AUTOMORPHISMS OF VERONESE SUBALGEBRAS OF POLYNOMIAL ALGEBRAS AND FREE POISSON ALGEBRAS

Bakhyt Aitzhanova[1], **Leonid Makar-Limanov**[2], and **Ualbai Umirbaev**[3]

Abstract. The Veronese subalgebra A_0 of degree $d \geq 2$ of the polynomial algebra $A = K[x_1, x_2, \ldots, x_n]$ over a field K in the variables $x_1, x_2, \ldots, x_n$ is the subalgebra of A generated by all monomials of degree d and the Veronese subalgebra P_0 of degree $d \geq 2$ of the free Poisson algebra $P = P\langle x_1, x_2, \ldots, x_n\rangle$ is the subalgebra spanned by all homogeneous elements of degree kd, where $k \geq 0$.

If $n \geq 2$ then every derivation and every locally nilpotent derivation of A_0 and P_0 over a field K of characteristic zero is induced by a derivation and a locally nilpotent derivation of A and P, respectively. Moreover, we prove that every automorphism of A_0 and P_0 over a field K closed with respect to taking all d-roots of elements is induced by an automorphism of A and P, respectively.

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1. Introduction

Let K be an arbitrary field and let $\mathbb{A}^n$ and $\mathbb{P}^n$ be the affine and the projective n-spaces over K, respectively. The *Veronese map* of degree d is the map

$$\nu_{n,d} : \mathbb{P}^n \to \mathbb{P}^m$$

that sends $[x_0 : \ldots : x_n]$ to all $m + 1$ possible monomials of total degree d, where

$$m = \binom{n+d}{d} - 1.$$

It is well known that the image $V_{n,d}$ of the Veronese map $\nu_{n,d}$ is a projective variety and is called the *Veronese variety* [7].

If $Y_{i_0\ldots i_n} = x_0^{i_0} \ldots x_n^{i_n}$, $i_0 + \ldots + i_n = d$, then the Veronese variety is determined by the set of quadratic relations

$$Y_{i_0\ldots i_n} Y_{j_0\ldots j_n} = Y_{k_0\ldots k_n} Y_{r_0\ldots r_n}, \tag{1}$$

where $i_0 + j_0 = k_0 + r_0, \ldots, i_n + j_n = k_n + r_n$ [19].

The *affine cone of* $V_{n,d}$ or the *affine Veronese variety* is the affine subvariety $\mathbb{A}^{m+1}$ determined by the set of relations (1). The algebra of polynomial functions on the affine

[1]Department of Mathematics, Wayne State University, Detroit, MI 48202, USA, e-mail: *aitzhanova.bakhyt01@gmail.com*

[2]Department of Mathematics, Wayne State University, Detroit, MI 48202, USA and Faculty of Mathematics and Computer Science, Weizmann Institute of Science, Rehovot, 7610001, Israel, e-mail: *lml@wayne.edu*

[3]Department of Mathematics, Wayne State University, Detroit, MI 48202, USA and Institute of Mathematics and Mathematical Modeling, Almaty, 050010, Kazakhstan, e-mail: *umirbaev@wayne.edu*

Veronese cone $V_{n,d}$ is isomorphic to the subalgebra of $K[x_0, \ldots, x_n]$ generated by all monomials of degree d [21].

Veronese surfaces play an important role in the description of quasihomogeneous affine surfaces given by M.H. Gizatullin [4] and V.L. Popov [18]. They form one of the main examples of the so-called *Gizatullin surfaces* [10]. The structure of the automorphism groups of Veronese surfaces are studied in [1, 2, 5, 6, 11, 12]. The derivations and locally nilpotent derivations of affine Veronese surfaces are described in [1].

Recently J. Kollar devoted two interesting papers [8, 9] to the study of automorphism groups of some more general affine varieties. In particular, he described the group of automorphisms of all affine Veronese varieties.

Let $A = K[x_1, x_2, \ldots, x_n]$ be the polynomial algebra over a field K in the variables $x_1, x_2, \ldots, x_n$. Consider the grading

$$A = A_0 \oplus A_1 \oplus \ldots \oplus A_{d-1},$$

where $d \geq 2$ and A_i is the subspace of A generated by all monomials of degree $kd + i$ for all $k \geq 0$. This is a $\mathbb{Z}_d$-grading of A, i.e., $A_iA_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{Z}_d$. The subalgebra A_0 is called the *Veronese subalgera of A of degree d.*

Similarly, let $P = P\langle x_1, x_2, \ldots, x_n\rangle$ be the free Poisson algebra over K in the variables $x_1, x_2, \ldots, x_n$. The grading

$$P = P_0 \oplus P_1 \oplus \ldots \oplus P_{d-1},$$

where P_i is the subspace of P generated by all homogeneous elements of degree $kd + i$ is a $\mathbb{Z}_d$-grading of P, i.e., $P_iP_j \subseteq P_{i+j}$ and $\{P_i, P_j\} \subseteq P_{i+j}$ for all $i, j \in \mathbb{Z}_d$. The subalgebra P_0 is called the *Veronese subalgera of P of degree d.*

We prove that every derivation and every locally nilpotent derivation of P_0 (of A_0) over a field K of characteristic zero is induced by a derivation and a locally nilpotent derivation of P (of A), respectively. Moreover, we prove that every automorphism of P_0 (of A_0) over a field K, closed with respect to taking all d-roots of elements, is induced by an automorphism of P (of A).

