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Chern character for infinity vector bundles

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Coherent sheaves on general complex manifolds do not necessarily have resolutions by finite complexes of vector bundles. However, D Toledo and Y L L Tong showed that one can resolve coherent sheaves by objects analogous to chain complexes of holomorphic vector bundles, whose cocycle relations are governed by a coherent infinite system of homotopies. In modern language, such objects are obtained by the ∞–sheafification of the simplicial presheaf of chain complexes of holomorphic vector bundles. We define a Chern character as a map of simplicial presheaves, whereby the connected components of its sheafification recover the Chern character of Toledo and Tong. As a consequence, our construction extends O'Brian, Toledo and Tong's definition of the Chern character to the settings of stacks and in particular the equivariant setting. Even in the classical setting of complex manifolds, the induced maps on higher homotopy groups provide new Chern–Simons, and higher Chern–Simons, invariants for coherent sheaves.

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1 Introduction

The celebrated Hirzebruch–Riemann–Roch theorem (HRR) [Hirzebruch 1954] is a generalization of the classical Riemann–Roch theorem for holomorphic line bundles on compact Riemann surfaces. In HRR, the setting of a line bundle on a Riemann surface is generalized to an arbitrary holomorphic bundle E on a smooth projective variety X over the complex numbers. The main tool in proving HRR is a resolution of the diagonal $X \to X \times X$, thought of as a coherent sheaf on $X \times X$, by a finite chain complex of vector bundles.

The Atiyah–Singer index theorem [1968], and the theory of elliptic and pseudoelliptic differential operators, can further be thought of as a far-reaching generalization of HRR and other high-powered theorems, such as the Gauss–Bonnet theorem, to a much vaster context. For example, using the Atiyah–Singer index theorem one can readily extend HRR to holomorphic bundles on even compact complex manifolds that are not necessarily algebraic (see for example [Freed 2021] for an exposition).

Unfortunately, such techniques, found in work by Atiyah, Bott and Patodi [1973] as well as by Gilkey [1973], use differential geometric methods that heavily rely on an auxiliary choice of a Hermitian metric on the manifold as well as the bundle. For example, one uses the metric to establish a heat flow and smooth out the diagonal de Rham current $X \to X \times X$ into a differential form (the heat kernel). However, generally, in complex geometry choosing a metric can be thought of as unnatural and out of context unless within the very specialized realm of Kähler geometry.

Casting this as a deficiency is not only a matter of taste but concerns applications of these ideas to settings where local automorphisms are involved, such as the equivariant as well as the "stacky" discussion. One would therefore desire an intrinsic complex geometric discussion, whereby one establishes HRR, and similar theorems, for general complex manifolds and holomorphic vector bundles outside metric geometry.

Toledo and Tong [1976; 1978a; 1978b; 1986] and O'Brian, Toledo and Tong [1981a; 1981b; 1981c] made several remarkable conceptual breakthroughs by providing local Čech cohomological proofs of HRR [1981b] and Grothendieck–Riemann–Roch (GRR) [1981a]. Through the modern lens, one may interpret their work as a hands-on theory of infinity stacks, which only much more recently has been made into a full-fledged mathematical theory. One of the key constructions by O'Brian, Toledo and Tong [1981c] is that of the Chern class for a coherent analytic sheaf on a complex manifold. While their construction is the one we focus on here, there is also another approach to calculating Chern classes for coherent analytic sheaves, as shown in [Green 1980; Toledo and Tong 1986] and later formalized by Timothy Hosgood [2020; 2023; 2024].

To get a taste for the type of math Toledo and Tong invented and utilized, consider the question of resolving the diagonal $X \to X \times X$, or more generally an arbitrary coherent sheaf, on a complex manifold, by a finite chain complex of vector bundles. One knows that when the complex manifold admits a positive line bundle such resolutions always exist (see [Griffiths and Harris 1978, page 705]). While in the algebraic setting the canonical line bundle provides such a line bundle, general complex manifolds may not support them. Toledo and Tong obviated such difficulties by resolving the problems in a homotopical setting in which strict identities are replaced with a coherent infinite system of homotopies. For instance, as a complex vector bundle is a bunch of transition functions satisfying the familiar cocycle conditions, they showed that, by requiring the cocycle condition to hold up to an infinite system of homotopies, not only could every coherent sheaf on a complex manifold be resolved by these more general objects, but also all of the necessary complex geometric arguments would remain valid.

Let us be more specific and start with a coherent sheaf on a complex manifold. Choose a good Stein cover for the manifold on which the coherent sheaf can be locally resolved by a chain complex of vector bundles; such a cover always exists. By restricting these resolutions to double intersections, we obtain two resolutions for the same coherent sheaf on that intersection which, by the uniqueness of resolutions over Stein manifolds, are then related by a quasi-isomorphism. On triple intersections, the three relevant quasi-isomorphisms may not fit to give you a chain complex of vector bundles, but the discrepancy can be killed by a homotopy. These assigned homotopies to triple intersections may not satisfy the required compatibilities on quadruple intersections but the discrepancy can be killed by a higher homotopy. Repeating this pattern ad infinitum gives rise to an infinite system of homotopies.

Historically, the use of coherent infinite systems of homotopy in a different context was known to some algebraic topologists almost 30 years prior but even there it was considered rather esoteric. Jim Stasheff [1963a; 1963b] showed how the based loop space of a pointed space was an A_∞ monoid. Nowadays these mathematical objects are inescapable and it is common knowledge among a large group of algebraic topologists that A_∞ algebras are just as good as differential associative algebras and have the same homotopy theories [Lefèvre-Hasegawa 2003]. Similarly, Toledo and Tong showed that these generalized objects are just as good as chain complexes of vector bundles as far as coherent cohomologies were concerned. While they did not make a formal claim about their corresponding homotopy theories, they showed how Ext and Tor of such generalized objects can be defined, calculated and, subsequently, be used to prove duality theorems à la Grothendieck and establish HRR and GRR.

Surprisingly, since their work very little has been done to formalize the homotopy theory of these objects. For example, in *Descente pour les n–champs*, André Hirschowitz and Carlos Simpson [1998] write:

> Dans les travaux de O'Brian, Toledo et Tong consacrés à une autre question issue de SGA 6, celle des formules de Riemann–Roch, on trouve des calculs de Čech qui sont certainement un exemple de situation de descente pour les complexes. Un meilleur cadre général pour ces calculs pourrait contribuer à notre compréhension des formules de Riemann–Roch.

This roughly translates to the following:

> In the work of O'Brian, Toledo and Tong devoted to another question arising from SGA 6 regarding the Riemann–Roch formulas, one can find Čech calculations that are an example of descent for complexes. A better general framework for these calculations could contribute to our understanding of the Riemann–Roch formulas.

Here we have taken the first step in providing a homotopy-theoretic framework for some of Toledo and Tong's mathematical objects. By simply finding the right homotopy-theoretic setting, their constructions extend far beyond what they had intended and point to new and exciting advances. For example, their construction of a Chern character for coherent sheaves in Hodge cohomology is easily generalized to the equivariant setting, or even to the setting of stacks. In addition, secondary and higher Chern characters are now an inseparable part of the discussion.

The inherent inclusion of these higher Chern characters points to the possibility of proving a version of GRR as a commutative diagram of spaces such that, after applying π_0 to the diagram, one would obtain a diagram of sets which is O'Brian, Toledo and Tong's GRR. Note that classical objects such as K–groups and cohomology groups are sets with additional algebraic structures.

In Section 2 we begin by defining the simplicial presheaves $\mathbf{IVB}$ and $\mathbf{\Omega}$, which will be the domain and codomain of our Chern map, respectively. For a fixed complex manifold $U \in \mathbb{C}\mathrm{Man}$, we first consider the dg-category $\mathrm{Perf}^{\nabla}(U)$ of finite chain complexes of holomorphic bundles with connection,[1] where there is no requirement that morphisms be compatible with connections. Taking the maximal Kan complex of the dg-nerve, we obtain a simplicial set $\mathbf{Perf}(U)$. Applying this construction objectwise over $\mathbb{C}\mathrm{Man}$ and noting that maps $f \in \mathbb{C}\mathrm{Man}^{\mathrm{op}}(U,V)$ induce maps of Kan complexes $\mathbf{Perf}(U) \xrightarrow{f^*} \mathbf{Perf}(V)$ via pullbacks, we obtain a simplicial presheaf $\mathbf{Perf}$ which is fibrant in the (global) projective model structure. Since the simplices $\mathbf{Perf}(U)_n = \mathrm{sSet}(\Delta^n, \mathbf{Perf}(U))$ lack the cyclic structure we will need later on to construct our trace map, we define a weakly equivalent (see Proposition 2.10) simplicial presheaf $\mathbf{IVB}(U)_n := \mathrm{sSet}(\widehat{\Delta}^n, \mathbf{Perf}(U))$ given by mapping the cyclic sets $\widehat{\Delta}^n$ into $\mathbf{Perf}(U)$. Here, $\widehat{\Delta}^n$ is the nerve of the category whose set of objects is $\mathbb{Z}/(n+1)\mathbb{Z}$ and all hom-sets have a single morphism (see Example 2.8). Next, we define $\mathbf{\Omega}$ in the same way we did in our previous paper [2022]; more precisely, $\mathbf{\Omega}(U)$ is the simplicial set whose k–simplices are decorations of all i–dimensional faces of the standard k–simplex with sequences of forms, all even for i even, and all odd for i odd, in such a way that the alternating sum of all forms sitting on the $(i-1)$–dimensional faces of any i–dimensional face add up to 0.

The Chern map $\mathbf{Ch}\colon \mathbf{IVB} \to \mathbf{\Omega}$ is then defined in Section 3 as follows. An n–simplex in $\mathbf{IVB}(U)_n$ consists of $n+1$ dg-bundles with connection $(\mathcal{E}_i, d_i, \nabla_i)$, and a set of maps $g = \{(g_{(i_0\dots i_k)}\colon \mathcal{E}_{i_k} \to \mathcal{E}_{i_0})\}_{(i_0,\dots,i_k)\in\widehat{\Delta}^n}$ satisfying the Maurer–Cartan condition (see Definition 3.3). First, in Definition 3.7, we define a trace map Tr_g similar to that of O'Brian, Toledo and Tong [1981c, Proposition 3.2], satisfying the condition

$$\mathrm{Tr}_g \circ (\hat{\delta} + D + [g,-]) = \delta \circ \mathrm{Tr}_g. \tag{3-8}$$

Using this trace map, $\mathbf{Ch}$ is then defined (in Definition 3.13) by assigning to an n–simplex in $\mathbf{IVB}(U)_n$ as above decorations of the nondegenerate k–faces of Δ^n given by the elements in $\mathbf{\Omega}(U)_n$

$$\mathrm{Tr}_g(A^k)_\alpha \cdot \frac{u^k}{k!} = \mathrm{Tr}_g\big((\nabla(d+g))^k\big)_\alpha \cdot \frac{u^k}{k!} = \sum \pm \mathrm{tr}(g\cdot\nabla(d+g)\cdot\nabla(d+g)\cdots\nabla(d+g))_\alpha \cdot \frac{u^k}{k!} \tag{3-11}$$

[1] The use of Perf^{∇} is meant to allude to the study of perfect complexes.

for $k>0$ and for $k=0$ we assign the Euler characteristic. Our first main result is that this provides a map of (objectwise Kan) simplicial presheaves:

Theorem 3.14 *The Chern character* $\mathbf{Ch}\colon \mathbf{IVB}\to\mathbf{\Omega}$ *defined above is a map of simplicial presheaves.*

In Section 4 we construct what we call the Čech sheafification, $\mathbf{Ch}^{\check{\dagger}}\colon \mathbf{IVB}^{\check{\dagger}}\to\mathbf{\Omega}^{\check{\dagger}}$ of the Chern map. Given a simplicial presheaf $\boldsymbol{F}$, the idea is that, for each open cover $(U_i\to X)_{i\in I}$, we can form the Čech nerve simplicial presheaf, $\check{N}U_\bullet$, and then compute the homotopy limit induced by the simplicial mapping space $\underline{\mathrm{sPre}}(\check{N}U_\bullet,\boldsymbol{F})=\mathrm{holim}_i\prod_{\alpha_0,\dots,\alpha_i}\boldsymbol{F}(U_{\alpha_0\dots\alpha_i})$ by taking the totalization of the induced cosimplicial simplicial set $\boldsymbol{F}(\check{N}U_\bullet)$ defined in (4-1). The Čech sheafification $F^{\check{\dagger}}$ is then defined (Definition 4.1) by taking the colimit over all covers:

$$\boldsymbol{F}^{\check{\dagger}}(X):=\operatorname*{colim}_{(U_\bullet\to X)\in\check{S}}\mathrm{Tot}(\boldsymbol{F}(\check{N}U_\bullet)).\tag{4-2}$$

As the construction is functorial in simplicial presheaves and preserves Kan complexes, we obtain a sheafified Chern map, $\mathbf{Ch}^{\check{\dagger}}\colon \mathbf{IVB}^{\check{\dagger}}\to\mathbf{\Omega}^{\check{\dagger}}$, which is a map of Kan complexes. The rest of the section is devoted to showing how $\mathbf{Ch}^{\check{\dagger}}$ is related to the Chern character map of O'Brian, Toledo and Tong [1981c], which begins with Theorem 4.9, stating that the twisting cochains of [loc. cit.] include into the vertices of $\mathbf{IVB}^{\check{\dagger}}$. The full correspondence is given in Theorem 4.18, which shows that, if we restrict **IVB** to the simplicial presheaf **CohSh** considering only nonpositively graded chain complexes whose homology is concentrated in degree zero, then we fully recover the data from the Chern map in [loc. cit.] by the connected components of our sheafified Chern map:

Theorem 4.18 *For a given coherent sheaf, the formula for the Chern character* (4-15) *from* [loc. cit.] *is given by the terms in the formula* (4-14) *of the Chern character map*

$$\{\text{isomorphism classes of coherent sheaves}\}\simeq\pi_0(\mathbf{CohSh}^{\check{\dagger}})\xrightarrow{\pi_0(\mathbf{Ch}^{\check{\dagger}})}\pi_0(\mathbf{\Omega}^{\check{\dagger}})\simeq\bigoplus_{\substack{p,q\\ p+q\ \text{even}}}H^p(\Omega^q)\tag{4-16}$$

applied to the corresponding twisting cochain interpreted (*by Theorem 4.9*) *as a* 0*–simplex in* $\mathbf{CohSh}^{\check{\dagger}}$.

