# On Classification of $\mathbb{Q}$-Fano 3-folds
# of Gorenstein index 2. III

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*Dedicated to Professor Yujiro Kawamata on the occasion of his 70th birthday*

ABSTRACT. We classified prime $\mathbb{Q}$-Fano 3-folds $X$ with only $1/2(1, 1, 1)$-singularities and with $h^0(-K_X) \geq 4$ a long time ago. The classification was undertaken by blowing up each $X$ at one $1/2(1, 1, 1)$-singularity and constructing a Sarkisov link. The purpose of this paper is to reveal the geometries behind the Sarkisov links for $X$ in 5 classes. The main result asserts that any $X$ in the 5 classes can be embedded as linear sections into bigger dimensional $\mathbb{Q}$-Fano varieties called key varieties, where the key varieties are constructed by extending partially the Sarkisov link in higher dimensions.

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## 1. INTRODUCTION

1.1. **Background.** In this paper, we work over $\mathbb{C}$, the complex number field.

This is a continuation of the papers [Tak1] after a long time. A projective variety $X$ is called a ℚ-*Fano variety* if $X$ has only terminal singularities and $-K_X$ is ample. A ℚ-Fano variety $X$ is called *prime* if $-K_X$ generates the group of numerical equivalence classes of ℚ-Cartier divisors on $X$. In [Tak1], we classified prime ℚ-Fano 3-folds $X$ with only $1/2(1,1,1)$-singularities and with $h^0(-K_X) \geq 4$. In this paper, we further study $X$ in the 5 classes No.1.1, 1.4, 1.9, 1.10, and 1.13 among [Tak1, Table 1].

1.2. **Prime ℚ-Fano 3-fold and Sarkisov link.** In this subsection, we explain our method of the classification of prime ℚ-Fano 3-folds in [Tak1] only in the five classes. The result is presented in the following table:

| No. | $g(X)$ | $N$ | $e$ | $\deg C$ | $g(C)$ | $X'$ |
|-----|--------|-----|-----|----------|--------|------|
| 1.1 | 4 | 2 | 7 | 7 | 8 | $\mathbb{P}(1^3, 2)$ |
| 1.4 | 5 | 1 | 6 | 9 | 9 | $\mathbb{P}^3$ |
| 1.9 | 6 | 1 | 6 | 3 | 0 | $B_3$ |
| 1.10 | 6 | 1 | 5 | 9 | 6 | $Q^3$ |
| 1.13 | 8 | 1 | 4 | 7 | 2 | $B_5$ |

The number $g(X)$ in the second column of the table is the *genus* of $X$ defined to be $h^0(-K_X) - 2$. The number $N$ in the third column is the number of $1/2(1,1,1)$-singularities of $X$. We explain the data in 4th–7th column below. For $X$'s in the 5 classes, we classify them by constructing the following Sarkisov links:

(1.1)

$$Y \dashrightarrow Y'$$

$$f \swarrow \qquad \searrow f'$$

$$X \qquad\qquad\qquad X',$$

where $f\colon Y \to X$ is the blow-up of $X$ at a $1/2(1,1,1)$-singularity, $Y \dashrightarrow Y'$ is a flop, and $f'$ is the blow-up of a ℚ-Fano 3-fold $X'$ along a smooth curve $C$ in $X' \setminus \operatorname{Sing} X'$ with the genus $g(C)$ and the degree $\deg C$ as in the 5th and 6th column of the table, where the degree of $C$ is measured by the primitive *Cartier* divisor on $X'$. We denote by $E$ the $f$-exceptional divisor and by $\widetilde{E}$ the strict transform on $Y'$ of $E$. The flop $Y \dashrightarrow Y'$ is the $E$-flop in the sense of [Ko]. The number $e$ in the

4th column is defined to be $E^3 - \widetilde{E}^3$, which roughly measures how many flopping curves the flop $Y \dashrightarrow Y'$ has (see [Tak1] for more details). In the 7th column, $B_3$ is a smooth cubic 3-fold in $\mathbb{P}^4$, $B_5$ is a codimension $3$ smooth linear section of $\mathrm{G}(2,5)$, and $Q^3$ is a smooth quadric 3-fold. Let $h\colon Y \to W$ be the $E$-flopping contraction. We call $W$ the *mid point* (for No. 1.1, however, see Caution in the subsection 2.3).

For simplicity, we call a $\mathbb{Q}$-Fano 3-fold $X$ as in the table as follows:

**Names of $\mathbb{Q}$-Fano 3-folds:** *a prime $\mathbb{Q}$-Fano 3-fold of genus 4, of genus 5, of genus 6 and* C*-type, of genus 6 and* Q*-type, of genus 8,* for $X$ of No.1.1, 1.4, 1.9, 1.10, and 1.13 respectively, where the word C-type (resp. Q-type) comes from the fact that $X'$ is isomorphic to the cubic 3-fold for No.1.9 (resp. the quadric 3-fold for No.1.10).

1.3. **Main result.** The main result of this paper is a classification of $\mathbb{Q}$-Fano 3-folds in the 5 classes in different nature to that in [Tak1]. The prototype of this result is the following: in [Gu], Gushel$'$ shows that any smooth prime Fano 3-fold of genus 8 is a linear section of $\mathrm{G}(2,6)$. In [Mu1, Mu2], Mukai shows that any prime smooth prime Fano 3-fold of genus 7, 9, 10 is a linear section of the orthogonal Grassmanian $\mathrm{OG}(5,10)$, the symplectic Grassmannian $\mathrm{Sp}(3,6)$, and the adjoint homogeneous variety of type $G_2$, respectively. Here we say that a projective variety $X$ is a *linear section* of a projective variety $\Sigma$ with respect to a linear system $|M|$ if it holds that $X = \Sigma \cap D_1 \cap \cdots \cap D_k$ for $k = \dim \Sigma - \dim X$ and some $D_1, \ldots, D_k \in |M|$. In the case of Gushel$'$ and Mukai, the linear system is the one generated by the primitive very ample divisor of a homogeneous space. We usually do not mention the linear system $|M|$ if $M$ generates the group of the numerical equivalence classes of $\mathbb{Q}$-Cartier divisors on $\Sigma$.

**Theorem 1.1** (Embedding theorem)**.** *For each one of the $5$ classes, there is a unique rational $\mathbb{Q}$-Fano variety $\Sigma$ of Picard number $1$ such that any prime $\mathbb{Q}$-Fano 3-fold $X$ in the class is a linear section of $\Sigma$. The $\mathbb{Q}$-Fano varieties $\Sigma$ are of $11$-, $12$-, $9$-, $8$-, and $5$-dimensional for $X$ of genus $4$, $5$, of genus $6$ and* Q*-type, of genus $6$ and* C*-type, and of genus $8$, respectively.*

For a prime $\mathbb{Q}$-Fano 3-fold $X$ in each of the 5 classes, we will call the variety $\Sigma$ *the key variety* for $X$. Theorem 1.1 is proved separately in each case; Theorem 3.7 (genus 8), Theorem 5.17 (genus 4, 6), and Theorem 6.15 (genus 5). We refer for more detailed descriptions of $\Sigma$ (constructions, birational geometries, singularities, etc) to the section 3 (genus 8), the sections 4 and 5 (genus 4, 6), and the section 6 (genus 5).

The equations of the key varieties are also available; see [R, Ex.6.8] in the genus 4 case, and [Ha] in the genus 5 case. In the genus 6 and 8 cases, we will publish them in separated papers (cf. [Tak4]).

1.4. **Structure of the paper.** The section 2: After showing some miscellaneous results in the subsection 2.1, we investigate in the subsections 2.2–2.5 the mid point of the Sarkisov link (1.1) in details in each of 5 cases.

The section 3: In this section we concentrate in studying the genus 8 case. In the subsection 3.1, we extend the mid point based upon the result in the subsection 2.5. In the subsection 3.2, we construct the key variety modifying birationally the extension of the mid point (Theorem 3.6). In the subsection 3.3, we show Theorem 1.1 in the genus 8 case (Theorem 3.7).

In the other cases, constructions of the key varieties and proofs of Theorem 1.1 are similar to those in the genus 8 case but are more involved.

The genus 4 and 6 cases are treated in a unified way in part in the sections 4 and 5; The mid points are extended in Propositions 4.3, 4.9 and 4.14. Together with compensations in the subsection 5.3, the key varieties are constructed in Theorem 5.11, and Theorem 1.1 is proved in Theorem 5.17.

The genus 5 case is treated in the section 6; The mid point is extended in the subsection 6.1, the key variety is constructed in the subsection 6.2, and Theorem 1.1 is proved in the subsection 6.3.

1.5. **Flow of the construction of the key variety .** Roughly speaking, the key variety in any case is constructed in the following manner: First we extend the mid point of the Sarkisov link (1.1) to an appropriate variety $\overline{\Sigma}$ (in the genus 4 case, the mid point is replaced by another 3-fold in the subsection 2.3). The extension $\overline{\Sigma}$ is found more or less naturally from the equation of the mid point. Second we construct a good resolution of $\overline{\Sigma}$. Except in the case of genus 6 and C-type, we can construct a small crepant resolution $\Sigma' \to \overline{\Sigma}$ with a projective bundle structure over certain Fano manifold. Except in the case of genus 4, the Fano manifold is $X'$ as in (1.1). The small resolution is actually an extension of $Y'$ in (1.1). Ideally, as the third step, we would construct a small birational map $\Sigma' \dashrightarrow \widetilde{\Sigma}$ such that $\widetilde{\Sigma}$ is an extension of $Y$ and find a contraction $\widetilde{\Sigma} \to \Sigma$ such that $\Sigma$ is the desired key variety. This strategy works in the genus 8 case. Even in the other cases except the case of genus 6 and C-type, this works but the construction of $\Sigma' \dashrightarrow \widetilde{\Sigma}$ is slightly involved (we refer a more detailed explanation of this to Remark 4.7). Hence we choose another resolution $\widehat{\Sigma} \to \overline{\Sigma}$ except in the genus 8 case. The variety $\widehat{\Sigma}$ has a structure of a projective bundle over certain Fanifold. Then we perform a small birational map $\Sigma' \dashrightarrow \widetilde{\Sigma}$ which is a composite of a flop and a flip such that $\widetilde{\Sigma}$ is an extension of $Y$ and find a contraction $\widetilde{\Sigma} \to \Sigma$ such that $\Sigma$ is the desired key variety. the advantage of this construction is that the flop and the flip can be described easily. Moreover, this works also in the genus 6 and C-type. Actually, the cases except the genus 5 case can be treated in a unified way in part. For unified treatement, we take a bit roundabout way. We refer for this to the sections 4 and 5. The case of genus 5 can be treated more or less in a straightforward way. We refer for this to the section 6.

1.6. **Future plan.** In this subsection, we use the notation as in the subsection 1.5. The construction of the key variety in each case is slightly involved but we have a significant application; in the forthcoming paper [Tak3], we construct a projective bundle which, in a certain sense, is dual to $\widehat{\Sigma}$ in the case of genus 6 and C-type, or is dual to $\Sigma'$ in the other cases. Using these dual varities, we can describe the cubic 3-fold $X'$ in the case of genus 6 and C-type, or the curve $C$ as in the Sarkisov link (1.1) in the other cases.

### Notation and Conventions

- *Conventions on projective bundle*: Let $\mathcal{E}$ be a vector bundle on a variety $X$, or a vector space. The notation $\mathbb{P}(\mathcal{E})$ is just the projectivization of $\mathcal{E}$ (*We don't use the Grothendieck notation*). Setting $\Sigma = \mathbb{P}(\mathcal{E})$, we often denote by $\mathcal{O}_\Sigma(1)$, or $H_\Sigma$ *the tautological line bundle associated to the vector bundle $\mathcal{E}$ without mentioning $\mathcal{E}$.*

- *Point of a projective space*: Let $V$ be a vector space. For a nonzero vector $\boldsymbol{x} \in V$ and a 1-dimensional subspace $V^1 \subset V$, we denote by $[\boldsymbol{x}]$ and $[V^1]$ the point of $\mathbb{P}(V)$ corresponding to $\boldsymbol{x}$ and $V^1$ respectively.
- *Cartier divisor and invertible sheaf*: We sometimes abuse notation of a Cartier divisor and an invertible sheaf. For example, we sometimes use the expression like $D = f^* \mathcal{O}_X(1)$.
- *Join $X_a * X_b$* : The projective variety in $\mathbb{P}(V_a \oplus V_b)$ which is the union of all the lines joining two projective varieties $X_a \subset \mathbb{P}(V_a \oplus 0)$ and $X_b \subset \mathbb{P}(0 \oplus V_b)$. If $X_b$ is a projective space, $X_a * X_b$ is just the cone over $X_a$ with the vertex $X_b$.
- *Flopping contraction of Atiyah type*: The small contraction $f \colon X \to Y$ such that, for a sufficiently small analytic neighborhood $U$ of any point $y$ of $Y$ in the image of the $f$-exceptional locus, $f|_{f^{-1}(U)} \colon f^{-1}(U) \to U$ is isomorphic to the product of some open subset $U_0 \subset \mathbb{C}^{n-3}$ with a small resolution $X_0 \to \{xy + zw = 0\} \subset \mathbb{C}^4$.
- $1/2\,(1^n)$-*singularity*: The singularity of an $n$-dimensional variety analytically isomorphic to that of the origin of the quotient of $\mathbb{C}^n$ by the involution defined by $\mathbb{C}^n \ni \boldsymbol{x} \mapsto -\boldsymbol{x} \in \mathbb{C}^n$. We often call this singularity a *1/2-singularity* for simplicity.
- $Q^n$: The *smooth* quadric $n$-fold.

## 2. Preliminaries

### 2.1. Miscellaneous results.
The results in this subsection are frequently used in the sequel. Proofs for them are omitted since they are elementary.

**Lemma 2.1.** *Let $V$ be a vector space, and $V = V^1 \oplus V'$ be a direct sum decomposition with a 1-dimensional subspace $V^1$ and a complementary subspace $V'$. The vector space $\wedge^2 V$ has the following direct sum decomposition:*

$$\wedge^2 V = V' \oplus \wedge^2 V',$$

*where we identify the subspace $V^1 \wedge V'$ with $V'$. Let $U \subset \wedge^2 V'$ be a subspace. For $\boldsymbol{x} \in V'$ and $\boldsymbol{y} \in U$, the following are equivalent:*

*(1) $\boldsymbol{x} + \boldsymbol{y} \in \mathrm{G}(2, V) \cap \mathbb{P}(V' \oplus U)$.*
*(2) There exists a 2-dimensional subspace $V^2 \subset V'$ such that $\wedge^2 V^2 \subset U$, $\boldsymbol{x} \in V^2$ and $\boldsymbol{y} \in \wedge^2 V^2$.*

**Lemma 2.2.** *Let $S$ be a projective manifold and $\mathcal{A}, \mathcal{B}$ vector bundles on $S$ whose dual bundles are globally generated. Let $U_{\mathcal{A}} := H^0(S, \mathcal{A}^*)^*$ and $U_{\mathcal{B}} := H^0(S, \mathcal{B}^*)^*$.*

*Let $p\colon \mathbb{P}_S(\mathcal{A} \oplus \mathcal{B}) \to S$ be the natural morphism and $\mu\colon \mathbb{P}_S(\mathcal{A} \oplus \mathcal{B}) \to \mathbb{P}(U_{\mathcal{A}} \oplus U_{\mathcal{B}})$ the morphism defined by the tautological linear system $|H_{\mathbb{P}(\mathcal{A} \oplus \mathcal{B})}|$. The following assertions hold:*

*(1) The projective bundle $\mathbb{P}_S(\mathcal{A} \oplus \mathcal{B})$ is contained in $\mathbb{P}(U_{\mathcal{A}} \oplus U_{\mathcal{B}}) \times S$ as a subbundle, and the morphism $\mu$ is nothing but the composite $\mathbb{P}_S(\mathcal{A} \oplus \mathcal{B}) \hookrightarrow \mathbb{P}(U_{\mathcal{A}} \oplus U_{\mathcal{B}}) \times S \to \mathbb{P}(U_{\mathcal{A}} \oplus U_{\mathcal{B}})$. The pull-back of $\mathcal{O}_{\mathbb{P}(U_{\mathcal{A}} \oplus U_{\mathcal{B}})}(1)$ by this morphism is the tautological line bundle of $\mathbb{P}_S(\mathcal{A} \oplus \mathcal{B})$.*

*(2) For a point $s \in S$, let $\mathcal{A}_s$ and $\mathcal{B}_s$ the fibers of $\mathcal{A}$ and $\mathcal{B}$ at $s$ respectively, which are subspaces of $U_{\mathcal{A}}$ and $U_{\mathcal{B}}$ respectively. The $\mu$-image coincides the locus*

$$\{[\boldsymbol{x} + \boldsymbol{y}] \in \mathbb{P}(U_{\mathcal{A}} \oplus U_{\mathcal{B}}) \mid \exists_{s \in S}, \boldsymbol{x} \in \mathcal{A}_s, \boldsymbol{y} \in \mathcal{B}_s\}$$

*and the $\mu$-fiber over a point $[\boldsymbol{x} + \boldsymbol{y}]$ coincides with the locus $\{s \in S \mid \boldsymbol{x} \in \mathcal{A}_s, \boldsymbol{y} \in \mathcal{B}_s\}$.*

Lemma 2.2 also holds for a direct sum of three or more vector bundles.

### 2.2. Indecomposability of the mid point .

In this subsection, we quickly review the classification of the mid point $W$ of the Sarkisov link (1.1) with a few compensation. An important concept for the classification is indecomposability of an effective divisor due to Mukai.

**Definition 2.3.** Let $X$ be a normal projective variety and $D$ a Weil divisor on $X$. We say that $D$ is *indecomposable* if there exists no Weil divisors $A$ and $B$ such that $D \sim A + B$ and $h^0(A) \geq 2$ and $h^0(B) \geq 2$. If $-K_X$ is indecomposable, then we say $X$ is indecomposable.

An indecomposable $\mathbb{Q}$-Fano variety generalizes a prime $\mathbb{Q}$-Fano variety for possibly non $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano variety. In our context, we have the following:

**Proposition 2.4.** *The mid point $W$ is indecomposable. In the genus 5,6, or 8 case, the anti-canonical divisor $-K_W$ is very ample. The image $\Pi$ of $E$ on $W$ is a plane.*

*Proof.* Assume by contradiction that there exist Weil divisors $A$ and $B$ such that $-K_W \sim A + B$ and $h^0(A) \geq 2$ and $h^0(B) \geq 2$. Then we have $-K_Z \sim h_*^{-1}(A) + h_*^{-1}(B) + \Delta$ with an effective $h$-exceptional divisor $\Delta$ (possibly equal to 0) since $g$ is crepant. Further we have $-K_X \sim g_* h_*^{-1}(A) + g_* h_*^{-1}(B) + g_*(\Delta)$ with $h^0(g_* h_*^{-1}(A)) \geq 2$ and $h^0(g_* h_*^{-1}(B)) \geq 2$. This is a contradiction since $X$ is prime.

In the genus 5,6, or 8 case, $W$ is Gorenstein. Hence, by [Mu2, Thm.6.5 (2) and Prop.7.8], $-K_W$ is very ample. Since $(-K_W)^2 E = 1$, we see that $\Pi$ is a plane. $\quad\square$

**Corollary 2.5.** *In the genus 5, 6, or 8 case, the rational map $W \dashrightarrow X'$ in the Sarkisov link (1.1) is the projection of $W$ from the plane $\Pi$.*

*Proof.* In the genus 5, 6, or 8 case, we have $E' \sim z(-K_{Y'}) - (z + 1)\widetilde{E}$, where $z + 1$ is the Fano index of $X'$ by [Tak1, Part I, Table 1]. Since $f'$ is the blow-up along the curve $C$, we have $-K_{Y'} = f'^*(-K_{X'}) - E'$. Therefore we obtain $-K_{Y'} - \widetilde{E} = f'^* H_{X'}$ where $H_{X'}$ is the primitive Cartier divisor. This implies the assertion since $-K_{Y'}$ is the pull-back of $\mathcal{O}_W(1)$. $\quad\square$

### 2.3. Mid point in the genus 4 case .

In the genus 4 case, we slightly modify the Sarkisov link (1.1) partially.

Let $g\colon Z \to X$ be the blow-up at the two $1/2$-singularities, and $E_1$ and $E_2$ be the $g$-exceptional divisors. By [Tak1, Part II, Thm.1.0], $-K_Z$ is nef and big. Let $h\colon Z \to \overline{Z}$ be the anti-canonical model and $\Pi_1$ and $\Pi_2$ the images on $\overline{Z}$ of $E_1$ and

$E_2$ respectively. In the same way as in the proof of Proposition 2.4, we see that $\overline{Z}$ is indecomposable, $-K_{\overline{Z}}$ is very ample and $\Pi_1$ and $\Pi_2$ are planes on $\overline{Z}$. By [Mu2, Thm.6.5 (2) and Prop.7.8], $\overline{Z}$ is the intersection of a quadric and a cubic.

**Proposition 2.6.** *The following assertions hold:*

(1) *It holds that $\Pi_1 \cap \Pi_2 = \emptyset$.*

(2) *$h\colon Z \to \overline{Z}$ is a crepant small contraction, and hence $\overline{Z}$ has only terminal singularities.*

*Proof.* First of all, we show that $h$ has no exceptional divisor $G$ such that $h(G)$ is a point. Assume by contraction that there is such a divisor $G$. Since $G$ is a crepant divisor while $-K_X$ is ample, we see that $G$ intersects $E_1$ or $E_2$. We may assume that $G \cap E_1 \neq \emptyset$. Then any irreducible component of $G \cap E_1$ is a curve which is at the same time numerically trivial and negative for $K_Y$, a contradiction.

Assume by contradiction that $\Pi_1 \cap \Pi_2 \neq \emptyset$. Let t be a point of $\Pi_1 \cap \Pi_2$. Let $\gamma_1 \cup \cdots \cup \gamma_k$ be the irreducible decomposition of the fiber of $Z \to W$ over t, where any $\gamma_l$ is a curve by the first paragraph. For any $l$, it holds that $E_1 \cdot \gamma_l > 0$ or $E_2 \cdot \gamma_l > 0$, and there exist $i$ and $j$ such that $E_1 \cdot \gamma_i > 0$ and $E_2 \cdot \gamma_j > 0$. We fix a curve $\gamma_j$ such that $E_2 \cdot \gamma_j > 0$. Since $E_2$ is mapped isomorphically onto the plane $\Pi_2$, we have $E_2 \cdot \gamma_j = 1$. Now we consider $Y$ in the Sarkisov link (1.1) as the target of the contraction of $E_2$ from $Z$ and $E$ as the image of $E_1$. Let $\gamma'_j$ be the image of $\gamma_j$ on $Y$. We have $-K_Y \cdot \gamma'_j = \frac{1}{2}$ since $E_2 \cdot \gamma_j = 1$. Let $\gamma''_j$ be the strict transform of $\gamma'_j$ on $Y'$. By a property of flop (cf.[Ko]), we have $-K_{Y'} \cdot \gamma''_j = -K_Y \cdot \gamma'_j = \frac{1}{2}$. By [Tak1, Part I, Table 1], we have $E' \sim 4(-K_{Y'}) - 5\widetilde{E}$. This implies that $E' \cdot \gamma''_j = 2 - 5\widetilde{E} \cdot \gamma''_j$. If $\widetilde{E} \cdot \gamma''_j > 0$, then $E' \cdot \gamma''_j \leq -3$ and hence $\gamma''_j \subset E'$. This is impossible since $E'$ does not contain the $\frac{1}{2}$-singularity of $Y'$ while $\gamma''_j$ contains it. Thus $\widetilde{E} \cdot \gamma''_j = 0$, and hence $E_1 \cdot \gamma_j = 0$ and $\gamma_j$ does not intersect any $\gamma_l$ such that $E_1 \cdot \gamma_l > 0$. This implies that the fiber of $Z \to W$ over t is disconnected, a contradiction.

If $h$ is a crepant divisorial contraction, then, by the first paragraph, the $h$-exceptional locus contains a prime divisor, say $G$, such that $h(G)$ is a curve, and $G$ intersects $E_1$ or $E_2$. Assume that $E_1 \cap G \neq \emptyset$ and $E_2 \cap G \neq \emptyset$. By the argument of the first paragraph, $E_i \cap G$ $(i = 1, 2)$ cannot contain an $h$-exceptional curve, and hence $E_i \cap G$ dominates $h(G)$. This implies that $\Pi_1 \cap \Pi_2 \neq \emptyset$, a contradiction to (1). Therefore, we may assume that $E_1 \cap G \neq \emptyset$ and $E_2 \cap G = \emptyset$. Again, we consider $Y$ in the Sarkisov link (1.1) as the target of the contraction of $E_2$ from $Z$ and $E$ as the image of $E_1$. Then the image of $G$ on $Y$ is a crepant divisor. This is impossible since $Y$ has no crepant divisorial contraction by [Tak1, Part I, Table 1]. $\qquad\square$

**Caution (change of notation):** In the genus 4 case, henceforth we set

$$W := \overline{Z}$$

for notational convenience. We also call this $W$ the *mid point* in the genus 4 case.

2.4. **Mid point in the genus 6 case .** By [Mu2, Thm.6.5 (2) and Prop.7.8], the mid point $W$ is a quadric section of a del Pezzo 4-fold $W_0$ with only canonical singularities. The indecomposability simplifies the situation as follows:

**Proposition 2.7.** *The del Pezzo 4-fold $W_0$ is smooth.*

*Proof.* Assume that $W_0$ is singular and is not a cone over the smooth quintic del Pezzo 3-fold $B_5$ (we do not exclude the possibility that $W_0$ is a cone over a singular quintic del Pezzo 3-fold). Then, by [Fuj3, p.160, (6)], $W_0$ contains a double point, say, t. By projecting $W_0$ from t, $W_0$ is mapped onto a non-degenerate cubic 4-fold $W_0'$ in $\mathbb{P}^6$. By the classification of $W_0'$, we see that $\mathcal{O}_{W_0'}(1)$ is decomposable. This implies that $\mathcal{O}_{W_0}(1)$ is, and hence $-K_W$ is decomposable, a contradiction.