This paper is organized as follows. In Section 2 we describe a basis of free Poisson algebras and give some elementary definitions that are necessary in the future. Derivations of the Veronese subalgebras are studied in Section 3 and automorphisms are studied in Section 4. All results are proven for Poisson algebras. Analogues of these results for polynomial algebras are formulated in Section 5. In the same section we give some counter-examples that show the analogous results do not hold for polynomial algebras in one variable and for free associative algebras.

2. Free Poisson algebra $P\langle x_1, \ldots, x_n\rangle$

A vector space P over a field K endowed with two bilinear operations $x \cdot y$ (a multiplication) and $\{x, y\}$ (a Poisson bracket) is called a *Poisson algebra* if P is a commutative associative algebra under $x \cdot y$, P is a Lie algebra under $\{x, y\}$, and P satisfies the following identity (the Leibniz identity):

$$\{x, y \cdot z\} = y \cdot \{x, z\} + \{x, y\} \cdot z.$$

Symplectic Poisson algebras P_n appear in many areas of algebra. For each natural $n \geq 1$ the symplectic Poisson algebra P_n of index n is a polynomial algebra $K[x_1, y_1, \ldots, x_n, y_n]$ endowed with the Poisson bracket defined by

$$\{x_i, y_j\} = \delta_{ij}, \quad \{x_i, x_j\} = 0, \quad \{y_i, y_j\} = 0,$$

where δ_{ij} is the Kronecker symbol and $1 \leq i, j \leq n$.

Let P be a Poisson algebra. A linear map $D : P \to P$ is called a *derivation* of the Poisson algebra P if

$$D(xy) = D(x)y + xD(y) \tag{2}$$

and

$$D(\{x, y\}) = \{D(x), y\} + \{x, D(y)\} \tag{3}$$

for all $x, y \in P$.

We refer to a linear map $D : P \to P$ satisfying (2) as an *associative derivation* of P and to a linear map $D : P \to P$ satisfying (3) as a *Lie derivation* of P. Thus, a derivation of a Poisson algebra is simultaneously an associative and a Lie derivation. We often refer to them as Poisson derivations.

The Leibniz identity implies that for any $x \in P$ the map

$$\mathrm{ad}_x : P \to P \ \ (y \mapsto \{x, y\})$$

is an associative derivation of P. The Jacobi identity implies that the map ad_x is also a Lie derivation of P. So for any $x \in P$ the map ad_x is a Poisson derivation of P.

Let L be a Lie algebra with Lie bracket $[\,,]$ over a field K and let $e_1, e_2 \ldots, e_k, \ldots$ be a linear basis of L. Then there exists a unique bracket $\{\,,\}$ on the polynomial algebra $K[e_1, e_2, \ldots, e_k, \ldots]$ defined by

$$\{e_i, e_j\} = [e_i, e_j]$$

for all i, j and satisfying the Leibniz identity. With this bracket

$$P(L) = \langle K[e_1, e_2, \ldots], \cdot, \{\,,\}\rangle$$

becomes a Poisson algebra. This Poisson algebra $P(L)$ is called the *Poisson enveloping algebra* [23] (or *Poisson symmetric algebra* [15]) of L. Note that the bracket $\{,\}$ of the algebra $P(L)$ depends on the structure of L but does not depend on a chosen basis.

Let $L = \mathrm{Lie}\langle x_1, \ldots, x_n\rangle$ be the free Lie algebra with free generators $x_1, \ldots, x_n$. It is well-known (see, for example [20]) that $P(L)$ is the free Poisson algebra over K in the variables $x_1, \ldots, x_n$. We denote it by $P\langle x_1, \ldots, x_n\rangle$.

Let us choose a multihomogeneous linear basis

$$x_1, \ldots, x_n, [x_1, x_2], \ldots, [x_1, x_n], \ldots, [x_{n-1}, x_n], [[x_1, x_2], x_3], \ldots$$

of a free Lie algebra L and denote the elements of this basis by

$$e_1, e_2, \ldots, e_s, \ldots. \tag{4}$$

The algebra $P = P\langle x_1, \ldots, x_n\rangle$ coincides with the polynomial algebra on the elements (4). Consequently, the monomials

$$u = e_{i_1} e_{i_2} \ldots e_{i_s}, \tag{5}$$

where $i_1 \leq i_2 \leq \ldots \leq i_s$ form a linear basis of P.

Denote by deg the standard degree function on P, i.e., $\deg(x_i) = 1$ for all $1 \leq i \leq n$. If u is an element of the form (5) then

$$\deg u = \deg e_{i_1} + \deg e_{i_2} + \ldots + \deg e_{i_s}.$$

Set also $d(u) = s$ and call it the *polynomial length* of u. Note that

$$\deg\{f, g\} = \deg f + \deg g$$

if f and g are homogeneous and $\{f, g\} \neq 0$.