Section 5 upgrades the results from the previous section to statements about (hyper)sheaves. Recall that a simplicial presheaf is a (hyper)sheaf if it is objectwise Kan and satisfies descent with respect to all hypercovers. By restricting our attention to simplicial presheaves of finite homotopy type taking values in Kan complexes, we prove in Proposition 5.2 that the aforementioned Čech sheafification construction computes the (hyper)sheafification. In particular, Proposition 5.12 states that, if we restrict to complex manifolds of bounded dimension, and restrict the homotopy type of **IVB**, then $\mathbf{Ch}^{\check{\dagger}}\colon \mathbf{IVB}^{\check{\dagger}}_{\le n}\to\mathbf{\Omega}^{\check{\dagger}}$ is a map of hypersheaves. If instead we consider again **CohSh**, we see that its sheafification is a classifying stack for coherent sheaves, $\mathbb{R}\mathrm{Hom}(X,\mathbf{CohSh})\simeq\mathbf{CohSh}^{\check{\dagger}}$:

Theorem 5.11 *The simplicial presheaf* **CohSh** *is a classifying prestack for coherent sheaves.*

Once again restricting to manifolds of bounded dimension, Theorem 5.13 states that our sheafified Chern map $\check{\mathbf{Ch}}^{\dagger}\colon \check{\mathbf{CohSh}}^{\dagger} \to \check{\mathbf{\Omega}}^{\dagger}$ is a map of (hyper)sheaves whose connected components yields the Chern map from [loc. cit.]. Finally, Remark 5.14 describes how our Chern character map generalizes to all stacks, with an eye towards future work in the equivariant setting.

Notation 1.1 The simplicial category is denoted by $\mathbf{\Delta}$. It has objects $[n] = \{0, \dots, n\}$ for $n \in \mathbb{N}_0$, and morphisms $\phi \in \mathbf{\Delta}([k],[n])$ that are nondecreasing maps $\phi\colon [k] \to [n]$, ie $\phi(i) \le \phi(j)$ for $i \le j$. The morphisms are generated by face maps $\delta_j\colon [n] \to [n+1]$ (the injection that skips the element j in $[n+1]$) and degeneracies $\sigma_j\colon [n] \to [n-1]$ (the surjection that maps j and $j+1$ to j).

Simplicial objects in a category $\mathcal{C}$ are functors $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$, where the induced face and degeneracy morphisms are denoted by d_j and s_j, respectively. We denote the category of simplicial sets by $\mathrm{sSet} = \mathrm{Set}^{\mathbf{\Delta}^{\mathrm{op}}}$. Cosimplicial objects in a category $\mathcal{C}$ are functors $\mathbf{\Delta} \to \mathcal{C}$.

Given an object X in a locally small category $\mathcal{C}$, we can consider its representable presheaf $yX := \mathcal{C}(-, X)$ given by the Yoneda embedding. Further, given a presheaf F on $\mathcal{C}$, we can consider its simplicially constant presheaf cF defined by $cF(Y)_n := F(Y)$. When context is clear we may drop the "y" or "c". For example, for an object X we might write X for the simplicial presheaf defined by $X(Y)_n := \mathcal{C}(Y, X)$.

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2 The simplicial presheaves IVB and $\mathbf{\Omega}$

We define two simplicial presheaves on the site of complex manifolds; first, $\mathbf{IVB}\colon \mathbb{C}\mathrm{Man}^{\mathrm{op}} \to \mathrm{sSet}$ is the presheaf which will later give rise to infinity vector bundles (see Definition 4.4), and $\mathbf{\Omega}\colon \mathbb{C}\mathrm{Man}^{\mathrm{op}} \to \mathrm{sSet}$ is the presheaf of holomorphic forms. In the next section we will then define the Chern character map as a map of simplicial presheaves $\mathbf{Ch}\colon \mathbf{IVB} \to \mathbf{\Omega}$.

Let $\mathbb{C}\mathrm{Man}$ be the category whose objects consist of complex manifolds, and morphisms are holomorphic maps. Furthermore, denote by dgCat the category of differential graded categories, ie categories $\mathcal{C}$ such that, for any two objects C_1 and C_2 of $\mathcal{C}$, the space of morphisms $\mathrm{Hom}(C_1, C_2)$ is a cochain complex, with the composition being a cochain map and the identity morphisms being closed.

Definition 2.1 Let $\mathrm{Perf}\colon \mathbb{C}\mathrm{Man}^{\mathrm{op}} \to \mathrm{dgCat}$ be given by setting $\mathrm{Perf}(U)$ to be the dg-category whose objects $\mathcal{E} = (E_\bullet, d, \nabla) \in \mathrm{Perf}(U)$ are finite chain complexes of holomorphic vector bundles $E_\bullet \to U$ over U with differential $d\colon E_\bullet \to E_{\bullet+1}$, and with a holomorphic connection ∇ on $E_\bullet$. Morphisms $\mathrm{Hom}(\mathcal{E}, \mathcal{E}')$ consist of graded morphisms of vector bundles $f\colon E_\bullet \to E'_\bullet$ which *need not have any special* *compatibility* with respect to the connections ∇ and ∇'. The dg structure on $\mathrm{Hom}(\mathcal{E},\mathcal{E}')$ is the induced one by the differential and gradings on $\mathcal{E}$ and $\mathcal{E}'$; in particular, the differential of an $f\in\mathrm{Hom}(\mathcal{E},\mathcal{E}')$ is defined to be $D(f):=f\circ d-(-1)^{|f|}d'\circ f$.

A holomorphic map $\varphi\colon U\to U'$ induces a functor $\mathrm{Perf}(\varphi)\colon\mathrm{Perf}(U')\to\mathrm{Perf}(U)$ by pulling back bundles via φ.

Since $\mathrm{Perf}(U)$ is a dg-category, we can apply the dg-nerve $\mathrm{dg}\mathcal{N}(\mathrm{Perf}(U))$, which gives a simplicial set.

Note 2.2 Explicitly, we can describe the simplicial structure of the dg-nerve $\mathrm{dg}\mathcal{N}(\mathcal{C})$ of a dg-category $\mathcal{C}$ (for us, it will always be $\mathcal{C}=\mathrm{Perf}(U)$) as follows; see [Lurie 2017, 1.3.1.6; Faonte 2017, Definition 2.2.8]:

(1) A 0–simplex in $\mathrm{dg}\mathcal{N}(\mathcal{C})_0$ consists of an object $\mathcal{E}$ of $\mathcal{C}$.

(2) A 1–simplex in $\mathrm{dg}\mathcal{N}(\mathcal{C})_1$ consists of $(\mathcal{E}_1,\mathcal{E}_0,g_{01})$, ie two objects $\mathcal{E}_0$ and $\mathcal{E}_1$ of $\mathcal{C}$ and a morphism $g_{01}\colon\mathcal{E}_1\to\mathcal{E}_0$ in $\mathcal{C}$ of degree 0, which is closed, ie $Dg_{01}=0$, where we denoted the differential in $\mathrm{Hom}_{\mathcal{C}}$ by D. (In the case of $\mathcal{C}=\mathrm{Perf}(U)$, the internal differential D is given by the differentials d and d' on E and E', respectively, via $Df=f\circ d-(-1)^{|f|}d'\circ f$, so that $Dg_{01}=0$ means that g_{01} is a chain map of dg-vector bundles.)

(3) A 2–simplex in $\mathrm{dg}\mathcal{N}(\mathcal{C})_2$ consists of $(\mathcal{E}_0,\mathcal{E}_1,\mathcal{E}_2,g_{01},g_{12},g_{02},g_{012})$, ie three objects $\mathcal{E}_0$, $\mathcal{E}_1$ and $\mathcal{E}_2$ of $\mathcal{C}$, three morphisms $g_{ij}\colon\mathcal{E}_j\to\mathcal{E}_j$ of degree 0, where $i,j\in\{0,1,2\}$ with $i<j$, and another morphism $g_{012}\colon\mathcal{E}_2\to\mathcal{E}_0$ of degree -1 satisfying $Dg_{012}=g_{01}\circ g_{12}-g_{02}$.

(4) An n–simplex in $\mathrm{dg}\mathcal{N}(\mathcal{C})_n$ consists of $n+1$ dg-vector bundles $\mathcal{E}_0,\dots,\mathcal{E}_n$ and morphisms
$$g_{i_0\dots i_k}\colon\mathcal{E}_{i_k}\to\mathcal{E}_{i_0}$$
of degree $1-k$ for each sequence $i_0,\dots,i_k\in\{0,\dots,n\}$ with $i_0<\dots<i_k$ and $k\geq 1$ such that
$$D(g_{i_0\dots i_k})=\sum_{j=1}^{k-1}(-1)^{j-1}g_{i_0\dots\hat{\imath}_j\dots i_k}+\sum_{j=1}^{k-1}(-1)^{k(j-1)+1}g_{i_0\dots i_j}\circ g_{i_j\dots i_k}. \tag{2-1}$$

(5) For a morphism $\phi\colon[n]\to[m]$ in $\mathbf{\Delta}$, there is an induced map $\phi^{\sharp}_{\mathrm{dg}\mathcal{N}}\colon\mathrm{dg}\mathcal{N}(\mathcal{C})_m\to\mathrm{dg}\mathcal{N}(\mathcal{C})_n$, given by mapping $(\mathcal{E}_i,g_{i_0\dots i_k})_{\text{all indices}}\in\mathrm{dg}\mathcal{N}(\mathcal{C})_m$ to $(\mathcal{E}_{\phi(i)},\tilde{g}_{i_0\dots i_k})_{\text{all indices}}\in\mathrm{dg}\mathcal{N}(\mathcal{C})_n$, which is defined by either $\tilde{g}_{i_0\dots i_k}=g_{\phi(i_0)\dots\phi(i_k)}$ if ϕ is injective on $\{i_0,\dots,i_k\}$, or $\tilde{g}_{i_0i_1}=\mathrm{id}_{E_{\phi(i_0)}}$ if $\phi(i_0)=\phi(i_1)$, or $\tilde{g}_{i_0\dots i_k}=0$ in all other cases, ie when $k\geq 2$ and $\phi(i_p)=\phi(i_{p+1})$ for some $p=0,\dots,k-1$.

In the later sections, we will use the dg-nerve of U as *local* building blocks of chain complexes of vector bundles on a complex manifold. To obtain a reasonable gluing, we will want the chain maps $g_{i_0i_1}$ to be *homotopy equivalences*. This can be achieved in a natural way by taking the maximal Kan subcomplex $\mathrm{dg}\mathcal{N}(\mathrm{Perf}(U))^{\circ}$ of $\mathrm{dg}\mathcal{N}(\mathrm{Perf}(U))$; see [Joyal 2002, Corollary 1.5].

Definition 2.3 Let $\mathbf{Perf}\colon\mathbb{C}\mathrm{Man}^{\mathrm{op}}\to\mathrm{sSet}$ be the simplicial presheaf given by $\mathbf{Perf}(U):=\mathrm{dg}\mathcal{N}(\mathrm{Perf}(U))^{\circ}$, ie the maximal Kan subcomplex of the dg-nerve of $\mathrm{Perf}(U)$.

We have the following characterization of the simplices of $\mathrm{dg}\mathcal{N}(\mathrm{Perf}(U))^\circ$ via [Joyal 2002, Theorem 2.2], for example:

Lemma 2.4 *An n–simplex in $\mathrm{dg}\mathcal{N}(\mathrm{Perf}(U))^\circ$ consists precisely of an n–simplex in $\mathrm{dg}\mathcal{N}(\mathrm{Perf}(U))$ as described in Note 2.2(4) above, with the extra condition that all morphisms $g_{i_0 i_1}\colon \mathcal{E}_{i_1} \to \mathcal{E}_{i_0}$ are **homotopy equivalences**.*

Now, all chain maps $g_{i_0 i_1}$ on the edges of all simplices of $\mathrm{dg}\mathcal{N}(\mathrm{Perf}(U))^\circ$ are homotopy equivalences. In order to be able to define the Chern character below, we will need to find homotopy inverses of these together with compatible higher homotopies. This can be achieved as follows. First, using the Yoneda lemma for simplicial sets, we know that the n–simplices of a simplicial set $X_\bullet$ are precisely the simplicial set maps from $\Delta^n := \mathbf{\Delta}(-,[n])$ into $X_\bullet$, ie $X_n = X([n]) \cong \mathrm{Nat}(\mathbf{\Delta}(-,[n]), X) = \mathrm{sSet}(\Delta^n, X)$. Thus, we define $\mathbf{Perf}^\Delta\colon \mathbb{C}\mathrm{Man}^{\mathrm{op}} \to \mathrm{sSet}$ by setting

$$\mathbf{Perf}^\Delta(U)_n := \mathrm{dg}\mathcal{N}(\mathrm{Perf}(U))^\circ_n = \mathrm{sSet}\big(\Delta^n, \mathrm{dg}\mathcal{N}(\mathrm{Perf}(U))^\circ\big). \tag{2-2}$$

More generally, we define:

Definition 2.5 Let Q be a cosimplicial simplicial set, ie $Q\colon \mathbf{\Delta} \to \mathrm{sSet}$. In more detail, we denote by $Q^n = Q([n]) \in \mathrm{sSet}$ the image of $[n] \in \mathbf{\Delta}$ under Q, which is itself a simplicial set, $Q^n_\bullet\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}$, $Q^n_k := Q^n([k]) \in \mathrm{Set}$. Then, define $\mathbf{Perf}^Q\colon \mathbb{C}\mathrm{Man}^{\mathrm{op}} \to \mathrm{sSet}$ by setting

$$\mathbf{Perf}^Q(U)_n := \mathrm{sSet}\big(Q^n, \mathrm{dg}\mathcal{N}(\mathrm{Perf}(U))^\circ\big). \tag{2-3}$$

Since $\{Q^n\}_n$ is a cosimplicial object in sSet, this induces, for each $(f\colon [n] \to [m]) \in \mathbf{\Delta}$, a map $\mathbf{Perf}^Q(U)_m \to \mathbf{Perf}^Q(U)_n$, making $\mathbf{Perf}^Q(U)$ into a simplicial set.

