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Master Thesis

in Physics

submitted by

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born in New York

2024

Radiative Corrections to

Nonlinear Compton Scattering

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Strahlungskorrekturen zur nichtlinearen Compton Streuung:

Diese Arbeit befasst sich mit Strahlungskorrekturen der nichtlinearen Compton Streuung, ein Prozess aus der Quantenelektrodynamik mit starken Hintergrundfeldern. Es wird ein allgemeiner Ausdruck für die Korrektur hergeleitet, die vom Polarisationsoperator in einer ebenen Welle versursacht wird. Weiterhin wird diese Korrektur im Rahmen einer Näherung untersucht, in der lokal ein konstantes Feld angenommen wird, und es wird im Falle eines inkohärenten Prozesses der Grenzfall von hohen Feldstärken untersucht. Inkohärente Prozesse sind solche, die mit der Gesamtdauer des Laserpulses skalieren. Für die gesamte Klasse von inkohärenten Korrekturen zur nichtlinearen Compton Streuung, welche nur aus Prozessen in erster Ordnung der Feinstrukturkonstante α bestehen, wird ebenfalls der Grenzfall von hohen Feldstärken untersucht.

Radiative Corrections to Nonlinear Compton Scattering:

This thesis deals with radiative corrections concerning the strong field QED process of nonlinear Compton scattering. A general expression for the corrections due to the polarization operator in a plane wave background field is computed and studied in the so called locally constant field approximation in which the background field is locally treated as a constant crossed field. Also, the high field limit which is the formal limit of having very high laser intensities, while keeping the electron energy constant, of the polarization correction and other coherent corrections is computed. Coherent processes are those that scale with the total time in which an electron is exposed to the background field. For all such coherent corrections to nonlinear Compton Scattering that are of first order in the fine structure constant α, as well as their resummed expression, the high field limit is computed.

Contents

1 Introduction

Strong field QED is the application of the fundamental theory of quantum electrodynamics (QED) to a system in which a high intensity background electromagnetic field is present. The theory of QED is regarded as one of the most successful theories of physics as it has been tested in experiment to astonishing precision. An important example is the electron magnetic moment for which measurements and theory have agreement on up to 10 significant figures [19]. Even so, there have been only few experiments so far to test the strong field regime of QED. The most notable strong field QED experiment to date has been carried out at SLAC in 1996 where a 46.6 GeV electron beam was collided with a laser of 10^{18} W/cm^2 peak intensity [11, 12]. The experiment succeeded at measuring the signature first order processes of strong field QED which are nonlinear Compton Scattering and nonlinear Breit-Wheeler pair production. Nonlinear Compton Scattering is the process in which an electron emits a single high energy photon while traveling through the laser field. Nonlinear Breit-Wheeler pair production is the process where a high energy photon, typically one that was emitted by an electron before, decays into an electron positron pair. Both these processes are ruled out in vacuum QED as they are incompatible with energy-momentum conservation and they only become possible in strong field QED due to the interaction of the electron with the background field. Fundamentally, even the background electromagnetic field consists of photons and is described by the laws of (vacuum) QED. However, if the photon field can be described by a coherent state of sufficiently high intensity, e.g. a high intensity laser field, it may be well described classically. This approximation holds as long as each mode of the photon field is occupied by a large amount $N_\gamma \gg 1$ of photons. The intensity I_0 needed for this condition is determined by the laser frequency ω_0 and must fulfill $I_0 \gg \omega_0^4$ [27], where natural units with $c = \hbar = 1$ are assumed. In the optical regime ($\omega_0 \sim 1$ eV), this corresponds to $I_0 \gg 6 \times 10^5$ W/cm^2. The success of QED is often accredited to the fact that the fine structure constant $\alpha = e^2/4\pi \approx 1/137$ is a small number, thus making it possible to apply perturbation theory. Conceptually it is easy to see that a sufficiently strong background field will change an electrons trajectory from that in the vacuum case enough so that it cannot be viewed as a small perturbation from the vacuum dynamics, and thus leading to the breakdown of conventional perturbation theory. This issue is typically overcome by employing the so called Furry picture in which one takes into account the interaction of the electron and the background field by finding exact solutions to the interacting Dirac equation. Thus by employing the Furry picture (typically a plane wave background as an approximation for a laser is assumed) one can make use of perturbation theory once again and expand observables in powers of the fine structure constant α. In this context, the first result of this thesis will be the calculation of an order α^2 radiative correction of nonlinear Compton Scattering which comes from the photon vacuum polarization. In strong field QED however, it turns out that there are even more instances where perturbation theory breaks down. For example, it is known that tree level probabilities calculated within the Furry picture exceed unity if the laser pulse length is long enough, thus indicating another breakdown of perturbation theory. This breakdown can be circumvented by taking into account so called damping effects due to incoherent processes. Yet another different instance where perturbation theory breaks down is found in the limit of extremely high laser intensities. According to the famous Ritus-Narozhny (RN) conjecture, in this regime the effective expansion parameter is $\alpha\chi^{2/3}$ where χ is the quantum nonlinearity parameter which is given by the field strength in units of the critical field $F_{cr} = m^2/|e|$ that an electron of mass m feels in its own rest frame. Thus according to the RN-conjecture there is another breakdown of perturbation theory when $\alpha\chi^{2/3} \approx 1$. Unlike in the other two cases, it is currently not known how to treat this regime of strong field QED, as in this case all radiative corrections of higher orders in α become relevant. In light of the RN conjecture it has become of theoretical interest to study the high field limit, corresponding to $\chi \gg 1$, of strong field QED processes. This will be the topic of the second part of the thesis where we will study the high field limit of the aforementioned incoherent processes that are responsible for damping effects in long laser pulses.

2 Strong Field QED

Quantum Electrodynamics (QED) is described by the following Lagrangian [9, 33]:

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\hat{\partial} - m)\psi - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - e\bar{\psi}\gamma_\mu\psi\mathcal{A}^\mu. \tag{2.1}$$

Here ψ denotes the electron/positron field which is given as a four component Dirac spinor with corresponding mass m. Dirac spinors can be acted on by the 4×4 gamma matrices γ^μ with $\mu = 0, 1, 2, 3$ which fulfill the anti-commutation relation $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, where $\eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ is the Minkowski metric. A specific representation of the gamma matrices is given in Appendix A. The gamma matrices are also used to define $\bar{\psi} \equiv \psi^\dagger\gamma^0$ and $\hat{\partial} \equiv \gamma^\mu\partial_\mu$ (for a general four-vector a^μ we define $\hat{a} = a_\mu\gamma^\mu$). The four-vector $\mathcal{A}^\mu$ describes the photon field and $\mathcal{F}^{\mu\nu} = \partial^\mu\mathcal{A}^\nu - \partial^\nu\mathcal{A}^\mu$ is the electromagnetic field tensor. In strong field QED we assume to have a classical background gauge field A^μ which can be taken into account by adding another interaction term to the Lagrangian:

$$\mathcal{L}_{\text{SFQED}} = \bar{\psi}(i\hat{\partial} - m)\psi - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - e\bar{\psi}\gamma_\mu\psi\mathcal{A}^\mu - e\bar{\psi}\gamma_\mu\psi A^\mu. \tag{2.2}$$

A kinetic term for the background field is omitted, since we already assume that it fulfills the free Maxwell's equations [21]. Calculations in vacuum QED are most often performed using perturbation theory. In the vacuum case this approach works well because of the small value of the fine-structure constant $\alpha = e^2/4\pi \approx 1/137$, where $e < 0$ is the electrons charge, making sure that interactions can be treated as small perturbations to the free vacuum dynamics. From the Lagrangian (2.2) it is clear that the coupling to the background field is proportional to its field amplitude, which means that perturbation theory breaks down for sufficiently strong background fields and interactions with the background field have to be treated non-perturbatively. A non-perturbative description of the dynamics of an electron in an external field $A^\mu(x)$ requires solving the interacting Dirac equation which is given by:

$$(i\hat{\partial} - e\hat{A} - m)\psi = 0, \tag{2.3}$$

which can be obtained by setting the radiation field $\mathcal{A}^\mu$ in Eq. (2.2) to zero and applying the Euler-Lagrange equations. The radiation field $\mathcal{A}^\mu$ will be quantized later on and its influence will be treated perturbatively. While there are no general analytical solutions of Eq. (2.3) for an arbitrary background field, there is a solution in the case of a plane wave background due to Volkov [45]. Plane waves are often used to approximate the conditions of a high intensity laser pulse [14]. When working with plane waves in particular, it is useful to introduce a new set of coordinates called light cone coordinates.

2.1 Light Cone Coordinates

When considering a plane wave background field in strong field QED it is convenient to employ light cone coordinates. For this, we define the following quantities:

$$n^\mu = (1, \boldsymbol{n}), \qquad \tilde{n}^\mu = (1/2)(1, -\boldsymbol{n}), \qquad e^\mu_j = (0, \boldsymbol{e}_j) \tag{2.4}$$

where $\boldsymbol{n}$ is an arbitrary unit vector which will be chosen such that it aligns with the propagation direction of the plane wave field. The vectors $\boldsymbol{e}_j$ with $j = 1, 2$ are unit vectors which are perpendicular to each other and to $\boldsymbol{n}$, such that $\boldsymbol{n} = \boldsymbol{e}_1 \times \boldsymbol{e}_2$. For an arbitrary four vector $v^\mu = (v^0, \boldsymbol{v})$ the light cone coordinates are defined as

$$\begin{aligned} v_+ &= (\tilde{n}v) = \left(v^0 + \boldsymbol{n} \cdot \boldsymbol{v}\right)/2 \\ v_- &= (nv) = v^0 - \boldsymbol{n} \cdot \boldsymbol{v} \\ v_j &= -(e_j v) = \boldsymbol{e}_j \cdot \boldsymbol{v} \end{aligned} \tag{2.5}$$

where we used the notation $a_\mu b^\mu = (ab)$ for the product of two arbitrary four-vectors. Any four-vector v^μ may be expressed in terms of its light cone components in the following way:

$$v^\mu = v_+ n^\mu + v_- \tilde{n}^\mu + v_1 e_1^\mu + v_2 e_2^\mu. \tag{2.6}$$

Also the metric may be decomposed in terms of the light cone basis:

$$\eta_{\mu\nu} = n_\mu \tilde{n}_\nu + \tilde{n}_\mu n_\nu - e_{1\mu} e_{1\nu} - e_{2\mu} e_{2\nu}. \tag{2.7}$$

By defining the perpendicular vector $\boldsymbol{v}_\perp = v_1 \boldsymbol{e}_1 + v_2 \boldsymbol{e}_2$, the scalar product of two four-vectors a^μ and b^μ is given by

$$(ab) = a_+ b_- + a_- b_+ - \boldsymbol{a}_\perp \cdot \boldsymbol{b}_\perp. \tag{2.8}$$

The four dimensional integration measure can be expressed in light cone coordinates as:

$$\int d^4 a = \int da_- da_+ da_\perp, \tag{2.9}$$

where $da_\perp = da_1 da_2$.

2.2 Plane Waves

Plane waves are the solutions to the Maxwell equations, which only depend on the parameter $\phi \equiv x_- = (nx)$. In Lorenz gauge ($\partial_\mu A^\mu = 0$) the field equations for the free electromagnetic field are given by [9]:

$$\partial^2 A^\mu = 0. \tag{2.10}$$

It can be seen that any four-vector field which only depends on $\phi = (nx)$ provides a solution to Eq. (2.10). Gauge invariance allows to impose the additional constraints $A^0 = A^3 = 0$ and therefore limiting the field to the physical transverse degrees of freedom. A general plane wave vector potential is then given by:

$$A^\mu(\phi) = \psi_1(\phi)a_1^\mu + \psi_2(\phi)a_2^\mu, \tag{2.11}$$

where ψ_j can be arbitrary functions with the physical requirement that they vanish at infinity such that $\psi_i(\pm\infty) = \psi_i'(\pm\infty) = 0$ (the prime denotes the derivative $d/d\phi$). Here we defined the four-vectors $a_j^\mu = (0, \boldsymbol{a}_j)$, where the $\boldsymbol{a}_j$ are constant vectors parallel to $\boldsymbol{e}_j$ that include the amplitude of the vector potential. We can therefore impose the normalization condition $\psi(\phi) \leq 1$. The field tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ corresponding to Eq. (2.11) is given by:

$$F^{\mu\nu}(\phi) = f_1^{\mu\nu}\psi_1'(\phi) + f_2^{\mu\nu}\psi_2'(\phi) \tag{2.12}$$

where $f_j^{\mu\nu} = n^\mu a_j^\nu - n^\nu a_j^\mu$. The electric field vector $E^i = -F^{0i}$ is given by

$$\boldsymbol{E}(\phi) = -\psi_1'(\phi)\boldsymbol{a}_1 - \psi_2'(\phi)\boldsymbol{a}_2. \tag{2.13}$$

2.3 Important Parameters of Strong Field QED

The classical intensity parameters are defined as:

$$\xi_i = \frac{|e|}{m}\sqrt{-a_i^2}, \qquad \xi = \sqrt{\xi_1^2 + \xi_2^2} = \frac{|eA_0|}{m}, \tag{2.14}$$

where A_0 is the amplitude of the vector potential. From its definition it can be seen that ξ quantifies the coupling of an electron to the external field (see the last term in the Lagrangian (2.2)). Thus, the perturbative approach to QED breaks down when $\xi \gtrsim 1$. Another important parameter of strong field QED is the critical field strength F_{cr}:

$$F_{cr} = \frac{m^2}{|e|}, \tag{2.15}$$

with the electrons mass m and the charge e. The critical field (also known as the Schwinger limit) gives a scale for the field strength at which the vacuum is unstable to spontaneous electron-positron pair creation [14], it must be noted however that in a plane wave this process cannot occur due to the fields symmetries [43]. By denoting the maximum amplitude of the electric field $E_0 = \omega_0\sqrt{|\boldsymbol{a}_1|^2 + |\boldsymbol{a}_2|^2}$, where ω_0 is the central frequency of the plane wave, the parameter ξ can be expressed as

$$\xi = \frac{m}{\omega_0}\frac{E_0}{F_{cr}} = \frac{|e|E_0}{m\omega_0}. \tag{2.16}$$

The quantum nonlinearity parameter χ is the field strength in units of the critical field in the electrons rest frame and is given by [14]:

$$\chi = \frac{p_-}{m}\frac{E_0}{F_{cr}} = \frac{|e|E_0 p_-}{m^3}. \tag{2.17}$$

At last, there is another important parameter of strong field QED given by:

$$\eta = \frac{\chi}{\xi} = \frac{\omega_0}{m}\frac{p_-}{m}. \tag{2.18}$$

For optical lasers ($\omega_0 \sim 1$ eV) the condition $\xi \gtrsim 1$ takes place at intensities of the order of 10^8 W/cm^2, which can be easily reached by modern laser facilities [14]. Record intensities that have been reached are of the order of 10^{23} W/cm^2 [47] which, in the optical regime, corresponds to $\xi \sim 230$, and even higher intensity lasers are currently in development, envisaging magnitudes of 10^{24} W/cm^2 [44]. Even so, intensities that correspond to the critical field strength, which is $I_{cr} \approx 10^{29}$ W/cm^2, are not feasible in the near future. There are however experimental efforts using high energy electron beams to reach and exceed these intensities in the electrons rest frame, corresponding to $\chi \gtrsim 1$ [4, 25, 28].

2.4 The Furry Picture

In vacuum QED one employs solutions of the free Dirac equation as modes of the quantum field which describes electrons and positrons. In the Furry picture one replaces the free modes with the solutions of the interacting Dirac Eq. (2.3). Before discussing the Volkov solutions, which are the solutions of the Dirac equation in a background plane wave electromagnetic field, we are going to review the solution of the free Dirac equation.

2.4.1 Free Dirac States

The free Dirac equation is given by [9, 33]:

$$(i\hat{\partial} - m)\psi = 0, \tag{2.19}$$

for which the solutions are given by (we use the normalisation conventions as in Ref. [33]):

$$\begin{aligned} \psi_{e^-,s,p}(x) &= e^{-i(px)}u_s(p) \\ \psi_{e^+,s,p}(x) &= e^{i(px)}v_s(p) \end{aligned} \tag{2.20}$$

corresponding to the electron and positron wave functions with four-momentum p^μ obeying $p^2 = m^2$. The spinors $u_s(p)$ and $v_s(p)$ are the solutions of

$$(\hat{p} - m)u_s(p) = 0 \quad \text{and} \quad (\hat{p} + m)v_s(p) = 0. \tag{2.21}$$

The equations (2.21) each have two independent solutions which are labeled by the spin quantum number $s = \pm 1$. The free spinors are normalized such that [33]:

$$u_s^\dagger(p)u_{s'}(p) = 2\varepsilon\delta_{ss'} \qquad v_s^\dagger(p)v_{s'}(p) = 2\varepsilon\delta_{ss'}, \tag{2.22}$$

where $\varepsilon = p^0$ is the energy, and they further obey the following relations:

$$\begin{aligned} \bar{u}_s(p)u_{s'}(p) &= 2m\delta_{ss'}, \quad \bar{u}_s(p)\gamma^\mu u_{s'}(p) = 2p^\mu\delta_{ss'} \\ \bar{v}_s(p)v_{s'}(p) &= -2m\delta_{ss'}, \quad \bar{v}_s(p)\gamma^\mu v_{s'}(p) = 2p^\mu\delta_{ss'}, \end{aligned} \tag{2.23}$$

where for an arbitrary spinor it is defined $\bar{w} = w^\dagger\gamma^0$. Also the following spin summations hold:

$$\begin{aligned} \sum_s u_s(p)\bar{u}_s(p) &= \hat{p} + m, \\ \sum_s v_s(p)\bar{v}_s(p) &= \hat{p} - m. \end{aligned} \tag{2.24}$$

2.4.2 Volkov States

We now turn to the case of having a plane wave background field. Solutions to the Dirac Equation (2.3) in a plane wave are called Volkov states [45] (See also Ref. [9]) and are given by:

$$\Psi_p(x) = E_{p,x}u_s(p), \tag{2.25}$$

where

$$E_{p,x} = \left[1 + \frac{e}{2p_-}\hat{n}\hat{A}(\phi)\right]e^{iS_p(x)} \tag{2.26}$$

is the Ritus matrix, and

$$S_p(x) = -(px) - \int_0^\phi d\phi' \left[\frac{e\left(pA\left(\phi'\right)\right)}{p_-} - \frac{e^2A^2\left(\phi'\right)}{2p_-}\right]. \tag{2.27}$$

Here $u_s(p)$ is the same spinor that appears in the free solution (2.20). Thus we can see that the Volkov solution recovers the free solution at $\phi \to \pm\infty$ as we expect the field $A^\mu(\phi)$ to vanish in this limit. It is also useful to define another Ritus matrix given by:

$$\bar{E}_{p,x} = \left[1 + \frac{e}{2p_-}\hat{A}\hat{n}(\phi)\right]e^{-iS_p(x)}. \tag{2.28}$$

The Ritus matrices have the following properties [39]:

$$\begin{aligned} &\int d^4x\bar{E}_{l,x}E_{l',x} = (2\pi)^4\delta^4\left(l - l'\right), \\ &\int \frac{d^4l}{(2\pi)^4}\bar{E}_{l,x}E_{l,y} = \delta^4(x-y), \\ &\left[i\hat{\partial} - e\hat{A}(\phi)\right]E_{l,x} = E_{l,x}\hat{l}. \end{aligned} \tag{2.29}$$

The Volkov propagator, which is the Green's function of the Dirac Eq. (2.3), is given by [27]:

$$iG(x,y) = i\int \frac{d^4p}{(2\pi)^4} E_{p,x}\frac{\hat{p}+m}{p^2-m^2+i0}\bar{E}_{p,y}. \tag{2.30}$$

2.4.3 Photon States

The equation of motion for the free photon field is [33]

$$\partial^2\mathcal{A}^\mu = 0, \tag{2.31}$$

where the Lorenz gauge $\partial_\mu\mathcal{A}^\mu$ is employed. Solutions to (2.31) that form a complete basis are given by the Fourier modes:

$$\mathcal{A}^\mu(x) = \epsilon_r^\mu(k)e^{-i(kx)}, \tag{2.32}$$

where $k^\mu = (\omega_k, \boldsymbol{k})$ is the photon four-momentum with $k^2 = 0$ and ϵ_r^μ with $r = 1, 2$ are the polarization vectors. While the equation of motion (2.31) is solved for any arbitrary polarization vector, the physical degrees of freedom correspond to transverse polarizations that satisfy $k_\mu\epsilon^\mu = 0$ and the additional gauge constraint $\epsilon^0 = 0$ [42]. Under these conditions the polarization vectors span the plane in three dimensional space perpendicular to the momentum $\boldsymbol{k}$ and thus have two degrees of freedom corresponding to $r = 1, 2$. The photon propagator (in Feynman gauge) is defined as [33]:

$$-iD_{\mu\nu}(x-y) = -i\int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)}\frac{\eta_{\mu\nu}}{k^2+i0}. \tag{2.33}$$

2.4.4 The Furry Picture

In vacuum QED the solutions (2.20) of the free Dirac equation become modes of the free quantum field. In the presence of a plane wave background field one can replace the free modes by the Volkov states (2.25). Thereby, Fourier modes $e^{-i(px)}$ are replaced by the Ritus matrices E_p and the propagator from vacuum QED is replaced by the Volkov propagator (2.30). The Volkov propagator fully takes into account all tree level interactions between the electron and the background field. In Feynman diagrams the Volkov propagator is drawn as a solid double line. The radiation photon field $\mathcal{A}^\mu$ in the Lagrangian (2.2) can be treated as a small perturbation after all, and can be quantized as in the vacuum case. The approach to strong field QED by quantizing the Volkov modes as well as the radiation photon field is known as the Furry picture. Computations within the Furry picture are most easily carried out by employing the position space Feynman rules from vacuum QED [9, 33] and by only replacing the vacuum propagator with the Volkov propagator (2.30) as well as incoming and outgoing fermion states with Volkov states. Feynman rules provide instructions for obtaining S-Matrix elements from Feynman diagrams. The position space Feynman rules of strong field QED can be summarized in the following way:

$$
\begin{array}{ll}
\text{Vertex} & -ie\gamma^{\mu} \\
\text{Photon propagator} & -iD_{\mu\nu}(x-y) \\
\text{Dirac propagator} & iG(x,y) \\
\text{Incoming fermion} & E_p(x)u_\sigma(p) \\
\text{Outgoing fermion} & \bar{u}_\sigma(p)\bar{E}_p(x) \\
\text{Incoming anti-fermion} & \bar{v}_\sigma(p)\bar{E}_{-p}(x) \\
\text{Outgoing anti-fermion} & E_{-p}(x)v_\sigma(p) \\
\text{Incoming photon} & \epsilon_r^{\mu}e^{-ikx} \\
\text{Outgoing photon} & \epsilon_r^{*\mu}e^{ikx}
\end{array}
\tag{2.34}
$$

When applying the position space Feynman rules (2.34) one has to integrate over the space-time positions of each vertex. Closed fermionic loops require a trace in the spinor indices and an additional minus sign.

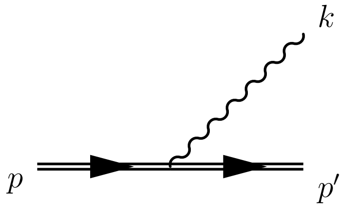
3 Corrections to Nonlinear Compton Scattering from the Polarization Operator

Nonlinear Compton scattering describes the emission of a photon by an electron when moving through an external electromagnetic field. We consider an incoming electron with four-momentum $p^\mu = (\varepsilon_p, \boldsymbol{p})$, an emitted photon $k^\mu = (\omega_k, \boldsymbol{k})$, and an outgoing electron $p'^\mu = (\varepsilon_{p'}, \boldsymbol{p}')$. The two polarization basis four-vectors of the outgoing photon are notated as $\epsilon_j^\mu(k)$ with $j = 1, 2$. We assume a background plane wave field according to Eq. (2.11) and we employ the Furry picture in which the electron is described by Volkov states. The tree level diagram for this process is shown in Fig. 3.1a. A calculation of the tree level scattering probability has been carried out in Ref. [16]. The work which is presented in this chapter is part of a larger effort to compute all the radiative corrections to nonlinear Compton Scattering to order α^2. In particular, here we consider the correction which comes from the polarization operator in the outgoing photon line (Fig. 3.1b). The polarization operator $i\mathcal{P}^{\mu\nu} = T^{\mu\nu}$ is related to the photons self energy, and in the context of Feynman diagrams, it is defined as the sum of all insertions into a photon line which cannot be subdivided by cutting a single line (See Ref. [9]). In the literature, diagrams which cannot be subdivided by cutting a single line are also referred to as 1-particle-irreducible [33]. To leading order, the polarization operator is represented by a single fermion loop. A full evalutation of the renormalized leading order polarization operator in the context of strong field QED, where fermion lines correspond to Volkov propagators, has been carried out in Ref. [27] from which the results will be used in the following calculations. The other corrections one has to consider are those from the mass operator and the vertex correction, either of which are not considered here. The mass operator is related to the electrons self energy and an evaluation in the context of strong field QED is given in Ref. [15]. The leading order correction to the scattering probability of nonlinear Compton scattering due to the mass operator has been computed in Ref. [34]. The vertex correction in strong field QED has been studied in Ref. [13].

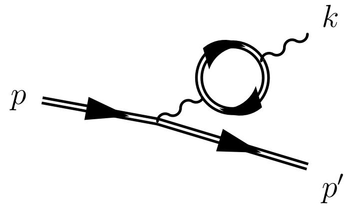
3.1 Calculations

The matrix element of nonlinear Compton scattering including first order loop corrections can be expressed as

$$S^{(1)}(p, p', k) = \left[\Gamma^\mu + \Gamma^{(M_1)\mu} + \Gamma^{(M_2)\mu} + \Gamma^{(P)\mu} + \Gamma^{(V)\mu}\right] \eta_{\mu\nu} \epsilon_l^{\nu *}(k). \tag{3.1}$$



(a)



(b)

Figure 3.1: Figure (a) depicts the tree level diagram of nonlinear Compton scattering. The double lines indicate that Volkov states are being used. Figure (b) depicts the polarization correction to nonlinear Compton scattering. This correction is characterized by a fermion loop in the outgoing photon line.

