Asymptotic equivariant real analytic torsions for compact locally symmetric spaces

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Abstract

In this paper, we study the asymptotics of the equivariant analytic torsions for a certain sequence of flat vector bundles over a compact locally symmetric space. Our approach is combining the twisted trace formula with an explicit geometric formula for the twisted orbital integrals. We show that the leading term of asymptotic equivariant analytic torsion is given in terms of W-invariants with oscillating coefficients.

Keywords: Equivariant analytic torsion; locally symmetric space; twisted orbital integral.

Contents

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1. Introduction

The purpose of this paper is to study the leading terms in the asymptotic expansions of the equivariant Ray-Singer real analytic torsions for compact locally symmetric spaces.

Let (M, g^{TM}) be a closed oriented Riemannian manifold, let $(F, \nabla^{F,f}, h^F)$ be a Hermitian flat vector bundle on M. Let $\mathbf{D}^{Z,F,2}$ be the de Rham-Hodge Laplacian associated with the de Rham complex $(\Omega^\bullet(M,F), d^{M,F})$. The real analytic torsion $\mathcal{T}(g^{TM}, \nabla^{F,f}, h^F)$ is a (graded) spectral invariant of $\mathbf{D}^{M,F,2}$ introduced by Ray and Singer [RS71, RS73]. When $\dim_{\mathbb{R}} M$ is odd, it does not depend on the choices of g^{TM}, h^F. The theorems of Cheeger [Che79] and Müller [Mül78] say that, for unitarily flat vector bundle F, this invariant coincides with the Reidemeister torsion, a topological invariant defined via CW complexes of M. Using the Witten deformation, Bismut and Zhang [BZ91, BZ92] gave an extension of the Cheeger-Müller theorem for arbitrary flat vector bundles.

Let Σ be a compact Lie group which acts on $(F, \nabla^{F,f}) \to M$ equivariantly. Then Σ acts on $(\Omega^\bullet(M,F), d^{M,F})$. In [LR91], Lott and Rothenberg introduced an equivariant version of Ray-Singer analytic torsion. If $\sigma \in \Sigma$, set

$$\vartheta_\sigma(g^{TM}, \nabla^{F,f}, h^F)(s) = -\mathrm{Tr_s}\big[N^{\Lambda^\cdot(T^*M)}\sigma^M(\mathbf{D}^{M,F,2})^{-s}\big]. \tag{1.0.1}$$

Then $\vartheta_\sigma(g^{TM}, \nabla^{F,f}, h^F)(s)$ extends to a meromorphic function of $s \in \mathbb{C}$, which is holomorphic at 0. The σ-equivariant Ray-Singer analytic torsion is defined as

$$\mathcal{T}_\sigma(g^{TM}, \nabla^{F,f}, h^F) = \frac{1}{2}\frac{\partial\vartheta_\sigma(g^{TM}, \nabla^{F,f}, h^F)}{\partial s}(0). \tag{1.0.2}$$

If $\sigma = \mathrm{Id}_G$, we just get the ordinary analytic torsion $\mathcal{T}(g^{TM}, \nabla^{F,f}, h^F)$.

When Σ is a finite group, in [Rot78], for a Σ-CW complex of M, Rothenberg constructed an equivariant version of the Reidemeister torsion. In [LR91], when F is unitarily flat, an extension of the Cheeger-Müller theorem was established by comparing the equivariant Reidemeister torsion and Ray-Singer analytic torsion. Then Bismut and Zhang [BZ94] generalized these results for arbitrary flat vector bundles with an equivariant action of a compact Lie group. Also Bunke [Bun99] showed that when F is unitarily flat, the equivariant analytic torsion can be determined by counting the cells of a Σ-CW decomposition of M, up to a locally constant function on Σ.

Now, let G be a connected linear real reductive Lie group with compact center, and let $X = G/K$ be the associated symmetric space. Let Γ be a cocompact torsion-free discrete subgroup of G. In this paper, we work on the compact locally symmetric space $M = \Gamma\backslash X$ equipped with a compact Lie group action generated by suitable $\sigma \in \mathrm{Aut}(G)$. We will consider a certain sequence of flat vector bundles F_d, $d \in \mathbb{N}$ on M, and we evaluate the leading term in the asymptotic expansion of $\mathcal{T}_\sigma(g^{TM}, \nabla^{F_d,f}, h^{F_d})$ as $d \to +\infty$.

Bergeron and Venkatesh [BV13] have considered the asymptotic behavior of the Ray-Singer analytic torsion under a tower of finite coverings of M, and then by Cheeger-Müller theorem, they studied the asymptotic growth of the torsions in homology. In [BL17], under finite coverings and acyclic base change, Bergeron and Lipnowski studied the asymptotic equivariant analytic torsions and then considered the growth of torsion cohomology under twisting action.

Müller [Mül12] initiated the study of the analytic torsion for symmetric powers of a given flat vector bundle on hyperbolic manifolds. Also Bismut-Ma-Zhang [BMZ11, BMZ17] and Müller-Pfaff [MP13b, MP13a] studied the case where one considers a sequence of flat vector bundles on M associated with multiples of a given highest weight of an irreducible G-representation. Moreover, Marshall-Müller [MM13] and Müller-Pfaff [MP14] applied the related results to study the asymptotic growth of torsion cohomology for a family of local systems on certain compact arithmetic manifolds.

Using methods of harmonic analysis, Ksenia Fedosova [Fed, Fed15] studied the asymptotic analytic torsions for compact hyperbolic orbifolds for a sequence of homogeneous flat vector bundles. Then in [Liu20], the author extended her results to arbitrary compact locally symmetric orbifolds of noncompact type via applying Bismut's explicit formula [Bis11] for orbital integrals.

Here, we introduce an equivariant analog to the settings in [BMZ17, Section 8] and [MP13a], and we study the asymptotics of the equivariant Ray-Singer analytic torsion for M. Let us give more details on the results of this paper.

Let $\theta \in \mathrm{Aut}(G)$ be the Cartan involution, whose fixed point set is the maximal compact subgroup K of G. Let $\mathfrak{g}$, $\mathfrak{k}$ denote the Lie algebras of G, K respectively. Then θ acts on $\mathfrak{g}$ and fixes $\mathfrak{k}$. Let $\mathfrak{p} \subset \mathfrak{g}$ be the eigenspace of θ associated with the eigenvalue -1. The Cartan decomposition of $\mathfrak{g}$ is

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}. \tag{1.0.3}$$

Let B be a G- and θ-invariant nondegenerate symmetric bilinear form on $\mathfrak{g}$, which is positive on $\mathfrak{p}$ and negative on $\mathfrak{k}$. When $\mathfrak{g}$ is not semisimple, we do not have a canonical choice of B such as the Killing form due to the nontrivial center of $\mathfrak{g}$, but here we always fix one choice once and for all. Let U be compact form of G with Lie algebra $\mathfrak{u} = \sqrt{-1}\mathfrak{p} \oplus \mathfrak{k}$. Then U is a compact linear Lie group. We extend the bilinear form B to $\mathfrak{u}$.

Let g^{TX} be the Riemannian metric on X induced from $B|_{\mathfrak{p}}$. Then the group G acts on X isometrically. Taking quotient by Γ, we get a compact locally symmetric manifold $(M = \Gamma\backslash X, g^{TM})$. Set $m = \dim \mathfrak{p} = \dim X = \dim M$.

Let $\sigma \in \mathrm{Aut}(G)$ be such that it commutes with θ and preserves B and Γ. Then it induces an isometry on X which descends to an isometry of M. Let $\Sigma^\sigma \subset \mathrm{Aut}(G)$ be the closure of the subgroup generated by σ, which is a compact Abelian subgroup. We assume that the action of σ on $\mathfrak{u}$ lifts to U. Set

$$G^\sigma = G \rtimes \Sigma^\sigma, \ U^\sigma = U \rtimes \Sigma^\sigma, \tag{1.0.4}$$

where $\rtimes$ denotes the semi-direct product.

If $\sigma \in \Sigma$, we define the σ-twisted conjugation C^σ so that if $h, \gamma \in G$,

$$C_\sigma(h)\gamma = h\gamma\sigma(h^{-1}). \tag{1.0.5}$$

Let $Z_\sigma(\gamma) \subset G$ be the σ-twisted centralizer of γ. Since σ preserves Γ, let $[\Gamma]_\sigma$ denote the set of σ-twisted conjugacy classes in Γ. If $\gamma \in \Gamma$ is such that $\gamma\sigma$ acting on X has fixed points, then we call $[\gamma]_\sigma \in [\Gamma]_\sigma$ an elliptic class. In this case, let $X(\gamma\sigma) \subset X$ denote the fixed point set of $\gamma\sigma$, which is a symmetric space associated with $Z_\sigma(\gamma)$.

Let (E, ρ^E, h^E) be an irreducible unitary representation of U^σ, then it extends uniquely to a representation of G^σ via unitary trick. This way, $(F = G \times_K E, h^F)$ becomes a Hermitian vector bundle on X equipped with an G^σ-invariant flat connection $\nabla^{F,f}$. It descends to a flat bundle on M equipped with an equivariant Σ^σ-action, so that $\mathcal{T}_\sigma(g^{TM}, \nabla^{F,f}, h^F)$ is well-defined.

In Theorem 3.3.2, we get several criteria to make $\mathcal{T}_\sigma(g^{TM}, \nabla^{F,f}, h^F)$ vanish. In particular, we show that if E is not irreducible as U-representation, then

$$\mathcal{T}_\sigma(g^{TM}, \nabla^{F,f}, h^F) = 0. \tag{1.0.6}$$

This theorem extends some classical results on the usual analytic torsions such as [MS91, Corollary 2.2], [Lot94, Proposition 9], [BL95, Theorem 3.26], [Bis11, Section 7.9], [BMZ17, Theorem 8.6], etc.

As a consequence of (1.0.6), we only need to focus on the irreducible U^σ-representations which are also irreducible when restricting to U. They correspond exactly the *essential* representations considered in [BL17]. In the context of [BMZ17, Section 8] and [MP13a], this condition means that we are concerned with a σ-fixed dominant weight λ of U with respect to a suitable root system.

Let N_λ be the flag manifold associated with λ, on which U^σ acts holomorphically. This U^σ-action also lifts to the canonical line bundle $L_\lambda \to N_\lambda$. The rigorous construction is given in Subsection 4.2. Then for each $d \in \mathbb{N}$, U^σ acts on $E_d = H^{(0,0)}(N_\lambda, L_\lambda^d)$. This way, we get a *canonical* sequence of irreducible unitary representations $(E_d, \rho^{E_d}, h^{E_d})$ of U^σ such that each (E_d, ρ^{E_d}) is the irreducible U-representation with highest weight $d\lambda$. It defines a sequence of flat vector bundles $\{F_d\}_{d\in\mathbb{N}}$ over M on which Σ^σ acts equivariantly.

For a nice spectral gap of the Hodge Laplacians, we also need to introduce a nondegeneracy condition on λ (Definition 4.1.2, Subsection 4.3). Equivalently, λ is called nondegenerate if (E_1, ρ^{E_1}) is not isomorphic to $(E_1, \rho^{E_1} \circ \theta)$ as U-representations. On a given closed Riemannian manifold, the W-invariant was introduced in [BMZ17]. Here, for a nondegenerate λ, it is a universally constructed G-invariant section W^λ of $\Lambda^\cdot(T^*X)$ (see Subsection 4.1). It is expressed in terms of the Duistermaat-Heckman integrals [DH82, DH83] associated with $L_\lambda \to N_\lambda$. Let $[W^\lambda]^{\max}$ denote the coefficient of the (oriented) volume element on X of norm 1 in W^λ. Since W^λ is G-invariant, $[W^\lambda]^{\max}$ here becomes a real constant. Put $n_\lambda = \dim_{\mathbb{C}} N_\lambda$. A result of [BMZ17] is that when the fundamental rank $\delta(G) = 1$, as $d \to +\infty$, we have

$$d^{-n_\lambda - 1}\mathcal{T}(g^{TM}, \nabla^{F_d,f}, h^{F_d}) = \mathrm{Vol}(M)[W^\lambda]^{\max} + \mathcal{O}(d^{-1}). \tag{1.0.7}$$

Then $\mathrm{Vol}(M)[W^\lambda]^{\max}$ is a topological invariant for M. Note that if $\delta(G) \neq 1$, we have $\mathcal{T}(g^{TM}, \nabla^{F_d,f}, h^{F_d}) = 0$, and the connected linear simple Lie groups with $\delta(G) = 1$ are completely classified (cf. [Bis11, Remark 7.9.2]).

As shown by the computations in [MP13b, MP13a], [BMZ17, Section 8], [Liu20, Subsections 7.3 & 7.4], given a concrete symmetric pair (G, K) with $\delta(G) = 1$ and a nondegenerate λ as above, the associated quantity $[W^\lambda]^{\max}$ can be evaluated explicitly in terms of λ and a root system of $\mathfrak{g}$. Then we can use these W-invariants to describe other geometric objects for symmetric spaces.

We now present the main result of this paper, where the sequence $\{F_d\}_{d\in\mathbb{N}}$ is constructed as above. Our notation will be made explicitly in Subsection 4.6. In particular, $E^{1,\max}_\sigma$ is a finite subset of elliptic classes in $[\Gamma]_\sigma$, and $\mathcal{J}(\gamma)^{\max}$ is a finite set determined by γ. Each $W^j_{\gamma\sigma}$ is a W-invariant for a symmetric space $X(\gamma\sigma)$ associated with a linear reductive Lie group of fundamental rank 1. The complex numbers $r_{\gamma,j}$ are all of modulo 1.

Theorem 1.0.1. *If $E^{1,\max}_\sigma \neq \emptyset$, there exists $m(\sigma) \in \mathbb{N}$ such that as $d \to +\infty$,*

$$\begin{aligned} &d^{-m(\sigma)-1}\mathcal{T}_\sigma(g^{TM}, \nabla^{F_d,f}, h^{F_d}) \\ &= \sum_{[\gamma]_\sigma \in E^{1,\max}_\sigma} \mathrm{Vol}(\Gamma \cap Z_\sigma(\gamma)\backslash X(\gamma\sigma)) \big(\sum_{j\in\mathcal{J}(\gamma)^{\max}} r^d_{\gamma,j}\varphi^j_\gamma [W^j_{\gamma\sigma}]^{\max} \big) + \mathcal{O}(d^{-1}), \end{aligned} \tag{1.0.8}$$

where the constants $r_{\gamma,j}$, φ^j_γ can be explicitly computed in terms of λ, σ and root data of $\mathfrak{u}$ (Proposition 4.2.6).

If $E^{1,\max}_\sigma = \emptyset$, then there exists constant $c > 0$, as $d \to +\infty$,

$$\mathcal{T}_\sigma(g^{TM}, \nabla^{F_d,f}, h^{F_d}) = \mathcal{O}(e^{-cd}). \tag{1.0.9}$$

Let ${}^\sigma M$ denote the fixed point set of σ in M. In Subsection 2.6, we show that ${}^\sigma M$ can be identified with a disjoint union of $\Gamma \cap Z_\sigma(\gamma)\backslash X(\gamma\sigma)$ associated with each elliptic class $[\gamma]_\sigma$ in $[\Gamma]_\sigma$. Therefore, (1.0.8) relates the asymptotic σ-equivariant analytic torsion of M to the W-invariants on ${}^\sigma M$.

A difference from the result (1.0.7) of [BMZ17, Section 8] is that the coefficients of $W^j_{\gamma\sigma}$ have oscillating factors $r^d_{\gamma,j}$ as d varies. Moreover, if σ is of finite order N_0, then each $r_{\gamma,j}$ is a N_0-th root of unity (cf. Corollary 4.6.3).

Note that in the asymptotic analytic torsions for compact locally symmetric orbifolds in [Fed15] and [Liu20], the oscillating coefficients also appear in the evaluation of elliptic orbital integrals. Here in (1.0.8), they come from the σ-twisted orbital integrals associated with elliptic classes $[\gamma]_\sigma \in [\Gamma]_\sigma$.

Now we explain our approach to Theorem 1.0.1. By (1.0.1) and Mellin transform, we need to study the asymptotic behavior of

$$\mathrm{Tr_s}[(N^{\Lambda^\bullet(T^*M)} - \frac{m}{2})\sigma^M \exp(-t\mathbf{D}^{M,F_d,2}/2)],\ t > 0, \tag{1.0.10}$$

where $\mathrm{Tr_s}[\cdot]$ denotes the supertrace with respect to the $\mathbb{Z}_2$-grading on $\Lambda^\bullet(T^*M)$.

At first, we apply the twisted Selberg's trace formula to $M = \Gamma\backslash X$. For $[\gamma]_\sigma \in [\Gamma]_\sigma$, let $\mathrm{Tr_s}^{[\gamma\sigma]}[(N^{\Lambda^\bullet(T^*X)} - \frac{m}{2})\exp(-t\mathbf{D}^{X,F_d,2}/2)]$ denote the associated twisted orbital integral (Subsection 2.5). Then

$$\begin{aligned}&\mathrm{Tr_s}[(N^{\Lambda^\bullet(T^*M)} - \frac{m}{2})\sigma^M \exp(-t\mathbf{D}^{M,F_d,2}/2)]\\ &= \sum_{[\gamma]_\sigma\in[\Gamma]_\sigma} \mathrm{Vol}(\Gamma\cap Z_\sigma(\gamma)\backslash X(\gamma\sigma))\mathrm{Tr_s}^{[\gamma\sigma]}[(N^{\Lambda^\bullet(T^*X)} - \frac{m}{2})\exp(-t\mathbf{D}^{X,F_d,2}/2)].\end{aligned} \tag{1.0.11}$$

In [Liu18, Section 5], using the Bismut's theory of hypoelliptic Laplacian for symmetric space, an explicit geometric formula was obtained for the twisted orbital integrals appeared in the right-hand side of (1.0.11).

In the sum of (1.0.11), if $[\gamma]_\sigma$ is elliptic with $\delta(Z_\sigma(\gamma)^0) \neq 1$, then its contribution is zero. For the case $\delta(Z_\sigma(\gamma)^0) = 1$, we can compute the leading terms in the asymptotics of $\mathrm{Tr_s}^{[\gamma\sigma]}[(N^{\Lambda^\bullet(T^*X)} - \frac{m}{2})\exp(-t\mathbf{D}^{X,F_d,2}/2)]$, so that, after Mellin transform, we obtain exactly an oscillating combination of some W-invariants for the compact locally symmetric space $\Gamma\cap Z_\sigma(\gamma)\backslash X(\gamma\sigma)$. The oscillating factor $r^d_{\gamma,j}$ comes from the action of $\gamma\sigma$ on $L^d_\lambda \to N_\lambda$ on the fixed points. To get exactly the asymptotic expansion in (1.0.8), in Theorem 4.4.1, we also obtain several important uniform estimate for the twisted orbital integrals when $t > 0$ is small and large.

The second step is to handle the contribution of the nonelliptic $[\gamma]_\sigma \in [\Gamma]_\sigma$, we use a spectral gap of $\mathbf{D}^{M,F_d,2}$ due to the nondegeneracy of λ. By [BMZ11, Théorème 3.2], [BMZ17, Theorem 4.4] which holds for a more general setting (cf. also [MP13a, Proposition 7.5, Corollary 7.6] for a proof by using representation theory), there exist constants $C > 0$, $c > 0$ such that for $d \in \mathbb{N}$,

$$\mathbf{D}^{M,F_d,2} \geq cd^2 - C. \tag{1.0.12}$$

Then for d large enough, F_d is acyclic flat vector bundle on M. Combining (1.0.12) with the fact that nonelliptic elements $\gamma\sigma$, $\gamma\in\Gamma$ admit a uniform positive lower bound for their displacement distances on X (Proposition 2.6.3), we prove that the contribution from nonelliptic classes of $[\Gamma]_\sigma$ to (1.0.10) is exponentially small as $d \to +\infty$. As a consequence, we get (1.0.9).

This paper is organized as follows. In Section 2, we describe our setting for the locally symmetric space with a twisting action of σ, and we recall the explicit formula for the twisted orbital integrals obtained in [Liu18, Section 5].

In Section 3, we consider the flat Hermitian vector bundle F on M defined from the unitary representations of U^σ, and we study the associated $\mathcal{T}_\sigma(g^{TM}, \nabla^{F,f}, h^F)$. In particular, we get a vanishing theorem for it.

Finally, in Section 4, for an irreducible U^σ-representation with a σ-fixed highest weight λ, we construct a canonical sequence of representations $\{E_d\}_{d\in\mathbb{N}}$ of U^σ. This way, we get a sequence of flat vector bundles F_d on M. We also recall the nondegeneracy condition for λ as in [BMZ17, Section 8]. At last, we prove Theorem 1.0.1.

The results contained in this paper are mainly from the second part of the author's thesis [Liu18] and were announced in [Liu19]. Note that (1.0.8) is a refinement of [Liu19, Theorem 4.5].

In the sequel, if V is a real vector space and if E is a complex vector space, we will denote by $V \otimes E$ the complex vector space $V \otimes_{\mathbb{R}} E$. We use the same convention for the tensor products of vector bundles. If $E = E^+ \oplus E^-$ is a $\mathbb{Z}_2$-graded vector space, if $A \in \mathrm{End}(E)$ has the diagonal elements $A^+ \in \mathrm{End}(E^+)$, $A^- \in \mathrm{End}(E^-)$, then the supertrace is defined as

$$\mathrm{Tr_s}^E[A] = \mathrm{Tr}^{E^+}[A^+] - \mathrm{Tr}^{E^-}[A^-]. \tag{1.0.13}$$

If H is a Lie group, let H^0 denote the connected component of identity.

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2. Twisted orbital integrals and locally symmetric spaces

In this section, we consider the action of a certain compact subgroup $\Sigma^\sigma \subset \mathrm{Aut}(G)$ on the symmetric space $X = G/K$, and we recall an explicit geometric formula for twisted orbital integrals obtained in the author's thesis [Liu18, Liu19]. Then, given a cocompact torsion-free discrete subgroup $\Gamma \subset G$ preserved by Σ^σ, we recall the twisted trace formula for $M = \Gamma \backslash X$.

As in the introduction, we always consider (G, θ, B) to be a connected linear real reductive Lie group with compact center. Set

$$m = \dim \mathfrak{p}, n = \dim \mathfrak{k}. \tag{2.0.1}$$

We also use the notation $\mathrm{Ad}(\cdot)$, $\mathrm{ad}(\cdot)$ for the adjoint actions of G, $\mathfrak{g}$ respectively.

2.1. Real reductive Lie group and symmetric space

The bilinear form B induces a symmetric bilinear form B^* on $\mathfrak{g}^*$, which extends to a bilinear form on $\Lambda^\cdot(\mathfrak{g}^*)$. The K-invariant bilinear form $\langle \cdot, \cdot \rangle = -B(\cdot, \theta \cdot)$ is a scalar product on $\mathfrak{g}$, which extends to a scalar product on $\Lambda^\cdot(\mathfrak{g}^*)$. We will use $|\cdot|$ to denote the norm under this scalar product.

Let $U\mathfrak{g}$ be the universal enveloping algebra of $\mathfrak{g}$. Let $C^{\mathfrak{g}} \in U\mathfrak{g}$ be the Casimir element associated with B, i.e., if $\{e_i\}_{i=1,\cdots,m+n}$ is a basis of $\mathfrak{g}$, and if $\{e_i^*\}_{i=1,\cdots,m+n}$ is the dual basis of $\mathfrak{g}$ with respect to B, then

$$C^{\mathfrak{g}} = -\sum e_i^* e_i. \tag{2.1.1}$$

We can identify $U\mathfrak{g}$ with the algebra of left-invariant differential operators over G, then $C^{\mathfrak{g}}$ is a second-order differential operator, which is $\mathrm{Ad}(G)$-invariant. Similarly, let $C^{\mathfrak{k}}$ denote the Casimir operator associated with $(\mathfrak{k}, B|_{\mathfrak{k}})$.