For a holomorphic map $\varphi\colon U \to U'$, the induced map $\mathbf{Perf}^Q(U') \to \mathbf{Perf}^Q(U)$ is given by composition with the map $\mathrm{Perf}(U') \to \mathrm{Perf}(U)$ from Definition 2.1, ie by pulling back via φ.

We are mainly interested in the following Examples 2.6 and 2.8.

Example 2.6 Let $\Delta\colon \mathbf{\Delta} \to \mathrm{sSet}$ be given by $\Delta^n := \mathbf{\Delta}(-,[n])$ be the standard simplicial n–simplex given by morphisms of $\mathbf{\Delta}$ into $[n]$, ie its k–simplices $\phi \in \Delta^n_k = \mathbf{\Delta}([k],[n])$ are nondecreasing maps from $[k]$ to $[n]$, ie if we set $i_j := \phi(j)$, these are sequences of indices $(i_0 \le \cdots \le i_k)$ with $i_0, \dots, i_k \in \{0, \dots, n\}$. Face maps are $d_j\colon \Delta^n_k \to \Delta^n_{k-1}$ that remove the j^{th} index i_j, and degeneracies $s_j\colon \Delta^n_k \to \Delta^n_{k+1}$ that repeat the j^{th} index i_j.

By Yoneda, any simplicial set map $\Delta^n \to X$ is completely determined by the image of its nondegenerate n–simplex. Thus, by (2-2), $\mathbf{Perf}^\Delta(U)$ has n–simplices given as described precisely by Lemma 2.4, ie by Note 2.2 with homotopy equivalences on edges.

Before we give our second main example for $\mathbf{Perf}^Q$, we record a useful lemma about simplicial set maps into the dg-nerve $\mathbf{Perf}^Q(U)$.

Lemma 2.7 *Let $X_\bullet$ be a simplicial set, and let $\mathcal{C}$ be a dg-category (for us, $\mathcal{C} = \mathrm{Perf}(U)$). Then a simplicial set map $X \to \mathrm{dg}\,\mathcal{N}(\mathcal{C})$ is precisely given by the following data:*

(1) *For each 0–simplex $\alpha \in X_0$, we have an object $\mathcal{E}_\alpha$ of $\mathcal{C}$.*

(2) *For each nondegenerate k–simplex $\alpha \in X_k$ with $k \geq 1$, there is a morphism $g_\alpha : \mathcal{E}_{\alpha(k)} \to \mathcal{E}_{\alpha(0)}$ of degree $1-k$ satisfying the compatibility condition*

$$D(g_\alpha) = \sum_{j=1}^{k-1} (-1)^{j-1} g_{\alpha(0,\dots,\hat{j},\dots,k)} + \sum_{j=1}^{k-1} (-1)^{k(j-1)+1} g_{\alpha(0,\dots,j)} \circ g_{\alpha(j,\dots,k)}. \tag{2-4}$$

Here, for a disjoint union decomposition $\{0,\dots,k\} = \{i_0,\dots,i_p\} \sqcup \{j_0,\dots,j_q\}$ with $i_0 < i_1 < \dots < i_p$ and $j_0 < j_1 < \dots < j_q$, we denote by $\alpha(i_0,\dots,i_p) := d_{j_0} \circ \dots \circ d_{j_q}(\alpha) \in X_p$ the face of α corresponding to indices $\{i_0,\dots,i_p\} \subseteq \{0,\dots,k\}$.

In particular, a simplicial set map $X \to \mathrm{dg}\,\mathcal{N}(\mathrm{Perf}(U))^\circ$ has the same data as given above with the extra condition that the maps g_α for $\alpha \in X_1$ are homotopy equivalences.

Proof Let $\mathcal{F}: X \to \mathrm{dg}\,\mathcal{N}(\mathcal{C})$ be a map of simplicial sets and, for $l \geq 0$, let $\alpha \in X_l$ be an l–simplex. Thus, $\mathcal{F}(\alpha) \in \mathrm{dg}\,\mathcal{N}(\mathcal{C})_l$, and, by Note 2.2, there are dg-vector spaces $\mathcal{E}_0^\alpha, \dots, \mathcal{E}_l^\alpha$, and for all $i_0, \dots, i_k \in \{0,\dots,l\}$, $k \geq 1$ with $i_0 < \dots < i_k$, there are maps $g^\alpha_{i_0\dots i_k} : \mathcal{E}^\alpha_{i_k} \to \mathcal{E}^\alpha_{i_0}$ satisfying (2-1). We claim that the data of the highest maps $g^\rho_{0\dots p}$ for all nondegenerate $\rho \in X_p$ is sufficient to recover all other maps $g^\alpha_{i_0\dots i_k}$: For $\alpha \in X_l$ and $i_0,\dots,i_k \in \{0,\dots,l\}$ with $i_0 < \dots < i_k$ with $k < l$, we use the commutative diagram for $\phi: [k] \to [l]$, $\phi(p) := i_p$,

$$\begin{array}{ccc} X_l & \xrightarrow{\mathcal{F}_l} & \mathrm{dg}\,\mathcal{N}(\mathcal{C})_l \\ {\scriptstyle \phi_X^\sharp}\downarrow & & \downarrow{\scriptstyle \phi^\sharp_{\mathrm{dg}\mathcal{N}}} \\ X_k & \xrightarrow{\mathcal{F}_k} & \mathrm{dg}\,\mathcal{N}(\mathcal{C})_k \end{array}$$

mapping the $i_0 < \dots < i_k$–component $g^\alpha_{i_0\dots i_k}$ of $\mathcal{F}_l(\alpha)$ under $\phi^\sharp_{\mathrm{dg}\mathcal{N}}$ to $\tilde{g}_{0\dots k} = g^\alpha_{i_0\dots i_k}$ (by Note 2.2(5), since ϕ is injective). Now, $\phi = \delta_{j_q} \circ \dots \circ \delta_{j_0}$ for $\{i_0,\dots,i_k\} \sqcup \{j_0,\dots,j_q\} = \{0,\dots,k\}$ with $j_0 < j_1 < \dots < j_q$, so that the left vertical map $\phi^\sharp_X$ maps $\phi^\sharp_X(\alpha) = d_{j_0} \circ \dots \circ d_{j_q}(\alpha) = \alpha(i_0,\dots,i_k)$. Then, $\mathcal{F}_k$ maps this to the $0 < \dots < k$–component $g^{\alpha(i_0,\dots,i_k)}_{0\dots k}$. By the commutativity of the diagram, we get that $g^\alpha_{i_0\dots i_k} = g^{\alpha(i_0,\dots,i_k)}_{0\dots k}$. This shows that the maps $g_\alpha := g^\alpha_{0\dots l}$ for all $\alpha \in X_l$ for $l \geq 1$ together with the implicit dg-vector spaces $\mathcal{E}_\alpha = \mathcal{F}_0(\alpha)$ for all 0–simplices $\alpha \in X_0$ give the complete data of the map of simplicial sets $\mathcal{F}: X \to \mathrm{dg}\,\mathcal{N}(\mathcal{C})$. Equation (2-1) for $g^\alpha_{0\dots l}$ using a fixed $\alpha \in X_l$ becomes precisely (2-4) via the identifications $g_\alpha = g^\alpha_{0\dots l}$ and $g^\alpha_{i_0\dots i_k} = g^{\alpha(i_0,\dots,i_k)}_{0\dots k}$.

Note moreover, by a similar argument, that degenerate simplices map to either the identity $g_{s_j(\alpha)} = \mathrm{id}_{E_\alpha}$ for $\alpha \in X_0$, or $g_{s_j(\alpha)} = 0$ for $\alpha \in X_l$ with $l \geq 1$.

Finally, $\mathcal{F}: X \to \mathrm{dg}\,\mathcal{N}(\mathrm{Perf}(U))^\circ$ lands in $\mathrm{dg}\,\mathcal{N}(\mathrm{Perf}(U))^\circ$ precisely if all maps g_α given by $\mathcal{F}(\alpha)$ for $\alpha \in X_1$ are homotopy equivalences by Lemma 2.4. □

Example 2.8 Let $\widehat{\Delta}\colon \mathbf{\Delta} \to \mathrm{sSet}$ be given as follows. Let $\widehat{\Delta}^n \in \mathrm{sSet}$ be the nerve of the category $E\mathbb{Z}_{n+1}^{\mathrm{Cat}}$, whose objects are elements of $\mathbb{Z}_{n+1} = \mathbb{Z}/(n+1)\mathbb{Z}$, and which has exactly one morphism between any two objects. More explicitly, $\widehat{\Delta}^n = E\mathbb{Z}_{n+1} = \mathcal{N}(E\mathbb{Z}_{n+1}^{\mathrm{Cat}})$ has k–simplices given by a sequence of k composable morphisms $[\![i_0]\!] \to [\![i_1]\!] \to \cdots \to [\![i_k]\!]$ where $[\![i_0]\!], \dots, [\![i_k]\!] \in \mathbb{Z}_{n+1}$, or, more concisely, the k–simplices $\widehat{\Delta}^n_k$ are sequences $(i_0, \dots, i_k)$ of indices $i_0, \dots, i_k \in \{0, \dots, n\}$, ie $\widehat{\Delta}^n_k \cong \{0, \dots, n\}^k$. Face maps $d_j\colon \widehat{\Delta}^n_k \to \widehat{\Delta}^n_{k-1}$ remove the j^{th} index i_j, and degeneracies $s_j\colon \widehat{\Delta}^n_k \to \widehat{\Delta}^n_{k+1}$ repeat the j^{th} index i_j. For example, for the simplicial set $\widehat{\Delta}^1$ a k–simplex consists of a sequence $(i_0, \dots, i_k)$ of 0's and 1's; a k–simplex is degenerate if and only if any two adjacent indices are equal, $i_j = i_{j+1}$; thus there are exactly two nondegenerate k–simplices: $(0,1,0,1,\dots)$ and $(1,0,1,0,\dots)$ for any k. The geometric realization of $\widehat{\Delta}^1$ is thus S^∞.

By Lemma 2.7, any simplicial set map $\widehat{\Delta}^n \to \mathrm{dg}\,\mathcal{N}(\mathrm{Perf}(U))$ is given by $n+1$ holomorphic dg-vector bundles with holomorphic connections $\mathcal{E}_0, \dots, \mathcal{E}_n$ together with maps $g_{i_0 \dots i_k}\colon E_{i_k} \to E_{i_0}$ for a nondegenerate k–simplex $\alpha = (i_0, \dots, i_k) \in \widehat{\Delta}^n_k = \{0, \dots, n\}^k$ without directly repeating indices, satisfying (2-4):

$$\begin{aligned} \text{(2-5)} \quad g_{i_0 \dots i_k} \circ d + (-1)^k \cdot d \circ g_{i_0 \dots i_k} &= D(g_{i_0 \dots i_k}) \\ &= \sum_{j=1}^{k-1} (-1)^{j-1} g_{i_0 \dots \hat{i}_j \dots i_k} + \sum_{j=1}^{k-1} (-1)^{k(j-1)+1} g_{i_0 \dots i_j} \circ g_{i_j \dots i_k}. \end{aligned}$$

Note furthermore that, for a *degenerate* simplex $(i_0, \dots, i_k)$ of $\widehat{\Delta}^n$ where the two consecutive indices $i_j = i_{j+1}$ are equal, we also have a map $g_{jj} = \mathrm{id}_{E_j}$ or $g_{i_0 \dots jj \dots i_k} = 0$ when $k \geq 2$ satisfying (2-5).

For a morphism $\phi\colon [n] \to [m]$ in $\mathbf{\Delta}$ we get an induced map of simplicial sets $\phi_\bullet\colon \widehat{\Delta}^n_\bullet \to \widehat{\Delta}^m_\bullet$ by mapping $\phi_k\colon \widehat{\Delta}^n_k \to \widehat{\Delta}^m_k$, $\phi_k(i_0, \dots, i_k) = (\phi(i_0), \dots, \phi(i_k))$. This gives the cosimplicial simplicial set $\widehat{\Delta}$. In particular, we can use Definition 2.5 to get the simplicial set $\mathbf{Perf}^{\widehat{\Delta}}(U)$, whose n–simplices are precisely $\mathbf{Perf}^{\widehat{\Delta}}(U)_n = \mathrm{sSet}\big(\widehat{\Delta}^n, \mathrm{dg}\,\mathcal{N}(\mathrm{Perf}(U))^\circ\big)$, ie simplicial set maps from $\widehat{\Delta}^n$ to $\mathrm{dg}\,\mathcal{N}(\mathrm{Perf}(U))^\circ$, which were described explicitly in the previous paragraph.

We note that, for the simplicial presheaf $\mathbf{Perf}^{\widehat{\Delta}}$, the "maximal Kan" condition follows automatically.

Lemma 2.9 *Simplicial set maps from $\widehat{\Delta}^n$ to $\mathrm{dg}\,\mathcal{N}(\mathrm{Perf}(U))$ take values in its maximal Kan subsimplex, ie*

$$\text{(2-6)} \qquad \mathbf{Perf}^{\widehat{\Delta}}(U)_n = \mathrm{sSet}\big(\widehat{\Delta}^n, \mathrm{dg}\,\mathcal{N}(\mathrm{Perf}(U))^\circ\big) = \mathrm{sSet}\big(\widehat{\Delta}^n, \mathrm{dg}\,\mathcal{N}(\mathrm{Perf}(U))\big).$$

Proof Any edge $g_{i_0 i_1}$ is automatically a homotopy equivalence with chain homotopy inverse $g_{i_1 i_0}$, since we have the homotopies $g_{i_0 i_1 i_0} \circ d + d \circ g_{i_0 i_1 i_0} = g_{i_0 i_0} - g_{i_0 i_1} \circ g_{i_1 i_0} = \mathrm{id}_{E_{i_0}} - g_{i_0 i_1} \circ g_{i_1 i_0}$ and $g_{i_1 i_0 i_1} \circ d + d \circ g_{i_1 i_0 i_1} = g_{i_1 i_1} - g_{i_1 i_0} \circ g_{i_0 i_1} = \mathrm{id}_{E_{i_1}} - g_{i_1 i_0} \circ g_{i_0 i_1}$. The claim follows from Lemma 2.4. □

Note that there is a map of cosimplicial simplicial sets $\Delta \to \widehat{\Delta}$, given by $\Delta^n_k \to \widehat{\Delta}^n_k$, $\Delta^n_k = \mathbf{\Delta}([k],[n]) \ni \phi \mapsto (i_0, \dots, i_k) := (\phi(0), \dots, \phi(k)) \in \widehat{\Delta}^n_k$. We thus get an induced map of simplicial sets $\mathbf{Perf}^{\widehat{\Delta}}(U) \to \mathbf{Perf}^{\Delta}(U)$.