Denote by $Q(P) = P(x_1, \ldots, x_n)$ the field of fractions of the polynomial algebra $K[e_1, e_2, \ldots]$ in the variables (4). The Poisson bracket $\{\cdot, \cdot\}$ on $K[e_1, e_2, \ldots] = P$ can be uniquely extended to a Poisson bracket on the field of its fractions $Q(P)$ and

$$\left\{\frac{a}{b}, \frac{c}{d}\right\} = \frac{\{a, c\}bd - \{a, d\}bc - \{b, c\}ad + \{b, d\}ac}{b^2d^2} \tag{6}$$

for all $a, b, c, d \in P$ with $bd \neq 0$.

The field $Q(P) = P(x_1, x_2, \ldots, x_n)$ with this Poisson bracket is called the *free Poisson field* over K in variables $x_1, \ldots, x_n$ [16].

Several combinatorial results on the structure of free Poisson algebras and free Poisson fields are proven in [13, 14, 15, 16, 17, 22].

We fix a grading

$$P = P_0 \oplus P_1 \oplus \ldots \oplus P_{d-1} \tag{7}$$

of the free Poisson algebra $P = P\langle x_1, \ldots, x_n\rangle$, where P_i is the linear span of all elements of degree $i + ds$, $i = 0, 1, \ldots, d-1$, and s is an arbitrary nonnegative integer. This is a $\mathbb{Z}_d$-grading of P, i.e.,

$$P_iP_j \subseteq P_{i+j}, \quad \{P_i, P_j\} \subseteq P_{i+j},$$

where $i, j \in \mathbb{Z}_d = \mathbb{Z}/d\mathbb{Z}$. For shortness we will refer to this grading as the d-grading.

An automorphism $\phi \in \operatorname{Aut} P$ is called a *graded automorphism* with respect to grading (7) if $\phi(x_1), \phi(x_2), \ldots, \phi(x_n) \in P_1$. A graded automorphism is called *graded tame* if it is a product of graded elementary automorphisms.

We will call a graded automorphism of P with respect to grading (7) a *d-graded automorphism* for shortness. Obviously, every d-graded automorphism induces an automorphism of the algebra P_0. A derivation D of P will be called a *d-graded derivation* if $D(x_1), D(x_2), \ldots, D(x_n) \in P_1$.

3. Derivations of P_0

In this section we assume that K is an arbitrary field of characteristic zero.

Lemma 1. *Every derivation of the Poisson algebra $P = P\langle x_1, x_2, \ldots, x_n\rangle$ over K can be uniquely extended to a derivation of the Poisson field $Q(P) = P(x_1, x_2, \ldots, x_n)$.*

Proof. Let D be an arbitrary derivation of the free Poisson algebra P. In particular, D is a derivation of the polynomial algebra $P = K[e_1, \ldots, e_s, \ldots]$. It is well known [25, p. 120] that D can be uniquely extended to an associative derivation S of the quotient field

$Q(P) = K(e_1, \dots, e_s, \dots)$. We will show that S is a Lie derivation of $Q(P)$, i.e., that S satisfies (3). We have to check that

$$S\left(\left\{\frac{a}{b}, \frac{c}{d}\right\}\right) = \left\{S\left(\frac{a}{b}\right), \frac{c}{d}\right\} + \left\{\frac{a}{b}, S\left(\frac{c}{d}\right)\right\} \tag{8}$$

for all $a, b, c, d \in P\langle x_1, \dots, x_n\rangle$ with $bd \neq 0$. Using (6) we get

$$\begin{aligned}
S\left(\left\{\frac{a}{b}, \frac{c}{d}\right\}\right) &= S\left(\frac{\{a,c\}bd - \{a,d\}bc - \{b,c\}ad + \{b,d\}ac}{b^2d^2}\right) \\
&= S\left(\frac{\{a,c\}}{bd}\right) - S\left(\frac{\{a,d\}c}{bd^2}\right) - S\left(\frac{\{b,c\}a}{b^2d}\right) + S\left(\frac{\{b,d\}ac}{b^2d^2}\right) \\
&= \frac{S(\{a,c\})bd - \{a,c\}S(bd)}{b^2d^2} - \frac{S(\{a,d\}c)bd^2 - \{a,d\}cS(bd^2)}{b^2d^4} \\
&\quad - \frac{S(\{b,c\}a)b^2d - \{b,c\}aS(b^2d)}{b^4d^2} + \frac{S(\{b,d\}ac)b^2d^2 - \{b,d\}acS(b^2d^2)}{b^4d^4} \\
&= \frac{S(\{a,c\})}{bd} - \frac{\{a,c\}S(b)}{b^2d} - \frac{\{a,c\}S(d)}{bd^2} - \frac{S(\{a,d\})c}{bd^2} - \frac{\{a,d\}S(c)}{bd^2} + \frac{\{a,d\}cS(b)}{b^2d^2} \\
&\quad + \frac{2\{a,d\}cS(d)}{bd^3} - \frac{S(\{b,c\})a}{b^2d} - \frac{\{b,c\}S(a)}{b^2d} + \frac{2\{b,c\}aS(b)}{b^3d} + \frac{\{b,c\}aS(d)}{b^2d^2} \\
&\quad + \frac{S(\{b,d\})ac}{b^2d^2} + \frac{\{b,d\}S(a)c}{b^2d^2} + \frac{\{b,d\}aS(c)}{b^2d^2} - \frac{2\{b,d\}acS(b)}{b^3d^2} - \frac{2\{b,d\}acS(d)}{b^2d^3} \\
&= \frac{\{S(a),c\}}{bd} + \frac{\{a,S(c)\}}{bd} - \frac{\{a,c\}S(b)}{b^2d} - \frac{\{a,c\}S(d)}{bd^2} - \frac{\{S(a),d\}c}{bd^2} \\
&\quad - \frac{\{a,S(d)\}c}{bd^2} - \frac{\{a,d\}S(c)}{bd^2} + \frac{\{a,d\}cS(b)}{b^2d^2} + \frac{2\{a,d\}cS(d)}{bd^3} - \frac{\{S(b),c\}a}{b^2d} \\
&\quad - \frac{\{b,S(c)\}a}{b^2d} - \frac{\{b,c\}S(a)}{b^2d} + \frac{2\{b,c\}aS(b)}{b^3d} + \frac{\{b,c\}aS(d)}{b^2d^2} + \frac{\{S(b),d\}ac}{b^2d^2} \\
&\quad + \frac{\{b,S(d)\}ac}{b^2d^2} + \frac{\{b,d\}S(a)c}{b^2d^2} + \frac{\{b,d\}aS(c)}{b^2d^2} - \frac{2\{b,d\}acS(b)}{b^3d^2} - \frac{2\{b,d\}acS(d)}{b^2d^3}.
\end{aligned}$$