Let $i = \sqrt{-1}$ denote one fixed square root of -1. Put

$$\mathfrak{u} = \sqrt{-1}\mathfrak{p} \oplus \mathfrak{k}. \tag{2.1.2}$$

If $a \in \mathfrak{p}$, we use notation $ia \in \sqrt{-1}\mathfrak{p} \subset \mathfrak{u}$ to denote the corresponding vector.

Then $\mathfrak{u}$ is a (real) Lie algebra, which is called the compact form of $\mathfrak{g}$. Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{u}_{\mathbb{C}}$. Let $G_{\mathbb{C}}$ be the complexification of G with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Then G is the analytic subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}$. Let $U \subset G_{\mathbb{C}}$ be the analytic subgroup associated with $\mathfrak{u}$. By [Kna86, Proposition 5.3], since G has compact center, then U is a compact Lie group and a maximal compact subgroup of $G_{\mathbb{C}}$.

Let $U\mathfrak{u}$, $U\mathfrak{g}_{\mathbb{C}}$ be the enveloping algebras of $\mathfrak{u}$, $\mathfrak{g}_{\mathbb{C}}$ respectively. Then $U\mathfrak{g}_{\mathbb{C}}$ can be identified with the left-invariant holomorphic differential operators on $G_{\mathbb{C}}$. Let $C^{\mathfrak{u}}$ be the Casimir operator of U associated with B, by (2.1.1), we have

$$C^{\mathfrak{u}} = C^{\mathfrak{g}} \in U\mathfrak{g} \cap U\mathfrak{u} \subset U\mathfrak{g}_{\mathbb{C}}. \tag{2.1.3}$$

Set

$$X = G/K. \tag{2.1.4}$$

Then X is a smooth manifold, and it is diffeomorphism to $\mathfrak{p}$ by the global Cartan decomposition of G.

Let $\omega^{\mathfrak{g}} \in \Omega^1(G, \mathfrak{g})$ be the canonical left-invariant 1-form on G. Then by the splitting (1.0.3), we write

$$\omega^{\mathfrak{g}} = \omega^{\mathfrak{p}} + \omega^{\mathfrak{k}}. \tag{2.1.5}$$

Let $p : G \to X$ denote the obvious projection. Then p is a K-principal bundle over X. Then $\omega^{\mathfrak{k}}$ is a connection form of this principal bundle. The associated curvature form

$$\Omega^{\mathfrak{k}} = d\omega^{\mathfrak{k}} + \frac{1}{2}[\omega^{\mathfrak{k}}, \omega^{\mathfrak{k}}] = -\frac{1}{2}[\omega^{\mathfrak{p}}, \omega^{\mathfrak{p}}]. \tag{2.1.6}$$

Moreover, the adjoint action of K on $\mathfrak{p}$ gives us exactly the tangent bundle

$$TX = G \times_K \mathfrak{p}. \tag{2.1.7}$$

The bilinear form B restricting to $\mathfrak{p}$ defines a Riemannian metric g^{TX}, and $\omega^{\mathfrak{k}}$ induces the associated Levi-Civita connection ∇^{TX}. Let $d(\cdot, \cdot)$ denote the Riemannian distance on X.

Let $\mathrm{Aut}(G)$ be the Lie group of automorphism of G [Hoc52, Theorem 2]. The semidirect product of G and $\mathrm{Aut}(G)$ is defined as

$$G \rtimes \mathrm{Aut}(G) := \{(g, \phi) \mid g \in G, \phi \in \mathrm{Aut}(G)\}, \tag{2.1.8}$$

with the group multiplication:

$$(g_1, \phi_1) \cdot (g_2, \phi_2) = (g_1 \phi_1(g_2), \phi_1 \phi_2). \tag{2.1.9}$$

In the sequel, we will often write $g\phi$ instead of $(g, \phi) \in G \rtimes \mathrm{Aut}(G)$.

Definition 2.1.1. Put

$$\Sigma := \{\phi \in \mathrm{Aut}(G) \ : \ \phi\theta = \theta\phi, \ \phi \text{ preserves the bilinear form } B\}. \tag{2.1.10}$$

Then Σ is a compact Lie subgroup of $\mathrm{Aut}(G)$. The action of Σ on G preserves K, and the induced action of Σ on $\mathfrak{g}$ preserves the splitting (1.0.3) and the scalar product of $\mathfrak{g}$. Note that Σ contains all the inner automorphisms defined by elements in K. Moreover, $G \rtimes \Sigma$ is a closed Lie subgroup of $G \rtimes \mathrm{Aut}(G)$.

Given $\sigma \in \Sigma$, the map $g \in G \to \sigma(g) \in G$ descends to a diffeomorphism of X, which we also denote by σ. By (2.1.7), (2.1.10), the derivative of σ is given by $(g, f) \to (\sigma(g), \sigma(f))$ with $g \in G, f \in \mathfrak{p}$. This way, $G \rtimes \Sigma$ acts on X isometrically and transitively, and we have the following identification,

$$X = (G \rtimes \Sigma)/(K \rtimes \Sigma). \tag{2.1.11}$$

2.2. Twisted conjugation

In the sequel, we fix an element $\sigma \in \Sigma$. If $g, g' \in G$, the σ-twisted conjugation of g on g' is defined as follows,

$$C_\sigma(g)g' := gg'\sigma(g^{-1}) \in G. \tag{2.2.1}$$

The map $C_\sigma(g)$ is not always a Lie group automorphism except $\sigma = \mathrm{Id}_G$. But $C_\sigma(\cdot)$ defines a left action of G on itself.

If $g \in G$, the stabilizer of C_σ-action at g is called σ-twisted centralizer of g in G, denoted by $Z_\sigma(g)$. More precisely, we have

$$Z_\sigma(g) = \{h \in G \mid C_\sigma(h)g = g.\} \tag{2.2.2}$$

It is a closed Lie subgroup of G. Let $\mathfrak{z}_\sigma(g) \subset \mathfrak{g}$ denote the Lie algebra of $Z_\sigma(g)$. If $\sigma = \mathrm{Id}_G$, then $Z_\sigma(g)$ is just the centralizer $Z(g)$ of g in G with Lie algebra $\mathfrak{z}(g)$. The orbit under this C_σ-action containing $g \in G$ is called the σ-twisted conjugacy class of g in G.

Since we already fix the element σ, we often use the word *twisted* instead of *σ-twisted* in the above terminologies.

2.3. Casimir operator and heat kernel

Let Σ^σ be the closure of the subgroup of Σ generated by σ, then it is a closed Lie subgroup of Σ. Set

$$G^\sigma = G \rtimes \Sigma^\sigma, \ K^\sigma = K \rtimes \Sigma^\sigma. \tag{2.3.1}$$

As in (2.1.11),

$$X = G^\sigma / K^\sigma. \tag{2.3.2}$$

If $k \in K$, then $k\sigma = (k, \sigma) \in K^\sigma$, and its adjoint action on $f \in \mathfrak{p}$ is given by

$$\mathrm{Ad}(k\sigma)(f) = \mathrm{Ad}(k)\sigma(f) \in \mathfrak{p}. \tag{2.3.3}$$

Then, analog to (2.1.7), we have

$$TX = G^\sigma \times_{K^\sigma} \mathfrak{p}. \tag{2.3.4}$$

If $\rho^E : K^\sigma \to \mathrm{U}(E, h^E)$ is a finite dimensional representation, then set

$$F = G^\sigma \times_{K^\sigma} E. \tag{2.3.5}$$

The metric h^E defines a Hermitian metric h^F on F. The action of Σ^σ lifts to $F \to X$, where σ-action is represented by $(g, v) \to (\sigma(g), \rho^E(\sigma)v)$, $g \in G$, $v \in E$.

If we restrict ρ^E to K, we can view (E, h^E) as a unitary representation of K. Then the above vector bundle F is equivalently defined as $G \times_K E \to X$. It is equipped with a unitary connection ∇^F induced by $\omega^{\mathfrak{k}}$.

Remark 2.3.1. An interesting question is what kind of representation of K can be extended to a representation of K^σ. For simplicity, we temporarily view σ just as an element in $\mathrm{Aut}(K)$. Let $\mathrm{Irr}(\cdot)$ denote the set of equivalent classes of irreducible (complex) representations of a compact Lie group. In [Liu18, Subsection 2.4], when K is semisimple, there exists an automorphism τ of K with finite order which lies in the connected component of $\mathrm{Aut}(K)$ containing σ. Moreover, τ acts on the set $P_{++}(K, T)$ of dominant weights of K for certain root system. Set $K^\tau = K \rtimes \langle\tau\rangle$. Then we proved the following bijections

$$\mathrm{Irr}(\Sigma^\sigma)\backslash\mathrm{Irr}(K^\sigma) \simeq \mathrm{Irr}(\langle\tau\rangle)\backslash\mathrm{Irr}(K^\tau) \simeq \langle\tau\rangle\backslash P_{++}(K, T). \tag{2.3.6}$$

We refer to the proof of Proposition 3.3.5 for understanding precisely the above bijections, where K is replaced by U.

Let $C^\infty(G, E)$ denote the set of smooth map from G into E. If $k \in K$, $s \in C^\infty(G, E)$, we define the dot-action of K by $(k.s)(g) = \rho^E(k)s(gk)$. Let $C^\infty_K(G, E)$ be the set of K-dot-invariant maps in $C^\infty(G, E)$. Let $C^\infty(X, F)$ denote the smooth sections of F over X. Then

$$C^\infty(X, F) = C^\infty_K(G, E). \tag{2.3.7}$$

Moreover, the left action of G^σ on $F \to X$ induces an action of G^σ on $C^\infty(X, F)$. Also ∇^F is invariant under this action of G^σ.

The Casimir operator $C^{\mathfrak{g}}$ acting on $C^\infty(G,E)$ preserves $C^\infty_K(G,E)$, so it induces an operator $C^{\mathfrak{g},X}$ acting on $C^\infty(X,F)$. Let $\Delta^{H,X}$ be the Bochner Laplacian acting on $C^\infty(X,F)$ given by ∇^F, and let $C^{\mathfrak{k},E} \in \mathrm{End}(E)$ be the action of the Casimir $C^{\mathfrak{k}}$ on E via ρ^E. The element $C^{\mathfrak{k},E}$ induces an self-adjoint section of $\mathrm{End}(F)$ over X. Then

$$C^{\mathfrak{g},X} = -\Delta^{H,X} + C^{\mathfrak{k},E}. \tag{2.3.8}$$

Let $C^{\mathfrak{k},\mathfrak{p}} \in \mathrm{End}(\mathfrak{p})$, $C^{\mathfrak{k},\mathfrak{k}} \in \mathrm{End}(\mathfrak{k})$ be the actions of $C^{\mathfrak{k}}$ on $\mathfrak{p}$, $\mathfrak{k}$ via the adjoint actions. Given $A \in \mathrm{End}(E)$ commuting with K^σ, we view it as a parallel section of $\mathrm{End}(F)$ over X. Let dx denote the Riemannian volume element of (X, g^{TX}).

Definition 2.3.2. Let $\mathcal{L}^{X,F}_A$ be the Bochner-like Laplacian acting on $C^\infty(X,F)$ given by

$$\mathcal{L}^{X,F}_A = \frac{1}{2} C^{g,X} + \frac{1}{16}\mathrm{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}] + \frac{1}{28}\mathrm{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}] + A. \tag{2.3.9}$$

If $A = 0$, we denote this operator simply by $\mathcal{L}^{X,F}$. For $t > 0$, $x, x' \in X$, let $p^X_t(x,x')$ denote its heat kernel with respect to dx'.

Since $C^{\mathfrak{g}}$ is invariant under the adjoint action of G^σ, the operator $\mathcal{L}^{X,F}_A$ commutes with G^σ-action on $C^\infty(X,F)$. Then $p^X_t(x,x')$ lifts to a function $p^X_t(g,g')$ on $G \times G$ valued in $\mathrm{End}(E)$ such that for $g'' \in G$, $k, k' \in K$,

$$\begin{aligned}
&p^X_t(g''g, g''g') = p^X_t(g,g'),\\
&p^X_t(gk, g'k') = \rho^E(k^{-1}) p^X_t(g,g') \rho^E(k'),\\
&p^X_t(\sigma(g),\sigma(g')) = \rho^E(\sigma) p^X_t(g,g') \rho^E(\sigma^{-1}).
\end{aligned} \tag{2.3.10}$$

Let $p^X_t(\cdot)$ be the smooth function on G valued in $\mathrm{End}(E)$ such that

$$p^X_t(g) = p^X_t(1,g). \tag{2.3.11}$$

In the sequel, we will often regard the heat kernel $p^X_t(x,x')$ and the function $p^X_t(g)$ as the same object.

2.4. Semisimple element

Recall that for $\gamma \in G$, $\gamma\sigma \in G^\sigma$ acts on X isometrically. The associated displacement function $d_{\gamma\sigma}$ is the function on X defined as

$$d_{\gamma\sigma}(x) = d(x, \gamma\sigma(x)) \ , \ x \in X. \tag{2.4.1}$$

Put $m_{\gamma\sigma} = \inf_{x\in X} d_{\gamma\sigma}(x) \in \mathbb{R}_{\geq 0}$.

Since X has nonpositive sectional curvature, by [Ebe96, Chapter 1, Example 1.6.6], $d_{\gamma\sigma}$ is a continuous nonnegative convex function on X, and $d^2_{\gamma\sigma}$ is a smooth convex function.

Definition 2.4.1. The element $\gamma\sigma \in G^\sigma$ is called semisimple if $d_{\gamma\sigma}(x)$ reaches its infimum $m_{\gamma\sigma}$ in X. An element $\gamma\sigma$ is called elliptic if it has fixed points in X, which is always semisimple by definition. If $\gamma\sigma$ is semisimple, put

$$X(\gamma\sigma) = \{x \in X \mid d_{\gamma\sigma}(x) = m_{\gamma\sigma}\}. \tag{2.4.2}$$

A semisimple element $\gamma\sigma$ as above shares many similar properties as a semisimple matrix in $\mathrm{GL}_n(\mathbb{R})$ or a semisimple element in a linear reductive Lie group [Ebe96, Section 2.19]. A detailed discussion can be found in the [Liu18, Section 1]. We recall some of the results in the sequel.

If $\gamma \in G$ and if $\gamma\sigma$ is semisimple, then there exists $g \in G$ such that

$$\gamma = g e^{a} k^{-1} \sigma(g^{-1}), a \in \mathfrak{p}, k \in K, \mathrm{Ad}(k^{-1})\sigma(a) = a. \tag{2.4.3}$$

An equivalent way to state the first identity in (2.4.3) is

$$\gamma\sigma = C_\sigma(g)(e^{a}k^{-1})\sigma = g(e^{a}k^{-1}\sigma)g^{-1} \in G^{\sigma}. \tag{2.4.4}$$

Moreover, we get

$$\begin{aligned} &Z_\sigma(\gamma) = g Z_\sigma(e^{a}k^{-1}) g^{-1} \subset G, \\ &X(\gamma\sigma) = g^{-1} X(e^{a}k^{-1}\sigma) \subset X,\ m_{\gamma\sigma} = m_{e^{a}k^{-1}\sigma} = |a|. \end{aligned} \tag{2.4.5}$$

Therefore, we may and we will focus on a semisimple element $\gamma\sigma$ such that

$$\gamma = e^{a}k^{-1}, a \in \mathfrak{p}, k \in K, \mathrm{Ad}(k^{-1})\sigma(a) = a. \tag{2.4.6}$$

Let $Z(a) \subset G$ be the centralizer of a under the adjoint action of G. Let $\mathfrak{z}(a)$ be its Lie algebra. Similar to the Jordan decomposition properties of a semisimple matrix, we have the following identities [Liu18, Proposition 1.3.5],

$$Z_\sigma(\gamma) = Z(e^{a}) \cap Z_\sigma(k^{-1}),\ Z(e^{a}) = Z(a). \tag{2.4.7}$$

Correspondingly, we have

$$\mathfrak{z}_\sigma(\gamma) = \mathfrak{z}(e^{a}) \cap \mathfrak{z}_\sigma(k^{-1}),\ \mathfrak{z}(e^{a}) = \mathfrak{z}(a). \tag{2.4.8}$$

The Cartan involution θ preserves $Z_\sigma(\gamma)$, $Z(e^{a})$ and $Z_\sigma(k^{-1})$, so that the corresponding Cartan decompositions of their Lie algebras hold true as in (1.0.3). In particular, by [Kna02, Proposition 7.25], $Z_\sigma(\gamma)$ is reductive.

Set

$$K_\sigma(\gamma) = Z_\sigma(\gamma) \cap K. \tag{2.4.9}$$

Moreover, $K_\sigma(\gamma)$ is a maximal compact subgroup of $Z_\sigma(\gamma)$, which meets every connected components of $Z_\sigma(\gamma)$.

Let $\mathfrak{k}_\sigma(\gamma) \subset \mathfrak{z}_\sigma(\gamma)$ be the Lie algebra of $K_\sigma(\gamma)$. Then

$$\mathfrak{k}_\sigma(\gamma) = \mathfrak{z}_\sigma(\gamma) \cap \mathfrak{k}. \tag{2.4.10}$$

Put

$$\mathfrak{p}_\sigma(\gamma) = \mathfrak{z}_\sigma(\gamma) \cap \mathfrak{p}. \tag{2.4.11}$$

Then the Cartan decomposition of $\mathfrak{z}(\gamma)$ with respect to θ is given by

$$\mathfrak{z}_\sigma(\gamma) = \mathfrak{k}_\sigma(\gamma) \oplus \mathfrak{p}_\sigma(\gamma). \tag{2.4.12}$$

Moreover, the bilinear form $B|_{\mathfrak{z}_\sigma(\gamma)}$ is positive on $\mathfrak{p}_\sigma(\gamma)$, and negative on $\mathfrak{k}_\sigma(\gamma)$. The splitting in (2.4.12) is orthogonal with respect to B.

The minimizing set $X(\gamma\sigma)$ is a totally geodesic submanifold of X, which is again a symmetric space. More precisely, by [Liu18, Lemma 1.4.6, Theorem 1.4.7], $Z_\sigma(\gamma)$ and its identity component $Z_\sigma(\gamma)^0$ act on $X(\gamma\sigma)$ transitively, and we have the following identifications,

$$X(\gamma\sigma) \simeq Z_\sigma(\gamma)/K_\sigma(\gamma) = Z_\sigma(\gamma)^0/K_\sigma(\gamma)^0. \tag{2.4.13}$$

Under the identification $X \simeq \mathfrak{p}$ via the global geodesic coordinate, we have $X(\gamma\sigma) \simeq \mathfrak{p}_\sigma(\gamma)$.

2.5. An explicit formula for twisted orbital integrals

In this subsection, we give an explicit geometric formula for twisted orbital integral $\mathrm{Tr}^{[\gamma\sigma]}[\exp(-t\mathcal{L}_A^X)]$ for p_t^X associated with a semisimple $\gamma\sigma$. We still assume that $\gamma \in G$ is given by (2.4.6).

Let dg be the left-invariant Haar measure on G induced by $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$. Since G is unimodular, then dg is also right-invariant. Let dk be the Haar measure on K induced by $-B|_{\mathfrak{k}}$, then

$$dg = dxdk. \tag{2.5.1}$$

Let dy be the Riemannian volume element of $X(\gamma\sigma)$, and let dz be the bi-invariant (positive) Haar measure on $Z_\sigma(\gamma)$ induced by $B|_{\mathfrak{z}_\sigma(\gamma)}$. Let $dk_\sigma(\gamma)$ be the Haar measure on $K_\sigma(\gamma)$ such that

$$dz = dydk_\sigma(\gamma). \tag{2.5.2}$$

Let $\mathrm{Vol}(K_\sigma(\gamma)\backslash K)$ be the volume of $K_\sigma(\gamma)\backslash K$ with respect to $dk, dk_\sigma(\gamma)$. Then

$$\mathrm{Vol}(K_\sigma(\gamma)\backslash K) = \frac{\mathrm{Vol}(K)}{\mathrm{Vol}(K_\sigma(\gamma))}. \tag{2.5.3}$$

Let dv be the G-right-invariant measure on $Z_\sigma(\gamma)\backslash G$ such that

$$dg = dzdv. \tag{2.5.4}$$

For $t > 0$, the twisted orbital integral $\mathrm{Tr}^{[\gamma\sigma]}[\exp(-t\mathcal{L}_A^{X,F})]$ is defined as

$$\mathrm{Tr}^{[\gamma\sigma]}[\exp(-t\mathcal{L}_A^{X,F})] = \frac{1}{\mathrm{Vol}(K_\sigma(\gamma)\backslash K)} \int_{Z_\sigma(\gamma)\backslash G} \mathrm{Tr}^E[\rho^E(\sigma)p_t^X(v^{-1}\gamma\sigma(v))]dv \tag{2.5.5}$$

By [Liu18, Propositions 4.2.1 & 4.4.1], the integral in (2.5.5) is well-defined. As indicated by the notation, it only depends on the conjugacy class $[\gamma\sigma]$ of $\gamma\sigma$ in G^σ, and then it only depends on the σ-twisted conjugacy class of γ in G. This kind of integrals play an important role in base change theory, we refer to [Lan80, Clo84, AC89, BL17] for more details.

In [Liu18, Subsection 4.2] [Liu19, Definition 2.1], a geometric formula for $\mathrm{Tr}^{[\gamma\sigma]}[\exp(-t\mathcal{L}_A^{X,F})]$ is established. We explain it as follows. Let $N_{X(\gamma\sigma)/X}$ be the orthogonal normal bundle of $X(\gamma\sigma)$ in X, and let $\mathcal{N}_{X(\gamma\sigma)/X}$ denote its total space. Then $\mathcal{N}_{X(\gamma\sigma)/X} \simeq X$ via the normal geodesics. For $x \in X(\gamma\sigma)$, let df be the Euclidean volume element on $N_{X(\gamma\sigma)/X,x}$. Then there exists a positive function $r(f)$ on $N_{X(\gamma\sigma)/X,x}$ such that $dx = r(f)dydf$. We have

$$\mathrm{Tr}^{[\gamma\sigma]}[\exp(-t\mathcal{L}_A^{X,F})] = \int_{N_{X(\gamma\sigma)/X,x}} \mathrm{Tr}^F[p_t^X(\exp_x(f), \gamma\sigma\exp_x(f))\gamma\sigma]r(f)df, \tag{2.5.6}$$

where the right-hand side of (2.5.6) does not depend on the choice of $x \in X(\gamma\sigma)$.

An explicit formula for $\mathrm{Tr}^{[\gamma\sigma]}[\exp(-t\mathcal{L}_A^{X,F})]$ was obtained in [Liu18, Theorem 5.2.1] [Liu19, Theorem 3.3] via the theory of hypoelliptic Laplacian developed by Bismut, which generalizes Bismut's formula for orbital integrals [Bis11, Theorem 6.1.1]. We now recall this formula.