Proposition 2.10 *For an object $U \in \mathbb{C}\mathrm{Man}$, the map of simplicial sets $\mathbf{Perf}^{\widehat{\Delta}}(U) \to \mathbf{Perf}^{\Delta}(U)$ is a weak equivalence.*

Proof Since $\mathrm{dg}\mathcal{N}(\mathrm{Perf}(U))^\circ$ is (by definition) a Kan complex, and by Definition 2.5 both $\mathbf{Perf}^{\widehat{\Delta}}(U) := \mathrm{sSet}\big(\widehat{\Delta}^\bullet, \mathrm{dg}\mathcal{N}(\mathrm{Perf}(U))^\circ\big)$ and $\mathbf{Perf}^{\Delta}(U) = \mathrm{sSet}\big(\Delta^\bullet, \mathrm{dg}\mathcal{N}(\mathrm{Perf}(U))^\circ\big)$, the proposition follows from Proposition A.1. □

In the later sections we mainly use $\mathbf{Perf}^{Q}$ for $Q = \widehat{\Delta}$, and we therefore make the following definition:

Definition 2.11 Denote by $\mathbf{IVB} := \mathbf{Perf}^{\widehat{\Delta}} : \mathbb{C}\mathrm{Man}^{\mathrm{op}} \to \mathrm{sSet}$, ie by (2-3),

$$\mathbf{IVB}(U)_n = \mathbf{Perf}^{\widehat{\Delta}}(U)_n = \mathrm{sSet}\big(\widehat{\Delta}^n, \mathrm{dg}\mathcal{N}(\mathrm{Perf}(U))^\circ\big). \tag{2-7}$$

For a motivation of this notation, see Definition 4.4.

The reason why we want to consider the cosimplicial simplicial set $\widehat{\Delta}$ is that it has an important additional cyclic structure which Δ is lacking, as we will explain now.

Definition 2.12 Let $\mathbf{\Delta} C$ be the cyclic category; see [Loday 1992, 6.1.1]. More precisely, $\mathbf{\Delta} C$ has the same objects $[n] = \{0, \dots, n\}$ for $n \in \mathbb{N}_0$ as $\mathbf{\Delta}$, and has morphisms generated by face maps δ_j and degeneracy maps σ_j (as in $\mathbf{\Delta}$; see Notation 1.1), together with an additional cyclic operator $\tau_n : [n] \to [n]$; see [Loday 1992, 6.1.1] for more details. It is convenient to represent morphisms $\phi \in \mathbf{\Delta} C([k], [n])$ by set maps $\phi : [k] \to [n]$ such that there exists a nondecreasing function $\tilde{\phi} : \{0, \dots, k\} \to \mathbb{N}_0$ satisfying $\tilde{\phi}(k) \le \tilde{\phi}(0) + n$ and $\phi(j) \equiv \tilde{\phi}(j) \pmod{\mathbb{Z}_n}$.

Then a cyclic object in a category $\mathcal{C}$ is a functor $X : \mathbf{\Delta} C^{\mathrm{op}} \to \mathcal{C}$. Since $\mathbf{\Delta} C \cong \mathbf{\Delta} C^{\mathrm{op}}$ are isomorphic [Loday 1992, 6.1.11], cyclic objects in $\mathcal{C}$ are cocyclic objects in $\mathcal{C}$ and vice versa. We denote the category of cyclic sets $X : \mathbf{\Delta} C \to \mathrm{Set}$ by cSet. Note that there is functor $\mathbf{\Delta} \to \mathbf{\Delta} C$, which makes every cyclic object into a simplicial object by precomposition $(\mathbf{\Delta} C^{\mathrm{op}} \xrightarrow{X} \mathcal{C}) \mapsto (\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{\Delta} C^{\mathrm{op}} \xrightarrow{X} \mathcal{C})$, and similarly every cocyclic object is a cosimplicial object. In particular, every cosimplicial cyclic set is a cosimplicial simplicial set.

Remark 2.13 The canonical cyclic sets $\Delta C^n := \mathbf{\Delta} C(-, [n])$ assemble for various n to a cocyclic cyclic set $\Delta C^\bullet : \mathbf{\Delta} C \to \mathrm{cSet}$. In particular, this is also a cosimplicial cyclic set $\mathbf{\Delta} \to \mathbf{\Delta} C \xrightarrow{\Delta C^\bullet} \mathrm{cSet}$, so that we also have a third example of a simplicial presheaf $\mathbf{Perf}^{\Delta C}$ using our setup from Definition 2.5. By Lemma 2.7, an n–simplex in $\mathbf{Perf}^{\Delta C}(U)$ is given by maps $g_{i_0 \dots i_k}$ for any "cyclic set of indices" $i_0 = \phi(0), \dots, i_k = \phi(k)$ for some $\phi \in \mathbf{\Delta} C([k], [n])$ (for example, for $n = 9$ we would have maps such as $g_{457034} : E_4 \to E_4$). Unfortunately, the analog of Proposition 2.10 does not hold, ie $\mathbf{Perf}^{\Delta C}(U)$ and $\mathbf{Perf}^{\Delta}(U)$ are in general not weakly equivalent. (For example, the nondegenerate simplices of $\mathbf{\Delta} C^1$ as sequences of indices are (0), (1), $(0,1)$, $(1,0)$, $(0,1,0)$, $(1,0,1)$ but no higher ones due to cyclicity, so that the geometric realization of $\mathbf{\Delta} C^1$ is the 2–sphere S^2.)

Now, while Δ^n is not a cyclic set, $\widehat{\Delta}^n$ is a cyclic set, and we will need to use the additional cyclic structure of $\widehat{\Delta}$ below to define our Chern character map.

Lemma 2.14 *The simplicial set $\widehat{\Delta}^n$ as described in the first paragraph of Example 2.8, together with the operator $t_k\colon \widehat{\Delta}^n_k \to \widehat{\Delta}^n_k$ given by $t_k(i_0,\dots,i_{k-1},i_k) = (i_k, i_0,\dots,i_{k-1})$, makes $\widehat{\Delta}^n$ into a cyclic set. This, in turn, makes $\widehat{\Delta}$ into a cosimplicial cyclic set.*

Proof One checks that t_k has the correct compatibility (see [Loday 1992, 6.1.2(b)–(c)]) with the face and degeneracy maps d_j and s_j. For a morphism $\phi\colon [n] \to [m]$ in $\mathbf{\Delta}$, the induced map of simplicial sets $\phi_\bullet\colon \widehat{\Delta}^n_\bullet \to \widehat{\Delta}^m_\bullet$, $\phi_k\colon \widehat{\Delta}^n_k \to \widehat{\Delta}^m_k$, $\phi_k(i_0,\dots,i_k) = (\phi(i_0),\dots,\phi(i_k))$, respects not only the face and degeneracy maps, but also the t_k operator, ie $\widehat{\Delta}\colon \mathbf{\Delta} \to \mathrm{cSet}$ is a cosimplicial cyclic set. □

We have thus defined the simplicial presheaf $\mathbf{IVB} = \mathbf{Perf}^{\widehat{\Delta}}$, which will be the domain of our Chern character map for holomorphic dg-vector bundles over U with connection. As for the range of the Chern character map, we use the same presheaf $\mathbf{\Omega}$ that we used in our previous work [2022, Definition 2.3] (for the Chern character map of holomorphic vector bundles that were not differential graded). For completeness sake, we will briefly review the definition of $\mathbf{\Omega}\colon \mathbb{C}\mathrm{Man}^{\mathrm{op}} \to \mathrm{sSet}$.

Definition 2.15 For an object $U \in \mathbb{C}\mathrm{Man}$, consider the (nonnegatively graded) cochain complex of holomorphic forms $\Omega^\bullet_{\mathrm{hol}}(U)$ on U with zero differential $d = 0$. Let u be a formal variable of degree $|u| = -2$, denote by $\Omega^\bullet_{\mathrm{hol}}(U)[u]$ polynomials in u, and by $\Omega^\bullet_{\mathrm{hol}}(U)[u]^{\bullet\le 0}$ its quotient by its positive degree part $\Omega^\bullet_{\mathrm{hol}}(U)[u]^{\bullet>0}$. Applying the Dold–Kan functor to this chain complex gives a simplicial abelian group $\mathrm{DK}(\Omega^\bullet_{\mathrm{hol}}(U)[u]^{\bullet\le 0})$, for which we consider its underlying simplicial set, denoted by an underline, ie $\mathbf{\Omega}(U) = \underline{\mathrm{DK}}(\Omega^\bullet_{\mathrm{hol}}(U)[u]^{\bullet\le 0})$:

$$\mathbf{\Omega}\colon \mathbb{C}\mathrm{Man}^{\mathrm{op}} \xrightarrow{\Omega^\bullet_{\mathrm{hol}}(-)[u]^{\bullet\le 0}} \mathrm{Ch}^{\le 0} \xrightarrow{\underline{\mathrm{DK}}} \mathrm{sSet}.$$

Since holomorphic forms pull back via a holomorphic map $\varphi\colon U \to U'$, this assignment defines a simplicial presheaf $\mathbf{\Omega}\colon \mathbb{C}\mathrm{Man}^{\mathrm{op}} \to \mathrm{sSet}$ by $\mathbf{\Omega} := \underline{\mathrm{DK}}(\Omega^\bullet_{\mathrm{hol}}(\,\cdot\,)[u]^{\bullet\le 0})\colon \mathbb{C}\mathrm{Man}^{\mathrm{op}} \to \mathrm{sSet}$.

Note 2.16 If $C = (C^{\bullet\le 0}, d_C)$ is a nonpositively graded chain complex, then the Dold–Kan functor $\mathrm{DK}(C) \in \mathrm{Ab}^{\mathbf{\Delta}^{\mathrm{op}}}$, which is a simplicial abelian group, can be described as follows; see our previous work [2022, Appendix B]. For $n \ge 0$, we may define $\mathrm{DK}(C)_n$ to be the abelian group (under addition) of cochain maps from the normalized cells of the standard simplex Δ^n to C, ie we may set

$$\mathrm{DK}(C)_n := \mathrm{Chain}(N(\mathbb{Z}\Delta^n), C). \tag{2-8}$$

Thus, this means that an element of $\mathrm{DK}(C)_n$ is given by a labeling of the nondegenerate cells of the standard simplex Δ^n by elements of C in such a way that, for a k–cell α of Δ^n whose boundary $(k-1)$–cells are $d_j(\alpha)$, we have

$$d_C(\alpha) = \sum_{j=0}^{k} (-1)^j \cdot d_j(\alpha). \tag{2-9}$$

In the situation of Definition 2.15, the chain complex $C = \Omega^\bullet_{\mathrm{hol}}(U)[u]^{\bullet\le 0}$ has a zero internal differential, ie $d_C = 0$.

3 Chern character $\mathbf{Ch}\colon \mathbf{IVB} \to \mathbf{\Omega}$

We now define a map of simplicial presheaves $\mathbf{Ch}\colon \mathbf{IVB} \to \mathbf{\Omega}$, where $\mathbf{IVB} = \mathbf{Perf}^{\widehat{\Delta}}$ from Definition 2.11 and $\mathbf{\Omega}$ is from Definition 2.15. We start by defining cochains on a simplicial set X with values in a dg-category $\mathcal{C}$ (for us $\mathcal{C} = \mathrm{Perf}(U)$), and, in the case when X is a cyclic set, its trace map. The main example to keep in mind for the following definitions is the cyclic set $X = \widehat{\Delta}^n$.

Definition 3.1 A *labeling* of a simplicial set X by a dg-category $\mathcal{C}$ is a set map from the vertices of X to the objects of $\mathcal{C}$, $L\colon X_0 \to \mathrm{Obj}(\mathcal{C})$, ie a choice of an object $\mathcal{E}_\alpha := L(\alpha)$ of $\mathcal{C}$ for each $\alpha \in X_0$.

Definition 3.2 Let X be a simplicial set, let $\mathcal{C}$ be dg-category, and let $L\colon X_0 \to \mathrm{Obj}(\mathcal{C})$ be a labeling such that we have a choice of objects $\mathcal{E}_\alpha$ for each $\alpha \in X_0$. We define the *cochains on X with values in $\mathcal{C}$* to be

$$C_L^\bullet(X,\mathcal{C}) := \prod_{p\geq 1}\prod_{\alpha\in X_p} \mathrm{Hom}_{\mathcal{C}}^\bullet(\mathcal{E}_{\alpha(p)}, \mathcal{E}_{\alpha(0)}), \tag{3-1}$$

where we used notation from Lemma 2.7 to denote the first and last vertices of $\alpha \in X_p$ by $\alpha(0)$ and $\alpha(p)$, respectively. In components, we will write $f \in C_L^\bullet(X,\mathcal{C})$ as $f = \{f_\alpha\}_{\alpha\in X}$, where, for $\alpha \in X_p$ and $p \geq 1$, we have $f_\alpha \in \mathrm{Hom}_{\mathcal{C}}^\bullet(\mathcal{E}_{\alpha(p)}, \mathcal{E}_{\alpha(0)})$.