Direct calculations give that

$$\begin{aligned}
\left\{S\left(\frac{a}{b}\right), \frac{c}{d}\right\} + \left\{\frac{a}{b}, S\left(\frac{c}{d}\right)\right\} &= \left\{\frac{S(a)b - aS(b)}{b^2}, \frac{c}{d}\right\} + \left\{\frac{a}{b}, \frac{S(c)d - cS(d)}{d^2}\right\} \\
&= \left\{\frac{S(a)}{b}, \frac{c}{d}\right\} - \left\{\frac{aS(b)}{b^2}, \frac{c}{d}\right\} + \left\{\frac{a}{b}, \frac{S(c)}{d}\right\} - \left\{\frac{a}{b}, \frac{cS(d)}{d^2}\right\} \\
&= \frac{\{S(a),c\}}{bd} - \frac{\{S(a),d\}c}{bd^2} - \frac{\{b,c\}S(a)}{b^2d} + \frac{\{b,d\}S(a)c}{b^2d^2} - \frac{\{a,c\}S(b)}{b^2d} \\
&\quad - \frac{\{S(b),c\}a}{b^2d} + \frac{\{a,d\}S(b)c}{b^2d^2} + \frac{\{S(b),d\}ac}{b^2d^2} + \frac{2\{b,c\}aS(b)}{b^3d} - \frac{2\{b,d\}aS(b)c}{b^3d^2} \\
&\quad + \frac{\{a,S(c)\}}{bd} - \frac{\{a,d\}S(c)}{bd^2} - \frac{\{b,S(c)\}a}{b^2d} + \frac{\{b,d\}aS(c)}{b^2d^2} - \frac{\{a,c\}S(d)}{bd^2} \\
&\quad - \frac{\{a,S(d)\}c}{bd^2} + \frac{2\{a,d\}cS(d)}{bd^3} + \frac{\{b,c\}S(d)a}{b^2d^2} + \frac{\{b,S(d)\}ac}{b^2d^2} - \frac{2\{b,d\}acS(d)}{b^2d^3}.
\end{aligned}$$

These two equalities imply (8). □

Consider the grading (7) of P. A Poisson derivation D of P will be called a *d-graded Poisson derivation* if $D(x_i) \in P_1$ for all $i = 1, \dots, n$. Obviously, every d-graded Poisson derivation of P induces a Poisson derivation of P_0. The reverse is also true.

Lemma 2. *Every Poisson derivation of P_0 can be uniquely extended to a d-graded Poisson derivation of $P = P\langle x_1, x_2, \dots, x_n\rangle$.*

Proof. Let D be a Poisson derivation of P_0. In particular, D is a derivation of the associative and commutative algebra P_0. Since P_0 is a domain, D can be uniquely extended [25, p. 120] to a derivation T of the field of fractions $Q(P_0)$ of P_0. The field extension

$$Q(P_0) \subseteq Q(P)$$

is algebraic since every e_i is a root of the polynomial $p(t) = t^d - e_i^d \in Q(P_0)[t]$ for all i. This extension is separable since K is a field of characteristic zero. By Corollaries 2 and $2'$ in [25, pages 124–125], the associative derivation T of the field $Q(P_0)$ can be uniquely extended to an associative derivation S of the field $Q(P)$.

Suppose that

$$S(e_j) = \frac{f_j}{g_j},$$

where $f_j \in P$, $0 \neq g_j \in P$, and the pairs f_j, g_j are relatively prime for all j. Notice that $e_j^d \in P_0$ and

$$D(e_j^d) = S(e_j^d) = de_j^{d-1}\frac{f_j}{g_j} \in P_0.$$

Consequently,

$$g_j | e_j^{d-1} \tag{9}$$

since the pair f_j, g_j is relatively prime, that is, g_j is a power of e_j.