Assume that $W_0$ is the cone over $B_5$. If $W$ do not contain the vertex of $W_0$, then, by projecting $W_0$ from the vertex, $B_5$ contains the plane which is the image of $\Pi$. This is absurd since $B_5$ does not contain a plane. Therefore $W$ contains the vertex of $W_0$, and hence $W$ has a non-hypersurface singularity at the vertex. This is again absurd since $W$ has only Gorenstein terminal singularities. $\square$

By [Fuj3], we can write $W_0 = \mathrm{G}(2, V) \cap \mathbb{P}(U^8)$ with $V \simeq \mathbb{C}^5$ and $U^8 \simeq \mathbb{C}^8 \subset \wedge^2 V$. We write $W = W_0 \cap Q$, where $Q$ is a quadric 6-fold in $\mathbb{P}(U^8)$. It is well-known that the 2-plane $\Pi$ has one of the following description as a subvariety of $\mathrm{G}(2, V)$:

(1)
$$\Pi = \{ [\mathbb{C}^2] \mid V^1 \subset \mathbb{C}^2 \subset V^4 \} \subset \mathrm{G}(2, V)$$
with some fixed vector subspaces $V^1 \simeq \mathbb{C}$ and $V^4 \simeq \mathbb{C}^4$ of $V$.

(2)
$$\Pi = \{ [\mathbb{C}^2] \mid \mathbb{C}^2 \subset V^3 \} \subset \mathrm{G}(2, V)$$
with a fixed vector subspace $V^3 \simeq \mathbb{C}^3$ of $V$.

**Proposition 2.8.** *If $X$ is of* Q-*type, then $\Pi$ satisfies* (1). *If $X$ is of* C-*type, then $\Pi$ satisfies* (2).

*Proof.* Since $W \dashrightarrow X'$ is the projection from $\Pi$ by Corollary 2.5, $X'$ is contained in the image $W_0'$ of the projection of $W_0$ from $\Pi$. By [Fuj3] (see also [IsP, Lem.3.4.4]), $W_0'$ is a smooth quadric 3-fold $Q^3$ if $\Pi$ satisfies (1), or is $\mathbb{P}^4$ if $\Pi$ satisfies (2). If $X$ is of C-type, then $X'$ is a smooth cubic 3-fold, hence $\Pi$ must satisfy (2). Assume by contradiction that $X$ is of Q-type and $\Pi$ satisfies (2). Then, since $X'$ is a smooth quadric 3-fold $Q^3$, we may choose the quadric 6-fold $Q$ as the cone over $Q^3$ with the vertex $\Pi$ . This implies that $W$ is singular along $\Pi$, a contradiction. $\square$

**Q-type**:

**Caution (change of notation)**: Hereafter, for notational convenience, we denote by $\Pi_0$ the 2-plane $\Pi$ only in this case.

Note that $\Pi_0$ satisfies (1). The notation $\Pi$ will denote the unique 3-plane

$$\Pi := \{ [\mathbb{C}^2] \mid V^1 \subset \mathbb{C}^2 \} \subset \mathrm{G}(2, V)$$

containing $\Pi_0$. It holds that $\Pi_0 = \Pi \cap \mathbb{P}(U^8)$. By a simple dimension count of linear subspaces, the linear hull $\mathbb{P}(U^9)$ of $\Pi \cup \mathbb{P}(U^8)$ is a hyperplane of $\mathbb{P}(\wedge^2 V)$. We set

$$A_{\mathbb{Q}} := \mathrm{G}(2, V) \cap \mathbb{P}(U^9).$$

Since $W_0$ is a linear section of $\mathrm{G}(2, V)$, so is $A_{\mathbb{Q}}$. Thus $A_{\mathbb{Q}}$ is not a cone over $W_0$ since otherwise $A_{\mathbb{Q}}$ cannot be contained in $\mathrm{G}(2, 5)$. We can show that $A_{\mathbb{Q}}$ is actually smooth in the same way as the proof of Proposition 2.7. Now we produce the situation as in Lemma 2.1. We choose a direct sum decomposition $V = V^1 \oplus V'$ with a complementary subspace $V'$ to $V^1$. Note that $\Pi = \mathbb{P}(V^1 \wedge V')$, which we identify with $\mathbb{P}(V')$. There exists a 5-dimensional subspace $U^5 \subset \wedge^2 V'$ such that $U^9 = V' \oplus U^5$.

The projection of $\mathrm{G}(2,V)$ from the 3-plane $\Pi$ induces the natural rational map $A_{\mathbb{Q}} \dashrightarrow \mathrm{G}(2,\mathrm{V}') \cap \mathbb{P}(U^5)$ and the target $\mathrm{G}(2,\mathrm{V}') \cap \mathbb{P}(U^5)$ is nothing but the smooth quadric 3-fold $Q^3$ as in the proof of Proposition 2.8.

By [Fuj3], the pair $(A_{\mathbb{Q}}, \Pi)$ is unique up to projective equivalence. We may take the following coordinates: Let $e_i\,(1 \leq i \leq 5)$ be a basis of $V$. We set $V^1 := \mathbb{C}e_1$, and $V' :=$ the subspace of $V$ generated by $e_2, \ldots, e_5$. Let $z_i\,(1 \leq i \leq 5)$ be the coordinate for $e_i$ and $x_{ij}$ the Plücker coordinate for $e_i \wedge e_j\,(1 \leq i < j \leq 5)$. We set

$$\Pi := \{x_{ij} = 0\,(2 \leq i \leq j \leq 5)\} \subset \mathbb{P}(\wedge^2 V),$$
$$U^9 := \{x_{24} - x_{35} = 0\} \subset \wedge^2 V,$$
$$U^5 := \{x_{24} - x_{35} = 0\} \subset \wedge^2 V'.$$

Moreover, dropping the coordinate $x_{24}$ by the equality $x_{24} = x_{35}$, we consider $A_{\mathbb{Q}}$ as a subvariety of $\mathbb{P}^8$ with coordinates $z_2, z_3, z_4, z_5$ and ${}^t\boldsymbol{x} := \begin{pmatrix} x_{23} & x_{25} & x_{34} & x_{35} & x_{45} \end{pmatrix}$ defined by the following equations:

**Equation of $A_{\mathbb{Q}}$**

(2.1) $$N_{\mathbb{Q}}\boldsymbol{x} = \boldsymbol{o},\; x_{23}x_{45} - x_{35}^2 + x_{25}x_{34} = 0,$$

where we set

$$N_{\mathbb{Q}} := \begin{pmatrix} z_4 & 0 & z_2 & -z_3 & 0 \\ z_5 & -z_3 & 0 & z_2 & 0 \\ 0 & -z_4 & 0 & z_5 & z_2 \\ 0 & 0 & z_5 & -z_4 & z_3 \end{pmatrix}.$$

In this situation,

$$\Pi = \{\boldsymbol{x} = \boldsymbol{o}\},$$
$$Q^3 = \{x_{23}x_{45} - x_{35}^2 + x_{25}x_{34} = 0\} \subset \mathbb{P}^4.$$

**$\mathbb{C}$-type:**

In this case, we set

$$A_{\mathbb{C}} := W_0.$$

Since $\Pi = \mathbb{P}(\wedge^2 V^3)$ in this case, we may write $U^8 = \wedge^2 V^3 \oplus U^5$ with some $U^5 \simeq \mathbb{C}^5$. The projection of $\mathrm{G}(2,V)$ from the 2-plane $\Pi$ induces the natural rational map $A_{\mathbb{C}} \dashrightarrow \mathbb{P}(U^5)$ and the target $\mathbb{P}(U^5)$ is nothing but $\mathbb{P}^4$ as in the proof of Proposition 2.8. Let $a\colon \widehat{A}_{\mathbb{C}} \to A_{\mathbb{C}}$ is the blow-up of $A_{\mathbb{C}}$ along $\Pi$. By a general property of linear projection, a morphism $b\colon \widehat{A}_{\mathbb{C}} \to \mathbb{P}(U^5)$ is induced. By [Fuj3, Sect.10], $b$ is the blow-up of $\mathbb{P}(U^5)$ along a twisted cubic $\gamma_{\mathbb{C}}$.

By [Fuj3, Sect.10] again, the pair $(A_{\mathbb{C}}, \Pi)$ is unique up to projective equivalence. For choices of coordinates $x_1, x_2, x_3$ of $\wedge^2 V^3$ and $y_1, \ldots, y_5$ of $U^5$, we may write the equation of $A_{\mathbb{C}}$ as follows:

**Equation of $A_{\mathbb{C}}$**

(2.2) $$\begin{pmatrix} y_4 & y_3 & y_2 \\ y_3 & y_2 & y_1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\; y_5\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_2^2 - y_1 y_3 \\ y_1 y_4 - y_2 y_3 \\ y_3^2 - y_2 y_4 \end{pmatrix},$$

where the twisted cubic $\gamma_{\mathbb{C}}$ is equal to

$$\left\{ y_2^2 - y_1 y_3 = y_1 y_4 - y_2 y_3 = y_3^2 - y_2 y_4 = y_5 = 0 \right\}.$$

2.5. **Mid point in the genus 8 case .** Let $V$ be a $6$-dimensional vector space. By [Mu2, Thm.6.5 (2) and Prop.7.8], $W$ is a codimension 5 linear section of $\mathrm{G}(2, V)$. We write $W = \mathrm{G}(2, V) \cap \mathbb{P}(U^{10})$ with a 10-dimensional subspace $U^{10} \subset \wedge^2 V$. Note that the image of a fiber of $E' \to C$ on $W$ in the Sarkisov link (1.1) is a line intersecting $\Pi$ since the equality $E' \sim -K_{Y'} - 2\widetilde{E}$ holds by [Tak1, Part I, Table 1]. Therefore, by [Tak2, Lem.5.3], the 2-plane $\Pi$ has the following description as a subvariety of $\mathrm{G}(2, V)$:

$$\Pi = \{ [\mathbb{C}^2] \mid V^1 \subset \mathbb{C}^2 \subset V^4 \} \subset \mathrm{G}(2, V)$$

with some fixed vector subspaces $V^1 \simeq \mathbb{C}$ and $V^4 \simeq \mathbb{C}^4$ of $V$.

## 3. Embedding theorem in the genus 8 case

3.1. **Extending the mid point .** The 2-plane $\Pi$ is contained in the following 4-plane

$$\overline{\Pi} := \{ [\mathbb{C}^2] \mid V^1 \subset \mathbb{C}^2 \} \subset \mathrm{G}(2, V),$$

and it holds that $\Pi = \overline{\Pi} \cap \mathbb{P}(U^{10})$. By a simple dimension count of linear subspaces, the linear hull $\mathbb{P}(U^{12})$ of $\overline{\Pi} \cup \mathbb{P}(U^{10})$ is a codimension 3 linear subspace of $\mathbb{P}(\wedge^2 V)$. We set

$$\overline{\Sigma} := \mathrm{G}(2, V) \cap \mathbb{P}(U^{12}).$$

Since $W$ is a linear section of $\mathrm{G}(2, V)$, so is $\overline{\Sigma}$. Now we produce the situation as in Lemma 2.1. We choose a direct sum decomposition $V = V^1 \oplus V'$ with a complementary subspace $V'$ to $V^1$. Note that $\Pi = \mathbb{P}(V^1 \wedge V')$, which we identify with $\mathbb{P}(V')$. There exists a 5-dimensional subspace $U^7 \subset \wedge^2 V'$ such that $U^{12} = V' \oplus U^7$.

The projection of $\mathrm{G}(2, V)$ from the 4-plane $\overline{\Pi}$ induces the natural rational map $\overline{\Sigma} \dashrightarrow \mathrm{G}(2, V') \cap \mathbb{P}(U^7)$. This also induces the rational map $W \dashrightarrow \mathrm{G}(2, V') \cap \mathbb{P}(U^7)$, which is the projection from $\Pi$. By Corollary 2.5, we have $\mathrm{G}(2, V') \cap \mathbb{P}(U^7) = X' \simeq B_5$.

3.2. **Construction of the key variety .**

**Definition 3.1.** Let $\mathcal{U}$ be the rank two universal subbundle on $\mathrm{G}(2, V') \simeq \mathrm{G}(2, 5)$. Set

$$\Sigma' := \mathbb{P}_{B_5}(\mathcal{U}|_{B_5} \oplus \mathcal{O}_{B_5}(-1)).$$

Note that, by a standard computation, it follows that

$$(3.1) \qquad\qquad\qquad -K_{\Sigma'} = 3H_{\Sigma'}.$$

To investigate the birational geometry of $\Sigma'$, we need the following beautiful classical result:

**Lemma 3.2.** *The natural morphism* $\mathbb{P}_{B_5}(\mathcal{U}|_{B_5}) \to \mathbb{P}(V')$ *to* $\mathbb{P}(V') \simeq \mathbb{P}^4$ *from the total space of lines in* $\mathbb{P}^4$ *parameterized by* $B_5 \subset \mathrm{G}(2, V')$ *is the blow-up along the projected Veronese surface* $\mathcal{V}$.

*Proof.* By [Il, Prop.2.4 (b)], $B_5$ parameterizes tri-secant lines of the projected second Veronese surface $\mathcal{V} \subset \mathbb{P}^4$. Let s be a point of $\mathbb{P}^4$. The fiber of $\mathbb{P}_{B_5}(\mathcal{U}|_{B_5}) \to \mathbb{P}(V')$ parameterizes the tri-secant lines of $\mathcal{V}$ through s. If $s \notin \mathcal{V}$, then there is a unique tri-secant line of $\mathcal{V}$ through s (this is classically known and follows by [L, Lem.8.1] for example), and if $s \in \mathcal{V}$, then tri-secant lines of $\mathcal{V}$ through s are parameterized by $\mathbb{P}^1$ ([Il, Prop.2.4 (a)]). Note that $-K_{\mathbb{P}_{B_5}(\mathcal{U}|_{B_5})} = 2H_{\mathbb{P}_{B_5}(\mathcal{U}|_{B_5})} + L$, where $L$ is the pull-back of $\mathcal{O}_{B_5}(1)$. Since a fiber of $\mathbb{P}_{B_5}(\mathcal{U}|_{B_5}) \to \mathbb{P}(V')$ is not contained in a fiber of $\mathbb{P}_{B_5}(\mathcal{U}|_{B_5}) \to B_5$, it is positive for $-K_{\mathbb{P}_{B_5}(\mathcal{U}|_{B_5})}$. Therefore, by [An, Thm.2.3], $\mathbb{P}_{B_5}(\mathcal{U}|_{B_5}) \to \mathbb{P}^4$ is the blow-up along $\mathcal{V}$. $\qquad\square$

**Proposition 3.3.** *The tautological linear system $|H_{\Sigma'}|$ defines a surjective birational morphism $\Sigma' \to \overline{\Sigma}$, which we will denote by $\varphi_{|H_{\Sigma'}|}$. It is a flopping contraction of Atiyah type. The image of the flopping locus on $\overline{\Sigma}$ is a projected second Veronese surface $\mathcal{V}$ in $\overline{\Pi}$.*

*Proof.* Take a point $\mathsf{p} := [\wedge^2 V^2] \in B_5 = \mathrm{G}(2, V') \cap \mathbb{P}(U^7)$, where $V^2 \subset V'$ is a 2-dimensional subspace such that $\wedge^2 V^2 \subset U^7$. The fiber of the projection $\Sigma' \to B_5$ over $\mathsf{p}$ is $\mathbb{P}(V^2 \oplus \wedge^2 V^2)$, which is a linear subspace of $\mathbb{P}(V' \oplus U^7)$.

For a 2-dimensional subspace $V^2 \subset V'$ such that $\wedge^2 V^2 \subset U^7$, we take a point $[\boldsymbol{x} + \boldsymbol{y}] \in \mathbb{P}(V^2 \oplus \wedge^2 V^2)$ with $\boldsymbol{x} \in V^2$ and $\boldsymbol{y} \in \wedge^2 V^2$. By Lemma 2.1, it holds that $[\boldsymbol{x} + \boldsymbol{y}] \in G(2, V) \cap \mathbb{P}(V' \oplus U^7) = \overline{\Sigma}$. Therefore the image of $\Sigma' \to \mathbb{P}(V' \oplus U^7)$ is contained in $\overline{\Sigma}$, and hence the desired morphism $\Sigma' \to \overline{\Sigma}$ is induced. By Lemma 2.2 (1), this is defined by the tautological linear system $|H_{\Sigma'}|$.

Let $\mathsf{t} := [\boldsymbol{x} + \boldsymbol{y}]$ be a point of $\overline{\Sigma}$ with $\boldsymbol{x} \in V'$ and $\boldsymbol{y} \in U^7$. Then the fiber of $\Sigma' \to \overline{\Sigma}$ over $\mathsf{t}$ is $\{\mathsf{t}\} \times \{[\wedge^2 V^2] \mid \boldsymbol{x} \in V^2, \boldsymbol{y} \in \wedge^2 V^2 \subset U^7\}$ by Lemma 2.2 (2), which is nonempty by Lemma 2.1. Therefore the morphism $\Sigma' \to \overline{\Sigma}$ is surjective. If $\boldsymbol{y} \neq \boldsymbol{o}$, then $V^2$ is uniquely determined by $\wedge^2 V^2 = \mathbb{C}\boldsymbol{y}$. Therefore the morphism $\Sigma' \to \overline{\Sigma}$ is birational.

If $\boldsymbol{y} = \boldsymbol{o}$, then $\mathsf{t}$ is a point of $\overline{\Pi}$. Note that the restriction of the morphism $\Sigma' \to \overline{\Sigma}$ over $\overline{\Pi}$ is $\mathbb{P}_{B_5}(\mathcal{U}|_{B_5} \oplus 0) \to \mathbb{P}(V' \oplus 0) = \overline{\Pi} \simeq \mathbb{P}^4$, which can be identified with the natural morphism $\mathbb{P}_{B_5}(\mathcal{U}|_{B_5}) \to \mathbb{P}(V')$ to $\mathbb{P}(V') \simeq \mathbb{P}^4$ from the total space of lines in $\mathbb{P}^4$ parameterized by $B_5 \subset \mathrm{G}(2, V')$. Let $l$ be the fiber of $\mathbb{P}_{B_5}(\mathcal{U}|_{B_5} \oplus 0) \to \overline{\Pi}$ over a point of the projected Veronese surface $\mathcal{V} \subset \mathbb{P}(V')$. We compute the normal bundle $\mathcal{N}_{l/\Sigma'}$. Since $\mathbb{P}_{B_5}(\mathcal{U}|_{B_5} \oplus 0) \to \overline{\Pi}$ is the blow-up of $\overline{\Pi}$ along $\mathcal{V}$ by Lemma 3.2, we see that $\mathcal{N}_{l/\mathbb{P}_{B_5}(\mathcal{U}|_{B_5} \oplus 0)} = \mathcal{O}_l^{\oplus 2} \oplus \mathcal{O}_l(-1)$. Let $L_{\Sigma'}$ be the pull-back of $\mathcal{O}_{B_5}(1)$ on $\Sigma'$. Since $\mathbb{P}_{B_5}(\mathcal{U}|_{B_5} \oplus 0)$ is linearly equivalent to $H_{\Sigma'} - L_{\Sigma'}$ in $\Sigma'$, we see that $\mathcal{N}_{\mathbb{P}_{B_5}(\mathcal{U}|_{B_5} \oplus 0)/\Sigma'}|_l = \mathcal{O}_l(-1)$. Therefore, by the normal bundle sequence $0 \to \mathcal{N}_{l/\mathbb{P}_{B_5}(\mathcal{U}|_{B_5} \oplus 0)} \to \mathcal{N}_{l/\Sigma'} \to \mathcal{N}_{\mathbb{P}_{B_5}(\mathcal{U}|_{B_5} \oplus 0)/\Sigma'}|_l \to 0$, we see that $\mathcal{N}_{l/\Sigma'} = \mathcal{O}_l^{\oplus 2} \oplus \mathcal{O}_l(-1)^{\oplus 2}$, and $\Sigma' \to \overline{\Sigma}$ is a flopping contraction of Atiyah type as desired. $\qquad\square$

Let $\Sigma' \dashrightarrow \widetilde{\Sigma}$ be the flop for this flopping contraction. Let $\Pi'$ and $\widetilde{\Pi}$ be the strict transforms of $\overline{\Pi}$ on $\Sigma'$ and $\widetilde{\Sigma}$ respectively. It is well-known that the flop can be constructed by the blow-up along the flopping locus and the blow-down of the exceptional divisor along the other direction. From this, we see that the restriction $\Pi' \dashrightarrow \widetilde{\Pi}$ of the flop is the blow-up $\Pi' \to \overline{\Pi}$ of $\overline{\Pi}$ along $\mathcal{V}$. Therefore $\widetilde{\Pi}$ is isomorphic to $\mathbb{P}^4$. Let $H_{\widetilde{\Sigma}}$ be the strict transform on $\widetilde{\Sigma}$ of $H_{\Sigma'}$.

**Lemma 3.4.** *The normal bundle $\mathcal{N}_{\widetilde{\Pi}/\widetilde{\Sigma}}$ is $\mathcal{O}_{\mathbb{P}^4}(-2)$.*

*Proof.* We have $H_{\widetilde{\Sigma}}|_{\widetilde{\Pi}} = \mathcal{O}_{\mathbb{P}^4}(1)$ since $\mathcal{O}_{\overline{\Sigma}}(1)|_{\overline{\Pi}} = \mathcal{O}_{\mathbb{P}^4}(1)$. Therefore we have $-K_{\widetilde{\Sigma}}|_{\widetilde{\Pi}} = \mathcal{O}_{\mathbb{P}^4}(3)$. Since $-K_{\widetilde{\Pi}} = \mathcal{O}_{\mathbb{P}^4}(5)$, we have $\mathcal{N}_{\widetilde{\Pi}/\widetilde{\Sigma}} \simeq \mathcal{O}_{\mathbb{P}^4}(-2)$ as desired.    $\square$

**Lemma 3.5.** $2H_{\widetilde{\Sigma}} + \widetilde{\Pi}$ *is semiample.*

*Proof.* We show that $2H_{\widetilde{\Sigma}}+\widetilde{\Pi}$ is nef. Assume that $(2H_{\widetilde{\Sigma}}+\widetilde{\Pi})\cdot\gamma < 0$ for an irreducible curve $\gamma$. Then $\widetilde{\Pi} \cdot \gamma < 0$ since $H_{\widetilde{\Sigma}}$ is nef, and hence $\gamma \subset \widetilde{\Pi}$. Since $\widetilde{\Pi}|_{\widetilde{\Pi}} = \mathcal{O}_{\mathbb{P}^4}(-2)$ and $H_{\widetilde{\Sigma}}|_{\widetilde{\Pi}} = \mathcal{O}_{\mathbb{P}^4}(1)$, we have $(2H_{\widetilde{\Sigma}}+\widetilde{\Pi})\cdot\gamma = 0$, a contradiction. Therefore $2H_{\widetilde{\Sigma}}+\widetilde{\Pi}$ is nef.

Since $-K_{\widetilde{\Sigma}}$ is nef and big, $2H_{\widetilde{\Sigma}} + \widetilde{\Pi}$ is semiample by the Kawamata-Shokurov base point free theorem ([KMM]).    $\square$

**Theorem 3.6.** *Let* $\mu\colon \widetilde{\Sigma} \to \Sigma$ *be the contraction defined by a sufficient multiple of* $2H_{\widetilde{\Sigma}} + \widetilde{\Pi}$. *The exceptional locus of this contraction is* $\widetilde{\Pi}$. *The image of* $\widetilde{\Pi}$ *on* $\Sigma$ *is a* $1/2$-*singularity.* $\Sigma$ *is a* 5-*dimensional rational* $\mathbb{Q}$-*Fano variety with only one* $1/2$-*singularity and with* $\rho(\Sigma) = 1$. *The image* $M_{\Sigma}$ *of* $H_{\widetilde{\Sigma}}$ *is a primitive Weil divisor and it holds that* $-K_{\Sigma} = 3M_{\Sigma}$.

*Proof.* As we have checked in the proof of Lemma 3.5, $2H_{\widetilde{\Sigma}} + \widetilde{\Pi}$ is numerical trivial for any curve in $\widetilde{\Pi}$. Thus the image of $\widetilde{\Pi}$ by $\widetilde{\Sigma} \to \Sigma$ is a $1/2$-singularity by Lemma 3.4. Assume by contradiction that $(2H_{\widetilde{\Sigma}} + \widetilde{\Pi}) \cdot \gamma = 0$ for an irreducible curve $\gamma \not\subset \widetilde{\Pi}$. Since $H_{\widetilde{\Sigma}}$ is nef and $\gamma \not\subset \widetilde{\Pi}$, we have $H_{\widetilde{\Sigma}} \cdot \gamma = \widetilde{\Pi} \cdot \gamma = 0$. By the condition that $H_{\widetilde{\Sigma}} \cdot \gamma = 0$, $\gamma$ is a flopped curve. This is absurd since $\widetilde{\Pi}$ is positive for a flopped curve.

We show that $\rho(\Sigma)=1$. Since $\Sigma' \to B_5$ is a projective bundle, we see that $\rho(\Sigma') = \rho(B_5) + 1 = 2$. Since $\Sigma' \dashrightarrow \widetilde{\Sigma}$ is a flop, we have $\rho(\widetilde{\Sigma}) = \rho(\Sigma') = 2$. Finally, since $\widetilde{\Sigma} \to \Sigma$ contracts a divisor, we have $\rho(\Sigma) \leq \rho(\widetilde{\Sigma}) - 1 = 1$. Hence we have $\rho(\Sigma) = 1$.

The equality $-K_{\Sigma} = 3M_{\Sigma}$ follows from (3.1). We show that $M_{\Sigma}$ is primitive. If $M_{\Sigma}$ were not primitive, then $M_{\Sigma}$ would be written as $M_{\Sigma} = \alpha M'_{\Sigma}$ with a primitive Weil divisor $M'_{\Sigma}$ and positive integer $\alpha \geq 2$. Since $\Sigma$ has only a $1/2$-singularity, $2M'_{\Sigma}$ is a Cartier divisor. Therefore we have $2H_{\widetilde{\Sigma}} + F_{\widetilde{\Sigma}} = \alpha\mu^*(2M'_{\Sigma})$ and hence there is a Cartier divisor $D$ on $\Sigma'$ such that $2H_{\Sigma'} + \Pi' = \alpha D$. Since $\Pi' \cdot l = -1$ for a flopping curve $l$ by the proof of Proposition 3.3, this implies that $\alpha D \cdot l = -1$, which is impossible if $\alpha \geq 2$. Therefore $M_{\Sigma}$ is primitive.

The rationality of $\Sigma$ follows since $\Sigma$ is birational to the projective bundle $\Sigma'$ over the rational Fano 3-fold $B_5$.    $\square$

3.3. **Embedding theorem .** Now we show Theorem 1.1 for a prime $\mathbb{Q}$-Fano 3-fold $X$ of genus $8$.