Note that $C_L^\bullet(X,\mathcal{C})$ is a dg-algebra:

(1) A cochain f of bidegree (p,q) assigns to a p–cell $\alpha \in X_p$ a degree q map $f_\alpha \in \mathrm{Hom}_{\mathcal{C}}^q(\mathcal{E}_{\alpha(p)}, \mathcal{E}_{\alpha(0)})$, and is zero elsewhere; in this case the total degree of f is $|f| = p+q$.

(2) A differential $\hat{\delta}\colon C_L^p(X,\mathcal{C}) \to C_L^{p+1}(X,\mathcal{C})$ is induced by the face maps $d_i : X^{p+1} \to X^p$, so that if $\alpha \in X_{p+1}$ is a $(p+1)$-simplex of X, then the deleted Čech differential of f, denoted by $\hat{\delta} f$, is defined by

$$(\hat{\delta} f)_\alpha := \sum_{i=1}^{p}(-1)^i f_{d_i(\alpha)} = \sum_{i=1}^{p}(-1)^i f_{\alpha(0,\dots,\hat{\imath},\dots,p+1)}. \tag{3-2}$$

Note that d_0 and d_{p+1} are not used in the differential, which ensures the terms in the sum are all maps in $\mathrm{Hom}_{\mathcal{C}}^q(\mathcal{E}_{\alpha(p+1)}, \mathcal{E}_{\alpha(0)})$.

(3) An internal differential $D\colon C_L^\bullet(X,\mathcal{C}) \to C_L^\bullet(X,\mathcal{C})$ is induced by the dg structure on $\mathcal{C}$, so that, if $\alpha \in X_p$ is a p–simplex and $f_\alpha \in \mathrm{Hom}_{\mathcal{C}}^q(\mathcal{E}_{\alpha(p)}, \mathcal{E}_{\alpha(0)})$, then $(Df)_\alpha := (-1)^{p+q+1}\cdot D(f_\alpha) = (-1)^p\cdot(d\circ f_\alpha - (-1)^q\cdot f_\alpha\circ d)$ as a homomorphism in $\mathrm{Hom}^{q+1}(\mathcal{E}_{\alpha(p)}, \mathcal{E}_{\alpha(0)})$.

(4) There is a product $f\cdot g$ on $C_L^\bullet(X,\mathcal{C})$, which, for $\alpha \in X_{p+r}$ is the extension of the maps $\mathrm{Hom}_{\mathcal{C}}^q(\mathcal{E}_{\alpha(p)}, \mathcal{E}_{\alpha(0)}) \times \mathrm{Hom}_{\mathcal{C}}^s(\mathcal{E}_{\alpha(p+r)}, \mathcal{E}_{\alpha(p)}) \to \mathrm{Hom}_{\mathcal{C}}^q(\mathcal{E}_{\alpha(p+r)}, \mathcal{E}_{\alpha(0)})$,

$$(f_{\alpha(0,\dots,p)}, g_{\alpha(p,\dots,p+r)}) \mapsto (f\cdot g)_{\alpha(0,\dots,p+r)} := (-1)^{q\cdot r}\cdot f_{\alpha(0,\dots,p)}\circ g_{\alpha(p,\dots,p+r)}, \tag{3-3}$$

on the components of $C_L^\bullet(X,\mathcal{C})$ to all of $C_L^\bullet(X,\mathcal{C})$.

We note that, in particular, $Df = d\cdot f - (-1)^{|f|} f\cdot d =: [d,f]$. It is well known (and straightforward to check) that with these definitions the cochains on X with values in $\mathcal{C}$, $C_L^\bullet(X,\mathcal{C})$, becomes a dg-algebra.

Definition 3.3 Given a simplicial set X, a dg-category $\mathcal{C}$ and a labeling L, we say an element $g \in C^\bullet_L(X,\mathcal{C})$ is a *Maurer–Cartan element* if

$$\hat{\delta} g + Dg + g \cdot g = 0 \tag{3-4}$$

Definition 3.4 Let $X_\bullet$ be a simplicial set, and let $\mathcal{C}$ be a dg-category. Then, by Lemma 2.7, a simplicial set map $\mathcal{F}\colon X \to \mathrm{dg}\mathcal{N}(\mathcal{C})$ induces objects $\mathcal{E}_\alpha$ for each 0–simplex $\alpha \in X_0$, and maps $g_\alpha : \mathcal{E}_{\alpha(p)} \to \mathcal{E}_{\alpha(0)}$ for every $\alpha \in X_p$ with $p \geq 1$ (for degenerate simplices, we take $g_\alpha = \mathrm{id}_{\mathcal{E}_{\alpha(0)}}$ when $\alpha \in X_1$, and $g_\alpha = 0$ when $\alpha \in X_p$ for $p \geq 2$). Thus, we can define a labeling $L := \mathcal{F}_0\colon X_0 \to \mathrm{dg}\mathcal{N}(\mathcal{C})_0 = \mathrm{Obj}(\mathcal{C})$ of X by $\mathcal{C}$ via $L(\alpha) := \mathcal{E}_\alpha$ for $\alpha \in X_0$. Moreover, the g_α for $\alpha \in X_p$ for $p \geq 1$, assemble to an element $g = \{g_\alpha\}_{\alpha \in X} \in C^\bullet_L(X,\mathcal{C})$.

Corollary 3.5 *The element $g \in C^\bullet_L(X,\mathcal{C})$ from Definition 3.4 is a Maurer–Cartan element, ie g satisfies* (3-4). *Moreover, g has components of bidegree $(p, 1-p)$ for $p \geq 1$, so that g is of total degree* 1.

Proof Each g_α for $\alpha \in X_p$ is of bidegree $(p, 1-p)$; see Lemma 2.7(2). For $\alpha \in X_{p+r}$ with $p, r \geq 1$, we have $g_{\alpha(0,\dots,p)} \cdot g_{\alpha(p,\dots,p+r)} = (-1)^{(1-p)(r-p)} g_{\alpha(0,\dots,p)} \circ g_{\alpha(p,\dots,p+r)}$, and since $(-1)^{(1-p)(r-p)} = (-1)^{(r+p)(p-1)}$, we see that (3-4) becomes exactly (2-4). □

Now consider the case $\mathcal{C} = \mathrm{Perf}(U)$. In this case, $C^\bullet_L(X,\mathcal{C})$ becomes a direct product of holomorphic sections, ie

$$C^\bullet_L(X,\mathrm{Perf}(U)) = \prod_{p\geq 1} \prod_{\alpha \in X_p} \Gamma_{\mathrm{hol}}(U, \mathrm{Hom}(E_{\alpha(p)}, E_{\alpha(0)})),$$

since morphisms $\mathrm{Hom}_{\mathcal{C}}(\mathcal{E}_1, \mathcal{E}_2)$, which are bundle maps, are in correspondence with holomorphic sections of the $\mathrm{Hom}(E_1, E_2)$–bundle. Since we want to include higher holomorphic forms as well, we will include this dg-algebra in a larger dg-algebra of all holomorphic forms $C^\bullet_L(X,\mathrm{Perf}(U)) \hookrightarrow C^\bullet_L(X, \Omega(U) \mathbin{\hat{\otimes}} \mathrm{Perf}(U))$, defined as follows.

Definition 3.6 Let X be a simplicial set and consider the dg-category $\mathrm{Perf}(U)$. Let $L : X_0 \to \mathrm{Obj}(\mathrm{Perf}(U))$ be a labeling as in Definition 3.2, ie $\mathcal{E}_\alpha = L(\alpha)$. We define the dg-algebra

$$C^\bullet_L(X, \Omega(U) \mathbin{\hat{\otimes}} \mathrm{Perf}(U)) := \prod_{p\geq 0} \prod_{\alpha \in X_p} \Omega^\bullet_{\mathrm{hol}}(U, \mathrm{Hom}^\bullet(E_{\alpha(p)}, E_{\alpha(0)})), \tag{3-5}$$

where we again denoted the first and last vertices of $\alpha \in X_p$ by $\alpha(0)$ and $\alpha(p)$, respectively. In components, we will write $f \in C^\bullet_L(X, \Omega(U) \mathbin{\hat{\otimes}} \mathrm{Perf}(U))$ as $f = \{f_\alpha\}_{\alpha \in X}$, where, for $\alpha \in X_p$, we have $f_\alpha \in \Omega^\bullet_{\mathrm{hol}}(U, \mathrm{Hom}^\bullet(E_{\alpha(p)}, E_{\alpha(0)}))$. Note that in (3-5) we included the 0–simplices ($p = 0$) when compared to (3-1).

The dg-algebra structure on $C^\bullet_L(X, \Omega(U) \mathbin{\hat{\otimes}} \mathrm{Perf}(U))$ is defined as follows:

(1) $f \in C^\bullet_L(X, \Omega(U)\hat{\otimes}\mathrm{Perf}(U))$ has triple degree (k,p,q) if it assigns to a p–cell $\alpha \in X_p$ a holomorphic k–form with values in the appropriate Hom-bundle of degree q, $f_\alpha \in \Omega^k_{\mathrm{hol}}(U, \mathrm{Hom}^q(E_{\alpha(p)}, E_{\alpha(0)}))$, and vanishes elsewhere; in this case the total degree of f is $|f| = k+p+q$.

(2) A differential $\hat{\delta}\colon C^\bullet_L(X, \Omega(U)\hat{\otimes}\mathrm{Perf}(U)) \to C^\bullet_L(X, \Omega(U)\hat{\otimes}\mathrm{Perf}(U))$, the deleted Čech differential, is defined just as in Definition 3.2(2), ie for $f \in C^\bullet_L(X, \Omega(U)\hat{\otimes}\mathrm{Perf}(U))$,

$$(\hat{\delta} f)_\alpha := \sum_{i=1}^{p} (-1)^i f_{d_i(\alpha)} = \sum_{i=1}^{p} (-1)^i f_{\alpha(0,\dots,\hat{\imath},\dots,p+1)}. \tag{3-6}$$

(3) A differential $D\colon C^\bullet_L(X, \Omega(U)\hat{\otimes}\mathrm{Perf}(U)) \to C^\bullet_L(X, \Omega(U)\hat{\otimes}\mathrm{Perf}(U))$, the internal differential, is defined similarly to Definition 3.2(3), ie if $f_\alpha \in \Omega^k_{\mathrm{hol}}(U, \mathrm{Hom}^q(E_{\alpha(p)}, E_{\alpha(0)}))$, then $(Df)_\alpha \in \Omega^k_{\mathrm{hol}}(U, \mathrm{Hom}^{q+1}(E_{\alpha(p)}, E_{\alpha(0)}))$,

$$(Df)_\alpha := (-1)^p \cdot (d_{\alpha(0)} \circ f_\alpha - (-1)^{k+q} \cdot f_\alpha \circ d_{\alpha(p)}),$$

where d_i denotes the differential of E_i.

(4) There is a product $f \cdot g$ similar to Definition 3.2(4). More explicitly, consider the maps

$$\begin{aligned} &\Omega^k_{\mathrm{hol}}(U, \mathrm{Hom}^q(\mathcal{E}_{\alpha(p)}, \mathcal{E}_{\alpha(0)})) \times \Omega^l_{\mathrm{hol}}(U, \mathrm{Hom}^s(\mathcal{E}_{\alpha(p+r)}, \mathcal{E}_{\alpha(p)})) \\ &\qquad\qquad \to \Omega^{k+l}_{\mathrm{hol}}(U, \mathrm{Hom}^q(\mathcal{E}_{\alpha(p+r)}, \mathcal{E}_{\alpha(0)})), \\ &(f_{\alpha(0,\dots,p)}, g_{\alpha(p,\dots,p+r)}) \mapsto (f\cdot g)_{\alpha(0,\dots,p+r)} := (-1)^{(k+q)\cdot r} \cdot f_{\alpha(0,\dots,p)} \circ g_{\alpha(p,\dots,p+r)}, \end{aligned} \tag{3-7}$$

where $\circ$ denotes wedging forms and composing Hom-spaces, and extend them from the components of $C^\bullet_L(X, \Omega(U)\hat{\otimes}\mathrm{Perf}(U))$ to the whole space.

We note that, again, $Df = d\cdot f - (-1)^{|f|} f \cdot d = [d, f]$. Just as in Definition 3.2, $C^\bullet_L(X, \Omega(U)\hat{\otimes}\mathrm{Perf}(U))$ becomes a dg-algebra, and the inclusion $C^\bullet_L(X, \mathrm{Perf}(U)) \hookrightarrow C^\bullet_L(X, \Omega(U)\hat{\otimes}\mathrm{Perf}(U))$ is a dg-algebra morphism. Note that this inclusion consists of two separate inclusions of holomorphic functions into holomorphic forms, $\Gamma_{\mathrm{hol}}(-) \hookrightarrow \Omega_{\mathrm{hol}}(-)$, as well as nonzero simplices into all simplices, $\prod_{p\geq 1}(-) \hookrightarrow \prod_{p\geq 0}(-)$. Note further, that $Df = d\cdot f - (-1)^{|f|} f\cdot d$, where $d = \{d_\alpha\}_{\alpha\in X} \in C^\bullet_L(X, \Omega(U)\hat{\otimes}\mathrm{Perf}(U))$ is given by the differentials $d_\alpha = d_{E_\alpha}$ for $\alpha \in X_0$ and $d_\alpha = 0$ for all other α.

Finally we remark that every Maurer–Cartan element in $C^\bullet_L(X, \mathrm{Perf}(U))$ is also a Maurer–Cartan element in the larger dg-algebra $C^\bullet_L(X, \Omega(U)\hat{\otimes}\mathrm{Perf}(U))$.

Now, for a vector bundle E, there is a trace map $\mathrm{tr}\colon \mathrm{Hom}(E,E) \to \mathbb{C}$. Following ideas of O'Brian, Toledo and Tong [1981c, page 238], we will define a trace map

$$\prod_{p\geq 0} \prod_{\alpha \in X_p} \Omega^\bullet_{\mathrm{hol}}(U, \mathrm{Hom}^\bullet(E_{\alpha(p)}, E_{\alpha(0)})) \to \prod_{p\geq 0} \prod_{\alpha\in X_p} \Omega^\bullet_{\mathrm{hol}}(U, \mathbb{C}).$$

Note that the left-hand side is $C^\bullet_L(X, \Omega(U)\hat{\otimes}\mathrm{Perf}(U))$. We denote the right-hand side by $C^\bullet(X, \Omega_{\mathrm{hol}}(U))$. To fit this into our current setting, we need an additional *cyclic* structure on X.