This expression includes the tree level term Γ^{μ}, two terms containing the Mass operator $\Gamma^{(M_1)\mu} + \Gamma^{(M_2)\mu}$, the vertex correction $\Gamma^{(V)\mu}$, and the term containing the polarization operator $\Gamma^{(P)\mu}$. In the following we will only consider the terms Γ^{μ} and $\Gamma^{(P)\mu}$ and thus we consider the following scattering matrix:

$$S_{fi} = \left(\Gamma^{\mu} + \Gamma^{(P)\mu}\right) \eta_{\mu\nu} \epsilon_l^{\nu *}(k). \tag{3.2}$$

According to the position space Feynman rules the terms Γ^{μ} and $\Gamma^{(P)\mu}$ are given by:

$$\Gamma^{\mu} = (-ie) \int d^4x \bar{u}\left(p'\right) \bar{E}_{p',x} \gamma^{\mu} E_{p,x} u(p) e^{ikx}, \tag{3.3}$$

$$\begin{aligned}\Gamma^{(P)\mu} = (-ie) \int d^4x \int d^4y \int d^4z \bar{u}\left(p'\right) \bar{E}_{p',x} \gamma^{\sigma} E_{p,x} u(p) (-i) \\ \times D_{\sigma\nu}(x-y) T^{\nu\mu}(y,z) e^{ikz},\end{aligned} \tag{3.4}$$

where the polarizaton operator in position space is defined as

$$T^{\mu\nu}(x,y) = (-ie)^2 \operatorname{tr}\left[\gamma^{\mu} G(x,y) \gamma^{\nu} G(y,x)\right]. \tag{3.5}$$

One can define the polarization operator in momentum space by taking the Fourier transform in both arguments:

$$T^{\mu\nu}(x,y) = \int \frac{d^4q_1}{(2\pi)^4} \int \frac{d^4q_2}{(2\pi)^4} e^{iq_1x} e^{-iq_2y}\, T^{\mu\nu}\left(q_1, q_2\right). \tag{3.6}$$

A full evaluation of the polarization operator $T^{\mu\nu}(q_1, q_2)$ in momentum space is given in Ref. [27]. The probability of the process described by S_{fi} is given by:

$$P^{(1)} = \frac{1}{2\varepsilon_p} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \int \frac{d^3p'}{(2\pi)^3} \frac{1}{2\varepsilon_{p'}} \frac{1}{2} \sum_{s,s',l} \left|S_{fi}\right|^2. \tag{3.7}$$

Here we perform a sum over the final spin $s' = \pm 1$ and the polarization of the outgoing photon $j = 1, 2$. We also perform an average over the initial spin $s = \pm 1$, which is why there appears a factor of $1/2$. For the following calculations it is convenient to express the metric as:

$$\eta^{\mu\nu} = \frac{n^\mu k^\nu + n^\nu k^\mu}{k_-} - \Lambda_1^\mu \Lambda_1^\nu - \Lambda_2^\mu \Lambda_2^\nu \tag{3.8}$$

where $k_- = (nk)$ and

$$\Lambda_j^\mu = e_j^\mu - \frac{(e_j k)\, n^\mu}{k_-}, \tag{3.9}$$

such that $(k\Lambda_i) = (n\Lambda_i) = 0$ and $(\Lambda_i \Lambda_j) = -\delta_{ij}$. This expansion of the metric is useful because due to the Ward identity [33, 27] it is $\Gamma^\mu k_\mu = \Gamma^{(P)\mu} k_\mu = 0$ and it is $k^\mu \epsilon_{l\mu}(k) = 0$, and thus we can write the scattering matrix as:

$$S_{fi} = \left(\Gamma^\mu + \Gamma^{(P)\mu}\right) \eta_{\mu\nu} \epsilon_l^{\nu *}(k) = - \sum_{j=1,2} \left(\Gamma^\mu \Lambda_{j\mu} + \Gamma^{(P)\mu} \Lambda_{j\mu}\right) \left(\Lambda_{j\nu} \epsilon_l^{\nu *}\right). \tag{3.10}$$

In order to calculate $\sum_{s,s',l} |S_{fi}|^2$ we can make use of the photon polarization sum (see e.g. Ref. [33])

$$\sum_l \epsilon_{l\mu}^* \epsilon_{l\nu} \to -\eta_{\mu\nu}. \tag{3.11}$$

and thus

$$\sum_{l=1,2} \left(\Lambda_j \epsilon_l^*\right) \left(\Lambda_i \epsilon_l\right) \to - \left(\Lambda_j \Lambda_i\right) = \delta_{ij}. \tag{3.12}$$

Thus we obtain

$$\frac{1}{2} \sum_{s,s',l} |S_{fi}|^2 = \frac{1}{2} \sum_{s,s'} \sum_j \left[|(\Gamma \Lambda_j)|^2 + \left|(\Gamma^{(P)} \Lambda_j)\right|^2 + 2\,\mathrm{Re}\left\{(\Gamma \Lambda_j)^* \left(\Gamma^{(P)} \Lambda_j\right)\right\}\right]. \tag{3.13}$$

The first term in Eq. (3.13) is the tree level contribution. The second term is of order α^3 and we will not consider this term from here on, as we are only interested in corrections up to order α^2. The third term is of order α^2 and thus we aim to evaluate it. The contribution to the probability which comes from this term is given by

$$\delta P = \frac{1}{2\varepsilon_p} \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_k} \int \frac{d^3 p'}{(2\pi)^3} \frac{1}{2\varepsilon_{p'}} \frac{1}{2} \sum_{s,s'} \sum_j 2\,\mathrm{Re}\left\{(\Gamma \Lambda_j)^* \left(\Gamma^{(P)} \Lambda_j\right)\right\}. \tag{3.14}$$

We continue with the following computations:

$$
\begin{aligned}
&\sum_{s,s'}\left(\Gamma\Lambda_j\right)^*\left(\Gamma^{(P)}\Lambda_j\right)=\sum_{s,s'}\left[(-ie)\int d^4x\bar{u}\left(p'\right)\bar{E}_{p',x}\hat{\Lambda}_jE_{p,x}u(p)e^{ikx}\right]^*\\
&\quad\times\left[(-ie)\int d^4x'\int d^4y\int d^4z\bar{u}\left(p'\right)\bar{E}_{p',x'}\gamma^\sigma E_{p,x'}u(p)(-i)D_{\sigma\nu}\left(x'-y\right)T^{\nu\mu}(y,z)\Lambda_{j\mu}e^{ikz}\right]\\
&=e^2\int d^4x\int d^4x'\int d^4y\int d^4ze^{-ikx}(-i)D_{\sigma\nu}\left(x'-y\right)T^{\nu\mu}(y,z)\Lambda_{j\mu}e^{ikz}\\
&\quad\times\sum_{s,s'}\bar{u}_s(p)\bar{E}_{p,x}\hat{\Lambda}_jE_{p',x}u_{s'}\left(p'\right)\bar{u}_{s'}\left(p'\right)\bar{E}_{p',x'}\gamma^\sigma E_{p,x'}u_s(p)\\
&=e^2\int d^4x\int d^4x'\int d^4y\int d^4ze^{-ikx}(-i)D_{\sigma\nu}\left(x'-y\right)T^{\nu\mu}(y,z)\Lambda_{j\mu}e^{ikz}\\
&\quad\times\operatorname{Tr}\left[(\hat{p}+m)\bar{E}_{p,x}\hat{\Lambda}_jE_{p',x}\left(\hat{p}'+m\right)\bar{E}_{p',x'}\gamma^\sigma E_{p,x'}\right].
\end{aligned}
\tag{3.15}
$$

The probability is therefore

$$
\begin{aligned}
\delta P=&\frac{e^2}{2\varepsilon_p}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\int\frac{d^3p'}{(2\pi)^3}\frac{1}{2\varepsilon_{p'}}\sum_j\int d^4x\int d^4x'\int d^4y\int d^4z\\
&\times\operatorname{Re}\bigg\{e^{-ik(x-z)}e^{i(S_p(x')-S_p(x))}e^{i\left(S_{p'}(x)-S_{p'}(x')\right)}(-i)D_{\sigma\nu}\left(x'-y\right)T^{\nu\mu}(y,z)\Lambda_{j,\mu}\\
&\times\operatorname{Tr}\left[(\hat{p}+m)\left(1-\frac{e}{2p_-}\hat{n}\hat{A}(\phi)\right)\hat{\Lambda}_j\left(1+\frac{e}{2p'_-}\hat{n}\hat{A}(\phi)\right)\left(\hat{p}'+m\right)\right.\\
&\left.\times\left(1-\frac{e}{2p'_-}\hat{n}\hat{A}\left(\phi'\right)\right)\gamma^\sigma\left(1+\frac{e}{2p_-}\hat{n}\hat{A}\left(\phi'\right)\right)\right]\bigg\}.
\end{aligned}
\tag{3.16}
$$

After writing the polarization operator in momentum space (Eq. (3.6)) one can perform the y and z integrations and thus we obtain:

$$
\begin{aligned}
\delta P=&\frac{e^2}{2\varepsilon_p}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\int\frac{d^3p'}{(2\pi)^3}\frac{1}{2\varepsilon_{p'}}\sum_j\int d^4x\int d^4x'\int\frac{d^4q}{(2\pi)^4}\\
&\times\operatorname{Re}\left\{e^{-ix(k-p+p')}e^{-ix'(q+p-p')}e^{-i\tilde{S}_p(\phi')}e^{i\tilde{S}_p(\phi)}e^{-i\tilde{S}_{p'}(\phi)}e^{i\tilde{S}_{p'}(\phi')}\right.\\
&\left.\times\frac{(-i)\eta_{\sigma\nu}}{q^2+i0}T^{\nu\mu}(-q,k)\Lambda_{j\mu}\operatorname{Tr}[\cdots]^\sigma\right\},
\end{aligned}
\tag{3.17}
$$

where we defined

$$
\tilde{S}_p\left(\phi\right)=\int_0^\phi d\phi'\left[\frac{e\left(pA\left(\phi'\right)\right)}{p_-}-\frac{e^2A^2\left(\phi'\right)}{2p_-}\right].
\tag{3.18}
$$

By performing the integrals in x_+ and $\boldsymbol{x}_\perp$ as well as x'_+ and $\boldsymbol{x}'_\perp$ we obtain the delta functions $(2\pi)^3\delta\left(k_- - p_- + p'_-\right)\delta\left(\boldsymbol{k}_\perp - \boldsymbol{p}_\perp + \boldsymbol{p}'_\perp\right)$ and $(2\pi)^3\delta\left(q_- + p_- - p'_-\right)\delta\left(\boldsymbol{q}_\perp + \boldsymbol{p}_\perp - \boldsymbol{p}'_\perp\right)$. In the following step we will additionally perform the transformation $q \to -q$:

$$\begin{aligned}\delta P =& \frac{e^2}{2\varepsilon_p}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\int\frac{d^3p'}{(2\pi)^3}\frac{1}{2\varepsilon_{p'}}\sum_j\int d\phi\int d\phi'\int\frac{d^4q}{(2\pi)^4} \\ &\times(2\pi)^3\delta\left(k_- - p_- + p'_-\right)\delta\left(\boldsymbol{k}_\perp - \boldsymbol{p}_\perp + \boldsymbol{p}'_\perp\right)(2\pi)^3\delta\left(-q_- + p_- - p'_-\right)\delta\left(-\boldsymbol{q}_\perp + \boldsymbol{p}_\perp - \boldsymbol{p}'_\perp\right) \\ &\times\mathrm{Re}\left\{e^{-i\phi\left(k_+ - p_+ + p'_+\right)}e^{-i\phi'\left(-q_+ + p_+ - p'_+\right)}e^{-i\tilde{S}_p(\phi')}e^{i\tilde{S}_p(\phi)}e^{-i\tilde{S}_{p'}(\phi)}e^{i\tilde{S}_{p'}(\phi')}\right. \\ &\left.\times\frac{(-i)\eta_{\sigma\nu}}{q^2+i0}T^{\nu\mu}(q,k)\Lambda_{j\mu}\,\mathrm{Tr}[\cdots]^\sigma\right\}.\end{aligned} \tag{3.19}$$

The polarization operator contains more delta functions in its definition [27] which we will extract, such that we can write

$$T^{\nu\mu}(q,k) = \delta\left(q_- - k_-\right)\delta\left(\boldsymbol{q}_\perp - \boldsymbol{k}_\perp\right)\tilde{T}^{\nu\mu}(q,k)\,, \tag{3.20}$$

and therefore

$$\begin{aligned}\delta P =& \frac{e^2}{2\varepsilon_p}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\int\frac{d^3p'}{(2\pi)^3}\frac{1}{2\varepsilon_{p'}}\sum_j\int d\phi\int d\phi'\int\frac{d^4q}{(2\pi)^4} \\ &\times(2\pi)^6\left[\delta\left(k_- - p_- + p'_-\right)\delta\left(\boldsymbol{k}_\perp - \boldsymbol{p}_\perp + \boldsymbol{p}'_\perp\right)\right]^2\delta\left(q_- - k_-\right)\delta\left(\boldsymbol{q}_\perp - \boldsymbol{k}_\perp\right) \\ &\times\mathrm{Re}\left\{e^{-i\phi\left(k_+ - p_+ + p'_+\right)}e^{-i\phi'\left(-q_+ + p_+ - p'_+\right)}e^{-i\tilde{S}_p(\phi')}e^{i\tilde{S}_p(\phi)}e^{-i\tilde{S}_{p'}(\phi)}e^{i\tilde{S}_{p'}(\phi')}\right. \\ &\left.\times\frac{(-i)\eta_{\sigma\nu}}{q^2+i0}\tilde{T}^{\nu\mu}(q,k)\Lambda_{j\mu}\,\mathrm{Tr}[\cdots]^\sigma\right\}.\end{aligned} \tag{3.21}$$

We can rewrite the first delta function in two different ways. Either we can write $Q'^\mu = p'^\mu + k^\mu$ and

$$\delta\left(p_- - Q'_-\right) = \frac{\varepsilon_p}{p_-}\delta\left(p_n - \tilde{p}_n\right) \qquad \text{and} \qquad \tilde{p}_n = \frac{m^2 + \boldsymbol{p}_\perp^2 - Q'^2_-}{2Q'_-} \tag{3.22}$$

or we write $Q^\mu = p^\mu - k^\mu$ and

$$\delta\left(p'_- - Q_-\right) = \frac{\varepsilon_{p'}}{p'_-}\delta\left(p'_n - \tilde{p}'_n\right) \qquad \text{and} \qquad \tilde{p}'_n = \frac{m^2 + \boldsymbol{p}'^2_\perp - Q^2_-}{2Q_-}. \tag{3.23}$$

where $p_n \equiv \boldsymbol{p}\cdot\boldsymbol{n}$ and $p'_n \equiv \boldsymbol{p}'\cdot\boldsymbol{n}$. Since we have $d^3p' = dp'_n d^2p'_\perp$ we can use Eq. (3.23) to simplify the dp'_n integral. In that case the delta function in Eq. (3.22) becomes:

$$\delta\left(p_n - \tilde{p}_n\right) \to \delta(0) = \frac{L}{2\pi} \tag{3.24}$$

where $L = 1$ corresponds to the quantization length in the direction $\boldsymbol{n}$. Then we can use the delta function $\delta\left(\boldsymbol{k}_\perp - \boldsymbol{p}_\perp + \boldsymbol{p}'_\perp\right)$ to simplify the $d^2p'_\perp$ integral. The remaining delta function becomes:

$$\delta\left(\boldsymbol{k}_\perp - \boldsymbol{p}_\perp + \boldsymbol{p}'_\perp\right) \to \delta^2(0) = \frac{A}{(2\pi)^2} \tag{3.25}$$

where $A = 1$ is the quantization area and $V = A \cdot L = 1$ is the quantization volume. Thus, by transforming the delta functions according to Eqs. (3.22) and (3.23) we were able to employ the usual procedure of squaring the delta functions by assuming a finite quantization volume as it is done in many text books (e.g. in Ref. [42]). Using that $d^4q = dq_+dq_-dq_\perp$ we can use the remaining delta functions in q to obtain the following expression for the probability:

$$\begin{aligned}\delta P =& \frac{1}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \frac{e^2}{4p_-p'_-} \int \frac{dq_+}{(2\pi)} \sum_j \int d\phi \int d\phi' \\ & \times \operatorname{Re}\left\{ e^{-i\phi\left(k_+ - p_+ + p'_+\right)} e^{-i\phi'\left(-q_+ + p_+ - p'_+\right)} e^{-i\tilde{S}_p(\phi')} e^{i\tilde{S}_p(\phi)} e^{-i\tilde{S}_{p'}(\phi)} e^{i\tilde{S}_{p'}(\phi')} \right. \\ & \left. \qquad \times \frac{(-i)\eta_{\sigma\nu}}{q^2 + i0} \tilde{T}^{\nu\mu}(q,k) \Lambda_{j\mu} \operatorname{Tr}[\cdots]^\sigma \right\},\end{aligned} \tag{3.26}$$

where it is implied that

$$\begin{aligned} p'_- &= p_- - k_-, \\ \boldsymbol{p}'_\perp &= \boldsymbol{p}_\perp - \boldsymbol{k}_\perp, \\ q_- &= k_-, \\ \boldsymbol{q}_\perp &= \boldsymbol{k}_\perp. \end{aligned}$$

Similar as in Ref. [16] we introduce the notation $\boldsymbol{\xi}_\perp = e\boldsymbol{A}_\perp/m$. We use that $A^0 = 0$ and $\boldsymbol{A}$ only has perpendicular components such that $\boldsymbol{A} = \boldsymbol{A}_\perp$. The phase can then be written as

$$\begin{aligned} & e^{-i\phi\left(k_+ - p_+ + p'_+\right)} e^{-i\phi'\left(-q_+ + p_+ - p'_+\right)} e^{-i\tilde{S}_p(\phi')} e^{i\tilde{S}_p(\phi)} e^{-i\tilde{S}_{p'}(\phi)} e^{i\tilde{S}_{p'}(\phi')} \\ =& \exp\Bigg(i\left(p_+ - p'_+\right)(\phi - \phi') - ik_+\phi + iq_+\phi' \\ & -im \int_{\phi'}^{\phi} d\phi'' \left[\frac{\boldsymbol{p}_\perp \cdot \boldsymbol{\xi}_\perp}{p_-} - \frac{\boldsymbol{p}'_\perp \cdot \boldsymbol{\xi}_\perp}{p'_-} + \frac{mk_-\boldsymbol{\xi}_\perp^2}{2p_-p'_-} \right] \Bigg) \\ =& \exp\Bigg(i\left(p_+ - p'_+ - k_+\right)(\phi - \phi') - ik_+\phi' + iq_+\phi' \\ & -im \int_{\phi'}^{\phi} d\phi'' \left[\frac{\boldsymbol{p}_\perp \cdot \boldsymbol{\xi}_\perp}{p_-} - \frac{\boldsymbol{p}'_\perp \cdot \boldsymbol{\xi}_\perp}{p'_-} + \frac{mk_-\boldsymbol{\xi}_\perp^2}{2p_-p'_-} \right] \Bigg) \\ =& \exp\left[-ik_+\phi' + i\frac{k_-m^2}{2p_-p'_-} \int_{\phi}^{\phi'} d\tilde{\phi} \left(1 + \left[\frac{\boldsymbol{p}_\perp}{m} - \frac{p_-}{k_-}\frac{\boldsymbol{k}_\perp}{m} - \boldsymbol{\xi}_\perp \right]^2 \right) + iq_+\phi' \right]. \end{aligned} \tag{3.27}$$

Therefore the probability can now be written as

$$
\begin{aligned}
\delta P =&\frac{1}{(2\pi)^3}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\frac{e^2}{4p_-p'_-}\int\frac{dq_+}{(2\pi)}\sum_j\int d\phi\int d\phi' \\
&\times \mathrm{Re}\left\{\exp\left[-ik_+\phi' + i\frac{k_-m^2}{2p_-p'_-}\int_{\phi}^{\phi'}d\tilde{\phi}\left(1+\left[\frac{\boldsymbol{p}_\perp}{m}-\frac{p_-}{k_-}\frac{\boldsymbol{k}_\perp}{m}-\boldsymbol{\xi}_\perp\right]^2\right)+iq_+\phi'\right]\right. \\
&\qquad\left.\times\frac{(-i)\eta_{\sigma\nu}}{q^2+i0}\tilde{T}^{\nu\mu}(q,k)\Lambda_{j\mu}\,\mathrm{Tr}[\cdots]^\sigma\right\}.
\end{aligned}
\tag{3.28}
$$

3.1.1 Calculation of the Trace

It is convenient to rewrite the trace by using manipulations such as

$$\left(1+\frac{e\hat{n}\hat{A}(\phi)}{2p_-}\right)(\hat{p}+m) = \frac{\hat{\pi}(\phi)+m}{2p_-}\hat{n}(\hat{p}+m) \tag{3.29}$$

and

$$(\hat{p}+m)\left(1-\frac{e\hat{n}\hat{A}(\phi)}{2p_-}\right) = (\hat{p}+m)\hat{n}\frac{\hat{\pi}(\phi)+m}{2p_-} \tag{3.30}$$

where

$$\pi^\mu(\phi) = p^\mu - eA^\mu(\phi) + \frac{e(pA(\phi))n^\mu}{p_-} - \frac{e^2A(\phi)^2n^\mu}{2p_-} \tag{3.31}$$

and for the momementum p' we define:

$$\pi'^\mu(\phi) = p'^\mu - eA^\mu(\phi) + \frac{e(p'A(\phi))n^\mu}{p'_-} - \frac{e^2A(\phi)^2n^\mu}{2p'_-} \tag{3.32}$$

and $\phi = (nx)$. Physically the quantity $\pi^\mu(\phi)$ corresponds to the classical kinetic four-momentum of a particle moving through the field A^μ with the initial condition $\pi^\mu(\phi=-\infty)=p^\mu$, as it is discussed e.g. in Ref. [21]. Here we only introduce these quantities for computational convenience. In the following calculations, trace expressions such as the following are going to be relevant:

$$
\begin{aligned}
T_{ji} =&\,\mathrm{Tr}\left[(\hat{p}+m)\left(1-\frac{e}{2p_-}\hat{n}\hat{A}(\phi)\right)\hat{\Lambda}_j\left(1+\frac{e}{2p'_-}\hat{n}\hat{A}(\phi)\right)(\hat{p}'+m)\right. \\
&\left.\times\left(1-\frac{e}{2p'_-}\hat{n}\hat{A}\left(\phi'\right)\right)\hat{\Lambda}_i\left(1+\frac{e}{2p_-}\hat{n}\hat{A}\left(\phi'\right)\right)\right] \\
&= \mathrm{Tr}\left[\hat{\Lambda}_j\frac{\hat{\pi}'(\phi)+m}{2p'_-}\hat{n}\left(\hat{p}'+m\right)\hat{n}\frac{\hat{\pi}'(\phi')+m}{2p'_-}\hat{\Lambda}_i\frac{\hat{\pi}(\phi')+m}{2p_-}\hat{n}(\hat{p}+m)\hat{n}\frac{\hat{\pi}(\phi)+m}{2p_-}\right].
\end{aligned}
$$

$$(3.33)$$

The different terms in (3.33) can be evaluated using the usual trace technology for γ-matrices, which are summarized in Appendix A. There are several identities related to the orthogonality of n^μ with A^μ and Λ_j^μ, that make the calculation easier, such as the following:

$$\begin{aligned}
&\hat{n}\hat{A} = -\hat{A}\hat{n} \\
&\hat{\pi}\hat{n} = 2p_- - \hat{n}\hat{\pi} \\
&\hat{n}\hat{n} = 0 \\
&\hat{n}\hat{\Lambda}_j = -\hat{\Lambda}_j\hat{n} \\
&\hat{\Lambda}_j\hat{\Lambda}_i = -2\delta_{ji} - \hat{\Lambda}_i\hat{\Lambda}_j .
\end{aligned} \tag{3.34}$$

The result of the trace calculation can be expressed as:

$$\begin{aligned}
\frac{T_{ji}}{2} =&\delta_{ji}\left[\frac{k_-^2 m^2}{p_-p'_-} - \frac{p_-}{p'_-}\left(\pi'(\phi)\pi'\left(\phi'\right)\right) - \frac{p'_-}{p_-}\left(\pi(\phi)\pi\left(\phi'\right)\right) + \left(\pi'(\phi)\pi(\phi')\right) + \left(\pi(\phi)\pi'\left(\phi'\right)\right)\right] \\
&+ \frac{p_-}{p'_-}\left[\left(\Lambda_j\pi'(\phi)\right)\left(\Lambda_i\pi'\left(\phi'\right)\right) - \left(\Lambda_i\pi'(\phi)\right)\left(\Lambda_j\pi'\left(\phi'\right)\right)\right] \\
&+ \frac{p'_-}{p_-}\left[\left(\Lambda_j\pi(\phi)\right)\left(\Lambda_i\pi\left(\phi'\right)\right) - \left(\Lambda_i\pi(\phi)\right)\left(\Lambda_j\pi\left(\phi'\right)\right)\right] \\
&+ \left(\Lambda_j\pi(\phi)\right)\left(\Lambda_i\pi'\left(\phi'\right)\right) + \left(\Lambda_i\pi(\phi)\right)\left(\Lambda_j\pi'\left(\phi'\right)\right) \\
&+ \left(\Lambda_j\pi'(\phi)\right)\left(\Lambda_i\pi(\phi')\right) + \left(\Lambda_i\pi'(\phi)\right)\left(\Lambda_j\pi(\phi')\right).
\end{aligned} \tag{3.35}$$

Further simplification leads to

$$\begin{aligned}
T_{ji} =& \frac{\delta_{ji}}{p_-p'_-}\left[k_-^2 m^2 + \left[p_-\boldsymbol{\pi}'_\perp(\phi) - p'_-\boldsymbol{\pi}_\perp(\phi)\right]\cdot\left[p_-\boldsymbol{\pi}'_\perp\left(\phi'\right) - p'_-\boldsymbol{\pi}_\perp\left(\phi'\right)\right]\right] \\
&+ \frac{1}{p_-p'_-}\left[p_-\left(\Lambda_j\pi'(\phi)\right) + p'_-\left(\Lambda_j\pi(\phi)\right)\right]\left[p'_-\left(\Lambda_i\pi\left(\phi'\right)\right) + p_-\left(\Lambda_i\pi'\left(\phi'\right)\right)\right] \\
&+ \frac{1}{p_-p'_-}\left[p_-\left(\Lambda_i\pi'(\phi)\right) - p'_-\left(\Lambda_i\pi(\phi)\right)\right]\left[p'_-\left(\Lambda_j\pi\left(\phi'\right)\right) - p_-\left(\Lambda_j\pi'\left(\phi'\right)\right)\right].
\end{aligned} \tag{3.36}$$

Additionally the following sum can be expressed in a more simple way

$$\begin{aligned}
T_{11} + T_{22} = &-8\left[m^2 - \frac{e^2}{4}\frac{p_-^2 + p_-'^2}{p_-p'_-}\Delta A^2 - \frac{1}{2k_-}\left[p_-\left(k\pi'(\phi')\right) + p'_-\left(k\pi(\phi')\right)\right]\right. \\
&\left.-\frac{1}{2k_-}\left[p_-\left(k\pi'(\phi)\right) + p'_-\left(k\pi(\phi)\right)\right]\right],
\end{aligned} \tag{3.37}$$

where

$$\Delta A^2 = \left(A^\mu(\phi') - A^\mu(\phi)\right)^2 = -\left(\boldsymbol{A}_\perp(\phi') - \boldsymbol{A}_\perp(\phi)\right)^2. \tag{3.38}$$

3.1.2 Inserting the Polarization Operator

For the following steps of the calculation it will become necessary to insert the full expression of $T^{\mu\nu}(q_1, q_2)$ from Ref. [27] Eq. (92). It is clear that only the products $\tilde{T}^{\mu\nu}(q,k)\Lambda_{j\nu}$ appear in the expression (3.28). From Ref. [27] these products can be explicitly written as:

$$\begin{aligned}\tilde{T}^{\mu\nu}(q,k)\Lambda_{1\nu} &= i\pi e^2 \int_{-1}^{1} dv \int_0^{\infty} \frac{d\tau}{\tau} \int_{-\infty}^{\infty} dz_- \left(b_2\Lambda_2^\mu + b_3\Lambda_1^\mu\right) e^{i\Phi} \\ \tilde{T}^{\mu\nu}(q,k)\Lambda_{2\nu} &= i\pi e^2 \int_{-1}^{1} dv \int_0^{\infty} \frac{d\tau}{\tau} \int_{-\infty}^{\infty} dz_- \left(b_1\Lambda_1^\mu + b_4\Lambda_2^\mu\right) e^{i\Phi}\end{aligned} \tag{3.39}$$

It is important to note that Ref. [27] does neither assume the incoming photon momentum q_1 nor the outgoing q_2 to be on-shell. In the present case the outgoing photon momentum which is k will be on-shell, i.e. $k^2 = 0$. We can use this fact to simplify the expression for the polarization operator by the shift $z_- \to z_- + \mu q_-$ in the z_--integral in Ref. [27], where $\mu = \tau(1 - v^2)/4$. After performing this shift the phase $e^{i\Phi}$ is given by:

$$e^{i\Phi} = \exp\left\{i\left[(k_+ - q_+)\, z_- - \tau m^2\right]\right\}, \tag{3.40}$$

which is simpler than the expression in Ref. [27] because a term in the exponent proportional to $k^2 = 0$ has vanished. The factors b_1, b_2, b_3, b_4 are given in Ref. [27] and we cite them here:

$$b_1 = 2m^2\xi_1\xi_2 \left(\frac{\tau}{4\mu}X_{12} - \frac{\tau v^2}{4\mu}X_{21}\right) e^{i\tau\beta} \tag{3.41}$$

$$b_2 = 2m^2\xi_1\xi_2 \left(\frac{\tau}{4\mu}X_{21} - \frac{\tau v^2}{4\mu}X_{12}\right) e^{i\tau\beta} \tag{3.42}$$

$$b_3 = -\left(\frac{i}{\tau} + \frac{qk}{2}\right)\left(e^{i\tau\beta} - 1\right) + 2m^2\left[\frac{\tau}{4\mu}\left(\xi_1^2 Z_1 + \xi_2^2 Z_2\right) + \xi_1^2 X_{11}\right] e^{i\tau\beta} \tag{3.43}$$

$$b_4 = -\left(\frac{i}{\tau} + \frac{qk}{2}\right)\left(e^{i\tau\beta} - 1\right) + 2m^2\left[\frac{\tau}{4\mu}\left(\xi_1^2 Z_1 + \xi_2^2 Z_2\right) + \xi_2^2 X_{22}\right] e^{i\tau\beta}. \tag{3.44}$$

The definitions of X_{ij}, Z_k $(i, j, k = 1, 2)$ and β are given in Section 3.2. Those expressions correspond to the same symbols in Ref. [27], however after the shift in the z_--integral. In the following calculations it will be convenient to define the second term of b_3 (b_4) as $\tilde{b}_3$ ($\tilde{b}_4$), such that:

$$b_{3/4} = -\left(\frac{i}{\tau} + \frac{qk}{2}\right)\left(e^{i\tau\beta} - 1\right) + \tilde{b}_{3/4}. \tag{3.45}$$