**Theorem 3.7.** *A* $\mathbb{Q}$-*Fano* 3-*fold* $X$ *of genus* $8$ *is a linear section of* $\Sigma$.

*Proof.* Note that $W\cap\mathrm{Sing}\,\overline{\Sigma}$ is $0$-dimensional since $W$ has only terminal singularities and $W$ is a linear section of $\overline{\Sigma}$ with respect to $|\mathcal{O}_{\overline{\Sigma}}(1)|$. Therefore, since $\widetilde{\Sigma} \to \overline{\Sigma}$ is crepant and small and nontrivial fibers are $1$-dimensional, the strict transform $W_{\widetilde{\Sigma}}$ of $W$ in $\widetilde{\Sigma}$ is a linear section of $\widetilde{\Sigma}$ with respect to $|H_{\widetilde{\Sigma}}|$ and hence the restriction $W_{\widetilde{\Sigma}} \to W$ of $\widetilde{\Sigma} \to \overline{\Sigma}$ over $W$ is also crepant and small. Since $W$ has only terminal singularities and $W_{\widetilde{\Sigma}} \to W$ is crepant, we see that $W_{\widetilde{\Sigma}}$ is normal and has only terminal singularities by [CKM, the proof of Prop.16.4]. Note that $\widetilde{\Pi}$ is relatively ample for $W_{\widetilde{\Sigma}} \to W$. Since $Y \to W$ is the unique small extraction such that the

strict transform of $\Pi$ is relatively ample, we see that $Y = W_{\widetilde{\Sigma}}$. Since we may write $Y = W_{\widetilde{\Sigma}} = \widetilde{H}_1 \cap \widetilde{H}_2$ with $\widetilde{H}_i \in |H_{\widetilde{\Sigma}}|$ $(i = 1, 2)$, we see that $X = M_1 \cap M_2$ with the images $M_i \in |M_\Sigma|$ of $\widetilde{H}_i$ as desired. $\qquad\square$

### 3.4. Extension of the Sarkisov link.
By the proof of Theorem 3.7, we have the following:

**Corollary 3.8.** *The following diagram is an extension of the Sarkisov link (1.1):*

(3.2)

$$\widetilde{\Sigma} \dashrightarrow^{\text{flop}} \Sigma'$$

$$\Sigma \qquad \overline{\Sigma} \qquad B_5,$$

*where $\Sigma$, $\widetilde{\Sigma}$, $\overline{\Sigma}$ and $\Sigma'$ are extensions of $X$, $Y$, $W$ and $Y'$ respectively.*

## 4. Extending the mid point in the genus 4 or 6 case

In this section, we extend the mid point $W$ to a certain variety $\overline{\Sigma}$ in the genus 4 or 6 case. We also construct a crepant small resolution of $\overline{\Sigma}$ in case of genus 4 or genus 6 and Q-type, which will be the main ingredient for Theorem **??** in each of these cases.

### 4.1. Genus 4 .

#### 4.1.1. *Extending the mid point.*
As we have seen in the subsection 2.3, the mid point $W$ is a complete intersection of a quadric and a cubic in $\mathbb{P}^5$ containing two disjoint planes $\Pi_1$ and $\Pi_2$.

To extend $W$, we start from mutually disjoint two planes $\Pi_1$ and $\Pi_2$ in $\mathbb{P}^5$. By a coordinate change, we may assume that $\Pi_1 = \{x_1 = x_2 = x_3 = 0\}$ and $\Pi_2 = \{y_1 = y_2 = y_3 = 0\}$ in $\mathbb{P}^5$ with coordinates $x_1, x_2, x_3, y_1, y_2, y_3$. Set ${}^t\boldsymbol{x} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$ and ${}^t\boldsymbol{y} = \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix}$. Take a quadric $\mathsf{Q}_0$ and a cubic $\mathsf{C}_0$ both of which contain $\Pi_1$ and $\Pi_2$. Then we may write

$$\mathsf{Q}_0 = \{{}^t\boldsymbol{y} M_0 \boldsymbol{x} = 0\}, \ \mathsf{C}_0 = \{{}^t\boldsymbol{y} M_1 \boldsymbol{x} = 0\},$$

where $M_0$ (resp. $M_1$) is a $3 \times 3$ matrix with constant entries (resp. linear entries).

Remarkably, the indecomposability simplifies the situation as follows:

**Lemma 4.1.** *It holds that* $\operatorname{rank} M_0 = 3$. *We may assume that $M_0$ is the identity matrix by a coordinate change, namely,*

$$\mathsf{Q}_0 = \{{}^t\boldsymbol{y}\boldsymbol{x} = 0\}.$$

*Proof.* If $\operatorname{rank} M_0 = 1$, then $\mathsf{Q}_0 \cap \mathsf{C}_0$ is reducible, a contradiction. Assume that $\operatorname{rank} M_0 = 2$. Then $\mathsf{Q}_0$ is the cone over $\mathbb{P}^1 \times \mathbb{P}^1$. Therefore $\mathsf{Q}_0$ contains two families of 3-planes $\{\mathsf{P}_a\}_{a \in \mathbb{P}^1}$ and $\{\mathsf{P}_b\}_{b \in \mathbb{P}^1}$ such that the sums $\mathsf{P}_a + \mathsf{P}_b$ are hyperplane sections of $\mathsf{Q}_0$. Hence $(\mathsf{P}_a + \mathsf{P}_b) \cap \mathsf{Q}_0 \cap \mathsf{C}_0$ are anti-canonical divisors, a contradiction to the indecomposability of $\mathsf{Q}_0 \cap \mathsf{C}_0$. Therefore $\operatorname{rank} M_0 = 3$. The latter assertion is obvious. $\qquad\square$

Moreover, subtracting $(1/3 \operatorname{Tr} M_1)$ times the equation of $\mathsf{Q}_0$ from the equation of $\mathsf{C}_0$, we may assume that

$$\operatorname{tr} M_1 = 0.$$

With the considerations as above, we will see that the mid point $W = Q_0 \cap C_0$ can be extended to the following 11-dimensional complete intersection $\overline{\Sigma}$ of a quadric and a cubic:

**Definition 4.2 (Extension of $W$).** Fix a 3-dimensional vector space $U^3$. Let $S^{1,0,-1}U^3$ be the 8-dimensional irreducible component of $U^3 \otimes (U^3)^*$ as representation space of $SL(U^3)$. We define $\overline{\Sigma}$ to be the complete intersection

$$Q \cap C \subset \mathbb{P}\left((U^3)^* \oplus U^3 \oplus S^{1,0,-1}U^3\right) \simeq \mathbb{P}^{13}$$

with

$$\text{the quadric } Q := \{(\boldsymbol{y}, \boldsymbol{x}) = 0\} \text{ and the cubic } C := \{\langle \boldsymbol{y}, M, \boldsymbol{x} \rangle = 0\},$$

where

- $\boldsymbol{x} \in U^3$, $\boldsymbol{y} \in (U^3)^*$, and $M \in S^{1,0,-1}U^3$,
- $(\ ,\ )$ is the dual pairing between $(U^3)^*$ and $U^3$, and
- $\langle\ ,\ ,\ \rangle$ is the natural tri-linear form induced by the contraction

$$(U^3)^* \times (U^3 \otimes (U^3)^*) \times U^3 \to \mathbb{C}.$$

We set

$$\overline{\Pi}_1 := \mathbb{P}\left((U^3)^* \oplus 0 \oplus S^{1,0,-1}U^3\right) = \{\boldsymbol{x} = \boldsymbol{o}\}, \quad \overline{\Pi}_2 := \mathbb{P}\left(0 \oplus U^3 \oplus S^{1,0,-1}U^3\right) = \{\boldsymbol{y} = \boldsymbol{o}\},$$

which are 10-planes contained in $\overline{\Sigma}$.

**Proposition 4.3.** *The mid point $W$ is a codimension 8 linear section of $\overline{\Sigma}$ such that $\overline{\Pi}_1 \cap W = \Pi_1$ and $\overline{\Pi}_2 \cap W = \Pi_2$.*

*Proof.* By taking a basis of $U^3$, we may describe $\overline{\Sigma}$ explicitly as follows:

Let $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ be a basis of $U^3$ and $x_1, x_2, x_3$ the coordinates of $U^3$ associated to this basis. Let $y_1, y_2, y_3$ be the coordinates of the dual space $(U^3)^*$ associated to the dual basis $\boldsymbol{e}_1^*, \boldsymbol{e}_2^*, \boldsymbol{e}_3^*$. Set ${}^t\boldsymbol{x} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$ and ${}^t\boldsymbol{y} = \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix}$. These notation will be compatible with the above. Let $z_{ij}$ $(1 \leq i, j \leq 3)$ be the coordinates of $U^3 \otimes (U^3)^*$ associated to the basis $\boldsymbol{e}_i \otimes \boldsymbol{e}_j^*$. Then

- $S^{1,0,-1}U^3$ is the subspace $\{\sum_{i=1}^3 z_{ii} = 0\}$ of $U^3 \otimes (U^3)^*$,
- $Q = \{{}^t\boldsymbol{y}\boldsymbol{x} = 0\}$,
- $C = \{{}^t\boldsymbol{y}M\boldsymbol{x} = 0\}$, where $M$ is the $3 \times 3$ matrix with the entries $(z_{ij})$, and
- $\overline{\Pi}_1 = \{\boldsymbol{x} = \boldsymbol{o}\}$, $\overline{\Pi}_2 = \{\boldsymbol{y} = \boldsymbol{o}\}$.

Therefore we have the assertion by the discussion above.                           $\square$

By elementary calculations, we obtain the singular locus of $\overline{\Sigma}$ as follows:

**Proposition 4.4.** *The singular locus of $\overline{\Sigma}$ is contained in $\overline{\Pi}_1 \sqcup \overline{\Pi}_2$ and is equal to*

$$\left\{[\boldsymbol{y}, \boldsymbol{o}, M] \in \overline{\Pi}_1 \mid \text{rank} \begin{pmatrix} {}^t\boldsymbol{y} \\ {}^t\boldsymbol{y}M \end{pmatrix} \leq 1\right\} \cup \left\{[\boldsymbol{o}, \boldsymbol{x}, M] \in \overline{\Pi}_2 \mid \text{rank} \begin{pmatrix} {}^t\boldsymbol{x} \\ {}^t\boldsymbol{x}\,{}^tM \end{pmatrix} \leq 1\right\}.$$

*In particular, $\overline{\Sigma}$ is Gorenstein and normal.*

4.1.2. ***Crepant small resolution*** . We take the coordinates of $U^3$ and $(U^3)^*$ as in the proof of Proposition 4.3. We consider

$$B_6 = \{{}^t\boldsymbol{y}\boldsymbol{x} = 0\} \subset \mathbb{P}((U^3)^*) \times \mathbb{P}(U^3).$$

We also identify $B_6$ with its image by the Segre embedding

$$S \colon \mathbb{P}((U^3)^*) \times \mathbb{P}(U^3) \hookrightarrow \mathbb{P}((U^3)^* \otimes U^3)$$
$$[\boldsymbol{y}] \times [\boldsymbol{x}] \mapsto [\boldsymbol{y} \otimes \boldsymbol{x}].$$

Then $B_6$ spans $\mathbb{P}(\mathrm{S}^{-1,0,1}U^3)$, where $\mathrm{S}^{-1,0,1}U^3$ is the 8-dimensional irreducible component of $(U^3)^* \otimes U^3$ as $\mathrm{SL}(U^3)$-representation space. We denote by $p_{ij}$ the coordinate of $\mathbb{P}((U^3)^* \otimes U^3)$ corresponding to $\boldsymbol{e}_i^* \otimes \boldsymbol{e}_j$. The subspace $\mathrm{S}^{-1,0,1}U^3$ is nothing but $\left\{\sum_{i=1}^3 p_{ii} = 0\right\}$. We denote the natural projections by $p_1 \colon B_6 \to \mathbb{P}((U^3)^*)$ and $p_2 \colon B_6 \to \mathbb{P}(U^3)$, and set $\mathcal{O}_{B_6}(1,0) := p_1^*\mathcal{O}_{\mathbb{P}((U^3)^*)}(1)$ and $\mathcal{O}_{B_6}(0,1) := p_2^*\mathcal{O}_{\mathbb{P}(U^3)}(1)$.

**Definition 4.5.** We set

$$\Sigma' := \mathbb{P}_{B_6}(\mathcal{O}_{B_6}(-1,0) \oplus \mathcal{O}_{B_6}(0,-1) \oplus \Omega^1_{\mathbb{P}(\mathrm{S}^{-1,0,1}U^3)}(1)|_{B_6}).$$

Note that, by a standard computation, it follows that $-K_{\Sigma'} = 9H_{\Sigma'}$.

**Proposition 4.6.** *The following assertions holds:*

*(1) The tautological linear system $|H_{\Sigma'}|$ defines a surjective birational morphism $\Sigma' \to \overline{\Sigma}$, which we will denote by $\varphi_{|H_{\Sigma'}|}$.*

*(2) The morphism $\varphi_{|H_{\Sigma'}|}$ is an isomorphism outside of $\mathrm{Sing}\,\overline{\Sigma}$ (note that $\overline{\Pi}_1 \cap \overline{\Pi}_2$ is contained in $\mathrm{Sing}\,\overline{\Sigma}$ by Proposition 4.4). Moreover, the $\varphi_{|H_{\Sigma'}|}$-fiber over a point $\mathsf{t} \in \mathrm{Sing}\,\overline{\Sigma}$ is*

$$\begin{cases} \mathbb{P}^1 : & \mathsf{t} \notin \mathrm{Sing}\,\overline{\Sigma} \setminus (\overline{\Pi}_1 \cap \overline{\Pi}_2), \\ \textit{a sextic del Pezzo surface:} & \mathsf{t} \in \overline{\Pi}_1 \cap \overline{\Pi}_2. \end{cases}$$

*(3) The morphism $\varphi_{|H_{\Sigma'}|}$ is a crepant small resolution.*

*Proof.* Take a point $\mathsf{p} := [W^1 \otimes U^1] \in B_6$, where $W^1 \subset (U^3)^*$ and $U^1 \subset U^3$ are 1-dimensional subspaces such that $W^1 \subset (U^1)^\perp$ with respect to the dual pairing. We set $(W^1 \otimes U^1)^\perp = (\mathrm{S}^{-1,0,1}U^3/(W^1 \otimes U^1))^*$. The fiber of the projective bundle $\Sigma' \to B_6$ over $\mathsf{p}$ is

$$\mathbb{P}(W^1 \oplus U^1 \oplus (W^1 \otimes U^1)^\perp),$$

which is a linear subspace of $\mathbb{P}((U^3)^* \oplus U^3 \oplus \mathrm{S}^{1,0,-1}U^3)$. By Lemma 2.2 (1), the tautological linear system defines a map $\Sigma' \to \mathbb{P}((U^3)^* \oplus U^3 \oplus \mathrm{S}^{1,0,-1}U^3)$. By the descriptions of fibers of $\Sigma' \to B_6$ and the definition of $\overline{\Sigma}$, we see that the image of this map is contained in $\overline{\Sigma}$.

Let $\mathsf{t} = [\boldsymbol{y} + \boldsymbol{x} + M]$ be a point of $\overline{\Sigma}$ with $\boldsymbol{y} \in (U^3)^*, \boldsymbol{x} \in U^3, M \in \mathrm{S}^{1,0,-1}U^3$ . By Lemma 2.2 (2), the fiber of $\Sigma' \to \overline{\Sigma}$ over $\mathsf{t}$ is

$$\{\mathsf{t}\} \times \left\{[W^1 \otimes U^1] \mid W^1 \subset (U^1)^\perp, \boldsymbol{y} \in W^1, \boldsymbol{x} \in U^1, M \in (W^1 \otimes U^1)^\perp\right\}.$$

We check the condition for $(\mathsf{t}, [W^1 \otimes U^1])$ to be in the fiber of $\Sigma' \to \overline{\Sigma}$ over $\mathsf{t}$. If $\mathsf{t} \notin \overline{\Pi}_1 \cup \overline{\Pi}_2$, then $W^1$ and $U^1$ are uniquely determined as $W^1 = \mathbb{C}\boldsymbol{y}$ and $U^1 = \mathbb{C}\boldsymbol{x}$. Therefore the morphism $\Sigma' \to \overline{\Sigma}$ is an isomorphism outside of $\overline{\Pi}_1 \cup \overline{\Pi}_2$. In particular, the morphism $\Sigma' \to \overline{\Sigma}$ is surjective and birational. Assume that $\mathsf{t} \in \overline{\Pi}_1 \setminus (\overline{\Pi}_1 \cap \overline{\Pi}_2)$, equivalently, $\boldsymbol{x} = \boldsymbol{o}$ and $\boldsymbol{y} \neq \boldsymbol{o}$. Then $W^1$ is uniquely determined as $W^1 = \mathbb{C}\boldsymbol{y}$. We

set $M_{\boldsymbol{y}} := \begin{pmatrix} {}^t\boldsymbol{y} \\ {}^t\boldsymbol{y}M \end{pmatrix}$. Note that, by Proposition 4.4, $\mathsf{t} \in (\mathrm{Sing}\,\overline{\Sigma}) \cap \overline{\Pi}_1$ if and only if rank $M_{\boldsymbol{y}} = 1$. The condition for $U^1$ is that $U^1 \subset \{\boldsymbol{z} \in U^3 \mid M_{\boldsymbol{y}}\boldsymbol{z} = \boldsymbol{0}\}$. Therefore, if rank $M_{\boldsymbol{y}} = 2$, then $U^3$ is uniquely determined, and if rank $M_{\boldsymbol{y}} = 1$, $U^1$'s are parameterized by $\mathbb{P}((W^1)^\perp) \simeq \mathbb{P}^1$. From this, the description of the fiber $\Sigma' \to \overline{\Sigma}$ over $\mathsf{t}$ follows. We can also describe the fiber over $\mathsf{t} \in \overline{\Pi}_2 \setminus (\overline{\Pi}_1 \cap \overline{\Pi}_2)$ in the same way. Finally, assume that $\mathsf{t} \in \overline{\Pi}_1 \cap \overline{\Pi}_2$. Since $\overline{\Pi}_1 \cap \overline{\Pi}_2 = \mathbb{P}(\mathrm{S}^{1,0,-1}U^3)$, its inverse image in $\Sigma'$ is the projective subbundle

$$S_{\Sigma'} := \mathbb{P}_{B_6}(\Omega^1_{\mathbb{P}(\mathrm{S}^{-1,0,1}U^3)}(1)|_{B_6}).$$

Note that $S_{\Sigma'} \subset \mathbb{P}(\mathrm{S}^{1,0,-1}U^3) \times B_6$, and $S_{\Sigma'} \to \mathbb{P}(\mathrm{S}^{1,0,-1}U^3)$ is the universal family of hyperplane sections of $B_6$, which is a fibration of sextic del Pezzo surfaces. Hence the description of the fiber of $\Sigma' \to \overline{\Sigma}$ over $\mathsf{t} \in \overline{\Pi}_1 \cap \overline{\Pi}_2$ follows.

The assertion (3) follows from (2) since it holds that $-K_{\Sigma'} = 9H_{\Sigma'}$.          □

*Remark 4.7.* It is possible to construct the flop for $\Sigma' \to \overline{\Sigma}$ but the construction is slightly involved (and produce singularities) since the flopping contraction $\Sigma' \to \overline{\Sigma}$ has jumping fibers as in Proposition 4.6 (2). We will see that the construction as in Proposition 5.4 is very close to and is easier than the construction of the flop for $\Sigma' \to \overline{\Sigma}$.

### 4.2. Genus 6. In this subsection, we use the notation as in the subsection 2.4.

#### 4.2.1. *Extending the mid point in the case of Q-type* .

**Definition 4.8 (Extension of $W$).** We denote by $Q_{\mathsf{q}}$ the quadric in the projective space $\mathbb{P}(V' \oplus U^5 \oplus (U^5)^*)$ defined by the dual pairing $U^5 \times (U^5)^* \to \mathbb{C}$. We set

$$\overline{\Sigma} := \big(A_{\mathsf{q}} * \mathbb{P}\big((U^5)^*\big)\big) \cap Q_{\mathsf{q}},$$

and

$$\overline{\Pi} := \mathbb{P}(V' \oplus 0 \oplus (U^5)^*) \subset \mathbb{P}(V' \oplus U^5 \oplus (U^5)^*).$$

Note that $\overline{\Pi} \simeq \mathbb{P}^8$ and $\overline{\Pi} \subset \overline{\Sigma}$.

**Proposition 4.9.** *The pair $(W, \Pi_0)$ is projectively equivalent to the pair of a linear section $W'$ of $\overline{\Sigma}$ and the 2-plane $\overline{\Pi} \cap W'$.*

*Proof.* We take coordinates $x_1, \ldots, x_4$ of $V'$ and $y_1, \ldots, y_5$ of $U^5$ respectively. Recall that $A_{\mathsf{q}} = \mathrm{G}(2, V) \cap \mathbb{P}(V' \oplus U^5)$, and $W$ is a quadric section of $A_{\mathsf{q}} \cap \mathbb{P}(U^8)$ with $U^8 \subset V' \oplus U^5$. We may assume that $\Pi_0 = \Pi \cap \{x_1 = 0\} = \{x_1 = y_1 = \cdots = y_5 = 0\} \subset \mathbb{P}(V' \oplus U^5)$. Then we may write $U^8 = (V' \oplus U^5) \cap \{l(x,y) = 0\}$ and $W = A_{\mathsf{q}} \cap \{l(x,y) = q(x,y) = 0\}$, where $l(x,y) = x_1 + l'(y)$ with a linear form $l'(y)$ and $q(x,y)$ is a quadratic form. Since $\Pi_0 \subset W$, we can write $q(x,y) = x_1 m(x) + q'(x,y)$ with a linear form $m(x)$ and a quadric form $q'(x,y) \in (y_1, \ldots, y_5)$. Replacing $q(x,y)$ with $q(x,y) - l(x,y)m(x) = -l'(y)m(x) + q'(x,y)$, we may assume that $\Pi \subset \{q(x,y) = 0\}$. Therefore we may write

$$q(x,y) = l_1 y_1 + \cdots + l_5 y_5$$

with linear forms $l_1, \ldots, l_5$.

Now we consider the projective space $\mathbb{P}(V' \oplus U^5 \oplus (U^5)^*)$ and the quadric $Q_{\mathsf{q}}$ as in Definition 4.8. Explicitly, let $z_1, \ldots, z_5$ be the coordinates of $(U^5)^*$ dual to $y_1, \ldots, y_5$. Then

(4.1)          $Q_{\mathsf{q}} = \{y_1 z_1 + \cdots + y_5 z_5 = 0\} \subset \mathbb{P}(V' \oplus U^5 \oplus (U^5)^*).$

Then, by the above construction, we see that the pair $(W, \Pi_0)$ is projectively equivalent to the pair of $W' := \overline{\Sigma} \cap \{z_1 - l_1 = \cdots = z_5 - l_5 = 0, l(x, y) = 0\}$ and $\overline{\Pi} \cap W' = \{x_1 = y_1 = \cdots = y_5 = 0, z_1 - l_1 = \cdots = z_5 - l_5 = 0\}$. □

Now we use the coordinates and the equation for $A_{\mathsf{q}}$ as in (2.1). We denote by $y_{23}, y_{25}, y_{34}, y_{35}, y_{45}$ be the coordinates of $(U^5)^*$ dual to that of $U^5$. Then $\overline{\Sigma}$ is defined by the equations of $A_{\mathsf{q}}$ and

$$Q_{\mathsf{q}} = \{x_{23}y_{23} + x_{25}y_{25} + x_{34}y_{34} + x_{35}y_{35} + x_{45}y_{45} = 0\}.$$

Using these, we obtain the following by an explicit calculation:

**Proposition 4.10.** *We set*

$$^t\boldsymbol{x} := \begin{pmatrix} x_{23} & x_{25} & x_{34} & x_{35} & x_{45} \end{pmatrix}, {}^t\boldsymbol{y} := \begin{pmatrix} y_{23} & y_{25} & y_{34} & y_{35} & y_{45} \end{pmatrix},$$

$$M_{\mathsf{q}} := \begin{pmatrix} 0 & z_2^2 & z_3^2 & z_2 z_3 & z_2 z_4 - z_3 z_5 \\ -z_2^2 & 0 & z_3 z_5 + z_2 z_4 & z_2 z_5 & -z_5^2 \\ -z_3^2 & -(z_3 z_5 + z_2 z_4) & 0 & -z_3 z_4 & -z_4^2 \\ -z_2 z_3 & -z_2 z_5 & z_3 z_4 & 0 & -z_4 z_5 \\ -(z_2 z_4 - z_3 z_5) & z_5^2 & z_4^2 & z_4 z_5 & 0 \end{pmatrix}.$$

*The singular locus of $\overline{\Sigma}$ is contained in $\overline{\Pi}$ and is equal to $\{\boldsymbol{x} = \boldsymbol{o}, M_{\mathsf{q}}\boldsymbol{y} = \boldsymbol{o}\}$. In particular, $\overline{\Sigma}$ is Gorenstein and normal.*

### 4.2.2. *Crepant small resolution of the mid point in the case of $\mathsf{q}$-type* .

**Definition 4.11.** We set

$$\Sigma' := \mathbb{P}_{Q^3}(\mathcal{U}|_{Q^3} \oplus \mathcal{O}_{Q^3}(-1) \oplus \Omega_{\mathbb{P}(U^5)}(1)|_{Q^3}) \to Q^3,$$

where $\mathcal{U}$ is the rank 2 universal subbundle on $G(2, V')$. By a standard computation, it follows that $-K_{\Sigma'} = 7H_{\Sigma'}$.

**Proposition 4.12.** *The following assertions hold:*

(1) The tautological linear system $|H_{\Sigma'}|$ defines a surjective birational morphism $\Sigma' \to \overline{\Sigma}$, which we will denote by $\varphi_{|H_{\Sigma'}|}$.

(2) The morphism $\varphi_{|H_{\Sigma'}|}$ is an isomorphism outside of $\operatorname{Sing} \overline{\Sigma}$ (note that $\mathbb{P}\left((U^5)^*\right) = \mathbb{P}\left(0 \oplus 0 \oplus (U^5)^*\right)$ is contained in $\operatorname{Sing} \overline{\Sigma}$ by Proposition 4.10). Moreover, the $\varphi_{|H_{\Sigma'}|}$-fiber over a point $\mathsf{t} \in \operatorname{Sing} \overline{\Sigma}$ is

$$\begin{cases} \mathbb{P}^1 : & \mathsf{t} \notin \operatorname{Sing} \overline{\Sigma} \setminus \mathbb{P}\left((U^5)^*\right), \\ \text{a quadric surface:} & \mathsf{t} \in \mathbb{P}\left((U^5)^*\right). \end{cases}$$

(3) The morphism $\varphi_{|H_{\Sigma'}|}$ is a crepant small resolution.