If $d | \deg e_j$ then $e_j \in P_0$ and $S(e_j) = D(e_j) \in P_0$, i.e., we may assume that $g_j = 1$. If $d \not| \deg e_j$ then there exist $1 \leq i \leq n$ and $1 \leq k < d$ such that $e_i = x_i \neq e_j$ and $x_i^k e_j \in P_0$. Then

$$D(x_i^k e_j) = S(x_i^k e_j) = kx_i^{k-1}\frac{f_i}{g_i}e_j + x_i^k\frac{f_j}{g_j} \in P_0.$$

Consequently,

$$g_i g_j | kx_i^{k-1} f_i g_j e_j + x_i^k f_j g_i.$$

This implies that $g_j | x_i^k f_j g_i$. Since f_j, g_j are relatively prime, we get $g_j | x_i^k g_i$. By (9) g_i is a power of $e_i = x_i$ and g_j is a power of e_j. Since $i \neq j$, this implies that $g_j \in K$ for all j. Consequently, $S(e_j) \in P$ and $S(P) \subseteq P$.

Let us now show that the restriction of S to P is a Lie derivation, i.e.,

$$S(\{u, v\}) = \{S(u), v\} + \{u, S(v)\} \tag{10}$$

for all u, v of the form (5). We prove (10) by induction on the polynomial length $d(u)+d(v)$. Suppose that $u = e_i$ and $v = e_j$. Since $e_i^d, e_j^d \in P_0$, we get

$$S(\{e_i^d, e_j^d\}) = D(\{e_i^d, e_j^d\}) = \{D(e_i^d), e_j^d\} + \{e_i^d, D(e_j^d)\}$$
$$= \{S(e_i^d), e_j^d\} + \{e_i^d, S(e_j^d)\} = \{de_i^{d-1}S(e_i), e_j\}de_j^{d-1} + de_i^{d-1}\{e_i, de_j^{d-1}S(e_j)\}$$
$$= d^2(d-1)e_i^{d-2}e_j^{d-1}S(e_i)\{e_i, e_j\} + d^2e_i^{d-1}e_j^{d-1}\{S(e_i), e_j\}$$
$$+d^2(d-1)e_i^{d-1}e_j^{d-2}S(e_j)\{e_i, e_j\} + d^2e_i^{d-1}e_j^{d-1}\{e_i, S(e_j)\}.$$

On the other hand,

$$\{e_i^d, e_j^d\} = d^2e_i^{d-1}e_j^{d-1}\{e_i, e_j\}$$

and

$$S(\{e_i^d, e_j^d\}) = S(d^2e_i^{d-1}e_j^{d-1}\{e_i, e_j\})$$
$$= d^2S(e_i^{d-1})e_j^{d-1}\{e_i, e_j\} + d^2e_i^{d-1}S(e_j^{d-1})\{e_i, e_j\} + d^2e_i^{d-1}e_j^{d-1}S(\{e_i, e_j\})$$
$$= d^2(d-1)e_i^{d-2}e_j^{d-1}S(e_i)\{e_i, e_j\} + d^2(d-1)e_i^{d-1}e_j^{d-2}S(e_j)\{e_i, e_j\} + d^2e_i^{d-1}e_j^{d-1}S(\{e_i, e_j\}).$$

Comparing two values of $S(\{e_i^d, e_j^d\})$, we get

$$S(\{e_i, e_j\}) = \{S(e_i), e_j\} + \{e_i, S(e_j)\}.$$

Suppose that $d(v) \geq 2$ and $v = v_1v_2$. Then

$$S(\{u, v\}) = S(\{u, v_1v_2\}) = S(v_1\{u, v_2\} + \{u, v_1\}v_2)$$
$$= S(v_1)\{u, v_2\} + v_1S(\{u, v_2\}) + S(\{u, v_1\})v_2 + \{u, v_1\}S(v_2).$$

By the induction proposition, (10) is true for pairs u, v_1 and u, v_2, i.e.,

$$S(\{u, v_1\}) = \{S(u), v_1\} + \{u, S(v_1)\}, \quad S(\{u, v_2\}) = \{S(u), v_2\} + \{u, S(v_2)\}.$$

Then

$$S(\{u, v\}) = S(v_1)\{u, v_2\} + v_1\{S(u), v_2\} + v_1\{u, S(v_2)\}$$
$$+\{S(u), v_1\}v_2 + \{u, S(v_1)\}v_2 + \{u, v_1\}S(v_2)$$
$$= \{S(u), v_1v_2\} + \{u, S(v_1)v_2 + v_1S(v_2)\} = \{S(u), v\} + \{u, S(v)\}.$$

Consequently, S is a derivation of a Poisson algebra P and induces D on P_0. □

Lemma 3. *Every locally nilpotent derivation of the Poisson algebra P_0 is induced by a locally nilpotent d-derivation of the Poisson algebra $P = P\langle x_1, x_2, \ldots, x_n\rangle$.*

Proof. Let D be a locally nilpotent derivation of P_0 and let S be a unique extension of D to P. We have to show that S is a locally nilpotent derivation of P. Notice that

$$P_0 \subset P$$

is an integral extension of domains since $e_i^d \in P_0$ for all $i \geq 1$. According to a result of W.V. Vasconcelos [24] (see also Proposition 1.3.37 from [3, p. 41]), S is locally nilpotent. □

4. Automorphisms of P_0

As we noticed above, every d-graded automorphism of $P\langle x_1, x_2, \ldots, x_n\rangle$ induces an automorphism of P_0. In this section we prove the reverse of this statement for $n > 1$.