To save the notation length, put

$$\mathfrak{z}_0 = \mathfrak{z}(a), \quad \mathfrak{p}_0 = \ker \mathrm{ad}(a) \cap \mathfrak{p}, \quad \mathfrak{k}_0 = \ker \mathrm{ad}(a) \cap \mathfrak{k}. \tag{2.5.7}$$

Let $\mathfrak{z}_0^\perp$, $\mathfrak{p}_0^\perp$, $\mathfrak{k}_0^\perp$ be the orthogonal subspaces to $\mathfrak{z}_0$, $\mathfrak{p}_0$, $\mathfrak{k}_0$ in $\mathfrak{g}, \mathfrak{p}, \mathfrak{k}$ with respect to B. Then

$$\mathfrak{z}_0 = \mathfrak{p}_0 \oplus \mathfrak{k}_0, \quad \mathfrak{z}_0^\perp = \mathfrak{p}_0^\perp \oplus \mathfrak{k}_0^\perp. \tag{2.5.8}$$

By (2.4.8),

$$\mathfrak{z}_\sigma(\gamma) = \mathfrak{z}_0 \cap \mathfrak{z}_\sigma(k). \tag{2.5.9}$$

Also $\mathfrak{p}_\sigma(\gamma)$, $\mathfrak{k}_\sigma(\gamma)$ are subspaces of $\mathfrak{p}_0$, $\mathfrak{k}_0$ respectively. Let $\mathfrak{z}_{\sigma,0}^\perp(\gamma)$, $\mathfrak{p}_{\sigma,0}^\perp(\gamma)$, $\mathfrak{k}_{\sigma,0}^\perp(\gamma)$ be the orthogonal spaces to $\mathfrak{z}_\sigma(\gamma)$, $\mathfrak{p}_\sigma(\gamma)$, $\mathfrak{k}_\sigma(\gamma)$ in $\mathfrak{z}_0$, $\mathfrak{p}_0$, $\mathfrak{k}_0$. Then

$$\mathfrak{z}_{\sigma,0}^\perp(\gamma) = \mathfrak{p}_{\sigma,0}^\perp(\gamma) \oplus \mathfrak{k}_{\sigma,0}^\perp(\gamma). \tag{2.5.10}$$

Also the action $\mathrm{ad}(a)$ gives an isomorphism between $\mathfrak{p}_0^\perp$ and $\mathfrak{k}_0^\perp$.

For $y \in \mathfrak{k}_\sigma(\gamma)$, $\mathrm{ad}(y)$ preserves $\mathfrak{p}_\sigma(\gamma), \mathfrak{k}_\sigma(\gamma), \mathfrak{p}_{\sigma,0}^\perp(\gamma), \mathfrak{k}_{\sigma,0}^\perp(\gamma)$, and it is an antisymmetric endomorphism with respect to the scalar product.

Recall that the function $\widehat{A}(x) = \frac{x/2}{\sinh(x/2)}$. Let H be a finite-dimensional Hermitian vector space. If $B \in \mathrm{End}(H)$ is self-adjoint, then $\dfrac{B/2}{\sinh(B/2)}$ is a self-adjoint positive endomorphism. Put

$$\widehat{A}(B) = \det{}^{1/2}\left[\frac{B/2}{\sinh(B/2)}\right]. \tag{2.5.11}$$

In (2.5.11), the square root is taken to be the positive square root.

If $y \in \mathfrak{k}_\sigma(\gamma)$, the following function $A(y)$ has a natural square root that is analytic in $y \in \mathfrak{k}_\sigma(\gamma)$,

$$A(y) = \frac{1}{\det(1-\mathrm{Ad}(k^{-1}\sigma))|_{\mathfrak{z}_{\sigma,0}^\perp(\gamma)}} \cdot \frac{\det(1-\exp(-i\mathrm{ad}(y))\mathrm{Ad}(k^{-1}\sigma))|_{\mathfrak{k}_{\sigma,0}^\perp(\gamma)}}{\det(1-\exp(-i\mathrm{ad}(y))\mathrm{Ad}(k^{-1}\sigma))|_{\mathfrak{p}_{\sigma,0}^\perp(\gamma)}}. \tag{2.5.12}$$

Its square root is denoted by

$$\left[\frac{1}{\det(1-\mathrm{Ad}(k^{-1}\sigma))|_{\mathfrak{z}^{\perp}_{\sigma,0}(\gamma)}}\cdot\frac{\det(1-\exp(-i\mathrm{ad}(y)\mathrm{Ad}(k^{-1}\sigma))|_{\mathfrak{k}^{\perp}_{\sigma,0}(\gamma)}}{\det(1-\exp(-i\mathrm{ad}(y)\mathrm{Ad}(k^{-1}\sigma))|_{\mathfrak{p}^{\perp}_{\sigma,0}(\gamma)}}\right]^{1/2}. \tag{2.5.13}$$

The value of (2.5.13) at $y=0$ is taken to be such that

$$\frac{1}{\det(1-\mathrm{Ad}(k^{-1})\sigma)|_{\mathfrak{p}^{\perp}_{\sigma,0}(\gamma)}}. \tag{2.5.14}$$

We recall an important function defined in [Liu18, Definition 5.1.2] [Liu19, Definition 3.2]

Definition 2.5.1. Let $J_{\gamma\sigma}(y)$ be the analytic function of $y\in\mathfrak{k}_{\sigma}(\gamma)$ given by

$$\begin{aligned}J_{\gamma\sigma}(y)=&\frac{1}{|\det(1-\mathrm{Ad}(\gamma\sigma))|_{\mathfrak{z}_0^{\perp}}|^{1/2}}\frac{\widehat{A}(i\mathrm{ad}(y)|_{\mathfrak{p}_{\sigma}(\gamma)})}{\widehat{A}(i\mathrm{ad}(y)|_{\mathfrak{k}_{\sigma}(\gamma)})}\\&\left[\frac{1}{\det(1-\mathrm{Ad}(k^{-1}\sigma))|_{\mathfrak{z}^{\perp}_{\sigma,0}(\gamma)}}\frac{\det(1-\exp(-i\mathrm{ad}(y))\mathrm{Ad}(k^{-1}\sigma))|_{\mathfrak{k}^{\perp}_{\sigma,0}(\gamma)}}{\det(1-\exp(-i\mathrm{ad}(y))\mathrm{Ad}(k^{-1}\sigma))|_{\mathfrak{p}^{\perp}_{\sigma,0}(\gamma)}}\right]^{1/2}.\end{aligned} \tag{2.5.15}$$

By (2.5.1), there exists $C_{\gamma\sigma}>0$, $c_{\gamma\sigma}>0$ such that if $y\in\mathfrak{k}_{\sigma}(\gamma)$,

$$|J_{\gamma\sigma}(y)|\leq C_{\gamma\sigma}e^{c_{\gamma\sigma}|y|}. \tag{2.5.16}$$

Put $p=\dim\mathfrak{p}_{\sigma}(\gamma)$, $q=\dim\mathfrak{k}_{\sigma}(\gamma)$. Then $r=\dim\mathfrak{z}_{\sigma}(\gamma)=p+q$. By [Liu18, Theorem 5.2.1] [Liu19, Theorem 3.3], for $t>0$, we have

$$\begin{aligned}&\mathrm{Tr}^{[\gamma\sigma]}[\exp(-t\mathcal{L}_A^{X,F})]\\&=\frac{1}{(2\pi t)^{p/2}}\int_{\mathfrak{k}_{\sigma}(\gamma)}J_{\gamma\sigma}(y)\mathrm{Tr}^{E}\left[\rho^{E}(k^{-1}\sigma)\exp(-i\rho^{E}(y)-tA)\right]e^{-|y|^2/2t}\frac{dy}{(2\pi t)^{q/2}}.\end{aligned} \tag{2.5.17}$$

Remark 2.5.2. In [BL17], under suitable conditions in base change setting (then σ is of finite order), Bergeron and Lipnowski managed to express certain twisted orbital integrals in terms of ordinary orbital integrals, where they can make use of Harish-Chandra's theory to compute them.

2.6. Compact locally symmetric space M with twisting action

Let Γ be a cocompact torsion-free discrete subgroup of G, which is preserved by σ. Even we do not require σ to be of finite order, the group Σ^{σ} descends to a finite Abelian subgroup of $\mathrm{Aut}(\Gamma)$. Also note that in many interesting cases such as the base change setting, or for G simple, σ will be of finite order.

By [Sel60, Lemmas 1,2], we have the following results.

Lemma 2.6.1. *If Γ is a cocompact discrete subgroup of G, if $\gamma \in \Gamma$, then it is semisimple in G, and $\Gamma \cap Z(\gamma)$ is a cocompact discrete subgroup of $Z(\gamma)$. Generally, if $\gamma \in \Gamma$, then $\gamma\sigma \in G^\sigma$ is also semisimple, and $\Gamma \cap Z_\sigma(\gamma)$ is a cocompact discrete subgroup of $Z_\sigma(\gamma)$.*

Definition 2.6.2. We denote by $[\Gamma]_\sigma$ the set of σ-twisted conjugacy classes in Γ. If $\gamma \in \Gamma$, let $[\gamma]_\sigma$ be the σ-twisted conjugacy class of γ in Γ. If $\gamma\sigma$ is elliptic, we say that $[\gamma]_\sigma$ is an elliptic class.

Let E_σ be the set of elliptic classes in $[\Gamma]_\sigma$. By [Liu18, Lemma 1.8.3], E_σ is a finite set. Note that $m_{\gamma\sigma} \in \mathbb{R}_{\geq 0}$ only depends on the class $[\gamma]_\sigma$ of $\gamma \in \Gamma$. Set

$$c_{\Gamma,\sigma} = \inf \{ m_{\gamma\sigma} \mid [\gamma]_\sigma \in [\Gamma]_\sigma \backslash E_\sigma \} \geq 0. \tag{2.6.1}$$

Proposition 2.6.3. *We have*

$$c_{\Gamma,\sigma} > 0. \tag{2.6.2}$$

Proof. Suppose that we have a sequence of $[\gamma_i]_\sigma \in [\Gamma]_\sigma \backslash E_\sigma$, $i \in \mathbb{N}$ such that $m_{\gamma_i\sigma} \to 0$ as $i \to +\infty$. Let $V \subset G$ be the compact connected fundamental domain for the quotient $\Gamma\backslash G$. Then for each class $[\gamma_i]_\sigma$, there exists $\gamma'_i \in [\gamma_i]_\sigma$, $x_i \in p(V)$ such that

$$d_{\gamma'_i\sigma}(x_i) = m_{\gamma_i\sigma}. \tag{2.6.3}$$

Since V is compact, we may and we will assume that $\{x_i\}_{i\in\mathbb{N}}$ is a convergent sequence with limit $x \in p(V)$. Then

$$d(x, \gamma'_i\sigma(x)) \leq d(x, x_i) + d(x_i, \gamma'_i\sigma(x_i)) + d(\gamma'_i\sigma(x_i), \gamma'_i\sigma(x)). \tag{2.6.4}$$

By the assumption, there exists $i_0 \in \mathbb{N}$ such that if $i \geq i_0$, then

$$d(x, \gamma'_i\sigma(x)) \leq 1/2. \tag{2.6.5}$$

Since Γ is discrete, there exists only finite number of γ'_i such that (2.6.5) holds. This contradicts the assumption that $m_{\gamma_i\sigma} \to 0$ as $i \to +\infty$, which completes our proof. ☐

Since Γ is torsion-free, a modification of the arguments in the proof of [MP13b, Proposition 3.2] shows the following lemma. Note that it is also a special case of [MM15, Eq.(3.19)].

Lemma 2.6.4. *There exist $c > 0$, $C > 0$ such that for $R > 0$, $x \in X$, we have*

$$\#\{\gamma \in \Gamma \mid \gamma\sigma \textit{ non-elliptic}, d_{\gamma\sigma}(x) \leq R\} \leq C \exp(cR). \tag{2.6.6}$$

Put $M = \Gamma\backslash X = \Gamma\backslash G/K$. The tangent vector bundle TX descends to the tangent vector bundle TM of M. Since Γ-action is isometric, g^{TX} induces a Riemannian metric g^{TM} on TM. Then M is a compact locally symmetric Riemannian manifold. Moreover, the Hermitian bundle (F, ∇^F, h^F) on X defined in Subsection 2.3 descends to a Hermitian vector bundle on M, which we still denote by the same notation.

Since σ preserves Γ, then Σ^σ acts on M isometrically, and this action lifts to Hermitian bundle F on M. We will use σ^M denote the action of σ on $F \to M$.

If $g \in G$, we denote by $[g]_X = pg$ (resp. $[g]_M$) the corresponding point in X (resp. M). If $A \subset X$, we denote by $[A]_M \subset M$ the image of $A \subset X$ under the quotient projection $X \to M$.

Let ${}^\sigma M \subset M$ be the fixed point set of σ in M. The following result is proved in [Liu18, Lemma 1.8.7].

Lemma 2.6.5. *If $\gamma_1, \gamma_2 \in \Gamma$ are σ-twisted conjugate in Γ, then*

$$[X(\gamma_1\sigma)]_M = [X(\gamma_2\sigma)]_M \subset M. \tag{2.6.7}$$

If $g \in G$, then $[g]_M \in {}^\sigma M$ if and only if there is $\gamma \in \Gamma$ such that $\gamma\sigma$ is elliptic and that $[g]_X \in X(\gamma\sigma) \subset X$. If $[\gamma_1]_\sigma$, $[\gamma_2]_\sigma \in E_\sigma$ are distinct classes, then

$$[X(\gamma_1\sigma)]_M \cap [X(\gamma_2\sigma)]_M = \emptyset. \tag{2.6.8}$$

By Lemma 2.6.5, we get that

$${}^\sigma M = \cup_{[\gamma]_\sigma \in E_\sigma} [X(\gamma\sigma)]_M. \tag{2.6.9}$$

Moreover, the right-hand side in (2.6.9) is a finite disjoint union.

By Lemma 2.6.1, $\Gamma \cap Z_\sigma(\gamma)$ is a cocompact discrete subgroup of $Z_\sigma(\gamma)$. Moreover, since Γ is torsion-free, so is $\Gamma \cap Z_\sigma(\gamma)$, and $\Gamma \cap Z_\sigma(\gamma) \backslash X(\gamma\sigma)$ is a compact locally symmetric manifold.

Take $[\gamma]_\sigma \in E_\sigma$, let $\gamma \in \Gamma$ be one representative of $[\gamma]_\sigma$. If $x \in X(\gamma\sigma)$, if $\gamma_0 \in \Gamma$ is such that $\gamma_0 x \in X(\gamma\sigma)$, then $\gamma_0 \in Z_\sigma(\gamma)$. Thus the projection $X \to M$ induces an identification between $\Gamma \cap Z_\sigma(\gamma) \backslash X(\gamma\sigma)$ and $[X(\gamma\sigma)]_M \subset M$. Then (2.6.9) can be rewritten as

$${}^\sigma M = \cup_{[\gamma]_\sigma \in E_\sigma} \Gamma \cap Z_\sigma(\gamma) \backslash X(\gamma\sigma). \tag{2.6.10}$$

Let $C^\infty(M, F)$ be the vector space of smooth sections of F on M, and let $C^\infty(X, F)^\Gamma$ be the subspace of $C^\infty(X, F)$ of Γ-invariant sections over X. Then we have a canonical identification

$$C^\infty(M, F) = C^\infty(X, F)^\Gamma. \tag{2.6.11}$$

By (2.3.7), (2.6.11), we get

$$C^\infty(M, F) = C^\infty_K(G, E)^\Gamma. \tag{2.6.12}$$

Recall that the Bochner-like Laplacian $\mathcal{L}_A^{X,F}$ is defined by (2.3.9). Since it commutes with G^σ, then it descends to a Bochner-like Laplacian $\mathcal{L}_A^{M,F}$ acting on $C^\infty(M, F)$ and commuting with Σ^σ.

For $t > 0$, let $p_t^M(z, z')$, $z, z' \in M$ be the heat kernel of $\mathcal{L}_A^{M,F}$ with respect to the Riemannian volume element dz'. If z, z' are identified with their lifts in X, then

$$p_t^M(z, z') = \sum_{\gamma \in \Gamma} \gamma p_t^X(\gamma^{-1} z, z') = \sum_{\gamma \in \Gamma} p_t^X(z, \gamma z')\gamma. \tag{2.6.13}$$

Theorem 2.6.6 (Twisted trace formula). *For $t > 0$, we have*

$$\mathrm{Tr}[\sigma^M \exp(-t\mathcal{L}_A^{M,F})] = \sum_{[\gamma]_\sigma \in [\Gamma]_\sigma} \mathrm{Vol}(\Gamma \cap Z_\sigma(\gamma)\backslash X(\gamma\sigma))\mathrm{Tr}^{[\gamma\sigma]}[\exp(-t\mathcal{L}_A^{X,F})]. \tag{2.6.14}$$

Here the convergences of the integrals and infinite sums are already guaranteed by the results in [Bis11, Chapters 2 & 4] and in [She18, Section 4D]. The proof to this formula can be found in many literature on base change theory, one can also find a detailed proof in [Liu18, Subsection 4.5].

3. Equivariant real analytic torsion for locally symmetric space

In this section, we explain how to make use of (2.5.17) and the twisted trace formula (2.6.14) to study the equivariant Ray-Singer analytic torsions of M.

We extend σ-action to $\mathfrak{g}_\mathbb{C}$ as a complex linear automorphism of $\mathfrak{g}_\mathbb{C}$, which preserves Lie subalgebra $\mathfrak{u}$. We also assume that σ-action on $\mathfrak{u}$ extends to an automorphism of U, this way, it acts on $G_\mathbb{C}$ bi-holomorphically. Set

$$U^\sigma = U \rtimes \Sigma^\sigma. \tag{3.0.1}$$

3.1. The de Rham operator associated with a flat bundle

In the sequel, we take (E, ρ^E, h^E) to be a unitary representation of U^σ. By Weyl's unitary trick, every irreducible unitary representation of U^σ extends uniquely to an irreducible representation of G^σ. We use the same notation ρ^E for the restrictions of this representation to G, to K and to K^σ. By (2.1.3),

$$C^{\mathfrak{u},E} = C^{\mathfrak{g},E} \in \mathrm{End}(E). \tag{3.1.1}$$

As in Subsection 2.3, put $F = G \times_K E$. Let ∇^F be the Hermitian connection induced by the connection form $\omega^\mathfrak{k}$. Then the map $(g, v) \in G \times_K E \to \rho^E(g)v \in E$ gives a canonical identification of vector bundles on X,

$$G \times_K E = X \times E. \tag{3.1.2}$$

Then F is equipped with a canonical flat connection $\nabla^{F,f}$ so that

$$\nabla^{F,f} = \nabla^F + \rho^E(\omega^\mathfrak{p}). \tag{3.1.3}$$

Since G has compact center, $(F, h^F, \nabla^{F,f})$ is a unimodular flat vector bundle.

Let $(\Omega_c^\bullet(X, F), d^{X,F})$ be the (compactly supported) de Rham complex associated with $(F, \nabla^{F,f})$. Let $d^{X,F,*}$ be the adjoint operator of $d^{X,F}$ with respect to the L_2-metric on $\Omega_c^\bullet(X, F)$. The Dirac operator $\mathbf{D}^{X,F}$ is

$$\mathbf{D}^{X,F} = d^{X,F} + d^{X,F,*}. \tag{3.1.4}$$

The Clifford algebras $c(TX)$, $\widehat{c}(TX)$ of (TX, g^{TX}) act on $\Lambda^\bullet(T^*X)$. We still use $e_1, \cdots, e_m$ to denote an orthonormal basis of $\mathfrak{p}$ or TX.

Let $\nabla^{\Lambda^\bullet(T^*X)\otimes F,u}$ be the unitary connection on $\Lambda^\bullet(T^*X)\otimes F$ induced by ∇^{TX} and ∇^F. Then the standard Dirac operator is given by

$$D^{X,F} = \sum_{j=1}^{m} c(e_j)\nabla^{\Lambda^\bullet(T^*X)\otimes F,u}_{e_j}. \tag{3.1.5}$$

By [BMZ17, Eq.(8.42)], we have

$$\mathbf{D}^{X,F} = D^{X,F} + \sum_{j=1}^{m} \widehat{c}(e_j)\rho^E(e_j). \tag{3.1.6}$$

In the same time, $C^{\mathfrak{g}}$ descends to an elliptic differential operator $C^{\mathfrak{g},X}$ acting on $C^\infty(X, \Lambda^\bullet(T^*X)\otimes F)$. Let $\kappa^{\mathfrak{g}} \in \Lambda^3(\mathfrak{g}^*)$ be such that if $a, b, c \in \mathfrak{g}$,

$$\kappa^{\mathfrak{g}}(a,b,c) = B([a,b],c). \tag{3.1.7}$$

Then $\kappa^{\mathfrak{g}}$ is a G^σ-invariant closed 3-form on G. The bilinear form B induces a corresponding bilinear form B^* on $\Lambda^\bullet(\mathfrak{g}^*)$. By [Bis11, Eq.(2.6.11)], we have

$$B^*(\kappa^{\mathfrak{g}},\kappa^{\mathfrak{g}}) = \frac{1}{2}\mathrm{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}] + \frac{1}{6}\mathrm{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}]. \tag{3.1.8}$$

Let $\mathcal{L}^{X,F}$ be the operator in Definition 2.3.2 but associated with the representation $\Lambda^\bullet(\mathfrak{p}^*)\otimes E$. By [BMZ17, Proposition 8.4] and (3.1.8), we have

$$\begin{aligned}\frac{\mathbf{D}^{X,F,2}}{2} &= \mathcal{L}^{X,F} - \frac{1}{2}C^{\mathfrak{g},E} - \frac{1}{8}B^*(\kappa^{\mathfrak{g}},\kappa^{\mathfrak{g}}),\\ &= \frac{1}{2}C^{\mathfrak{g},X} - \frac{1}{2}C^{\mathfrak{g},E}.\end{aligned} \tag{3.1.9}$$

3.2. Equivariant Ray-Singer real analytic torsions on M

Let Γ be a cocompact torsion-free discrete subgroup of G preserved by σ. Let $M = \Gamma\backslash X$ be the compact locally symmetric manifold considered in Subsection 2.6. The flat vector bundle F defined in last subsection descends to a flat vector bundle on M, which we still denote by F on which Σ^σ acts equivariantly.

Note that since X is contractible, then

$$\pi_1(M) = \Gamma. \tag{3.2.1}$$

When restricting the representation ρ^E to Γ, we associate it with a flat vector bundle (or a local system) $\Gamma\backslash(X\times E)$ on M. By (3.1.2), this is an equivalent way to define $(F, \nabla^{F,f})$.

The de Rham-Dirac operator $\mathbf{D}^{X,F}$ in (3.1.4) descends to the corresponding Dirac operator $\mathbf{D}^{M,F}$ on M, so that

$$\mathbf{D}^{M,F} = d^{M,F} + d^{M,F,*}. \tag{3.2.2}$$

Then $\mathbf{D}^{M,F}$ commutes with Σ^σ. Let $H^\bullet_{\mathrm{dR}}(M,F)$ be the de Rham cohomology group of $(\Omega^\bullet(M,F), d^{M,F})$. By Hodge theory,

$$\ker \mathbf{D}^{M,F} \simeq H^\bullet_{\mathrm{dR}}(M,F). \tag{3.2.3}$$

Let $N^{\Lambda^\bullet(T^*M)}$ denote the number operator on $\Omega^\bullet(M,F)$, i.e. multiplication by the degrees of forms. Let $(\mathbf{D}^{M,F,2})^{-1}$ be the inverse of $\mathbf{D}^{M,F,2}$ acting on the orthogonal space of $\ker \mathbf{D}^{M,F}$ in $\Omega^\bullet(M,F)$.

Definition 3.2.1. For $s \in \mathbb{C}$, $\mathrm{Re}(s)$ big enough, set

$$\vartheta_\sigma(g^{TM}, \nabla^{F,f}, h^F)(s) = -\mathrm{Tr_s}\big[N^{\Lambda^\cdot(T^*M)}\sigma^M(\mathbf{D}^{M,F,2})^{-s}\big]. \tag{3.2.4}$$

By standard heat equation methods [See67], $\vartheta_\sigma(g^{TM}, \nabla^{F,f}, h^F)(s)$ extends to a meromorphic function of $s \in \mathbb{C}$, which is holomorphic near $s = 0$.