*Proof.* Take a point $\mathsf{p} := [\wedge^2 W^2] \in Q^3 = G(2, V') \cap \mathbb{P}(U^5)$, where $W^2 \subset V'$ is a 2-dimensional subspace such that $\wedge^2 W^2 \subset U^5$. We set $(\wedge^2 W^2)^\perp = (U^5/\wedge^2 W^2)^*$. The fiber of $\Sigma' \to Q^3$ over $\mathsf{p}$ is $\mathbb{P}(W^2 \oplus \wedge^2 W^2 \oplus (\wedge^2 W^2)^\perp)$, which is a linear subspace of $\mathbb{P}(V' \oplus U^5 \oplus (U^5)^*)$. By Lemma 2.2 (1), the tautological linear system $|H_{\Sigma'}|$ defines a morphism $\Sigma' \to \mathbb{P}(V' \oplus U^5 \oplus (U^5)^*)$. By the descriptions of fibers of $\Sigma' \to Q^3$, Lemma 2.1 and Definition 4.8, we see that the image of this map is contained in $\overline{\Sigma}$.

Let $\mathsf{t} = [\boldsymbol{x} + \boldsymbol{y} + \boldsymbol{z}] \in \overline{\Sigma} \subset \mathbb{P}(V' \oplus U^5 \oplus (U^5)^*)$ be a point with $\boldsymbol{x} \in V'$, $\boldsymbol{y} \in U^5$ and $\boldsymbol{z} \in (U^5)^*$. By Lemma 2.2 (2), the $\varphi_{|H_{\Sigma'}|}$-fiber over $\mathsf{t}$ is

$$\{\mathsf{t}\} \times \left\{[\wedge^2 W^2] \in G(2, V') \mid \wedge^2 W^2 \subset U^5, \boldsymbol{x} \in W^2, \boldsymbol{y} \in \wedge^2 W^2, \boldsymbol{z} \in (\wedge^2 W^2)^\perp\right\}.$$

If $\boldsymbol{y} \neq \boldsymbol{o}$, namely, $\mathsf{t} \in \overline{\Sigma} \setminus \overline{\Pi}$, then $\boldsymbol{y}$ uniquely determines the 2-dimensional subspace $W^2 \subset V'$ by $\mathbb{C}\boldsymbol{y} = \wedge^2 W^2$. Then the fiber is nonempty by Lemma 2.1 and (4.1), and consists of one point. Therefore the morphism $\Sigma' \to \overline{\Sigma}$ is an isomorphism outside of $\overline{\Pi}$. In particular, the morphism $\Sigma' \to \overline{\Sigma}$ is surjective and birational. We assume that $\mathsf{t} \in \overline{\Pi} \setminus \mathbb{P}\big((U^5)^*\big)$, namely, $\boldsymbol{x} \neq \boldsymbol{o}$ and $\boldsymbol{y} = \boldsymbol{o}$. The condition of $W^2$ so that $\{\mathsf{t}\} \times \{[\wedge^2 W^2]\}$ is contained in the $\varphi_{|H_{\Sigma'}|}$-fiber over $\mathsf{t}$ is that $\boldsymbol{x} \in W^2$, $\wedge^2 W^2 \subset U^5$ and $\boldsymbol{z} \in (\wedge^2 W^2)^{\perp}$. Using this with the coordinates and the equation (2.1) for $A_\mathsf{Q}$ and the description of $\operatorname{Sing} \overline{\Sigma}$ as in Proposition 4.10, we see that the $\varphi_{|H_{\Sigma'}|}$-fiber over $\mathsf{t}$ consists of one point if $\mathsf{t} \in \overline{\Pi} \setminus \operatorname{Sing} \overline{\Sigma}$, or is isomorphic to $\mathbb{P}^1$ if $\mathsf{t} \in \operatorname{Sing} \overline{\Sigma} \setminus \mathbb{P}\big((U^5)^*\big)$. Finally, assume that $\mathsf{t} \in \mathbb{P}\big((U^5)^*\big)$. The inverse image in $\Sigma'$ of $\mathbb{P}\big((U^5)^*\big)$ is the projective subbundle

$$S_{\Sigma'} := \mathbb{P}_{Q^3}(\Omega^1_{\mathbb{P}((U^5)^*)}(1)|_{Q^3}).$$

Note that $S_{\Sigma'} \subset \mathbb{P}\big((U^5)^*\big) \times Q^3$, and $S_{\Sigma'} \to \mathbb{P}\big((U^5)^*\big)$ is the universal family of hyperplane sections of $Q^3$, which is a fibration of quadric surfaces. Hence the description of the fiber of $\Sigma' \to \overline{\Sigma}$ over $\mathsf{t} \in \mathbb{P}\big((U^5)^*\big)$ follows.

The assertion (3) follows from (2) since it holds that $-K_{\Sigma'} = 7H_{\Sigma'}$.  $\square$

### 4.2.3. *Extending the mid point in the case of* C-*type* .

**Definition 4.13** (**Extension of** $W$)**.** We denote by $Q_\mathsf{c}$ the quadric in the projective space $\mathbb{P}(\wedge^2 V^3 \oplus U^5 \oplus (U^5)^*)$ defined by the dual pairing $U^5 \times (U^5)^* \to \mathbb{C}$. We set

$$\overline{\Sigma} := \big(A_\mathsf{C} * \mathbb{P}\big((U^5)^*\big)\big) \cap Q_\mathsf{c},$$

and

$$\overline{\Pi} := \mathbb{P}\big(\wedge^2 V^3 \oplus 0 \oplus (U^5)^*\big) \subset \mathbb{P}\big(\wedge^2 V^3 \oplus U^5 \oplus (U^5)^*\big).$$

Note that $\overline{\Pi} \simeq \mathbb{P}^7$ and $\overline{\Pi} \subset \overline{\Sigma}$.

**Proposition 4.14.** *The pair* $(W, \Pi)$ *is projectively equivalent to the pair of a linear section* $W'$ *of* $\overline{\Sigma}$ *and the* 2-*plane* $\overline{\Pi} \cap W'$.

*Proof.* We can show this in a similar (and simpler) way to Proposition 4.9, hence we omit a proof.  $\square$

## 5. EMBEDDING THEOREM IN THE GENUS 4 AND 6 CASES

In this section, we show Theorem 1.1 in the genus 4 and 6 cases (Theorem 5.17). To show the theorem in a unified way, we proceed in the the following two subsections 5.1 and 5.2 under a more general setting.

### 5.1. **Basic set-up.** Let $A$ be a Fano manifold. We denote by $f_A$ the Fano index of $A$, and by $L_A$ the ample divisor such that $-K_A = f_A L_A$.

**Assumption 1.** We assume that $L_A$ is very ample, $\dim A \geq 4$, and

$$(5.1) \qquad\qquad d := f_A - (\dim A - 2) > 0.$$

We embed $A$ by $|L_A|$ the projective space denoted by $\mathbb{P}(U_A)$. Sometimes we also denote $L_A$ by $\mathcal{O}_A(1)$.

**Assumption 2.** We assume moreover that $A$ contains mutually disjoint codimension two linear spaces $\Pi_1 = \mathbb{P}(U_{(1)}), \ldots, \Pi_l = \mathbb{P}(U_{(l)})$, where $U_{(1)}, \ldots, U_{(l)}$ are linear subspaces of $U_A$ of dimension $\dim A - 1$.

We set

$$\Pi := \Pi_1 \sqcup \cdots \sqcup \Pi_l.$$

Let $a \colon \widehat{A} \to A$ be the blow-up along $\Pi$ and $F_a$ the $a$-exceptional divisor, which consists of $l$ connected components.

**Assumption 3.** We further assume that there exists a morphism $b' \colon \widehat{A} \to \mathbb{P}(U_B)$ to a projective space $\mathbb{P}(U_B) \simeq \mathbb{P}^N$ such that

$$b'^{*}\mathcal{O}_{\mathbb{P}(U_B)}(1) = a^*(dL_A) - F_a,$$

where $d$ is defined as in (5.1).

We denote by $B$ the image of $b'$ and by $b \colon \widehat{A} \to B$ the induced morphism. We also set $L_B := \mathcal{O}_{\mathbb{P}(U_B)}(1)|_B$. Therefore we have

$$(5.2) \qquad\qquad b^* L_B = a^*(dL_A) - F_a,$$

*Remark* 5.1. We can classify the situation as above but we omit a proof since we will just apply the construction in the section 5 to the situations appearing for prime $\mathbb{Q}$-Fano 3-folds of genus 4 or 6.

## 5.2. **Construction of the key varieties .**

**Definition 5.2.** We define

$$(5.3) \qquad\qquad \widehat{\Sigma} := \mathbb{P}_{\widehat{A}}(a^*\mathcal{O}_A(-1) \oplus b^*(\Omega_{\mathbb{P}(U_B)}(1)|_B)).$$

We denote by $\pi \colon \widehat{\Sigma} \to \widehat{A}$ the natural projection.

The linear system $|H_{\widehat{\Sigma}}|$ defines a morphism $\varphi_{|H_{\widehat{\Sigma}}|} \colon \widehat{\Sigma} \to \overline{\Sigma}$ since $\mathrm{Bs}\,|H_{\widehat{\Sigma}}| = \emptyset$. Note that $\overline{\Sigma} \subset \mathbb{P}(U_A \oplus (U_B)^*)$.

Summarizing the above constructions, we obtain the following diagram:

$$(5.4)$$

$$
\begin{array}{ccc}
\widehat{\Sigma} & \xrightarrow{\varphi_{|H_{\widehat{\Sigma}}|}} & \overline{\Sigma} \subset \mathbb{P}(U_A \oplus (U_B)^*) \\
\downarrow{\scriptstyle \pi=\text{proj. bundle}} & & \\
\widehat{A} & &
\end{array}
$$

$$
a=\text{bl.up along }\Pi \swarrow \qquad\qquad \searrow b
$$

$$\mathbb{P}(U_A) \supset A \qquad\qquad\qquad B \subset \mathbb{P}(U_B).$$

Using $-K_{\widehat{A}} = a^*(-K_A) - F_a$ and (5.2), we have

$$(5.5)$$
$$-K_{\widehat{\Sigma}} = (N+1)H_{\widehat{\Sigma}} - \pi^*(K_{\widehat{A}} + a^* L_A + b^* L_B) =$$
$$(N+1)H_{\widehat{\Sigma}} + (f_A - d - 1)\pi^* a^* L_A = (N+1)H_{\widehat{\Sigma}} + (\dim A - 3)\pi^* a^* L_A,$$

where we also use (5.1) in the last equality. By this calculation, we see that $\mathrm{Bs}\,|-K_{\widehat{\Sigma}}| = \emptyset$. We denote by $\nu \colon \widehat{\Sigma} \to \widehat{\Sigma}'$ the anti-canonical model.

The $\mathbb{Q}$-Fano variety $\Sigma$ will be constructed as in the following several steps summarized in the diagram:

(5.6)



where the divisors $E_{\widehat{\Sigma}}$ and $F_{\widehat{\Sigma}}$ are defined in the sequel. We will finally achieve the construction of $\Sigma$ in Theorem 5.11.

*Remark* 5.3. This is not a Sarkisov link since the relative Picard number of $b \circ \pi$ is greater than or equal to $3$ in any case. We will, however, explain later that this is close to an extension of the Sarkisov link for a $\mathbb{Q}$-Fano 3-fold of genus 4 or 6.

**Two divisors $E_{\widehat{\Sigma}}$ and $F_{\widehat{\Sigma}}$ on $\widehat{\Sigma}$.**

We define the following two divisors $E_{\widehat{\Sigma}}$ and $F_{\widehat{\Sigma}}$ on $\widehat{\Sigma}$:

$$E_{\widehat{\Sigma}} := \mathbb{P}_{\widehat{A}}(0 \oplus b^*\Omega_{\mathbb{P}(U_B)}(1)|_B), \; F_{\widehat{\Sigma}} := \pi^* F_a.$$

We remark that

(5.7) $$E_{\widehat{\Sigma}} \sim H_{\widehat{\Sigma}} - \pi^* a^* L_A, \text{ and } F_{\widehat{\Sigma}} \sim \pi^*(a^*(dL_A) - b^* L_B),$$

where the former is a standard equation as for projective subbundle and the latter follows from (5.2).

In the step by step construction of the birational map from $\widehat{\Sigma}$ to $\Sigma$ in the sequel, it is useful to describe how these two divisors $E_{\widehat{\Sigma}}$ and $F_{\widehat{\Sigma}}$ on $\widehat{\Sigma}$ are transformed by birational maps.

**Flop $\widehat{\Sigma} \dashrightarrow \Sigma^+$.**

**Proposition 5.4.** *The following assertions hold:*

*(1) The anti-canonical model $\nu \colon \widehat{\Sigma} \to \widehat{\Sigma}'$ is defined over $\overline{\Sigma}$, and is a flopping contraction of Atiyah type. The divisor $\pi^* b^* L_B$ is relatively ample for the flopping contraction.*

*(2) The anti-canonical model $E_{\widehat{\Sigma}} \to E'_{\widehat{\Sigma}}$ is the restriction of $\nu$ and is also a flopping contraction of Atiyah type. It is defined over $\mathbb{P}(0 \oplus (U_B)^*)$.*

*Proof.* (1). Let $l \subset \widehat{\Sigma}$ be an irreducible $\nu$-exceptional curve (the existence of such an $l$ will be verified below). By (5.5), we have $H_{\widehat{\Sigma}} \cdot l = \pi^* a^* L_A \cdot l = 0$, where we also use the assumption $\dim A \geq 4$. By the former condition, $\nu$ is defined over $\overline{\Sigma}$. Since $\pi \colon \widehat{\Sigma} \to \widehat{A}$ is a projective bundle and $H_{\widehat{\Sigma}} = \mathcal{O}(1)$ in a fiber, $l$ is not contracted by $\pi$. By the latter condition, $l$ is contracted also by $a \circ \pi \colon \widehat{\Sigma} \to A$. Therefore the image $\gamma$ of $l$ on $\widehat{A}$ is an exceptional curve of the blow-up $a \colon \widehat{A} \to A$ along $\Pi$. Thus $\gamma \simeq \mathbb{P}^1$ and is mapped isomorphically to a line $\gamma'$ on $B$ by the equation (5.2). Then, by (5.3), the restriction $\widehat{\Sigma}_\gamma$ of $\widehat{\Sigma}$ over $\gamma$ is isomorphic to $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}^{\oplus N} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$. Since $l$ is contained in $\widehat{\Sigma}_\gamma$ and the map defined by $|H_{\widehat{\Sigma}}|_{\widehat{\Sigma}_\gamma}|$ is the blow-up of $\mathbb{P}^{N+1}$ along a $(N-1)$-plane, we see that $l$ is an exceptional curve of this blow-up and hence $l \simeq \mathbb{P}^1$ (now the existence of the curve $l$ has been verified).

To show the first assertion, it suffices to show $\mathcal{N}_{l/\widehat{\Sigma}} \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus \dim A + N - 3} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$. Note that the normal bundle $\mathcal{N}_{\gamma/\widehat{A}}$ is $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus \dim A - 2}$ since $a \colon \widehat{A} \to A$ is the blow-up along $\Pi$ and $\gamma$ is one of its fiber. Therefore the restriction to $l$ of $\mathcal{N}_{\widehat{\Sigma}_\gamma/\widehat{\Sigma}}$ is also $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus \dim A - 2}$. Moreover, $\mathcal{N}_{l/\widehat{\Sigma}_\gamma} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus N - 1}$ since $l$ is a fiber of the blow-up $\widehat{\Sigma}_\gamma \to \mathbb{P}^{N+1}$ along a $(N-1)$-plane. Therefore, by the normal bundle sequence $0 \to \mathcal{N}_{l/\widehat{\Sigma}_\gamma} \to \mathcal{N}_{l/\widehat{\Sigma}} \to \mathcal{N}_{\widehat{\Sigma}_\gamma/\widehat{\Sigma}}|_l \to 0$, we see that $\mathcal{N}_{l/\widehat{\Sigma}} \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus \dim A + N - 3} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ as desired.

The divisor $L_B$ is relatively ample for the flopping contraction since the image of a flopping curve on $B$ is a line as we have seen above.

(2). Note that, by (5.5) and (5.7), we have

$$(5.8) \quad -K_{E_{\widehat{\Sigma}}} = \{(N+1)H_{\widehat{\Sigma}} + (\dim A - 3\pi^* a^* L_A\}|_{E_{\widehat{\Sigma}}} - (H_{\widehat{\Sigma}} - \pi^* a^* L_A)|_{E_{\widehat{\Sigma}}} = N(H_{E_{\widehat{\Sigma}}}) + (\dim A - 2)\pi^* a^* L_A.$$

Thus $\mathrm{Bs}\,|-K_{E_{\widehat{\Sigma}}}| = \emptyset$. Using these, we can show (2) in a similar way to (1). $\qquad \square$

Let $\widehat{\Sigma} \dashrightarrow \Sigma^+$ be the flop for this flopping contraction $\nu$. It is well-known that the flop can be constructed by the blow-up along the $\nu$-exceptional locus and the blow-down of the exceptional divisor along the other direction. Let $E_{\Sigma^+}$ be the strict transform on $\Sigma^+$ of $E_{\widehat{\Sigma}}$. By the construction of the flop, we see that the restriction $E_{\widehat{\Sigma}} \dashrightarrow E_{\Sigma^+}$ of $\widehat{\Sigma} \dashrightarrow \Sigma^+$ is also the flop.

We call a positive dimensional fiber of $\varphi_{|H_{\widehat{\Sigma}}|} \colon \widehat{\Sigma} \to \overline{\Sigma}$ a $\varphi_{|H_{\widehat{\Sigma}}|}$-*exceptional curve,* and the union of $\varphi_{|H_{\widehat{\Sigma}}|}$-exceptional curves *the* $\varphi_{|H_{\widehat{\Sigma}}|}$-*exceptional locus.* We can identify the $\varphi_{|H_{\widehat{\Sigma}}|}$-exceptional locus as follows:

**Lemma 5.5.** *The* $\varphi_{|H_{\widehat{\Sigma}}|}$-*exceptional locus is the union of the flopping locus for* $\widehat{\Sigma} \dashrightarrow \Sigma^+$ *and the divisor* $E_{\widehat{\Sigma}}$. *The flopping locus is contained in* $F_{\widehat{\Sigma}}$. *The* $\varphi_{|H_{\widehat{\Sigma}}|}$-*image of* $E_{\widehat{\Sigma}} = \mathbb{P}_{\widehat{A}}(0 \oplus b^* \Omega_{\mathbb{P}(U_B)}(1)|_B)$ *is* $\mathbb{P}(0 \oplus (U_B)^*)$ *and the* $\varphi_{|H_{\widehat{\Sigma}}|}$-*inverse image of* $\mathbb{P}(0 \oplus (U_B)^*)$ *coincides with* $E_{\widehat{\Sigma}}$.

*Proof.* Let $l$ be a $\varphi_{|H_{\widehat{\Sigma}}|}$-exceptional curve. Note that $H_{\widehat{\Sigma}} \cdot l = 0$. If $\pi^* a^* L_A \cdot l > 0$, then by (5.7), we have $E_{\widehat{\Sigma}} \cdot l < 0$ and hence $l \subset E_{\widehat{\Sigma}}$. If $\pi^* a^* L_A \cdot l = 0$, then, by the proof of Proposition 5.4, $l$ is an exceptional curve of the anti-canonical model $\widehat{\Sigma} \to \widehat{\Sigma}'$, namely, a flopping curve. Therefore the $\varphi_{|H_{\widehat{\Sigma}}|}$-exceptional locus is contained in the union of the flopping locus for $\widehat{\Sigma} \dashrightarrow \Sigma^+$ and the divisor $E_{\widehat{\Sigma}}$. Since the restriction of $\varphi_{|H_{\widehat{\Sigma}}|}$ to $E_{\widehat{\Sigma}}$ is $\mathbb{P}_{\widehat{A}}(0 \oplus b^* \Omega_{\mathbb{P}(U_B)}(1)|_B) \to \mathbb{P}(0 \oplus (U_B)^*)$, $E_{\widehat{\Sigma}}$ is contained in $\varphi_{|H_{\widehat{\Sigma}}|}$-exceptional locus. Thus the first assertion follows. By the second assertion of Proposition 5.4 (1) and (5.7), a flopping curve is negative for $F_{\widehat{\Sigma}}$, hence is contained in $F_{\widehat{\Sigma}}$. Therefore the second assertion follows. The final assertion obviously holds. $\qquad \square$

We denote by $a'$ and $b'$ the restriction to $F_a$ of $a$ and $b$ respectively. By (5.3), we have $F_{\widehat{\Sigma}} = \mathbb{P}_{F_a}(a'^*\mathcal{O}_\Pi(-1) \oplus b'^*\Omega_{\mathbb{P}(U_B)}(1)|_B)$.

$$
\begin{array}{c}
F_{\widehat{\Sigma}} \\
\pi_F \downarrow \\
F_a \\
{}^{a'}\swarrow \qquad \searrow^{b'} \\
\Pi \qquad\qquad B.
\end{array}
$$

**Lemma 5.6.** *The following assertions hold:*

(1) *Let $\nu_F$ be the restriction to $F_{\widehat{\Sigma}}$ of the flopping contraction $\nu$ (note that $\nu_F$ is the contraction over $\Pi$ by the proof of Proposition 5.4). The $\nu_F$-image of $F_{\widehat{\Sigma}}$ is isomorphic to $\mathbb{P}_\Pi(\mathcal{O}_\Pi(-1) \oplus (U_B)^* \otimes \mathcal{O}_\Pi) = \sqcup_{i=1}^l \mathbb{P}_{\Pi_i}(\mathcal{O}_{\Pi_i}(-1) \oplus (U_B)^* \otimes \mathcal{O}_{\Pi_i})$. The image of $\mathbb{P}_{\Pi_i}(\mathcal{O}_{\Pi_i}(-1) \oplus (U_B)^* \otimes \mathcal{O}_{\Pi_i})$ on $\overline{\Sigma}$ is $\mathbb{P}(U_{(i)} \oplus (U_B)^*)$, which we denote by $\overline{\Pi}_i$.*

(2) *Let $F_{\Sigma^+}$ be the strict transform on $\Sigma^+$ of $F_{\widehat{\Sigma}}$. The restriction $F_{\widehat{\Sigma}} \dashrightarrow F_{\Sigma^+}$ of the flop $\widehat{\Sigma} \dashrightarrow \Sigma^+$ is identified with the contraction $F_{\widehat{\Sigma}} \to \mathbb{P}_\Pi(\mathcal{O}_\Pi(-1) \oplus (U_B)^* \otimes \mathcal{O}_\Pi)$.*

*Proof.* (1). As we have seen in the proof of Proposition 5.4, a fiber of $F_{\widehat{\Sigma}} \to \Pi$ ($\widehat{\Sigma}_\gamma$ in the proof) is $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}^{\oplus N} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$. By the proof of Proposition 5.4 again, the restriction of $\nu_F$ to a fiber of $F \to \Pi$ is the blow-up of $\mathbb{P}^N$ along a $(N-1)$-plane. Hence $\nu_F(F_{\widehat{\Sigma}})$ is the $\mathbb{P}^N$-bundle $\mathbb{P}_\Pi(\mathcal{E}_{F_{\widehat{\Sigma}}})$ with $\mathcal{E}_{F_{\widehat{\Sigma}}} := \{a'_*\pi_{F*}\mathcal{O}_{F_{\widehat{\Sigma}}}(H_{F_{\widehat{\Sigma}}})\}^*$. We have

$$a'_*\pi_{F*}\mathcal{O}_{F_{\widehat{\Sigma}}}(H_{F_{\widehat{\Sigma}}}) \simeq a'_*(a'^*\mathcal{O}_\Pi(1) \oplus b'^* T_{\mathbb{P}(U_B)}(-1)|_B) \simeq \mathcal{O}_\Pi(1) \oplus a'_* b'^* T_{\mathbb{P}(U_B)}(-1)|_B.$$

To compute $a'_* b'^* T_{\mathbb{P}(U_B)}(-1)|_B$, we consider the restriction to $B$ of the Euler sequence of $\mathbb{P}(U_B)$:

$$0 \to \mathcal{O}_B(-1) \to U_B \otimes \mathcal{O}_B \to T_{\mathbb{P}(U_B)}(-1)|_B \to 0.$$

Since $a'_* b'^* \mathcal{O}_B(-1) = R^1 a'_* b'^* \mathcal{O}_B(-1) = 0$ by (5.2), we have $a'_* b'^* T_{\mathbb{P}(U_B)}(-1)|_B \simeq U_B \otimes \mathcal{O}_\Pi$. Therefore we have $\mathcal{E}_{F_{\widehat{\Sigma}}} \simeq \mathcal{O}_\Pi(-1) \oplus (U_B)^* \otimes \mathcal{O}_\Pi$ as desired. The final assertion obviously holds.

(2). The assertion follows from the explicit construction of the flop of Atiyah type. $\qquad\square$

**Lemma 5.7.** *The following assertions hold:*

(1) $E_{\Sigma^+} \cap F_{\Sigma^+} = \mathbb{P}_\Pi((U_B)^* \otimes \mathcal{O}_\Pi) \simeq \Pi \times \mathbb{P}((U_B)^*)$.

(2) The exceptional locus of $\Sigma^+ \to \overline{\Sigma}$ is the union of the divisor $E_{\Sigma^+}$ and the flopped locus.

*Proof.* (1). By definition, the intersection $E_{\widehat{\Sigma}} \cap F_{\widehat{\Sigma}}$ is equal to $\mathbb{P}_{F_a}(b'^*(\Omega^1_{\mathbb{P}(U_B)}(1)|_B))$. By the contraction $F_{\widehat{\Sigma}} \to \mathbb{P}_\Pi(\mathcal{O}_\Pi(-1) \oplus (U_B)^* \otimes \mathcal{O}_\Pi) \simeq F_{\Sigma^+}$, $E_{\widehat{\Sigma}} \cap F_{\widehat{\Sigma}}$ is mapped onto $\mathbb{P}_\Pi((U_B)^* \otimes \mathcal{O}_\Pi)$. By the explicit construction of the flop $\widehat{\Sigma} \dashrightarrow \Sigma^+$, we see that $E_{\Sigma^+} \cap F_{\Sigma^+} = \mathbb{P}_\Pi((U_B)^* \otimes \mathcal{O}_\Pi)$.

(2). The assertion follows from Lemma 5.5. $\qquad\square$

In the following steps, we separate $E_{\Sigma^+}$ and $F_{\Sigma^+}$ by a flip, and finally contract their strict transforms.

**Flip $\Sigma^+ \dashrightarrow \widetilde{\Sigma}$.**

Let $H_{\Sigma^+}$ and $L_A^+$ be the strict transforms on $\Sigma^+$ of $H_{\widehat{\Sigma}}$ and $\pi^* a^* L_A$ respectively.