Using the conditions $q_- = k_-$, $\boldsymbol{q}_\perp = \boldsymbol{k}_\perp$ and $k^2 = 0$, the product (qk) can be written as:

$$(qk) = q_+ k_- - k_+ k_- . \tag{3.46}$$

After inserting Eqs. (3.39) into (3.28) one can take the q_+-integral. The appearing integrals are of the following form and they can be solved with complex contour integration:

$$\int_{-\infty}^{\infty} dq_+ \frac{e^{iq+(\phi'-z_-)}}{q^2 + i0} = \frac{(-\pi i)}{k_-} e^{-ik_+(z_- - \phi')} \theta\left(z_- - \phi'\right), \tag{3.47}$$

$$\int_{-\infty}^{\infty} dq_+ \frac{q_+ e^{iq_+(\phi'-z_-)}}{q^2 + i0} = \frac{\pi}{k_-} e^{-ik_+(z_- - \phi')} \left[-ik_+ \theta\left(z_- - \phi'\right) + \delta\left(z_- - \phi'\right)\right]. \tag{3.48}$$

Now we can evaluate the following expression:

$$\begin{aligned}
& e^{-ik_+\phi'} \sum_j \mathrm{Tr}[\cdots]_{j\mu} \frac{(-i)}{(2\pi)} \int dq_+ \frac{e^{iq_+\phi'}}{q^2 + i0} \tilde{T}^{\mu v}(q,k) \Lambda_{j\nu} \\
=& \frac{e^2}{2} \int_{-1}^{1} dv \int_0^\infty \frac{d\tau}{\tau} \int_{-\infty}^{\infty} dz_- e^{-i\tau m^2} \\
\times & \left[\left(T_{21} b_1 + T_{12} b_2 + T_{11} \tilde{b}_3 + T_{22} \tilde{b}_4 \right) \frac{(-\pi i)}{k_-} \theta\left(z_- - \phi'\right) \right. \\
& \left. + \left(T_{11} + T_{22}\right) \left(e^{i\tau\beta} - 1\right) \left[-\frac{1}{\tau} \frac{\pi}{k_-} \theta\left(z_- - \phi'\right) - \frac{\pi}{2} \delta\left(z_- - \phi'\right) \right] \right] \\
=& \frac{e^2}{2} \int_{-1}^{1} dv \int_0^\infty \frac{d\tau}{\tau} e^{-i\tau m^2} \left\{ \frac{(-\pi i)}{k_-} \int_{\phi'}^{\infty} dz_- \left(T_{21} b_1 + T_{12} b_2 + T_{11} \tilde{b}_3 + T_{22} \tilde{b}_4 \right) \right. \\
& \left. + \left(T_{11} + T_{22}\right) \left[-\frac{1}{\tau} \frac{\pi}{k_-} \int_{\phi'}^{\infty} dz_- \left(e^{i\tau\beta} - 1\right) - \frac{\pi}{2} \left(e^{i\tau\beta} - 1\right)\Big|_{z_- = \phi'} \right] \right\}.
\end{aligned} \tag{3.49}$$

The probability is then given by:

$$\begin{aligned}
\delta P =& \frac{1}{(2\pi)^3} \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_k} \frac{e^4}{8 p_- p'_-} \int d\phi \int d\phi' \\
& \times \mathrm{Re} \left\{ \exp\left[i \frac{k_- m^2}{2 p_- p'_-} \int_\phi^{\phi'} d\tilde{\phi} \left(1 + \left[\frac{\boldsymbol{p}_\perp}{m} - \frac{p_-}{k_-} \frac{\boldsymbol{k}_\perp}{m} - \boldsymbol{\xi}_\perp \right]^2 \right) \right] \right. \\
& \quad \times \int_{-1}^{1} dv \int_0^\infty \frac{d\tau}{\tau} e^{-i\tau m^2} \left\{ \frac{(-\pi i)}{k_-} \int_{\phi'}^{\infty} dz_- \left(T_{21} b_1 + T_{12} b_2 + T_{11} \tilde{b}_3 + T_{22} \tilde{b}_4 \right) \right. \\
& \quad \left. \left. + \left(T_{11} + T_{22}\right) \left[-\frac{1}{\tau} \frac{\pi}{k_-} \int_{\phi'}^{\infty} dz_- \left(e^{i\tau\beta} - 1\right) - \frac{\pi}{2} \left(e^{i\tau\beta} - 1\right)\Big|_{z_- = \phi'} \right] \right\} \right\}.
\end{aligned} \tag{3.50}$$

By using the definition $\alpha = e^2/4\pi$ and by renaming $b_1 = b_{21}, b_2 = b_{12}, \tilde{b}_3 = b_{11}$ and $\tilde{b}_4 = b_{22}$ we obtain

$$\begin{aligned}
\delta P =& \frac{\alpha^2}{4} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_k p_- p'_- k_-} \int d\phi d\phi' \,\mathrm{Re} \Bigg\{ \exp\left[i \frac{k_- m^2}{2p_- p'_-} \int_\phi^{\phi'} d\tilde{\phi} \left(1 + \left[\frac{\boldsymbol{p}_\perp}{m} - \frac{p_-}{k_-} \frac{\boldsymbol{k}_\perp}{m} - \boldsymbol{\xi}_\perp \right]^2 \right) \right] \\
& \times \int_{-1}^{1} dv \int_0^\infty \frac{d\tau}{\tau} e^{-i\tau m^2} \Bigg\{ (-i) \int_{\phi'}^\infty dz_- \left(\frac{T_{21}}{2} b_{21} + \frac{T_{12}}{2} b_{12} + \frac{T_{11}}{2} b_{11} + \frac{T_{22}}{2} b_{22} \right) \\
& - \frac{T_{11} + T_{22}}{4} \left[\frac{2}{\tau} \int_{\phi'}^\infty dz_- \left(e^{i\tau\beta} - 1 \right) + k_- \left(e^{i\tau\beta} - 1 \right)\Big|_{z_- = \phi'} \right] \Bigg\} \Bigg\}.
\end{aligned} \tag{3.51}$$

Insertion of the trace expression yields the following probability

$$\begin{aligned}
\delta P =& \frac{\alpha^2}{4} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_k p_- p'_- k_-} \int d\phi d\phi' \,\mathrm{Re} \Bigg\{ e^{i \frac{k_- m^2}{2p_- p'_-} \int_\phi^{\phi'} d\tilde{\phi} \left(1 + \left[\frac{\boldsymbol{p}_\perp}{m} - \frac{p_-}{k_-} \frac{\boldsymbol{k}_\perp}{m} - \boldsymbol{\xi}_\perp \right]^2 \right)} \int_{-1}^{1} dv \int_0^\infty \frac{d\tau}{\tau} e^{-i\tau m^2} \\
& \Bigg(\frac{(-i)}{p_- p'_-} \int_{\phi'}^\infty dz_- \sum_{j,i} b_{ji} \left\{ \left[p_- \left(\Lambda_j \pi'(\phi) \right) + p'_- \left(\Lambda_j \pi(\phi) \right) \right] \left[p'_- \left(\Lambda_i \pi(\phi') \right) + p_- \left(\Lambda_i \pi'(\phi') \right) \right] \right. \\
& \qquad \left. + \left[p_- \left(\Lambda_i \pi'(\phi) \right) - p'_- \left(\Lambda_i \pi(\phi) \right) \right] \left[p'_- \left(\Lambda_j \pi\left(\phi'\right) \right) - p_- \left(\Lambda_j \pi'\left(\phi'\right) \right) \right] \right\} \\
& + \frac{(-i)}{p_- p'_-} \int_{\phi'}^\infty dz_- \left(b_{11} + b_{22} \right) \left[k_-^2 m^2 + \left[p_- \boldsymbol{\pi}'_\perp(\phi) - p'_- \boldsymbol{\pi}_\perp(\phi) \right] \cdot \left[p_- \boldsymbol{\pi}'_\perp\left(\phi'\right) - p'_- \boldsymbol{\pi}_\perp\left(\phi'\right) \right] \right] \\
& + 2 \left[m^2 - \frac{e^2}{4} \frac{p_-^2 + p'^2_-}{p_- p'_-} \Delta A^2 - \frac{1}{2k_-} \left[p_- \left(k\pi'(\phi') \right) + p'_- \left(k\pi(\phi') \right) \right] \right. \\
& \qquad \left. - \frac{1}{2k_-} \left[p_- \left(k\pi'(\phi) \right) + p'_- \left(k\pi(\phi) \right) \right] \right] \\
& \qquad \times \left[\frac{2}{\tau} \int_{\phi'}^\infty dz_- \left(e^{i\tau\beta} - 1 \right) + k_- \left(e^{i\tau\beta} - 1 \right)\Big|_{z_- = \phi'} \right] \Bigg) \Bigg\}.
\end{aligned} \tag{3.52}$$

3.1.3 Gauge Invariance Trick

Due to the gauge invariance the contraction of Eq. (3.3) with k_μ is zero (Ward identity):

$$\int d^4x \bar{u}\left(p'\right) \bar{E}_{p',x} k_\mu \gamma^\mu E_{p,x} u(p) e^{ikx} = 0. \tag{3.53}$$

Starting from the above condition it can be shown that

$$\int d\phi \left[\pi_+(\phi) - k_+ - \pi'_+(\phi) \right] e^{-i \int_0^\phi d\tilde{\phi} \left[\pi_+(\tilde{\phi}) - k_+ - \pi'_+(\tilde{\phi}) \right]} = 0. \tag{3.54}$$

We note that the phase in Eq. (3.54) already appears in the phase of Eq. (3.53) since:

$$S_p(x) - S_{p'}(x) + (kx) = -\int_0^\phi d\tilde{\phi} \left[\pi_+(\tilde{\phi}) - k_+ - \pi'_+(\tilde{\phi})\right] + k_- x_+ - \boldsymbol{k}_\perp \cdot \boldsymbol{x}_\perp. \quad (3.55)$$

The condition (3.54) is equivalent to the statement:

$$i \int_{-\infty}^{\infty} d\phi \frac{d}{d\phi} e^{-i \int_0^\phi d\tilde{\phi} \left[\pi_+ - k_+ - \pi'_+\right]} = \left[e^{-i \int_0^\phi d\tilde{\phi} \left[\pi_+ - k_+ - \pi'_+\right]}\right]_{-\infty}^{\infty} = 0. \quad (3.56)$$

In any case, with the following scalar products

$$(\pi - k - \pi')(\pi + k + \pi') = (\pi_+ - k_+ - \pi')(p_- + k_- + p') = -2k\pi' \quad (3.57)$$

$$(\pi - k - \pi')(\pi - k + \pi') = \left(\pi_+ - k_+ - \pi'_+\right)\left(p_- - k_- + p'_-\right) = -2k\pi \quad (3.58)$$

we see that

$$\delta^{(-,\perp)}(p - k - p') \int d\phi e^{-i \int_0^\phi d\tilde{\phi} \left[\pi_+(\tilde{\phi}) - k_+ - \pi'_+(\tilde{\phi})\right]} (k\pi(\phi)) = 0. \quad (3.59)$$

and

$$\delta^{(-,\perp)}(p - k - p') \int d\phi e^{-i \int_0^\phi d\tilde{\phi} \left[\pi_+(\tilde{\phi}) - k_+ - \pi'_+(\tilde{\phi})\right]} (k\pi'(\phi)) = 0. \quad (3.60)$$

This lets us eliminate the term $-\frac{1}{2k_-}\left[p_-\left(k\pi'(\phi)\right) + p'_-\left(k\pi(\phi)\right)\right]$ in δP.

3.2 Final Result

Here we give the final result for the correction to the probability of nonlinear Compton scattering due to the polarization operator. We consider the process with an incoming electron with four-momentum $p^\mu = (\varepsilon_p, \boldsymbol{p})$, an emitted photon $k^\mu = (\omega_k, \boldsymbol{k})$, and an outgoing electron $p'^\mu = (\varepsilon_{p'}, \boldsymbol{p}')$. The expression contains many light cone components of four-vectors for which the notation in Sec. 2.1 is used. All other definitions and notations are given below. The final expression is:

$$
\begin{aligned}
\delta P =& \frac{\alpha^2}{4} \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\omega_k p_- p'_- k_-} \int d\phi d\phi' \\
& \times \mathrm{Re}\Bigg\{ e^{i \frac{k_- m^2}{2p_- p'_-} \int_\phi^{\phi'} d\tilde{\phi} \left(1 + \left[\frac{\boldsymbol{p}_\perp}{m} - \frac{p_-}{k_-}\frac{\boldsymbol{k}_\perp}{m} - \boldsymbol{\xi}_\perp(\tilde{\phi})\right]^2\right)} \int_{-1}^{1} dv \int_0^\infty \frac{d\tau}{\tau} e^{-i\tau m^2} \\
& \times \Bigg(\frac{(-i)}{p_- p'_-} \int_{\phi'}^{\infty} dz_- \sum_{j,i} b_{ji} \left[p_- \left(\Lambda_j \pi'(\phi)\right) + p'_- \left(\Lambda_j \pi(\phi)\right)\right] \left[p'_- \left(\Lambda_i \pi(\phi')\right) + p_- \left(\Lambda_i \pi'(\phi')\right)\right] \\
& + \frac{(-i)}{p_- p'_-} \int_{\phi'}^{\infty} dz_- \sum_{j,i} b_{ji} \left[p_- \left(\Lambda_i \pi'(\phi)\right) - p'_- \left(\Lambda_i \pi(\phi)\right)\right] \left[p'_- \left(\Lambda_j \pi\left(\phi'\right)\right) - p_- \left(\Lambda_j \pi'\left(\phi'\right)\right)\right] \\
& + \frac{(-i)}{p_- p'_-} \int_{\phi'}^{\infty} dz_- \left(b_{11} + b_{22}\right) \left[k_-^2 m^2 + \left[p_- \boldsymbol{\pi}'_\perp(\phi) - p'_- \boldsymbol{\pi}_\perp(\phi)\right] \cdot \left[p_- \boldsymbol{\pi}'_\perp\left(\phi'\right) - p'_- \boldsymbol{\pi}_\perp\left(\phi'\right)\right]\right] \\
& + 2\left[m^2 - \frac{e^2}{4}\frac{p_-^2 + p'^2_-}{p_- p'_-} \Delta A^2 - \frac{1}{2k_-}\left[p_- \left(k\pi'(\phi')\right) + p'_- \left(k\pi(\phi')\right)\right]\right] \\
& \quad \times \left[\frac{2}{\tau} \int_{\phi'}^{\infty} dz_- \left(e^{i\tau\beta} - 1\right) + k_- \left(e^{i\tau\beta} - 1\right)\Big|_{z_- = \phi'}\right] \Bigg) \Bigg\}.
\end{aligned}
\tag{3.61}
$$

In this expression we impose the following conservation laws:

$$
\begin{aligned}
p'_- &= p_- - k_- \\
\boldsymbol{p}'_\perp &= \boldsymbol{p}_\perp - \boldsymbol{k}_\perp .
\end{aligned}
$$

We used the following definition:

$$
\Lambda_j^\mu = e_j^\mu - \frac{\left(e_j k\right) n^\mu}{k_-}, \tag{3.62}
$$

where $j = 1, 2$ and $n^\mu = (1, \boldsymbol{n})$, $e_j^\mu = (0, \boldsymbol{e}_j)$, $\boldsymbol{n} = \boldsymbol{e}_1 \times \boldsymbol{e}_2$ and the plane wave which propagates along the unit vector $\boldsymbol{n}$ is given by:

$$
A^\mu(\phi) = a_1^\mu \psi_1(\omega_0 \phi) + a_2^\mu \psi_2(\omega_0 \phi), \tag{3.63}
$$

with $a_j^\mu = (0, \boldsymbol{a}_j)$ and where $\boldsymbol{a}_j \parallel \boldsymbol{e}_j$ and $|\psi(\omega_0 \phi)| \leq 1$ such that

$$
A^\mu = (0, \boldsymbol{A}_\perp), \tag{3.64}
$$

$$
\boldsymbol{\xi}_\perp = \frac{e\boldsymbol{A}_\perp}{m}, \tag{3.65}
$$

$$
\xi_i = \frac{1}{m}\sqrt{-a_i^2 e^2}, \tag{3.66}
$$

and

$$\Delta A^2 = (A^\mu(\phi') - A^\mu(\phi))^2 = -(\boldsymbol{A}_\perp(\phi') - \boldsymbol{A}_\perp(\phi))^2. \tag{3.67}$$

Furthermore we have defined

$$\pi^\mu(\phi) = p^\mu - eA^\mu(\phi) + \frac{e(pA(\phi))n^\mu}{p_-} - \frac{e^2 A(\phi)^2 n^\mu}{2p_-} \tag{3.68}$$

$$\pi'^\mu(\phi) = p'^\mu - eA^\mu(\phi) + \frac{e(p'A(\phi))n^\mu}{p'_-} - \frac{e^2 A(\phi)^2 n^\mu}{2p'_-}. \tag{3.69}$$

The functions $b_{ij}(z_-)$ are defined as

$$b_{12} = 2m^2\xi_1\xi_2\left(\frac{\tau}{4\mu}X_{21} - \frac{\tau v^2}{4\mu}X_{12}\right)e^{i\tau\beta} \tag{3.70}$$

$$b_{21} = 2m^2\xi_1\xi_2\left(\frac{\tau}{4\mu}X_{12} - \frac{\tau v^2}{4\mu}X_{21}\right)e^{i\tau\beta} \tag{3.71}$$

$$b_{11} = 2m^2\left[\frac{\tau}{4\mu}\left(\xi_1^2 Z_1 + \xi_2^2 Z_2\right) + \xi_1^2 X_{11}\right]e^{i\tau\beta} \tag{3.72}$$

$$b_{22} = 2m^2\left[\frac{\tau}{4\mu}\left(\xi_1^2 Z_1 + \xi_2^2 Z_2\right) + \xi_2^2 X_{22}\right]e^{i\tau\beta} \tag{3.73}$$

and

$$e^{i\tau\beta} = \exp\left[i\tau m^2 \sum_{i=1,2}\xi_i^2\left(I_i^2 - J_i\right)\right], \tag{3.74}$$

with

$$\mu = \frac{1}{4}\tau\left(1 - v^2\right). \tag{3.75}$$

Furthermore

$$I_i = \int_0^1 d\lambda\psi_i(\omega_0 z_- + 2\omega_0\lambda\mu k_-) \tag{3.76}$$

$$J_i = \int_0^1 d\lambda\psi_i^2(\omega_0 z_- + 2\omega_0\lambda\mu k_-) \tag{3.77}$$

$$X_{ij} = \left[I_i - \psi_i\left(\omega_0 z_- + 2\omega_0\mu k_-\right)\right]\left[I_j - \psi_j\left(\omega_0 z_-\right)\right] \tag{3.78}$$

$$Z_i = \frac{1}{2}\left[\psi_i\left(\omega_0 z_-\right) - \psi_i\left(\omega_0 z_- + 2\omega_0\mu k_-\right)\right]^2. \tag{3.79}$$

4 Formulation in the Locally-Constant-Field Approximation

The locally constant field approximation (LCFA) describes the limit in which the central frequency ω_0 of the external field goes to zero: $\omega_0 \to 0$ [16, 21]. Recalling the classical intensity parameter

$$\xi = \frac{|e|E_0}{m\omega_0} \tag{4.1}$$

it can be seen that the LCFA corresponds to the limit $\xi \to \infty$ while keeping the field strength E_0 fixed. By taking the LCFA it is physically assumed that the formation length of strong field QED processes is much smaller than the wavelength of the background field, such that within the formation region of the photon emission (or pair production) the field is well approximated by a constant crossed field. Indeed, scattering probabilities calculated within the LCFA are typically identical to the corresponding expression in a constant crossed field if one performs an additional integral over the laser phase. It was shown in Ref. [16] (and as was previously known, see e.g. Ref. [41] p. 519) that the formation phase of photon emission is $\varphi_f \equiv \omega_0\phi_f \sim 1/\xi$, thus making this interpretation compatible with the limit of $\xi \gg 1$. Observables in the LCFA are characterized by the quantum nonlinearity paramaters defined by:

$$\kappa_i = \xi_i \frac{(k_0k)}{m^2} = \frac{|e|E_ik_-}{m^3}, \tag{4.2}$$

where $k_0^\mu = \omega_0(1, \boldsymbol{n})$ and

$$\kappa = \sqrt{\kappa_1^2 + \kappa_2^2} = \xi\frac{(k_0k)}{m^2} = \frac{|e|E_0k_-}{m^3}. \tag{4.3}$$

The quantum nonlinearity parameter for the electron is given by:

$$\chi = \frac{|e|E_0p_-}{m^3}, \tag{4.4}$$

and when taking the LCFA one keeps κ and χ finite. The LCFA is a widely used approximation, as it simplifies expressions significantly. It is commonly used to model photon emission and pair production in semi-classical numerical simulations, see Ref. [30] for a recent implementation (See also [21] for a general overview). The LCFA expression of the tree level expression for the probability of nonlinear Compton scattering has been derived in Ref. [16]. In Refs. [36, 37] the LCFA was used to obtain the scattering probabilities of nonlinear Compton scattering and nonlinear Breit-Wheeler pair production including particle damping effects. We will study these damping effects, which are due to the total pulse length of the laser, in Chapter 5. For now, it is important that when expanding the result from Refs. [36, 37], one obtains the LCFA version of the incoherent terms of the scattering probability (3.61) computed in the previous chapter. In the following, we will apply the LCFA to the result (3.61) and compare the outcome with the literature.

4.1 Applying the LCFA to the Tree Level Probability

The expression (3.61) gives a correction to the scattering probability of nonlinear Compton Scattering. Before applying the LCFA there, we will first apply the LCFA to the tree level amplitude of nonlinear Compton scattering. The S-Matrix element for the tree level process is given by $S_{fi} = \Gamma^\mu \epsilon^*_{l\nu}$ where Γ^μ is given in Eq. (3.3) and ϵ^μ_l is the photon polarization four-vector. For the calculation of the probability, Eq. (3.7) can be used and the preceding calculation steps are similar to the steps in the previous chapter. As a result, the tree-level probability for non-linear Compton Scattering is given by:

$$\begin{aligned} P_0 =& \frac{1}{2}\int \frac{d^3k}{16\pi^2}\frac{\alpha}{p_-p'_-\omega_k}\int d\phi d\phi'(T_{11}+T_{22}) \\ &\times \exp\left[i\frac{k_-m^2}{2p_-p'_-}\int_\phi^{\phi'} d\tilde{\phi}\left(1+\left[\frac{\boldsymbol{p}_\perp}{m}-\frac{p_-}{k_-}\frac{\boldsymbol{k}_\perp}{m}-\boldsymbol{\xi}_\perp(\tilde{\phi})\right]^2\right)\right] \end{aligned} \tag{4.5}$$

A computation of the tree level probability as well the expression in the LCFA can also be found in Ref. [16]. When applying the LCFA it is common to perform a variable transformation according to:

$$\begin{aligned} \phi_+ &= \frac{\phi+\phi'}{2}, \\ \phi_- &= \phi-\phi'. \end{aligned} \tag{4.6}$$

such that $d\phi d\phi' = d\phi_+ d\phi_-$. Thus the probability can be written as:

$$\begin{aligned} P_0 =& \frac{1}{2}\int \frac{d^3k}{16\pi^2}\frac{\alpha}{p_-p'_-\omega_k}\int d\phi_+ d\phi_-(T_{11}+T_{22}) \\ &\times \exp\left[-i\frac{k_-m^2}{2p_-p'_-}\int_{\phi_+-\phi_-/2}^{\phi_++\phi_-/2} d\tilde{\phi}\left(1+\left[\frac{\boldsymbol{p}_\perp}{m}-\frac{p_-}{k_-}\frac{\boldsymbol{k}_\perp}{m}-\boldsymbol{\xi}_\perp(\tilde{\phi})\right]^2\right)\right]. \end{aligned} \tag{4.7}$$

As discussed in Ref. [16], by writing $P_0 = \int d\phi_+ dP_0/d\phi_+$ the quantity $dP_0/d\phi_+$ can be interpreted as the probability per unit phase ϕ_+ of the plane wave field. This interpretation assumes that the formation region of the ϕ_- integration is localized around 0, such that $\omega_0\phi_- \sim 1/\xi \ll 1$, as discussed before. The usual approach of taking the LCFA is to expand the fields appearing in the preexponent in (4.7) to first order in ϕ_- and to expanded the phase to third order, which is what also has been done in Ref. [36]. This approach is similar to applying the method of stationary phase [2]. Before expanding the phase it can be rewritten in the following way:

$$\begin{aligned}
&\exp\left[-i\frac{m^2k_-}{2p_-p'_-}\int_{\phi_+-\phi_-/2}^{\phi_++\phi_-/2}d\tilde{\phi}\left(1+\left[\frac{\boldsymbol{p}_\perp}{m}-\frac{p_-}{k_-}\frac{\boldsymbol{k}_\perp}{m}-\boldsymbol{\xi}_\perp(\tilde{\phi})\right]^2\right)\right]\\
&=\exp\left[-i\frac{m^2k_-\phi_-}{2p_-p'_-}\left(\frac{\boldsymbol{p}_\perp}{m}-\frac{p_-}{k_-}\frac{\boldsymbol{k}_\perp}{m}-\frac{1}{\phi_-}\int_{\phi_+-\phi_-/2}^{\phi_++\phi_-/2}d\tilde{\phi}\boldsymbol{\xi}_\perp(\tilde{\phi})\right)^2\right]\\
&\times\exp\left[-i\frac{m^2k_-}{2p_-p'_-}\left(\phi_--\frac{1}{\phi_-}\left(\int_{\phi_+-\phi_-/2}^{\phi_++\phi_-/2}d\tilde{\phi}\boldsymbol{\xi}_\perp(\tilde{\phi})\right)^2+\int_{\phi_+-\phi_-/2}^{\phi_++\phi_-/2}d\tilde{\phi}\boldsymbol{\xi}_\perp(\tilde{\phi})^2\right)\right].
\end{aligned}\tag{4.8}$$

It can be seen that the middle line in Eq. (4.8) may be used to perform a Gaussian integral in $\boldsymbol{k}_\perp$, after which the term only contributes to the preexponent. Thus, in line with the other terms in the preexponent, we may expand this term to first order in ϕ_-. After performing the expansion of the phase the resulting tree level probability of nonlinear Compton scattering within the LCFA is given by:

$$\begin{aligned}
P_0 =&\frac{1}{2}\int\frac{d^3k}{16\pi^2}\frac{\alpha}{p_-p'_-\omega_k}\int d\phi_+d\phi_-(T_{11}+T_{22})\\
&\times\exp\left[-i\frac{k_-m^2}{2p_-p'_-}\left\{\left(1+\left[\frac{\boldsymbol{p}_\perp}{m}-\frac{p_-}{k_-}\frac{\boldsymbol{k}_\perp}{m}-\boldsymbol{\xi}_\perp(\phi_+)\right]^2\right)\phi_-+\boldsymbol{\xi}'^2_\perp(\phi_+)\frac{\phi_-^3}{12}\right\}\right].
\end{aligned}\tag{4.9}$$

Within the LCFA one must also expand the trace expressions T_{jj} to first order in ϕ_-, which will be done in section 4.3. To be able to compare the expression (4.9) to the literature, it is useful to redefine the variables $\phi \leftrightarrow \phi'$ after which one would obtain another minus sign in the exponent. This would give the expression for the phase that can be found in Ref. [36].