Definition 3.2.2. Put

$$\mathcal{T}_\sigma(g^{TM}, \nabla^{F,f}, h^F) = \frac{1}{2}\frac{\partial \vartheta_\sigma(g^{TM}, \nabla^{F,f}, h^F)}{\partial s}(0). \tag{3.2.5}$$

The quantity in (3.2.5) is called the equivariant Ray-Singer real analytic torsion.

We now explain a method to compute $\mathcal{T}_\sigma(g^{TM}, \nabla^{F,f}, h^F)$ by Mellin transform. For $t > 0$, as in [BL08, Eq.(1.8.5)], put

$$b_t(F,h^F) = \frac{1}{2}(1 + 2t\frac{\partial}{\partial t})\mathrm{Tr_s}\big[\big(N^{\Lambda^\bullet(T^*M)} - \frac{m}{2}\big)\sigma^M \exp(-t\mathbf{D}^{M,F,2}/4)\big]. \tag{3.2.6}$$

Put

$$\begin{aligned} \chi_\sigma(M,F) &= \sum_{j=0}^m (-1)^j \mathrm{Tr}^{H^j_{\mathrm{dR}}(M,F)}[\sigma], \\ \chi'_\sigma(M,F) &= \sum_{j=0}^m (-1)^j j\, \mathrm{Tr}^{H^j_{\mathrm{dR}}(M,F)}[\sigma]. \end{aligned} \tag{3.2.7}$$

By [BL08, Eqs.(1.8.7),(1.8.8)], we have

$$b_t(F,h^F) = \begin{cases} \mathcal{O}(\sqrt{t}) & \text{as } t \to 0, \\ \frac{1}{2}\chi'_\sigma(M,F) - \frac{m}{4}\chi_\sigma(M,F) + \mathcal{O}(1/\sqrt{t}) & \text{as } t \to +\infty, \end{cases} \tag{3.2.8}$$

where $\mathcal{O}(\cdot)$ is the big-O convention.

Set

$$b_\infty(F,h^F) = \frac{1}{2}\chi'_\sigma(M,F) - \frac{m}{4}\chi_\sigma(M,F). \tag{3.2.9}$$

Let $\Gamma(s)$ be the Gamma function. By [BL08, Eq.(1.8.11)], we have

$$\begin{aligned} \mathcal{T}_\sigma(g^{TM}, \nabla^{F,f}, h^F) = -\int_0^1 b_t(F,h^F)\frac{dt}{t} - \int_1^{+\infty}(b_t(F,h^F) - b_\infty(F,h^F))\frac{dt}{t} \\ -(\Gamma'(1) + 2(\log 2 - 1))b_\infty(F,h^F). \end{aligned} \tag{3.2.10}$$

3.3. A vanishing theorem on the equivariant analytic torsions

Let T be a maximal torus of K with Lie algebra $\mathfrak{t}$, put

$$\mathfrak{b} = \{f \in \mathfrak{p} \,:\, [f, \mathfrak{t}] = 0\}. \tag{3.3.1}$$

Put $\mathfrak{h} = \mathfrak{b} \oplus \mathfrak{t}$, then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Let H be analytic subgroup of G associated with $\mathfrak{h}$, then it is also a Cartan subgroup of G. Moreover, $\dim \mathfrak{t}$ is just the complex rank of K, and $\dim \mathfrak{h}$ is the complex rank of G.

Definition 3.3.1. Using the above notations, the deficiency of G, or the fundamental rank of G is defined as

$$\delta(G) = \mathrm{rk}_{\mathbb{C}} G - \mathrm{rk}_{\mathbb{C}} K = \dim_{\mathbb{R}} \mathfrak{b}. \tag{3.3.2}$$

The integer $m - \delta(G)$ is even.

We assume at first that $\gamma\sigma$ is a semisimple element given by (2.4.6), i.e.,

$$\gamma = e^a k^{-1},\ a \in \mathfrak{p},\ k \in K,\ \mathrm{Ad}(k^{-1})\sigma(a) = a. \tag{3.3.3}$$

Let S be a maximal torus of $K_\sigma(\gamma)^0$ with Lie algebra $\mathfrak{s} \subset \mathfrak{k}_\sigma(\gamma)$. Set

$$\mathfrak{b}_\sigma(\gamma) = \{f \in \mathfrak{p}_\sigma(k^{-1}) \mid [f, \mathfrak{s}] = 0\}. \tag{3.3.4}$$

Then

$$a \in \mathfrak{b}_\sigma(\gamma),\ \ \dim_{\mathbb{R}} \mathfrak{b}_\sigma(\gamma) \geq \delta(Z_\sigma(\gamma)^0). \tag{3.3.5}$$

In general, for $\gamma \in \Gamma$, if γ is C_σ-conjugate to $e^a k^{-1}$ as in (3.3.3), put

$$\varepsilon(\gamma\sigma) = \dim \mathfrak{b}_\sigma(e^a k^{-1}). \tag{3.3.6}$$

Note that

$$\varepsilon(\gamma\sigma) \geq \delta(Z_\sigma(\gamma)^0). \tag{3.3.7}$$

In particular, if $\gamma\sigma$ is elliptic, then $\varepsilon(\gamma\sigma) = \delta(Z_\sigma(\gamma)^0)$; if $\gamma\sigma$ is non-elliptic, then $\varepsilon(\gamma\sigma) \geq \delta(Z_\sigma(\gamma)^0) \geq 1$. The integer $\varepsilon(\gamma\sigma)$ depends only on the class $[\gamma]_\sigma \in [\Gamma]_\sigma$.

We now state a vanishing theorem on $\mathcal{T}_\sigma(g^{TM}, \nabla^{F,f}, h^F)$ as follows. For simplicity, we assume that the representation (E, ρ^E, h^E) of U^σ is irreducible.

Theorem 3.3.2. *If one of the following four assumptions is verified:*

(i) *m is even and σ preserves the orientation of $\mathfrak{p}$;*

(ii) *m is odd and σ does not preserve the orientation of $\mathfrak{p}$;*

(iii) *(E, ρ^E) is irreducible as U^σ-representation, but not irreducible when restricting to U;*

(iv) *For $\gamma \in \Gamma$, $\varepsilon(\gamma\sigma) \neq 1$, or $\delta(Z_\sigma(\gamma)^0) \neq 1$;*

then we have

$$\mathcal{T}_\sigma(g^{TM}, \nabla^{F,f}, h^F) = 0 \tag{3.3.8}$$

Before proving the above theorem, we need to do some computations on the twisted orbital integrals in order to evaluate the right-hand side of (3.2.6).

Let $N^{\Lambda^\bullet(\mathfrak{p}^*)}$, $N^{\Lambda^\bullet(T^*X)}$ be the number operators on $\Lambda^\bullet(\mathfrak{p}^*)$, $\Lambda^\bullet(T^*X)$.

Proposition 3.3.3. *Assume γ is given by (3.3.3). For $t>0$, we have*

$$\begin{aligned}&\mathrm{Tr_s}^{[\gamma\sigma]}\Big[\big(N^{\Lambda^\bullet(T^*X)}-\frac{m}{2}\big)\exp(-t\mathbf{D}^{X,F,2}/2)\Big]\\&=\frac{\exp(-|a|^2/2t)}{(2\pi t)^{p/2}}\exp\big(\frac{t}{48}\mathrm{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}]+\frac{t}{16}\mathrm{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}]\big)\int_{\mathfrak{k}_\sigma(\gamma)}J_{\gamma\sigma}(y)\\&\qquad\mathrm{Tr_s}^{\Lambda^\bullet(\mathfrak{p}^*)\otimes E}\Big[\big(N^{\Lambda^\bullet(\mathfrak{p}^*)}-\frac{m}{2}\big)\rho^{\Lambda^\bullet(\mathfrak{p}^*)\otimes E}(k^{-1}\sigma)\\&\qquad\qquad\exp\big(-i\rho^{\Lambda^\bullet(\mathfrak{p}^*)\otimes E}(y)+\frac{t}{2}C^{\mathfrak{g},E}\big)\Big]\exp(-|y|^2/2t)\frac{dy}{(2\pi t)^{q/2}}.\end{aligned}\tag{3.3.9}$$

If m is even and σ acting on $\mathfrak{p}$ preserves the orientation, or m is odd and σ does not preserve the orientation of $\mathfrak{p}$, or if $\dim\mathfrak{b}_\sigma(\gamma)\geq 2$, then (3.3.9) vanishes.

Proof. The identity (3.3.9) follows from (2.5.17), (3.1.9).

Inside the integrand in (3.3.9), the supertrace term splits as the product of the supertrace on $\Lambda^\bullet(\mathfrak{p}^*)$ and the trace on E. By a direct computation on matrix, we get that under the conditions listed in our proposition, for $y\in\mathfrak{k}_\sigma(\gamma)$,

$$\mathrm{Tr_s}^{\Lambda^\bullet(\mathfrak{p}^*)}\big[\big(N^{\Lambda^\bullet(\mathfrak{p}^*)}-\frac{m}{2}\big)\rho^{\Lambda^\bullet(\mathfrak{p}^*)}(k^{-1}\sigma)\exp(-i\rho^{\Lambda^\bullet(\mathfrak{p}^*)}(y))\big]=0.\tag{3.3.10}$$

This way, we complete the proof to our proposition. □

Corollary 3.3.4. *If $\gamma=k^{-1}\in K$, i.e., $\gamma\sigma$ is elliptic, and if $\dim\mathfrak{b}_\sigma(\gamma)=0$, then for $t>0$,*

$$\mathrm{Tr_s}^{[\gamma\sigma]}\big[\big(N^{\Lambda^\bullet(T^*X)}-\frac{m}{2}\big)\exp(-t\mathbf{D}^{X,F,2}/2)\big]=0.\tag{3.3.11}$$

Proof. Note that when $\gamma=k^{-1}\in K$, $\mathfrak{b}_\sigma(\gamma)\oplus\mathfrak{s}$ is a Cartan subalgebra of $\mathfrak{z}_\sigma(\gamma)$. If $\dim\mathfrak{b}_\sigma(\gamma)=0$, then $\dim\mathfrak{p}_\sigma(\gamma)$ is even. If σ preserves the orientation of $\mathfrak{p}$, then $\dim\mathfrak{p}_\sigma^\perp(\gamma)$ is even. If σ does not preserves the orientation of $\mathfrak{p}$, then $\dim\mathfrak{p}_\sigma^\perp(\gamma)$ is odd. By Proposition 3.3.3, we get (3.3.11). □

Recall that $\mathrm{Irr}(\cdot)$ denotes the set of equivalent classes of irreducible (complex) representations of a compact Lie group.

Proposition 3.3.5. *If $(E,\rho^E)\in\mathrm{Irr}(U^\sigma)$ and if the restriction of (E,ρ^E) to U is not irreducible, then for $k\in U$, we have*

$$\mathrm{Tr}^E[\rho^E(\sigma)\rho^E(k)]=0.\tag{3.3.12}$$

Moreover, in this case, if $\gamma\in G$ is such that $\gamma\sigma$ is semisimple, then for $t>0$,

$$\mathrm{Tr_s}^{[\gamma\sigma]}\big[\big(N^{\Lambda^\bullet(T^*X)}-\frac{m}{2}\big)\exp(-t\mathbf{D}^{X,F,2}/2)\big]=0.\tag{3.3.13}$$

Proof. We firstly assume that U is semisimple. Let $\mathrm{Inn}(U)$ denote the inner automorphism group of U. The outer automorphism group of U is

$$\mathrm{Out}(U) = \mathrm{Aut}(U)/\mathrm{Inn}(U). \tag{3.3.14}$$

By fixing a maximal torus T_U of U and a positive root system R^+, $\mathrm{Out}(U)$ can be realized as a finite subgroup of $\mathrm{Aut}(U)$ whose elements preserve T_U and R^+ [Bou04, Chapter VIII, §4.4 and Chapter IX, §4.10]. Moreover,

$$\mathrm{Aut}(U) = \mathrm{Inn}(U) \rtimes \mathrm{Out}(U). \tag{3.3.15}$$

Take $k_0 \in U$, $\tau \in \mathrm{Out}(U)$ such that for $k \in U$,

$$\sigma(k) = k_0 \tau(k) k_0^{-1}. \tag{3.3.16}$$

Let U^τ be the subgroup of $U \rtimes \mathrm{Out}(U)$ generated by U and τ. We claim that there exists $c_\tau \in \mathbb{C}$ such that if set

$$\rho^{E,\prime}(\tau) = c_\tau \rho^E(k_0^{-1})\rho^E(\sigma), \;\; \rho^{E,\prime}(k) = \rho^E(k), \tag{3.3.17}$$

then $(E, \rho^{E,\prime})$ is an irreducible representation of U^τ. Note that such number c_τ is not unique, it depends on the order of τ and the choice of k_0.

Indeed, set

$$A = \rho^E(k_0^{-1})\rho^E(\sigma) \in \mathrm{End}(E). \tag{3.3.18}$$

Let $N_0 \geq 1$ be the order of τ in $\mathrm{Out}(U)$. Set

$$\widehat{k} = k_0 \tau(k_0) \cdots \tau^{N_0 - 1}(k_0) \in U. \tag{3.3.19}$$

Then

$$\sigma(\widehat{k}) = \widehat{k} \in U, \; \sigma^{N_0} = \mathrm{Ad}(\widehat{k}) \in \mathrm{Inn}(U). \tag{3.3.20}$$

Also we have

$$A^{N_0} = \rho^E(\widehat{k}^{-1})\rho^E(\sigma^{N_0}). \tag{3.3.21}$$

We can verify directly that A^{N_0} commutes with U^σ. Since (E, ρ^E) is irreducible as U^σ-representation, then A^{N_0} is a non-zero scalar endomorphism of E, then we take $c_\tau \in \mathbb{C}$ such that $c_\tau^{N_0} A^{N_0} = \mathrm{Id}_E$.

We define $\rho^{E,\prime}$ as in (3.3.17). Then for $k \in U$,

$$\rho^{E,\prime}(\tau)\rho^{E,\prime}(k)\rho^{E,\prime}(\tau^{-1}) = \rho^{E,\prime}(\tau(k)). \tag{3.3.22}$$

Therefore, $(E, \rho^{E,\prime})$ become an irreducible representation of U^τ.

For proving (3.3.12), it is enough to prove that for $k \in U$, one has

$$\mathrm{Tr}^E[\rho^{E,\prime}(\tau)\rho^{E,\prime}(k)] = 0. \tag{3.3.23}$$

Let P_{++} be the dominant weights for the pair (U, T_U) with respect to R^+. Then τ acts on P_{++}. If $\lambda \in P_{++}$, let $V_\lambda \in \mathrm{Irr}(U)$ denote the one with the highest weight λ.

Now we take a dominant weight $\lambda \in P_{++}$ such that V_λ embeds into (E, ρ^E) as a U-subrepresentation. Let $\{\tau^i(\lambda)\}_{i=0}^{d-1} \subset P_{++}$ be the orbit of λ under the action of τ. Note that $d \geq 1$ is the length of the orbit and $d \,|\, N_0$. By the description of all the irreducible representations of non-connected compact Lie groups in [DK00, Corollary 4.13.2 and Proposition 4.13.3], we get that the representation $(E, \rho^{E,\prime})$ restricting on U is of the form

$$\oplus_{i=0}^{d-1} V_{\tau^i(\lambda)}. \tag{3.3.24}$$

Moreover, the action $\rho^{E,\prime}(\tau)$ on E sends the component $V_{\tau^i(\lambda)}$ to $V_{\tau^{i+1}(\lambda)}$.

If (E, ρ^E) restricting to U is not irreducible, then $d \geq 2$, and (3.3.23) holds, so does (3.3.12). The identity (3.3.13) follows from (3.3.9) and (3.3.12).

If U is not semisimple, let Z_U^0 be the identity component of the center of U, and let U_{ss} be the analytic subgroup of U associated with the semisimple subalgebra $\mathfrak{u}_{\mathrm{ss}} = [\mathfrak{u}, \mathfrak{u}]$. Then $Z_U^0 \times U_{\mathrm{ss}}$ is a finite cover of U. Note that Z_U^0 is a torus, the action of σ on it is of finite order. Then if we proceed as in the above for U_{ss}, we can still apply [DK00, Corollary 4.13.2 and Proposition 4.13.3] to get (3.3.12). This completes the proof of our proposition. □

Proof to Theorem 3.3.2. If m and σ verify either of the first two cases in our theorem, then by (2.6.14) and Proposition 3.3.3, for $t > 0$,

$$\mathrm{Tr_s}\big[\big(N^{\Lambda^\cdot(T^*Z)} - \frac{m}{2}\big)\sigma^Z \exp(-t\mathbf{D}^{Z,F,2}/4)\big] = 0. \tag{3.3.25}$$

By (3.2.6), (3.3.25), the function $b_t(F, g^F)$ vanishes identically. In particular,

$$b_\infty(F, g^F) = 0. \tag{3.3.26}$$

Then by (3.2.10), we get (3.3.8).

If $(E, \rho^E) \in \mathrm{Irr}(U^\sigma)$ is not irreducible when restricting to U, then by Proposition 3.3.5, we get that (3.3.25), (3.3.26) still hold. Then (3.3.8) follows.

If $\gamma \in \Gamma$ is such that $\gamma\sigma$ is nonelliptic, then $\varepsilon(\gamma\sigma) \geq 1$. If the fourth assumption is verified, then by Theorem 3.3.3, Corollary 3.3.4, the identity (3.3.25) still holds, which implies (3.3.8). This completes the proof to our theorem. □

4. The asymptotics of the equivariant real analytic torsion

In this section, we compute the asymptotics of the equivariant Ray-Singer analytic torsions associated with a certain sequence of flat vector bundles on a compact locally symmetric space $M = \Gamma\backslash X$. We extend the results of [MP13a], [BMZ17, Section 8] to the equivariant setting.

This section is organized as follows. In Subsection 4.1, we recall the construction of the W-invariant on $X = G/K$ under a nondegeneracy condition. This construction will be applied to $X(\gamma\sigma) = Z_\sigma(\gamma)^0/K_\sigma(\gamma)^0$ with $\gamma \in K$.

In Subsection 4.2, for an irreducible U^σ-representation with a σ-fixed highest weight λ, we construct a canonical sequence of representations E_d, $d \in \mathbb{N}$ of U^σ. This way, we get a sequence of flat vector bundles F_d on X or M.

In Subsection 4.3, we show that the nondegeneracy condition of λ for G implies the nondegeneracy condition of λ for $Z_\sigma(\gamma)^0$ with $\gamma \in K$.

In Subsection 4.4, when $\gamma \in K$ and $\dim \mathfrak{b}_\sigma(\gamma) = 1$, for $t > 0$, we compute the asymptotics as $d \to +\infty$ of $\mathrm{Tr_s}^{[\gamma\sigma]}\big[\big(N^{\Lambda^\bullet(T^*X)} - \frac{m}{2}\big)\exp(-t\mathbf{D}^{X,F_d,2}/2d^2)\big]$.

In Subsection 4.5, we recall some results on the spectral gap of Hodge Laplacian obtained in [BMZ17, Section 4] under the nondegeneracy condition.

Finally, in Subsection 4.6, we give a proof to Theorem 1.0.1.

4.1. The forms e_t, d_t and the W-invariant

Let $S\mathfrak{g}$ be the symmetric algebra of $\mathfrak{g}$, which can be identified with the algebra of real differential operators with constant coefficients on $\mathfrak{g}$. By Poincaré-Birkhoff-Witt theorem, let $\boldsymbol{\sigma} : U\mathfrak{g} \to S\mathfrak{g}$ be the symbol map of $U\mathfrak{g}$, which is an isomorphism of vector spaces. Let $\widehat{\mathfrak{p}}$ be another copy of $\mathfrak{p}$. Together with the symbol map of Clifford algebras, we get a symbol map

$$\boldsymbol{\sigma} : \widehat{c}(\widehat{\mathfrak{p}}) \otimes U\mathfrak{g} \to \Lambda^\bullet(\widehat{\mathfrak{p}}^*) \otimes S\mathfrak{g}, \tag{4.1.1}$$

which is an identification of filtered $\mathbb{Z}_2$-graded vector spaces.

Let $e_1, \cdots, e_m$ be an orthonormal basis of $\mathfrak{p}$, then $\widehat{e}_1, \cdots, \widehat{e}_m$ is a basis of $\widehat{\mathfrak{p}}$, and let $\widehat{e}^1, \cdots, \widehat{e}^m$ be the corresponding dual basis of $\widehat{\mathfrak{p}}^*$. Put

$$\beta = \sum_{i=1}^m \widehat{e}^i e_i \in \widehat{\mathfrak{p}}^* \otimes \mathfrak{g}. \tag{4.1.2}$$

By [BMZ17, Eq.(1.8)], $\beta^2 \in \Lambda^2(\widehat{\mathfrak{p}}^*) \otimes \mathfrak{k}$ is given by

$$\beta^2 = \frac{1}{2}[\beta,\beta] = \frac{1}{2}\widehat{e}^i\widehat{e}^j[e_i, e_j]. \tag{4.1.3}$$

Let $\underline{\beta}$ be the corresponding element of β in $\Lambda^\bullet(\widehat{\mathfrak{p}}^*)\otimes U\mathfrak{g}$. Then $\underline{\beta}^2 \in \Lambda^2(\widehat{\mathfrak{p}}^*)\otimes U\mathfrak{g}$ coincides with β^2 in (4.1.3). Let $\Delta^{\mathfrak{p}}$ be the Laplacian of Euclidean vector space $\mathfrak{p}$. Set

$$|\beta|^2 = \sum_{i=1}^m e_i^2 = \Delta^{\mathfrak{p}} \in S\mathfrak{g},\ |\underline{\beta}|^2 = \sum_{i=1}^p \underline{\beta}(\widehat{e}_i)^2 \in U\mathfrak{g}. \tag{4.1.4}$$

By [BMZ17, Eqs.(1.10), (1.14)], we have

$$\begin{aligned} &|\beta|^2 \in S^2\mathfrak{g} \cap S^2\mathfrak{u},\ |\beta|^2 = -|i\beta|^2 \in S^2\mathfrak{g}_{\mathbb{C}}, \\ &|\underline{\beta}|^2 \in U\mathfrak{g} \cap U\mathfrak{u},\ |\underline{\beta}|^2 = -|i\underline{\beta}|^2 \in U\mathfrak{g}_{\mathbb{C}}. \end{aligned} \tag{4.1.5}$$

Then

$$\boldsymbol{\sigma}(|\underline{\beta}|^2) = |\beta|^2. \tag{4.1.6}$$

Set

$$\widehat{c}(\underline{\beta}) = \sum_{i=1}^m \widehat{c}(\widehat{e}_i)\underline{\beta}(\widehat{e}_i) \in \widehat{c}(\widehat{\mathfrak{p}}) \otimes U\mathfrak{g}. \tag{4.1.7}$$

Then we have

$$\boldsymbol{\sigma}(\widehat{c}(\underline{\beta})) = \beta. \tag{4.1.8}$$

Let $\mathfrak{g}_r$ be a copy of the vector bundle $G \times_K \mathfrak{g}$ on X but equipped with the Lie bracket on the fibre. Similarly, put

$$U\mathfrak{g}_r = G \times_K U\mathfrak{g},\ S\mathfrak{g}_r = G \times_K S\mathfrak{g}. \tag{4.1.9}$$

Let $\widehat{TX}$ (resp. $\widehat{T^*X}$) be another copies of TX (resp. T^*X) on X. Recall that ∇^{TX} is the Levi-Civita connection of TX. Let $\nabla^{\mathfrak{g}_r,\widehat{u}}$ be the connections on $\widehat{T^*X} \otimes \mathfrak{g}_r$ induced by the connection form $\omega^{\mathfrak{k}}$, and let $\nabla^{U\mathfrak{g}_r,\widehat{u}}$ be the connections on $\widehat{T^*X} \otimes U\mathfrak{g}_r$ induced by $\omega^{\mathfrak{k}}$. We still denote by $\nabla^{U\mathfrak{g}_r,\widehat{u}}$ the corresponding connection on $\widehat{c}(\widehat{TX}) \otimes U\mathfrak{g}_r$.