Theorem 1. *Let K be a field closed with respect to taking all d-roots of elements. Then every automorphism of P_0 over K is induced by a d-graded automorphism of $P = P\langle x_1, x_2, \ldots, x_n\rangle$ if $n > 1$.*

Proof. Let α be an automorphism of P_0. Denote the extension of α to the quotient field $Q(P_0)$ by the same symbol. We have $\frac{x_2}{x_1} \in Q(P_0)$. Suppose that

$$\alpha\left(\frac{x_2}{x_1}\right) = \frac{f_2}{f_1}, \tag{11}$$

where f_1, f_2 are relatively prime. Then

$$\alpha\left(\frac{x_2^d}{x_1^d}\right) = \alpha\left(\frac{x_2}{x_1}\right)^d = \frac{f_2^d}{f_1^d}.$$

Since f_1, f_2 are relatively prime it follows that $\alpha(x_1^d) = vf_1^d$ and $\alpha(x_2^d) = vf_2^d$ for some $v \in P$. Moreover, $\alpha(x_1^i x_2^{d-i}) = vf_1^i f_2^{d-i}$ for all $0 \leq i < d$.

We have $vf_1^d, vf_2^d \in P_0$. If K is a field of characteristic $p > 0$ and p divides d, then $f_1^d, f_2^d \in P_0$. Consequently, $v \in P_0$. Assume that K is a field of charateristic 0 or of characteristic $p > 0$ and p does not divide d. Let ϵ be a primitive d-root of unity. Consider the automorphism ε of $Q(P)$ such that $\varepsilon(x_i) = \epsilon x_i$ for all i. Notice that for any $f \in Q(P)$ we have $f \in Q(P_0)$ if and only if $\varepsilon(f) = f$. Then

$$\frac{f_2}{f_1} = \varepsilon(\frac{f_2}{f_1}) = \frac{\varepsilon(f_2)}{\varepsilon(f_1)}$$

and $\varepsilon(f_1), \varepsilon(f_2)$ are relatively prime. Hence $f_1\varepsilon(f_2) = \varepsilon(f_1)f_2$ and, say, f_1 divides $\varepsilon(f_1)$ and $\varepsilon(f_1)$ divides f_1. Consequently f_1 and $\varepsilon(f_1)$ are proportional which is possible only if f_1 is a d-homogeneous element. Similarly, f_2 is a d-homogeneous element. Then $f_1^d, f_2^d \in P_0$ and, consequently, $v \in P_0$.

This implies that

$$x_1^d = \alpha^{-1}(v)\alpha^{-1}(f_1^d), \quad x_2^d = \alpha^{-1}(v)\alpha^{-1}(f_2^d).$$

Since x_1^d is irreducible in P_0, this is possible only if $v \in K$. Let $\mu \in K$ be a d-root of v, i.e., $\mu^d = v$. Replacing f_1 and f_2 by μf_1 and μf_2, we may assume that

$$\alpha(x_1^d) = f_1^d, \quad \alpha(x_2^d) = f_2^d. \tag{12}$$

By (11) and (12), we get

$$\alpha(x_1^{i_1}x_2^{i_2}) = f_1^{i_1}f_2^{i_2}, \text{ if } x_1^{i_1}x_2^{i_2} \in P_0.$$

Consider an arbitrary e_i with $i \geq 3$. Suppose that $\deg e_i = s$. Then $y_i = \frac{e_i}{x_1^s} \in Q(P_0)$. Suppose that

$$\alpha(y_i) = \alpha\left(\frac{e_i}{x_1^s}\right) = \frac{f_i}{g_i},$$

where f_i, g_i are relatively prime. Then

$$\alpha\left(\frac{e_i^d}{x_1^{sd}}\right) = \alpha\left(\frac{e_i}{x_1^s}\right)^d = \frac{f_i^d}{g_i^d}.$$

Again $\alpha(e_i^d) = vf_i^d$ and $\alpha(x_1^{sd}) = vg_i^d$ for some $v \in P$. As above, we get that $f_i^d, g_i^d \in P_0, v \in K$, and we can assume that

$$\alpha(e_i^d) = f_i^d, \quad \alpha(x_1^{sd}) = g_i^d.$$

Then $f_1^{sd} = g_i^d$ and $g_i = \lambda f_1^s$, where λ is a d-root of unity. After rescaling, we can assume that $g_i = f_1^s$ and

$$\alpha(e_i^d) = f_i^d, \quad \alpha(y_i) = \alpha\left(\frac{e_i}{x_1^s}\right) = \frac{f_i}{f_1^s}, \tag{13}$$

where $s = \deg e_i$ and $i \geq 3$. This is true for $i = 2$ by (11) and (12).