4.2 The Polarization Correction in the LCFA

In the remainder of this chapter we evaluate the radiative correction (3.61) in the LCFA. For this, we will consider the expression (3.51) and we will separate it into two parts:

$$\delta P = \delta P_{\mathrm{incoh}} + \delta P_{\mathrm{coh}},\tag{4.10}$$

where

$$\begin{aligned}\delta P_{\text{coh}} =& -\frac{\alpha^2}{4}\int \frac{d^3k}{(2\pi)^3}\frac{1}{\omega_k p_- p'_-}\int d\phi d\phi' \\ &\times \operatorname{Re}\left\{e^{i\Omega}\int_{-1}^{1}dv\int_0^\infty \frac{d\tau}{\tau}e^{-i\tau m^2}\frac{T_{11}+T_{22}}{4}\left[\left(e^{i\tau\beta}-1\right)\big|_{z_-=\phi'}\right]\right\},\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}\delta P_{\text{incoh}} =& \frac{\alpha^2}{4}\int \frac{d^3k}{(2\pi)^3}\frac{1}{\omega_k p_- p'_- k_-}\int d\phi d\phi' \operatorname{Re}\left\{e^{i\Omega}\int_0^\infty \frac{d\tau}{\tau}e^{-i\tau m^2}\right. \\ &\times \int_{\phi'}^\infty dz_- \frac{(-i)}{2}\int_{-1}^{1}dv\left[\left(b_{11}+\frac{(-i)}{\tau}\left(e^{i\tau\beta}-1\right)\right)T_{11}\right. \\ &\left.\left.+\left(b_{22}+\frac{(-i)}{\tau}\left(e^{i\tau\beta}-1\right)\right)T_{22}\right]\right\},\end{aligned} \tag{4.12}$$

and where

$$e^{i\Omega} = \exp\left[i\frac{k_- m^2}{2p_- p'_-}\int_\phi^{\phi'} d\tilde{\phi}\left(1+\left[\frac{\boldsymbol{p}_\perp}{m}-\frac{p_-}{k_-}\frac{\boldsymbol{k}_\perp}{m}-\boldsymbol{\xi}_\perp\right]^2\right)\right]. \tag{4.13}$$

In this chapter we just consider the case of a linearly polarized plane wave, i.e, we set $\psi_2(\phi)=0$. Thus we have already set $b_{12}=b_{21}=0$ in the expressions above. We have chosen a decomposition such that δP_{incoh} contains all the terms which include the integral over z_-, while δP_{coh} is given by the remaining term. The variable z_- corresponds to the minus component of a space time integral. Thus we expect this integral to scale linearly with the total phase $\Delta\phi$ over which the pulse shape function is supported, e.g. the standard deviation of a Gaussian pulse. A total dimensionless phase can be defined by $\Phi=\omega_0\Delta\phi$. The terms containing the z_- integral thus have a formation length that is of the same order of the laser pulse and we interpret these terms as belonging to the incoherent processes, in which the emitted photon travels a significant distance before the pair creation occurs. The remaining term without the z_- integral can be interpreted as the coherent process, in which the pair creation takes place almost immediately after the photon emission. First we are going to derive the LCFA of the incoherent process. The coherent process will be studied in Sec. 4.7. In order to apply the LCFA we need to apply the limit $\omega_0\to 0$ to each of the terms defined below Eq. (3.61). Since all b_{ij} involve the quantity ξ^2 it is necessary to expand the other preexponents to second order in ω_0 such that we obtain a result for the b_{ij} which is of order zero in ω_0. In the following we will write the pulse shape functions as functions of the dimensionless phase $\varphi=\omega_0\phi$, i.e. as $\psi_i(\omega_0\phi)$. The prime now defines the derivative with respect to φ i.e $\psi_i'(\varphi)=d\psi_i(\varphi)/d\varphi$, such that the Taylor expansion will clearly show which terms are of what order in ω_0. By performing the Taylor expansions we obtain:

$$I_i^2-J_i=-\frac{1}{3}\mu^2\left(\omega_0 k_-\right)^2\psi_i'^2\left(\omega_0 z_-\right)+O(\omega_0^3) \tag{4.14}$$

$$\xi_i^2(I_i^2 - J_i) = -\frac{\mu^2}{3} m^4 \kappa_i^2 \psi_i'^2 (\omega_0 z_-) + O(\omega_0) \tag{4.15}$$

$$X_{ij} = -\mu^2 (\omega_0 k_-)^2 \psi_i' (\omega_0 z_-) \psi_j' (\omega_0 z_-) + O\left(\omega_0^3\right) \tag{4.16}$$

$$\xi_l \xi_k X_{ij} = -\mu^2 m^4 \kappa_l \kappa_k \psi_i' (\omega_0 z_-) \psi_j' (\omega_0 z_-) + O(\omega_0) \tag{4.17}$$

$$Z_i = 2\mu^2 (\omega_0 k_-)^2 \psi_i'^2 (\omega_0 z_-) + O\left(\omega_0^3\right) \tag{4.18}$$

$$\xi_k^2 Z_i = 2\mu^2 m^4 \kappa_k^2 \psi_i'^2 (\omega_0 z_-) + O(\omega_0) . \tag{4.19}$$

With these expansions we can evaluate the prefactors in the probability. First of all we obtain the following phase:

$$e^{i\tau\beta} = \exp\left[-i\frac{\tau^3 \left(1 - v^2\right)^2}{48} m^6 \sum_i \kappa_i^2 \psi_i'^2 (\omega_0 z_-) + O(\omega_0)\right] . \tag{4.20}$$

Next we can evaluate the functions b_{ij}:

$$\begin{aligned}
b_{11} &= m^6 \tau\mu \sum_i \kappa_i^2 \psi_i'^2 (\omega_0 z_-) e^{i\tau\beta} - 2m^6 \mu^2 \kappa_1^2 \psi_1'^2 (\omega_0 z_-) e^{i\tau\beta}, \\
b_{22} &= m^6 \tau\mu \sum_i x_i^2 \psi_i'^2 (\omega_0 z_-) e^{i\tau\beta} - 2m^6 \mu^2 \kappa_2^2 \psi_2'^2 (\omega_0 z_-) e^{i\tau\beta}, \\
b_{12} &= b_{21} = -2m^6 \kappa_1 \kappa_2 \psi_1' (\omega_0 z_-) \psi_2' (\omega_0 z_-) \mu^2 e^{i\tau\beta}.
\end{aligned} \tag{4.21}$$

From here on we will limit the analysis to the special case of a linearly polarized wave for which $\psi_2 = 0$ and we rename $\psi_1 = \psi$. In the same way we have $\xi = \xi_1$ and $\kappa = \kappa_1$. It can be seen that $b_{12} = b_{21} = 0$ in this case since $\xi_2 = 0$. For the other quantities we obtain:

$$b_{11} = m^6 \kappa^2 \psi'^2 (\omega_0 z_-) \frac{\tau^2}{8} \left(1 - v^2\right) \left(1 + v^2\right) e^{i\tau\beta} \tag{4.22}$$

$$b_{22} = m^6 \kappa^2 \psi'^2 (\omega_0 z_-) \frac{\tau^2}{4} \left(1 - v^2\right) e^{i\tau\beta} \tag{4.23}$$

$$e^{i\tau\beta} = \exp\left[-i\frac{\tau^3\left(1-v^2\right)^2}{48}m^6\kappa^2\psi'^2\left(\omega_0 z_-\right) + O\left(\omega_0\right)\right]. \tag{4.24}$$

In order to simplify expressions, we perform a substitution $\tau^3 b = t^3$:

$$\int_0^\infty \frac{d\tau}{\tau^2}e^{-i\tau m^2}\left(e^{i\tau\beta}-1\right) = \int_0^\infty \frac{dt}{t^2}\cdot b^{1/3}e^{-itm^2 b^{-1/3}}\left(e^{-\frac{i}{3}t^3}-1\right), \tag{4.25}$$

where

$$b = m^6\kappa^2\psi'^2(\omega_0 z_-)\left[\frac{1}{4}\left(1-v^2\right)\right]^2, \tag{4.26}$$

such that

$$i\tau\beta = -\frac{i}{3}\tau^3 b. \tag{4.27}$$

Next, as it is done in Ref. [27] we define the following function:

$$f(x) = i\int_0^\infty dt\exp\left[-i\left(tx+\frac{1}{3}t^3\right)\right] = \pi\mathrm{Gi}(x) + i\pi\mathrm{Ai}(x), \tag{4.28}$$

in which Ai is the Airy function and Gi is the Scorer function (see Appendix B). The derivative of the function f is given by:

$$f'(x) = \int_0^\infty t dt\exp\left[-i\left(tx+\frac{1}{3}t^3\right)\right]. \tag{4.29}$$

Furthermore we define the following two functions:

$$f_1(x) = \int_0^\infty \frac{dt}{t}\exp(-itx)\left[\exp\left(-\frac{i}{3}t^3\right)-1\right] = \int_x^\infty dt\left[f(t)-\frac{1}{t}\right], \tag{4.30}$$

and

$$f_2(x) = \int_0^\infty \frac{dt}{t^2}\exp(-itx)\left[\exp\left(-\frac{i}{3}t^3\right)-1\right] = -i\left[xf_1(x)+f'(x)\right]. \tag{4.31}$$

These functions obey the following differential equations:

$$f''(x) = xf(x)-1, \quad f_1'(x) = \frac{1}{x}-f(x) = -\frac{1}{x}f''(x). \tag{4.32}$$

Having these definitions we can write:

$$\begin{aligned}&\int_{-1}^1 dv\int_0^\infty \frac{d\tau}{\tau}e^{-i\tau m^2}\frac{1}{\tau}\left(e^{i\tau\beta}-1\right)\\ &= (-i)\left[m^2\int_{-1}^1 dv f_1\left(m^2 b^{-1/3}\right) + \int_{-1}^1 dv b^{1/3}f'\left(m^2 b^{-1/3}\right)\right].\end{aligned} \tag{4.33}$$

For $\rho(v) = m^2 b(v)^{-1/3}$ we can use the following integral identity:

$$\int_{-1}^{+1} dv g(v) f_1[\rho(v)] = -\int_{-1}^{+1} dv \left[\frac{G(v)}{\rho(v)}\right]' f'[\rho(v)] \tag{4.34}$$

which is valid if $\rho(v) \to \infty$ for $v \to \pm 1$ and for any finite function g and where $G'(v) = g(v)$. In our case

$$\left[\frac{G(v)}{\rho(v)}\right]' = \left[v b^{1/3}/m^2\right]' = \frac{b^{1/3}}{m^2}\left(1 - \frac{4}{3}\frac{v^2}{1-v^2}\right). \tag{4.35}$$

Then we obtain

$$\begin{aligned}
&\int_{-1}^{1} dv \int_0^\infty \frac{d\tau}{\tau} e^{-i\tau m^2} \frac{1}{\tau}\left(e^{i\tau\beta} - 1\right) \\
=&(-i)\int_{-1}^{1} dv \frac{4}{3}\frac{v^2}{1-v^2} b^{1/3} f'\left(m^2 b^{-1/3}\right) \\
=&(-i)\int_{-1}^{1} dv \int_0^\infty \frac{d\tau}{\tau} e^{-i\tau m^2} \frac{m^6}{12}\kappa^2 \psi'^2(\omega_0 z_-) e^{i\tau\beta}\tau^2\left(1-v^2\right)v^2.
\end{aligned} \tag{4.36}$$

Now we can insert this term as well as the prefactors from before into the expression for the probability:

$$\begin{aligned}
\delta P_{\text{incoh}} =& \frac{\alpha^2}{4}\int \frac{d^3k}{(2\pi)^3}\frac{1}{\omega_k p_- p'_- k_-}\int d\phi d\phi' \,\mathrm{Re}\left\{\exp[\cdots]\int_0^\infty \frac{d\tau}{\tau} e^{-i\tau m^2}\int_{\phi'}^\infty dz_-\right. \\
&\left.\times(-i)\int_0^1 dv \frac{m^6}{24}\kappa^2\psi'^2\left(\omega_0 z_-\right)\tau^2\left(1-v^2\right)\left[\left(3+v^2\right)T_{11} + \left(6-2v^2\right)T_{22}\right]e^{i\tau\beta}\right\}.
\end{aligned} \tag{4.37}$$

From here on, one can proceed just like in the tree level case and we can switch to new variables given by $\phi_+ = (\phi + \phi')/2$ and $\phi_- = \phi - \phi'$ and then expand the expressions T_{jj} to first order and the phase in third order in ϕ_- around $\phi_- = 0$. Because we have already expanded the terms arising from the polarization operator it is sufficient to replace in the dz_- integral $\phi' \to \phi_+$. Thus we arrive at

$$\begin{aligned}
\delta P_{\text{incoh}} =& \frac{\alpha^2}{4}\int \frac{d^3k}{(2\pi)^3}\frac{1}{\omega_k p_- p'_- k_-}\int d\phi_+ d\phi_- \,\mathrm{Re}\Bigg\{ \\
&\times e^{-i\frac{k_- m^2}{2p_- p'_-}\left\{\left(1+\left[\frac{\boldsymbol{p}_\perp}{m} - \frac{p_-}{k_-}\frac{\boldsymbol{k}_\perp}{m} - \boldsymbol{\xi}_\perp(\phi_+)\right]^2\right)\phi_- + \boldsymbol{\xi}'^2_\perp(\phi_+)\frac{\phi_-^3}{12}\right\}}\int_0^\infty \frac{d\tau}{\tau}e^{-i\tau m^2}\int_{\phi_+}^\infty dz_- \\
&\times(-i)\int_0^1 dv \frac{m^6}{24}\kappa^2\psi'^2\left(\omega_0 z_-\right)\tau^2\left(1-v^2\right)\left[\left(3+v^2\right)T_{11} + \left(6-2v^2\right)T_{22}\right]e^{i\tau\beta}\Bigg\}.
\end{aligned} \tag{4.38}$$

The explicit expansion of the T_{jj} will be carried out in section 4.3. We now introduce the following notation:

$$\kappa(\varphi) = \kappa|\psi'(\varphi)|. \tag{4.39}$$

With the substitution $u = m^2\tau$ we can rewrite the following expression:

$$e^{-i\tau m^2} e^{i\tau\beta} = e^{-iu\left[1+\frac{(1-v^2)^2}{48}\kappa^2(\varphi)u^2\right]}. \tag{4.40}$$

Also we define the following quantities, which are known in the literature as the transverse components of the polarization operator in a constant crossed field [27, 37]:

$$P_1(k,\phi) = \frac{\alpha}{48\pi} m\kappa^2(\phi) \int_0^\infty du u \int_0^1 dv e^{-iu\left[1+\frac{\left(1-v^2\right)^2}{48}\kappa^2(\phi)u^2\right]} \left(1-v^2\right)\left(3+v^2\right), \tag{4.41}$$

$$P_2(k,\phi) = \frac{\alpha}{48\pi} m\kappa^2(\phi) \int_0^\infty du u \int_0^1 dv e^{-iu\left[1+\frac{\left(1-v^2\right)^2}{48}\kappa^2(\phi)u^2\right]} \left(1-v^2\right)\left(6-2v^2\right). \tag{4.42}$$

Now the probability can be written in the following way:

$$\begin{aligned}
\delta P_{\text{incoh}} =& \int \frac{d^3k}{16\pi^2} \frac{\alpha}{p_- p'_- \omega} \int d\phi_+ \operatorname{Re}\Bigg\{(-i) \int d\phi_- \exp[\cdots] \int_{\phi_+}^\infty d\varphi \\
& \frac{m}{k_-} \frac{\alpha}{48\pi} m\kappa^2(\varphi) \int_0^\infty u du \int_0^1 dv e^{-iu\left[1+\frac{(1-v^2)^2}{48}\kappa^2(\varphi)u^2\right]} \\
& \left[\left(1-v^2\right)\left(3+v^2\right)T_{11} + \left(1-v^2\right)\left(6-2v^2\right)T_{22}\right]\Bigg\} \\
=& \int \frac{d^3k}{16\pi^2} \frac{\alpha}{p_- p'_- \omega} \int d\phi_+ \operatorname{Im}\Bigg\{\int d\phi_- \exp[\cdots] \int_{\phi_+}^\infty d\varphi \\
& \left[\frac{m}{k_-} P_1(k,\varphi) T_{11} + \frac{m}{k_-} P_2(k,\varphi) T_{22}\right]\Bigg\}
\end{aligned} \tag{4.43}$$

From the tree level probability we know that the expression

$$\int d\phi_- e^{-i\frac{k_- m^2}{2p_- p'_-}\left\{\left(1+\left[\frac{\boldsymbol{p}_\perp}{m} - \frac{p_-}{k_-}\frac{\boldsymbol{k}_\perp}{m} - \boldsymbol{\xi}_\perp(\phi_+)\right]^2\right)\phi_- + \boldsymbol{\xi}'^2_\perp(\phi_+)\frac{\phi_-^3}{12}\right\}} T_{jj} \tag{4.44}$$

has to be real within the LCFA. This is because, up to prefactors, this quantity may be interpreted as the probability per unit phase of the external field. Therefore we can exclude this term from the imaginary part and we obtain:

$$
\begin{aligned}
\delta P_{\text{incoh}} =& \frac{1}{2}\sum_j \int \frac{d^3k}{16\pi^2}\frac{\alpha}{p_- p'_- \omega}\int d\phi_+ 2\,\mathrm{lm}\left\{\int_{\phi_+}^{\infty} d\varphi \frac{m}{k_-} P_j(k,\varphi)\right\} \\
& \int d\phi_- e^{-i\frac{k_- m^2}{2p_- p'_-}\left\{\left(1+\left[\frac{\boldsymbol{p}_\perp}{m}-\frac{p_-}{k_-}\frac{\boldsymbol{k}_\perp}{m}-\boldsymbol{\xi}_\perp(\phi_+)\right]^2\right)\phi_- + \boldsymbol{\xi}'^2_\perp(\phi_+)\frac{\phi_-^3}{12}\right\}} T_{jj}.
\end{aligned}
\tag{4.45}
$$

Note that T_{jj} (compare Eq. (3.33)) is the exact expression one gets if one takes the trace expression in Ref. [36] Eq. (52) (including the factor $1/4$) and takes the sum over the spins s and s'. The factor $1/2$ in Eq. (4.45) accounts for the averaging over the spin s. This factor is missing in Ref. [36] because the spin sums as well as the spin average have not been taken.

4.3 Trace Expansion and Transverse Integration

This section will follow the treatment of the trace terms as well as the transverse momentum integrals in Ref. [37] very closely. In the following we evaluate the trace expression (3.35) for $i = j$ in the case of a linearly polarized plane wave field. Subsequently we will apply the LCFA, by expanding the trace expressions to first order in ϕ_-. For $i = j$ the trace expressions are given by:

$$
\begin{aligned}
\frac{T_{jj}}{2} =& \frac{k_-^2 m^2}{p_- p'_-} - \frac{p_-}{p'_-}\left(\pi'(\phi)\pi'(\phi')\right) - \frac{p'_-}{p_-}\left(\pi(\phi)\pi(\phi')\right) \\
& + \left(\pi'(\phi)\pi(\phi')\right) + \left(\pi(\phi)\pi'(\phi')\right) \\
& + 2\left(\Lambda_j\pi(\phi)\right)\left(\Lambda_j\pi'(\phi')\right) + 2\left(\Lambda_j\pi'(\phi)\right)\left(\Lambda_j\pi(\phi')\right).
\end{aligned}
\tag{4.46}
$$

Unlike the trace expressions in Refs. [36, 37] the ones above have already included a summation over the initial and final spins s, s'. With this being the only difference, in the following calculation we will obtain at every step the results from Ref. [37] summed over the spins $s, s' = \pm 1$. As mentioned before, the first step, is to evaluate the trace, for the case of a linearly polarized plane wave. We thus have $\psi_2(\phi) \equiv 0$ and we redifine $\psi_1(\phi) = \psi(\phi)$, such that

$$
A^\mu(\phi) = A_0\psi(\phi)e_1^\mu = A_1(\phi)e_1^\mu, \tag{4.47}
$$

where A_0 is the amplitude of the vector potential A^μ and $|\psi(\phi)| \le 1$ is normalized. Now each of the four-vector products in Eq. (4.46) has to be evaluated, which is a lengthy calculation. The result is the following:

$$
\begin{aligned}
T_{jj} =& -4m^2 + 4(pp') + 8\left(p_j - \frac{p_-}{k_-}k_j\right)^2 \\
& + 2\left[p_1\left(1-\frac{p_-}{p'_-}+\frac{k_-}{p_-}-4\delta_{1j}\right) - k_1\left(1-\frac{p_-}{p'_-}-4\delta_{1j}\frac{p_-}{k_-}\right)\right]\left[eA_1(\phi)+eA_1(\phi')\right] \\
& + 2\left(-\left[1-\frac{p_-}{p'_-}+\frac{k_-}{p_-}\right]+4\delta_{1j}\right)e^2A_1(\phi)A_1(\phi')
\end{aligned}
$$

$$(4.48)$$

where δ_{ji} is the Kronecker delta. Next, we will apply the LCFA by first substituting:

$$\phi_+ = \frac{\phi + \phi'}{2}, \qquad \phi_- = \phi - \phi'. \tag{4.49}$$

Thus the inverse transformation is given by:

$$\phi = \phi_+ + \frac{\phi_-}{2}, \qquad \phi' = \phi_+ - \frac{\phi_-}{2}. \tag{4.50}$$

Then we must expand the fields in the trace to first order in ϕ_- such that:

$$\begin{aligned} A_1(\phi) &\simeq A_1(\phi_+) + A_1'(\phi_+)\frac{\phi_-}{2} \\ A_1(\phi') &\simeq A_1(\phi_+) - A_1'(\phi_+)\frac{\phi_-}{2}. \end{aligned} \tag{4.51}$$

Substituting the above expression gives:

$$\begin{aligned} T_{jj} =& -4m^2 + 4(pp') + 8\left(p_j - \frac{p_-}{k_-}k_j\right)^2 \\ &+ 4\left[p_1\left(1 - \frac{p_-}{p'_-} + \frac{k_-}{p_-} - 4\delta_{1j}\right) - k_1\left(1 - \frac{p_-}{p'_-} - 4\delta_{1j}\frac{p_-}{k_-}\right)\right] eA_1(\phi_+) \\ &+ 2\left(-\left[1 - \frac{p_-}{p'_-} + \frac{k_-}{p_-}\right] + 4\delta_{1j}\right)\left[e^2A_1(\phi_+)^2 - e^2A_1'(\phi_+)^2\frac{\phi_-^2}{4}\right]. \end{aligned} \tag{4.52}$$

A few more manipulations lead to:

$$\begin{aligned} T_{11} =& 4(pp') - 4m^2 + 8\left(p_1 - \frac{p_-}{k_-}k_1\right)^2 \\ &- 4\left[p_1\left(4 + \frac{k_-^2}{p_-p'_-}\right) - k_1\left(4\frac{p_-}{k_-} + \frac{k_-}{p'_-}\right)\right] eA_1(\phi_+) \\ &+ \left(8 + 2\frac{k_-^2}{p_-p'_-}\right)\left[e^2A_1(\phi_+)^2 - e^2A_1'(\phi_+)^2\frac{\phi_-^2}{4}\right], \end{aligned} \tag{4.53}$$

$$\begin{aligned} T_{22} =& 4(pp') - 4m^2 + 8\left(p_2 - \frac{p_-}{k_-}k_2\right)^2 \\ &- 4\left[p_1\frac{k_-^2}{p_-p'_-} - k_1\frac{k_-}{p'_-}\right] eA_1(\phi_+) \\ &+ 2\frac{k_-^2}{p_-p'_-}\left[e^2A_1(\phi_+)^2 - e^2A_1'(\phi_+)^2\frac{\phi_-^2}{4}\right]. \end{aligned} \tag{4.54}$$

In the next step, we recall, that the trace expressions will be multiplied with the phase factor Eq. (4.8) and are integrated over ϕ_-. Thus we can neglect certain terms in the trace, which will vanish after the ϕ_- integral is taken. In particular such terms are given by those which are proportional to the derivative of the phase with respect to ϕ_-:

$$\frac{\partial\Phi(\phi_+,\phi_-)}{\partial\phi_-} = -i\frac{k_-m^2}{2p_-p'_-}\left(1+\left[\frac{\boldsymbol{p}_\perp}{m}-\frac{p_-}{k_-}\frac{\boldsymbol{k}_\perp}{m}-\boldsymbol{\xi}_\perp(\phi_+)\right]^2+\boldsymbol{\xi}'^2_\perp(\phi_+)\frac{\phi_-^2}{4}\right) \tag{4.55}$$

where $\boldsymbol{\xi}_\perp = e\boldsymbol{A}_\perp/m$. For those terms the integral in ϕ_- gives:

$$\begin{aligned}\int d\phi_-\frac{\partial\Phi(\phi_+,\phi_-)}{\partial\phi_-}e^{i\Phi(\phi_+,\phi_-)} &= -i\int d\phi_-\frac{\partial}{\partial\phi_-}e^{i\Phi(\phi_+,\phi_-)}\\ &= -i\left[e^{i\Phi(\phi_+,\phi_-)}\right]_{-\infty}^{\infty}=0,\end{aligned} \tag{4.56}$$

which is equivalent to Eq. (3.54). Thus by leaving out the terms which are proportional to $\partial\Phi(\phi_+,\phi_-)/\partial\phi_-$ we obtain the following expressions for the trace:

$$T_{11} = -8m^2-4\left(4+\frac{q_-^2}{p_-p'_-}\right)A'_1(\phi_+)^2\frac{\phi_-^2}{4}-8\left(p_2-\frac{p_-}{q_-}q_2\right)^2, \tag{4.57}$$

$$T_{22} = -4\frac{q_-^2}{p_-p'_-}A'_1(\phi_+)^2\frac{\phi_-^2}{4}+8\left(p_2-\frac{p_-}{q_-}q_2\right)^2. \tag{4.58}$$

It can be seen that after eliminating all the terms proportional to $\partial\Phi(\phi_+,\phi_-)/\partial\phi_-$ the traces only depend on the field derivative A'^μ but not on the fields A^μ, thus we have achieved manifest gauge invariance. Before performing the transverse $\boldsymbol{k}_\perp$ integration in Eq. (4.45) the integration measure can be transformed as follows:

$$d^3k = \frac{\omega_k}{k_-}dk_-d^2k_\perp, \tag{4.59}$$

where $d^2k_\perp = dk_1dk_2$. The resulting transverse integrals are Gaussian, which are also found in Ref. [37]:

$$\int d^2k_\perp e^{i\frac{m^2k_-}{2p_-p'_-}\boldsymbol{\pi}^2_\perp(\phi_+)\phi_-} = 2\pi i\frac{k_-p'_-}{p_-(\phi_-+i0)}, \tag{4.60}$$

$$\int d^2k_\perp\left(p_2-\frac{p_-}{k_-}k_2\right)^2 e^{i\frac{m^2k_-}{2p_-p'_-}\boldsymbol{\pi}^2_{\perp,}(\phi_+)\phi_-} = -2\pi\frac{p'^2_-}{(\phi_-+i0)^2}, \tag{4.61}$$

where

$$\boldsymbol{\pi}_\perp(\phi_+) = \frac{\boldsymbol{p}_\perp}{m}-\frac{p_-}{k_-}\frac{\boldsymbol{k}_\perp}{m}-\boldsymbol{\xi}_\perp(\phi_+). \tag{4.62}$$

For the ϕ_- integral we use the integral representation of the Airy function [32]:

$$\mathrm{Ai}(z) = \int_{-\infty}^{\infty} \frac{d\tilde{\phi}}{2\pi} e^{iz\tilde{\phi} + \frac{i\tilde{\phi}^3}{3}}. \tag{4.63}$$

As a result of these integrations one obtains much simpler expressions for the tree and loop level probabilities. Applying the integrals to the tree level probability (4.9) results in:

$$P_0 = -\frac{\alpha}{2} \frac{m^2}{\omega_0 p_-^2} \int_0^{p_-} dq_- \int d\varphi_+ (T_1 + T_2). \tag{4.64}$$

where

$$T_1 = \mathrm{Ai}_1(z) + \left[3 + \frac{q_-^2}{p_-(p_- - q_-)}\right] \frac{\mathrm{Ai}'(z)}{z} \tag{4.65}$$

$$T_2 = \mathrm{Ai}_1(z) + \left[1 + \frac{q_-^2}{p_-\,(p_- - q_-)}\right] \frac{\mathrm{Ai}'(z)}{z} \tag{4.66}$$

where

$$z = \left[\frac{q_-}{(p_- - q_-)\chi_p\,(\phi_+)}\right]^{2/3}, \tag{4.67}$$

for which we define

$$\chi_p(\phi) := \frac{|e|E_0 p_-}{m^3} \left|\frac{\psi'(\phi)}{\omega_0}\right|, \tag{4.68}$$

and $\mathrm{Ai}_1(z) = \int_z^\infty \mathrm{Ai}(x)dx$. For the incoherent part of the polarization correction Eq. (4.45) we obtain:

$$\delta P_{\mathrm{incoh}} = -\frac{1}{2} \frac{\alpha m^2}{p_-^2 \omega_0^2} \int_0^{p_-} dq_- \int d\varphi_+ \sum_j 2\,\mathrm{Im}\left\{\int_{\varphi_+}^{\infty} d\varphi \frac{m}{q_-} P_j(q,\varphi)\right\} T_j(\varphi_+). \tag{4.69}$$