Then $\omega^{\mathfrak{p}}$ can be considered as a section of $T^*X \otimes \mathfrak{g}_r$, and β, $\underline{\beta}$ can be considered as a section of $\widehat{T^*X} \otimes \mathfrak{g}_r$, $\widehat{T^*X} \otimes U\mathfrak{g}_r$ respectively. By [BMZ17, Eq.(1.41)], we have

$$\nabla^{\mathfrak{g}_r,\widehat{u}}\beta = 0,\ \nabla^{U\mathfrak{g}_r,\widehat{u}}\underline{\beta} = 0. \tag{4.1.10}$$

Definition 4.1.1. For $t \geq 0$, let $\mathcal{A}_t$ be the superconnection

$$\mathcal{A}_t = \nabla^{U\mathfrak{g}_r,\widehat{u}} + \sqrt{t}\widehat{c}(\underline{\beta}). \tag{4.1.11}$$

By [BMZ17, Def. 1.2], $\mathcal{A}_t^2$ is a smooth section of $[\Lambda^\bullet(T^*X)\widehat{\otimes}\widehat{c}(\widehat{TX})]^{\text{even}} \otimes U\mathfrak{g}_r$, so that $\boldsymbol{\sigma}(\mathcal{A}_t^2)$ is a smooth section of $[\Lambda^\bullet(T^*X)\widehat{\otimes}\Lambda^\bullet(\widehat{T^*X})]^{\text{even}} \otimes S\mathfrak{g}_r$.

If $\mu, \nu \in \Lambda^\bullet(\mathfrak{p}^*)$ or $\Lambda^\bullet(\widehat{\mathfrak{p}}^*)$, $a, b \in \mathfrak{k}$, we define

$$\langle \mu \otimes a, \nu \otimes b\rangle' = \mu \wedge \nu\langle a, b\rangle. \tag{4.1.12}$$

By [BMZ17, Theorem 1.3 & Eq.(8.70)], we have

$$\boldsymbol{\sigma}(\mathcal{A}_t^2) = -\frac{1}{2}\langle \omega^{\mathfrak{p},2}, \beta^2\rangle' - \omega^{\mathfrak{p},2} + t|\beta|^2 + t\beta^2. \tag{4.1.13}$$

Let N be a compact complex manifold, and let η^N be a smooth real closed nondegenerate $(1,1)$-form on N. We assume that U acts holomorphically on N and preserves the form η^N. Let $\mu : N \to \mathfrak{u}^*$ be the moment map associated with the action of U and η^N.

If $y \in \mathfrak{u}$, set

$$\widetilde{R}(y) = \int_N \exp(2\pi i\langle \mu, y\rangle + \eta^N). \tag{4.1.14}$$

Then $\widetilde{R}$ is U-invariant function, we can extend it to a holomorphic function $\mathfrak{u}_{\mathbb{C}} \to \mathbb{C}$. If $y \in \mathfrak{u}_{\mathbb{C}}$, let $\mathrm{Im}(y)$ denote the component of y in $i\mathfrak{u}$.

The algebra $S\mathfrak{u}$ acts on $\widetilde{R}(y)$, by [BMZ17, Eq.(1.24)],

$$\exp(-t|\beta|^2)\widetilde{R}(y) = \int_N \exp(-4\pi^2 t|\langle \mu, i\beta\rangle|^2 + 2\pi i\langle \mu, y\rangle + \eta_N). \tag{4.1.15}$$

We regard $\mathfrak{k}^*$ as a subspace of $\mathfrak{u}^*$ by the metric dual of $\mathfrak{k} \subset \mathfrak{u}$.

Definition 4.1.2. We say that (N, μ) is nondegenerate (with respect to $\omega^{\mathfrak{p}}$) if

$$\mu(N) \cap \mathfrak{k}^* = \emptyset. \tag{4.1.16}$$

Equivalently, there exists $c > 0$ such that

$$|\langle \mu, i\beta \rangle|^2 \geq c. \tag{4.1.17}$$

By [BMZ17, Eq.(1.27)], if (N, μ) is nondegenerate, there exists $C_0 > 0$, $C_1 > 0$ such that, if $y \in \mathfrak{u}_{\mathbb{C}}$,

$$|\exp(-t|\beta|^2)\widetilde{R}(y)| \leq C_0 \exp(-tc + C_1|\mathrm{Im}(y)|). \tag{4.1.18}$$

If there is no confusion, we also say that the function $\widetilde{R}$ is nondegenerate with respect to $\omega^{\mathfrak{p}}$.

Definition 4.1.3. The Berezin integral $\int^{\widehat{B}} : \Lambda^\bullet(T^*X)\widehat{\otimes}\Lambda^\bullet(\widehat{T^*X}) \to \Lambda^\bullet(T^*X)$ is a linear map such that, if $\alpha \in \Lambda^\bullet(T^*X)$, $\alpha' \in \Lambda^\bullet(\widehat{T^*X})$,

$$\begin{aligned} &\int^{\widehat{B}} \alpha\alpha' = 0, \text{ if } \deg \alpha' < m; \\ &\int^{\widehat{B}} \alpha\widehat{e}^1 \wedge \cdots \wedge \widehat{e}^m = \frac{(-1)^{m(m+1)/2}}{\pi^{m/2}}\alpha. \end{aligned} \tag{4.1.19}$$

More generally, let $o(\widehat{\mathfrak{p}})$ be the orientation line of $\widehat{\mathfrak{p}}$, which can be identified with $o(\mathfrak{p})$. Then $\int^{\widehat{B}}$ defines a map from $\Lambda^\bullet(T^*X)\widehat{\otimes}\Lambda^\bullet(\widehat{T^*X})$ into $\Lambda^\bullet(T^*X)\widehat{\otimes}o(\widehat{\mathfrak{p}})$.

Let ψ be the endomorphism of $\Lambda^\bullet(T^*X)\otimes_{\mathbb{R}}\mathbb{C}$ which maps $\alpha \in \Lambda^k(T^*X)\otimes_{\mathbb{R}}\mathbb{C}$ into $(2\pi i)^{-k/2}\alpha$. Set

$$L = \sum_{i=1}^m e^i \wedge \widehat{e}^i. \tag{4.1.20}$$

Definition 4.1.4. For $t \geq 0$, set

$$\begin{aligned} d_t &= -(2\pi i)^{m/2}\psi \int^{\widehat{B}} \sqrt{t}\frac{\omega^{\mathfrak{p}} \wedge \beta}{2} \exp(-\boldsymbol{\sigma}(\mathcal{A}_t^2))\widetilde{R}(0), \\ e_t &= (2\pi i)^{m/2}\psi \int^{\widehat{B}} \frac{L}{4\sqrt{t}} \exp(-\boldsymbol{\sigma}(\mathcal{A}_t^2))\widetilde{R}(0). \end{aligned} \tag{4.1.21}$$

Then d_t, e_t are smooth real forms on X.

Note that the action of G on X lifts to $\mathfrak{g}_r$, $U\mathfrak{g}_r$ and $S\mathfrak{g}_r$. Then the sections $\omega^{\mathfrak{p}}$, β, $\underline{\beta}$ are G-invariant. Therefore, e_t, d_t are G-invariant forms, so that they are determined by their values at the point $p1 \in X$.

Let Ψ be the canonical element of norm 1 in $\Lambda^m(\mathfrak{p}^*) \otimes o(\mathfrak{p})$ (respectively, a section of norm 1 of $\Lambda^m(T^*X) \otimes o(TX)$). For $\alpha \in \Lambda^\bullet(\mathfrak{p}^*) \otimes o(\mathfrak{p})$ (respectively $\Lambda^\bullet(T^*X) \otimes o(TX)$), for $0 \leq l \leq m$, let $\alpha^{(l)}$ be the component of α of degree l. We define $[\alpha]^{\max} \in \mathbb{R}$ by

$$\alpha^{(m)} = [\alpha]^{\max}\Psi. \tag{4.1.22}$$

Then $[d_t]^{\max}$, $[e_t]^{\max}$ are constant on X. By [BMZ17, Theorem 2.10],

$$(1 + 2t\frac{\partial}{\partial t})[e_t]^{\max} = [d_t]^{\max}. \tag{4.1.23}$$

Also if (N, μ) is nondegenerate, there exists $c > 0$ such that, on X, as $t \to +\infty$,

$$d_t = \mathcal{O}(e^{-ct}), \quad e_t = \mathcal{O}(e^{-ct}). \tag{4.1.24}$$

Definition 4.1.5. If (N, μ) is nondegenerate, set

$$W = -\int_0^{+\infty} d_t \frac{dt}{t}. \tag{4.1.25}$$

Then W is a G-invariant smooth form on X with values in $o(TX)$, so that $[W]^{\max}$ is a real constant.

As explained in Introduction, in [BMZ17], the authors showed that W appears naturally as the leading term in the asymptotic analytic torsions of M. The quantity $\mathrm{Vol}(M)[W]^{\max}$ is called a W-invariant, we refer to [MP13b, MP13a], [BMZ17, Section 8], [Liu20, Subsections 7.3 & 7.4] for more concrete computations on them. Here, we use abusively this name for the form W.

The purpose of the rest of this paper is to develop an analog of [BMZ17, Section 8] in the context of the equivariant analytic torsions. If $(E, \rho^E) \in \mathrm{Irr}(U^\sigma)$ is not irreducible when restricting to U, then by Proposition 3.3.2,

$$\mathcal{T}_\sigma(g^{TM}, \nabla^{F,f}, h^F) = 0 \tag{4.1.26}$$

Then the only non-trivial case is that (E, ρ^E) is also a U-irreducible representation, then it will correspond to a σ-fixed dominant weight λ of U. In the next subsections, we will construct a sequence of flat vector bundles F_d, $d \in \mathbb{N}$ associated with this λ and ρ^E. In Subsection 4.6, we will show that the leading term of asymptotic $\mathcal{T}_\sigma(g^{TM}, \nabla^{F_d,f}, h^{F_d})$ as $d \to +\infty$ is described in terms of W-invariants of ${}^\sigma M$, the fixed point set of σ in M.

4.2. A sequence of unitary representations of U^σ

Let $\mathfrak{u}^{\mathrm{reg}}$ be the set of regular elements in $\mathfrak{u}$. Recall that $\mathfrak{u}_{\mathrm{ss}} = [\mathfrak{u}, \mathfrak{u}]$ is semisimple and that U_{ss} is the associated analytic subgroup of U. By [DK00, Lemma (3.15.4)], $[\mathfrak{u}, \mathfrak{u}](\sigma)$ contains regular elements in $[\mathfrak{u}, \mathfrak{u}]$. Then there exists $v \in \mathfrak{u}(\sigma) \cap \mathfrak{u}^{\mathrm{reg}}$. If $\mathfrak{t}_U = \mathfrak{u}(v)$, then $\mathfrak{t}_U$ is a Cartan subalgebra of $\mathfrak{u}$. Let $T_U \subset U$ be the corresponding maximal torus. Let R_U be the associated (real) root system, and let W_U be the associated Weyl group. Let $\mathfrak{c} \subset \mathfrak{t}_U$ be the Weyl chamber containing v, and let $R_U^+(\mathfrak{c})$ denote the corresponding positive root system (i.e. the root $\alpha \in R_U$ such that $\alpha(v) > 0$). Let $P_{++}(\mathfrak{c})$ be the set of the dominant weights on $\mathfrak{u}$ with respect to $\mathfrak{c}$. Then σ acts on $\mathfrak{t}_U$ and on its dual, which preserves $R_U^+(\mathfrak{c})$ and $P_{++}(\mathfrak{c})$.

If $(E, \rho^E) \in \mathrm{Irr}(U^\sigma)$ is irreducible as U-representation with highest weight $\lambda \in P_{++}(\mathfrak{c})$, then σ fixes λ, i.e., $\lambda \in \mathfrak{a}^*$. Actually, the converse also holds true, i.e., if $\lambda \in P_{++}(\mathfrak{c})$ is fixed by σ-action, then the corresponding irreducible (complex) U-representation $(E_\lambda, \rho^{E_\lambda})$ extends to a representation of U^σ. As explained in Remark 2.3.1, such extension is not unique, they are different by twisting with elements in $\mathrm{Irr}(\Sigma^\sigma)$.

From now on, we fix a $\lambda \in P_{++}(\mathfrak{c})$ such that $\sigma\lambda = \lambda$, then we construct a sequence of irreducible representations (E_d, ρ^{E_d}), $d \in \mathbb{N}$ of U^σ such that each (E_d, ρ^{E_d}) is an irreducible U-representation with highest weight $d\lambda$. In general, such sequence is not unique. Here, we use the flag manifold N_λ to get a canonical construction in the sense that it is determined uniquely by $(E_1, \rho^{E_1}) \in \mathrm{Irr}(U^\sigma)$.

More precisely, set

$$U^\sigma(\lambda) = \{u \in U^\sigma \mid \mathrm{Ad}(u)\lambda = \lambda\}, \;\; U(\lambda) = U^\sigma(\lambda) \cap U. \tag{4.2.1}$$

Then

$$U^\sigma(\lambda) = U(\lambda) \rtimes \Sigma^\sigma. \tag{4.2.2}$$

By [Wal73, Lemma 6.2.2], $U(\lambda)$ is a connected. Moreover, $T_U \subset U(\lambda)$.

Note that T_U is also a maximal torus of $U(\lambda)$. Let $R_{U(\lambda)}$ be the associated (real) root system of $U(\lambda)$, then $R_{U(\lambda)} = \{\alpha \in R_U \mid \langle \alpha, \lambda \rangle = 0\}$. Let $\mathfrak{c}_1$ denote the Weyl chamber containing v for $(\mathfrak{u}(\lambda)_\mathbb{C}, \mathfrak{t}_U)$. Then $R^+_{U(\lambda)}(\mathfrak{c}_1) = R^+_U(\mathfrak{c}) \cap R_{U(\lambda)}$ is the corresponding positive root system of $R_{U(\lambda)}$. Note that λ is also a dominant weight for $(U(\lambda), T_U)$ with respect to $R^+_{U(\lambda)}(\mathfrak{c}_1)$.

If $\alpha \in R^+_U(\mathfrak{c}) \backslash R^+_{U(\lambda)}(\mathfrak{c}_1)$, $\beta \in R_{U(\lambda)}$ and $\alpha + \beta$ is a (real) root, then $\langle \alpha, \lambda \rangle > 0$ so that $\alpha + \beta \in R^+_U(\mathfrak{c}) \backslash R^+_{U(\lambda)}(\mathfrak{c}_1)$. Set

$$\mathfrak{b}_+ = \sum_{\alpha \in R^+_U(\mathfrak{c}) \backslash R^+_{U(\lambda)}(\mathfrak{c}_1)} \mathfrak{u}_\alpha, \tag{4.2.3}$$

then

$$[\mathfrak{u}(\lambda), \mathfrak{b}_+] \subset \mathfrak{b}_+, \;\; [\mathfrak{b}_+, \mathfrak{b}_+] \subset \mathfrak{b}_+. \tag{4.2.4}$$

Moreover, σ preserves $\mathfrak{b}_+$.

Set

$$N_\lambda = U/U(\lambda) = U^\sigma / U^\sigma(\lambda). \tag{4.2.5}$$

Then by [Wal73, Lemma 6.2.13], N_λ has a complex structure such that the holomorphic tangent bundle TN_λ is

$$TN_\lambda = U \times_{U(\lambda)} \mathfrak{b}_+ = U^\sigma \times_{U^\sigma(\lambda)} \mathfrak{b}_+. \tag{4.2.6}$$

Moreover, U^σ acts holomorphically on N_λ. Put $n_\lambda = \dim_\mathbb{C} N_\lambda$.

Lemma 4.2.1. *Let $(V^\lambda, \rho^{V^\lambda}) \in \mathrm{Irr}(U(\lambda))$ be the one with highest weight λ. Then $\dim_\mathbb{C} V^\lambda = 1$, and for $u \in U(\lambda)$,*

$$\rho^{V^\lambda}(\sigma(u)) = \rho^{V^\lambda}(u). \tag{4.2.7}$$

Therefore, after tensoring $(V^\lambda, \rho^{V^\lambda})$ with any element in $\mathrm{Irr}(\Sigma^\sigma)$, we extend it as a representation of $U^\sigma(\lambda)$.

Proof. Note that if $\alpha \in R^+_{U(\lambda)}(\mathfrak{c}_1)$, then $\langle \alpha, \lambda \rangle = 0$. Set

$$\rho_{\mathfrak{u}(\lambda)} = \frac{1}{2} \sum_{\alpha \in R^+_{U(\lambda)}(\mathfrak{c}_1)} \alpha. \tag{4.2.8}$$

By the dimension formula [BtD85, Chapter VI, Theorem (1.7)], we have

$$\dim_{\mathbb{C}} V^\lambda = \prod_{\alpha \in R^+_{U(\lambda)}(\mathfrak{c}_1)} \frac{\langle \alpha, \lambda + \rho_{\mathfrak{u}(\lambda)} \rangle}{\langle \alpha, \rho_{\mathfrak{u}(\lambda)} \rangle} = 1. \tag{4.2.9}$$

Since σ fixes λ, we get $(V^\lambda, \rho^{V^\lambda}) \simeq (V^\lambda, \rho^{V^\lambda} \circ \sigma) \in \mathrm{Irr}(U(\lambda))$. Then (4.2.7) follows, so that it extends to $U^\sigma(\lambda)$. This completes the proof of our lemma. □

We fix an extension $(V^\lambda, \rho^{V^\lambda}) \in \mathrm{Irr}(U^\sigma(\lambda))$ as in Lemma 4.2.1. Put

$$L_\lambda = U^\sigma \times_{U^\sigma(\lambda)} V^\lambda. \tag{4.2.10}$$

By [Wal73, Proposition 6.3.3], L_λ is a holomorphic line bundle on N_λ on which U^σ acts holomorphically. If $d \in \mathbb{N}_{>0}$, put

$$E_d = H^{(0,0)}(N_\lambda, L^d_\lambda). \tag{4.2.11}$$

Then each (E_d, ρ^{E_d}) is a unitary representation of U^σ, which is also an irreducible representation of U with highest weight $d\lambda \in P_{++}(\mathfrak{c})$.

Remark 4.2.2. Let $(E, \rho^E) \in \mathrm{Irr}(U^\sigma)$ be irreducible as U-representation with highest weight $\lambda \in P_{++}(\mathfrak{c})$. Let $E^{\mathfrak{b}_+} \subset E$ be the vector space

$$E^{\mathfrak{b}_+} = \{ w \in E \ : \ \text{if } f \in \mathfrak{b}_+, \text{ then } \rho^E(f) w = 0 \}. \tag{4.2.12}$$

Then $E^{\mathfrak{b}_+}$ is preserved by $U^\sigma(\lambda)$, which is exactly the irreducible representation of $U(\lambda)$ with highest weight λ. Then by the dimension formula (4.2.9), we get $\dim_{\mathbb{C}} E^{\mathfrak{b}_+} = 1$, so that it is just the highest weight line (λ-eigenspace) of (E, ρ^E). In (4.2.10), if we take $V^\lambda = E^{\mathfrak{b}_+}$ to define L_λ, then by [Wal73, Theorem 6.3.7], we have $(E_1, \rho^{E_1}) = (E, \rho^E)$ as U^σ-representation.

Let χ_d be the character of (E_d, ρ^{E_d}) on U^σ. In the sequel, we study the asymptotics of $\chi_d(u_0 \sigma e^{y/d})$ as $d \to +\infty$ for $u_0 \in U$, $y \in \mathfrak{u}_\sigma(u_0)$.

Set $U(\sigma) = U_\sigma(1)$ and $\mathfrak{u}(\sigma) = \mathfrak{u}_\sigma(1)$. Put $\mathfrak{a} = \mathfrak{t}_U \cap \mathfrak{u}(\sigma)$. Then $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{u}(\sigma)$. Let $A \subset U(\sigma)^0$ be the corresponding maximal torus. If $u_0 \in U$, then by [Seg68, Proposition I.4], there exists $u \in U$, $t_0 \in A$ such that

$$u_0 = u t_0 \sigma(u^{-1}). \tag{4.2.13}$$

Put $\underline{Z} = U_\sigma(u_0)$, the σ-twisted centralizer of u_0 in U. Let $\underline{\mathfrak{z}} \subset \mathfrak{u}$ be its Lie algebra. By (4.2.13), we get

$$\underline{Z} = u U_\sigma(t_0) u^{-1}, \ \ \underline{Z}^0 = u U^0_\sigma(t_0) u^{-1}. \tag{4.2.14}$$

Then $\mathrm{Ad}(u)(A)$ is a maximal torus of $\underline{Z}^0$.

Let $N_U(T_U)$ be the normalizer of T_U in U. Put

$$N_U(T_U)(\sigma) = \{g \in N_U(T_U) \mid \mathrm{Ad}(g)|_{\mathfrak{t}_U} \text{ commutes with } \sigma|_{\mathfrak{t}_U}\}. \tag{4.2.15}$$

Let $N_U(A)$ be the normalizer of A in U, then

$$N_U(T_U)(\sigma) = N_U(A). \tag{4.2.16}$$

If $g \in N_U(A)$, then

$$\mathrm{Ad}(g)\lambda \in \mathfrak{a}^*. \tag{4.2.17}$$

Let ${}^{u_0\sigma}N_\lambda$ be the fixed point set of $u_0\sigma$ in N_λ, which is a complex submanifold (it may have several connected components). If $u' \in U$, it depends to a point $[u']_\lambda \in N_\lambda$. Recall that $v \in \mathfrak{c} \cap \mathfrak{u}^{\mathrm{reg}} \cap \mathfrak{u}(\sigma)$.

Lemma 4.2.3. *We have*

$${}^{u_0\sigma}N_\lambda = \underline{Z}^0 u N_U(A) U(\lambda)/U(\lambda) \subset N_\lambda. \tag{4.2.18}$$

Let $\mathcal{J}(u_0)$ denote the index set for connected components of ${}^{u_0\sigma}N_\lambda$, then $\mathcal{J}(u_0)$ is a finite set.

If $u' \in uN_U(A)$, and if we take the Weyl chamber of $(\underline{\mathfrak{z}}, \mathrm{Ad}(u)\mathfrak{a})$ containing $\mathrm{Ad}(u')v$, then $\mathrm{Ad}(u')\lambda$ is a dominant weight for $\underline{Z}^0$. Then the connected component of $[u']_\lambda$ is isomorphic to the flag manifold $\underline{Z}^0/\underline{Z}^0(\mathrm{Ad}(u')\lambda)$ as complex manifolds. Under this identification, $H^{(0,0)}(\underline{Z}^0/\underline{Z}^0(\mathrm{Ad}(u')\lambda), L_\lambda)$ is the irreducible representation of $\underline{Z}^0$ with highest weight $\mathrm{Ad}(u')\lambda$.