Let $u = e_{i_1} \ldots e_{i_k}$ be an arbitrary element of P of the form (5). We have

$$u = x_1^s \frac{e_{i_1}}{x_1^{s_{i_1}}} \ldots \frac{e_{i_k}}{x_1^{s_{i_k}}} = x_1^s y_{i_1} \ldots y_{i_k}, \tag{14}$$

where $s = s_{i_1} + \ldots + s_{i_k}$. We have $d|s$ since $u \in P_0$. Then

$$\alpha(u) = f_1^s \frac{f_{i_1}}{f_1^{s_1}} \ldots \frac{f_{i_k}}{f_1^{s_k}} = f_{i_1} \ldots f_{i_k}$$

by (11), (12), and (13).

Consequently, the polynomial endomorphism β of P, determined by $\beta(e_i) = f_i$ for all $i \geq 1$, induces α on P_0. First we show that β is a polynomial automorphism of P. The elements (4) are algebraically independent and, consequently, the elements $e_1^d, \ldots, e_s^d, \ldots$ are algebraically independent. Since α is an automorphism and $\alpha(e_i^d) = f_i^d$ for all i by (12) and (13), the elements $f_1^d, \ldots, f_s^d, \ldots$ are algebraically independent. Therefore the elements $f_1, \ldots, f_s, \ldots$ are algebraically independent and β is an injective endomorphism. Then β can be uniquely extended to an endomorphism of the quotient field $P(x_1, x_2, \ldots, x_n)$ and we denote this extension also by β.

The restriction of β on $Q(P_0)$ is an automorphism since it coincides with the α. Consider the space

$$V = Q(P_0)P\langle x_1, x_2, \ldots, x_n\rangle.$$

By (14) every element $f \in P$ can be written as

$$f = f_0 + f_1 x_1 + \ldots + f_{d-1} x_1^{d-1},$$

where $f_0, f_1, \ldots, f_{d-1} \in K[t, y_2, \ldots, y_s, \ldots]$ and $t = x_1^d$. Hence V is the $Q(P_0)$-span of the elements $1, x_1, x_1^2, \ldots, x_1^{d-1}$. If

$$V = b_1 Q(P_0) \oplus \ldots \oplus b_k Q(P_0),$$

then

$$\beta(V) = \beta(b_1)Q(P_0) + \ldots + \beta(b_k)Q(P_0)$$

since $\beta(Q(P_0)) = Q(P_0)$. Notice that $\beta(V) \subseteq V$. If $\beta(V) \neq V$ then $\dim_{Q(P_0)} V < k$ and $\mathrm{Ker}\beta \neq 0$. It is impossible for nonzero field endomorphisms. Consequently, $\beta(V) = V$

and $e_i \in \beta(V)$ for all i. Therefore β is an automorphism of the field $P(x_1, x_2, \ldots, x_n)$ and of the polynomial algebra $P\langle x_1, x_2, \ldots, x_n\rangle$.

It remains to show that β is a Lie automorphism of P, i.e.,

$$\beta(\{u, v\}) = \{\beta(u), \beta(v)\} \tag{15}$$

for all u, v of the form (5). We prove (15) by induction on the polynomial length $d(u)+d(v)$. Suppose that $u = e_i$ and $v = e_j$. Since $e_i^d, e_j^d \in P_0$, we get

$$\begin{aligned}\beta(\{e_i^d, e_j^d\}) &= \alpha(\{e_i^d, e_j^d\}) = \{\alpha(e_i^d), \alpha(e_j^d)\} = \{\beta(e_i^d), \beta(e_j^d)\}\\ &= \{\beta(e_i)^d, \beta(e_j)^d\} = d^2\beta(e_i)^{d-1}\beta(e_j)^{d-1}\{\beta(e_i), \beta(e_j)\}.\end{aligned}$$

On the other hand,

$$\beta(\{e_i^d, e_j^d\}) = \beta(d^2 e_i^{d-1} e_j^{d-1}\{e_i, e_j\}) = d^2\beta(e_i)^{d-1}\beta(e_j)^{d-1}\beta(\{e_i, e_j\}).$$

Comparing two values of $\beta(\{e_i^d, e_j^d\})$, we get that (15) holds for $u = e_i$ and $v = e_j$.

Suppose that $d(v) \geq 2$ and $v = v_1 v_2$. Then

$$\begin{aligned}\beta(\{u, v\}) &= \beta(\{u, v_1 v_2\}) = \beta(v_1\{u, v_2\} + \{u, v_1\}v_2)\\ &= \beta(v_1)\beta(\{u, v_2\}) + \beta(\{u, v_1\})\beta(v_2).\end{aligned}$$

By the induction proposition, we may assume that (15) is true for pairs u, v_1 and u, v_2. Then

$$\begin{aligned}\beta(\{u, v\}) &= \beta(v_1)\{\beta(u), \beta(v_2)\} + \{\beta(u), \beta(v_1)\}\beta(v_2)\\ &= \{\beta(u), \beta(v_1)\beta(v_2)\} = \{\beta(u), \beta(v)\}.\end{aligned}$$

Consequently, β is an automorphism of P and induces α on P_0. □

Let $\mathrm{Aut}_d P$ be the group of all d-graded automorphisms of the free Poisson algebra P.