**Proposition 5.8.**

(1) Let $\Gamma$ be a fiber of $E_{\Sigma^+} \cap F_{\Sigma^+} \to \mathbb{P}((U_B)^*)$, which is a copy of $\Pi$ by Lemma 5.7 (1). It holds that
$$\mathcal{N}_{\Gamma/\Sigma^+} = \mathcal{O}_{\mathbb{P}^{\dim A - 2}}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^{\dim A - 2}}^{\oplus N}.$$

(2) There exists a small contraction $\Sigma^+ \to (\Sigma^+)'$ contracting $E_{\Sigma^+} \cap F_{\Sigma^+} \simeq \Pi \times \mathbb{P}((U_B)^*)$ onto $\mathbb{P}((U_B)^*)$.

*Proof.* (1). We set $G = E_{\Sigma^+} \cap F_{\Sigma^+}$. To determine the normal bundle $\mathcal{N}_{\Gamma/\Sigma^+}$, let us consider the normal bundle sequence $0 \to \mathcal{N}_{\Gamma/F_{\Sigma^+}} \to \mathcal{N}_{\Gamma/\Sigma^+} \to \mathcal{N}_{F_{\Sigma^+}/\Sigma^+}|_\Gamma \to 0$. Since $G = \mathbb{P}_\Pi((U_B)^* \otimes \mathcal{O}_\Pi) \subset \mathbb{P}_\Pi(\mathcal{O}_\Pi(-1) \oplus (U_B)^* \otimes \mathcal{O}_\Pi) = F_{\Sigma^+}$, we have $\mathcal{N}_{G/F_{\Sigma^+}} = (H_{F_{\Sigma^+}} - L_\Pi)|_G$, where $L_\Pi$ is the pull-back of $\mathcal{O}_\Pi(1)$. Therefore we have $\mathcal{N}_{\Gamma/F_{\Sigma^+}} \simeq \mathcal{O}_{\mathbb{P}^{\dim A - 2}}(-1) \oplus \mathcal{O}_{\mathbb{P}^{\dim A - 2}}^{\oplus N}$. By (5.5), we have $-K_{\Sigma^+} = (N+1)H_{\Sigma^+} + (\dim A - 3)L_A^+$. Since $H_{\Sigma^+}|_\Gamma = 0$ and $L_A^+|_\Gamma = \mathcal{O}_{\mathbb{P}^{\dim A - 2}}(1)$, we have $-K_{\Sigma^+}|_\Gamma = \mathcal{O}_{\mathbb{P}^{\dim A - 2}}(\dim A - 3)$. Therefore $\deg \mathcal{N}_{\Gamma/\Sigma^+} = \dim A - 3 - (\dim A - 1) = -2$ and hence by the above normal bundle sequence, $F_{\Sigma^+}|_\Gamma = \mathcal{O}_{\mathbb{P}^{\dim A - 2}}(-1)$ and then we have $\mathcal{N}_{\Gamma/\Sigma^+} = \mathcal{O}_{\mathbb{P}^{\dim A - 2}}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^{\dim A - 2}}^{\oplus N}$.

(2). We show that $L_A^+ + F_{\Sigma^+}$ is nef over $\overline{\Sigma}$ and numerically trivial only for fibers of $G \to \mathbb{P}((U_B)^*)$. Assume that $(L_A^+ + F_{\Sigma^+}) \cdot \gamma \leq 0$ for an exceptional curve $\gamma$ for $\Sigma^+ \to \overline{\Sigma}$. It is enough to show that $\gamma$ is contained in a fiber of $G \to \mathbb{P}((U_B)^*)$, and $(L_A^+ + F_{\Sigma^+}) \cdot \gamma = 0$. By Lemma 5.7 (2), $\gamma$ is a flopped curve or is contained in $E_{\Sigma^+}$. Assume that $\gamma$ is a flopped curve. Let $\gamma' \subset \widehat{\Sigma}$ be the corresponding flopping curve. Since $\pi^* a^* L_A \cdot \gamma' = 0$ on $\widehat{\Sigma}$, we have $L_A^+ \cdot \gamma = 0$ on $\Sigma^+$. By the proof of Proposition 5.4, we have $F_{\widehat{\Sigma}} \cdot \gamma' = -1$, hence we have $F_{\Sigma^+} \cdot \gamma = 1$ by a property of the flop of Atiyah type. Therefore $(L_A^+ + F_{\Sigma^+}) \cdot \gamma = 1 > 0$, a contradiction. Thus we have $\gamma \subset E_{\Sigma^+}$. If $F_{\Sigma^+} \cdot \gamma < 0$, then $\gamma \subset F_{\Sigma^+}$, hence $\gamma \subset F_{\Sigma^+} \cap E_{\Sigma^+} = \Lambda$. Since $\gamma$ is exceptional over $\overline{\Sigma}$, $\gamma$ must be contained in a fiber of $G \to \mathbb{P}((U_B)^*)$. To compute $(L_A^+ + F_{\Sigma^+}) \cdot \gamma$, we may assume that $\gamma$ is a line. Then we have $L_A^+ \cdot \gamma = 1$, and $F_{\Sigma^+} \cdot \gamma = -1$ by the proof of (1). Therefore $(L_A^+ + F_{\Sigma^+}) \cdot \gamma = 0$ as desired. Since we are already done if $F_{\Sigma^+} \cdot \gamma < 0$, we may assume that $F_{\Sigma^+} \cdot \gamma \geq 0$ in the sequel. Then, since $L_A^+$ is nef, we have $(L_A^+ + F_{\Sigma^+}) \cdot \gamma \geq 0$, hence $L_A^+ \cdot \gamma = F_{\Sigma^+} \cdot \gamma = 0$ by the assumption that $(L_A^+ + F_{\Sigma^+}) \cdot \gamma \leq 0$. By $F_{\Sigma^+} \cdot \gamma = 0$, $\gamma$ cannot be a flopped curve. Therefore its strict transform $\gamma'$ on $\widehat{\Sigma}$ satisfies $\pi^* a^* L_A \cdot \gamma' = 0$ and $H_{\widehat{\Sigma}} \cdot \gamma' = 0$. However, this implies that $\gamma'$ is a flopping curve by the proof of Proposition 5.4, a contradiction. Now we have shown that $L_A^+ + F_{\Sigma^+}$ is nef over $\overline{\Sigma}$ and numerically trivial only for fibers of $G \to \mathbb{P}((U_B)^*)$.

Note that $-K_{\Sigma^+}$ is nef and big since so is $-K_{\widehat{\Sigma}}$ by Proposition 5.4 and $\widehat{\Sigma} \dashrightarrow \Sigma^+$ is a flop. Therefore, $L_A^+ + F_{\Sigma^+}$ is semiample by the Kawamata-Shokurov base point free theorem (cf.[KMM]). Thus the contraction over $\overline{\Sigma}$ defined by $L_A^+ + F_{\Sigma^+}$ is the desired one. $\square$

By Proposition 5.8 (1), the contraction $\Sigma^+ \to \overline{\Sigma}^+$ is of flipping type, and the flip can be constructed by the blow-up along $G$ and the blow-down of the exceptional

divisor along the other direction (this is a so called a family of standard flips [Ka1]). Let $\Sigma^+ \dashrightarrow \widetilde{\Sigma}$ be the flip. By Proposition 5.8 (1) again, the flipped locus is a $\mathbb{P}^1$-bundle over $\mathbb{P}((U_B)^*)$. We denote by $E_{\widetilde{\Sigma}}$, $F_{\widetilde{\Sigma}}$ and $H_{\widetilde{\Sigma}}$ be the strict transforms on $\widetilde{\Sigma}$ of $E_{\Sigma^+}$, $F_{\Sigma^+}$ and $H_{\widehat{\Sigma}}$ respectively.

**Contracting $E_{\widetilde{\Sigma}}$ and $F_{\widetilde{\Sigma}}$.**

By the constructions of the flop $\widehat{\Sigma} \dashrightarrow \Sigma^+$ and the flip $\Sigma^+ \dashrightarrow \widetilde{\Sigma}$, and the description of $E_{\Sigma^+} \cap F_{\Sigma^+}$ as in Lemma 5.7 (1), we see that $E_{\widetilde{\Sigma}} \cap F_{\widetilde{\Sigma}} = \emptyset$.

By the construction of the flip, we see that the restriction $F_{\Sigma^+} \dashrightarrow F_{\widetilde{\Sigma}}$ of the flip $\Sigma^+ \dashrightarrow \widetilde{\Sigma}$ is the contraction $F_{\Sigma^+} \to \sqcup_{i=1}^l \mathbb{P}(U_{(i)} \oplus U_B^*)$, where we recall that $U_{(i)}$ are defined in the beginning of the subsection 5.1. Thus $F_{\widetilde{\Sigma}}$ is the disjoint union of $\widetilde{F}_i := \mathbb{P}(U_{(i)} \oplus U_B^*) \simeq \mathbb{P}^{\dim A + N - 1}$ $(i = 1, \ldots, l)$.

**Lemma 5.9.** *The normal bundle $\mathcal{N}_{\widetilde{F}_i/\widetilde{\Sigma}}$ is $\mathcal{O}_{\mathbb{P}^{\dim A + N - 1}}(-2)$ for $i = 1, \ldots, l$.*

*Proof.* Since $E_{\widetilde{\Sigma}} \cap F_{\widetilde{\Sigma}} = \emptyset$, we see that the restriction to $F_{\widetilde{\Sigma}}$ of the strict transform on $\widetilde{\Sigma}$ of $\pi^* a^* L_A$ is linearly equivalent to $H_{\widetilde{\Sigma}}|_{F_{\widetilde{\Sigma}}}$ by (5.7). Therefore, by (5.5), we see that $-K_{\widetilde{\Sigma}}|_{\widetilde{F}_i} = \mathcal{O}_{\mathbb{P}^{\dim A + N - 1}}(N + \dim A - 2)$. Since $-K_{\widetilde{F}_i} = \mathcal{O}_{\mathbb{P}^{\dim A + N - 1}}(\dim A + N)$, we have $\mathcal{N}_{\widetilde{F}_i/\widetilde{\Sigma}} = \mathcal{O}_{\mathbb{P}^{\dim A + N - 1}}(-2)$ as desired. $\qquad\square$

**Lemma 5.10.** $2H_{\widetilde{\Sigma}} + F_{\widetilde{\Sigma}}$ *is semiample.*

*Proof.* We show that $2H_{\widetilde{\Sigma}} + F_{\widetilde{\Sigma}}$ is nef. Assume that $(2H_{\widetilde{\Sigma}} + F_{\widetilde{\Sigma}}) \cdot \gamma < 0$ for an irreducible curve $\gamma$. Then $F_{\widetilde{\Sigma}} \cdot \gamma < 0$ since $H_{\widetilde{\Sigma}}$ is nef, and hence $\gamma \subset F_{\widetilde{\Sigma}}$. Since $F_{\widetilde{\Sigma}}|_{\widetilde{F}_i} = \mathcal{O}_{\mathbb{P}^{\dim A + N - 1}}(-2)$ and $H_{\widetilde{\Sigma}}|_{\widetilde{F}_i} = \mathcal{O}_{\mathbb{P}^{\dim A + N - 1}}(1)$, we have $(2H_{\widetilde{\Sigma}} + F_{\widetilde{\Sigma}}) \cdot \gamma = 0$, a contradiction. Therefore $2H_{\widetilde{\Sigma}} + F_{\widetilde{\Sigma}}$ is nef.

To show $2H_{\widetilde{\Sigma}} + F_{\widetilde{\Sigma}}$ is semiample, we have only to show $m(2H_{\widetilde{\Sigma}} + F_{\widetilde{\Sigma}}) - K_{\widetilde{\Sigma}}$ is nef and big for $m \gg 0$ by the Kawamata-Shokurov base point free theorem. Since $-K_{\widehat{\Sigma}}$ is nef and big, and $\widehat{\Sigma} \dashrightarrow \Sigma^+$ is a flop, we see that $-K_{\Sigma^+}$ is also nef and big. Since $\Sigma^+ \dashrightarrow \widetilde{\Sigma}$ is a flip, we see that $-K_{\widetilde{\Sigma}}$ is big and is negative only for flipped curves. Let $\gamma$ be a flipped curve. Then $-K_{\widetilde{\Sigma}} \cdot \gamma = -(\dim A + N - 2)$ and $F_{\widetilde{\Sigma}} \cdot \gamma = 1$ by the construction of the flip, we have $(m(2H_{\widetilde{\Sigma}} + F_{\widetilde{\Sigma}}) - K_{\widetilde{\Sigma}}) \cdot \gamma > 0$ for $m \gg 0$. Therefore $m(2H_{\widetilde{\Sigma}} + F_{\widetilde{\Sigma}}) - K_{\widetilde{\Sigma}}$ is nef for $m \gg 0$. The bigness is clear since $2H_{\widetilde{\Sigma}} + F_{\widetilde{\Sigma}}$ is nef and $-K_{\widetilde{\Sigma}}$ is big. $\qquad\square$

**Theorem 5.11.** *Let $\mu \colon \widetilde{\Sigma} \to \Sigma$ be the contraction defined by a sufficient multiple of $2H_{\widetilde{\Sigma}} + F_{\widetilde{\Sigma}}$. We recall that $\mathbb{P}(U_B) \simeq \mathbb{P}^N$. The following assertion holds:*

*(1) The $\mu$-exceptional locus is the union of the two divisors $E_{\widetilde{\Sigma}}$ and $F_{\widetilde{\Sigma}}$.*

*(2) The $\mu$-image of $F_{\widetilde{\Sigma}}$ consists of $l$ 1/2-singularities.*

*(3) The discrepancy of $E_{\widetilde{\Sigma}}$ is $\dim A - 3$ and $\mu(E_{\widetilde{\Sigma}}) \simeq \mathbb{P}((U_B)^*)$. In particular $\Sigma$ has Gorenstein terminal singularities along $\mathbb{P}((U_B)^*)$.*

*(4) If $\rho(A) = 1$, then the $(\dim A + N)$-dimensional variety $\Sigma$ is a $\mathbb{Q}$-Fano variety with only terminal singularities and with $\rho(\Sigma) = 1$.*

*(5) The image $M_\Sigma$ of $H_\Sigma$ is a primitive integral ample Weil divisor $M_\Sigma$ and it holds that $-K_\Sigma = (\dim A + N - 2)M_\Sigma$.*

*Proof.* As we have checked in the proof of Lemma 5.10, $2H_{\widetilde{\Sigma}} + F_{\widetilde{\Sigma}}$ is numerical trivial for any curve in $F_{\widetilde{\Sigma}}$. Thus the image of $F_{\widetilde{\Sigma}}$ by $\widetilde{\Sigma} \to \Sigma$ consists of $l$ $1/2$-singularities by Lemma 5.9. Since $E_{\widetilde{\Sigma}} \cap F_{\widetilde{\Sigma}} = \emptyset$, we have $(2H_{\widetilde{\Sigma}} + F_{\widetilde{\Sigma}})|_{E_{\widetilde{\Sigma}}} = 2H_{\widetilde{\Sigma}}|_{E_{\widetilde{\Sigma}}}$. Therefore $E_{\widetilde{\Sigma}}$ is $\mu$-exceptional and $\mu(E_{\widetilde{\Sigma}})$ is isomorphic to $\mathbb{P}((U_B)^*)$.

We show that the $\mu$-exceptional locus is the union of $E_{\widetilde{\Sigma}}$ and $F_{\widetilde{\Sigma}}$. Assume by contradiction that $(2H_{\widetilde{\Sigma}} + F_{\widetilde{\Sigma}}) \cdot \gamma = 0$ for an irreducible curve $\gamma \not\subset E_{\widetilde{\Sigma}} \cup F_{\widetilde{\Sigma}}$. Since $H_{\widetilde{\Sigma}}$ is nef and $\gamma \not\subset F_{\widetilde{\Sigma}}$, we have $H_{\widetilde{\Sigma}} \cdot \gamma = F_{\widetilde{\Sigma}} \cdot \gamma = 0$. Then, by Lemma 5.7 (2) and the conditions that $H_{\widetilde{\Sigma}} \cdot \gamma = 0$ and $\gamma \not\subset E_{\widetilde{\Sigma}}$, $\gamma$ is a flipped curve or the strict transform of a flopped curve. If $\gamma$ is a flipped curve, then $F_{\widetilde{\Sigma}} \cdot \gamma > 0$, a contradiction. Assume that $\gamma$ is the strict transform of a flopped curve. If $\gamma$ is disjoint from flipping curves, then $F_{\widetilde{\Sigma}} \cdot \gamma > 0$ since a flopped curve is positive for $F_{\Sigma^+}$, a contradiction. Therefore $\gamma$ intersects a flipped curve. Let $\gamma'$ be the strict transform of $\gamma$ on $\Sigma^+$ ($\gamma'$ is a flopped curve). Since $\gamma$ intersects a flipped curve and the flipping locus is $E_{\Sigma^+} \cap F_{\Sigma^+}$, we see that $\gamma'$ intersects $E_{\Sigma^+}$. Since $E_{\Sigma^+} \cdot \gamma' = 0$, this implies that $\gamma' \subset E_{\Sigma^+}$ and hence $\gamma \subset E_{\widetilde{\Sigma}}$, a contradiction.

We compute the discrepancy of $E_{\widetilde{\Sigma}}$. By (5.5) and (5.7), we have $-K_{\widetilde{\Sigma}} = (N + 1 + \dim A - 3)H_{\widetilde{\Sigma}} - (\dim A - 3)E_{\widetilde{\Sigma}}$. Since $H_{\widetilde{\Sigma}} \sim_{\mathbb{Q}} \mu^* M_{\Sigma} - \frac{1}{2}F_{\widetilde{\Sigma}}$, we have

$$(5.9) \quad -K_{\widetilde{\Sigma}} \sim_{\mathbb{Q}} (N+1+\dim A-3)\mu^* M_{\widetilde{\Sigma}} - \frac{N + 1 + \dim A - 3}{2}F_{\widetilde{\Sigma}} - (\dim A - 3)E_{\widetilde{\Sigma}}.$$

Therefore the discrepancy of $E_{\widetilde{\Sigma}}$ is equal to $\dim A - 3$. Since this is a positive integer, $\Sigma$ has only Gorenstein terminal singularities along $\mu(E_{\widetilde{\Sigma}})$. By (5.9), we have $-K_{\Sigma} = (\dim A + N - 2)M_{\Sigma}$.

We show that $\rho(\Sigma)=1$. Since $\widehat{A} \to A$ is the blow-up along $l$ disjoint projective spaces, and $\widehat{\Sigma} \to \widehat{A}$ is a projective bundle, we see that $\rho(\widehat{\Sigma}) = \rho(A) + l + 1$. Since $\widehat{\Sigma} \dashrightarrow \widetilde{\Sigma}$ is small, we have $\rho(\widetilde{\Sigma}) = \rho(\widehat{\Sigma}) = \rho(A) + l + 1$. Finally, since $\widetilde{\Sigma} \to \Sigma$ contracts $l + 1$ disjoint divisors, we have $\rho(\Sigma) \leq \rho(\widetilde{\Sigma}) - (l + 1) = \rho(A) = 1$. Hence $\rho(\Sigma) = 1$.

Finally, we show that $M_{\Sigma}$ is primitive. If $M_{\Sigma}$ were not primitive, then $M_{\Sigma}$ would be written as $M_{\Sigma} = \alpha M'_{\Sigma}$ with a primitive Weil divisor $M'_{\Sigma}$ and positive integer $\alpha \geq 2$. Since $2M'_{\Sigma}$ are Cartier divisors by (2) and (3), we have $2H_{\widetilde{\Sigma}} + F_{\widetilde{\Sigma}} = \alpha \mu^*(2M'_{\Sigma})$. Hence there is a Cartier divisor $D$ on $\widehat{\Sigma}$ such that $2H_{\widehat{\Sigma}} + F_{\widehat{\Sigma}} = \alpha D$. Let $l$ be a flopping curve for $\widehat{\Sigma} \dashrightarrow \Sigma^+$. By (5.2) and the proof of Proposition 5.4, we have $F_{\widehat{\Sigma}} \cdot l = -1$. This implies that $\alpha D \cdot l = -1$. This is impossible for $\alpha \geq 2$. Therefore $M_{\Sigma}$ is primitive. $\qquad\square$

### 5.3. **Application to the three cases .** In this subsection, we produce the situation as in the subsections 5.1 and 5.2 for a $\mathbb{Q}$-Fano 3-fold of genus 4 or 6.

5.3.1. *Genus 4 .* In this case, we set

$$A := Q^4 \subset \mathbb{P}((U^3)^* \oplus U^3)$$

with the same equation as that of $Q$ in Definition 4.2, and

$$\Pi_1 := \mathbb{P}((U^3)^*), \ \Pi_2 := \mathbb{P}(U^3),$$

which are certainly contained in $Q^4$. We also set

$$B := B_6 = \mathbb{P}(\Omega^1_{\mathbb{P}^2}(1))$$

with the equation as in the subsection 4.1.2, and

$$\widehat{A} := \mathbb{P}_{B_6}(\mathcal{O}_{B_6}(-1,0) \oplus \mathcal{O}_{B_6}(0,-1)).$$

Finally, we set $b$ as the projection morphism

$$\mathbb{P}_{B_6}(\mathcal{O}_{B_6}(-1,0) \oplus \mathcal{O}_{B_6}(0,-1)) \to B_6.$$

**Lemma 5.12.** *There exists a morphism* $a\colon \widehat{A} \to A$ *which is the blow-up of* $\widehat{A}$ *along* $\Pi = \Pi_1 \sqcup \Pi_2$ *and whose exceptional divisor* $F_a$ *is* $\mathbb{P}_{B_6}(\mathcal{O}_{B_6}(-1,0) \oplus 0) \sqcup \mathbb{P}_{B_6}(0 \oplus \mathcal{O}_{B_6}(0,-1))$. *The pull-back of* $\mathcal{O}_A(1)$ *on* $\widehat{A}$ *is the tautological line bundle associated with* $\mathcal{O}_{B_6}(-1,0) \oplus \mathcal{O}_{B_6}(0,-1)$. *The triplet* $(Q^4, \Pi = \Pi_1 \sqcup \Pi_2, B_6)$ *satisfies the condition of* $(A, \Pi, B)$ *as in the subsection 5.1 by setting* $d = l = 2$.

*Proof.* Take a point $\mathsf{p} := [W^1 \otimes U^1] \in B_6$, where $W^1 \subset (U^3)^*$ and $U^1 \subset U^3$ are 1-dimensional subspaces such that $U^1 \subset (V^1)^\perp$ with respect to the dual pairing. The fiber of the projection $\widehat{A} \to B_6$ over $\mathsf{p}$ is $\mathbb{P}(W^1 \oplus U^1)$, which is a linear subspace of $\mathbb{P}((U^3)^* \oplus U^3)$. Note that, for a point $[\boldsymbol{y} + \boldsymbol{x}] \in \mathbb{P}(W^1 \oplus U^1)$ with $\boldsymbol{y} \in W^1$ and $\boldsymbol{x} \in U^1$, it holds that ${}^t\boldsymbol{y}\boldsymbol{x} = 0$ since $U^1 \subset (W^1)^\perp$. Therefore the image of $\widehat{A} \to \mathbb{P}((U^3)^* \oplus U^3)$ is contained in $Q^4$. We denote by $a$ the induced morphism $\widehat{A} \to A = Q^4$. By Lemma 2.2 (1), the second assertion follows.

Let $\mathsf{q} := [\boldsymbol{y} + \boldsymbol{x}]$ be a point of $Q^4$ with $\boldsymbol{y} \in (U^3)^*$ and $\boldsymbol{x} \in U^3$. By Lemma 2.2 (2), the fiber of $\widehat{A} \to Q^4$ over $\mathsf{q}$ is

$$\{\mathsf{q}\} \times \left\{ [W^1 \otimes U^1] \mid U^1 \subset (W^1)^\perp, \boldsymbol{y} \in W^1, \boldsymbol{x} \in U^1 \right\}.$$

If $\boldsymbol{y} \neq \boldsymbol{o}$ and $\boldsymbol{x} \neq \boldsymbol{o}$, then $W^1$ and $U^1$ are uniquely determined as $W^1 = \mathbb{C}\boldsymbol{y}$ and $U^1 = \mathbb{C}\boldsymbol{x}$ (since ${}^t\boldsymbol{y}\boldsymbol{x} = 0$, it holds that $U^1 \subset (W^1)^\perp$). Therefore the morphism $\widehat{A} \to Q^4$ is dominant, hence is surjective, and is an isomorphism outside of $\Pi$. The fiber of $\widehat{A} \to Q^4$ over $\boldsymbol{o} + \boldsymbol{x} \in \Pi_1$ is isomorphic to $\mathbb{P}((\mathbb{C}\boldsymbol{x})^\perp) \simeq \mathbb{P}^1$, and the fiber of $\widehat{A} \to Q^4$ over $\boldsymbol{y} + \boldsymbol{o} \in \Pi_2$ is isomorphic to $\mathbb{P}((\mathbb{C}\boldsymbol{y})^\perp) \simeq \mathbb{P}^1$. Note that, since $-K_{\widehat{A}} = 2H_{\widehat{A}} + b^* L_B$, we see that $-K_{\widehat{A}}$ is relatively ample for $\widehat{A} \to Q^4$. Therefore, by [An, Thm.2.3], $\widehat{A} \to Q^4$ is the blow-up of $Q^4$ along $\Pi$ and the $a$-exceptional divisor is $\mathbb{P}_{B_6}(\mathcal{O}_{B_6}(-1,0) \oplus 0) \sqcup \mathbb{P}_{B_6}(0 \oplus \mathcal{O}_{B_6}(0,-1))$.