Proof. Let $\mathcal{O}_\lambda \subset \mathfrak{u}^*$ denote the orbit of λ by the adjoint action of U. Then $N_\lambda \simeq \mathcal{O}_\lambda$. Then the fixed point set of $u_0\sigma$ is just $\mathcal{O}_\lambda \cap \underline{\mathfrak{z}}^*$. Let η_λ be the canonical symplectic form on $\mathcal{O}_\lambda \subset \mathfrak{u}^*$ [BGV04, Sections 7.5, 8.2]. Then

$$c_1(L_\lambda, g^{L_\lambda}) = \eta_\lambda. \tag{4.2.19}$$

The corresponding moment map $\mu : N_\lambda \to \mathfrak{u}^*$ associated with the U-action is just the embedding $i : \mathcal{O}_\lambda \subset \mathfrak{u}^*$.

If λ is regular, then (4.2.18) follows exactly from [DHV84, I.2 : Lemme (7)] and [Bou87, Lemme 6.1.1]. In general, (4.2.18) is a consequence of [Bou87, Lemma 7.2.2]. This proves the first part of our lemma.

Fix $u' \in uN_U(A)$ and $x = [u']_\lambda \in {}^{u_0\sigma}N_\lambda$. The stabilizer of x under the action of $\underline{Z}^0$ is $\underline{Z}^0(\mathrm{Ad}(u')\lambda)$. Then we can identify the connected component of $[u']_\lambda$ in ${}^{u_0\sigma}N_\lambda$ with the quotient $\underline{Z}^0/\underline{Z}^0(\mathrm{Ad}(u')\lambda)$ as $\underline{Z}^0$-manifolds.

Let $N_U(\mathfrak{c})$ be the normalizer of $\mathfrak{c}$ in U. Put

$$t = (u')^{-1}u_0\sigma(u') \in N_U(A). \tag{4.2.20}$$

A direct computation shows that $t \in N_U(\mathfrak{c}) \cap U(\lambda)$. Then the action of $u_0\sigma$ on T_xN_λ is identified with the adjoint action of $t\sigma$ on $\mathfrak{b}_+$, so that

$$T_x{}^{u_0\sigma}N_\lambda = \mathfrak{b}_+(t\sigma). \tag{4.2.21}$$

Note that $\mathrm{Ad}(u')\mathfrak{b}_+(t\sigma) \subset \underline{\mathfrak{z}}_{\mathbb{C}}$. By taking the Weyl chambers containing $\mathrm{Ad}(u')v$ for $\underline{\mathfrak{z}}$ and $\underline{\mathfrak{z}}(\mathrm{Ad}(u')(\lambda))$ with respect to $\mathrm{Ad}(u')\mathfrak{a}$, similar to (4.2.3) - (4.2.6), we get a complex structure on $\underline{Z}^0/\underline{Z}^0(\mathrm{Ad}(u')\lambda)$ such that the holomorphic tangent bundle is given by $\mathrm{Ad}(u')\mathfrak{b}_+(t\sigma)$, which is exactly the same one inherited from the complex structure of N_λ.

Since $\lambda \in P_{++}(\mathfrak{c})$, $\mathrm{Ad}(u')\lambda$ is a dominant weight for $\underline{Z}^0$ with respect to the above Weyl chamber. Using the identification $N_\lambda \simeq \mathcal{O}_\lambda$, we get that L_λ restricting to the connected component $\underline{Z}^0/\underline{Z}^0(\mathrm{Ad}(u')\lambda)$ is just the canonical line bundle associated with the dominant weight $\mathrm{Ad}(u')\lambda$. The last assertion follows from the Borel-Weil theorem. This completes our proof. □

If $j \in \mathcal{J}(u_0)$, let ${}^{u_0\sigma}N_\lambda^j$ denote the corresponding connected component of ${}^{u_0\sigma}N_\lambda$. Let $\mu : N_\lambda \to \mathfrak{u}^*$ be the moment map associated with the action of U on $L_\lambda \to N_\lambda$. As explained in the proof of Lemma 4.2.3, the restriction of μ to each ${}^{u_0\sigma}N_\lambda^j$ is just the moment map associated with the action of $\underline{Z}^0$ on $L_\lambda \to {}^{u_0\sigma}N_\lambda^j$.

Definition 4.2.4. If $y \in \underline{\mathfrak{z}}$, $j \in \mathcal{J}(u_0)$, set

$$R^j_{u_0,\lambda}(y) = \int_{{}^{u_0\sigma}N_\lambda^j} \exp\left(2\pi i\langle \mu, y\rangle + c_1(L_\lambda|_{{}^{u_0\sigma}N_\lambda}, g^{L_\lambda|_{{}^{u_0\sigma}N_\lambda}})\right). \tag{4.2.22}$$

Note that $R^j_{u_0,\lambda}(y)$ is a function of the same type as the one given in (4.1.14). We can verify that $R^j_{u_0,\lambda}$ is a $\underline{Z}^0$-invariant function on $\underline{\mathfrak{z}}$. Also $R^j_{u_0,\lambda}(y)$ can be computed by the localization formulas in [DH82, DH83], [BGV04, Chapter 7]. Let $\Delta^{\underline{\mathfrak{z}}}$ be the standard Laplacian on $\underline{\mathfrak{z}}$, then by [BMZ17, Eq.(8.146)], we have

$$\Delta^{\underline{\mathfrak{z}}} R^j_{u_0,\lambda} = -4\pi^2|\lambda|^2 R^j_{u_0,\lambda}. \tag{4.2.23}$$

Let $\underline{\mathfrak{q}}$ be the orthogonal subspace of $\underline{\mathfrak{z}}$ in $\mathfrak{u}$ with respect to B. If $u' \in \underline{Z}^0 u N_U(A)$, let $\underline{\mathfrak{z}}(\mathrm{Ad}(u')\lambda)$ be Lie algebra of $\underline{Z}^0(\mathrm{Ad}(u')\lambda)$, and let $\underline{\mathfrak{z}}^\perp(\mathrm{Ad}(u')\lambda)$ be the orthogonal of $\underline{\mathfrak{z}}(\mathrm{Ad}(u')\lambda)$ in $\underline{\mathfrak{z}}$. Put

$$\underline{\mathfrak{q}}(\mathrm{Ad}(u')\lambda) = \underline{\mathfrak{q}} \cap \mathfrak{u}(\mathrm{Ad}(u')\lambda). \tag{4.2.24}$$

Let $\underline{\mathfrak{q}}^\perp(\mathrm{Ad}(u')\lambda)$ be the orthogonal of $\underline{\mathfrak{q}}(\mathrm{Ad}(u')\lambda)$ in $\underline{\mathfrak{q}}$. Then

$$\mathfrak{u}(\mathrm{Ad}(u')\lambda) = \underline{\mathfrak{z}}(\mathrm{Ad}(u')\lambda) \oplus \underline{\mathfrak{q}}(\mathrm{Ad}(u')\lambda). \tag{4.2.25}$$

By Lemma 4.2.3, the (real) vector space $\underline{\mathfrak{q}}^\perp(\mathrm{Ad}(u')\lambda)$ can be identified with the holomorphic normal vector space of ${}^{u_0\sigma}N_\lambda$ at $[u']_\lambda$, so that it inherits a complex structure $J_{u'}$ and $u_0\sigma$ acts on it as a complex linear map. Set

$$\varphi_{u_0}(u') = \frac{1}{\det_{\mathbb{C}}(1 - \mathrm{Ad}(u_0\sigma)^{-1})|_{(\underline{\mathfrak{q}}^\perp(\mathrm{Ad}(u')\lambda), J_{u'})}}. \tag{4.2.26}$$

Lemma 4.2.5. *If $x = [u']_\lambda \in {}^{u_0\sigma}N_\lambda$ with $u' \in \underline{Z}^0 u N_U(A)$, then the map $x \mapsto \varphi_{u_0}(u')$ defines a locally constant function $\varphi_{u_0}(x)$ on ${}^{u_0\sigma}N_\lambda$. In particular, for $j \in \mathcal{J}(u_0)$, let $\varphi^j_{u_0} \in \mathbb{C}$ denote the value of φ_{u_0} on the component ${}^{u_0\sigma}N_\lambda^j$.*

Proof. By (4.2.18), (4.2.26), $\varphi_{u_0}(x)$ is well-defined on ${}^{u_0\sigma}N_\lambda$. If $h \in \underline{Z}^0$, then

$$\underline{\mathfrak{q}}^\perp(\mathrm{Ad}(hu')\lambda) = \mathrm{Ad}(h)\underline{\mathfrak{q}}^\perp(\mathrm{Ad}(u')\lambda). \tag{4.2.27}$$

Since h acts on N_λ holomorphically and commutes with $u_0\sigma$, then $\varphi_{u_0}(x)$ is a $\underline{Z}^0$-invariant function on ${}^{u_0\sigma}N_\lambda$. This completes the proof of our lemma. □

Put

$$n(u_0\sigma) = \max\{\dim_{\mathbb{C}} {}^{u_0\sigma}N_\lambda^j \mid j \in \mathcal{J}(u_0)\}. \tag{4.2.28}$$

We call $n(u_0\sigma)$ the (maximal) dimension of ${}^{u_0\sigma}N_\lambda$. Let $\mathcal{J}(u_0)^{\max}$ be the subset of $\mathcal{J}(u_0)$ of the index j with $\dim_{\mathbb{C}} {}^{u_0\sigma}N_\lambda^j = n(u_0\sigma)$, i.e. the index set for the connected component of ${}^{u_0\sigma}N_\lambda$ of the maximal dimension.

Proposition 4.2.6. *For $j \in \mathcal{J}(u_0)$, if $u_j \in U$ is such that $x_j \in [u_j]_\lambda \in {}^{u_0\sigma}N_\lambda^j$, then $u_j^{-1}u_0\sigma(u_j) \in U(\lambda)$, and $r_{u_0,j} = \rho^{V^\lambda}(u_j^{-1}u_0\sigma(u_j)) \in \mathbb{S}^1$ only depends on $j \in \mathcal{J}(u_0)$. The action of $u_0\sigma$ on fibre L_{λ,x_j} is given by the multiplication of the number $r_{u_0,j}$.*

If $y \in \underline{\mathfrak{z}}$, as $d \to +\infty$, then

$$\chi_d(u_0\sigma e^{y/d}) = d^{n(u_0\sigma)} \sum_{j\in\mathcal{J}(u_0)^{\max}} r_{u_0,j}^d \varphi_{u_0}^j R_{u_0,\lambda}^j(y) + \mathcal{O}(d^{n(u_0\sigma)-1}). \tag{4.2.29}$$

Proof. The first part of our proposition follows from the definition of N_λ, L_λ and (4.2.18). We will use a fixed point formula of Berline and Vergne [BV85, Theorem 3.23] to get (4.2.29). If B is a complex (q,q) matrix, Let $\mathrm{Td}(B)$ denote the Todd function of B [BMZ17, Subsection 3.4]. Set

$$e(B) = \det B. \tag{4.2.30}$$

Let ∇^{TN_λ} be the Chern connection on TN_λ, and let R^{TN_λ} be its curvature. If $y \in \mathfrak{u}$, let y^{N_λ} be the associated real vector field on N_λ let $\mathrm{L}_y^{TN_\lambda}$ be the natural action of y on the smooth sections of TN_λ. Let $\nu^{TN_\lambda}(y)$ be the map given by

$$2\pi i\nu^{TN_\lambda}(y) = \nabla^{TN_\lambda}_{y^{N_\lambda}} - \mathrm{L}_y^{TN_\lambda}. \tag{4.2.31}$$

If $x \in {}^{u_0\sigma}N_\lambda$, let $e^{i\theta_1}, \cdots, e^{i\theta_l}$, $0 \le \theta_j < 2\pi$ be the distinct eigenvalues of $u_0\sigma$ acting on T_xN_λ. Since $u_0\sigma$ is parallel, these eigenvalues are locally constant on ${}^{u_0\sigma}N_\lambda$. Then $TN_\lambda|_{{}^{u_0\sigma}N_\lambda}$ splits holomorphically as an orthogonal sum of the subbundles $TN_\lambda^{\theta_j}$. The Chern connection $\nabla^{TN_\lambda|_{{}^{u_0\sigma}N_\lambda}}$ also splits as the sum of the Chern connection on $TN_\lambda^{\theta_j}$. Let R^{θ_j} denote the corresponding curvature.

If $y \in \underline{\mathfrak{z}}$, let $\nu^{TN_\lambda|_{{}^{u_0\sigma}N_\lambda}}(y)$ be the restriction of $\nu^{TN_\lambda}(y)$ to ${}^{u_0\sigma}N_\lambda$, which is given by the same formula as in (4.2.31) with respect to the action of $\underline{Z}^0$ on $TN_\lambda|_{{}^{u_0\sigma}N_\lambda}$. The action of $\nu^{TN_\lambda|_{{}^{u_0\sigma}N_\lambda}}(y)$ preserves the splitting of $TN_\lambda|_{{}^{u_0\sigma}N_\lambda}$. The equivariant Todd genus is given by

$$\begin{aligned}&\mathrm{Td}_y^{u_0\sigma}(TN_\lambda|_{{}^{u_0\sigma}N_\lambda}, g^{TN_\lambda|_{{}^{u_0\sigma}N_\lambda}})\\ &= \mathrm{Td}\big(-\frac{R^0}{2\pi i} + \nu^{TN_\lambda^0}(y)\big) \prod_{\theta_j\neq 0} \big(\frac{\mathrm{Td}}{e}\big)\big(-\frac{R^{\theta_j}}{2\pi i} + \nu^{TN_\lambda^{\theta_j}}(y) + i\theta_j\big).\end{aligned} \tag{4.2.32}$$

We will denote by $\mathrm{Td}_y^{u_0\sigma}(TN_\lambda|_{^{u_0\sigma}N_\lambda})$ the its equivariant cohomology class. We refer to [BV85], [BGV04, Chapter 7] for more details.

Let $r_{u_0} \in \mathbb{S}^1$ denote the action of $u_0\sigma$ on $L_\lambda|_{^{u_0\sigma}N_\lambda}$, which is equal to $r_{u_0,j}$ on ${}^{u_0\sigma}N_\lambda^j$. The equivariant Chern character form of $L_\lambda^d|_{^{u_0\sigma}N_\lambda}$ is given by

$$\mathrm{ch}_y^{u_0\sigma}(L_\lambda^d|_{^{u_0\sigma}N_\lambda}, g^{L_\lambda^d|_{^{u_0\sigma}N_\lambda}}) = r_{u_0}^d \exp\big(2\pi i d\langle\mu, y\rangle + dc_1(L_\lambda|_{^{u_0\sigma}N_\lambda}, g^{L_\lambda|_{^{u_0\sigma}N_\lambda}})\big). \tag{4.2.33}$$

By [BV85, Theorem 3.23], if y is in a small neighborhood of $\underline{\mathfrak{z}}$, we have

$$\begin{aligned}\chi_d(u_0\sigma e^y) = \int_{^{u_0\sigma}N_\lambda} & \mathrm{Td}_y^{u_0\sigma}(TN_\lambda|_{^{u_0\sigma}N_\lambda}) r_{u_0}^d \\ & \exp\big(2\pi i d\langle\mu, y\rangle + dc_1(L_\lambda|_{^{u_0\sigma}N_\lambda}, g^{L_\lambda|_{^{u_0\sigma}N_\lambda}})\big).\end{aligned} \tag{4.2.34}$$

For $y \in \underline{\mathfrak{z}}$, when taking the asymptotics of $\chi_d(u_0\sigma e^{y/d})$ as $d \to +\infty$, only the maximal dimensional components of ${}^{u_0\sigma}N_\lambda$ contribute to the leading term. For the leading term, $\mathrm{Td}_{y/d}^{u_0\sigma}(TN_\lambda|_{^{u_0\sigma}N_\lambda})$ only contributes the degree 0 component of $\mathrm{Td}_0^{u_0\sigma}(TN_\lambda|_{^{u_0\sigma}N_\lambda})$, which is just φ_{u_0} defined in Lemma 4.2.5. If $j \in \mathcal{J}(u_0)^{\max}$, the integration of $\exp\big(2\pi i\langle\mu, y\rangle + dc_1(L_\lambda|_{^{u_0\sigma}N_\lambda}, g^{L_\lambda|_{^{u_0\sigma}N_\lambda}})\big)$ on ${}^{u_0\sigma}N_\lambda^j$ is just $d^{n(u_0\sigma)}R_{u_0,\lambda}^j(y)$. Then we get (4.2.29). This completes the proof of our proposition. □

4.3. *The nondegeneracy condition on λ*

Recall that N_λ is identified with the coadjoint orbit $\mathcal{O}_\lambda$ in $\mathfrak{u}^*$. Dual to (2.1.2),

$$\mathfrak{u}^* = \sqrt{-1}\mathfrak{p}^* \oplus \mathfrak{k}^*. \tag{4.3.1}$$

Then the nondegeneracy condition defined in Definition 4.1.2 is equivalent to that each vector $v \in \mathcal{O}_\lambda$ has a nonzero component in $\sqrt{-1}\mathfrak{p}^*$.

Take $k \in K$, then $k\sigma$ is an elliptic element in G^σ. We can also consider it as an element in U^σ. Recall that $U_\sigma(k)^0$ denotes the identity component of σ-twisted centralizer of k in U. Then it is the compact form of $Z_\sigma(k)^0$. By the discussion in Subsection 2.4, $Z_\sigma(k)^0$ is still a linear reductive group with the Cartan involution induced by θ, and $K_\sigma(k)^0$ is the corresponding maximal compact subgroup of $Z_\sigma(k)^0$. Recall that

$$X(k\sigma) = Z_\sigma(k)^0/K_\sigma(k)^0. \tag{4.3.2}$$

Let $\omega^{\mathfrak{z}_\sigma(k)} = \omega^{\mathfrak{k}_\sigma(k)} + \omega^{\mathfrak{p}_\sigma(k)}$ be the canonical 1-form on $Z_\sigma(k)^0$ as in (2.1.5).

By Lemma 4.2.3, we have

$${}^{k\sigma}N_\lambda = \cup_{j\in\mathcal{J}(k)}{}^{k\sigma}N_\lambda^j. \tag{4.3.3}$$

If $j \in \mathcal{J}(k)$, the function $R_{k,\lambda}^j$ is defined in (4.2.22). Then proceeding as in Subsection 4.1, by using instead $(Z_\sigma(k)^0, K_\sigma(k)^0)$ and $\omega^{\mathfrak{p}_\sigma(k)}$, associated with $R_{k,\lambda}^j$, we get the invariant differential forms $e_{k,t}^j$, $d_{k,t}^j$, $t > 0$ on $X(k\sigma)$.

If the function $R^j_{k,\lambda}$ satisfies the nondegeneracy condition with respect to $\omega^{\mathfrak{p}_\sigma(k)}$, then there exists $c_j > 0$ such that, as $t \to +\infty$,

$$e^j_{k,t} = \mathcal{O}(e^{-c_j t}), \;\; d^j_{k,t} = \mathcal{O}(e^{-c_j t}). \tag{4.3.4}$$

Put

$$W^j_{k\sigma} = -\int_0^{+\infty} d^j_{k,t} \frac{dt}{t}. \tag{4.3.5}$$

We say that $W^j_{k\sigma}$ is the W-invariant associated with $k\sigma$ and $R^j_{k,\lambda}$.

Proposition 4.3.1. *Take $k \in K$. If (N_λ, μ) defined in Subsection 4.2 is nondegenerate with respect to $\omega^{\mathfrak{p}}$, then for $j \in \mathcal{J}(k)$, $({}^{k\sigma}N^j_\lambda, \mu|_{{}^{k\sigma}N^j_\lambda})$ is nondegenerate with respect to $\omega^{\mathfrak{p}_\sigma(k)}$.*

Proof. As in the proof of Lemma 4.2.3, we have

$${}^{k\sigma}N_\lambda \simeq \mathcal{O}_\lambda \cap \mathfrak{u}_\sigma(k)^*. \tag{4.3.6}$$

The splitting (4.3.1) induces a splitting of $\mathfrak{u}_\sigma(k)^*$,

$$\mathfrak{u}_\sigma(k)^* = \sqrt{-1}\mathfrak{p}_\sigma(k)^* \oplus \mathfrak{k}_\sigma(k)^*. \tag{4.3.7}$$

By Definition 4.1.2, if (N_λ, μ) is nondegenerate, then $\mu(N_\lambda) \cap \mathfrak{k}^* = \emptyset$, so that $\mu({}^{k\sigma}N^j_\lambda) \cap \mathfrak{k}_\sigma(k)^* = \emptyset$, which says that $({}^{k\sigma}N^j_\lambda, \mu|_{{}^{k\sigma}M^j_\lambda})$ is nondegenerate with respect to $\omega^{\mathfrak{p}_\sigma(k)}$. This completes the proof of our proposition. □

In the sequel, we always assume that (N_λ, μ) is nondegenerate with respect to $\omega^{\mathfrak{p}}$ (or for short, λ is nondegenerate). For a general elliptic element $\gamma\sigma$ with $\gamma \in G$, we can not always regard it as an element of U^σ. But there exists $g \in G$ such that $k = g\gamma\sigma(g^{-1}) \in K$. Then we construct the corresponding triplets $(r_{k,j}, \varphi^j_k, R^j_{k,\lambda})_{j\in\mathcal{J}(k)}$ and the associated invariant forms $e^j_{k,t}$, $d^j_{k,t}$, $W^j_{k\sigma}$, $j \in \mathcal{J}(k)$ on $X(k\sigma)$.

If we take another $g' \in G$ such that $k' = g'\gamma\sigma(g'^{-1}) \in K$, then $Z_\sigma(k)$ and $Z_\sigma(k')$ can be identified by the conjugation of $h = g'g^{-1} \in G$. But we still use the Cartan involution on $Z_\sigma(k')$ induced from θ to define the associated forms $e^j_{k',t}$, $d^j_{k',t}$, $W^j_{k'\sigma}$, $j \in \mathcal{J}(k')$ on $X(k'\sigma)$.

Lemma 4.3.2. *Let $\gamma \in G$, $k, k' \in K$ be as above. Then we have $n(k\sigma) = n(k'\sigma)$. Moreover, there is an identification between $\mathcal{J}(k)$ and $\mathcal{J}(k')$ such that if $j \in \mathcal{J}(k) = \mathcal{J}(k')$, we have*

$$r_{k,j} = r_{k',j}, \;\; \varphi^j_k = \varphi^j_{k'}, \;\; [W^j_{k\sigma}]^{\max} = [W^j_{k'\sigma}]^{\max}. \tag{4.3.8}$$

Proof. By the Cartan decomposition of G, there exist unique $f \in \mathfrak{p}$, $k_0 \in K$ such that $h = k_0 e^f$. Since $hk\sigma(h^{-1}) = k'$, we get $k_0 k\sigma(k_0^{-1}) = k'$. Moreover,

$$k_0 Z_\sigma(k) k_0^{-1} = Z_\sigma(k'), \;\; k_0 U_\sigma(k) k_0^{-1} = U_\sigma(k'). \tag{4.3.9}$$

The fixed point sets of $k\sigma$ and $k'\sigma$ in N_λ are identified via the action of k_0. This way, we identify $\mathcal{J}(k)$ with $\mathcal{J}(k')$. On the fixed point sets, the actions of $k\sigma$ on the bundles TN_λ, L_λ are identified with the corresponding actions of $k'\sigma$. Therefore, the data $(r_{k',j}, \varphi^j_{k'}, R^j_{k',\lambda})_{j\in\mathcal{J}(k')}$ can be identified with $(r_{k,j}, \varphi^j_k, R^j_{k,\lambda})_{j\in\mathcal{J}(k)}$ via the action of k_0. In particular, $n(k\sigma) = n(k'\sigma)$ and $r_{k,j} = r_{k',j}$, $\varphi^j_k = \varphi^j_{k'}$. The last identity of (4.3.8) follows from (4.3.9) and the fact that the Cartan involutions on $Z_\sigma(k)$, $Z_\sigma(k')$ that we use to define the W-invariants are identified by the conjugation of k_0. □

4.4. Asymptotics of the elliptic twisted orbital integrals

Let (E_d, ρ^{E_d}), $d \in \mathbb{N}$ be the sequence of irreducible unitary representations of U^σ constructed in Subsection 4.2 with the nondegenerate σ-fixed $\lambda \in P_{++}(\mathfrak{c})$. We extend it to a sequence of representations of G^σ. Then we get a family of flat homogeneous vector bundles $F_d = G \times_K E_d$, $d \in \mathbb{N}$ on X. Recall that $\mathbf{D}^{X,F_d,2}$ is the Hodge Laplacian associated with F_d.