4.4 Formulation in Dimensionless Quantities

For a formulation in dimensionless quantities we are going to express the pulse shape function as a function of the phase $\varphi \equiv \omega_0 \phi$. Accordingly we will change the notation such that $\psi(\phi) \to \psi(\varphi)$. We define the phase dependent quantum nonlinearity parameters as:

$$\chi(\varphi) = \chi_0 |\psi'(\varphi)|, \tag{4.70}$$

$$\kappa(\varphi) = \kappa_0 |\psi'(\varphi)|. \tag{4.71}$$

where the prime now denotes the derivative $d/d\varphi$. Now the pulse shape functions are normalized such that $|\psi(\varphi)| \leq 1$ and $|\psi'(\varphi)| \leq 1$. In terms of dimensionless quantities the tree level probability can be written as

$$P_0 = -\frac{1}{2}\frac{\alpha}{\eta}\int_0^1 d\tau \int d\varphi_+ (T_1 + T_2), \tag{4.72}$$

$$T_1 = \mathrm{Ai}_1(z) + \left[3 + \frac{\tau^2}{(1-\tau)}\right]\frac{\mathrm{Ai}'(z)}{z} \tag{4.73}$$

$$T_2 = \mathrm{Ai}_1(z) + \left[1 + \frac{\tau^2}{(1-\tau)}\right]\frac{\mathrm{Ai}'(z)}{z} \tag{4.74}$$

and

$$z = \left[\frac{\tau}{(1-\tau)\chi(\varphi_+)}\right]^{2/3}. \tag{4.75}$$

Here the substitutin $\tau = q_-/p_-$ was used. It is possible to write the probability in terms of modified Bessel-function of the second kind. To do this we can make use of the following identities [1]:

$$\mathrm{Ai}(x) = \frac{1}{\pi}\sqrt{\frac{x}{3}}K_{1/3}\left(\frac{2}{3}x^{3/2}\right), \tag{4.76}$$

$$\mathrm{Ai}'(x) = -\frac{x}{\pi\sqrt{3}}K_{2/3}\left(\frac{2}{3}x^{3/2}\right), \tag{4.77}$$

which are valid for $x > 0$. Also, in order to obtain a compact expression the following identity is useful:

$$\int_0^\infty dz\left(\nu\frac{g(z)}{z} + g'(z)\right)K_\nu(z) = \int_0^\infty dz g(z)K_{\nu+1}(z), \tag{4.78}$$

which is valid for any function $g(z)$ for which $g(0) = 0$. The following expression for the tree level probability can be obtained:

$$P_0 = \frac{1}{3\sqrt{3}}\frac{\alpha}{\eta}\int d\varphi_+ \int_0^\infty du \frac{5u^2 + 7u + 5}{(1+u)^3}K_{2/3}\left(\frac{2u}{3\chi(\varphi_+)}\right). \tag{4.79}$$

This is the expression that can be found in Ref. [6]. The incoherent part of the polarization correction in dimensionless quantities may be written as:

$$\delta P_{\mathrm{incoh}} = -\frac{\alpha}{\eta^2}\int_0^1 \frac{ds}{s}\int d\varphi_+ \int_{\varphi_+}^\infty d\varphi \left(T_1(\varphi_+)\frac{\operatorname{Im} P_1(\varphi)}{m} + T_2(\varphi_+)\frac{\operatorname{Im} P_2(\varphi)}{m}\right). \tag{4.80}$$

4.5 Incoherent Trident Pair Creation

The probability δP_{incoh} may be written as:

$$\delta P_{\text{incoh}} = -\sum_{j=1,2} \int_0^{p_-} dq_- \int d\phi \, \frac{dP_j^{\text{Tree}}}{dq_- d\phi} \int_\phi^\infty d\varphi \, \text{Im} \left\{ -\frac{2m}{q_-} P_j(q, \varphi) \right\} \tag{4.81}$$

where

$$\frac{dP_j^{\text{Tree}}}{dq_- d\phi} = -\frac{\alpha}{2} \frac{m^2}{p_-^2} T_j(p_-, q_-, \chi) \tag{4.82}$$

is the differential tree level probability for the emmission of a photon with the momentum component q_- and polarization quantum number j. Note that when taking the sum over the polarizations $j = 1, 2$ and the integrals in q_- and ϕ one obtains the total tree level probability (4.64). According to the optical theorem the quantity $\text{Im}\{-(2m/q_-)P_j(q, \varphi)\}$ is equal to the probability that a photon of polarization j creates an electron-positron pair (Breit-Wheeler pair creation) (as discussed in Refs. [26, 37, 5]). Thus, up to a minus sign, δP_{incoh} can be interpreted as the probability that the electron emits an on-shell photon which later creates an electron-positron pair. This process is known the trident process [21, 6]. The appearance of the trident probability in the polarization correction of nonlinear Compton Scattering has been discussed in Ref. [40] where it was also shown that when applying the optical theorem to the polarization correction of the mass operator, all incoherent contributions will cancel out. These considerations allow for an interpretation of δP_{incoh} as the correction to nonlinear Compton scattering which is due to the possibility that the outgoing photon may decay into an electron-positron pair at a later time in the laser pulse. The suppression of the total probability due to incoherent processes are commonly referred to as damping effects. In Refs. [36, 37] the probabilities of nonlinear Compton Scattering and nonlinear Breit-Wheeler pair creation which include damping effects due to arbitrary many first order mass and polarization corrections were derived. The correction δP_{incoh} is one of the first order corrections one obtains when expanding the result in Ref. [37] Eq. (13).

4.6 Plots

Fig. 4.1 shows plots of the differential probabilities dP_0/dk_- (the tree level probability) and $d\delta P_{\text{incoh}}/dk_-$ (the incoherent polarization correction) in the LCFA. The results were achieved by performing a numerical integration of the phase integrals using Wolfram Mathematica 13.3. In both cases we assumed an electron of energy $\epsilon = 10$ GeV and the pulse shape function:

$$\psi(\phi) = e^{-\left(\frac{\phi}{\Delta\phi}\right)^2} \sin(\omega_0 \phi), \tag{4.83}$$

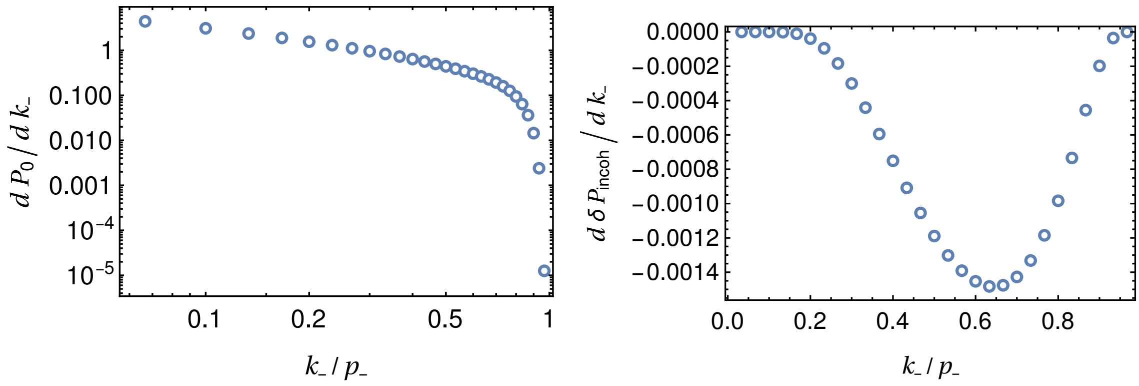


Figure 4.1: This figure contains the differential plots of the LCFA expressions of dP_0/dk_- (the tree level probability) and $d\delta P_{\text{incoh}}/dk_-$ (the incoherent polarization correction). The parameters used are $\epsilon = 10$ GeV, $\chi = 2$, and $\omega_0 = 5$ fs corresponding to $\eta \approx 0.12$ and $\xi \approx 16.85$.

with $\omega_0 = 1.55$ eV and $\Delta\phi = 5$ fs, corresponding to a total phase of $\Phi = \omega_0\Delta\phi \approx 11.77$. The specifications of the electron energy and the wavelength correspond to $\eta \approx 0.12$. The field amplitude was chosen such that $\chi = 2$ and accordingly one has $\xi = \chi/\eta \approx 16.85$, thus being within the regime of the LCFA.

4.7 The Coherent Part of the Polarization Correction

The coherent part of the polarization correction is given by:

$$\begin{aligned}\delta P_{\text{coh}} = &-\frac{\alpha^2}{4}\int \frac{d^3k}{(2\pi)^3}\frac{1}{\omega_k p_- p'_-}\int d\phi d\phi' \\ &\times \operatorname{Re}\left\{ e^{i\Omega}\int_{-1}^{1} dv \int_0^\infty \frac{d\tau}{\tau} e^{-i\tau m^2}\frac{T_{11}+T_{22}}{4}\left[\left(e^{i\tau\beta}-1\right)\big|_{z_-=\phi'}\right]\right\},\end{aligned} \tag{4.84}$$

where

$$e^{i\Omega} = \exp\left[i\frac{k_- m^2}{2p_- p'_-}\int_\phi^{\phi'} d\tilde{\phi}\left(1+\left[\frac{\boldsymbol{p}_\perp}{m}-\frac{p_-}{k_-}\frac{\boldsymbol{k}_\perp}{m}-\boldsymbol{\xi}_\perp\right]^2\right)\right]. \tag{4.85}$$

We can perform similar calculation steps as in the incoherent case, by making use of the function defined in Eq. (4.28) and thus we can write:

$$\begin{aligned}&\int_0^\infty \frac{d\tau}{\tau} e^{-i\tau m^2}\left(e^{i\tau\beta}-1\right) \\ &= \int_0^\infty \frac{dt}{t} e^{-itm^2 b^{-1/3}}\left(e^{-\frac{i}{3}t^3}-1\right) \\ &= f_1(m^2 b^{-1/3}),\end{aligned} \tag{4.86}$$

where

$$b = m^6 \kappa^2(\phi) \left[\frac{1}{4} \left(1 - v^2\right) \right]^2, \tag{4.87}$$

Also we can make use of the relation (4.34) which we can use to rewrite the integral of f_1 into an integral of f'. Thus we arrive at the following:

$$\int_{-1}^{1} dv \int_0^{\infty} \frac{d\tau}{\tau} e^{-i\tau m^2} \left(e^{i\tau\beta} - 1\right) = -\int_{-1}^{1} dv \frac{b^{1/3}}{m^2} \left(1 - \frac{4}{3} \frac{v^2}{1 - v^2}\right) f'\left(\frac{m^2}{b^{1/3}}\right). \tag{4.88}$$

Since the intergrand is symmetric in v we can write

$$\int_{-1}^{1} dv = 2 \int_0^1 dv. \tag{4.89}$$

Thus for the coherent probability we obtain:

$$\begin{aligned} \delta P_{\text{coh}} = &- \frac{\alpha^2}{8} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_k p_- p'_-} \int d\phi d\phi' \\ &\times \operatorname{Re}\left\{ e^{i\Omega} (T_{11} + T_{22}) \int_0^1 dv \frac{b^{1/3}}{m^2} \left(1 - \frac{4}{3} \frac{v^2}{1 - v^2}\right) f'\left(\frac{m^2}{b^{1/3}}\right) \right\}. \end{aligned} \tag{4.90}$$

Next, we will perform the following substitution:

$$\begin{aligned} u &= \frac{1}{1 - v^2}, \qquad v = \sqrt{1 - \frac{1}{u}}, \\ dv &= \frac{du}{2u\sqrt{u(u-1)}}. \end{aligned} \tag{4.91}$$

After the substitution we obtain the following integration boundaries:

$$u(v = 0) = 1, \qquad u(v = 1) = \infty. \tag{4.92}$$

After substitution we have

$$\frac{m^2}{b^{1/3}} = \frac{(4u)^{2/3}}{\kappa(\phi')^{2/3}}, \qquad \frac{v^2}{1 - v^2} = u - 1. \tag{4.93}$$

Performing the substitution in the probability gives:

$$\begin{aligned} \delta P_{\text{coh}} = &\frac{\alpha^2}{8} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_k p_- p'_-} \int d\phi d\phi' \\ &\times \operatorname{Re}\left\{ e^{i\Omega} (T_{11} + T_{22}) \frac{\kappa(\phi')^{2/3}}{12 \cdot 2^{1/3}} \int_1^{\infty} \frac{du(7 - 4u)}{u^{5/3}\sqrt{u(u-1)}} f'\left(\frac{(4u)^{2/3}}{\kappa(\phi')^{2/3}}\right) \right\}. \end{aligned} \tag{4.94}$$

As before we transform the phase integrals:

$$\phi_+ = \frac{\phi + \phi'}{2}, \qquad \phi_- = \phi - \phi'. \tag{4.95}$$

The expansion of the phase expression to third order in ϕ_- and the expansion of the trace expression are once again identical to the tree level case. The terms that come from the polarization operator have already been expanded in powers of ω_0 such that it is sufficient to only keep the leading order in ϕ_-. Thus, for those terms we just replace $\phi' \to \phi_+$. And so we obtain:

$$\begin{aligned}\delta P_{\text{coh}} =& \frac{\alpha^2}{8} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_k p_- p'_-} \int d\phi_+ \int d\phi_- \\ & \times \operatorname{Re}\left\{ e^{i\Omega}(T_{11} + T_{22}) \frac{\kappa(\phi_+)^{2/3}}{12 \cdot 2^{1/3}} \int_1^\infty \frac{du(7-4u)}{u^{5/3}\sqrt{u(u-1)}} f'\left(\frac{(4u)^{2/3}}{\kappa(\phi_+)^{2/3}}\right) \right\}.\end{aligned} \tag{4.96}$$

Next, we can express the d^3k integral in lightcone coordinates by:

$$d^3k = \frac{\omega_k}{k_-} dk_- d^2\boldsymbol{k}_\perp. \tag{4.97}$$

Then we can rewrite the probability as:

$$\begin{aligned}\delta P_{\text{coh}} =& \frac{\alpha}{2\pi} \int_0^{p_-} dk_- \int d\phi_+ \operatorname{Re}\left\{ \frac{1}{2} \int \frac{d^2\boldsymbol{k}_\perp}{16\pi^2} \frac{\alpha}{p_- p'_- k_-} \int d\phi_- (T_{11} + T_{22}) e^{i\Omega} \right. \\ & \left. \times \frac{\kappa(\phi_+)^{2/3}}{12 \cdot 2^{1/3}} \int_1^\infty \frac{du(7-4u)}{u^{5/3}\sqrt{u(u-1)}} f'\left(\frac{(4u)^{2/3}}{\kappa(\phi_+)^{2/3}}\right) \right\}.\end{aligned} \tag{4.98}$$

Now we recognize that the expression above contains the differential probability for the tree level process which is given by:

$$\frac{dP_0}{dk_- d\phi_+} = \frac{1}{2} \int \frac{d^2\boldsymbol{k}_\perp}{16\pi^2} \frac{\alpha}{p_- p'_- k_-} \int d\phi_- (T_{11} + T_{22}) e^{i\Omega}. \tag{4.99}$$

We know that the above quantity is real, which means that we can take it out of the real part operator. Then we can use $\operatorname{Re} f'(x) = \pi \operatorname{Gi}'(x)$ (See Appendix B). Thus we arrive at:

$$\delta P_{\text{coh}} = \frac{\alpha}{2} \int_0^{p_-} dk_- \int d\phi_+ \frac{dP_0}{dk_- d\phi_+} \frac{\kappa(\phi_+)^{2/3}}{12 \cdot 2^{1/3}} \int_1^\infty \frac{du(7-4u)}{u^{5/3}\sqrt{u(u-1)}} \operatorname{Gi}'\left(\frac{(4u)^{2/3}}{\kappa(\phi_+)^{2/3}}\right). \tag{4.100}$$

Next, we can rewrite the result with dimensionless quantities. For this we introduce the phase $\varphi_+ = \omega_0 \phi_+$ and the variable $\tau = k_-/p_-$. Thus the final result is:

$$\delta P_{\text{coh}} = \int d\varphi_+ \int_0^1 d\tau \frac{dP_0}{d\tau d\varphi_+} \frac{\alpha\kappa(\varphi_+)^{2/3}}{24 \cdot 2^{1/3}} \int_1^\infty \frac{du(7-4u)}{u^{5/3}\sqrt{u(u-1)}} \operatorname{Gi}'\left(\frac{(4u)^{2/3}}{\kappa(\varphi_+)^{2/3}}\right).$$

$$(4.101)$$

After the transverse momentum integral is taken the expression $dP_0/d\tau d\varphi_+$ (this is the differential tree level probability) is given by:

$$\frac{dP_0}{d\tau d\varphi_+} = -\frac{1}{2}\frac{\alpha}{\eta}(T_1 + T_2) \tag{4.102}$$

where

$$\begin{aligned} T_1 &= \mathrm{Ai}_1(z) + \left[3 + \frac{\tau^2}{(1-\tau)}\right]\frac{\mathrm{Ai}'(z)}{z} \\ T_2 &= \mathrm{Ai}_1(z) + \left[1 + \frac{\tau^2}{(1-\tau)}\right]\frac{\mathrm{Ai}'(z)}{z} \end{aligned} \tag{4.103}$$

and

$$z = \left[\frac{\tau}{(1-\tau)\chi\left(\varphi_+\right)}\right]^{2/3}. \tag{4.104}$$

When taking the integral in τ one has to have in mind that $\kappa = \tau\chi$.

5 The High Field Limit for Incoherent Processes

High Field limit corresponds to the limit $\chi \gg 1$ while keeping the electron energy fixed. Studies of the high field limit of various strong field QED processes and radiative corrections have been of high interest ever since the Ritus-Narozhny conjecture was formulated in the 1970s. The Ritus-Narozhny (RN) conjecture states that in the regime $\chi \gg 1$ the effective expansion parameter of QED in a constant crossed field is given by $\alpha\chi^{2/3}$ [31, 20]. From the RN conjecture it follows that when $\alpha\chi^{2/3} \approx 1$, corresponding to $\chi \approx 1600$, all orders of perturbation theory become equally important and thus the perturbative approach to strong field QED breaks down. It has to be noted that the RN conjecture has not been rigorously proven but instead it was postulated by Ritus and Narozhny after determining the scaling of the radiative correction to the mass and polarization operators in a constant crossed field up to order α^3. Ref. [20] provides a comprehensive summary of the diagrams that have been studied and what their scaling is. Since the RN conjecture has not been proven, the high field behaviour of radiative corrections is still an active area of research. As an example in Ref. [29] the RN conjecture has recently been confirmed to be valid in a constant crossed field for a subset of possible corrections, the so called bubble-type polarization corrections, to the mass operator. While Ritus and Narozhny only considered the case of a constant crossed field, the validity of the RN conjecture can be extended to include all plane wave fields of sufficiently low frequency by means of the locally constant field approximation (LCFA), as within the LCFA, differential probabilities of the plane wave phase are assumed to be identical to the ones in the constant crossed field case. Beyond the LCFA it has been shown on multiple occasions that the RN conjecture does not hold [23, 18]. It has to be noted that all the above references have assumed that the limit $\chi \gg 1$ is achieved by increasing the field strength of the background field while keeping the electron energy constant, i.e. it really is the high field limit and not the high energy limit. In this way, since the parameter ξ also scales linearly with the field strength, it is always guaranteed that $\xi \gg \chi$ and that $\eta \ll 1$ is a constant in this limit, which is in accordance with the LCFA. Indeed, in Ref. [35] it was shown that the scaling of the one-loop corrections differs depending on whether the high field limit or the high energy limit was taken. Here it was concluded that the limit of large electron energies is in contradiction with the LCFA, which means that the RN conjecture in a plane wave can only be tested by fixing the electron energy and increasing the laser intensity. Similar conclusions can be found in Ref. [24]. Experimentally the regime $\alpha\chi^{2/3} \sim 1$ has not been achieved yet, however several experimental proposals to test QED at such high intensities have been put forward [46, 10, 7, 8, 17]. In this chapter we specifically study the high field limit of the incoherent radiative corrections to nonlinear Compton Scattering. The starting point will be the result Eq. (4.80) from the last chapter, for which the high field limit will be computed. This will be followed up by a discussion of effects of the pulse length on the probabilities of single photon emission. As will be discussed, having a large enough pulse length is another way of invalidating perturbation theory. The radiative corrections which are associated to these pulse length effects as well as their high field behaviour will be the topic of the rest of the chapter.

5.1 High Field Limit of the Incoherent Polarization Correction

In the following calculation the large χ asymptotic of the incoherent part of the polarization correction Eq. (4.80) is computed. First we state again the explicit definition of the components P_j of the polarization operator:

$$P_1(\varphi) = \frac{\alpha}{48\pi} m\kappa^2(\varphi) \int_0^\infty duu \int_0^1 dve^{-iu\left[1+\frac{\left(1-v^2\right)^2}{48}\kappa^2(\varphi)u^2\right]} \left(1-v^2\right)\left(3+v^2\right), \tag{5.1}$$

$$P_2(\varphi) = \frac{\alpha}{48\pi} m\kappa^2(\varphi) \int_0^\infty duu \int_0^1 dve^{-iu\left[1+\frac{\left(1-v^2\right)^2}{48}\kappa^2(\varphi)u^2\right]} \left(1-v^2\right)\left(6-2v^2\right). \tag{5.2}$$

These may also be written in terms of Airy and Scorer functions.

$$P_1(\varphi) = \frac{am\kappa^{2/3}(\varphi)}{6\cdot 2^{1/3}} \int_0^1 dv \frac{(3+v^2)}{\left(1-v^2\right)^{1/3}} (\mathrm{Gi}'(y) + i\,\mathrm{Ai}'\,(y)) \tag{5.3}$$

$$P_2(\varphi = \frac{\alpha m\kappa^{2/3}(\varphi)}{6\cdot 2^{1/3}} \int_0^1 dv \frac{(6-2v^2)}{\left(1-v^2\right)^{1/3}} (\mathrm{Gi}'(y) + i\,\mathrm{Ai}'\,(y)) \tag{5.4}$$

where

$$y = \left[\frac{4}{(1-v^2)\kappa(\varphi)}\right]^{2/3} = \left[\frac{4}{(1-v^2)\tau\chi(\varphi)}\right]^{2/3}, \tag{5.5}$$

with $\tau = k_-/p_-$ as before. Using the expressions from above we have

$$\begin{aligned}&\int_0^1 \frac{d\tau}{\tau}\left(T_1(\varphi_+)\frac{\operatorname{Im} P_1(\varphi)}{m} + T_2(\varphi_+)\frac{\operatorname{Im} P_2(\varphi)}{m}\right)\\ &= \frac{\alpha m\chi^{2/3}(\varphi)}{6\cdot 2^{1/3}}\left\{\frac{1}{\chi^{2/3}(\varphi_+)}M + \chi^{2/3}(\varphi_+)N + \chi^{2/3}(\varphi_+)L\right\}\end{aligned} \tag{5.6}$$

where

$$
\begin{aligned}
M = &\int_0^1 \frac{dv(9-v^2)}{(1-v^2)^{1/3}} \int_0^1 \frac{d\tau \cdot \tau^{1/3}}{(1-\tau)^{2/3}} \mathrm{Ai}'\left(\frac{2^{4/3}}{(1-v^2)^{2/3}\,(\tau\chi(\varphi))^{2/3}}\right) \\
&\times \int_1^\infty dw\, \mathrm{Ai}\left(\frac{w\tau^{2/3}}{(1-\tau)^{2/3}\chi^{2/3}(\varphi_+)}\right),
\end{aligned}
\tag{5.7}
$$

$$
\begin{aligned}
N = &\int_0^1 \frac{dv\,(15+v^2)}{(1-v^2)^{1/3}} \int_0^1 \frac{d\tau(1-\tau)^{2/3}}{\tau} \mathrm{Ai}'\left(\frac{2^{4/3}}{(1-v^2)^{2/3}\,(\tau\chi(\varphi))^{2/3}}\right) \\
&\times \mathrm{Ai}'\left(\frac{\tau^{2/3}}{(1-\tau)^{2/3}\chi^{2/3}(\varphi_+)}\right),
\end{aligned}
\tag{5.8}
$$

$$
\begin{aligned}
L = &\int_0^1 \frac{dv\,(9-v^2)}{(1-v^2)^{1/3}} \int_0^1 \frac{d\tau \cdot \tau}{(1-\tau)^{1/3}} \mathrm{Ai}'\left(\frac{2^{4/3}}{(1-v^2)^{2/3}\,(\tau\chi(\varphi))^{2/3}}\right) \\
&\times \mathrm{Ai}'\left(\frac{\tau^{2/3}}{(1-\tau)^{2/3}\chi^{2/3}(\varphi_+)}\right).
\end{aligned}
\tag{5.9}
$$

In the following we will not distinguish between $\chi(\varphi_+)$ and $\chi(\varphi)$ anymore. Instead we just use the notation χ for both and we assume this quantity to be large. In order to find an expansion of the probability for large χ values we need to find the leading order terms of the expressions (5.7),(5.8),(5.9). For M and L the integrand is finite everywhere, so one can just replace the Airy functions by their leading order Taylor expansions which are given by:

$$
\begin{aligned}
\mathrm{Ai}(x) &= \frac{1}{3^{2/3}\Gamma(2/3)} + \mathcal{O}(x) \\
\mathrm{Ai}'(x) &= -\frac{1}{3^{1/3}\Gamma(1/3)} + \mathcal{O}(x^2)
\end{aligned}
\tag{5.10}
$$

after which the integration can be done. For the N this straightforward procedure cannot be done, because the remaining integrals would diverge. Instead one can divide the τ integral into two regions separated at an arbitrary value τ_0 such that $1/\chi \ll \tau_0 \ll 1$. Subsequently one may apply appropriate approximations. For example we may consider the following integral:

$$
\begin{aligned}
&\int_0^{\tau_0} \frac{d\tau(1-\tau)^{2/3}}{\tau} \operatorname{Ai}'\left(\frac{2^{4/3}}{\left(1-v^2\right)^{2/3}(\tau\chi(\varphi))^{2/3}}\right) \operatorname{Ai}'\left(\frac{\tau^{2/3}}{(1-\tau)^{2/3}\chi^{2/3}(\varphi_+)}\right) \\
&+\int_{\tau_0}^{1} \frac{d\tau(1-\tau)^{2/3}}{\tau} \operatorname{Ai}'\left(\frac{2^{4/3}}{\left(1-v^2\right)^{2/3}(\tau\chi(\varphi))^{2/3}}\right) \operatorname{Ai}'\left(\frac{\tau^{2/3}}{(1-\tau)^{2/3}\chi^{2/3}(\varphi_+)}\right) \\
\simeq &-\frac{1}{3^{1/3}\Gamma(1/3)} \int_0^{\tau_0} \frac{d\tau}{\tau} \operatorname{Ai}'\left(\frac{2^{4/3}}{\left(1-v^2\right)^{2/3}(\tau\chi(\varphi))^{2/3}}\right) \\
&+\frac{1}{3^{2/3}\Gamma(1/3)^2} \int_{\tau_0}^{1} \frac{d\tau(1-\tau)^{2/3}}{\tau} \\
\simeq &\frac{1}{3^{2/3}\Gamma\left(\frac{1}{3}\right)^2}\left(-\gamma+\frac{\sqrt{3}}{12}\pi+\frac{1}{4}\ln(3)-\ln\left[\frac{4}{\left(1-v^2\right)\chi}\right]+\ln(\tau_0)\right) \\
&+\frac{1}{3^{2/3}\Gamma\left(\frac{1}{3}\right)^2}\left(-\frac{3}{2}-\frac{\sqrt{3}}{6}\pi+\frac{3}{2}\ln(3)-\ln(\tau_0)\right) \\
= &\frac{1}{3^{2/3}\Gamma\left(\frac{1}{3}\right)^2}\left(-\frac{3}{2}-\gamma-\frac{\sqrt{3}}{12}\pi+\frac{7}{4}\ln(3)-\ln\left[\frac{4}{\left(1-v^2\right)\chi}\right]\right),
\end{aligned}
\tag{5.11}
$$

where γ is the Euler–Mascheroni constant. For the integrals that appear in the calculation above and the subsequent series expansion to first order we used Wolfram Mathematica. In the end it is important that the result does not depend on the arbitrary value τ_0 which is the case in this calculation. This calculation is the first step to finding the asymptotic of the term N. The subsequent integral in v can be taken without running into problems. We then arrive at the following asymptotics for large χ:

$$
M \simeq \frac{6^{2/3}\pi\chi^{2/3}}{\Gamma\left(-\frac{5}{6}\right)\Gamma\left(\frac{13}{6}\right)}
\tag{5.12}
$$

$$
L \simeq \frac{10\pi^{3/2}\Gamma\left(\frac{2}{3}\right)}{9\cdot 3^{1/6}\Gamma\left(\frac{1}{3}\right)^2\Gamma\left(\frac{13}{6}\right)}
\tag{5.13}
$$

$$
N \simeq \frac{\left(\frac{3}{2}\right)^{1/3}\pi}{14\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{13}{6}\right)}\left[-632-84\gamma+49\sqrt{3}\pi+168\ln(2)+147\ln(3)+84\ln\left(\frac{\chi}{4}\right)\right]
\tag{5.14}
$$

By inserting these expressions into Eq. (5.6) we obtain

$$
\begin{aligned}
&\int_0^1 \frac{d\tau}{\tau}(T_1 \operatorname{Im} P_1 + T_2 \operatorname{Im} P_2) \\
\simeq &\frac{\alpha m\chi^{4/3}\Gamma\left(\frac{2}{3}\right)}{336\cdot 6^{2/3}\Gamma\left(\frac{7}{6}\right)\Gamma\left(\frac{13}{6}\right)}[147\pi+\sqrt{3}(-590-84\gamma+147\ln(3)+84\ln(\chi))].
\end{aligned}
\tag{5.15}
$$

Now that we have the final result we may reinstate the dependence of χ on φ_+ and φ. In fact, the χ inside the logarithm originates from the polarization operator and it therefore depends on φ. Also we will include the coefficients from Eq. (4.80). Therefore we obtain an asymptotic expansion of the integrand in (4.80):

$$\begin{aligned}\frac{d\delta P_{\text{incoh}}}{d\varphi_+ d\varphi} &= -\frac{\alpha}{\eta^2}\int_0^1 \frac{d\tau}{\tau}\left(T_1(\varphi_+)\frac{\operatorname{Im} P_1(\varphi)}{m} + T_2(\varphi_+)\frac{\operatorname{Im} P_2(\varphi)}{m}\right) \\ &\simeq -\frac{\alpha^2}{\eta^2}\frac{\chi(\varphi_+)^{2/3}\chi(\varphi)^{2/3}\Gamma\left(\frac{2}{3}\right)}{336\cdot 6^{2/3}\Gamma\left(\frac{7}{6}\right)\Gamma\left(\frac{13}{6}\right)}[147\pi + \sqrt{3}(-590 - 84\gamma + 147\ln(3) + 84\ln(\chi(\varphi)))].\end{aligned} \tag{5.16}$$

The expression shows that the correction due to the polarization operator in this process scales as $\chi^{4/3}\ln(\chi)$. From the above calculation it is also clear that the term with the logarithm, which is the term that scales the most with χ, comes from the τ values which are close to 0, corresponding to small photon k_-.