Corollary 1. *Let K be a field closed with respect to taking all d-roots of elements and let $E = \{\lambda \mathrm{id} | \lambda^d = 1, \lambda \in K\}$, where* id *is the identity automorphism of P. Then*

$$\mathrm{Aut}\, P_0 \cong \mathrm{Aut}_d P/E.$$

Proof. Consider the homomorphism

$$\psi : \mathrm{Aut}_d P \to Aut\, P_0 \tag{16}$$

defined by $\psi(\alpha) = \overline{\alpha}$, where $\overline{\alpha}$ is the automorphism of P_0 induced by the d-graded automorphism α of P.

By Theorem 1, ψ is an epimorphism. Let $\alpha \in \mathrm{Ker}\,\psi$. Then $\alpha(x_1)^d = x_1^d$. Consequently, $\alpha(x_1) = \lambda x_1$ for some dth root of unity $\lambda \in K$. Extending α to $Q(P_0)$, we get $\alpha(x_i/x_1) = x_i/x_1$. Consequently, $\alpha(x_i) = \lambda x_i$ for all i and $\alpha = \lambda \mathrm{id}$, i.e., $\alpha \in E$. Obviously, $E \subseteq \mathrm{Ker}\,\psi$. □

5. Veronese subalgebras of polynomial algebras

Let $A = K[x_1, x_2, \ldots, x_n]$ be the polynomial algebra over a field K in the variables $x_1, x_2, \ldots, x_n$. Consider the grading

$$A = A_0 \oplus A_1 \oplus \ldots \oplus A_{d-1},$$

where $d \geq 2$ and A_i is the subspace of A generated by all monomials of degree $kd + i$ for all $k \geq 0$. This is a $\mathbb{Z}_d$-grading of A, i.e., $A_iA_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{Z}_d$. The subalgebra A_0 is called the *Veronese subalgera of A of degree d*.

Corollary 2. *Let $A = K[x_1, x_2, \ldots, x_n]$ be the polynomial algebra over a field K of characteristic zero in $n \geq 2$ variables $x_1, x_2, \ldots, x_n$. Then every derivation of the Veronese subalgebra A_0 can be uniquely extended to a d-graded derivation of $K[x_1, x_2, \ldots, x_n]$.*

Corollary 3. *Let $A = K[x_1, x_2, \ldots, x_n]$ be the polynomial algebra over a field K of characteristic zero in $n \geq 2$ variables $x_1, x_2, \ldots, x_n$. Then every locally nilpotent derivation of the Veronese subalgebra A_0 is induced by a locally nilpotent d-derivation of the polynomial algebra $K[x_1, x_2, \ldots, x_n]$.*

Corollary 4. *Let $A = K[x_1, x_2, \ldots, x_n]$ be the polynomial algebra in $n \geq 2$ variables $x_1, x_2, \ldots, x_n$ over a field K closed with respect to taking all d-roots of elements. Then every automorphism of the Veronese subalgebra A_0 of degree d is induced by a d-graded automorphism of $K[x_1, x_2, \ldots, x_n]$.*

This result is also proven in [8].

Let $\mathrm{Aut}_d A$ be the group of all d-graded automorphisms of the polynomial algebra A.

Corollary 5. *Let K be a field closed with respect to taking all d-roots of elements and let $E = \{\lambda \mathrm{id} | \lambda^d = 1, \lambda \in K\}$, where* id *is the identity automorphism of A. Then*

$$\mathrm{Aut}\, A_0 \cong \mathrm{Aut}_d A/E.$$

The proofs of Corollary 2, Corollary 3, Corollary 4, and Corollary 5 repeat the polynomial parts of the proofs of Lemma 2, Lemma 3, Theorem 1, and Corollary 1, respectively.

Notice that these statements are not true for the polynomial algebra $A = K[x]$ in one variable x. In this case, the Veronese subalgebra A_0 of degree d is the polynomial algebra in one variable x^d. Then the locally nilpotent derivation of A_0 determined by

$$x^d \mapsto 1$$

cannot be induced by any derivation of A and the automorphism of A_0 determined by

$$x^d \mapsto x^d + 1$$

cannot be induced by any automorphism of A.

In addition, analogues of these results are not true for free associative algebras. In fact, if $B = K\langle x, y\rangle$ is the free associative algebra in the variables x, y and $d = 2$ then the Veronese subalgebra B_0 of degree d is the free associative algebra in the variables x^2, xy, yx, y^2. It is easy to check that the locally nilpotent derivation of B_0 determined by

$$x^2 \mapsto 1, xy \mapsto 0, yx \mapsto 0, y^2 \mapsto 0$$

cannot be induced by any derivation of B and the automorphism of B_0 determined by

$$x^2 \mapsto x^2 + 1, xy \mapsto xy, yx \mapsto yx, y^2 \mapsto y^2$$

cannot be induced by any automorphism of B.

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