In this subsection, we consider the case $\gamma = k^{-1} \in K$. Then

$$\mathfrak{u}_\sigma(\gamma) = \sqrt{-1}\mathfrak{p}_\sigma(\gamma) \oplus \mathfrak{k}_\sigma(\gamma). \tag{4.4.1}$$

As explained in the beginning of Subsection 4.2, there exists $v' \in \mathfrak{k}_\sigma(\gamma) \cap \mathfrak{k}^{\mathrm{reg}}$. If $\mathfrak{t} = \mathfrak{k}(v')$, then $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$. Let T be the corresponding maximal torus of K. Put

$$\mathfrak{s} = \mathfrak{t} \cap \mathfrak{k}_\sigma(\gamma). \tag{4.4.2}$$

Then $\mathfrak{s}$ is a Cartan subalgebra of $\mathfrak{k}_\sigma(\gamma)$. Recall that $\mathfrak{b}_\sigma(\gamma) \subset \mathfrak{p}$ is defined in (3.3.4), then $\mathfrak{b}_\sigma(\gamma) \oplus \mathfrak{s}$ is a Cartan subalgebra of $\mathfrak{z}_\sigma(\gamma)$. By Theorem 3.3.3, Corollary 3.3.4, the twisted orbital integral in (3.3.9) associated with this $\gamma\sigma$ vanishes except the case $\dim \mathfrak{b}_\sigma(\gamma) = 1$.

In the sequel, we also assume that $\dim \mathfrak{b}_\sigma(\gamma) = 1$, then $\dim \mathfrak{p}_\sigma(\gamma)$ is odd and $\delta(Z_\sigma(\gamma)^0) = 1$. This assumption also implies that $\delta(G) \geq 1$, which is a necessary condition for the existence of the nondegenerate (N_λ, μ) discussed in Subsections 4.2 and 4.3.

Recall that for the triplets $(r_{\gamma,j}, \varphi^j_\gamma, R^j_{\gamma,\lambda})_{j\in\mathcal{J}(\gamma)}$ and the associated invariant forms $e^j_{\gamma,t}$, $d^j_{\gamma,t}$, $W^j_{\gamma\sigma}$, $j \in \mathcal{J}(\gamma)$ on $X(\gamma\sigma)$ are constructed in Subsection 4.2, 4.3. The main results of this subsection are as follows.

Theorem 4.4.1. *Suppose that* $\dim \mathfrak{b}_\sigma(\gamma) = 1$. *For* $t > 0$, *as* $d \to +\infty$,

$$\begin{aligned}
&d^{-n(\gamma\sigma)-1}\mathrm{Tr_s}^{[\gamma\sigma]}\Big[\big(N^{\Lambda^\bullet(T^*X)} - \frac{m}{2}\big)\exp(-t\mathbf{D}^{X,F_d,2}/2d^2)\Big]\\
&= 2\sum_{j\in\mathcal{J}(\gamma)^{\max}} r^d_{\gamma,j}\varphi^j_\gamma[e^j_{\gamma,t/2}]^{\max} + \mathcal{O}(d^{-1}),\\
&d^{-n(\gamma\sigma)-1}\mathrm{Tr_s}^{[\gamma\sigma]}\Big[\big(N^{\Lambda^\bullet(T^*X)} - \frac{m}{2}\big)\big(1 - \frac{t\mathbf{D}^{X,F_d,2}}{d^2}\big)\exp(-t\mathbf{D}^{X,F_d,2}/2d^2)\Big]\\
&= 2\sum_{j\in\mathcal{J}(\gamma)^{\max}} r^d_{\gamma,j}\varphi^j_\gamma[d^j_{\gamma,t/2}]^{\max} + \mathcal{O}(d^{-1}).
\end{aligned} \tag{4.4.3}$$

There exists $C'>0$ such that for $d>1$, we have

$$\begin{aligned}\Big| d^{-n(\gamma\sigma)-1}\int_1^d \mathrm{Tr_s}^{[\gamma\sigma]}\big[\big(N^{\Lambda^\bullet(T^*X)}-\frac{m}{2}\big)\exp(-t\mathbf{D}^{X,F_d,2}/2d^2)\big]\frac{dt}{t}\\ -2\sum_{j\in\mathcal{J}(\gamma)^{\max}} r^d_{\gamma,j}\varphi^j_\gamma\int_1^d[e^j_{\gamma,t/2}]^{\max}\frac{dt}{t}\Big|\leq\frac{C'}{d}.\end{aligned} \tag{4.4.4}$$

There exists $C>0$ such that for $d\in\mathbb{N}_{>0}$, $0<t\leq 1$,

$$\Big|d^{-n(\gamma\sigma)-1}\mathrm{Tr_s}^{[\gamma\sigma]}\big[\big(N^{\Lambda^\bullet(T^*X)}-\frac{m}{2}\big)\exp(-t\mathbf{D}^{X,F_d,2}/2d^2)\big]\Big|\leq C/\sqrt{t}$$

$$\begin{aligned}\Big|d^{-n(\gamma\sigma)-1}\mathrm{Tr_s}^{[\gamma\sigma]}\big[\big(N^{\Lambda^\bullet(T^*X)}-\frac{m}{2}\big)(1-t\mathbf{D}^{X,F_d,2}/d^2)\\ \exp(-t\mathbf{D}^{X,F_d,2}/2d^2)\big]\Big|\leq C\sqrt{t}.\end{aligned} \tag{4.4.5}$$

There exists $c>0$, $c'>0$ such that for $t\geq 1$, d large enough, we have

$$\Big|d^{-n(\gamma\sigma)-1}\mathrm{Tr_s}^{[\gamma\sigma]}\big[\big(N^{\Lambda^\bullet(T^*X)}-\frac{m}{2}\big)\exp(-t\mathbf{D}^{X,F_d,2}/2d^2)\big]\Big|\leq ce^{-c't}. \tag{4.4.6}$$

Proof. Note that (4.4.3) is an extension of [BMZ17, Theorem 8.14].

Recall that $p=\dim_\mathbb{R}\mathfrak{p}_\sigma(\gamma)$, $q=\dim_\mathbb{R}\mathfrak{k}_\sigma(\gamma)$. By (3.3.9), for $d\in\mathbb{N}_{>0}$, we get

$$\begin{aligned}&\mathrm{Tr_s}^{[\gamma\sigma]}\Big[\big(N^{\Lambda^\bullet(T^*X)}-\frac{m}{2}\big)\exp(-t\mathbf{D}^{X,F_d,2}/2d^2)\Big]\\ &=\frac{d^p}{(2\pi t)^{p/2}}\exp\big(\frac{t}{48d^2}\mathrm{Tr}^\mathfrak{k}[C^{\mathfrak{k},\mathfrak{k}}]+\frac{t}{16d^2}\mathrm{Tr}^\mathfrak{p}[C^{\mathfrak{k},\mathfrak{p}}]\big)\int_{\mathfrak{k}_\sigma(\gamma)}J_{\gamma\sigma}(\sqrt{t}y/d)\\ &\cdot\mathrm{Tr_s}^{\Lambda^\bullet(\mathfrak{p}^*)}\big[\big(N^{\Lambda^\bullet(\mathfrak{p}^*)}-\frac{m}{2}\big)\rho^{\Lambda^\bullet(\mathfrak{p}^*)}(k^{-1}\sigma)\exp(-i\rho^{\Lambda^\bullet(\mathfrak{p}^*)}(\sqrt{t}y/d))\big]\\ &\cdot\mathrm{Tr}^{E_d}\big[\rho^{E_d}(k^{-1}\sigma)\exp(-i\rho^{E_d}(\sqrt{t}y/d)+\frac{t}{2d^2}C^{\mathfrak{g},E_d})\big]\exp\big(-\frac{|y|^2}{2}\big)\frac{dy}{(2\pi)^{q/2}}.\end{aligned} \tag{4.4.7}$$

In this proof, we denote by C or c a positive constant independent of the variables d, t and $y\in\mathfrak{k}_\sigma(\gamma)$. We use the symbol $\mathcal{O}_{\mathrm{ind}}$ to denote the big-O convention which does not depend on d, t and y. Set $\langle y\rangle=\sqrt{1+|y|^2}$.

By (2.5.15), for $d\geq 1$, $t>0$ and $y\in\mathfrak{k}_\sigma(\gamma)$, we have

$$J_{\gamma\sigma}(\sqrt{t}y/d)=\frac{1}{\det(1-\mathrm{Ad}(k^{-1}\sigma))|_{\mathfrak{p}^\perp_\sigma(\gamma)}}+\mathcal{O}_{\mathrm{ind}}\big(\frac{\sqrt{t}|y|}{d}e^{C\frac{\sqrt{t}|y|}{d}}\big). \tag{4.4.8}$$

Let $\mathfrak{b}^\perp_\sigma(\gamma)\subset\mathfrak{p}_\sigma(\gamma)$ be the space orthogonal to the one-dimensional line $\mathfrak{b}_\sigma(\gamma)$ in $\mathfrak{p}_\sigma(\gamma)$. If $y\in\mathfrak{s}$, by [BMZ17, Eq.(8.133)], we have

$$\begin{aligned}&\mathrm{Tr_s}^{\Lambda^\bullet(\mathfrak{p}^*)}\big[\big(N^{\Lambda^\bullet(\mathfrak{p}^*)}-\frac{m}{2}\big)\rho^{\Lambda^\bullet(\mathfrak{p}^*)}(k^{-1}\sigma)\exp(-i\rho^{\Lambda^\bullet(\mathfrak{p}^*)}(\sqrt{t}y/d))\big]\\ &=-\det(1-\exp(-i\mathrm{ad}(\sqrt{t}y/d)))|_{\mathfrak{b}^\perp_\sigma(\gamma)}\\ &\qquad\cdot\det(1-\mathrm{Ad}(k^{-1}\sigma)\exp(-i\mathrm{ad}(\sqrt{t}|y|/d)))|_{\mathfrak{p}^\perp_\sigma(\gamma)}.\end{aligned} \tag{4.4.9}$$

By (4.4.9) if $y \in \mathfrak{s}$, $d \geq 1$ and if $t > 0$, we get

$$\frac{d^{p-1}}{t^{(p-1)/2}} \mathrm{Tr_s}^{\Lambda^\bullet(\mathfrak{p}^*)}[(N^{\Lambda^\bullet(\mathfrak{p}^*)} - \frac{m}{2})\rho^{\Lambda^\bullet(\mathfrak{p}^*)}(k^{-1}\sigma)\exp(-i\rho^{\Lambda^\bullet(\mathfrak{p}^*)}(\sqrt{t}y/d))]$$
$$= -\det(i\mathrm{ad}(y))|_{\mathfrak{b}_\sigma^\perp(\gamma)} \det(1 - \mathrm{Ad}(k^{-1}\sigma))|_{\mathfrak{p}_\sigma^\perp(\gamma)} + \mathcal{O}_{\mathrm{ind}}(\frac{\sqrt{t}|y|}{d} e^{C\frac{\sqrt{t}|y|}{d}}). \tag{4.4.10}$$

Let $\widehat{\Omega}^{\mathfrak{z}_\sigma(\gamma)} \in \Lambda^2(\widehat{\mathfrak{p}}_\sigma(\gamma)^*) \otimes \mathfrak{k}_\sigma(\gamma)$ be a copy of $\Omega^{\mathfrak{z}_\sigma(\gamma)}$. Let L and the Berezin integral be the ones as in (4.1.10) and (4.1.19) associated with $\mathfrak{p}_\sigma(\gamma)$. Note that $\dim \mathfrak{p}_\sigma(\gamma)$ is odd, then by (4.1.19), we have

$$\pi^{-p/2}\det(i\mathrm{ad}(y))|_{\mathfrak{b}_\sigma^\perp(\gamma)} = -\Big[\int^{\widehat{B}} L\exp(\langle y, \Omega^{\mathfrak{z}_\sigma(\gamma)} + \widehat{\Omega}^{\mathfrak{z}_\sigma(\gamma)}\rangle)\Big]^{\max}. \tag{4.4.11}$$

Combining (4.4.10) and (4.4.11), we get that if $y \in \mathfrak{s}$, $d \geq 1$ and if $t > 0$,

$$\pi^{-p/2}(\frac{d}{\sqrt{t}})^{p-1} \mathrm{Tr_s}^{\Lambda^\bullet(\mathfrak{p}^*)}[(N^{\Lambda^\bullet(\mathfrak{p}^*)} - \frac{m}{2})\rho^{\Lambda^\bullet(\mathfrak{p}^*)}(k^{-1}\sigma)\exp(-i\rho^{\Lambda^\bullet(\mathfrak{p}^*)}(\sqrt{t}y/d))]$$
$$= \Big[\int^{\widehat{B}} L\exp(\langle y, \Omega^{\mathfrak{z}_\sigma(\gamma)} + \widehat{\Omega}^{\mathfrak{z}_\sigma(\gamma)}\rangle)\Big]^{\max} \det(1 - \mathrm{Ad}(k^{-1}\sigma))|_{\mathfrak{p}_\sigma^\perp(\gamma)}$$
$$+ \mathcal{O}_{\mathrm{ind}}(\frac{\sqrt{t}|y|}{d} e^{C\frac{\sqrt{t}|y|}{d}}). \tag{4.4.12}$$

Using the adjoint invariance, the equation (4.4.12) extends to $y \in \mathfrak{k}_\sigma(\gamma)$.

Note that since (N_λ, μ) is nondegenerate, then there exists a small constant $\epsilon \in]0, \frac{|\lambda|}{2}]$ such that on N_λ, for $y \in \mathfrak{k}_\sigma(\gamma)$

$$|\langle \mu, y\rangle| \leq (|\lambda| - \epsilon)|y|. \tag{4.4.13}$$

Then by (4.2.22), we have

$$|R^j_{\gamma,\lambda}(-i\sqrt{t}y)| \leq Ce^{2\pi\sqrt{t}(|\lambda|-\epsilon)|y|}. \tag{4.4.14}$$

By (4.2.34) and (4.4.13), we get that for $d \geq 1$, $t > 0$ and $y \in \mathfrak{k}_\sigma(\gamma)$,

$$d^{-n(\gamma\sigma)}\mathrm{Tr}^{E_d}[\rho^{E_d}(k^{-1}\sigma)\exp(-i\rho^{E_d}(\sqrt{t}y/d))]$$
$$= \sum_{j \in \mathcal{J}(\gamma)^{\max}} r^d_{\gamma,j}\varphi^j_\gamma R^j_{\gamma,\lambda}(-i\sqrt{t}y) + \mathcal{O}_{\mathrm{ind}}\big(\frac{(\sqrt{t}+1)\langle y\rangle}{d} e^{2\pi\sqrt{t}(|\lambda|-\epsilon)|y| + C\frac{\sqrt{t}}{d}|y|}\big). \tag{4.4.15}$$

Combining (4.4.8), (4.4.12), (4.4.14) and (4.4.15), we get that for $d \geq 1$,

$t > 0$ and $y \in \mathfrak{k}_\sigma(\gamma)$,

$$\begin{aligned}
&\frac{d^{p-1-n(\gamma\sigma)}}{\pi^{p/2}t^{(p-1)/2}} J_{\gamma\sigma}\Big(\frac{\sqrt{t}y}{d}\Big) \mathrm{Tr_s}^{\Lambda^\bullet(\mathfrak{p}^*)}\Big[\Big(N^{\Lambda^\bullet(\mathfrak{p}^*)} - \frac{m}{2}\Big)\rho^{\Lambda^\bullet(\mathfrak{p}^*)}(k^{-1}\sigma)e^{-i\rho^{\Lambda^\bullet(\mathfrak{p}^*)}(\sqrt{t}y/d)}\Big] \\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot \mathrm{Tr}^{E_d}[\rho^{E_d}(k^{-1}\sigma)\exp(-i\rho^{E_d}(\sqrt{t}y/d))] \\
&= \sum_{j \in \mathcal{J}(\gamma)^{\max}} r^d_{\gamma,j}\varphi^j_\gamma \Big[\int^{\widehat{B}} L\exp(\langle y, \Omega^{\mathfrak{z}_\sigma(\gamma)} + \widehat{\Omega}^{\mathfrak{z}_\sigma(\gamma)}\rangle)\Big]^{\max} R^j_{\gamma,\lambda}(-i\sqrt{t}y) \\
&\qquad\qquad + \mathcal{O}_{\mathrm{ind}}\Big(\frac{(\sqrt{t}+1)\langle y\rangle}{d} e^{2\pi\sqrt{t}(|\lambda|-\epsilon)|y| + C\frac{\sqrt{t}}{d}|y|}\Big).
\end{aligned} \tag{4.4.16}$$

Let $\rho_\mathfrak{u}$ be the half of the sum of the positive roots in $R^+_U(\mathfrak{c})$. By [Bis11, Proposition 7.5.2] and (3.1.8), we have

$$\frac{t}{48d^2}\mathrm{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}] + \frac{t}{16d^2}\mathrm{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}] + \frac{t}{2d^2}C^{\mathfrak{g},E} = -\frac{2\pi^2 t|d\lambda + \rho_\mathfrak{u}|^2}{d^2}. \tag{4.4.17}$$

Using exactly the arguments as in [BMZ17, Eqs.(8.143)-(8.154)], we get that for each $j \in \mathcal{J}(\gamma)^{\max}$,

$$\begin{aligned}
\frac{e^{-2\pi^2 t|\lambda|^2}}{2^{p/2}\sqrt{t}} \int_{\mathfrak{k}_\sigma(\gamma)} &\Big[\int^{\widehat{B}} L\exp(\langle y, \Omega^{\mathfrak{z}_\sigma(\gamma)} + \widehat{\Omega}^{\mathfrak{z}_\sigma(\gamma)}\rangle)\Big]^{\max} R^j_{\gamma,\lambda}(-i\sqrt{t}y) \\
&\cdot \exp(-|y|^2/2)\frac{dy}{(2\pi)^{q/2}} = 2[e^j_{\gamma,t/2}]^{\max}.
\end{aligned} \tag{4.4.18}$$

By (4.4.7), (4.4.16), (4.4.18), we get

$$\begin{aligned}
&d^{-n(\gamma\sigma)-1}\mathrm{Tr_s}^{[\gamma\sigma]}\Big[\Big(N^{\Lambda^\bullet(T^*X)} - \frac{m}{2}\Big)\exp(-t\mathbf{D}^{X,F_d,2}/2d^2)\Big] \\
&= 2\exp\Big(-\frac{2\pi^2 t|d\lambda+\rho_\mathfrak{u}|^2}{d^2} + 2t\pi^2|\lambda|^2\Big) \sum_{j\in\mathcal{J}(\gamma)^{\max}} r^d_{\gamma,j}\varphi^j_\gamma[e^j_{\gamma,t/2}]^{\max} + R(t,d).
\end{aligned} \tag{4.4.19}$$

Here $R(t,d)$ is an error term such that

$$|R(t,d)| \le C\frac{\sqrt{t}+1}{\sqrt{t}}\frac{e^{-\frac{2\pi^2 t}{d^2}|d\lambda+\rho_\mathfrak{u}|^2}}{d}\int_{\mathfrak{k}_\sigma(\gamma)} \langle y\rangle e^{2\pi\sqrt{t}(|\lambda|-\epsilon)|y| + C\frac{\sqrt{t}|y|}{d} - |y|^2/2} dy. \tag{4.4.20}$$

Set $c_0 = 2\pi^2(|\lambda|^2 - (|\lambda|-\epsilon)^2) > 0$. Then for $d \ge 1$, $t > 0$,

$$|R(t,d)| \le C\frac{\sqrt{t}+1}{\sqrt{t}}\frac{t^{q/2}}{d}e^{-c_0 t + \frac{ct}{d}}. \tag{4.4.21}$$

By (4.4.19), (4.4.21), we get the first identity in (4.4.3). By (4.1.23), we get the second identity in (4.4.3). Then using (4.4.19), (4.4.21) for $0 \le t \le 1$ and by (4.1.21), we get the first estimate of (4.4.5). Note that there exists $d_0 \in \mathbb{N}$ large enough such that if $d \geq d_0$, then

$$\frac{c}{d} \leq \frac{1}{4} c_0 \tag{4.4.22}$$

Recall that $e^j_{\gamma,t/2}$ has the exponential decay of as $t \to +\infty$ described in (4.1.24). Together with (4.4.19), (4.4.21) and (4.4.22), we get (4.4.6). Note that for $d > 1$

$$\int_1^d |R(t,d)| \frac{dt}{t} \leq \frac{C'}{d}. \tag{4.4.23}$$

Then (4.4.4) follows from (4.4.19) and (4.4.23).

We now prove the second estimate in (4.4.3). If $y \in \mathfrak{k}_\sigma(\gamma)$, set

$$\begin{aligned} f(y) = J_{\gamma\sigma}(y) \det(1 - \mathrm{Ad}(k^{-1}\sigma) \exp(-i\mathrm{ad}(y)))|_{\mathfrak{p}^\perp_\sigma(\gamma)} \\ \cdot d^{-n(\gamma\sigma)} \mathrm{Tr}^{E_d}[\rho^{E_d}(k^{-1}\sigma) \exp(-i\rho^{E_d}(y))]. \end{aligned} \tag{4.4.24}$$

Then $f(y)$ is an analytic function on $\mathfrak{k}_\sigma(\gamma)$. If $y \in \mathfrak{s}$, by (4.4.9), (4.4.24),

$$\begin{aligned} \frac{d^{p-n(\gamma\sigma)-1}}{t^{p/2}} J_{\gamma\sigma}(\frac{\sqrt{t}y}{d}) \mathrm{Tr_s}^{\Lambda^\bullet(\mathfrak{p}^*)}\big[\big(N^{\Lambda^\bullet(\mathfrak{p}^*)} - \frac{m}{2}\big) \mathrm{Ad}(k^{-1}\sigma) \exp(-i\rho^{\Lambda^\bullet(\mathfrak{p}^*)}(\frac{\sqrt{t}y}{d}))\big] \\ \mathrm{Tr}^{E_d}[\rho^{E_d}(k^{-1}\sigma) \exp(-i\rho^{E_d}(\sqrt{t}y/d))] \\ = \frac{1}{\sqrt{t}} f(\frac{\sqrt{t}y}{d}) \frac{d^{p-1}}{t^{(p-1)/2}} \det(1 - \exp(-i\mathrm{ad}(\sqrt{t}y/d)))|_{\mathfrak{b}^\perp_\sigma(\gamma)}. \end{aligned} \tag{4.4.25}$$

Let $\nabla f(y)$ be the gradient of f on $\mathfrak{k}_\sigma(\gamma)$ with respect to the Euclidean scalar product of $\mathfrak{k}_\sigma(\gamma)$. Put

$$\begin{aligned} I(t,y,d) = \frac{\partial}{\partial t}\bigg(f(\frac{\sqrt{t}y}{d}) \frac{d^{p-1}}{t^{(p-1)/2}} \det(1 - \exp(-i\mathrm{ad}(\sqrt{t}y/d)))|_{\mathfrak{b}^\perp_\sigma(\gamma)}\bigg) \\ = \frac{1}{t} \langle \nabla f(\frac{\sqrt{t}y}{d}), \frac{\sqrt{t}y}{2d}\rangle \frac{d^{p-1}}{t^{(p-1)/2}} \det(1 - \exp(-i\mathrm{ad}(\sqrt{t}y/d)))|_{\mathfrak{b}^\perp_\sigma(\gamma)} \\ + f(\frac{\sqrt{t}y}{d}) \frac{\partial}{\partial t}\bigg(\frac{d^{p-1}}{t^{(p-1)/2}} \det(1 - \exp(-i\mathrm{ad}(\sqrt{t}y/d)))|_{\mathfrak{b}^\perp_\sigma(\gamma)}\bigg). \end{aligned} \tag{4.4.26}$$

Since $\mathrm{Tr}^{\mathfrak{b}^\perp_\sigma(\gamma)}[\mathrm{ad}(y)] = 0$, then there exists $c' > 0$, $C' > 0$ such that for $d \in \mathbb{N}_{>0}$, $0 < t \leq 1$, and $y \in \mathfrak{s}$,

$$\frac{\partial}{\partial t}\bigg(\frac{d^{p-1}}{t^{(p-1)/2}} \det(1 - \exp(-i\mathrm{ad}(\sqrt{t}y/d)))|_{\mathfrak{b}^\perp_\sigma(\gamma)}\bigg) \leq C' \exp(c'|y|). \tag{4.4.27}$$

Also since $\dim \mathfrak{b}^\perp_\sigma(\gamma)$ is even, when taking the Taylor expansion of the function as follows

$$\frac{1}{t} \langle \nabla f(\frac{\sqrt{t}y}{d}), \frac{\sqrt{t}y}{2d}\rangle \frac{d^{p-1}}{t^{(p-1)/2}} \det(1 - \exp(-i\mathrm{ad}(\sqrt{t}y/d)))|_{\mathfrak{b}^\perp_\sigma(\gamma)}, \tag{4.4.28}$$

the terms of even power of y have no negative powers of the parameter t in their coefficient. Using the conjugation invariance of the left-hand side of (4.4.25), the above consideration holds for $y \in \mathfrak{k}_\sigma(\gamma)$.