Assumptions 1–2 are clearly satisfied. We check Assumption 3, equivalently, the relation (5.2) with $d = 2$. Take a hyperplane $L \subset \mathrm{S}^{-1,0,1}U^3$. We consider elements of $\mathrm{S}^{1,0,-1}U^3$ and $\mathrm{S}^{-1,0,1}U^3$ as $3 \times 3$ traceless matrices. The dual pairing between $U^3 \otimes (U^3)^*$ and $(U^3)^* \otimes U^3$ induces a natural dual pairing between $\mathrm{S}^{1,0,-1}U^3$ and $\mathrm{S}^{-1,0,1}U^3$. Explicitly, for $Z = (z_{ij}) \in \mathrm{S}^{-1,0,1}U^3$ and $P = (p_{ij}) \in \mathrm{S}^{1,0,-1}U^3$, the dual pairing is defined as $(Z, P) \mapsto \sum z_{ij} p_{ij}$. For $L$, there exists $M = (m_{ij}) \in \mathrm{S}^{1,0,-1}U^3$ such that $L = \left\{ \sum_{1 \leq i,j \leq 3} m_{ij} z_{ij} = 0 \right\}$. The above construction show that $Q^4 \setminus \Pi \to B_6$ is defined by $[\boldsymbol{y} + \boldsymbol{x}] \mapsto [\boldsymbol{y} \otimes \boldsymbol{x}]$. Therefore we see that $a_* b^* (B_6 \cap \mathbb{P}(L)) = Q^4 \cap \{{}^t\boldsymbol{y}M\boldsymbol{x} = 0\}$, which is a quadric section of $Q^4$ containing $\Pi$. We can explicitly check that a general $a_* b^* (B_6 \cap \mathbb{P}(L))$ is generically smooth along $\Pi$. Since $b^* (B_6 \cap \mathbb{P}(L))$ does not contain the $a$-exceptional divisor $F_a$, it is the strict transform of $a_* b^* (B_6 \cap \mathbb{P}(L))$ for a general $L$. Therefore the relation (5.2) holds with $d = 2$. $\qquad \square$

In the following subsections 5.3.2 and 5.3.3, we use the notation as in the subsection 2.4.

5.3.2. **Genus 6, $\mathbb{Q}$-type** . In this case, we set

$$A := A_{\mathbb{Q}} = \mathrm{G}(2, V) \cap \mathbb{P}(V' \oplus U^5)$$

and $\Pi$ the same as in the subsection 2.4. We recall that the projection of $\mathrm{G}(2, V)$ from the 3-plane $\Pi$ induces the natural rational map $A_{\mathbb{Q}} \dashrightarrow \mathrm{G}(2, V') \cap \mathbb{P}(U^5)$ and the target $\mathrm{G}(2, V') \cap \mathbb{P}(U^5)$ is the smooth quadric 3-fold $Q^3$. We set

$$B := Q^3 = \mathrm{G}(2, V') \cap \mathbb{P}(U^5),$$

and

$$\widehat{A} := \widehat{A}_{\mathbb{Q}} := \mathbb{P}_{Q^3}(\mathcal{U}|_{Q^3} \oplus \mathcal{O}_{Q^3}(-1)),$$

where $\mathcal{U}$ is the rank two universal subbundle on $\mathrm{G}(2, V')$. Finally we set $b$ as the projection morphism

$$\mathbb{P}_{Q^3}(\mathcal{U}|_{Q^3} \oplus \mathcal{O}_{Q^3}(-1)) \to Q^3.$$

**Lemma 5.13.** *There exists a morphism $\widehat{A}_{\mathbb{Q}} \to A_{\mathbb{Q}}$ which is the blow-up of $\widehat{A}_{\mathbb{Q}}$ along $\Pi$ and whose exceptional divisor is $\mathbb{P}_{Q^3}(\mathcal{U}|_{Q^3} \oplus 0)$. The pull-back of $\mathcal{O}_{A_{\mathbb{Q}}}(1)$ on $\widehat{A}_{\mathbb{Q}}$ is the tautological line bundle associated with $\mathcal{U}|_{Q^3} \oplus \mathcal{O}_{Q^3}(-1)$. The triplet $(A_{\mathbb{Q}}, \Pi, Q^3)$ satisfies the condition of $(A, \Pi, B)$ as in the subsection 5.1 by setting $d = l = 1$.*

*Proof.* Since we can show the first two assertions in a quite similar way to Lemma 3.3, we only show that $\widehat{A}_{\mathbb{Q}} \to A_{\mathbb{Q}}$ is the blow-up along $\Pi$. Note that the restriction of the morphism $\widehat{A}_{\mathbb{Q}} \to A_{\mathbb{Q}}$ over $\Pi$ is $\mathbb{P}_{Q^3}(\mathcal{U}|_{Q^3} \oplus 0) \to \mathbb{P}(V' \oplus 0) = \Pi \simeq \mathbb{P}^3$, which can be identified with the natural morphism $\mathbb{P}_{Q^3}(\mathcal{U}|_{Q^3}) \to \mathbb{P}(V')$ to $\mathbb{P}(V') \simeq \mathbb{P}^3$ from the total space of lines in $\mathbb{P}^3$ parameterized by $Q^3 \subset \mathrm{G}(2, V')$. By [SW, Prop.3.4], $\mathbb{P}_{Q^3}(\mathcal{U}|_{Q^3}) \to \mathbb{P}^3$ is the projectivization of the null-correlation bundle. Note that, since $-K_{\widehat{A}_{\mathbb{Q}}} = 2H_{\widehat{A}_{\mathbb{Q}}} + L_{Q^3}$ where $L_{Q^3}$ is the pull-back of $\mathcal{O}_{Q^3}(1)$, we see that $-K_{\widehat{A}}$ is relatively ample for $\widehat{A}_{\mathbb{Q}} \to A_{\mathbb{Q}}$ Therefore, by [An, Thm.2.3], $\widehat{A}_{\mathbb{Q}} \to A_{\mathbb{Q}}$ is the blow-up of $A_{\mathbb{Q}}$ along $\Pi$ and the exceptional divisor is $\mathbb{P}_{Q^3}(\mathcal{U} \oplus 0)$. By the construction, the pull-back of $\mathcal{O}_{A_{\mathbb{Q}}}(1)$ on $\widehat{A}_{\mathbb{Q}}$ is the tautological line bundle associated with $\mathbb{P}_{Q^3}(\mathcal{U}|_{Q^3} \oplus \mathcal{O}_{Q^3}(-1))$.

Assumptions 1–2 are clearly satisfied. Since $A_{\mathbb{Q}} \dashrightarrow Q^3$ is the restriction of the projection from $\Pi$, the relation (5.2) follows. $\square$

5.3.3. **Genus 6, $\mathbb{C}$-type** . In this case, we set

$$A := A_{\mathbb{C}} = \mathrm{G}(2, V) \cap \mathbb{P}(U^8),$$

$\Pi$ the same as in the subsection 2.4, and $a\colon \widehat{A} \to A$ the blow-up of $A$ along $\Pi$. We recall that the projection of $\mathrm{G}(2, V)$ from the 2-plane $\Pi$ induces the natural rational map $A_{\mathbb{C}} \dashrightarrow \mathbb{P}(U^5)$. We set

$$B := \mathbb{P}(U^5) \simeq \mathbb{P}^4,$$

and $b\colon \widehat{A} \to B$ the naturally induced morphism.

**Lemma 5.14.** *The triplet $(A, \Pi, \mathbb{P}^4)$ satisfies the condition of $(A, \Pi, B)$ as in the subsection 5.1 by setting $d = l = 1$.*

*Proof.* The assertion is almost clear. $\square$

5.3.4. *Rationality of the key varieties.*

**Corollary 5.15.** *In the genus $4$ or $6$ case, $\Sigma$ is rational.*

*Proof.* The assertion follows since $\Sigma$ is birational to a projective bundle over a rational Fano manifold in the genus 4 or 6 case. $\qquad\qquad\square$

## 5.4. Coincidence between $\overline{\Sigma}$'s in the subsection 5.2 and in the section 4.

**Lemma 5.16.** *In the genus 4 or 6 case, the variety $\overline{\Sigma}$ as in the subsection 5.2 is the same as the variety $\overline{\Sigma}$ defined as in the section 4. The morphism $\varphi_{|H_{\widehat{\Sigma}}|}\colon \widehat{\Sigma} \to \overline{\Sigma}$ is birational. The $\varphi_{|H_{\widehat{\Sigma}}|}$-image of $E_{\widehat{\Sigma}}$ on $\overline{\Sigma}$ is disjoint from $W$.*

*Proof.* The variety $\overline{\Sigma}$ as in the subsection 5.2 defined for the triplet $(A, \Pi, B)$ is contained in $\mathbb{P}(U_A \oplus U_B^*)$ by the fact that $H^0(a^*\mathcal{O}_A(1)) = U_A$ and $H^0(b^*(\Omega^1_{\mathbb{P}(U_B)}(1)|_B)) = U_B^*$. Temporarily, we denote by $\overline{\Sigma}'$ the variety $\overline{\Sigma}$ as in the subsection 4, which is also contained in $\mathbb{P}(U_A \oplus U_B^*)$ in each case .

First we show that $\overline{\Sigma} \subset \overline{\Sigma}'$. For this, it suffices to check the $\varphi_{|H_{\widehat{\Sigma}}|}$-image of a general point of $\widehat{\Sigma}$ is mapped in $\overline{\Sigma}'$ since $\widehat{\Sigma}$ is irreducible. Since $\widehat{A} \subset A \times B$ by Lemma 2.2 (1) and the results in the subsection 5.3, we can express a point of $\widehat{A}$ as $([\boldsymbol{x}], [\boldsymbol{y}])$ with $\boldsymbol{x} \in U_A$ and $\boldsymbol{y} \in U_B$. Note that $\boldsymbol{x}$ satisfies the equation of $A$. The fiber of $\pi\colon \widehat{\Sigma} \to \widehat{A}$ over $\mathsf{p}$ is $\mathbb{P}(\mathbb{C}\boldsymbol{x} \oplus (U_B/\mathbb{C}\boldsymbol{y})^*)$. We choose a point $\mathsf{p} := ([\boldsymbol{x}], [\boldsymbol{y}]) \in \widehat{A}$ such that $[\boldsymbol{x}] \notin \Pi$. In this case, it holds that $[\boldsymbol{y}] = [b(a^{-1}(\boldsymbol{x}))]$. In each of the three cases, we check that the fiber $\pi^{-1}(\mathsf{p})$ is mapped by $\varphi_{|H_{\widehat{\Sigma}}|}$ into $\overline{\Sigma}'$ in the sequel.

Genus 4: We can express a point of $\widehat{A}$ as $([\boldsymbol{x}_1 + \boldsymbol{x}_2], [\boldsymbol{y}])$ with $\boldsymbol{x}_1 \in (U^3)^*$, $\boldsymbol{x}_2 \in U^3, \boldsymbol{y} \in \mathrm{S}^{-1,0,1}U^3$. We are choosing a point $\mathsf{p} := ([\boldsymbol{x}_1 + \boldsymbol{x}_2], [\boldsymbol{y}]) \in \widehat{A}_{\mathsf{Q}}$ such that $[\boldsymbol{x}_1 + \boldsymbol{x}_2] \notin \Pi$, namely, $\boldsymbol{x}_1 \neq \boldsymbol{o}$ and $\boldsymbol{x}_2 \neq \boldsymbol{o}$. In this case, it holds that $[\boldsymbol{y}] = [\boldsymbol{x}_2 \otimes \boldsymbol{x}_1]$ by the proof of Lemma 5.12 and hence the $\pi$-fiber over $\mathsf{p}$ is $\mathbb{P}(\mathbb{C}(\boldsymbol{x}_1 + \boldsymbol{x}_2) \oplus (U^5/\mathbb{C}(\boldsymbol{x}_2 \otimes \boldsymbol{x}_1))^*)$. Therefore, by the definition of $\overline{\Sigma}'$ as in 4.2, we see that $\overline{\Sigma} \subset \overline{\Sigma}'$.

Genus 6, Q-type: We can express a point of $\widehat{A}_{\mathsf{Q}}$ as $([\boldsymbol{x}_1 + \boldsymbol{x}_2], [\boldsymbol{y}])$ with $\boldsymbol{x}_1 \in V'$, $\boldsymbol{x}_2, \boldsymbol{y} \in U^5$. We are choosing a point $\mathsf{p} := ([\boldsymbol{x}_1 + \boldsymbol{x}_2], [\boldsymbol{y}]) \in \widehat{A}_{\mathsf{Q}}$ such that $[\boldsymbol{x}_1 + \boldsymbol{x}_2] \notin \Pi$, namely, $\boldsymbol{x}_2 \neq \boldsymbol{o}$. In this case, it holds that $[\boldsymbol{y}] = [\boldsymbol{x}_2]$ since $A_{\mathsf{Q}} \dashrightarrow Q^3$ is the projection from $\Pi$, and hence the $\pi$-fiber over $\mathsf{p}$ is $\mathbb{P}(\mathbb{C}(\boldsymbol{x}_1 + \boldsymbol{x}_2) \oplus (U^5/\mathbb{C}\boldsymbol{x}_2)^*)$. Therefore, by the definition of $\overline{\Sigma}'$ as in 4.8, we see that $\overline{\Sigma} \subset \overline{\Sigma}'$.

We can show that $\overline{\Sigma} \subset \overline{\Sigma}'$ in the case of genus 6 and C-type in a similar way to the case of genus 6 and Q-type, so we omit a proof.

Now we check that $\overline{\Sigma} = \overline{\Sigma}'$. Since $\dim \widehat{\Sigma} = \dim \overline{\Sigma}'$, it suffices to show a general $\varphi_{|H_{\widehat{\Sigma}}|}$-fiber consists of one point. This also implies that $\varphi_{|H_{\widehat{\Sigma}}|}\colon \widehat{\Sigma} \to \overline{\Sigma}$ is birational. We take a point $\mathsf{t} := [\boldsymbol{t}_1 + \boldsymbol{t}_2] \in \overline{\Sigma} \setminus \overline{\Pi}$ with $\boldsymbol{t}_1 \in U_A \setminus \{\boldsymbol{o}\}$ and $\boldsymbol{t}_2 \in U_B^*$. Since $\pi^{-1}(F_a) = \varphi_{|H_{\widehat{\Sigma}}|}^{-1}(\overline{\Pi})$ , we have

$$\varphi_{|H_{\widehat{\Sigma}}|}^{-1}(\mathsf{t}) = \{\mathsf{t}\} \times \left\{ ([\boldsymbol{x}], [b(a^{-1}(\boldsymbol{x}))], \mathsf{t}) \in \widehat{\Sigma} \mid [\boldsymbol{x}] \notin \Pi, \mathsf{t}_1 \in \mathbb{C}\boldsymbol{x}, \boldsymbol{t}_2 \in (U_B/(\mathbb{C}b(a^{-1}(\boldsymbol{x}))))^*) \right\}$$

by Lemma 2.2 (2). This is nonempty since we take $\mathsf{t}$ in the $\varphi_{|H_{\widehat{\Sigma}}|}$-image of $\widehat{\Sigma}$. Moreover, it consist of one point as desired since $[\boldsymbol{x}] = [\boldsymbol{t}_1]$.

We show the last assertion. Note that the $\varphi_{|H_{\widehat{\Sigma}}|}$-image of $E_{\widehat{\Sigma}} = \mathbb{P}(0 \oplus b^*(\Omega^1_{\mathbb{P}(U_B)}(1)|_B))$ on $\overline{\Sigma}$ coincides with $\mathbb{P}(0 \oplus U_B^*)$. In the genus 4 case, this is equal to $\overline{\Pi}_1 \cap \overline{\Pi}_2 =$

$\mathbb{P}(0 \oplus 0 \oplus \mathrm{S}^{1,0,-1}U^3)$ and is disjoint from $W$ since $W \cap (\overline{\Pi}_1 \cap \overline{\Pi}_2) = \Pi_1 \cap \Pi_2 = \emptyset$ by Proposition 2.6. In the genus 6 case, $\overline{\Sigma}$ has non-hypersurface singularities along $\mathbb{P}(0 \oplus U_B^*)$, hence must be disjoint from $W$. $\hfill\square$

### 5.5. Embedding theorem .

Now we show Theorem 1.1 for a prime $\mathbb{Q}$-Fano 3-fold $X$ of genus 4 or 6.

**Theorem 5.17.** *A $\mathbb{Q}$-Fano 3-fold $X$ of genus $4$ or $6$ is a linear section of $\Sigma$.*

*Proof.* The proof which will be given below is more or less the same as that of Theorem 3.7 but is slightly involved, so we write it for readers' convenience.

Note that $W \cap \mathrm{Sing}\,\overline{\Sigma}$ is 0-dimensional since $W$ has only terminal singularities and $W$ is a linear section of $\overline{\Sigma}$ with respect to $|\mathcal{O}_{\overline{\Sigma}}(1)|$. By Lemmas 5.16, $W$ is disjoint from the image of $E_{\widehat{\Sigma}}$. Therefore, since $\widetilde{\Sigma} \to \overline{\Sigma}$ is crepant and small and nontrivial fibers are 1-dimensional over $W$, the strict transform $W_{\widehat{\Sigma}}$ of $W$ in $\widehat{\Sigma}$ is a linear section of $\widehat{\Sigma}$ with respect to $|H_{\widehat{\Sigma}}|$ and hence the restriction $W_{\widehat{\Sigma}} \to W$ of $\widehat{\Sigma} \to \overline{\Sigma}$ over $W$ is also crepant and small. Since $W$ has only terminal singularities and $W_{\widehat{\Sigma}} \to W$ is crepant, we see that $W_{\widehat{\Sigma}}$ is normal and has only terminal singularities by [CKM, the proof of Prop.16.4]. Note that $F_{\widehat{\Sigma}}|_{W_{\widehat{\Sigma}}}$ is the strict transform of $\Pi$ and is relatively ample for $W_{\widehat{\Sigma}} \to W$. Since $Y \to W$ in the genus 6 case (resp. $Z \to W$ in the genus 4 case) is the unique small extraction such that the strict transform of $\Pi$ is relatively ample, we see that $Y = W_{\widehat{\Sigma}}$ in the genus 6 case (resp. $Z = W_{\widehat{\Sigma}}$ in the genus 4 case). Since we may write $Y = W_{\widehat{\Sigma}} = \widetilde{H}_1 \cap \cdots \cap \widetilde{H}_{\dim\widetilde{\Sigma}-3}$ with $\widetilde{H}_i \in |H_{\widehat{\Sigma}}|$ $(1 \le i \le \dim\widetilde{\Sigma}-3)$, we see that $X = M_1 \cap \cdots \cap M_{\dim\widetilde{\Sigma}-3}$ with the image $M_i \in |M_\Sigma|$ of $\widetilde{H}_i$ as desired. $\hfill\square$

### 5.6. Extension of the Sarkisov link.

We have obtained the following diagram in the genus 4 or 6 case:

(5.10)
$$
\begin{array}{ccc}
\widetilde{\Sigma} \overset{\text{anti-flip}}{\dashrightarrow} \Sigma^+ \overset{\text{flop}}{\dashrightarrow} \widehat{\Sigma} & & \\
{\scriptstyle\mu}\swarrow \qquad \downarrow \qquad \searrow {\scriptstyle b\circ\pi} & & \\
\Sigma \qquad \overline{\Sigma} \qquad B. & &
\end{array}
$$

**Corollary 5.18.** *In the case of genus 4, the restriction of (5.10) to 3-folds, we obtain the following diagram:*

(5.11)
$$
\begin{array}{ccc}
Z \overset{\text{flop}}{\dashrightarrow} Z' & & \\
{\scriptstyle g}\swarrow \quad \searrow \swarrow \quad \searrow {\scriptstyle g'} & & \\
X \qquad W \qquad B_6, & &
\end{array}
$$

*where $Z$ and $W$ defined as in the subsection 2.3 are linear sections of $\widetilde{\Sigma}$ and $\overline{\Sigma}$ with respect to $|H_{\widetilde{\Sigma}}|$ and $|\mathcal{O}_{\overline{\Sigma}}(1)|$ respectively; $Z'$ is defined as the corresponding linear section of $\widehat{\Sigma}$ with respect to $|H_{\widehat{\Sigma}}|$; the restriction of the anti-flip to $Z$ is the identity. Moreover, the following assertions hold:*
*(1) The morphism $g' \colon Z' \to B_6$ is the blow-up of $B_6$ along a smooth curve $C'$ of genus $8$ isomorphic to $C$.*

*(2) The curve $C'$ is the complete intersection of the strict transforms of the $g$-exceptional divisors, which are divisors of types $(2,1)$ and $(1,2)$.*

*In the case of genus 6, (5.10) is an extension of (1.1), where $Y$,$W$ and $Y'$ are linear sections of $\widetilde{\Sigma}$, $\overline{\Sigma}$ and $\widehat{\Sigma}$ with respect to $|H_{\widetilde{\Sigma}}|$, $|\mathcal{O}_{\overline{\Sigma}}(1)|$ and $|H_{\widehat{\Sigma}}|$ respectively, and $X' = B = Q^3$ in the case of $\mathbb{Q}$-type (resp. $X'$ is a cubic 3-fold in $\mathbb{P}(U^5)$ in the case of $\mathbb{C}$-type). The restriction of the anti-flip to $Y$ is the identity.*

*Proof.* The genus 4 case: The restriction of the anti-flip to $Z$ is the identity since the flipped locus in $\widetilde{\Sigma}$ is contained in $E_{\widetilde{\Sigma}}$ by Proposition 5.8 (2), and $E_{\widetilde{\Sigma}}$ is disjoint from $Z'$ by Lemma 5.16. The rest of the assertion except (1) and (2) easily follow from the proof of Theorem 5.17.

(1). Since $\widehat{\Sigma} \to B_6$ is a $\mathbb{P}^8$-bundle, and $Z'$ is a linear section of $\widehat{\Sigma}$ with respect to $|H_{\widehat{\Sigma}}|$ of codimension 8, we see that $Z' \to B_6$ is birational and $-K_{Z'} = \pi^*a^*L_{Q^4}|_{Z'}$ by (5.5). Since $E_{\widehat{\Sigma}}$ is disjoint from $Z'$, we have $H_{\widehat{\Sigma}}|_{Z'} \sim (\pi^*a^*L_A)|_{Z'}$ by (5.7). Therefore we have $-K_{Z'} = H_{\widehat{\Sigma}}|_{Z'}$. Since $H_{\widehat{\Sigma}}$ is relatively ample over $B_6$, so is $-K_{Z'}$. Since $\rho(Z') = 3$ and $\rho(B_6) = 2$, the relative Picard number of the morphism $Z' \to B_6$ is 1. Therefore, by [Mo], $Z' \to B_6$ is the blow-up of $B_6$ at a point or along a curve $C'$. Comparing the Intermediate Jacobians of $Z'$ and the 3-fold obtained by blowing up of $Y'$ at the $1/2$-singularity, we see that $Z' \to B_6$ is the blow-up of $B_6$ along a curve $C'$ such that $C \simeq C'$ as desired.

(2). Since the images of $E_1$ and $E_2$ on $W$ are disjoint by Proposition 2.6, the strict transforms $E_1'$ and $E_2'$ on $Z'$ of $E_1$ and $E_2$ respectively are also disjoint. Therefore $C'$ is set-theoretically the intersection between the strict transforms $E_1''$ and $E_2''$ on $B_6$ of $E_1$ and $E_2$. Moreover, since $Z' \to B_6$ is the blow-up along $C'$, and $E_1' \cap E_2' = \emptyset$, it holds that $C'$ is the complete intersection of $E_1'$ and $E_2'$. Note that the anticanonical morphism $Z' \to W$ is induced from the restriction of $\widehat{\Sigma} \to Q^4$ since $-K_{Z'} = \pi^*a^*L_{Q^4}|_{Z'}$ as we saw in the proof of (1). Therefore we have $F_{\widehat{\Sigma}}|_{Z'} = E_1' \sqcup E_2'$. From this, we obtain $(2\pi^*a^*L_{Q^4})|_{Z'} - E_1' - E_2' = (\pi^*b^*L_{B_6})|_{Z'}$ by the equation (5.2). Since $-K_{Z'} = H_{\widehat{\Sigma}}|_{Z'}$ as we saw in the proof of (1), we have $-2K_{Z'} - E_1' - E_2' = (\pi^*b^*L_{B_6})|_{Z'}$. On the other hand, we have $-K_{Z'} = (2\pi^*b^*L_{B_6})|_{Z'} - E_{C'}$, where $E_{C'}$ is the exceptional divisor of the blow-up $Z' \to B_6$. Therefore we obtain $E_1'' + E_2'' = 3L_{B_6}$. For the curve $C'$ of genus 4 to be the complete intersection of $E_1''$ and $E_2''$, it must holds that $E_1''$ and $E_2''$ are of types $(2, 1)$ and $(1, 2)$.

The genus 6 case: We can show the assertions in a similar and simpler way as in the genus 4 case. $\qquad\square$

## 5.7. Singularity of $\Sigma$ along $\mathbb{P}((U_B)^*)$.

By Theorem 5.11, the birational morphism $\mu\colon \widetilde{\Sigma} \to \Sigma$ contracts $E_{\widetilde{\Sigma}}$ onto $\mathbb{P}((U_B)^*)$. In this subsection, we describe the morphism $\mu|_{E_{\widetilde{\Sigma}}}\colon E_{\widetilde{\Sigma}} \to \mathbb{P}((U_B)^*)$. This follows by studying how fibers of the morphism $E_{\widehat{\Sigma}} \to \mathbb{P}((U_B)^*)$ are transformed by the flop $\widehat{\Sigma} \dashrightarrow \Sigma^+$ and the flip $\Sigma^+ \dashrightarrow \widetilde{\Sigma}$.

We note that the natural morphism $\mathbb{P}(\Omega_{\mathbb{P}(U_B)}(1)) \to \mathbb{P}((U_B)^*)$ is the universal family of hyperplanes of $\mathbb{P}(U_B)$. Therefore the naturally induced morphism $E_{\widehat{\Sigma}} \to \mathbb{P}((U_B)^*)$ is the universal family of the members of $|b^*L_B|$. In particular, $E_{\widehat{\Sigma}} \to \mathbb{P}((U_B)^*)$ is flat.

Note that the restrictions of the flopping and flipping contractions on the strict transforms of $E_{\widehat{\Sigma}}$ are defined over $\mathbb{P}((U_B)^*)$. The strict transform $E_{\Sigma^+}$ on $\Sigma^+$ of $E_{\widehat{\Sigma}}$ is smooth since $E_{\widehat{\Sigma}} \dashrightarrow E_{\Sigma^+}$ is also a flop of Atiyah type by Proposition 5.4. By Proposition 5.8 and the construction of the flip $\Sigma^+ \dashrightarrow \widetilde{\Sigma}$, we see that $E_{\widetilde{\Sigma}}$ is smooth

since $E_{\Sigma^+} \dashrightarrow E_{\widetilde{\Sigma}}$ is the blow-up along $l$ $\mu|_{E_{\widetilde{\Sigma}}}$-sections whose exceptional divisor is $\mathbb{P}((U_B)^*) \times \Pi$.

By the description of $E_{\widehat{\Sigma}} \to \mathbb{P}((U_B)^*)$ and $E_{\widehat{\Sigma}} \dashrightarrow E_{\Sigma^+} \dashrightarrow E_{\widetilde{\Sigma}}$ as above, the morphism $E_{\widetilde{\Sigma}} \to \mathbb{P}((U_B)^*)$ is also flat.