Definition 3.7 Let X be a cyclic set. Let $\alpha \in X_p$ be a p–simplex, ie by our convention $\alpha = \alpha(0, \dots, p)$, then, using the additional operator $\tau_p\colon [p] \to [p]$, we denote the induced map $t_p\colon X_p \to X_p$ by $\alpha(p, 0, \dots, p-1) := t_p(\alpha)$.

Now let $L\colon X_0 \to \mathrm{Obj}(\mathrm{Perf}(U))$ be a labeling, and let g be a Maurer–Cartan element of $C^\bullet_L(X, \mathrm{Perf}(U))$. Then we define the *trace* map

$$\mathrm{Tr}_g : C^\bullet_L(X, \Omega(U) \mathbin{\widehat{\otimes}} \mathrm{Perf}(U)) \to C^\bullet(X, \Omega_{\mathrm{hol}}(U)),$$
$$(\mathrm{Tr}_g(f))_{\alpha \in X_s} := \sum_{0 \le k \le l \le s} (-1)^{(k+1)\cdot s + l - k} \cdot \mathrm{tr}(g_{\alpha(l,\dots,s,0,\dots,k)} \circ f_{\alpha(k,\dots,l)}).$$

Note that the trace on the right makes sense, since it is applied to $\mathrm{Hom}(E_{\alpha(l)}, E_{\alpha(l)})$.

The following proposition follows the arguments from [loc. cit., Proposition 3.2]:

Proposition 3.8 *Let X be a cyclic set with labeling L, and let g be a Maurer–Cartan element in $C^\bullet_L(X, \mathrm{Perf}(U))$. Then the trace map Tr_g satisfies*

$$\mathrm{Tr}_g \circ (\hat{\delta} + D + [g, -]) = \delta \circ \mathrm{Tr}_g, \tag{3-8}$$

where δ is the (full) Čech differential including first and last term, ie $(\delta f)_\alpha := \sum_{j=0}^{p+1} (-1)^j f_{\alpha(0,\dots,\hat{j},\dots,p+1)}$ for $\alpha \in X_{p+1}$.

Proof Let $f \in C^\bullet_L(X, \Omega(U) \mathbin{\widehat{\otimes}} \mathrm{Perf}(U))$, and let $\alpha \in X_s$. Then

$$\big(\delta(\mathrm{Tr}_g(f))\big)_\alpha = \sum_{j=0}^{s} (-1)^j \cdot \mathrm{Tr}_g(f)_{\alpha(0,\dots,\hat{j},\dots,s)} = A + B + C$$

equals the sum of the three terms

$$A := \sum_{0 \le k \le l \le s} \sum_{j=k+1}^{l-1} (-1)^{j + (k+1)(s-1) + l - k - 1} \cdot \mathrm{tr}(g_{\alpha(l,\dots,s,0,\dots,k)} \circ f_{\alpha(k,\dots,\hat{j},\dots,l)}),$$

$$B := \sum_{0 \le k \le l \le s} \sum_{j=0}^{k-1} (-1)^{j + k(s-1) + l - k} \cdot \mathrm{tr}(g_{\alpha(l,\dots,s,0,\dots,\hat{j},\dots,k)} \circ f_{\alpha(k,\dots,l)}),$$

$$C := \sum_{0 \le k \le l \le s} \sum_{j=l+1}^{s} (-1)^{j + (k+1)(s-1) + l - k} \cdot \mathrm{tr}(g_{\alpha(l,\dots,\hat{j},\dots,s,0,\dots,k)} \circ f_{\alpha(k,\dots,l)}).$$

The first term A in the above sum is equal to

$$\begin{aligned} A &= \sum_{0 \le k \le l \le s} \sum_{j=k+1}^{l-1} (-1)^{j + (k+1)s + l} \cdot \mathrm{tr}(g_{\alpha(l,\dots,s,0,\dots,k)} \circ f_{\alpha(k,\dots,\hat{j},\dots,l)}) \\ &= \sum_{0 \le k \le l \le s} (-1)^{(k+1)s + l - k} \cdot \mathrm{tr}(g_{\alpha(l,\dots,s,0,\dots,k)} \circ (\hat{\delta} f)_{\alpha(k,\dots,l)}) = \big(\mathrm{Tr}_g(\hat{\delta}(f))\big)_\alpha. \end{aligned}$$

To evaluate $B+C$, note that

$$\sum_{0\le k\le l\le s}(-1)^{(k+1)s+1}\cdot \operatorname{tr}((\hat{\delta}(g))_{\alpha(l,\dots,s,0,\dots,k)}\circ f_{\alpha(k,\dots,l)}) \tag{3-9}$$
$$= \sum_{0\le k\le l\le s}\sum_{j=l+1}^{s}(-1)^{(k+1)s+1+j-l}\cdot \operatorname{tr}(g_{\alpha(l,\dots,\hat{j},\dots,s,0,\dots,k)}\circ f_{\alpha(k,\dots,l)})$$
$$+\sum_{0\le k\le l\le s}\sum_{j=0}^{k-1}(-1)^{(k+1)s+1+s-l+1+j}\cdot \operatorname{tr}(g_{\alpha(l,\dots,s,0,\dots,\hat{j},\dots,k)}\circ f_{\alpha(k,\dots,l)})$$
$$= C+B.$$

We claim that this is equal to $\big(\operatorname{Tr}_g(D(f)+[g,-](f))\big)_\alpha$, which we evaluate now. By Definition 3.6, we may write $D(f)=d\cdot f-(-1)^{|f|}f\cdot d=[d,f]$, where $|f|$ denotes the total degree of f. Thus, if we define $\tilde{g}:=d+g$, ie for $\alpha\in X_0$, $\tilde{g}_\alpha=d_\alpha$, and for $\alpha\in X_k$ with $k\ge 1$, $\tilde{g}_\alpha=g_\alpha$, then $D(f)+[g,-](f)=[d+g,f]=[\tilde{g},f]$. With this, we write $(\operatorname{Tr}_g([\tilde{g},f]))_\alpha=\big(\operatorname{Tr}_g(\tilde{g}\cdot f-(-1)^{|\tilde{g}|\cdot|f|}f\cdot\tilde{g})\big)_\alpha=E+F$, which are given as follows. First,

$$E:=\operatorname{Tr}_g(\tilde{g}\cdot f)_\alpha=\sum_{0\le j\le l\le s}(-1)^{(j+1)s+l-j}\cdot \operatorname{tr}(g_{\alpha(l,\dots,s,0,\dots,j)}\circ(\tilde{g}\cdot f)_{\alpha(j,\dots,l)})$$
$$=\sum_{0\le j\le k\le l\le s}(-1)^{(j+1)s+l-j+(1-k+j)(l-j)}\cdot \operatorname{tr}(g_{\alpha(l,\dots,s,0,\dots,j)}\circ\tilde{g}_{\alpha(j,\dots,k)}\circ f_{\alpha(k,\dots,l)}),$$

where we used that the (de Rham, Čech, Hom)–triple degree of $\tilde{g}_{\alpha(j,\dots,k)}$ is $(0,k-j,1-k+j)$. For the second term, we get

$$F:=\operatorname{Tr}_g(-(-1)^{|\tilde{g}|\cdot|f|}f\cdot\tilde{g})_\alpha$$
$$=\sum_{0\le k\le j\le s}(-1)^{|f|+1+(k+1)s+j-k}\cdot \operatorname{tr}(g_{\alpha(j,\dots,s,0,\dots,k)}\circ(f\cdot\tilde{g})_{\alpha(k,\dots,j)})$$
$$=\sum_{0\le k\le l\le j\le s}(-1)^{|f|+1+(k+1)s+j-k+(|f|-l+k)(j-l)}\cdot \operatorname{tr}(g_{\alpha(j,\dots,s,0,\dots,k)}\circ f_{\alpha(k,\dots,l)}\circ\tilde{g}_{\alpha(l,\dots,j)})$$
$$=\sum_{0\le k\le l\le j\le s}(-1)^{|f|+1+(k+1)s+j-k+(|f|-l+k)(j-l)+(|f|+1-1-s+j-l)(1-j+l)}\cdot \operatorname{tr}(\tilde{g}_{\alpha(l,\dots,j)}\circ g_{\alpha(j,\dots,s,0,\dots,k)}\circ f_{\alpha(k,\dots,l)}),$$

where we used that $\operatorname{tr}(h\circ k)=(-1)^{a\cdot b}\cdot\operatorname{tr}(k\circ h)$ when the (Hom-degree) + (de Rham degree) = (total degree) − (Čech degree) of h and k is a and b, respectively, and that the Čech-degree of any $h_{\alpha(j,\dots,s,0,\dots,l)}$ is $1+s-j+l$. With this, we obtain

$$\sum_{0\le k\le l\le s}(-1)^{(k+1)s}\cdot \operatorname{tr}((\tilde{g}\cdot\tilde{g})_{\alpha(l,\dots,s,0,\dots,k)}\circ f_{\alpha(k,\dots,l)}) \tag{3-10}$$
$$=\sum_{0\le k\le l\le j\le s}(-1)^{(k+1)s+(1-j+l)(1+s-j+k)}\cdot \operatorname{tr}(\tilde{g}_{\alpha(l,\dots,j)}\circ g_{\alpha(j,\dots,s,0,\dots,k)}\circ f_{\alpha(k,\dots,l)})$$
$$+\sum_{0\le j\le k\le l\le s}(-1)^{(k+1)s+(l-s-j)(k-j)}\cdot \operatorname{tr}(g_{\alpha(l,\dots,s,0,\dots,j)}\circ\tilde{g}_{\alpha(j,\dots,k)}\circ f_{\alpha(k,\dots,l)})$$
$$=F+E,$$

where we used that $\tilde{g} = d + g$ and $d \cdot d = 0$, and that the (de Rham, Čech, Hom)–triple degree of $g_{\alpha(l,\dots,s,0,\dots,j)}$ is $(0, 1+s-l+j, l-s-j)$. Comparing the left-hand sides of (3-9) and (3-10), and using that g is a Maurer–Cartan element, so that $\hat{\delta} g = -(Dg + g \cdot g) = -\tilde{g} \cdot \tilde{g}$, we obtain that

$$B + C = \text{(3-9)} = \text{(3-10)} = E + F = (\mathrm{Tr}_g([\tilde{g}, f]))_\alpha = \big(\mathrm{Tr}_g(D(f) + [g, -](f))\big)_\alpha. \qquad \square$$

Remark 3.9 The trace map of O'Brian, Toledo and Tong [1981c, Section 3] satisfies some additional properties which carry over to our trace map from Definition 3.7 . For example, following the algebraic proof from [loc. cit., Proposition 3.8], Tr_g vanishes on graded commutators: for a Maurer–Cartan element g and cocycles $u, v \in C^\bullet_L(X, \Omega(U) \mathbin{\hat{\otimes}} \mathrm{Perf}(U))$, $\mathrm{Tr}_g(u \cdot v)$ and $\mathrm{Tr}_g(v \cdot u)$ are cohomologous (up to sign) in $C^\bullet(X, \Omega_{\mathrm{hol}}(U))$.

We have one further structure on $C^\bullet_L(X, \Omega(U) \mathbin{\hat{\otimes}} \mathrm{Perf}(U))$ coming from the holomorphic connections ∇ of the objects $\mathcal{E}$ of $\mathrm{Perf}(U)$. Note that there is an induced connection on the Hom-bundle $\mathrm{Hom}^\bullet(E, E')$ of two graded bundles E and E' with connections, which we also denote by $\nabla \colon \Omega^\bullet_{\mathrm{hol}}(U, \mathrm{Hom}^\bullet(E, E')) \to \Omega^{\bullet+1}_{\mathrm{hol}}(U, \mathrm{Hom}^\bullet(E, E'))$, and which is a graded derivation with respect to the wedge composition $\circ$ using the total degree of $\Omega^\bullet_{\mathrm{hol}}(U, \mathrm{Hom}^\bullet(E, E'))$.

Definition 3.10 Define $\nabla \colon C^\bullet_L(X, \Omega(U) \mathbin{\hat{\otimes}} \mathrm{Perf}(U)) \to C^\bullet_L(X, \Omega(U) \mathbin{\hat{\otimes}} \mathrm{Perf}(U))$ to be given in components by the maps $(-1)^p \cdot \nabla \colon \Omega^k_{\mathrm{hol}}(U, \mathrm{Hom}^q(\mathcal{E}_{\alpha(p)}, \mathcal{E}_{\alpha(0)})) \to \Omega^{k+1}_{\mathrm{hol}}(U, \mathrm{Hom}^q(\mathcal{E}_{\alpha(p)}, \mathcal{E}_{\alpha(0)}))$. More explicitly, for $f \in C^\bullet_L(X, \Omega(U) \mathbin{\hat{\otimes}} \mathrm{Perf}(U))$, $f = \{f_\alpha\}_{\alpha \in X}$, we define $\nabla f = \{(\nabla f)_\alpha\}_{\alpha \in X}$ to be given by $(\nabla f)_\alpha := (-1)^p \cdot \nabla(f_\alpha)$ when $\alpha \in X_p$.

One can check that $\nabla \circ \hat{\delta} = -\hat{\delta} \circ \nabla$, and that $\nabla(f \cdot g) = \nabla(f) \cdot g + (-1)^{|f|} f \cdot \nabla(g)$, where $|f|$ is the total degree of the triple grading.