5.1.1 The Case of a Constant Crossed Field

We now consider the case of a constant crossed field that is switched on for only a finite time, which corresponds to a finite total Phase Φ. This case can be modeled by having $\chi = \chi_0$ be constant and by adjusting the boundaries of the phase integrals in the following way:

$$\int_{-\infty}^{\infty} d\varphi_+ \to \int_0^{\Phi} d\varphi_+, \tag{5.17}$$

and

$$\int_{\varphi_+}^{\infty} d\varphi \to \int_{\varphi_+}^{\Phi} d\varphi. \tag{5.18}$$

Since χ is constant now, the phase integral is simply:

$$\int_0^{\Phi} d\varphi_+ \int_{\varphi_+}^{\Phi} d\varphi = \frac{1}{2}\Phi^2. \tag{5.19}$$

Thus we can obtain the high χ asymptotic for the total correction δP_{incoh} by multiplying (5.16) by the factor $\Phi^2/2$:

$$\delta P_{\text{incoh}} \simeq -\frac{1}{2}\Phi^2\frac{\alpha^2}{\eta^2}\frac{\chi^{4/3}\Gamma\left(\frac{2}{3}\right)[147\pi + \sqrt{3}(-590 - 84\gamma + 147\ln(3) + 84\ln(\chi))]}{336\cdot 6^{2/3}\Gamma\left(\frac{7}{6}\right)\Gamma\left(\frac{13}{6}\right)}. \tag{5.20}$$

For the case of a constant field it is useful to quantify the pulse length Φ in terms of the total time T that an electron is exposed to the field. Assuming an electron with four-momentum $p^\mu = (\varepsilon, \boldsymbol{p})$ and $\phi \equiv x_-$ it is

$$\frac{d\phi}{dt} = \frac{p_-}{\varepsilon}, \tag{5.21}$$

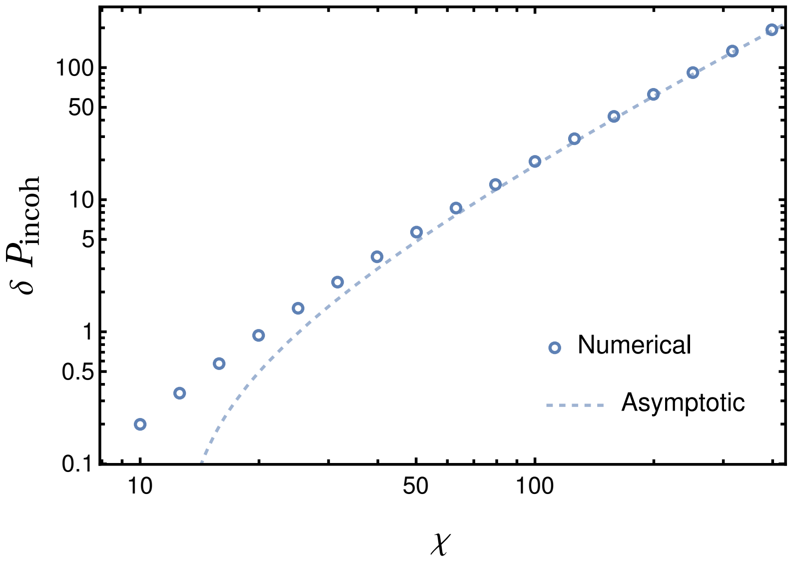


Figure 5.1: For the case of a constant crossed field, a comparison between the exact values (circles) of δP_{incoh} which were computed numerically from the expression (4.80) and the asymptotic (5.20) (dashed line) is shown. The parameters being used are $\eta = 0.1$ and $\Phi = 2\pi$.

and thus we can estimate

$$\frac{\Phi}{\omega_0} = \frac{p_-}{\varepsilon} T. \tag{5.22}$$

In the actual case of a constant crossed field, there is no frequency associated with the background field, and thus the frequency ω_0 is arbitrary. In the case of a propagating plane wave field within the LCFA, there will always be a finite value of ω_0 associated, thus the definition of the total phase makes sense in this case. As mentioned before the correction δP_{incoh} is equal to minus the probability of the incoherent part of the trident process. Indeed the probability of this process is presented in Ref. [6] and the large χ asymptotic for the incoherent process is given in Eq. (6.63) of Ref. [6], which is in agreement with our result (5.20). A comparison between the asymptotic and the exact values of δP_{incoh} is given in Fig. 5.1.

5.2 The Damping of Particle States

The formation length of nonlinear Compton scattering is of the order of $\varphi_f \sim 1/\xi$ [16, 41]. If $\Phi/\varphi_f > 1$ where Φ is the total length of the laser pulse, there is the possibility that an emitted photon decays into an electron-positron pair and incoming/outgoing electrons emit more photons while traversing the laser pulse. Thus the probability of a single photon emission becomes smaller when increasing the pulse length Φ. As it turns out, these pulse length effects reduce the probability of a single photon emission exponentially, as it was known for a long time [22] and was derived from first principles in Refs. [36, 37]. The probabilities derived in Refs. [36, 37] are valid within the LCFA and they depend on the expressions for the mass and polarization operators within the LCFA. In the following we will summarize these expression and introduce the notatian that will be used in the subsequent sections. Subsequently we are going to cite the findings from Refs. [36, 37] regarding the nonlinear Compton Scattering process, as these results will be used as a starting point when computing the high field limit of the incoherent corrections to nonlinear Compton scattering. We will denote the incoming electron momentum as p^μ, the outgoing electron momentum as p'^μ and the outgoing photon momentum as q^μ. Due to the plane wave symmetries there is momentum conservation in the minus and transverse components such that $p'_- = p_- - q_-$. In order to distinguish between nonlinearity parameters of incoming and outgoing momenta we will use the following definitions:

$$\chi_p(\varphi) = \frac{|e|E_0 p_-}{m^3}|\psi'(\varphi)|, \tag{5.23}$$

$$\chi_{p'}(\varphi) = \frac{|e|E_0 p'_-}{m^3}|\psi'(\varphi)|, \tag{5.24}$$

and

$$\kappa_q(\varphi) = \frac{|e|E_0 q_-}{m^3}|\psi'(\varphi)|, \tag{5.25}$$

where the prime denotes the derivatve $d/d\varphi$ and $\varphi = \omega_0\varphi$ corresponds to the dimensionless phase.

5.2.1 The Mass Operator in the LCFA

The leading order corrections to scattering probabilities are given by the mass operator and the polarization operator. In the following we will consider these operators within the linear constant field approximation (LCFA). The mass operator in a constant crossed field was studied in Refs. [38, 39]. Within the LCFA one can use the expressions from a constant crossed field and make the replacement $\chi \to \chi(\varphi)$. By using the same notation as in Ref. [37], the mass operator within the LCFA is given by:

$$\begin{aligned} M_s(p,\varphi) =& \frac{\alpha m}{2\pi}\int_0^\infty du \int_0^\infty \frac{dv}{(1+v)^3} e^{-iu\left[1+\frac{1}{3}\frac{\chi_p^2(\varphi)}{v^2}u^2\right]} \\ & \times \left[\frac{5+7v+5v^2}{3}\frac{\chi_p^2(\varphi)}{v^2}u - is\chi_p(\varphi)\right]. \end{aligned} \tag{5.26}$$

The expression for the mass operator can be simplified by absorbing the u-integral into Airy and Scorer functions after which one obtains:

$$M_s(p,\varphi)=\frac{\alpha m}{2}\int_0^\infty \frac{dv}{(1+v)^3}\left(\frac{\chi_p^{2/3}}{3}\frac{(5+7v+5v^2)}{v^{2/3}}\left[\mathrm{Gi}'\left(\frac{v^{2/3}}{\chi_p^{2/3}}\right)+i\,\mathrm{Ai}'\left(\frac{v^{2/3}}{\chi_p^{2/3}}\right)\right]\right.$$
$$\left.-s\chi_p^{1/3}v^{2/3}\left[\mathrm{Gi}\left(\frac{v^{2/3}}{\chi_p^{2/3}}\right)+i\,\mathrm{Ai}\left(\frac{v^{2/3}}{\chi_p^{2/3}}\right)\right]\right). \tag{5.27}$$

For the calculation of scattering probabilities only the imaginary part of the mass operator will be relevant. We therefore compute the following quantity:

$$\frac{2m}{\omega_0 p_-}\,\mathrm{Im}\left\{M_s(p,\varphi)\right\}=\frac{\alpha}{\eta_0}\int_0^\infty \frac{dv}{(1+v)^3}\left[\frac{\chi_p^{2/3}}{3}\frac{(5+7v+5v^2)}{v^{2/3}}\,\mathrm{Ai}'\left(\frac{v^{2/3}}{\chi_p^{2/3}}\right)\right.$$
$$\left.-s\chi_p^{1/3}v^{2/3}\,\mathrm{Ai}\left(\frac{v^{2/3}}{\chi_p^{2/3}}\right)\right], \tag{5.28}$$

which according to the optical theorem is equal to the differential tree level probability per unit phase $\varphi=\omega_0\phi$ of nonlinear Compton scattering of an electron with initial spin s. Further we write this expression as

$$\frac{2m}{\omega_0 p_-}\,\mathrm{Im}\left\{M_s(p,\varphi)\right\}=\frac{\alpha\chi_p^{2/3}}{\eta_0}\int_0^\infty dv\left[p_1(v)\,\mathrm{Ai}'\left(\frac{v^{2/3}}{\chi_p^{2/3}}\right)-\frac{s}{\chi_p^{1/3}}p_2(v)\,\mathrm{Ai}\left(\frac{v^{2/3}}{\chi_p^{2/3}}\right)\right], \tag{5.29}$$

where we define

$$p_1(v)=\frac{5+7v+5v^2}{3v^{2/3}(1+v)^3}, \tag{5.30}$$

$$p_2(v)=\frac{v^{2/3}}{(1+v)^3}. \tag{5.31}$$

Now by introducing the variable $\tau=q_-/p_-$ and using the momentum conservation $p'_-=p_--q_-$ we can write $p'_-=(1-\tau)p_-$ and therefore $\chi_{p'}=(1-\tau)\chi_p$. We then have:

$$\tilde{M}_{s'}(\tau):=\frac{2m}{\omega_0 p'_-}\,\mathrm{Im}\left\{M_{s'}(p',\varphi)\right\}=\frac{1}{(1-\tau)^{1/3}}\frac{\alpha\chi_p^{2/3}}{\eta_0}\int_0^\infty dv\left[p_1(v)\,\mathrm{Ai}'\left(\frac{v^{2/3}}{(1-\tau)^{2/3}\chi_p^{2/3}}\right)\right.$$
$$\left.-\frac{s'}{(1-\tau)^{1/3}\chi_p^{1/3}}p_2(v)\,\mathrm{Ai}\left(\frac{v^{2/3}}{(1-\tau)^{2/3}\chi_p^{2/3}}\right)\right]. \tag{5.32}$$

With this definition we also have

$$\tilde{M}_s(0)=\frac{2m}{\omega_0 p_-}\,\mathrm{Im}\left\{M_s(p,\varphi)\right\}. \tag{5.33}$$

5.2.2 The Polarization Operator in the LCFA

The polarization operator in a constant crossed field is studied in Refs. [39]. We note that the polarization operator can be expressed by one longitudinal and two transverse components. When dealing with on-shell photons, as it is the case in the incoherent radiative corrections studied in Refs, [36, 37] only the transverse components have to be considered. Within the LCFA one can use the expressions from the constant crossed field and replace $\kappa \to \kappa(\varphi)$. Using the notation from Ref. [37] the transverse components of the polarization operator are given by:

$$P_j(q,\varphi) = \frac{\alpha}{48\pi} m j \kappa_q^2(\varphi) \int_0^\infty du u \int_0^1 dv e^{-iu\left[1+\frac{\left(1-v^2\right)^2}{48}\kappa_q^2(\varphi)u^2\right]} \left(1-v^2\right)\left[3-(-1)^j v^2\right], \tag{5.34}$$

with $j = 1, 2$. The expression for the polarization operator can be simplified by absorbing the u-integral into the definition of the Airy and Scorer functions through Eq. (4.28), and thus one obtains:

$$P_j(q,\varphi) = \frac{\alpha m \kappa_q^{2/3}(\varphi)}{24 \cdot 2^{1/3}} \int_1^\infty \frac{du\,(8u+1+3(-1)^j)}{u^{5/3}\sqrt{u(u-1)}} \left(\mathrm{Gi}'(y) + i\,\mathrm{Ai}'(y)\right) \tag{5.35}$$

and

$$y = \frac{(4u)^{2/3}}{\kappa_q^{2/3}}. \tag{5.36}$$

With the definition $\tau := q_-/p_-$ it is $\kappa_q = \tau\chi$. We also define the following quantity:

$$\tilde{P}_j(\tau) := \frac{2m}{\omega_0 q_-} \mathrm{Im}\{P_j(q,\varphi)\} = \frac{\alpha\chi^{2/3}}{\eta}\frac{1}{\tau^{1/3}} \int_1^\infty du q_j(u)\,\mathrm{Ai}'\left(\frac{(4u)^{2/3}}{\tau^{2/3}\chi^{2/3}}\right) \tag{5.37}$$

with

$$q_j(u) = \frac{8u+1+3(-1)^j}{12 \cdot 2^{1/3} u^{5/3}\sqrt{u(u-1)}}, \tag{5.38}$$

which according to the optical theorem is equal to the differential tree level probability per unit phase $\varphi = \omega_0\phi$ of Breit-Wheeler pair production of an initial photon of polarization quantum number j.

5.2.3 Nonlinear Compton-Scattering including the damping of particle states in the LCFA

The damping effects which come from the extended pulse length of the background field are derived from first principles in Refs. [36, 37]. It turns out, the damping effects are described by all incoherent radiative corrections that are obtained by attaching arbitrarily many first order mass or polarization operators to the external electron and photon lines. First order means in this case that the mass and polarization operators are proportional to α, however higher order one particle irreducible diagrams, e.g. α^2, are not taken into account. The electron and photon states which include these damping effects can be found by solving the Dyson-Schwinger equations for the electron and photon wave functions. These wave functions may be used to compute the probabilities of nonlinear Compton scattering and Breit-Wheeler pair production by means of perturbation theory. Indeed, while the states found in Ref. [36] treat the pulse length effects in a non-perturbative way, the result is nonetheless perturbative with respect to α. Here we cite the result for the probability of nonlinear Compton scattering with an electron of inital spin $s = 1, -1$ and final spin $s' = 1, -1$ and a photon of polarization $j = 1, 2$ from Ref. [37]:

$$\begin{aligned} P_{j,s,s'} = & -\frac{\alpha m^2}{4p_-^2} \int_0^{p_-} dq_- \int d\phi_+ \tilde{T}_{j,s,s'} \\ & \times e^{2\,\mathrm{Im}\left\{ \frac{m}{p_-} \int_{-\infty}^{\phi_+} d\phi M_s(p,\phi) + \int_{\phi_+}^{\infty} d\phi \left[\frac{m}{p'_-} M_{s'}(p',\phi) + \frac{m}{q_-} P_j(q,\phi) \right] \right\}}, \end{aligned} \tag{5.39}$$

where the mass and polarization operators are defined as in the previous section. Here the phase $\phi \equiv x_- \equiv (nx)$ has units of inverse energy. The trace expressions are given by:

$$\begin{aligned} \tilde{T}_{1,s,s'} = & \left[1 + ss' \left(1 - \frac{q_-^2}{2p_-(p_- - q_-)} \right) \right] \mathrm{Ai}_1(z) \\ & + \left[3 + \frac{q_-^2}{p_-(p_- - q_-)} + ss' \left(3 + \frac{q_-^2}{2p_-(p_- - q_-)} \right) \right] \\ & \times \frac{\mathrm{Ai}'(z)}{z} + (s + s') \left(2\frac{q_-}{p_-} + \frac{q_-^2}{p_-(p_- - q_-)} \right) \\ & \times \frac{\mathrm{Ai}(z)}{\sqrt{z}} \operatorname{sgn}\left(\psi'\left(\phi_+\right)\right), \end{aligned} \tag{5.40}$$

$$\begin{aligned} \tilde{T}_{2,s,s'} = & \left[1 + ss' \left(1 + \frac{q_-^2}{2p_-(p_- - q_-)} \right) \right] \mathrm{Ai}_1(z) \\ & + \left[1 + \frac{q_-^2}{p_-(p_- - q_-)} + ss' \left(1 - \frac{q_-^2}{2p_-(p_- - q_-)} \right) \right] \\ & \times \frac{\mathrm{Ai}'(z)}{z} + (s' - s) \frac{q_-^2}{p_-(p_- - q_-)} \frac{\mathrm{Ai}(z)}{\sqrt{z}} \operatorname{sgn}\left(\psi'\left(\phi_+\right)\right), \end{aligned} \tag{5.41}$$

and

$$z = \left[\frac{q_-}{(p_- - q_-)\chi_p(\phi_+)} \right]^{2/3}. \tag{5.42}$$

5.2.4 Formulation in Dimensionless Quantities

We introduce the quantity $\tau = q_-/p_-$ and thus $p'_- = (1-\tau)p_-$. Also we introduce the dimensionless phases $\varphi_+ = \omega_0\phi_+$ and $\varphi = \omega_0\phi$. Expressions that were given in terms of ϕ before will now be written in terms of φ by straightforward substitution. The scattering probability for nonlinear Compton scattering including the damping of particle states is then give by:

$$\begin{aligned} P_{j,s,s'} = &- \frac{\alpha}{4\eta_0}\int_0^1 d\tau \int d\varphi_+ \tilde{T}_{j,s,s'} \\ &\times e^{2\,\mathrm{Im}\left\{\frac{m}{\omega_0 p_-}\int_{-\infty}^{\varphi_+} d\varphi M_s(p,\varphi) + \int_{\varphi_+}^{\infty} d\varphi \left[\frac{m}{\omega_0 p'_-} M_{s'}(p',\varphi) + \frac{m}{\omega_0 q_-} P_j(q,\varphi)\right]\right\}} . \end{aligned} \tag{5.43}$$

The exponent may be written in terms of dimensionless quantities by using the definitions (5.32) and (5.37). The trace terms can be written as:

$$\begin{aligned} \tilde{T}_{1,s,s'} = &\left[1 + ss'\left(1 - \frac{\tau^2}{2(1-\tau)}\right)\right] \mathrm{Ai}_1(z) \\ &+ \left[3 + \frac{\tau^2}{1-\tau} + ss'\left(3 + \frac{\tau^2}{2(1-\tau)}\right)\right] \frac{\mathrm{Ai}'(z)}{z} \\ &+ (s+s')\left(2\tau + \frac{\tau^2}{1-\tau}\right) \frac{\mathrm{Ai}(z)}{\sqrt{z}} \,\mathrm{sgn}\left(\psi'\left(\varphi_+\right)\right), \end{aligned} \tag{5.44}$$

$$\begin{aligned} \tilde{T}_{2,s,s'} = &\left[1 + ss'\left(1 + \frac{\tau^2}{2(1-\tau)}\right)\right] \mathrm{Ai}_1(z) \\ &+ \left[1 + \frac{\tau^2}{1-\tau} + ss'\left(1 - \frac{\tau^2}{2(1-\tau)}\right)\right] \frac{\mathrm{Ai}'(z)}{z} \\ &+ (s'-s)\frac{\tau^2}{1-\tau} \frac{\mathrm{Ai}(z)}{\sqrt{z}} \,\mathrm{sgn}\left(\psi'\left(\varphi_+\right)\right), \end{aligned} \tag{5.45}$$

and where

$$z = \left[\frac{\tau}{(1-\tau)\chi_p\left(\varphi_+\right)}\right]^{2/3} . \tag{5.46}$$

5.3 High Field Limit for Arbitrarily Many First Order Loop Corrections

The expression (5.43) includes all the incoherent radiative corrections that can be obtained by attaching arbitrarily many first order mass or polarization operators to the external electron and photon lines. In this sense the expression already provides a resummation of all the diagrams of this category. If one is interested in the specific contributions of each of these diagrams, one can use the series expansion for each of the three exponentials and consider each resulting term separately. Indeed, by expanding the exponentials to first order and by only considering the term containing one polarization operator and no mass operators, and if one sums over initial and final quantum numbers, one obtains the correction given by Eq. (4.80), which is the incoherent part of the polarization correction. The high field limit corresponding to $\chi \gg 1$ for the case of the one polarization operator correction has already been computed in Sec. 5.20. We are now interested in finding the asymptotic of every other term in the exponential series. It is important to note that, because the expression (5.43) is a probability, it is related to the square of S-Matrix element. Thus, each term of the exponential series contains not only the square of single diagrams (non-exchange terms) but also the exchange terms which are products of two different diagrams. Thus, one cannot map each term of the exponential series to a single process because each term will include multiple different exchange and non-exchange terms. Still, in the context of perturbation theory, it is the sum of all of these exchange and non-exchange terms at a given order of Φ and α that can be measured, and thus it is sensible to study the sum of these expressions. When writing the exponentials as a series expansion one obtains the following expression for the probability:

$$\begin{aligned} P_{j,s,s'} = &- \frac{\alpha}{4\eta_0} \int_0^1 d\tau \int d\varphi_+ \tilde{T}_{j,s,s'} \\ &\times \sum_{l=0}^{\infty} \frac{1}{l!} \left(\int_{-\infty}^{\varphi_+} d\varphi \frac{2m}{\omega_0 p_-} \operatorname{Im}\left\{ M_s(p,\varphi) \right\} \right)^l \\ &\times \sum_{n=0}^{\infty} \frac{1}{n!} \left(\int_{\varphi_+}^{\infty} d\varphi \frac{2m}{\omega_0 p'_-} \operatorname{Im}\left\{ M_s(p',\varphi) \right\} \right)^n \\ &\times \sum_{k=0}^{\infty} \frac{1}{k!} \left(\int_{\varphi_+}^{\infty} d\varphi \frac{2m}{\omega_0 q_-} \operatorname{Im}\{ P_j(q,\varphi) \} \right)^k \end{aligned} \tag{5.47}$$

We now define each of the terms in the series as follows:

$$\begin{aligned} P^{(l,n,k)}_{j,s,s'} = &- \frac{\alpha}{4\eta_0} \int_0^1 d\tau \int d\varphi_+ \tilde{T}_{j,s,s'} \\ &\times \frac{1}{l!} \left(\int_{-\infty}^{\varphi_+} d\varphi \frac{2m}{\omega_0 p_-} \operatorname{Im}\left\{ M_s(p,\varphi) \right\} \right)^l \\ &\times \frac{1}{n!} \left(\int_{\varphi_+}^{\infty} d\varphi \frac{2m}{\omega_0 p'_-} \operatorname{Im}\left\{ M_s(p',\varphi) \right\} \right)^n \\ &\times \frac{1}{k!} \left(\int_{\varphi_+}^{\infty} d\varphi \frac{2m}{\omega_0 q_-} \operatorname{Im}\{ P_j(q,\varphi) \} \right)^k \end{aligned} \tag{5.48}$$

In the following we consider the case of a constant crossed field, which gets switched on at $\varphi_+ = 0$ and gets switched off at $\varphi_+ = \Phi$. In this case the parameter χ can be taken as constant and the φ integrals simplify:

$$\begin{aligned}P^{(l,n,k)}_{j,s,s'} =& -\frac{\alpha}{4\eta_0}\int_0^1 d\tau\int_0^\Phi d\varphi_+\tilde{T}_{j,s,s'}\\ &\times\frac{1}{l!}\Big(\varphi_+\tilde{M}_s(0)\Big)^l\\ &\times\frac{1}{n!}\Big((\Phi-\varphi_+)\tilde{M}_{s'}(\tau)\Big)^n\\ &\times\frac{1}{k!}\Big((\Phi-\varphi_+)\tilde{P}_j(\tau)\Big)^k.\end{aligned}\tag{5.49}$$

5.3.1 The Case of k Polarization Operators and $ss'=1$

Wie first consider the case $l=n=0$ and $k\geq 2$. The latter requirement will become apparent in the subsequent derivation. The expression is:

$$P^{(0,0,k)}_{j,s,s'} = -\frac{\alpha}{4\eta}\int_0^1 d\tau\int_0^\Phi d\varphi_+\tilde{T}_{j,s,s'}\frac{1}{k!}\Big((\Phi-\varphi_+)\tilde{P}_j(\tau)\Big)^k\tag{5.50}$$

We will ignore the φ_+ integral for now and conser the differential probability:

$$\frac{dP^{(0,0,k)}_{j,s,s'}}{d\varphi_+} = -\frac{\alpha}{4\eta}\frac{1}{k!}(\Phi-\varphi_+)^k\int_0^1 d\tau\tilde{T}_{j,s,s'}(\tau)\tilde{P}_j(\tau)^k.\tag{5.51}$$

As in the case of $k=1$ (Sec. 5.20) the main contribution of this integral for asymptotically large χ comes from the region of $\tau\sim 1/\chi\ll 1$ which asymptotically approaches 0. This is because at $\tau=1/\chi$ the Airy function in the polarization operator does not cause an exponential suppression and when $ss'=1$ the integrand gets amplified by the poles of the trace expressions $\tilde{T}_{j,s,s'}(\tau)$ at $\tau=0$. While the trace expressions also have poles at $\tau=1$, they are of lesser degree than those at $\tau=0$, and contributions from τ close to 1 get suppressed by the Airy function in the trace at large χ. We therefore approximate the trace expression by evaluating it around $\tau=0$ and only keeping those terms which are of highest order in χ. Thus we approximate the trace by the following expression:

$$\tilde{T}_{j,s,s'}\simeq -\frac{2(5-2j)}{3^{1/3}\Gamma(1/3)}\frac{\chi^{2/3}}{\tau^{2/3}},\tag{5.52}$$

where we have expanded the Airy function as $\mathrm{Ai}'(z)\simeq\mathrm{Ai}'(0)=-1/(3^{1/3}\Gamma(1/3))$. After performing the substitution $\rho=\chi\tau$ we obtain:

$$\begin{aligned}\frac{dP^{(0,0,k)}_{j,s,s'}}{d\varphi_+}\simeq&(\Phi-\varphi_+)^k\frac{\alpha}{\eta}\chi^{1/3}\left(\frac{\alpha\chi}{\eta}\right)^k\frac{1}{4k!}\frac{2(5-2j)}{3^{1/3}\Gamma(1/3)}\\ &\times\int_0^\chi d\rho\rho^{-\frac{2+k}{3}}\left[\int_1^\infty q_j(u)\,\mathrm{Ai}'\left(\frac{(4u)^{2/3}}{\rho^{2/3}}\right)\right]^k\end{aligned}\tag{5.53}$$

One can now obtain the asymptotic by simply setting $\chi \to \infty$ in the upper boundary of the ρ integral. This step is of course only possible if the resulting integral converges. This is not the case for $k = 1$, but for $k \geq 2$ the integral does converge. Thus we obtain:

$$\begin{aligned}\frac{dP^{(0,0,k)}_{j,s,s'}}{d\varphi_+} \simeq &(\Phi - \varphi_+)^k \frac{\alpha}{\eta}\chi^{1/3}\left(\frac{\alpha\chi}{\eta}\right)^k \frac{1}{4k!}\frac{2(5-2j)}{3^{1/3}\Gamma(1/3)} \\ &\times \int_0^\infty d\rho\rho^{-\frac{2+k}{3}}\left[\int_1^\infty q_j(u)\,\mathrm{Ai}'\left(\frac{(4u)^{2/3}}{\rho^{2/3}}\right)\right]^k.\end{aligned} \tag{5.54}$$

Taking the integral in φ_+ we obtain:

$$\begin{aligned}P^{(0,0,k)}_{j,s,s'} \simeq &\frac{1}{\chi^{2/3}}\left(\frac{\Phi\alpha\chi}{\eta}\right)^{k+1}\frac{1}{4(k+1)!}\frac{2(5-2j)}{3^{1/3}\Gamma(1/3)} \\ &\times \int_0^\infty d\rho\rho^{-\frac{2+k}{3}}\left[\int_1^\infty q_j(u)\,\mathrm{Ai}'\left(\frac{(4u)^{2/3}}{\rho^{2/3}}\right)\right]^k\end{aligned} \tag{5.55}$$

This expression is indeed an asymptotic for large χ, because the integrals in ρ and u have no χ-dependence anymore and are thus numerical constants. All the χ dependence is now in the prefactor. As mentioned before, the integral is only convergent for $k \geq 2$. For $k = 1$ there would be a logarithmic divergence, which was treated in Sec. 5.1. An important difference in Sec. 5.1 is that the result is the sum of the contributions of all spin and polarization quantum numbers, while here we resolve each spin and polarization configuration individually.