We denote by $\widehat{\Gamma}$, $\Gamma^+$, and $\widetilde{\Gamma}$ a general fiber of $E_{\widehat{\Sigma}} \to \mathbb{P}((U_B)^*)$ and its strict transforms on $\Sigma^+$ and $\widetilde{\Sigma}$ respectively. By the argument as above, we see that the restriction $\widehat{\Gamma} \dashrightarrow \Gamma^+$ to $\widehat{\Gamma}$ of the flop $\widehat{\Sigma} \dashrightarrow \Sigma^+$ is also a flop of Atiyah type, and the restriction $\Gamma^+ \dashrightarrow \widetilde{\Gamma}$ to $\Gamma^+$ of the flip $\Sigma^+ \dashrightarrow \widetilde{\Sigma}$ is the blow-up of $\widetilde{\Gamma}$ at $l$ smooth points whose exceptional divisor is $\Pi$. Moreover, $\Gamma^+$ and $\widetilde{\Gamma}$ are general fibers of $E_{\Sigma^+} \to \mathbb{P}((U_B)^*)$ and $E_{\widetilde{\Sigma}} \to \mathbb{P}((U_B)^*)$ respectively. We set $F_{\widehat{\Gamma}} := F_{\widehat{\Sigma}}|_{\widehat{\Gamma}}$ and $F_{\Gamma^+} := F_{\Sigma^+}|_{\Gamma^+}$.

Hereafter we consider separately in each case and determine $\widetilde{\Gamma}$.

### 5.7.1. *Genus 4*.

**Proposition 5.19.** *A general fiber $\widetilde{\Gamma}$ of the morphism $E_{\widetilde{\Sigma}} \to \mathbb{P}(S^{1,0,-1}U^3)$ is $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.*

*Proof.* Since a general fiber of $\mathbb{P}(\Omega_{\mathbb{P}(S^{-1,0,1}U^3)}(1)|_{B_6}) \to B_6$ is a smooth sextic del Pezzo surface $S$, $\widehat{\Gamma}$ is isomorphic to $\mathbb{P}_S(\mathcal{O}_S(-1,0) \oplus \mathcal{O}_S(0,-1))$ by Lemma 5.12. The divisor $F_{\widehat{\Gamma}}$ of $\widehat{\Gamma}$ consists of $\widehat{G}_1 := \mathbb{P}_S(\mathcal{O}_S(-1,0) \oplus 0) \simeq S$ and $\widehat{G}_2 := \mathbb{P}_S(0 \oplus \mathcal{O}_S(0,-1)) \simeq S$. It is easy to see the assertion as in the following steps:

- Let $\widehat{\Gamma} \to \widehat{\Gamma}'$ be the flopping contraction, which is the restriction of $\widehat{\Sigma} \to \widehat{\Sigma}'$. This induce the morphisms $\widehat{G}_1 \to \mathbb{P}(U_3^* \oplus 0)$ and $\widehat{G}_2 \to \mathbb{P}(0 \oplus U_3)$, each of which is a contraction of three $(-1)$-curves. These can be identified with the restrictions of $\widehat{G}_1 \dashrightarrow G_1^+$ and $\widehat{G}_2 \dashrightarrow G_2^+$ respectively, where $G_1^+$ and $G_2^+$ are the strict transforms of $\widehat{G}_1$ and $\widehat{G}_2$ respectively on $\Gamma^+$.

- The restriction $\Gamma^+ \dashrightarrow \widetilde{\Gamma}$ to $\Gamma^+$ of the flip $\Sigma^+ \dashrightarrow \widetilde{\Sigma}$ is the blow-up of $\widetilde{\Gamma}$ at two smooth points whose exceptional divisor consists of $G_1^+$ and $G_2^+$.

- $\widetilde{\Gamma}$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Indeed, it holds that $\rho(\widetilde{\Gamma}) = 3$ since $\rho(\widetilde{\Gamma}) + 2 = \rho(\Gamma^+) = \rho(\widehat{\Gamma}) = 5$. Moreover we see that $\widetilde{\Gamma}$ is a sextic del Pezzo 3-folds as follows: it holds that $(-K_{\widetilde{\Gamma}})^3 = (-K_{\Gamma^+})^3 + 2 = (-K_{\widehat{\Gamma}})^3 + 2 = 6$. Since $-K_{\widehat{\Gamma}} = 2H_{\widehat{\Gamma}}$, we have $-K_{\widetilde{\Gamma}} = 2H_{\widetilde{\Gamma}}$ where $H_{\widetilde{\Gamma}}$ is the strict transform of $H_{\widehat{\Gamma}}$. Since $-K_{\widehat{\Sigma}}$ is nef and big and is numerically trivial only for flopping curves, $-K_{\Gamma^+}$ is nef and big and is numerically trivial only for flopped curves. Therefore, $-K_{\widetilde{\Sigma}}$ is ample since flopped curves is numerically positive for the exceptional divisor $G_1^+ \cup G_2^+$ of the blow-up $\Gamma^+ \to \widetilde{\Gamma}$. Since $\widetilde{\Gamma}$ is a sextic del Pezzo 3-folds of $\rho(\widetilde{\Gamma}) = 3$, $\widetilde{\Gamma} \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ by [Fuj1, Thm.5.16].

$\square$

### 5.7.2. *Genus 6, $\mathbb{Q}$-type*.

**Proposition 5.20.** *A general fiber $\widetilde{\Gamma}$ of the morphism $E_{\widetilde{\Sigma}} \to \mathbb{P}((U^5)^*)$ is $\mathbb{P}^2 \times \mathbb{P}^2$.*

*Proof.* Since a general fiber of $\mathbb{P}(\Omega_{\mathbb{P}(U^5)}(1)|_{Q^3}) \to Q^3$ is $\mathbb{P}^1 \times \mathbb{P}^1$, $\widehat{\Gamma}$ is isomorphic to $\mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1) \oplus \mathcal{O}(-1,-1))$ by the definition of $\widehat{A}_{\mathbb{Q}}$ as in the subsection 5.3.2. The divisor $F_{\widehat{\Gamma}}$ of $\widehat{\Gamma}$ is $\mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1) \oplus 0)$. Referring to [Fuk, Thm.5.1] for details, we see the assertion as in the following steps:

- The flopping locus of the flop $\widehat{\Gamma} \dashrightarrow \Gamma^+$ is $\mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}(-1,0) \oplus 0 \oplus 0) \sqcup \mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(0 \oplus \mathcal{O}(0,-1) \oplus 0)$. The divisor $F_{\Gamma^+}$ of $\Sigma^+$ is $\mathbb{P}^3$.
- The restriction $\Gamma^+ \dashrightarrow \widetilde{\Gamma}$ to $\Gamma^+$ of the flip $\Sigma^+ \dashrightarrow \widetilde{\Sigma}$ is the blow-up of $\widetilde{\Gamma}$ at a smooth point whose exceptional divisor is $F_{\Gamma^+} \simeq \mathbb{P}^3$.
- $\widetilde{\Gamma}$ is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^2$.

<div align="right">□</div>

### 5.7.3. *Genus 6, C-type.*

**Proposition 5.21.** *A general fiber $\widetilde{\Gamma}$ of the morphism $E_{\widehat{\Sigma}} \to \mathbb{P}((U^5)^*)$ is $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.*

*Proof.* By [Fuj3], $b \colon \widehat{A}_{\mathsf{C}} \to \mathbb{P}(U^5)$ is the blow-up of $\mathbb{P}(U^5)$ along a twisted cubic $\gamma$. Since a general fiber $H$ of $\mathbb{P}(\Omega_{\mathbb{P}(U^5)}(1)) \to \mathbb{P}((U^5)^*)$ is a hyperplane of $\mathbb{P}(U^5)$, $\widehat{\Gamma}$ is isomorphic to the 3-fold obtained by blowing up $H \simeq \mathbb{P}^3$ along $H \cap \gamma$ which consists of three points $\mathsf{p}_1, \mathsf{p}_2, \mathsf{p}_3$ in a general position. Referring to [Fuk, Thm.4.1] for details, we see the assertion as in the following steps:

- The flopping locus of the flop $\widehat{\Gamma} \dashrightarrow \Gamma^+$ consists of the strict transforms of three lines $l_{ij}$ through $\mathsf{p}_i$ and $\mathsf{p}_j$ $(1 \leq i < j \leq 3)$. The divisor $F_{\Gamma^+}$ of $\Sigma^+$ is $\mathbb{P}^2$.
- The restriction $\Gamma^+ \dashrightarrow \widetilde{\Gamma}$ to $\Gamma^+$ of the flip $\Sigma^+ \dashrightarrow \widetilde{\Sigma}$ is the blow-up of $\widetilde{\Gamma}$ at a smooth point whose exceptional divisor is $F_{\Gamma^+} \simeq \mathbb{P}^2$.
- $\widetilde{\Gamma}$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

<div align="right">□</div>

### 5.7.4. *Comparison of $\Sigma'$ and $\widehat{\Sigma}$.*

In this subsection, we clarify the relationship between $\widehat{\Sigma}$ as in the subsection 5.2 and $\Sigma'$ as in the subsections 4.1.2 and 4.2.2. Setting

$$\mathcal{F} := \begin{cases} \mathcal{O}_{B_6}(-1,0) \oplus \mathcal{O}_{B_6}(0,-1) : & \text{genus 4,} \\ \mathcal{U}|_{Q^3} \oplus \mathcal{O}_{Q^3}(-1) : & \text{genus 6, Q-type,} \end{cases}$$

we may write $\widehat{A} = \mathbb{P}_B(\mathcal{F})$ and $\Sigma' = \mathbb{P}_B(\mathcal{F} \oplus (\Omega^1_{\mathbb{P}(U_B)}(1)|_B))$ in each case.

**Proposition 5.22.** *There exists a naturally induced birational morphism $\tau \colon \widehat{\Sigma} \to \Sigma'$ over $\overline{\Sigma}$ and its exceptional locus coincides with $E_{\widehat{\Sigma}}$. The morphism $\tau$ is the blow-up of $\Sigma'$ along $\mathbb{P}_B(0 \oplus \Omega^1_{\mathbb{P}(U_B)}(1)|_B)$.*

*Proof.* By Lemmas 5.12 and 5.13, we have a surjection $b^*\mathcal{F}^* \to a^*\mathcal{O}_A(1)$, which induces the following natural morphism $\tau$:

$$\begin{aligned} \widehat{\Sigma} = \mathbb{P}_{\widehat{A}}(a^*\mathcal{O}_A(-1) &\oplus b^*(\Omega^1_{\mathbb{P}(U_B)}(1)|_B)) \\ &\hookrightarrow \mathbb{P}_{\widehat{A}}(b^*(\mathcal{F} \oplus \Omega^1_{\mathbb{P}(U_B)}(1)|_B)) := \Sigma'' \\ &\to \mathbb{P}_B(\mathcal{F} \oplus \Omega^1_{\mathbb{P}(U_B)}(1)|_B) = \Sigma', \end{aligned}$$

where the former is the inclusion morphism of projective bundles, and the latter is a $\mathbb{P}^1$-bundle since it is the base change of the $\mathbb{P}^1$-bundle $b \colon \widehat{A} = \mathbb{P}_B(\mathcal{F}) \to B$. By this construction, the $\tau$-pull-back of $\mathcal{O}_{\Sigma'}(1)$ coincides with $\mathcal{O}_{\widehat{\Sigma}}(1)$. Therefore the composite of the morphism $\tau$ and $\varphi_{|H_{\Sigma'}|} \colon \Sigma' \to \overline{\Sigma}$ coincides with $\varphi_{|H_{\widehat{\Sigma}}|}$. This implies that $\tau$ is birational since so is $\varphi_{|H_{\widehat{\Sigma}}|}$ by Lemma 5.16.

By Lemma 5.5, $E_{\widehat{\Sigma}}$ is contracted by $\tau$ since $\varphi_{|H_{\Sigma'}|}$ is small. By the description of $\varphi_{|H_{\widehat{\Sigma}}|}$-fibers as in Lemma 5.5, and the description of $\varphi_{|H_{\Sigma'}|}$-fibers as in Propositions 4.6 and 4.12, $\tau$ is isomorphic outside $E_{\widehat{\Sigma}}$. Note that $\tau$ induces $E_{\widehat{\Sigma}} = \mathbb{P}_{\widehat{A}}(b^*(\Omega^1_{\mathbb{P}(U_B)}(1)|_B)) \to \mathbb{P}_B(\Omega^1_{\mathbb{P}(U_B)}(1)|_B)$ and this is a $\mathbb{P}^1$-bundle. Moreover, by (5.5), $-K_{\widehat{\Sigma}}$ is $\tau$-ample for the morphism. Therefore $\tau$ is the blow-up of $\Sigma'$ along $\mathbb{P}_B(\Omega^1_{\mathbb{P}(U_B)}(1)|_B)$ by [An, Thm.2.3]. $\square$

## 6. Embedding theorem in the genus 5 case

In this section, we treat the genus 5 case. The overall story is similar to the one of the section 5 though details are different. We develop the discussion in this section while keeping in mind the flow of discussion of the section 5.

6.1. **Extending the mid point .** By [Mu2, Thm.6.5 (2) and Prop.7.8], $W$ is a complete intersection of three quadrics in $\mathbb{P}^6$. Let $x_1, \ldots, x_7$ be coordinates of $\mathbb{P}^6$. We may assume that the plane $\Pi$ in $W$ is equal to $\{x_1 = \cdots = x_4 = 0\}$. In this situation, the equation of $W$ is of the following form:

$$
\begin{pmatrix} l_{11} & l_{12} & l_{13} & l_{14} \\ l_{21} & l_{22} & l_{23} & l_{24} \\ l_{31} & l_{32} & l_{33} & l_{34} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},
$$

where $l_{ij}$ are linear forms of $x_1, \ldots, x_7$.

Assume by contradiction that the dimension of the vector space generated by the linear forms $l_{ij}(0, 0, 0, 0, x_5, x_6, x_7)$ is less than or equal to 2. Then $W$ is the cone over a complete intersection of three quadrics in $\mathbb{P}^5$ with a point $\mathsf{v}$ in $\Pi$ as the vertex. Then the Zariski tangent space of $W$ at $\mathsf{v}$ is dimension 6. This is absurd since $W$ has only Gorenstein terminal singularities. Therefore, by a coordinate change keeping the equation of $\Pi$ if necessary, we may assume that some three of $l_{ij}$, say, $l_{i_1 j_1}, l_{i_2 j_2}, l_{i_3 j_3}$ are equal to $x_5, x_6, x_7$ respectively.

**Definition 6.1** (Extension of $W$)**.** In the projective space $\mathbb{P}^{15}$ with coordinates $x_1, \ldots, x_4$ and $y_{ij}$ ($1 \le i \le 3, 1 \le j \le 4$), let $\overline{\Sigma}$ be the the following complete intersection of three quadrics:

$$
\overline{\Sigma} := \left\{ [M_y, \boldsymbol{x}] \in \mathbb{P}^{15} \mid M_y \boldsymbol{x} = \boldsymbol{o} \right\},
$$

where

$$
M_y := \begin{pmatrix} y_{11} & y_{12} & y_{13} & y_{14} \\ y_{21} & y_{22} & y_{23} & y_{24} \\ y_{31} & y_{32} & y_{33} & y_{34} \end{pmatrix}, {}^t\boldsymbol{x} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}.
$$

We set

$$
\overline{\Pi} := \{x_1 = x_2 = x_3 = x_4 = 0\} \subset \mathbb{P}^{15}.
$$

**Proposition 6.2.** *The pair $(W, \Pi)$ is projectively equivalent to the pair of a linear section $W'$ of $\overline{\Sigma}$ and the 2-plane $\overline{\Pi} \cap W'$.*

*Proof.* By writing $y_{i_1 j_1}, y_{i_2 j_2}, y_{i_3 j_3}$ as $x_5, x_6, x_7$ respectively, it holds that the pair $(W, \Pi)$ is projectively equivalent to the pair of

$$
W' = \overline{\Sigma} \cap \{y_{ij} = l_{ij} \text{ for } (i, j) \neq (i_1, j_1), (i_2, j_2), (i_3, j_3)\},
$$

and $\overline{\Pi} \cap W'$. $\square$

We use the following notation:

$$\overline{\Sigma}_i := \left\{ [M_y, \boldsymbol{o}] \in \overline{\Sigma} \mid \operatorname{rank} M_y \leq i \right\} \subset \overline{\Pi}.$$

By elementary calculations, we determine the singular locus of $\overline{\Sigma}$ as follows:

**Proposition 6.3.** *The singular locus of $\overline{\Sigma}$ is contained in $\overline{\Pi}$ and is equal to $\overline{\Sigma}_2$. In particular, $\overline{\Sigma}$ is Gorenstein and normal.*

*Remark* 6.4. The variety $\overline{\Sigma}$ is an example of a variety of a complex (cf. [Tan, Sect. 5]).

## 6.2. **Construction of the key variety .**

**Definition 6.5.** For $U^3 \simeq \mathbb{C}^3$ and $U^4 \simeq \mathbb{C}^4$, we set

$$\Sigma' := \mathbb{P}_{\mathbb{P}(U^4)}(U^3 \otimes \Omega^1_{\mathbb{P}(U^4)}(1) \oplus \mathcal{O}_{\mathbb{P}(U^4)}(-1)).$$

Note that, by a standard computation, it follows that $-K_{\Sigma'} = 10H_\Sigma$.

Under the situation of the subsection 6.1, we consider $\boldsymbol{x}$ as a coordinate vector of $U^4$ and $M_y$ as a coordinate matrix of $U^3 \otimes (U^4)^*$. Then we can regard $\overline{\Sigma}$ as a subvariety of $\mathbb{P}(U^3 \otimes (U^4)^* \oplus U^4)$. With this identification, we have the following proposition:

**Proposition 6.6.** *The following assertions hold:*

*(1) The tautological linear system $|H_{\Sigma'}|$ defines a surjective and birational morphism $\Sigma' \to \overline{\Sigma}$, which we denote by $\varphi_{|H_{\Sigma'}|}$.*

*(2) The morphism $\varphi_{|H_{\Sigma'}|}$ is an isomorphism outside of $\overline{\Sigma}_2 = \operatorname{Sing} \overline{\Sigma}$.*

*(3) The $\varphi_{|H_{\Sigma'}|}$-fiber over a point $\mathsf{t} \in \overline{\Sigma}_2$ is*

$$\begin{cases} \mathbb{P}^1 : & \mathsf{t} \notin \overline{\Sigma}_1, \\ \mathbb{P}^2 : & \mathsf{t} \in \overline{\Sigma}_1. \end{cases}$$

*The morphism $\varphi_{|H_{\Sigma'}|}$ is a crepant small resolution.*

*Proof.* Let $\mathrm{p} := [U^1] \in \mathbb{P}(U^4)$ be a point, where $U^1 \subset U^4$ is a 1-dimensional vector space. The fiber of the projective bundle $\Sigma' \to \mathbb{P}(U^4)$ over $\mathrm{p}$ is

$$\mathbb{P}(U^3 \otimes (U^4/U^1)^* \oplus U^1),$$

which is a linear subspace of $\mathbb{P}(U^3 \otimes (U^4)^* \oplus U^4)$. By Lemma 2.2 (1), the tautological linear system $|H_{\Sigma'}|$ defines a morphism $\Sigma' \to \mathbb{P}(U^3 \otimes (U^4)^* \oplus U^4)$. By the description of fibers of $\Sigma' \to \mathbb{P}(U^4)$ as above and the definition of $\overline{\Sigma}$, we see that the image of this map is contained in $\overline{\Sigma}$.

Let $\mathsf{t} = [M_y, \boldsymbol{x}]$ be a point of $\overline{\Sigma}$. By Lemma 2.2 (2), the fiber of $\Sigma' \to \overline{\Sigma}$ over $\mathsf{t}$ is

$$\{\mathsf{t}\} \times \left\{ [U^1] \mid M_y \in U^3 \otimes (U^4/U^1)^*, \boldsymbol{x} \in U^1 \right\}.$$

If $\mathsf{t} \notin \overline{\Pi}$, namely, $\boldsymbol{x} \neq \boldsymbol{o}$, then $U^1$ are uniquely determined as $U^1 = \mathbb{C}\boldsymbol{x}$ (since $\mathsf{t} \in \overline{\Sigma}$, it holds that $M_y \in U^3 \otimes (U^4/\mathbb{C}\boldsymbol{x})^*$). Therefore the morphism $\Sigma' \to \overline{\Sigma}$ is an isomorphism outside of $\overline{\Pi}$. In particular, the morphism $\Sigma' \to \overline{\Sigma}$ is birational. Assume that $\mathsf{t} \in \overline{\Pi}$, equivalently, $\boldsymbol{x} = \boldsymbol{o}$. The condition for $U^1$ is that $U^1 \subset \left\{ \boldsymbol{x} \in U^4 \mid M_y\boldsymbol{x} = \boldsymbol{0} \right\} \simeq \mathbb{C}^{4-\operatorname{rank} M_y}$. Therefore the fiber of $\Sigma' \to \overline{\Sigma}$ over $\mathsf{t}$ is isomorphic to $\mathbb{P}^{3-\operatorname{rank} M_y}$. From this, the description of the fiber $\Sigma' \to \overline{\Sigma}$ over $\mathsf{t}$ follows. The morphism $\Sigma' \to \overline{\Sigma}$ is crepant since it holds that $-K_{\Sigma'} = 10H_{\Sigma'}$. $\quad\square$

Let $\Sigma_1' \subset \Sigma'$ be the inverse image of $\overline{\Sigma}_1$. Since $\overline{\Sigma}_1 = \mathbb{P}(U^3) \times \mathbb{P}((U^4)^*)$, we see that

$$(6.1) \qquad\qquad \Sigma_1' = \mathbb{P}(U^3) \times \mathbb{P}(\Omega^1_{\mathbb{P}(U^4)}(1)).$$

Let $\tau \colon \widehat{\Sigma} \to \Sigma'$ be the blow-up of $\Sigma'$ along $\Sigma_1'$ , and $E_{\widehat{\Sigma}}$ the $\tau$-exceptional divisor. We denote by $\Pi'$ and $\widehat{\Pi}$ the strict transforms of $\overline{\Pi}$ on $\Sigma'$ and $\widehat{\Sigma}$ respectively. Note that

$$(6.2) \qquad\qquad \Pi' = \mathbb{P}(U^3 \otimes \Omega^1_{\mathbb{P}(U^4)}(1) \oplus 0).$$

**Lemma 6.7.** *Let* $\mathsf{t} = [U^1] \times [W^1]$ *be a point of* $\overline{\Sigma}_1 = \mathbb{P}(U^3) \times \mathbb{P}((U^4)^*)$, *where* $U^1$ *and* $W^1$ *are* 1-*dimensional subspaces of* $U^3$ *and* $(U^4)^*$ *respectively. The following assertions hold:*

*(1) The fiber of* $\Sigma_1' \to \overline{\Sigma}_1$ *over* $\mathsf{t}$ *can be identified with the fiber of* $\mathbb{P}(\Omega^1_{\mathbb{P}(U^4)}(1)) \to \mathbb{P}((U^4)^*)$ *over the point* $[W^1]$ *and then with* $\mathbb{P}(W^{1,\perp})$, *where* $W^{1,\perp}$ *is the subspace of* $U^4$ *orthogonal to* $W^1$ *with respect to the dual pairing. Let* $E_{\mathsf{t}}$ *be the fiber of* $E_{\widehat{\Sigma}} \to \overline{\Sigma}_1$ *over* $\mathsf{t}$. *It holds that*

$$E_{\mathsf{t}} = \mathbb{P}_{\mathbb{P}(W^{1,\perp})}((U^3/U^1) \otimes \Omega^1_{\mathbb{P}(W^{1,\perp})}(1) \oplus \mathcal{O}_{\mathbb{P}(W^{1,\perp})}(-1)),$$

*and*

$$E_{\mathsf{t}} \cap \widehat{\Pi} = \mathbb{P}_{\mathbb{P}(W^{1,\perp})}((U^3/U^1) \otimes \Omega^1_{\mathbb{P}(W^{1,\perp})}(1) \oplus 0).$$

*(2) We identify an element of* $(U^3/U^1) \otimes (W^{1,\perp})^*$ *with a* $2 \times 3$ *matrix. The linear system* $|H_{E_{\mathsf{t}}}|$ *defines a morphism* $E_{\mathsf{t}} \to \mathbb{P}((U^3/U^1) \otimes (W^{1,\perp})^* \oplus W^{1,\perp})$, *and the image is*

$$\overline{E}_{\mathsf{t}} := \left\{ [M, \boldsymbol{x}] \mid M \in (U^3/U^1) \otimes (W^{1,\perp})^*, \boldsymbol{x} \in W^{1,\perp}, M\boldsymbol{x} = \boldsymbol{o} \right\},$$

*which is a complete intersection of two quadrics.*

*(3) The singular locus of* $\overline{E}_{\mathsf{t}}$ *is*

$$\left\{ [M, \boldsymbol{o}] \mid M \in (U^3/U^1) \otimes (W^{1,\perp})^*, \operatorname{rank} M \leq 1 \right\},$$

*which is* $\mathbb{P}(U^3/U^1) \times \mathbb{P}((W^{1,\perp})^*) \simeq \mathbb{P}^1 \times \mathbb{P}^2$. *The morphism* $E_{\mathsf{t}} \to \overline{E}_{\mathsf{t}}$ *is an isomorphism outside of* $\operatorname{Sing} \overline{E}_{\mathsf{t}}$, *and the fiber over a point of* $\operatorname{Sing} \overline{E}_{\mathsf{t}}$ *is* $\mathbb{P}^1$. *The morphism* $E_{\mathsf{t}} \to \overline{E}_{\mathsf{t}}$ *is a crepant small resolution.*

*(4) The induced morphism* $E_{\mathsf{t}} \cap \widehat{\Pi} \to \mathbb{P}((U^3/U^1) \otimes (W^{1,\perp})^* \oplus 0) \simeq \mathbb{P}^5$ *is the blow-up of* $\mathbb{P}^5$ *along* $\operatorname{Sing} \overline{E}_{\mathsf{t}} \simeq \mathbb{P}^1 \times \mathbb{P}^2$. *Let* $L_{\mathbb{P}(W^{1,\perp})}$ *be the pull-back to* $E_{\mathsf{t}} \cap \widehat{\Pi}$ *of a line in* $\mathbb{P}(W^{1,\perp}) \simeq \mathbb{P}^2$. *The exceptional divisor of the blow-up of* $\mathbb{P}^5$ *along* $\mathbb{P}^1 \times \mathbb{P}^2$ *is linearly equivalent to* $2H_{E_{\mathsf{t}} \cap \widehat{\Pi}} - L_{\mathbb{P}(W^{1,\perp})}$.