Definition 3.11 Let X be a cyclic set and let $\mathcal{F} \colon X \to \mathrm{dg}\mathcal{N}(\mathrm{Perf}(U))$ be a simplicial set map. By Definition 3.4, we get a labeling $L \colon X_0 \to \mathrm{Obj}(\mathrm{Perf}(U))$, and a Maurer–Cartan element

$$g \in C^\bullet_L(X, \mathrm{Perf}(U)) \hookrightarrow C^\bullet_L(X, \Omega(U) \mathbin{\hat{\otimes}} \mathrm{Perf}(U)).$$

For a vertex $\alpha \in X_0$, denote by d_{E_α} the internal differential of the chain complex of vector bundles $\mathcal{E}_\alpha$, out of which we build the element $d = \{d_\alpha\}_{\alpha \in X} \in C^\bullet_L(X, \Omega(U) \mathbin{\hat{\otimes}} \mathrm{Perf}(U))$, given by $d_\alpha := d_{E_\alpha}$, and which has triple degree $(0, 0, 1)$. Then $d + g \in C^\bullet_L(X, \Omega(U) \mathbin{\hat{\otimes}} \mathrm{Perf}(U))$, and we call

$$A := \nabla(d + g) \in C^\bullet_L(X, \Omega(U) \mathbin{\hat{\otimes}} \mathrm{Perf}(U))$$

the *Atiyah class*, which is concentrated in degrees $(1, k, 1-k)$ for $k \geq 0$.

Proposition 3.12 *We have* $(\hat{\delta} + D + [g, -])(A) = 0$, *and thus*

$$\delta(\mathrm{Tr}_g(A^k)) = 0 \quad \textit{for all } k \geq 0.$$

Proof We apply ∇ to the Maurer–Cartan equation (3-4), ie to $\hat{\delta}g + Dg + g\cdot g = 0$. Using $\nabla\hat{\delta}g = -\hat{\delta}\nabla g$, and $\nabla(g\cdot g) = \nabla g\cdot g - g\cdot\nabla g = -[g,\nabla g]$ together with

$$\nabla Dg = \nabla(d\cdot g + g\cdot d) = \nabla d\cdot g - d\cdot\nabla g + \nabla g\cdot d - g\cdot\nabla d = -D(\nabla g) - [g,\nabla d],$$

we obtain

$$0 = \nabla(\hat{\delta}g + Dg + g\cdot g) = -\hat{\delta}(\nabla g) - D(\nabla g) - [g,\nabla d] - [g,\nabla g] = -(\hat{\delta} + D + [g,-])(\nabla g + \nabla d).$$

In the last equality, we also used that $\hat{\delta}(\nabla d) = 0$ (since the deleted Čech differential vanishes on 0–simplices), and from $d^2 = 0$ it follows that $0 = \nabla(d\cdot d) = \nabla d\cdot d - d\cdot\nabla d = -D(\nabla d)$. This shows that, for $A = \nabla(d+g)$, we have $(\hat{\delta} + D + [g,-])(A) = 0$.

Since $(\hat{\delta} + D + [g,-])$ is a derivation on $C^\bullet_L(X, \Omega(U)\mathbin{\hat{\otimes}}\mathrm{Perf}(U))$, the k^{th} powers of a also satisfy $(\hat{\delta} + D + [g,-])(A^k) = 0$. Thus,

$$\delta(\mathrm{Tr}_g(A^k)) \overset{(3\text{-}8)}{=} \mathrm{Tr}_g((\hat{\delta} + D + [g,-])(A^k)) = 0. \qquad \square$$

We are now ready to define our Chern character map $\mathbf{Ch}\colon \mathbf{IVB}\to\mathbf{\Omega}$, which is a map of simplicial presheaves, as shown in Theorem 3.14 below.

Definition 3.13 We define the *Chern character* as a map $\mathbf{Ch}\colon\mathbf{IVB}\to\mathbf{\Omega}$; that is, for a complex manifold U and $k\geq 0$, we define a map $\mathbf{Ch}(U)_n\colon\mathbf{IVB}(U)_n\to\mathbf{\Omega}(U)_n$.

For an n–simplex $\mathcal{F}\in\mathbf{IVB}(U)_n = \mathrm{sSet}\big(\widehat{\Delta}^n, \mathrm{dg}\mathcal{N}(\mathrm{Perf}(U))\big)$, we have (by Definition 2.5 and Example 2.8) the data of $n+1$ dg-vector bundles $\mathcal{E}_0,\dots,\mathcal{E}_n$, and maps $g_{i_0\dots i_k}\colon E_{i_k}\to E_{i_0}$, so $g = \{g_{(i_0\dots i_k)}\}_{(i_0,\dots,i_k)\in\widehat{\Delta}^n}$ satisfies the Maurer–Cartan equation by Corollary 3.5. To this we associate $\mathbf{Ch}(U)_n(\mathcal{F})\in\mathbf{\Omega}(U)_n$, which is a labeling of the nondegenerate cells of Δ^n by elements in $\Omega^\bullet_{\mathrm{hol}}(U)[u]^{\bullet\leq 0}$ (by Definition 2.15 and Note 2.16). Consider a nondegenerate k–cell of Δ^n given by the vertices $i_0,\dots,i_k$ of Δ^n with $i_0<\dots<i_k$.

If $k=0$, then we assign the Euler characteristic $\chi(E_{i_0})$ to this cell. If $k>0$, then we use $\alpha = (i_0,\dots,i_k)\in\widehat{\Delta}^n_k$ to assign the following expression to this cell:

$$\mathrm{Tr}_g(A^k)_\alpha\cdot\frac{u^k}{k!} = \mathrm{Tr}_g\big((\nabla(d+g))^k\big)_\alpha\cdot\frac{u^k}{k!} = \sum \pm\mathrm{tr}(g\cdot\nabla(d+g)\cdot\nabla(d+g)\cdots\nabla(d+g))_\alpha\cdot\frac{u^k}{k!}. \tag{3-11}$$

For example, here are the assignments for simplicial degrees 0, 1 and 2:

***n* = 0** A 0–simplex $\mathcal{F}\in\mathbf{IVB}(U)_0$ is just the data of one object $\mathcal{E} = (E\to U,\nabla)$ of $\mathrm{Perf}(U)$. Then $\mathbf{Ch}(U)_0(\mathcal{F})$ is the labeling of the Δ^0 by Euler characteristic of $\mathcal{E}$, denoted by $\chi(E)\in\Omega^0_{\mathrm{hol}}(U)[u]^{\bullet\leq 0}$.

***n* = 1** A 1–simplex $\mathcal{F}\in\mathbf{IVB}(U)_1$ consists of bundles $\mathcal{E}_0$ and $\mathcal{E}_1$ and sequences of morphisms $g_{0101\dots}$ and $g_{1010\dots}$. Then $\mathbf{Ch}(U)_1(\mathcal{F})$ is the labeling of Δ^1 given by $\chi(\mathcal{E}_i)$ on the vertices of Δ^1, and on the edge of Δ^1 we place the labeling $\mathrm{Tr}_g(\nabla(d+g))_{(0,1)}\cdot u\in\Omega^1_{\mathrm{hol}}(U)[u]^{\bullet\leq 0}$, where $(0,1)\in\widehat{\Delta}^1$:

$$\overset{\chi(E_0)}{\bullet} \xleftarrow{\quad\mathrm{Tr}_g(\nabla(d+g))_{(0,1)}\cdot u\quad} \overset{\chi(E_1)}{\bullet}$$

Explicitly, the trace has terms (using $g_i = d_{E_i}$ for the internal differential of E_i)

$$\mathrm{Tr}_g(\nabla(d+g))_{(0,1)} = \mathrm{tr}(g_{101}\nabla g_1 - g_{010}\nabla g_0 + g_{10}\nabla g_{01})$$

***n* = 2** A 2–simplex $\mathcal{F} \in \mathbf{IVB}(U)_2$ consists of bundles $\mathcal{E}_0$, $\mathcal{E}_1$ and $\mathcal{E}_2$ and sequences of morphisms $g_{i_0 i_1 \dots i_p}$ for $p \geq 1$ and $i_l \in \{0,1,2\}$ for any $0 \leq l \leq p$. Then, $\mathbf{Ch}(U)_2(\mathcal{F})$ is the labeling of Δ^2 given by $\chi(\mathcal{E}_i) \in \Omega^0_{\mathrm{hol}}(U)[u]^{\bullet \leq 0}$ on the vertices, $\mathrm{Tr}_g(\nabla(d+g))_{(i,j)} \cdot u \in \Omega^1_{\mathrm{hol}}(U)[u]^{\bullet\leq 0}$ on the edge of Δ^1 we place the labeling $\mathrm{Tr}_g(\nabla(d+g)\cdot\nabla(d+g))_{(0,1,2)} \cdot u^2/2! \in \Omega^2_{\mathrm{hol}}(U)[u]^{\bullet\leq 0}$ on the nondegenerate 2–cell, where $(0,1,2) \in \widehat{\Delta}^2$:

$$\begin{tikzcd}
 & \chi(E_1) \arrow[dl, "\mathrm{Tr}_g(\nabla(d+g))_{(0,1)}\cdot u"'] & \\
\chi(E_0) & \mathrm{Tr}_g(\nabla(d+g)\cdot\nabla(d+g))_{(0,1,2)}\cdot u^2/2! & \chi(E_2) \arrow[ul, "\mathrm{Tr}_g(\nabla(d+g))_{(1,2)}\cdot u"'] \arrow[ll, "\mathrm{Tr}_g(\nabla(d+g))_{(0,2)}\cdot u"]
\end{tikzcd}$$

Explicitly, we have (again using $g_i = d_{E_i}$ for the internal differential of E_i)

$$\begin{aligned}
\mathrm{Tr}_g(\nabla(d+g)\cdot\nabla(d+g))_{(0,1,2)} = \mathrm{tr}&(g_{20}\nabla g_0 \nabla g_{012} + g_{20}\nabla g_{01}\nabla g_{12} + g_{20}\nabla g_{012}\nabla g_2) \\
&- \mathrm{tr}(g_{201}\nabla g_1 \nabla g_{12} + g_{201}\nabla g_{12}\nabla g_2) \\
&- \mathrm{tr}(g_{120}\nabla g_0 \nabla g_{01} + g_{120}\nabla g_{01}\nabla g_1) \\
&+ \mathrm{tr}(g_{2012}\nabla g_2 \nabla g_2 + g_{1201}\nabla g_1\nabla g_1 + g_{0120}\nabla g_0 \nabla g_0).
\end{aligned}$$

Theorem 3.14 *The Chern character* $\mathbf{Ch}\colon \mathbf{IVB} \to \mathbf{\Omega}$ *defined above is a map of simplicial presheaves.*

Proof We use the notation from Definition 3.13. First, we note that $\mathbf{Ch}(U)_n(\mathcal{F})$ is a well-defined element of $\mathbf{\Omega}(U)_n$, ie we still need to show that the labeling satisfies (2-9). Since the internal differential vanishes for $\Omega^\bullet_{\mathrm{hol}}(\cdot)[u]^{\bullet\leq 0}$, this amounts to showing that, for each p–cell given by $\alpha = (i_0, \dots, i_p)$, the sum of the labelings on the boundary cells vanishes. This follows since

$$\sum_{j=0}^{k} (-1)^j \cdot d_j\left((\mathrm{Tr}_g(A^k))_\alpha \cdot \frac{u^k}{k!}\right) = \sum_{j=0}^{k} (-1)^j \cdot (\mathrm{Tr}_g(A^k))_{\alpha(0,\dots,\hat{j},\dots,k)} \cdot \frac{u^k}{k!} = \delta\big((\mathrm{Tr}_g(A^k))_\alpha\big) \cdot \frac{u^k}{k!} = 0,$$

using Proposition 3.12 for the last equality. Next, we show that $\mathbf{Ch}(U)\colon \mathbf{IVB}(U) \to \mathbf{\Omega}(U)$ is a map of simplicial sets, ie that it respects the face and degeneracy maps. If $\delta_j : [n] \to [n+1]$ is the j^{th} face map, then $d_j : \mathbf{IVB}(U)_{n+1} \to \mathbf{IVB}(U)_n$ is given by precomposition with $\widehat{\Delta}^n \to \widehat{\Delta}^{n+1}$, $\{0,\dots,n\}^k \ni (i_0,\dots,i_k) \mapsto (\delta_j(i_0),\dots,\delta_j(i_k)) \in \{0,\dots,n+1\}^k$. Thus, for $\mathcal{F} \in \mathbf{IVB}(U)_{n+1}$ with corresponding Maurer–Cartan element g, we have $\mathbf{Ch}(U)_n \circ d_j(\mathcal{F})|_{\alpha=(i_0<\dots<i_k)} = \mathrm{Tr}_g(A^k)_{(\delta_j(i_0)<\dots<\delta_j(i_k))} \cdot u^k/k!$. This is equal to taking $\mathbf{Ch}(U)_{n+1}(\mathcal{F}) \in \mathrm{DK}(C)_{n+1} = \mathrm{Chain}(N(\mathbb{Z}\Delta^{n+1}), C)$, where $C = \Omega^\bullet_{\mathrm{hol}}(U)[u]^{\bullet\leq 0}$, after applying $d_j : \mathrm{DK}(C)_{n+1} \to \mathrm{DK}(C)_n$ to it, and looking at the labeling of the cell $i_0 < \dots < i_k$ of Δ^n. Similarly, if $\sigma_j : [n] \to [n-1]$ is the j^{th} degeneracy, and $s_j : \mathbf{IVB}(U)_{n-1} \to \mathbf{IVB}(U)_n$ is the induced map, then, for

$\mathcal{F} \in \mathbf{IVB}(U)_{n-1}$ with corresponding Maurer–Cartan element g, we get $\mathbf{Ch}(U)_n \circ s_j(\mathcal{F})|_{\alpha=(i_0<\cdots<i_k)} = \mathrm{Tr}_g(A^k)_{(\sigma_j(i_0)\le\cdots\le\sigma_j(i_k))} \cdot u^k/k!$. Now, if σ is injective on $\{i_0,\ldots,i_k\}$, then, by Note 2.2(5), this is equal to $\mathrm{Tr}_g(A^k)_{(\sigma_j(i_0)<\cdots<\sigma_j(i_k))} \cdot u^k/k!$, which is the labeling of $s_j \circ \mathbf{Ch}(U)_{n-1}(\mathcal{F})$ at $i_0 < \cdots < i_k$. In the case where σ_j is not injective on $\{i_0,\ldots,i_k\}$, we get that $g_{\sigma_j(i_0)\ldots\sigma_j(i_k)}$ is either the identity or zero, so, in either case, $\nabla g_{\sigma_j(i_0)\ldots\sigma_j(i_k)} = 0$, and thus $\mathbf{Ch}(U)_n \circ s_j(\mathcal{F})|_{\alpha=(i_0<\cdots<i_k)} = 0$, which is equal to the degeneracy $s_j : \mathrm{DK}(C)_{n-1} \to \mathrm{DK}(C)_n$ applied to $\mathbf{Ch}(U)_{n-1}(\mathcal{F})$ at the cell $i_0 < \cdots < i_k$.