We now turn to the case of $ss' = -1$. This case is very different because now the leading terms of the trace are given by:

$$\tilde{T}_{j,s,s'} \simeq \frac{2+(-1)^j}{2}\frac{\tau^{4/3}}{(1-\tau)^{1/3}}\chi^{2/3}\,\mathrm{Ai}'(z). \tag{5.56}$$

In fact, the trace terms do not have a pole at $\tau = 0$ anymore. As a result the integrand of the τ integral does not dominate at $\tau = 0$ anymore and the entire domain of integration becomes relevant. For $k \leq 6$ we can approximate the Airy function in the polarization operator by setting the argument 0 (since $\chi \to \infty$). In this case we obtain for $ss' = -1$ and $k \leq 6$:

$$\begin{aligned}P^{(0,0,k)}_{j,s,s'} \simeq &\frac{\Phi^{k+1}}{4(k+1)!}\frac{2+(-1)^j}{2}\frac{1}{3^{1/3}\Gamma\left(\frac{1}{3}\right)}\left(\frac{\alpha\chi^{2/3}}{\eta}\right)^{k+1} \\ &\times\left[\mathrm{Ai}'(0)\int_1^\infty q_j(u)\right]^k\int_0^1 d\tau\frac{\tau^{\frac{4-k}{3}}}{(1-\tau)^{1/3}}.\end{aligned} \tag{5.57}$$

The integrals appearing in the expression above can be computed analytically and they give:

$$\int_0^1 d\tau\frac{\tau^{\frac{4-k}{3}}}{(1-\tau)^{1/3}} = \frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{7}{3}-\frac{k}{3}\right)}{\Gamma\left(3-\frac{k}{3}\right)} \qquad \text{for } k < 7, \tag{5.58}$$

$$\mathrm{Ai}'(0)\int_1^\infty q_j(u)du = -\frac{(5+(-1)^j)\sqrt{\pi}\Gamma(2/3)}{6\cdot 6^{1/3}\Gamma(1/3)\Gamma(13/6)}. \tag{5.59}$$

Thus in the case $k \leq 6$ we were able to derive the asymptotic by taking the point wise limit $\chi \to \infty$ of the integrand. Point wise means that the limit can be taken for each value of τ in the same way, effectively treating τ as a constant. This approach was not possible in the previous case $ss' = 1$ because, if done so, the resulting integrals would diverge. For the case of $k \geq 7$ the τ integral in (5.57) would not converge and thus one has to treat these cases separately. Specifically for the case $k \geq 8$ one can proceed similar as to the case $ss' = 1$. This is because in this case the polarization operators provide enough powers of τ in the denominator such that the asymptotic is only determind by the integral in the region around $\tau = 0$. We can use the expansion of the trace (5.56), perform the substitution $\rho = \chi\tau$, and again put the integration limit of the ρ-integration to infinity. Thus for $ss' = -1$ and $k \geq 8$ we obtain:

$$\begin{aligned} P_{j,s,s'}^{(0,0,k)} \simeq &-\frac{\Phi^{k+1}}{4(k+1)!}\frac{2+(-1)^j}{2}\,\mathrm{Ai}'(0)\left(\frac{\alpha\chi^{2/3}}{\eta}\right)^{k+1}\chi^{\frac{k-7}{3}} \\ &\times\int_0^\infty d\rho\rho^{\frac{4-k}{3}}\left[\int_1^\infty q_j(u)\,\mathrm{Ai}'\left(\frac{(4u)^{2/3}}{\rho^{2/3}}\right)\right]^k. \end{aligned} \tag{5.60}$$

For the case $k = 7$ there would be again a logarithmic divergence, which needs to be treated separately. As this case does not seem to be of special interest we will omit this procedure.

5.3.2 The Case of n Mass Operators

For the case of n mass operators in the outgoing fermion line the differential probability is given by

$$\frac{dP_{j,s,s'}^{(0,n,0)}}{d\varphi_+} = -\frac{\alpha}{4\eta_0}\frac{1}{n!}(\Phi-\varphi_+)^n\int_0^1 d\tau\tilde{T}_{j,s,s'}\tilde{M}_{s'}(\tau)^n. \tag{5.61}$$

If $n \geq 3$, the dominant contribution comes from the integral around $\tau = 1 - 1/\chi$. This is because the mass operator has additional powers of $1/(1-\tau)$ in its definition and at $\tau = 1-1/\chi$ the Airy functions in the mass operator don't suppress the result exponentially. Thus for the purpose of finding the asymptotic for large χ we may expand the trace expression around $\tau = 1$ and thus we obtain:

$$\begin{aligned} \tilde{T}_{j,s,s'} \simeq &\frac{\gamma_{jss'}^{(1)}}{(1-\tau)^{1/3}}\chi^{2/3}\,\mathrm{Ai}'\left(\frac{1}{(1-\tau)^{2/3}\chi^{2/3}}\right) + \frac{\gamma_{jss'}^{(2)}}{(1-\tau)^{2/3}}\chi^{1/3}\,\mathrm{Ai}\left(\frac{1}{(1-\tau)^{2/3}\chi^{2/3}}\right) \\ &+(-1)^j\frac{ss'}{2(1-\tau)}\,\mathrm{Ai}_1\left(\frac{1}{(1-\tau)^{2/3}\chi^{2/3}}\right), \end{aligned}$$

$$(5.62)$$

with the constants

$$\gamma^{(1)}_{jss'} = \frac{2-(-1)^j ss'}{2}, \qquad \gamma^{(2)}_{jss'} = s' - (-1)^j s. \tag{5.63}$$

Even though the first term has the highest scaling in χ it turns out that all three terms are of equal importance for the limit. This is because when taking the integral each power $1/(1-\tau)$ will contribute another power of χ. Indeed, by substituting $\rho = \chi(1-\tau)$ we obtain:

$$\begin{aligned}\frac{dP^{(0,n,0)}_{j,s,s'}}{d\varphi_+} \simeq &-\frac{1}{4n!}(\Phi-\varphi_+)^n \frac{\alpha}{\eta}\left(\frac{\alpha\chi}{\eta}\right)^n \\ &\times \int_0^\chi d\rho \left[\frac{\gamma^{(1)}_{jss'}}{\rho^{1/3}}\,\mathrm{Ai}'\left(\frac{1}{\rho^{2/3}}\right) + \frac{\gamma^{(2)}_{jss'}}{\rho^{2/3}}\,\mathrm{Ai}\left(\frac{1}{\rho^{2/3}}\right) + (-1)^j\frac{ss'}{2\rho}\,\mathrm{Ai}_1\left(\frac{1}{\rho^{2/3}}\right)\right] \\ &\times \left(\int_0^\infty dv \left[\frac{1}{\rho^{1/3}}p_1(v)\,\mathrm{Ai}'\left(\frac{v^{2/3}}{\rho^{2/3}}\right) - \frac{s'}{\rho^{2/3}}p_2(v)\,\mathrm{Ai}\left(\frac{v^{2/3}}{\rho^{2/3}}\right)\right]\right)^n .\end{aligned} \tag{5.64}$$

And thus we obtain the asymptotic by setting $\chi \to \infty$ as the upper bound of the ρ-integral. By doing this and also taking the phase integral we obtain:

$$\begin{aligned}P^{(0,n,0)}_{j,s,s'} \simeq &-\frac{\Phi^{n+1}}{4(n+1)!}\frac{\alpha}{\eta}\left(\frac{\alpha\chi}{\eta}\right)^n \\ &\times \int_0^\infty d\rho \left[\frac{\gamma^{(1)}_{jss'}}{\rho^{1/3}}\,\mathrm{Ai}'\left(\frac{1}{\rho^{2/3}}\right) + \frac{\gamma^{(2)}_{jss'}}{\rho^{2/3}}\,\mathrm{Ai}\left(\frac{1}{\rho^{2/3}}\right) + (-1)^j\frac{ss'}{2\rho}\,\mathrm{Ai}_1\left(\frac{1}{\rho^{2/3}}\right)\right] \\ &\times \left(\int_0^\infty dv \left[\frac{1}{\rho^{1/3}}p_1(v)\,\mathrm{Ai}'\left(\frac{v^{2/3}}{\rho^{2/3}}\right) - \frac{s'}{\rho^{2/3}}p_2(v)\,\mathrm{Ai}\left(\frac{v^{2/3}}{\rho^{2/3}}\right)\right]\right)^n .\end{aligned} \tag{5.65}$$

Again, this result is valid for $n \geq 3$. For $n = 2$ there would be a logarithmic divergence in the ρ integral which has to be treated in a similar way as the case $k = 1$ in Sec. 5.20 from which one can obtain the scaling $P^{(0,2,0)} \sim \Phi^3\alpha^3\chi^2 \ln\chi$. For the case $n = 1$ one can simply take the point wise limit of the integrand resulting a scaling $P^{(0,1,0)} \sim \Phi^2\alpha^2\chi^{4/3}$.

5.3.3 Plots

In order to confirm the results for the asymptotics derived in this section, they were compared to the exact numerical evaluations of the probabilities $P^{(0,n,k)}_{j,s,s'}$ from Eq. (5.49) for different values of χ. A direct comparison of the numerical values of the probabilities and the corresponding asymptotic expressions for different combinations of n, k, j, s and s' is shown in Fig. 5.2. Note that each plot shows the absolute value of the probabilities and their asymptotic expressions in order make use of the logarithmic scale. While these plots make it evident that the asymptotics work well, due to the logarithmic scale it is difficult to quantify how good the convergence is quantitatively at a given scale of χ. Thus for a better quantification we define the following ratio:

$$R_{j,s,s'}^{(l,n,k)} = \frac{\left(P_{j,s,s'}^{(l,n,k)}\right)_{\text{asymptotic}}}{\left(P_{j,s,s'}^{(l,n,k)}\right)_{\text{exact}}}, \tag{5.66}$$

where in the numerator are the asymptotic expressions of the probability and in the denominator are the exact values that are obtained numerically. Thus one can see how well the asymptotics converge, by looking at how fast this ratio approaches unity. Plots of the ratios for different combinations of n, k, j, s and s' are shown in Fig. 5.3.

5.3.4 Discussion of the Perturbative Results

Because the exponent with the incoming mass operator does not depend on τ, it is clear that each power l contributes with $\alpha\chi^{2/3}/\eta$ according to the point wise limit. As can be seen from the previous calculations, it is in fact the τ-integration, i.e. the integral over the final photon momenta, that provides additional powers of χ. These additional powers of χ are an interesting result because both the polarization and mass operators are often cited to have a scaling of $\alpha\chi^{2/3}$ (see e.g. Ref. [20]). Here we have shown that when considering total probabilities, after performing the outgoing photon momentum integral, the scaling in χ can be even larger if multiple mass and polarization operators are involved. For both cases, where there is either a chain of many polarization operators in the outgoing photon line, or where there is a chain of mass operators in the outgoing electron line, we have shown that the effective scaling, that each of these operators contributes, is $\Phi\alpha\chi/\eta$. And thus we have shown that the effective expansion parameter for incoherent processes of strong field QED is $\Phi\alpha\chi/\eta = \Phi\alpha\xi$. This shows that large field strengths enhance the importance of incoherent radiative corrections (i.e. damping effects) even more than one would expect from the individual asymptotics of the mass and polarization operators (the naive expectation would be that the expansion parameter is $\Phi\alpha\chi^{2/3}/\eta$). From the condition $\Phi\alpha\xi = 1$ one can estimate that damping effects to all orders become important when $\Phi\xi = 1/\alpha \approx 137$. In light of the Ritus-Narozhny conjecture it is important to ask whether these results are relevant for the higher order radiative corrections of the mass and polarization operators. According to the optical theorem, any corrections to the mass and polarization operators are related to corrections of scattering probabilites by the so called cutting rules [33, 26]. Indeed, the set of bubble-type polarization corrections which are studied in Ref. [29] are exactly

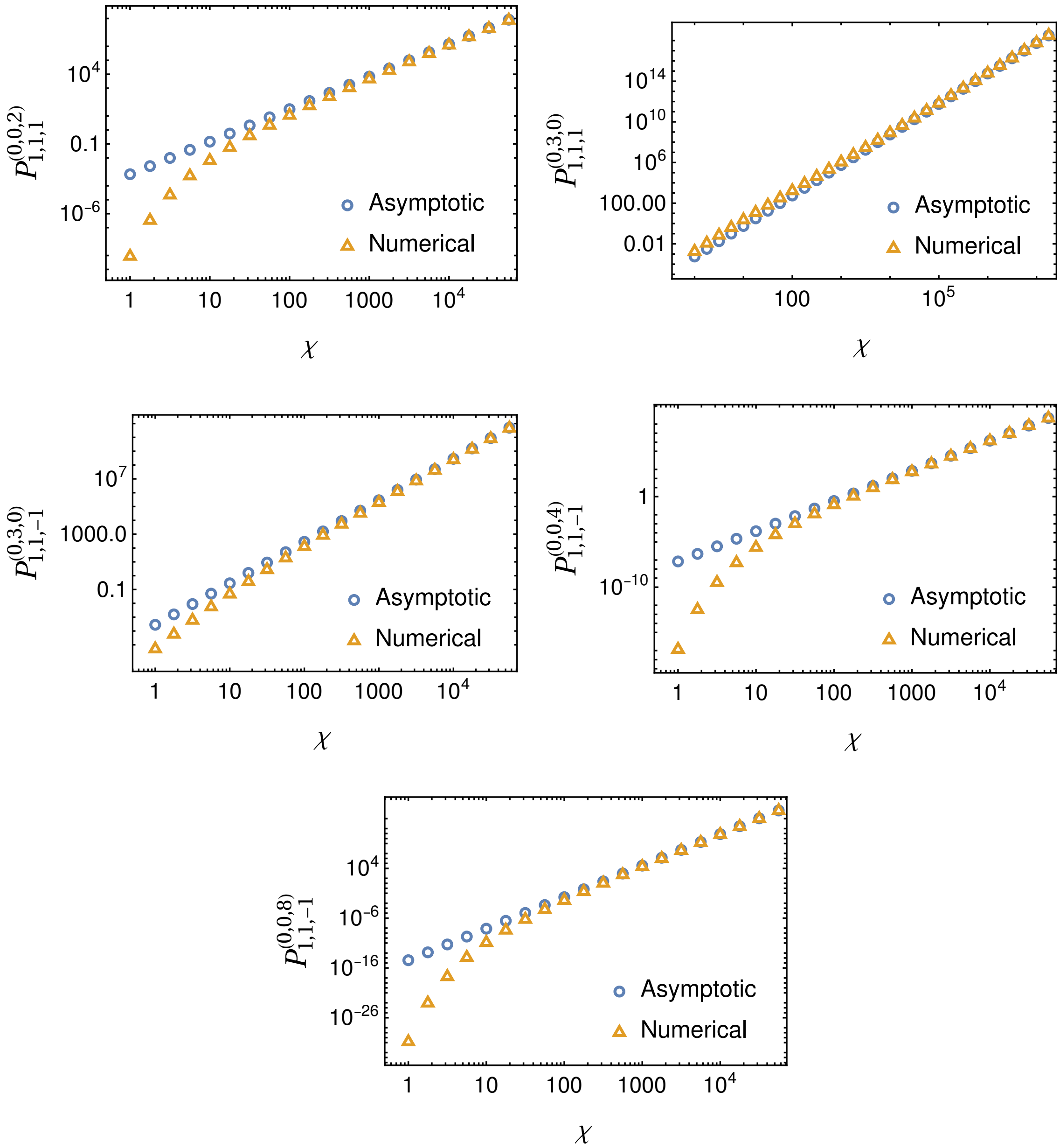


Figure 5.2: This figure contains several plots of the absolute values of the probabilities $P^{(0,n,k)}_{j,s,s'}$ as well as the corresponding asymptotics as a function of χ. The plots were made for $j = 1$ and different values of n, k, s and s' in the case of having a constant crossed field of total phase Φ. The numerical parameters that were used are: $\eta = 0.1$ and $\Phi = 2\pi$.

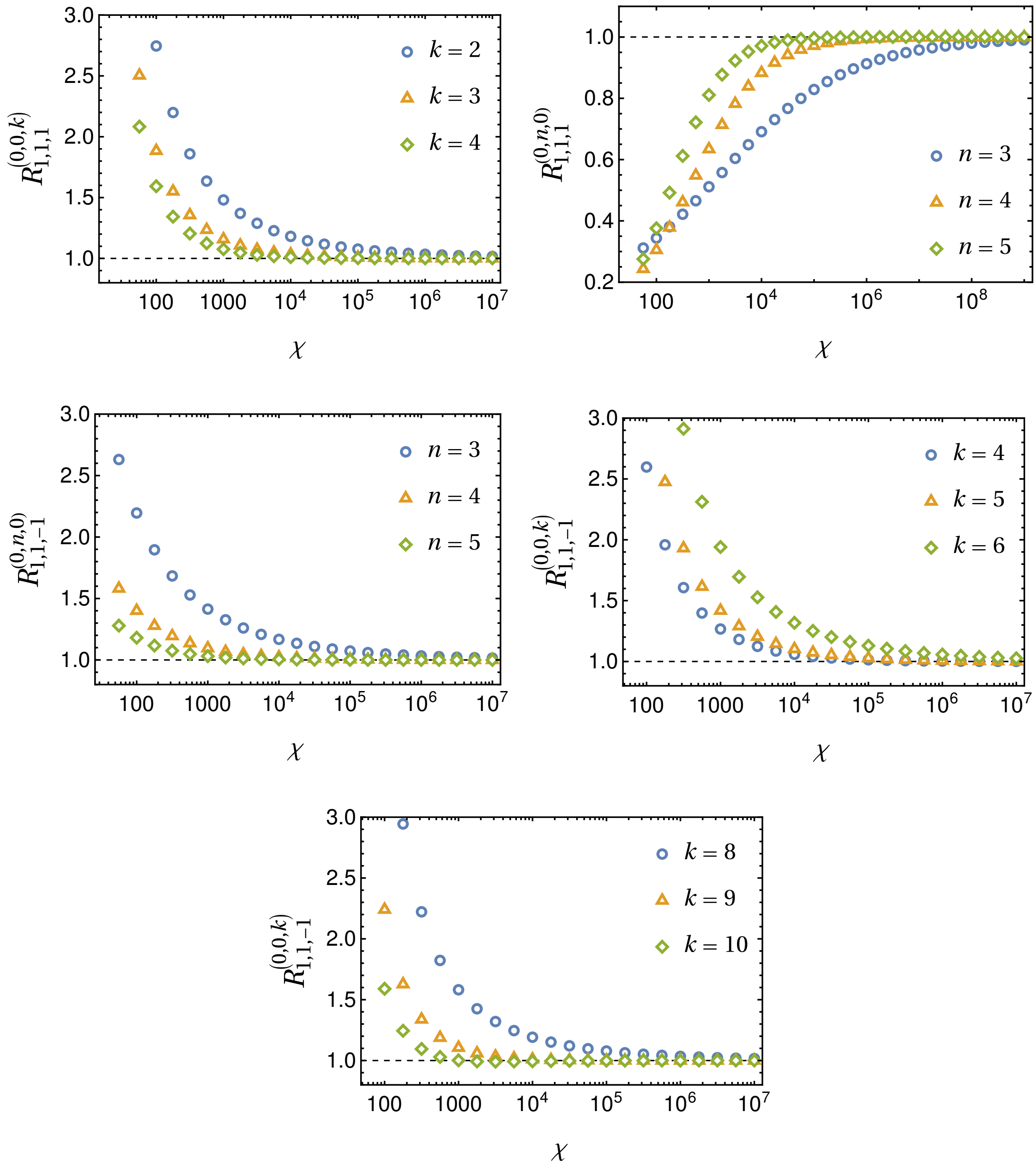


Figure 5.3: This figure contains several plots of the ratios $R_{j,s,s'}^{(0,n,k)}$ which indicate how well the asymptotic expressions converge (when the value is 1). The ratios were plotted as a function of χ for $j=1$ and different values of n, k, s and s' in the case of having a constant crossed field of total phase Φ. The numerical parameters that were used are: $\eta = 0.1$ and $\Phi = 2\pi$.

those diagrams, that should give the probabilities $P^{(0,0,k)}_{j,s,s'}$ from Eq. (5.50) when taking the imaginary part, i.e. cutting them. In Ref. [29] it was found however that these bubble-type diagrams have the scaling that is expected from the RN conjecture and not the larger scaling that we found in this chapter. The reason for this difference is presumably that when applying the cutting rules, one not only obtains corrections to single photon emission, but one also obtains corrections to the trident process. The incoherent parts from the trident process will then exactly cancel the incoherent parts from the photon emission. This is what has been explicitly shown when applying the optical theorem to the single polarization correction of the mass operator in Ref. [40] which was also discussed in Sec. 4.5. More generally it is physically sensible to assume that radiative corrections to closed loops only contain coherent contributions, since closed loops are expected to be localized in space and time. In contrast, incoherent processes are typically those in which intermediate particles propagate freely for a while, and are thus extended over space and time. The mathematical details of how this cancellation of the incoherent parts in the mass operator occurs, could be an interesting topic of future work.

5.4 The Non-Perturbative Case

We consider the case of a constant field. The probability (5.43) can be written as:

$$\frac{dP_{j,s,s'}}{d\varphi_+} = -\frac{\alpha}{4\eta}\int_0^1 d\tau \tilde{T}_{j,s,s'} e^{\varphi_+ \tilde{M}_s(0)+(\Phi-\varphi_+)\left(\tilde{M}_{s'}(\tau)+\tilde{P}_j(\tau)\right)}. \tag{5.67}$$

The ϕ_+ integral can be taken analytically and we obtain:

$$\begin{aligned} P_{jss'} &= -\frac{\alpha}{4\eta}\int_0^1 d\tau \tilde{T}_{j,s,s'} \frac{e^{\Phi\tilde{M}_s(0)} - e^{\Phi(\tilde{M}_{s'}(\tau)+\tilde{P}_j(\tau))}}{\tilde{M}_s(0)-\tilde{M}_{s'}(\tau)-\tilde{P}_j(\tau)} \\ &= -\frac{\alpha}{4\eta} e^{\Phi\tilde{M}_s(0)} \int_0^1 d\tau \tilde{T}_{j,s,s'} \frac{1-e^{-\Phi(\tilde{M}_s(0)-\tilde{M}_{s'}(\tau)-\tilde{P}_j(\tau))}}{\tilde{M}_s(0)-\tilde{M}_{s'}(\tau)-\tilde{P}_j(\tau)}. \end{aligned} \tag{5.68}$$

We note that the asymptotic of $e^{\Phi\tilde{M}_s(0)}$ is obtained by inserting the asymptotic of $\tilde{M}_s(0)$ which is given in Eq. (5.89). This asymptotic is straightforward to obtain because it is not affected by the τ-integration. As in the perturbative case, it is in fact the integral in τ that makes the asymptotic non-trivial.

5.4.1 The Case $ss' = 1$

For the treatment of the non-perturbative case we are going to make use of several asymptotic expressions of the mass and polarization operators. First we are going to consider the asymptotic of the probability of pair production for the case that $\tau\chi \ll 1$:

$$\tilde{P}_j(\tau) \simeq -\frac{\alpha\chi}{\eta}\sqrt{\frac{3}{8}}\frac{j}{4}e^{-\frac{8}{3\tau\chi}} \qquad \text{for } \tau\chi \ll 1. \tag{5.69}$$

This asymptotic can be obtained by making use of the known asymptotic expression of the Airy function for large arguments $x \gg 1$ [32]:

$$\mathrm{Ai}'(x) = e^{-\frac{2}{3}x^{3/2}} \left(-\frac{x^{1/4}}{2\sqrt{\pi}} + \mathcal{O}\left(\frac{1}{x^{5/4}}\right)\right), \tag{5.70}$$

and taking the u integral in Eq. (5.37). Furthermore we are going to need the large χ asymptotic of the pair production probability given in Eq. (5.32), within the regime $\tau \ll 1$. For this, since $1-\tau \simeq 1$ we can take the limit of $\chi \to \infty$ inside the Airy functions and obtain:

$$\tilde{M}_{s'}(\tau) \simeq \frac{1}{(1-\tau)^{1/3}} \frac{\alpha\chi^{2/3}}{\eta} \int_0^\infty dv \left[p_1(v)\, \mathrm{Ai}'(0) - \frac{s'}{(1-\tau)^{1/3}\chi^{1/3}} p_2(v)\, \mathrm{Ai}(0) \right] \tag{5.71}$$

In the probability expression (5.68) it appears the difference $\tilde{M}_s(0) - \tilde{M}_s(\tau)$. For this difference we obtain for $\tau \ll 1$:

$$\tilde{M}_s(0) - \tilde{M}_s(\tau) \simeq a \frac{\alpha\chi^{2/3}}{\eta} \left(1 - \frac{1}{(1-\tau)^{1/3}}\right) \simeq a \frac{\alpha\chi^{2/3}}{\eta} \left(-\frac{1}{3}\tau\right) \tag{5.72}$$

where a is a constant that is given in Eq. (5.92), that will however not be relevant in the following calculation. We have neglected the spin terms in (5.71) since they are sub-leading in χ. For the trace we will use the following expansion which is valid for $ss' = 1$ and $\tau \ll 1$:

$$\tilde{T}_{j,s,s'} \simeq -\frac{2(5-2j)}{3^{1/3}\Gamma(1/3)} \frac{\chi^{2/3}}{\tau^{2/3}}. \tag{5.73}$$

In order to see that most contributions come from the small τ region, it is important to consider the following function:

$$f(\tau) := \frac{1 - e^{-\Phi(\tilde{M}_s(0) - \tilde{M}_{s'}(\tau) - \tilde{P}_j(\tau))}}{\tilde{M}_s(0) - \tilde{M}_{s'}(\tau) - \tilde{P}_j(\tau)}. \tag{5.74}$$

Because the polarization operator $\tilde{P}_j$ also behaves exponentially (Eq. 5.69), this function has a double exponential behaviour in τ. It is $f(0) = \Phi$ and f has a sharp drop at a certain value τ^*. In order to find τ^* we impose the following condition:

$$f(\tau^*) = \frac{\Phi}{2\rho}, \tag{5.75}$$

where $\rho > 1/2$ is a constant we introduce, to keep the calculation as general as possible. We will later show that the value of ρ does not change the resulting asymptotic. To solve this condition for τ^* we make two assumptions:

1. $\tau^* \ll 1/\chi$,

2. $|\tilde{M}_s(0) - \tilde{M}_{s'}(\tau^*)| \ll |\tilde{P}_j(\tau^*)|$.