By (4.4.26), (4.4.27), there exist $C > 0$ such that for $d \in \mathbb{N}_{>0}$, $0 < t \leq 1$,

$$\left| \int_{\mathfrak{k}_\sigma(\gamma)} I(t, y, d) \exp(-|y|^2/2) dy \right| \leq C. \tag{4.4.29}$$

Using the fact that the two quantities in (4.4.5) are related by the operator $1 + 2t\frac{\partial}{\partial t}$, and by (4.4.25), (4.4.29), we get the second estimate in (4.4.5). This completes the proof of our theorem. □

4.5. A lower bound for the Hodge Laplacian on X

We use the notation as in Subsection 3.1. Recall that $e_1, \cdots, e_m$ is an orthogonal basis of TX or $\mathfrak{p}$. Set

$$C^{\mathfrak{g},H} = -\sum_{i=1}^{m} e_i^2 \in U\mathfrak{g}. \tag{4.5.1}$$

Let $C^{\mathfrak{g},H,E}$ be its action on E. Then

$$C^{\mathfrak{g},E} = C^{\mathfrak{g},H,E} + C^{\mathfrak{k},E}. \tag{4.5.2}$$

Let $\Delta^{H,X}$ be the Bochner-Laplace operator on bundle $\Lambda^\cdot(T^*X) \otimes F$. Put

$$\begin{aligned} \Theta(E) = \frac{S^X}{4} - \frac{1}{8} \langle R^{TX}(e_i, e_j)e_k, e_\ell \rangle c(e_i)c(e_j)\widehat{c}(e_k)\widehat{c}(e_\ell) \\ -C^{\mathfrak{g},H,E} + \frac{1}{2}\big(c(e_i)c(e_j) - \widehat{c}(e_i)\widehat{c}(e_j)\big) R^F(e_i, e_j). \end{aligned} \tag{4.5.3}$$

Then $\Theta(E)$ is a self-adjoint section of $\mathrm{End}(\Lambda^\cdot(T^*X) \otimes F)$, which is parallel with respect to $\nabla^{\Lambda^\cdot(T^*X)\otimes F,u}$. By [BMZ17, Eq.(8.39)], we have

$$\mathbf{D}^{X,F,2} = -\Delta^{H,X} + \Theta(E). \tag{4.5.4}$$

Let $\langle \cdot, \cdot \rangle_{L_2}$ be the L_2 scalar product of $\Omega^\cdot_c(X, F)$. If $s \in \Omega^\cdot_c(X, F)$, we have

$$\langle \mathbf{D}^{X,F,2} s, s \rangle_{L_2} \geq \langle \Theta(E) s, s \rangle_{L_2}. \tag{4.5.5}$$

Let $\Delta^{H,X,i}$ denote the Bochner-Laplace operator acting on $\Omega^i(X, F)$, and let $p_t^{H,i}(x, x')$ be the kernel of $\exp(t\Delta^{H,X,i}/2)$ on X with respect to dx'. We denote by $p_t^{H,i}(g) \in \mathrm{End}(\Lambda^i(\mathfrak{p}^*) \otimes E)$ its lift to G explained in Subsection 2.3.

Let Δ_0^X be the scalar Laplacian on X with the heat kernel $p_t^{X,0}$. Let $||p_t^{H,i}(g)||$ be the operator norm of $p_t^{H,i}(g)$ in $\mathrm{End}(\Lambda^i(\mathfrak{p}^*) \otimes E)$. By [MP13b, Proposition 3.1], if $g \in G$, then

$$||p_t^{H,i}(g)|| \leq p_t^{X,0}(g). \tag{4.5.6}$$

Let p_t^H be the kernel of $\exp(t\Delta^{H,X}/2)$, then

$$p_t^H = \oplus_{i=1}^p p_t^{H,i}. \tag{4.5.7}$$

Let $q_t^{X,F}$ be the heat kernel associated with $\mathbf{D}^{X,F,2}/2$, by (4.5.4), for $x, x' \in X$,

$$q_t^{X,F}(x,x') = \exp(-t\Theta(E)/2)p_t^H(x,x'). \tag{4.5.8}$$

Now we consider the representations (E_d, ρ^{E_d}), $d \in \mathbb{N}$ constructed in Subsection 4.2 for a nondegenerate λ. By [BMZ17, Theorem 4.4 & Remark 4.5], there exist $c > 0$, $C > 0$ such that, for $d \in \mathbb{N}$,

$$\Theta(E_d) \geq cd^2 - C. \tag{4.5.9}$$

By (4.5.4), (4.5.5), (4.5.9), we get

$$\mathbf{D}^{X,F_d,2} \geq cd^2 - C. \tag{4.5.10}$$

Lemma 4.5.1. *There exists $d_0 \in \mathbb{N}$ and $c_0 > 0$ such that if $d \geq d_0$, $x, x' \in X$*

$$||q_t^{X,F_d}(x,x')|| \leq e^{-c_0 d^2 t} p_t^{X,0}(x,x'). \tag{4.5.11}$$

Proof. By (4.5.9), there exist $d_0 \in \mathbb{N}$, $c' > 0$ such that if $d \geq d_0$,

$$\Theta(E_d) \geq c'd^2. \tag{4.5.12}$$

Then if $t > 0$,

$$||\exp(-t\Theta(E_d)/2)|| \leq e^{-c'd^2t/2}. \tag{4.5.13}$$

By (4.5.6) - (4.5.8), (4.5.13), we get (4.5.11). This completes our proof. □

Recall that Γ is a cocompact torsion-free discrete subgroup of G preserved by σ. For $t > 0$, $x \in X$, $\gamma \in \Gamma$, set

$$v_t(F_d, \gamma\sigma, x) = \frac{1}{2}\mathrm{Tr_s}^{\Lambda^{\cdot}(T^*X)\otimes F_d}\left[\left(N^{\Lambda^{\cdot}(T^*X)} - \frac{m}{2}\right)q_{t/2}^{X,F_d}(x, \gamma\sigma(x))\gamma\sigma\right]. \tag{4.5.14}$$

Then Lemma 4.5.1 implies the following lemma.

Lemma 4.5.2. *There exist $C_0 > 0$, $c_0 > 0$ such that if d is large enough, for $t > 0$, $x \in X$, $\gamma \in \Gamma$,*

$$|v_t(F_d, \gamma\sigma, x)| \leq C_0 \dim(E_d) e^{-c_0 d^2 t} p_{t/2}^{X,0}(x, \gamma\sigma(x)). \tag{4.5.15}$$

Proposition 4.5.3. *There exist constants $C > 0$, $c > 0$ such that if $x \in X$, $t \in]0,1]$, then*

$$\sum_{\gamma\in\Gamma, \gamma\sigma \text{ nonelliptic}} p_t^{X,0}(x, \gamma\sigma(x)) \leq C\exp(-c/t). \tag{4.5.16}$$

Proof. By [Don79, Theorem 3.3], there exists $C_0 > 0$ such that when $0 < t \leq 1$,

$$p_t^{X,0}(x,x') \leq C_0 t^{-m/2} \exp\left(-\frac{d^2(x,x')}{4t}\right). \tag{4.5.17}$$

By Proposition 2.6.3, we have $c_{\Gamma,\sigma} > 0$. Then using Lemma 2.6.4, (4.5.17), and the arguments as in the proof of [MP13b, Proposition 3.2], we get (4.5.16). □

4.6. A proof to Theorem 1.0.1

Now we work on $M = \Gamma\backslash X$. The flat vector bundle F_d in previous subsection descends to M, which we still denote by F_d. The action of Σ^σ lifts to F_d so that the de Rham-Dirac operator $\mathbf{D}^{M,F_d}$ commutes with its action.

By (4.5.10), we have

$$\mathbf{D}^{M,F_d,2} \geq cd^2 - C. \tag{4.6.1}$$

Then if d is large enough, we have

$$H^{\bullet}_{\mathrm{dR}}(M, F_d) = 0. \tag{4.6.2}$$

By (3.2.5), (3.2.7), if d is large enough, we have

$$\chi_\sigma(M, F_d) = 0, \quad \chi'_\sigma(M, F_d) = 0. \tag{4.6.3}$$

Recall that the function $b_t(F_d, h^{F_d})$ is defined in (3.2.6). Then by (3.2.9), (3.2.10), (4.6.3), we have

$$\mathcal{T}_\sigma(g^{TM}, \nabla^{F_d,f}, h^{F_d}) = -\int_0^{+\infty} b_t(F_d, h^{F_d}) \frac{dt}{t}. \tag{4.6.4}$$

Recall that E_σ is the finite set of elliptic classes in $[\Gamma]_\sigma$. Set

$$E^1_\sigma = \{[\gamma]_\sigma \in E_\sigma \mid \delta(Z_\sigma(\gamma)^0) = 1\}. \tag{4.6.5}$$

Proposition 4.6.1. *There exists $c > 0$ such that for d large enough,*

$$\begin{aligned}\mathcal{T}_\sigma(g^{TM}, \nabla^{F_d,f}, h^{F_d}) &= -\frac{1}{2} \sum_{[\gamma]_\sigma \in E^1_\sigma} \mathrm{Vol}(\Gamma \cap Z_\sigma(\gamma)\backslash X(\gamma\sigma)) \\ \cdot \int_0^d \mathrm{Tr_s}^{[\gamma\sigma]}&\left[\big(N^{\Lambda^\bullet(T^*X)} - \frac{m}{2}\big)\Big(1 - \frac{t}{2d^2}\mathbf{D}^{X,F_d,2}\Big) e^{-\frac{t}{4d^2}\mathbf{D}^{X,F_d,2}}\right] \frac{dt}{t} + \mathcal{O}(e^{-cd}).\end{aligned} \tag{4.6.6}$$

If $E^1_\sigma = \emptyset$, as $d \to +\infty$,

$$\mathcal{T}_\sigma(g^{TM}, \nabla^{F_d,f}, h^{F_d}) = \mathcal{O}(e^{-cd}). \tag{4.6.7}$$

Proof. By (4.6.4), we have

$$\mathcal{T}_\sigma(g^{TM}, \nabla^{F_d,f}, h^{F_d}) = -\int_{1/d}^{+\infty} b_t(F_d, h^{F_d}) \frac{dt}{t} - \int_0^d b_{t/d^2}(F_d, h^{F_d}) \frac{dt}{t}. \tag{4.6.8}$$

By (4.6.1) and using the same arguments as in [BMZ17, Subsection 7.2], we can get that there exists $c > 0$ such that

$$\int_{1/d}^{+\infty} b_t(F_d, h^{F_d}) \frac{dt}{t} = \mathcal{O}(e^{-cd}). \tag{4.6.9}$$

By (3.2.6), (4.5.14), we get

$$b_t(F_d, h^{F_d}) = (1 + 2t\frac{\partial}{\partial t}) \int_M \sum_{\gamma\in\Gamma} v_t(F_d, \gamma\sigma, z)dz. \tag{4.6.10}$$

We split the sum in (4.6.10) into two parts:

$$\sum_{\gamma\in\Gamma, \gamma\sigma \text{ elliptic}} + \sum_{\gamma\in\Gamma, \gamma\sigma \text{ nonelliptic}} \tag{4.6.11}$$

When writing down the integrals explicitly with the heat kernels, the integral of the first part in (4.6.11) is just the sum of the twisted orbital integrals associated with the elliptic classes in E_σ. If $[\gamma]_\sigma \in E_\sigma$ and if $[\gamma]_\sigma \notin E^1_\sigma$, then by Theorem 3.3.3, Corollary 3.3.4, we get that for $t > 0$,

$$\mathrm{Tr_s}^{[\gamma\sigma]}\Big[\big(N^{\Lambda^\cdot(T^*X)} - \frac{m}{2}\big)\exp(-t\mathbf{D}^{X,F_d,2}/4d^2)\Big] = 0. \tag{4.6.12}$$

This gives the the first sum in the right-hand side of (4.6.6).

If $x \in X$, put

$$h_t(F_d, h^{F_d}, x) = \sum_{\gamma\in\Gamma, \gamma\sigma \text{ nonelliptic}} v_t(F_d, \gamma\sigma, x). \tag{4.6.13}$$

Then it is enough to prove that

$$\int_0^d (1 + 2t\frac{\partial}{\partial t}) \int_M h_{t/d^2}(F_d, h^{F_d}, z)dz \frac{dt}{t} = \mathcal{O}(e^{-cd}). \tag{4.6.14}$$

Indeed, using Lemma 4.5.2 and Proposition 4.5.3, there exists $C > 0$, $c' > 0$, $c'' > 0$ such that if d is large enough, $0 < t \leq d$, then

$$|h_{t/d^2}(F_d, h^{F_d}, x)| \leq C \dim(E_d) e^{-c't} \exp(-c''d^2/t). \tag{4.6.15}$$

Recall that $n_\lambda = \dim_{\mathbb{C}} N_\lambda$. By (4.2.34), there exists $C_0 > 0$ such that

$$\dim(E_d) \leq C_0 d^{n_\lambda}. \tag{4.6.16}$$

By (4.6.15), (4.6.16), we have

$$\begin{aligned} &\Big|\int_0^1 h_{t/d^2}(F_d, h^{F_d}, x)\frac{dt}{t}\Big| \leq Ce^{-c''d^2/2} \dim(E_d) \int_0^1 e^{-c''d^2/2t}\frac{dt}{t} = \mathcal{O}(e^{-cd}), \\ &\Big|\int_1^d h_{t/d^2}(F_d, h^{F_d}, x)\frac{dt}{t}\Big| \leq Ce^{-c''d} \dim(E_d) \int_1^d e^{-c't}\frac{dt}{t} = \mathcal{O}(e^{-cd}). \end{aligned} \tag{4.6.17}$$

By (4.6.15) - (4.6.17), we get (4.6.14). The equation (4.6.7) follows from (4.6.6). This completes the proof of our proposition. □

By (2.6.10), ${}^\sigma M$ has different connected components, each component is also a compact locally symmetric space associated with an elliptic class $[\gamma]_\sigma$. Then (4.6.6) says that only the components with the fundamental rank 1 contribute to the leading terms of the asymptotics of $\mathcal{T}_\sigma(g^{TM}, \nabla^{F_d,f}, h^{F_d})$ as $d \to +\infty$.

Now suppose that $E^1_\sigma \neq \emptyset$. We use the notation of Subsection 4.2. As explained in the end of Subsection 4.3, for each $[\gamma]_\sigma \in E^1_\sigma$, we fix $g \in G$, $k \in K$ such that $k^{-1} = g\gamma\sigma(g^{-1})$. Then put $\mathcal{J}(\gamma) = \mathcal{J}(k^{-1})$. For $j \in \mathcal{J}(\gamma)$, let $n(\gamma\sigma)$, $R^j_{\gamma,\lambda}$, $r_{\gamma,j}$, φ^j_γ, $e^j_{\gamma,t}$, $d^j_{\gamma,t}$ $W^j_{\gamma\sigma}$ be the ones associated with k^{-1}. By Lemma 4.3.2, these quantities do not depend on the choice of g or k. Set

$$\begin{aligned} &m(\sigma) = \max\{n(\gamma\sigma) \mid [\gamma]_\sigma \in E^1_\sigma\}, \\ &E^{1,\max}_\sigma = \{[\gamma]_\sigma \in E^1_\sigma \mid n(\gamma\sigma) = m(\sigma)\}. \end{aligned} \tag{4.6.18}$$

Theorem 4.6.2. *If $E^1_\sigma \neq \emptyset$, as $d \to +\infty$,*

$$\begin{aligned} &d^{-m(\sigma)-1}\mathcal{T}_\sigma(g^{TM}, \nabla^{F_d,f}, h^{F_d}) \\ &= \sum_{[\gamma]_\sigma \in E^{1,\max}_\sigma} \mathrm{Vol}(\Gamma \cap Z_\sigma(\gamma)\backslash X(\gamma\sigma))\big[\sum_{j \in \mathcal{J}(\gamma)^{\max}} r^d_{\gamma,j}\varphi^j_\gamma[W^j_{\gamma\sigma}]^{\max}\big] + \mathcal{O}(\frac{1}{d}). \end{aligned} \tag{4.6.19}$$

Proof. For $[\gamma]_\sigma \in E^1_\sigma$, γ is C^σ-conjugate to k^{-1}. Then

$$\begin{aligned} &\mathrm{Tr_s}^{[k^{-1}\sigma]}\big[\big(N^{\Lambda^\bullet(T^*X)} - \frac{m}{2}\big)\exp(-t\mathbf{D}^{X,F_d,2}/2)\big] \\ &= \mathrm{Tr_s}^{[\gamma\sigma]}\big[\big(N^{\Lambda^\bullet(T^*X)} - \frac{m}{2}\big)\exp(-t\mathbf{D}^{X,F_d,2}/2)\big]. \end{aligned} \tag{4.6.20}$$

If $[\gamma]_\sigma \in E^1_\sigma$, by Theorem 4.4.1 and by (4.6.12), (4.6.20), as $d \to +\infty$,

$$\begin{aligned} &\int_0^1 \mathrm{Tr_s}^{[\gamma\sigma]}\big[\big(N^{\Lambda^\bullet(T^*X)} - \frac{m}{2}\big)(1 - t\mathbf{D}^{X,F_d,2}/2d^2)\exp(-t\mathbf{D}^{X,F_d,2}/4d^2)\big]\frac{dt}{t} \\ &= 2d^{n(\gamma\sigma)+1}\sum_{j \in \mathcal{J}(\gamma)^{\max}} r^d_{\gamma,j}\varphi^j_\gamma \int_0^1 [d^j_{\gamma,t/4}]^{\max}\frac{dt}{t} + \mathcal{O}(d^{n(\gamma\sigma)}). \end{aligned} \tag{4.6.21}$$

Note that

$$\begin{aligned} &\int_1^d \mathrm{Tr_s}^{[\gamma\sigma]}\big[\big(N^{\Lambda^\bullet(T^*X)} - \frac{m}{2}\big)(1 - \frac{t}{2d^2}\mathbf{D}^{X,F_d,2})\exp(-\frac{t}{4d^2}\mathbf{D}^{X,F_d,2})\big]\frac{dt}{t} \\ &= \int_1^d \mathrm{Tr_s}^{[\gamma\sigma]}\big[\big(N^{\Lambda^\bullet(T^*X)} - \frac{m}{2}\big)\exp(-\frac{t}{4d^2}\mathbf{D}^{X,F_d,2})\big]\frac{dt}{t} \\ &\quad + 2\mathrm{Tr_s}^{[\gamma\sigma]}\big[\big(N^{\Lambda^\bullet(T^*X)} - \frac{m}{2}\big)\exp(-\frac{1}{4d}\mathbf{D}^{X,F_d,2})\big] \\ &\qquad - 2\mathrm{Tr_s}^{[\gamma\sigma]}\big[\big(N^{\Lambda^\bullet(T^*X)} - \frac{m}{2}\big)\exp(-\frac{1}{4d^2}\mathbf{D}^{X,F_d,2})\big]. \end{aligned} \tag{4.6.22}$$

Similarly, by (4.1.23), we have

$$\int_1^d [d^j_{\gamma,t/4}]^{\max}\frac{dt}{t} = \int_1^d [e^j_{\gamma,t/4}]^{\max}\frac{dt}{t} + 2[e^j_{\gamma,d/4}]^{\max} - 2[e^j_{\gamma,1/4}]^{\max}. \tag{4.6.23}$$

Also we have

$$\int_d^{+\infty} d^j_{\gamma,t/4}\frac{dt}{t} = \mathcal{O}(e^{-cd}). \tag{4.6.24}$$

Combining (4.4.3) - (4.4.6) with (4.6.22) - (4.6.24), we get that as $d \to +\infty$

$$d^{-n(\gamma\sigma)-1}\int_1^d \mathrm{Tr}_{\mathrm{s}}^{[\gamma\sigma]}\big[\big(N^{\Lambda^\bullet(T^*X)} - \frac{m}{2}\big)\big(1 - \frac{t\mathbf{D}^{X,F_d,2}}{2d^2}\big)\exp(-t\mathbf{D}^{X,F_d,2}/4d^2)\big]\frac{dt}{t}$$
$$= 2\sum_{j\in\mathcal{J}(\gamma)^{\max}} r^d_{\gamma,j}\varphi^j_\gamma \int_1^{+\infty}[d^j_{\gamma,t/4}]^{\max}\frac{dt}{t} + \mathcal{O}(\frac{1}{d}). \tag{4.6.25}$$

By (4.3.5), (4.6.6), (4.6.18), (4.6.21), (4.6.25), we get (4.6.19). This completes the proof of our theorem. □

Combing Proposition 4.6.1 and Theorem 4.6.2, we get Theorem 1.0.1.

Corollary 4.6.3. *If $\sigma \in \Sigma$ is of finite order N_0 and preserves Γ, then each number $r_{\gamma,j}$ appeared in right-hand side of (4.6.19) is a N_0-th root of unity.*

Proof. If $\gamma \in \Gamma$ is such that $\gamma\sigma$ is elliptic, then $(\gamma\sigma)^{N_0} \in \Gamma$ is elliptic. Since Γ is torsion-free, then

$$(\gamma\sigma)^{N_0} = 1. \tag{4.6.26}$$

Let $k^{-1} \in K$ be an element that is C_σ-conjugate to γ. Then we also have $(k^{-1}\sigma)^{N_0} = 1$. By Proposition 4.2.6, $r_{\gamma,j}$ represents the unitary action of $k^{-1}\sigma$ on the fiber L_λ at its fixed points, thus it must be a N_0-th root of unity. □

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