*Proof.* We show the assertion (1). The first assertion of (1) easily follows from (6.1). For the second assertion of (2), we have only to determine the restriction to $\mathbb{P}(W^{1,\perp})$ of the normal bundle $\mathcal{N}_{\Sigma_1'/\Sigma'}$. Since $\Sigma_1'$ is a sub $\mathbb{P}^2 \times \mathbb{P}^2$-bundle of the $\mathbb{P}^8$-bundle $\Pi'$ over $\mathbb{P}(U^4)$ by (6.1) and (6.2), we see that

$$(6.3) \qquad\qquad \mathcal{N}_{\Sigma_1'/\Pi'} \simeq T_{\mathbb{P}(U^3)} \otimes T_{\mathbb{P}(\Omega^1_{\mathbb{P}(U^4)}(1))/\mathbb{P}(U^4)}$$

relativising the normal bundle of the Segre embedded $\mathbb{P}^2 \times \mathbb{P}^2$ in $\mathbb{P}^8$. Let $p \colon \mathbb{P}(\Omega^1_{\mathbb{P}(U^4)}(1)) \to \mathbb{P}(U^4)$ be the natural morphism. We consider $\mathbb{P}(W^{1,\perp})$ as the $p$-fiber over $[W^1] \in$

$\mathbb{P}((U^4)^*)$. Restricting to $\mathbb{P}(W^{1,\perp})$ the relative Euler sequence

$$0 \to \mathcal{O}(-H_{\mathbb{P}(\Omega^1_{\mathbb{P}(U^4)}(1))}) \to p^*\Omega^1_{\mathbb{P}(U^4)}(1) \to T_{\mathbb{P}(\Omega^1_{\mathbb{P}(U^4)}(1))/\mathbb{P}(U^4)}(-H_{\mathbb{P}(\Omega^1_{\mathbb{P}(U^4)}(1))}) \to 0,$$

we obtain the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}(W^{1,\perp})} \to \Omega^1_{\mathbb{P}(W^{1,\perp})}(1) \oplus \mathcal{O}_{\mathbb{P}(W^{1,\perp})} \to T_{\mathbb{P}(\Omega^1_{\mathbb{P}(U^4)}(1))/\mathbb{P}(U^4)}|_{\mathbb{P}(W^{1,\perp})} \to 0$$

since $H_{\mathbb{P}(\Omega^1_{\mathbb{P}(U^4)}(1))}|_{\mathbb{P}(W^{1,\perp})} = 0$. Therefore we have $T_{\mathbb{P}(\Omega^1_{\mathbb{P}(U^4)}(1))/\mathbb{P}(U^4)}|_{\mathbb{P}(W^{1,\perp})} \simeq$ $\Omega^1_{\mathbb{P}(W^{1,\perp})}(1)$. We also note that $T_{\mathbb{P}(U^3)}|_{[U^1]} \simeq (U^3/U^1) \otimes (U^1)^*$. Hence, by (6.3), we obtain

$$\mathcal{N}_{\Sigma'_1/\Pi'}|_{\mathbb{P}(W^{1,\perp})} \simeq (U^3/U^1) \otimes (U^1)^* \otimes \Omega^1_{\mathbb{P}(W^{1,\perp})}(1) \simeq (U^3/U^1) \otimes \Omega^1_{\mathbb{P}(W^{1,\perp})}(1).$$

Let $L_{\mathbb{P}(U^4)}$ be the pull-back to $\Sigma'$ of a hyperplane of $\mathbb{P}(U^4)$. Since

$$(6.4) \qquad\qquad\qquad\qquad \Pi' \sim H_{\Sigma'} - L_{\mathbb{P}(U^4)},$$

we have $\mathcal{N}_{\Pi'/\Sigma'}|_{\mathbb{P}(W^{1,\perp})} \simeq \mathcal{O}_{\mathbb{P}(W^{1,\perp})}(-1)$. Therefore, by the normal bundle sequence $0 \to \mathcal{N}_{\Sigma'_1/\Pi'} \to \mathcal{N}_{\Sigma'_1/\Sigma'} \to \mathcal{N}_{\Pi'/\Sigma'}|_{\Sigma'_1} \to 0$, we obtain

$$\mathcal{N}_{\Sigma'_1/\Sigma'}|_{\mathbb{P}(W^{1,\perp})} \simeq (U^3/U^1) \otimes \Omega^1_{\mathbb{P}(W^{1,\perp})}(1) \oplus \mathcal{O}_{\mathbb{P}(W^{1,\perp})}(-1),$$

and hence the assertion (1) follows.

The assertions (2)–(4) can be proved in a similar way to Proposition 6.6 due to structural similarity between $E_t$ and $\Sigma'$, so we omit a proof. $\qquad\square$

Note that

$$(6.5) \quad -K_{\widehat{\Sigma}} = \tau^*(-K_{\Sigma'}) - 4E_{\widehat{\Sigma}} = 10\tau^*H_{\Sigma'} - 4E_{\widehat{\Sigma}} = 2\tau^*H_{\Sigma'} + 4(2\tau^*H_{\Sigma'} - E_{\widehat{\Sigma}}).$$

By Proposition 6.6, $H_{\Sigma'}$ is nef and big and, since $\overline{\Sigma}_1$ is the intersection of quadrics, $\mathrm{Bs}\,|2\tau^*H_{\Sigma'} - E_{\widehat{\Sigma}}| = \emptyset$. Therefore $-K_{\widehat{\Sigma}}$ is nef and big. Let $\nu\colon \widehat{\Sigma} \to \widehat{\Sigma}'$ be the anti-canonical model.

**Flop** $\widehat{\Sigma} \dashrightarrow \Sigma^+$.

**Proposition 6.8.** *The anti-canonical model $\nu\colon \widehat{\Sigma} \to \widehat{\Sigma}'$ is defined over $\overline{\Sigma}$ and is a $\widehat{\Pi}$-negative flopping contraction of Atiyah type. The morphism $\nu|_{E_{\widehat{\Sigma}}}$ is also a flopping contraction of Atiyah type.*

*Proof.* Let $l \subset \widehat{\Sigma}$ be an irreducible $\nu$-exceptional curve. By (6.5) and $-K_{\widehat{\Sigma}} \cdot l = 0$, we have $\tau^*H_{\Sigma'} \cdot l = (2\tau^*H_{\Sigma'} - E_{\widehat{\Sigma}}) \cdot l = 0$, and hence $\tau^*H_{\Sigma'} \cdot l = E_{\widehat{\Sigma}} \cdot l = 0$. By $\tau^*H_{\Sigma'} \cdot l = 0$, the morphism $\nu$ is defined over $\overline{\Sigma}$. Since $\Pi'$ is smooth, we have $\widehat{\Pi} = \tau^*\Pi' - E_{\widehat{\Sigma}}$. Therefore $\widehat{\Pi}$ is negative for any $\nu$-exceptional curve since $\Pi'$ is negative for any exceptional curve for $\Sigma' \to \overline{\Sigma}$ by (6.4). By Proposition 6.6 (2), $l$ is contained in the union of $E_{\widehat{\Sigma}}$ and the strict transform $\widehat{\Sigma}_2$ of $\overline{\Sigma}_2$. By Proposition 6.6 (3), the $\nu$-fiber over a point $\mathsf{s} \in \widehat{\Sigma}_2 \setminus E_{\widehat{\Sigma}}$ is $\mathbb{P}^1$. Note that $E_t$ as in Lemma 6.7 (1) has two nontrivial contractions; one is the morphism $E_t \to \mathbb{P}(W^{1,\perp})$, and nontrivial fibers of another morphism are $\mathbb{P}^1$ by Lemma 6.7 (3). Since $\nu|_{E_t}$ cannot be the morphism $E_t \to \mathbb{P}(W^{1,\perp})$, we see that the nontrivial $\nu$-fiber over a point $\mathsf{s} \in E_{\widehat{\Sigma}}$ is also $\mathbb{P}^1$. Therefore any nontrivial fiber of the morphism $\nu|_{\widehat{\Pi}}\colon \widehat{\Pi} \to \nu(\widehat{\Pi})$ is $\mathbb{P}^1$. In particular, this implies that the relative Picard number of $\nu|_{\widehat{\Sigma}}$ is one. Note that, since $\widehat{\Pi}$ is $\nu$-negative , $-K_{\widehat{\Pi}}$ is $\nu$-ample. Therefore, by [An, Thm.2.3], $\nu(\widehat{\Pi})$ is smooth and $\nu|_{\widehat{\Pi}}$ is the blow-up of $\nu(\widehat{\Pi})$ along a smooth subvariety of $\nu(\widehat{\Pi})$ which is the strict

transform of $\overline{\Sigma}_2$. This implies that $\mathcal{N}_{l/\widehat{\Pi}} \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus 9} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, and $\widehat{\Pi} \cdot l = K_{\widehat{\Pi}} \cdot l = -1$. Therefore, by the normal bundle sequence $0 \to \mathcal{N}_{l/\widehat{\Pi}} \to \mathcal{N}_{l/\widehat{\Sigma}} \to \mathcal{N}_{\widehat{\Pi}/\widehat{\Sigma}}|_l \to 0$, we have $\mathcal{N}_{l/\widehat{\Sigma}} \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus 9} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$, and hence $\nu$ is a flopping contraction of Atiyah type.

Since $E_{\widehat{\Sigma}} \cdot l = 0$, $\nu|_{E_{\widehat{\Sigma}}}$ is also a flopping contraction of Atiyah type. $\qquad \square$

Let $\widehat{\Sigma} \dashrightarrow \Sigma^+$ be the flop for the flopping contraction $\nu$. It is well-known that the flop can be constructed by the blow-up along the $\nu$-exceptional locus and the blow-down of the exceptional divisor along the other direction. We denote by $\Pi^+$ and $E_{\Sigma^+}$ the strict transforms on $\Sigma^+$ of $\widehat{\Pi}$ and $E_{\widehat{\Sigma}}$ respectively. By the construction of $\widehat{\Sigma} \dashrightarrow \Sigma^+$, we see that the induced map $E_{\widehat{\Sigma}} \dashrightarrow E_{\Sigma^+}$ is also the flop and the induced map $\widehat{\Pi} \dashrightarrow \Pi^+$ is identified with $\nu|_{\widehat{\Pi}} \colon \widehat{\Pi} \to \nu(\widehat{\Pi})$, which is the blow-up of $\nu(\widehat{\Pi})$ along the strict transform of $\overline{\Sigma}_2$ on $\nu(\widehat{\Pi})$ by the proof of Proposition 6.8.

In the following steps, we separate $E_{\Sigma^+}$ and $\Pi^+$ by a flip, and finally contract their strict transforms.

**Flip $\Sigma^+ \dashrightarrow \widetilde{\Sigma}$.**
We set
$$G := E_{\Sigma^+} \cap \Pi^+.$$

**Lemma 6.9.** *The exceptional locus of $\Pi^+ \to \overline{\Pi}$ is $G$ and $G$ is a $\mathbb{P}^5$-bundle over $\mathbb{P}(U^3) \times \mathbb{P}((U^4)^*)$.*

*Proof.* As we note above, we may identify $\Pi^+ \to \overline{\Pi}$ with $\nu(\widehat{\Pi}) \to \overline{\Pi}$. Therefore the assertion follows by Lemma 6.7 (4) and the construction of the flop. $\qquad \square$

**Proposition 6.10.**

(1) Let $\Gamma \simeq \mathbb{P}^5$ be a fiber of $G \to \mathbb{P}(U^3) \times \mathbb{P}((U^4)^*)$. It holds that
$$\mathcal{N}_{\Gamma/\Sigma^+} = \mathcal{O}_{\mathbb{P}^5}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^5}^{\oplus 5}.$$

(2) There exists a small contraction $\Sigma^+ \to \overline{\Sigma}^+$ contracting $E_{\Sigma^+} \cap \Pi^+$ onto $\mathbb{P}(U^3) \times \mathbb{P}((U^4)^*)$.

*Proof.* (1). Let $l$ be a general line in a fiber of the $\mathbb{P}^3$-bundle $E_{\mathrm{t}} \cap \widehat{\Pi} \to \mathbb{P}(W^{1,\perp})$ as in Lemma 6.7 (1) and $l^+$ the strict transform of $l$ on $\Sigma^+$. Since $l$ is contained in a fiber of the blow-up $\tau \colon \widehat{\Sigma} \to \Sigma'$ and $E_{\widehat{\Sigma}}$ is the $\tau$-exceptional divisor, we have $-K_{\widehat{\Sigma}} \cdot l = 4$ and $E_{\widehat{\Sigma}} \cdot l = -1$. Since both $-K_{\widehat{\Sigma}}$ and $E_{\widehat{\Sigma}}$ are numerically trivial for flopping curves by the proof of Proposition 6.8, we have $-K_{\Sigma^+} \cdot l^+ = 4$ and $E_{\Sigma^+} \cdot l^+ = -1$. From the latter equality, we have $(G \cdot l^+)_{\Pi^+} = E_{\Pi^+} \cdot l^+ = -1$. Therefore, by the normal bundle sequence $0 \to \mathcal{N}_{\Gamma/G} \to \mathcal{N}_{\Gamma/\Pi^+} \to \mathcal{N}_{G/\Pi^+}|_\Gamma \to 0$ and $\mathcal{N}_{\Gamma/G} \simeq \mathcal{O}_\Gamma^{\oplus 5}$, we have $\mathcal{N}_{\Gamma/\Pi^+} \simeq \mathcal{O}_{\mathbb{P}^5}(-1) \oplus \mathcal{O}_{\mathbb{P}^5}^{\oplus 5}$. Since $-K_{\Sigma^+} \cdot l^+ = 4$ and $\Gamma \simeq \mathbb{P}^5$, we have $\deg \mathcal{N}_{\Gamma/\Sigma^+} = -2$. Therefore, by the normal bundle sequence $0 \to \mathcal{N}_{\Gamma/\Pi^+} \to \mathcal{N}_{\Gamma/\Sigma^+} \to \mathcal{N}_{\Pi^+/\Sigma^+}|_\Gamma \to 0$, we have $\mathcal{N}_{\Gamma/\Sigma^+} \simeq \mathcal{O}_{\mathbb{P}^5}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^5}^{\oplus 5}$ as desired.

(2). Let $H_{\Sigma^+}$ be the strict transform on $\Sigma^+$ of $\tau^* H_{\Sigma'}$. We can show that $(2H_{\Sigma^+} - E_{\Sigma^+}) + \Pi^+$ is nef over $\overline{\Sigma}$ and numerically trivial only for fibers of $G \to \mathbb{P}(U^3) \times \mathbb{P}((U^4)^*)$. A proof is quite similar to the one of Proposition 5.8 (2), so we omit it. $\qquad \square$

By Proposition 6.10 (1), the contraction $\Sigma^+ \to \overline{\Sigma}^+$ is of flipping type, and the flip can be constructed by the blow-up along $\Lambda$ and the blow-down of the exceptional divisor along the other direction . Let $\Sigma^+ \dashrightarrow \widetilde{\Sigma}$ be the flip. By Proposition 6.10 (1) again, the flipped locus is a $\mathbb{P}^1$-bundle over $\mathbb{P}(U^3) \times \mathbb{P}((U^4)^*)$. We denote by $E_{\widetilde{\Sigma}}$, $\widetilde{\Pi}$ and $H_{\widetilde{\Sigma}}$ the strict transforms on $\widetilde{\Sigma}$ of $E_{\Sigma^+}$, $\Pi^+$ and $H_{\Sigma^+}$ respectively.

The following lemma will describe a part of singularities of the key variety $\Sigma$ which we are going to construct:

**Lemma 6.11.** *Let* $\mathsf{t} = [U^1] \times [W^1]$ *be a point of* $\overline{\Sigma}_1 = \mathbb{P}(U^3) \times \mathbb{P}((U^4)^*)$ *as in Lemma 6.7. The fiber* $E_{\mathsf{t}}^+$ *of* $E_{\Sigma^+} \to \mathbb{P}(U^3) \times \mathbb{P}((U^4)^*)$ *over* $\mathsf{t}$ *is the blow-up of the Grassmannian* $\mathrm{G}(2, (U^3/U^1) \oplus ((W^1)^\perp)^*) \simeq \mathrm{G}(2,5)$ *at the point* $[\wedge^2(U^3/U^1)]$. *The fiber of* $E_{\widetilde{\Sigma}} \to \mathbb{P}(U^3) \times \mathbb{P}((U^4)^*)$ *over* $\mathsf{t}$ *is* $\mathrm{G}(2, (U^3/U^1) \oplus ((W^1)^\perp)^*)$.

*Proof.* For simplicity of notation, we set $\overline{U}^2 := U^3/U^1$, $\mathrm{Gr} := \mathrm{G}(2, \overline{U}^2 \oplus (W^{1,\perp})^*)$ and denote by $\widetilde{\mathrm{Gr}}$ the blow-up of $\mathrm{Gr}$ at the point $[\wedge^2\overline{U}^2]$. By Lemma 6.7, the fiber $E_{\mathsf{t}}$ of $E_{\widehat{\Sigma}} \to \mathbb{P}(U^3) \times \mathbb{P}((U^4)^*)$ over $\mathsf{t}$ has two contractions, one of which is the $\mathbb{P}^4$-bundle $E_{\mathsf{t}} \to \mathbb{P}(W^{1,\perp})$ and another of which is the flopping contraction of Atiyah type $E_{\mathsf{t}} \to \overline{E}_{\mathsf{t}}$. Note that $\overline{E}_{\mathsf{t}}$ has another unique small resolution different from $E_{\mathsf{t}} \to \overline{E}_{\mathsf{t}}$, which we denote by $E_{\mathsf{t}}^+ \to \overline{E}_{\mathsf{t}}$. Therefore, to show $E_{\mathsf{t}}^+ \simeq \widetilde{\mathrm{Gr}}$, it suffices to show that $\widetilde{\mathrm{Gr}}$ has a small contraction onto $\overline{E}_{\mathsf{t}}$ (note that $\widetilde{\mathrm{Gr}}$ is different from $E_{\mathsf{t}}$ since $E_{\mathsf{t}}$ has no contraction onto $\mathrm{G}(2,5)$). We note the following decomposition:

$$\wedge^2\left(\overline{U}^2 \oplus (W^{1,\perp})^*\right) = \wedge^2\overline{U}^2 \oplus \overline{U}^2 \wedge (W^{1,\perp})^* \oplus \wedge^2(W^{1,\perp})^*$$

$$\simeq \wedge^2\overline{U}^2 \oplus \overline{U}^2 \otimes (W^{1,\perp})^* \oplus W^{1,\perp}.$$

Therefore, the linear projection from the point $[\wedge^2\overline{U}^2]$ maps $\mathrm{Gr}$ into the projective space $\mathbb{P}\left(\overline{U}^2 \wedge (W^{1,\perp})^* \oplus W^{1,\perp}\right)$. Let $e_1, e_2$ and $e_3, e_4, e_5$ be basis of $\overline{U}^2$ and $(W^{1,\perp})^*$ respectively, and $p_{ij}$ the Plücker coordinates associated to the basis $e_1, \dots, e_5$ of $\overline{U}^2 \oplus (W^{1,\perp})^*$. The equation of $\mathrm{Gr}$ is given by $\wedge^2(\sum_{1 \le i < j \le 5} p_{ij} e_i \wedge e_j) = o$. We can check explicitly that the image of the projection of $\widetilde{\mathrm{Gr}}$ is defined by

$$(6.6) \qquad \left(\sum_{i=1,2,j=3,4,5} p_{ij} e_i \wedge e_j\right) \wedge \left(\sum_{3 \le i < j \le 5} p_{ij} e_i \wedge e_j\right) = o$$

in $\mathbb{P}\left(\wedge^2\overline{U}^2 \oplus \overline{U}^2 \wedge (W^{1,\perp})^* \oplus \wedge^2(W^{1,\perp})^*\right)$, and the projection is birational onto the image. We identify $\wedge^2(W^{1,\perp})^*$ with $W^{1,\perp}$ by regarding $e_3 \wedge e_4$, $e_3 \wedge e_5$, and $e_4 \wedge e_5$ as $e_5^*, -e_4^*$, and $e_3^* \in W^{1,\perp}$ respectively, and also $\overline{U}^2 \wedge (W^{1,\perp})^*$ with $\overline{U}^2 \otimes (W^{1,\perp})^*$ by regarding $e_i \wedge e_j$ with $e_i \otimes e_j$. Then the equation (6.6) defines $\overline{E}_{\mathsf{t}}$ in $\mathbb{P}\left(\overline{U}^2 \wedge (W^{1,\perp})^* \oplus W^{1,\perp}\right)$. By a standard property of linear projection, a natural morphism $\rho^+ \colon \widetilde{\mathrm{Gr}} \to \overline{E}_{\mathsf{t}}$ is induced. Let $E_{\mathrm{Gr}}$ be the exceptional divisor of the blow-up $\widetilde{\mathrm{Gr}} \to \mathrm{Gr}$, and $L_{\mathrm{Gr}}$ the total transform on $\widetilde{\mathrm{Gr}}$ of a hyperplane section of $\mathrm{Gr}$. We have $-K_{\widetilde{\mathrm{Gr}}} = 5(L_{\mathrm{Gr}} - E_{\mathrm{Gr}})$ and $L_{\mathrm{Gr}} - E_{\mathrm{Gr}}$ is the total transform of a hyperplane section of $\overline{E}_{\mathsf{t}}$. Therefore the morphism $\rho^+$ is crepant, and hence must be small since $\overline{E}_{\mathsf{t}}$ has only terminal singularities. Now we have shown that $\widetilde{\mathrm{Gr}}$ has a small contraction onto $\overline{E}_{\mathsf{t}}$ as desired.

The description of the fiber of $E_{\widetilde{\Sigma}} \to \mathbb{P}(U^3) \times \mathbb{P}((U^4)^*)$ over t follows from the above by the construction of the flip $\Sigma^+ \dashrightarrow \widetilde{\Sigma}$. $\qquad\square$

**Contracting $E_{\widetilde{\Sigma}}$ and $\widetilde{\Pi}$.**

By the constructions of the flop $\widehat{\Sigma} \dashrightarrow \Sigma^+$ and the flip $\Sigma^+ \dashrightarrow \widetilde{\Sigma}$, and the description of $E_{\Sigma^+} \cap \Pi^+$ as in Lemma 6.9, we see that $E_{\widetilde{\Sigma}} \cap \widetilde{\Pi} = \emptyset$.

By the construction of the flip, we see that $\Sigma^+ \dashrightarrow \widetilde{\Sigma}$ induces the contraction $\Pi^+ \to \mathbb{P}(U^3 \otimes (U^4)^*)$. Thus $\widetilde{\Pi} \simeq \mathbb{P}^{11}$.

**Lemma 6.12.** *The normal bundle $\mathcal{N}_{\widetilde{\Pi}/\widetilde{\Sigma}}$ is $\mathcal{O}_{\mathbb{P}^{11}}(-2)$ .*

*Proof.* A proof is similar to those of Lemmas 3.4 and 5.9, so we omit it. $\qquad\square$

**Lemma 6.13.** $2H_{\widetilde{\Sigma}} + \widetilde{\Pi}$ *is semiample.*

*Proof.* A proof is quite similar to the one of Lemma 5.10, so we omit it. $\qquad\square$

**Theorem 6.14.** *Let $\mu\colon \widetilde{\Sigma} \to \Sigma$ be the contraction defined by a sufficient multiple of $2H_{\widetilde{\Sigma}} + \widetilde{\Pi}$. The following assertions hold:*

*(1) The $\mu$-exceptional locus is the union of $E_{\widetilde{\Sigma}}$ and $\widetilde{\Pi}$.*

*(2) $\mu(\widetilde{\Pi})$ is a $1/2$-singularity.*

*(3) The discrepancy of $E_{\widetilde{\Sigma}}$ is $4$ and $\mu(E_{\widetilde{\Sigma}}) \simeq \mathbb{P}(U^3) \times \mathbb{P}((U^4)^*)$. Any fiber of $E_{\widetilde{\Sigma}} \to \mu(E_{\widetilde{\Sigma}})$ is $\mathrm{G}(2,5)$. In particular $\Sigma$ has Gorenstein terminal singularities along $\mathbb{P}(U^3) \times \mathbb{P}((U^4)^*)$.*

*(4) The variety $\Sigma$ is a $12$-dimensional rational $\mathbb{Q}$-Fano variety with $\rho(\Sigma) = 1$.*

*(5) The image $M_\Sigma$ of $H_\Sigma$ is a primitive integral ample Weil divisor $M_\Sigma$ and it holds that $-K_\Sigma = 10 M_\Sigma$.*

*Proof.* The assertion (3) follows from Lemma 6.11. The rest assertion can be proved similarly to Theorem 5.11, so we omit a proof. $\qquad\square$

6.3. **Embedding theorem .** Now we arrive at Theorem 1.1 for a prime $\mathbb{Q}$-Fano 3-fold $X$ of genus 5.

**Theorem 6.15.** *A $\mathbb{Q}$-Fano $3$-fold $X$ of genus $5$ is a linear section of $\Sigma$.*

*Proof.* We only remark that $W$ is disjoint from $\overline{\Sigma}_1$ since $\overline{\Sigma}$ has non-hypersurface singularities along $\overline{\Sigma}_1$. The rest of the proof is similar to the one of Theorem 5.17, so we omit it. $\qquad\square$

6.4. **Extension of the Sarkisov link.** We have obtained the following diagram:

$$(6.7) \qquad \widetilde{\Sigma} \xrightarrow[\text{anti-flip}]{} \Sigma^+ \xrightarrow[\text{flop}]{} \widehat{\Sigma}$$

$$\mu \qquad\qquad\qquad\qquad\qquad$$

$$\Sigma \qquad\qquad \overline{\Sigma} \qquad\qquad \mathbb{P}^3.$$

By the proof of Theorem 6.15, we obtain the following:

**Corollary 6.16.** *The diagram (6.7) is an extension of (1.1) in the case of genus 5, where $Y, W$ and $Y'$ are linear sections of $\widetilde{\Sigma}$, $\overline{\Sigma}$ and $\widehat{\Sigma}$ with respect to $|H_{\widetilde{\Sigma}}|$, $|\mathcal{O}_{\overline{\Sigma}}(1)|$ and $|H_{\widehat{\Sigma}}|$ respectively. The restriction of the anti-flip to $Y$ is the identity.*

*Proof.* The restriction of the anti-flip to $Y$ is the identity since the image in $\overline{\Sigma}$ of the flipped locus in $\widetilde{\Sigma}$ is contained in the image of $E_{\widetilde{\Sigma}}$ on $\overline{\Sigma}$ by Proposition 6.10 (2), and the latter is disjoint from $W$ as we remarked in the proof of Theorem 6.15. The rest follows easily.                                                                                    □

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