Assumption 1 allows us to use the expansion (5.69). Assumption 2 lets us neglect the mass operators when calculating τ^*. After finding τ^* we have to come back to see if these assumptions are met and verify self-consistency. By using assumption 2 and neglecting the mass operators the condition (5.75) becomes:

$$\frac{1 - e^{-\Phi|\tilde{P}_j(\tau^*)|}}{\Phi|\tilde{P}_j(\tau^*)|} = \frac{1}{2\rho}, \tag{5.76}$$

Because the above equation depends only on the product $\Phi|\tilde{P}_j(\tau^*)|$ it is solved for a specific value

$$\Phi|\tilde{P}_j(\tau^*)| = y(\rho) \tag{5.77}$$

where y depends only on the constant ρ. Since ρ is a constant, also y is a constant. By assumption 1 we can use the expansion (5.69) and the condition is:

$$c_j(\chi)e^{-\frac{8}{3\tau^*\chi}} = \frac{y}{\Phi}, \tag{5.78}$$

with

$$c_j(\chi) := \frac{\alpha\chi}{\eta}\sqrt{\frac{3}{8}\frac{j}{4}}. \tag{5.79}$$

Solving for τ^* gives:

$$\tau^* = \frac{8}{3\chi\left[\ln\left(\Phi c_j\right) - \ln\left(y\right)\right]}. \tag{5.80}$$

It can be seen that τ^* fulfills Assumption 1, as c_j is also linear in χ. We can also verify Assumption 2 by making use of the expansion (5.72):

$$|\tilde{M}_s(0) - \tilde{M}_{s'}(\tau^*)| \sim \frac{1}{\chi^{1/3}\ln(\chi)}, \tag{5.81}$$

while $|\tilde{P}_j(\tau^*)|$ is constant in χ according to Eq. (5.77). Thus both assumptions are fulfilled in the asymptotic regime of large χ, which is what we are interested in. Proceeding, we can perform the integral (5.68) on the domain $\tau \in [0, \tau^*]$ and to use the Taylor expansion of f which is given by:

$$f(\tau) \simeq \Phi - \frac{1}{2}\Phi^2(\tilde{M}_s(0) - \tilde{M}_{s'}(\tau) - \tilde{P}_j(\tau)). \tag{5.82}$$

The issue here is, making use of the Taylor expansion requires the condition

$$\Phi|\tilde{M}_s(0) - \tilde{M}_{s'}(\tau) - \tilde{P}_j(\tau)| \ll 1 \tag{5.83}$$

which is not fulfilled for $\tau = \tau^*$ since by construction (Eq. (5.77)) $\Phi\tilde{P}_j(\tau^*)$ is of order 1. We note however that the following condition is fulfilled:

$$|\tilde{P}_j((1-\epsilon)\tau^*)| \sim \left(\frac{1}{\chi}\right)^{\epsilon/(1-\epsilon)} \tag{5.84}$$

for $\epsilon \in [0,1)$. Therefore, for $\epsilon > 0$ and at sufficiently large χ, we are allowed to use the Taylor expansion (5.82) if we integrate over $[0,(1-\epsilon)\tau^*]$. By making ϵ a function of χ we can show that the remaining integral over $[(1-\epsilon)\tau^*, \tau^*]$ is negligible. We need to watch out however that a choice of $\epsilon(\chi)$ fullfills the condition $|\tilde{P}_j((1-\epsilon)\tau^*)| \ll 1$ for sufficiently large χ. One possibility would be $\epsilon(\chi) = 1/\sqrt{\ln(\chi)}$. Then we will conclude that we can can use the expansion (5.82) in an integral over the full intervall $\tau \in [0,\tau^*]$. Indeed it can be seen that

$$\int_{(1-\epsilon)\tau^*}^{\tau^*} \frac{d\tau}{\tau^{2/3}} f(\tau) \simeq \epsilon\tau^* \frac{f(\tau^*)}{\tau^{*2/3}} \sim \frac{\epsilon\chi^{1/3}}{\ln^{1/3}(\chi)}, \tag{5.85}$$

since $f(\tau^*) = \text{const}$. When taking the integral over $[0,(1-\epsilon)\tau^*]$, according to our assumptions, we only need to keep the leading term in the expansion (5.82):

$$\begin{aligned} &\int_0^{(1-\epsilon)\tau^*} \frac{d\tau}{\tau^{2/3}} f(\tau) \simeq \Phi \int_0^{(1-\epsilon)\tau^*} \frac{d\tau}{\tau^{2/3}} \\ &= \Phi \int_0^{\tau^*} \frac{d\tau}{\tau^{2/3}} - \Phi \int_{(1-\epsilon)\tau^*}^{\tau^*} \frac{d\tau}{\tau^{2/3}} \\ &\sim \frac{\chi^{1/3}}{\ln^{1/3}(\chi)} + \frac{\epsilon\chi^{1/3}}{\ln^{1/3}(\chi)}. \end{aligned} \tag{5.86}$$

We see that any effects of integrating only up to $(1-\epsilon)\tau^*$ instead of τ^* are suppressed by a factor of ϵ. We conclude that we can use the Taylor expansion (5.82) on the full interval $[0,\tau^*]$. We can now proceed to use all of these approximations to calculate the asymptotic of the full probability:

$$\begin{aligned} &\int_0^{\tau^*} d\tau \tilde{T}_{j,s,s'} \frac{1 - e^{-\Phi(\tilde{M}_s(0) - \tilde{M}_{s'}(\tau) - \tilde{P}_j(\tau))}}{\tilde{M}_s(0) - \tilde{M}_{s'}(\tau) - \tilde{P}_j(\tau)} \\ &\simeq \int_0^{\tau^*} d\tau (-) \frac{2(5-2j)}{3^{1/3}\Gamma(1/3)} \frac{\chi^{2/3}}{\tau^{2/3}} \Phi \\ &\simeq (-) \frac{2(5-2j)}{3^{1/3}\Gamma(1/3)} \chi^{2/3} \int_0^{\tau^*} \frac{d\tau}{\tau^{2/3}} \Phi \\ &= (-) \frac{2(5-2j)}{3^{1/3}\Gamma(1/3)} \chi^{2/3} \Phi 3 \tau^{*1/3} \\ &= \frac{12(5-2j)\,\mathrm{Ai}'(0)\Phi\chi^{1/3}}{3^{1/3}[\ln(\Phi c_j) - \ln(y)]^{1/3}}. \end{aligned} \tag{5.87}$$

In the above calculation we neglected the next to leading order term of expansion (5.82), according to our considerations above. In the following step we need to rely on the fact that the τ-integral on the interval $[0, \tau^*]$ is the most significant contribution to the full integral (5.68). This is supported by the fact that the function $f(\tau)$, due to its double exponential behaviour, has a plateau at $\tau = 0$ with $f(0) = \Phi$ and $f'(0) = 0$ and a sharp drop around $\tau = \tau^*$ with the attribute $f'(\tau^*) \to -\infty$ for $\chi \to \infty$. Also the integral on the domain $[1/\chi, 1]$ can be taken in the limit $\chi \to \infty$ by simply computing the point wise limit of the integrand. This calculation is similar to the one presented in the next section for the case $ss' = -1$ and as result this integral will have a constant scaling in χ, and therefore will only contribute to lower order than the result (5.87). Thus by assuming that the integral (5.87) gives the leading order contribution to the integral, the asymptotic of the full probability is then given by:

$$P_{jss'} \simeq \frac{\alpha}{\eta} e^{\Phi \tilde{M}_s(0)} \frac{3^{1/3}(5-2j)\Phi\chi^{1/3}}{\Gamma(1/3)\left[\ln\left(\frac{\Phi\alpha\chi}{\eta}\sqrt{\frac{3}{8}\frac{j}{4}}\right) - \ln(y)\right]^{1/3}}. \tag{5.88}$$

Since y is a constant and c_j grows linearly with χ, the term $\ln(y)$ may be disregarded, thus proving that the asymptotic does not depend on the exact value of ρ in Eq. (5.75).

5.4.2 The case $ss' = -1$

The case $ss' = -1$ is simpler than the case $ss' = 1$, because there are no significant contributions at very small $\tau < 1/\chi$. This is because, as was discussed in the perturbative case, when $ss' = -1$ the trace expression does not have a pole at $\tau = 0$ anymore. The asymptotic can be calculated by taking the point wise limit $\chi \to \infty$ in the integrand of (5.68). For this we consider the following asymptotics for a fixed $\tau \in (0, 1)$:

$$\tilde{M}_s(0) \to a\frac{\alpha\chi^{2/3}}{\eta} + sc\frac{\alpha\chi^{1/3}}{\eta}, \tag{5.89}$$

$$\tilde{M}_{s'}(\tau) \to \frac{a}{(1-\tau)^{1/3}}\frac{\alpha\chi^{2/3}}{\eta} + \frac{s'c}{(1-\tau)^{2/3}}\frac{\alpha\chi^{1/3}}{\eta}, \tag{5.90}$$

$$\tilde{P}_j(\tau) \to \frac{b_j\chi^{2/3}}{\tau^{1/3}}, \tag{5.91}$$

with

$$a = \mathrm{Ai}'(0)\int_0^\infty p_1(v)dv = -\frac{28\pi}{9\cdot 3^{5/6}\Gamma(1/3)}, \tag{5.92}$$

$$b_j = \mathrm{Ai}'(0) \int_1^\infty q_j(u)du = -\frac{(5+(-1)^j)\sqrt{\pi}\Gamma(2/3)}{6 \cdot 6^{1/3}\Gamma(1/3)\Gamma(13/6)}, \tag{5.93}$$

$$c = -\,\mathrm{Ai}'(0) \int_0^\infty p_2(v)dv. \tag{5.94}$$

Then we have

$$\begin{aligned} &\tilde{M}_s(0) - \tilde{M}_{s'}(\tau) - \tilde{P}_j(\tau) \\ \simeq &\frac{\alpha\chi^{2/3}}{\eta} \frac{a\tau^{1/3}((1-\tau)^{1/3}-1) - b(1-\tau)^{1/3}}{\tau^{1/3}(1-\tau)^{1/3}} \\ &+ c\frac{\alpha\chi^{1/3}}{\eta}\left(s - \frac{s'}{(1-\tau)^{2/3}}\right). \end{aligned} \tag{5.95}$$

For the limit of $\chi \to \infty$ only the middle line of the equation above which scales as $\chi^{2/3}$ will be important. It is easily verified that this term is positive for all $\tau \in (0,1)$. For this reason we can conclude:

$$e^{-\Phi(\tilde{M}_s(0)-\tilde{M}_{s'}(\tau)-\tilde{P}_j(\tau))} \to 0 \qquad \text{for } \chi \to \infty \tag{5.96}$$

for all $\tau \in (0,1)$. The dominating terms in the trace terms will be given by the middle lines of Eqs. (5.44) and (5.45), such that:

$$\tilde{T}_{jss'} \to \chi^{2/3}\frac{(1-\tau)^{2/3}}{\tau^{2/3}}\,\mathrm{Ai}'(0)\left[(5-2j)(1+ss') + \left(1-(-1)^j\frac{ss'}{2}\right)\frac{\tau^2}{1-\tau}\right] \tag{5.97}$$

Since we only consider the case $ss' = -1$ here, we can explicitly insert this condition into the trace and obtain:

$$\tilde{T}_{jss'} \to \chi^{2/3}\,\mathrm{Ai}'(0)\left(1+\frac{1}{2}(-1)^j\right)\frac{\tau^{4/3}}{(1-\tau)^{1/3}}. \tag{5.98}$$

Inserting everything into the expression of the full probability, we see that the $\chi^{2/3}$ from the trace will cancel the $\chi^{2/3}$ from the denominator and we obtain:

$$P_{jss'} \simeq -\frac{1}{4}\,\mathrm{Ai}'(0)e^{\Phi\tilde{M}_s(0)}\int_0^1 d\tau \frac{(1+\frac{1}{2}(-1)^j)\tau^{5/3}}{-a\tau^{1/3}(1-(1-\tau)^{1/3}) - b_j(1-\tau)^{1/3}}. \tag{5.99}$$

The τ-integral does not depend on χ anymore and thus the entire χ dependence is contained in the exponential prefactor $e^{\Phi\tilde{M}_s(0)}$. By evaluating the remaining τ-integral in (5.99) numerically, we obtain:

$$P_{jss'} = e^{\Phi\tilde{M}_s(0)} \cdot \begin{cases} 0.0185375, & j=1 \\ 0.0472879, & j=2 \end{cases}. \tag{5.100}$$

5.4.3 Plots

In order to confirm the results for the asymptotics derived in this section, they were compared to the exact numerical evaluations of the probabilities $P_{j,s,s'}$ from Eq. (5.68) for different values of χ. A direct comparison of the numerical values of the probabilities and the corresponding asymptotic expressions for different combinations of j, s and s' is shown in Fig. 5.4. While these plots make it evident that the asymptotics work well, due to the logarithmic scale it is difficult to quantify how good the convergence is quantitatively at a given scale of χ. Thus for a better quantification we define the following ratio:

$$R_{(j,s,s')} = \frac{(P_{j,s,s'})_{\text{asymptotic}}}{(P_{j,s,s'})_{\text{exact}}} \tag{5.101}$$

where in the numerator are the asymptotic expressions of the probability and in the denominator are the exact values that are obtained numerically. Thus one can see how well the asymptotics converge, by looking at how fast this ratio approaches unity. Plots of the ratios for different combinations of j, s and s' are shown in Fig. 5.5.

5.4.4 Discussion of the Non-Perturbative Results

In the treatment of the perturbative cases we have seen that the integral over outgoing photon momenta makes the scaling of incoherent processes with respect to χ nontrivial. The result was certainly different from the "naive" scaling according to which each mass and polarization operator in the expression contributes a factor of $\alpha\chi^{2/3}$. For the non-perturbative case, i.e. the resummation of all the contributions from the perturbative case, we have found that at least in the more dominant case $ss' = 1$ there are also nontrivial contributions from the photon momentum integral. Just like in the perturbative case, these nontrivial contributions are those which arise from the region of very small q_- for which $\tau = q_-/p_- < 1/\chi$. In the case $ss' = -1$ the asymptotics could be obtained by taking the point wise limit $\chi \to \infty$ of the integrand, which shows that no such nontrivial contributions play a role. As a result the probabilities for the case of $ss' = -1$ are sub-leading in the high field limit. It must be pointed out however, that these asymptotic probabilities will be difficult to verify experimentally. For one, the exponential prefactor diminishes the values of the probabilities in both cases $ss' = \pm 1$ very quickly with increasing χ, as incoherent processes become more and more likely. Also, as discussed before, the most significant contribution comes from photons which have a very small momentum minus component. Since the photon is massless, this corresponds to the situation that the photon gets emitted in a direction that is very close to the propagation direction $\boldsymbol{n}$ of the plane wave field. In an experiment with a laser, these photons would be very difficult to distinguish from the laser photons. It also has to be noted that at $\chi \sim 1600$ higher order radiative corrections to the first order mass and po-

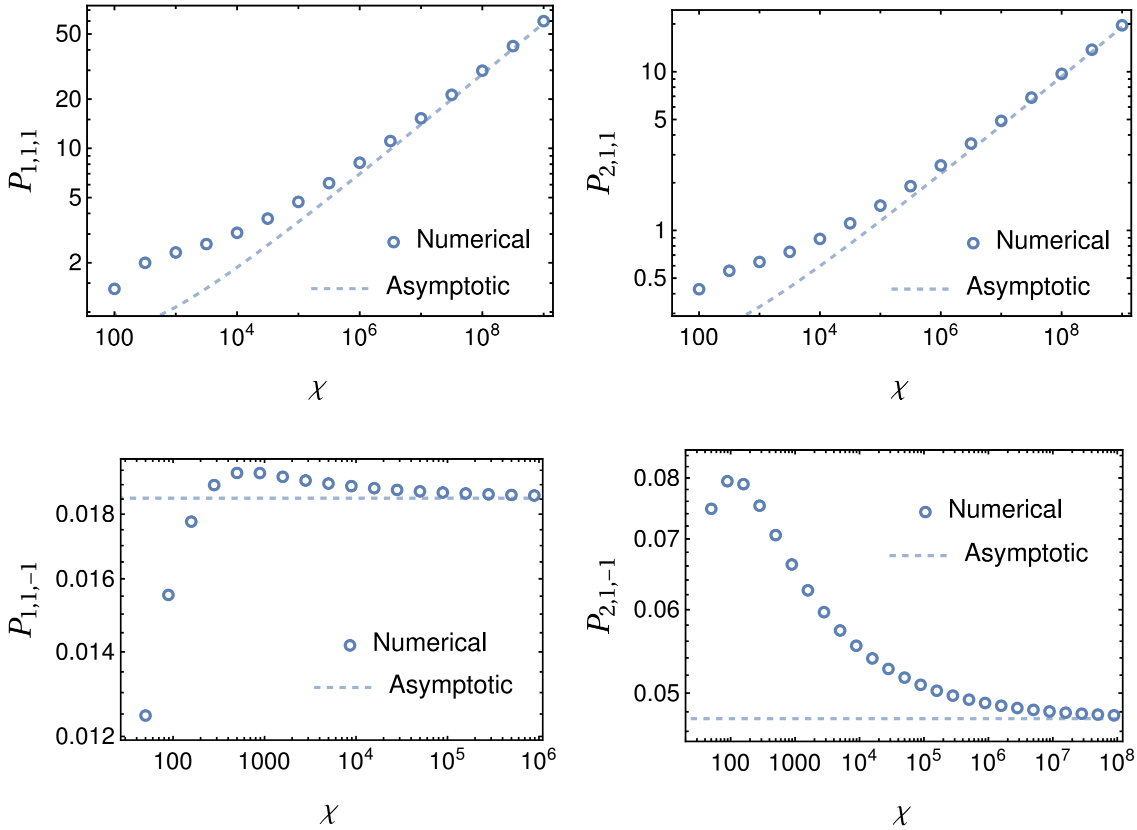


Figure 5.4: This figure contains plots of the probabilities $P_{j,s,s'}$ and their asymptotics as a function of χ without the exponential prefactor. The plots were made for different values of j, s and s' in the case of having a constant crossed field of total phase Φ. The numerical parameters that were used are: $\eta = 0.5$ and $\Phi = 2\pi$.

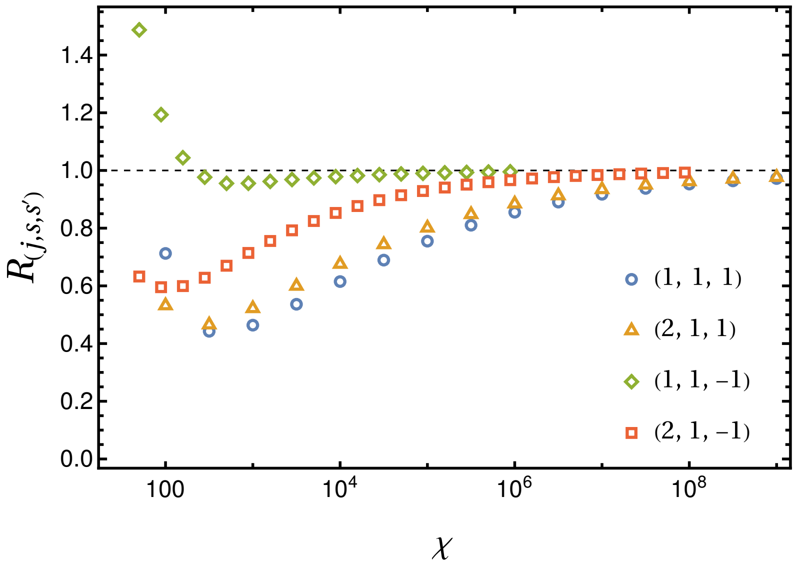


Figure 5.5: This plot shows the ratios $R_{(j,s,s')}$ which indicate how well the asymptotic expressions converge (which is when the value gets close to 1). The ratios were plotted as a function of χ for different values of j, s and s' in the case of having a constant crossed field of total phase Φ. The parameters given in the legend correspond to the values (j, s, s'). The numerical parameters that were used are: $\eta = 0.5$ and $\Phi = 2\pi$.

larization operators will become important (Ritus-Narozhny conjecture) which are not accounted for in the expression of the probability Eq. (5.43).

6 Summary and Conclusion

In the first part of the thesis we have computed a general expression for the first order correction to nonlinear Compton scattering due to the polarization operator. We have computed the locally constant field approximation of this general expression. We found that this correction includes an incoherent contribution that has the exact negative value of the incoherent trident process. Thus, our result agrees with the known result that incoherent contributions to higher order loop corrections will cancel each other out. For this incoherent correction, we computed the high field limit which is characterized by $\chi \ll 1$, where χ is the quantum nonlinearity parameter. The expression we found also agrees with the corresponding result for the trident process from the literature. The expression of the incoherent correction is also a first order contribution to the damping effects due to long laser pulse lengths. If the laser pulse in a strong field experiment is significantly larger than the formation length, it becomes likely for in- and outgoing electrons and photons to decay into different particle states, and thus the probability of single photon emission is damped. We recalled from the literature the expression for the probability of nonlinear Compton scattering, which includes all such damping effects due to processes that are of first order in α. After expanding this expression we identified that each order must correspond to certain incoherent radiative corrections with a given amount of mass and polarization operators. For each of these terms, we proceeded to find the high field limit. We found that terms containing multiple mass or polarization operators in the outgoing particle lines, exhibit a larger scaling in χ than is expected from the known scaling of single mass and polarization operators. We continued to find the high field limit of the full expression which contains all such contributions, and we found that only in the case where the incoming and outgoing electron share the same spin there is a non-trivial scaling in χ. From the asymptotics of the perturbative case we concluded that in the high field limit for incoherent processes, there is an effective expansion parameter given by $\Phi\alpha\chi$, where Φ is the total pulse length. This is an interesting result because the scaling in powers of χ is bigger than one would expect from the Ritus-Narozhny conjecture, which states that the expansion parameter for radiative corrections in the high field limit in a constant crossed field is $\alpha\chi^{2/3}$. We discussed however that in closed loop diagrams the incoherent contributions are expected to cancel out, just like it explicitly could be seen in the case of the first order polarization correction. Thus our result is not in contradiction with the Ritus-Narozhny conjecture, if applied to coherent processes and loop corrections only. For future work it would be interesting to consider the cancellation of incoherent processes in closed loops in more detail. Also while the incoherent processes were studied very extensively in this thesis, it would be interesting to also study the coherent correction from the polarization operator more closely.

Part I

Appendix

A Gamma Matrices

The gamma matrices are four 4×4 matrices which are defined by the condition

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}, \tag{A.1}$$

where $\mu, \nu = 0, 1, 2, 3$ and where $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is the Minkowski metric. A possible representation of the Gamma matrices is the Weyl representation in which they are specifically given by [42]:

$$\gamma^0 = \begin{pmatrix} & & 1 & 0 \\ & & 0 & 1 \\ 1 & 0 & & \\ 0 & 1 & & \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} & & 0 & 1 \\ & & 1 & 0 \\ 0 & -1 & & \\ -1 & 0 & & \end{pmatrix},$$
$$\gamma^2 = \begin{pmatrix} & & 0 & -i \\ & & i & 0 \\ 0 & i & & \\ -i & 0 & & \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} & & 1 & 0 \\ & & 0 & -1 \\ -1 & 0 & & \\ 0 & 1 & & \end{pmatrix}. \tag{A.2}$$

For a four-vector a^μ we define

$$\hat{a} = a_\mu \gamma^\mu. \tag{A.3}$$

In QED computations one often has to compute traces of products of gamma matrices. For these computations it is convenient to know that the trace of an uneven number of gamma matrices is always zero. For traces of even numbers of gamma matrices the following identities are helpful:

$$\mathrm{tr}\left(\gamma^\mu\gamma^\nu\right) = 4\eta^{\mu\nu}, \tag{A.4}$$

$$\mathrm{tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) = 4\left(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right), \tag{A.5}$$

and

$$\begin{aligned} &\mathrm{tr}\left(\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) \\ =&4g^{\kappa\lambda}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right) \\ &- 4g^{\kappa\mu}\left(g^{\lambda\nu}g^{\rho\sigma} - g^{\lambda\rho}g^{\nu\sigma} + g^{\lambda\sigma}g^{\nu\rho}\right) \\ &+ 4g^{\kappa\nu}\left(g^{\lambda\mu}g^{\rho\sigma} - g^{\lambda\rho}g^{\mu\sigma} + g^{\lambda\sigma}g^{\mu\rho}\right) \\ &- 4g^{\kappa\rho}\left(g^{\lambda\mu}g^{\nu\sigma} - g^{\lambda\nu}g^{\mu\sigma} + g^{\lambda\sigma}g^{\mu\nu}\right) \\ &+ 4g^{\kappa\sigma}\left(g^{\lambda\mu}g^{\nu\rho} - g^{\lambda\nu}g^{\mu\rho} + g^{\lambda\rho}g^{\mu\nu}\right). \end{aligned} \tag{A.6}$$

The above trace identities follow from the condition (A.1) and from the Leibniz rule:

$$\left\{\gamma^\kappa, \gamma^\lambda\gamma^\mu\gamma^\nu\right\} = \left\{\gamma^\kappa, \gamma^\lambda\right\}\gamma^\mu\gamma^\nu - \gamma^\lambda\left\{\gamma^\kappa, \gamma^\mu\right\}\gamma^\nu + \gamma^\lambda\gamma^\mu\left\{\gamma^\kappa, \gamma^\nu\right\}. \tag{A.7}$$

B Airy Functions

Airy functions are the solutions to the differential equation [3]

$$\frac{d^2y}{dx^2} - xy = 0. \tag{B.1}$$

There are two independent solutions to this equation given by [1]:

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) dt, \tag{B.2}$$

and

$$\mathrm{Bi}(x) = \frac{1}{\pi}\int_0^\infty \left[\exp\left(-\frac{t^3}{3} + xt\right) + \sin\left(\frac{t^3}{3} + xt\right)\right] dt. \tag{B.3}$$

There is another integral representation of the Airy function given by [3]:

$$\mathrm{Ai}(z) = \int_{-\infty}^{\infty} \frac{d\tilde{\phi}}{2\pi} e^{iz\tilde{\phi} + i\frac{\tilde{\phi}^3}{3}}. \tag{B.4}$$

The derivative of the Airy function is given by:

$$\mathrm{Ai}'(z) = i\int_{-\infty}^{\infty} \frac{d\tilde{\phi}}{2\pi} \tilde{\phi} e^{iz\tilde{\phi} + i\frac{\tilde{\phi}^3}{3}}. \tag{B.5}$$

Also the following function has been relevant for calculations:

$$\mathrm{Ai}_1(z) = \int_z^\infty dx\,\mathrm{Ai}(x) = i\int_{-\infty}^{\infty} \frac{d\tilde{\phi}}{2\pi} \frac{1}{\tilde{\phi} + i0} e^{iz\tilde{\phi} + i\frac{\tilde{\phi}^3}{3}}. \tag{B.6}$$

Evaluating the Airy functions and its derivative at zero, one obtains [3]:

$$\begin{aligned} \mathrm{Ai}(0) &= \frac{1}{3^{2/3}\Gamma\left(\frac{2}{3}\right)} \\ \mathrm{Ai}'(0) &= -\frac{1}{3^{1/3}\Gamma\left(\frac{1}{3}\right)} \end{aligned} \tag{B.7}$$

Another function that is related to the Airy functions is the Scorer function defined by:

$$\mathrm{Gi}(x) = \frac{1}{\pi}\int_0^\infty \sin\left(\frac{t^3}{3} + xt\right) dt. \tag{B.8}$$

From the definitions it follows that:

$$f(x) = i\int_0^\infty dt \exp\left[-i\left(tx + \frac{1}{3}t^3\right)\right] = \pi\mathrm{Gi}(x) + i\pi\mathrm{Ai}(x), \tag{B.9}$$

where we have defined the function f which is often used for QED calculations in a constant crossed field. For large arguments $x \gg 1$ the derivative of the Airy function has the following asymptotic expression [32].

$$\mathrm{Ai}'(x) = e^{-\frac{2}{3}x^{3/2}}\left(-\frac{x^{1/4}}{2\sqrt{\pi}} + \mathcal{O}\left(\frac{1}{x^{5/4}}\right)\right). \tag{B.10}$$

C Lists

C.1 List of Figures

D Bibliography

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Acknowledgments
I would like to thank my supervisor Antonino Di Piazza for all the support he has shown me over the past years. Antonino has been an incredible mentor and I was able to learn from him in many aspects that have helped me grow as a physicist. Among many things, Antonino taught me the subject of QED, introduced me into research in strong field QED and taught me how to write scientific papers. Thank you!

I would like to thank Tobias Podszus for working with me on some of the topics in this thesis and helping me out whenever I had questions.

I would like to share my condolences to family members and friends of our collaborator Victor Dinu who passed away recently, and I would like to thank Victor for the helpful discussions and his help in numerical tasks.

I would like to thank my colleagues Lorenzo, Giulio, Sam, Mateo, Michael, Zoia, Cedric, Hirikesh, Junis, Konstantin, Igor, Sasha, Miriam and everyone else in the office for their company and support. Also I would like to thank Sibel Babacan and Kevin Zink for all organizational and IT matters.

Part of this work was completed at the University of Rochester. I would like to thank Antonino for making the trip to Rochester possible.

I would like to thank Joerg Jaeckel for being a second advisor for this thesis.

I would like to thank my friends and family outside of MPIK for all the support while working on the thesis.