2)$. The case $V = \frac{1}{2}\log(2)$ does seem to give a basis, though, and the regularisation $t^{*,V=\frac{1}{2}\log(2)}(1) = \frac{1}{2}\log(2)$ is also a very natural one to take, based on the expression for $t(n) = \frac{1}{2}(\zeta(n) - \zeta(\overline{n}))$, extended to $n = 1$. A more durable conjecture would be as follows.

Conjecture 3.2 *For any rational $\lambda \in \mathbb{Q}$, with $0 < \lambda < 1$, the stuffle-regularised alternating MtVs with $V = \lambda \log(2)$, of the following form:*

$$\{t^{*,V}(w) \mid w \in \{1, \overline{1}\}^{\times}\}$$

are linearly independent and form a basis for the space of alternating MtVs.

Those rational λ for which the regularisation $t^{*,V}$, $V = \lambda \log(2)$ does not give a basis in Conjecture 3.2 should be termed *singular regularisation parameters*. The following $V = \lambda \log(2)$ are singular regularisation parameters, first appearing at the indicated weight.

N	1	2	3	4	5	6	7	8	9	10	11
λ	0	1	-2	$\frac{13}{11}$	$-\frac{220}{203}$	$\frac{4971}{3911}$	$-\frac{428854}{506177}$	$\frac{8829285}{6699031}$	$-\frac{12070249400}{16117649299}$	$\frac{91040059801}{67506970721}$	$-\frac{917750647910294}{1321840200143647}$

A new singular regularisation parameter λ appears in each weight, corresponding to a reduction of

$$t^{*,V}(\{\bar{1}\}^n, 1) = \sum_{i=0}^{n-1} c_i t(\{\bar{1}\}^i, 1, \{\bar{1}\}^{n-i}),$$

for some $c_i \in \mathbb{Q}$. For example, when $V = \frac{13}{11}\log(2)$ in weight 4, we have

$$t^{*,V=\frac{13}{11}\log(2)}(\bar{1}, \bar{1}, \bar{1}, 1) = \frac{41}{33}t(\bar{1}, \bar{1}, 1, \bar{1}) - \frac{15}{11}t(\bar{1}, 1, \bar{1}, \bar{1}) + \frac{15}{11}t(1, \bar{1}, \bar{1}, \bar{1}),$$

as can be verified with the evaluation in Theorem 1.1. In weight $N+1$, this reduction can also be obtained directly from the identity in Theorem 1.1, when written in matrix form with rows indexed by c_i and columns by $\zeta(2r+1)$ and $\beta(2r)$; the factor $(\frac{\pi}{4})^i$ is fixed by weight, so can be discarded. The singular regularisation parameter in weight $N+1$ then corresponds to the determinant of the resulting matrix vanishing.

The sequence $(\lambda_i)_{i=1}^\infty$ of singular regularisation parameters appears to satisfy a number of properties.

Conjecture 3.3 *The sequence* $(\lambda_i)_{i=1}^\infty = \left(0, 1, -2, \frac{13}{11}, -\frac{220}{203}, \ldots\right)$ *of singular regularisation parameters satisfies the following, discounting the $\lambda_1 = 0$ term:*

(i) *the sign of λ_i is $(-1)^i$ for all $i > 1$,*

(ii) *the odd-indexed and even-indexed subsequences are increasing: $\lambda_{2i+2} > \lambda_{2i}$ and $\lambda_{2i+3} > \lambda_{2i+1}$ for all $i \geq 1$.*

(iii) *the odd-indexed and even-indexed subsequences are bounded as follows: $\lambda_{2i} \leq \frac{3}{2}$ and $\lambda_{2i+1} \leq -\frac{1}{2}$ for all $i \geq 1$.*

(iii′) *the odd-indexed and even-indexed subsequences have the following limits: $\lim_{i\to\infty} \lambda_{2i} = \frac{3}{2}$ and $\lim_{i\to\infty} \lambda_{2i+1} = -\frac{1}{2}$*

As a final observation, based on the expression of the alternating MtVs as real/imaginary parts of coloured MZVs of level $N = 4$, depending on the parity of the number of $\varepsilon_i = -1$, and the expected motivic results related to this, we should have the following.

Conjecture 3.4 *Any relation between alternating MtVs is homogeneous in the number of signs $\varepsilon_i = -1$, counted modulo 2.*

Example 3.5 In weight 5, and with $V = \frac{1}{2}\log(2)$, we have the numerically checked identity

$t(\overline{1}, 2, \overline{2})$

$$
\begin{aligned}
&\overset{?}{=} -\tfrac{143368}{3215} t(\overline{1}, \overline{1}, 1, \overline{1}, \overline{1}) + \tfrac{121464}{3215} t(\overline{1}, \overline{1}, \overline{1}, 1, \overline{1}) + \tfrac{183472}{16075} t^{*, V = \frac{1}{2} \log(2)}(\overline{1}, \overline{1}, \overline{1}, \overline{1}, 1) \\
&\quad - 48 t(1, \overline{1}, \overline{1}, \overline{1}, \overline{1}) + 48 t(\overline{1}, 1, \overline{1}, \overline{1}, \overline{1}) - \tfrac{24}{5} t^{*, V = \frac{1}{2} \log(2)}(1, \overline{1}, \overline{1}, 1, 1) \\
&\quad + \tfrac{24}{5} t^{*, V = \frac{1}{2} \log(2)}(\overline{1}, 1, \overline{1}, 1, 1) - \tfrac{24}{25} t^{*, V = \frac{1}{2} \log(2)}(\overline{1}, \overline{1}, 1, 1, 1) \\
&\quad + 4 t^{*, V = \frac{1}{2} \log(2)}(1, \overline{1}, 1, \overline{1}, 1) - 4 t^{*, V = \frac{1}{2} \log(2)}(\overline{1}, 1, 1, \overline{1}, 1) \,.
\end{aligned}
$$

One sees immediately that each term has either 2 or 4 barred entries, corresponding to 2 or 4 arguments with associated sign $\varepsilon_i = -1$.

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The corresponding author states that there is no conflict of interest.

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