Finally, we show that $\mathbf{Ch}\colon \mathbf{IVB} \to \mathbf{\Omega}$ is a map of simplicial presheaves, ie that under a holomorphic map $\varphi\colon U \to U'$, the following diagram commutes:

$$\begin{tikzcd}
\mathbf{IVB}(U') \arrow[r, "\mathbf{Ch}(U')"] \arrow[d, "\mathbf{IVB}(\varphi)"'] & \mathbf{\Omega}(U') \arrow[d, "\mathbf{\Omega}(\varphi)"] \\
\mathbf{IVB}(U) \arrow[r, "\mathbf{Ch}(U)"] & \mathbf{\Omega}(U)
\end{tikzcd}$$

This follows, since both compositions are given by pullback via φ, ie for $\mathcal{F}' \in \mathbf{IVB}(U')$ with induced Maurer–Cartan element g' and induced differential d' on E'_α, we have

$$\begin{aligned}
\mathbf{Ch}(U)_n \circ \mathbf{IVB}_n(\varphi)(\mathcal{F}')|_{\alpha=(i_0<\cdots<i_k)} &= \mathrm{Tr}_{\varphi^* g'}\big(\big((\varphi^*\nabla)(\varphi^*(d'+g'))\big)^k\big)_\alpha \cdot \frac{u^k}{k!} \\
&= \varphi^*\big(\mathrm{Tr}_{g'}\big((\nabla(d'+g'))^k\big)_\alpha\big) \cdot \frac{u^k}{k!} \\
&= \mathbf{\Omega}(\varphi)_n \circ \mathbf{Ch}(U')_n(\mathcal{F}')|_{\alpha=(i_0<\cdots<i_k)}.
\end{aligned}$$

□

4 A higher Chern character for coherent sheaves

In this section, we apply a construction, which we will call Čech sheafification, to the Chern character map $\mathbf{Ch}\colon \mathbf{IVB} \to \mathbf{\Omega}$ from Definition 3.13. More precisely, an endofunctor on simplicial presheaves $\boldsymbol{F} \mapsto \boldsymbol{F}^{\check{\dagger}}$ is defined as the colimit over all Čech covers of the totalization of the presheaf applied to the cover (see Definition 4.1), and then an explicit interpretation is offered for the induced map $\mathbf{Ch}^{\check{\dagger}}\colon \mathbf{IVB}^{\check{\dagger}} \to \mathbf{\Omega}^{\check{\dagger}}$. Theorem 4.9 states that 0–simplices of $\mathbf{IVB}^{\check{\dagger}}$ are twisting cochains (up to equivalence) in the sense of O'Brian, Toledo and Tong [1981c], and Theorem 4.18 states that the induced Chern character $\mathbf{Ch}^{\check{\dagger}}$ recovers the Chern character from [loc. cit.].

To fix some notation, let $(U_i \to X)_{i\in I}$ be an open cover, which is a particular diagram in $\mathbb{C}\mathrm{Man}$. To this cover we associate the augmented simplicial presheaf $\check{N}U_\bullet \to X$ whose p–simplices are coproducts of representable presheaves given by $(p+1)$–fold intersections of the cover,

$$\check{N}U_p = \coprod_{i_0,\ldots,i_p\in I} yU_{i_0,\ldots,i_p},$$

where yU denotes the Yoneda functor applied to U, ie $yU : \mathbb{C}\mathrm{Man}^{\mathrm{op}} \to \mathrm{Set}$, $V \mapsto \mathbb{C}\mathrm{Man}(V,U)$, interpreted as a constant simplicial set. Given another simplicial presheaf $\boldsymbol{F}$ we abuse notation by writing $\boldsymbol{F}(\check{N}U_\bullet)$

for the cosimplicial simplicial set with

$$\boldsymbol{F}(\check{N}U_\bullet)^l_p := \prod_{i_0,\dots,i_l} \boldsymbol{F}(U_{i_0,\dots,i_l})_p. \tag{4-1}$$

Definition 4.1 Given a simplicial presheaf $\boldsymbol{F} : \mathbb{C}\mathrm{Man}^{\mathrm{op}} \to \mathrm{sSet}$, define its *Čech sheafification* on a test manifold $X \in \mathbb{C}\mathrm{Man}$ to be the simplicial set given by

$$\boldsymbol{F}^{\check{\dagger}}(X) := \operatorname*{colim}_{(U_\bullet \to X) \in \check{S}} \mathrm{Tot}(\boldsymbol{F}(\check{N}U_\bullet)), \tag{4-2}$$

where $\check{S}$ is the category of all Čech covers, and Tot is the totalization, which is reviewed in Appendix B. (For further details about the totalization, see our previous paper [2022, Appendix D.1] and [Hirschhorn 2003, Definition 18.6.3]; specific examples of Tot are worked out in Note 4.5 below, as well as in our previous paper [2022, Proof of Proposition 3.16].)

While $\boldsymbol{F}^{\check{\dagger}}$ may not be a hypersheaf in general, Section 5 discusses the sheaf property and there the above definition is justified.

Proposition 4.2 *If $\boldsymbol{F}$ is a simplicial presheaf which takes values in Kan complexes, then its Čech sheafification is a Kan complex.*

Proof By Proposition C.1, for an open cover $U_\bullet$ of X, $\mathrm{Tot}(\mathbf{IVB}(\check{N}U_\bullet))$ is a Kan complex. Now, since our colimit over Čech covers is directed once we pass to simplicial presheaves $\check{N}U_\bullet$, one can check by hand that the colimit in $\mathbf{IVB}^{\check{\dagger}}(X)$ sends a diagram of projectively fibrant objects to a projectively fibrant object (ie $\mathbf{IVB}^{\check{\dagger}}$ takes values in Kan complexes). □

Definition 4.3 The *Čech sheafified Chern character map* $\mathbf{Ch}^{\check{\dagger}} : \mathbf{IVB}^{\check{\dagger}} \to \boldsymbol{\Omega}^{\check{\dagger}}$ is the map obtained by applying Čech sheafifications to the Chern character map from Definition 3.13.

4.1 Čech sheafification of IVB as twisting cochains

In this subsection, the vertices of the simplicial presheaf, $\mathbf{IVB}^{\check{\dagger}}$, are examined and shown in Theorem 4.9 to be precisely the twisting cochains of O'Brian, Toledo and Tong [1981c] up to equivalence. We thus define:

Definition 4.4 An *infinity vector bundle* over a complex manifold X is a 0–simplex of $\mathbf{IVB}^{\check{\dagger}}(X)$.

The following note looks at the k–simplices of $\mathbf{IVB}^{\check{\dagger}}(X)$ in general, before focusing more specifically on the 0–simplices:

Note 4.5 Fix a complex manifold X. Definition 4.1 applied to $\boldsymbol{F} = \mathbf{IVB}$ yields

$$\mathbf{IVB}^{\check{\dagger}}(X) = \operatorname*{colim}_{(U_\bullet \to X) \in \check{S}} \mathrm{Tot}(\mathbf{IVB}(\check{N}\mathcal{U}_\bullet)). \tag{4-3}$$

Now fix a Čech cover, $U_\bullet \to X$, and denote by $K_\bullet^\bullet$ the cosimplicial simplicial set whose l–cosimplices are given by

$$K^l := \mathbf{IVB}(\check{N}\mathcal{U}_l) = \mathbf{Perf}^{\widehat{\Delta}}(\check{N}\mathcal{U}_l).$$

Following (B-2), a k–simplex in $\mathrm{Tot}(K)$ consists of a collection $\{x^{(k,l)}\}_{l\geq 0}$ with

$$\begin{aligned} x^{(k,l)} \in \mathrm{sSet}(\Delta^k \times \Delta^l, K^l) &= \mathrm{sSet}(\Delta^k \times \Delta^l, \mathbf{Perf}^{\widehat{\Delta}}(\check{N}\mathcal{U}_l)) \\ &\overset{(2\text{-}3)}{=} \mathrm{sSet}\big(\Delta^k \times \Delta^l, \mathrm{sSet}\big(\widehat{\Delta}, \mathrm{dg}\,\mathcal{N}(\mathrm{Perf}(\check{N}\mathcal{U}_l))^\circ\big)\big) \\ &= \mathrm{sSet}\big(\operatorname*{colim}_{\Delta^p \to \Delta^k \times \Delta^l} \widehat{\Delta}^p, \mathrm{dg}\,\mathcal{N}(\mathrm{Perf}(\check{N}\mathcal{U}_l))^\circ\big), \end{aligned}$$

where in the last equality the calculation from (B-8) is used. Thus, according to Appendix B, page 4985, these are given by p–cells

$$x^{(k,l)}_{\left[\begin{smallmatrix}\alpha_0 & \cdots & \alpha_p \\ \beta_0 & \cdots & \beta_p\end{smallmatrix}\right]} \in \mathrm{dg}\,\mathcal{N}(\mathrm{Perf}(\check{N}\mathcal{U}_l))^\circ_p \tag{4-4}$$

for certain paths $\left[\begin{smallmatrix}\alpha_0 & \cdots & \alpha_p \\ \beta_0 & \cdots & \beta_p\end{smallmatrix}\right]$ in the $(k+1)\times(l+1)$ grid (ie for any path within the indices of a nondecreasing path). Given such a path $\left[\begin{smallmatrix}\alpha_0 & \cdots & \alpha_p \\ \beta_0 & \cdots & \beta_p\end{smallmatrix}\right]$, and a choice of a component $i_0,\dots,i_l \in \check{N}\mathcal{U}_l$ describing an $(l+1)$–fold intersection, the p–cell (4-4) decorates each index $\left[\begin{smallmatrix}\alpha_j \\ \beta_j\end{smallmatrix}\right]$ with a bundle-with-connection

$$E^{(k,l)}_{\left[\begin{smallmatrix}\alpha_0 & \cdots & \alpha_p \\ \beta_0 & \cdots & \beta_p\end{smallmatrix}\right];\left[\begin{smallmatrix}\alpha_j \\ \beta_j\end{smallmatrix}\right];i_0,\dots,i_l} \to U_{i_0,\dots,i_l},$$

and decorates subpaths $\left[\begin{smallmatrix}\tilde\alpha_0 & \cdots & \tilde\alpha_q \\ \tilde\beta_0 & \cdots & \tilde\beta_q\end{smallmatrix}\right]$ of these indices with maps between them. To be precise, before taking into account any simplicial or coherence conditions, the p–cell (4-4) is itself (by Example 2.8 and Lemma 2.9) given by the data

$$x^{(k,l)} = \{x^{(k,l)}_{\left[\begin{smallmatrix}\alpha_0 & \cdots & \alpha_p \\ \beta_0 & \cdots & \beta_p\end{smallmatrix}\right]}\}, \tag{4-5}$$

where we vary over the components $i_0,\dots,i_l$ of $\check{N}\mathcal{U}_l$ and

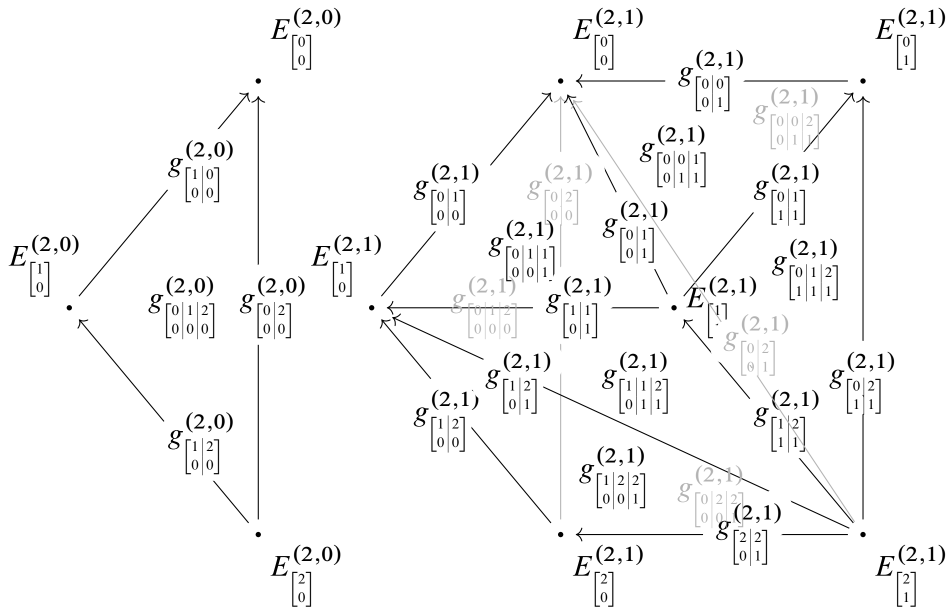
$$\begin{aligned} x^{(k,l)}_{\left[\begin{smallmatrix}\alpha_0 & \cdots & \alpha_p \\ \beta_0 & \cdots & \beta_p\end{smallmatrix}\right]} = \Big(& E^{(k,l)}_{\left[\begin{smallmatrix}\alpha_0 & \cdots & \alpha_p \\ \beta_0 & \cdots & \beta_p\end{smallmatrix}\right];\left[\begin{smallmatrix}\alpha_j \\ \beta_j\end{smallmatrix}\right];i_0,\dots,i_l} \to U_{i_0,\dots,i_l}, \nabla^{(k,l)}_{\left[\begin{smallmatrix}\alpha_0 & \cdots & \alpha_p \\ \beta_0 & \cdots & \beta_p\end{smallmatrix}\right];\left[\begin{smallmatrix}\alpha_j \\ \beta_j\end{smallmatrix}\right];i_0,\dots,i_l}, \\ & g^{(k,l)}_{\left[\begin{smallmatrix}\alpha_0 & \cdots & \alpha_p \\ \beta_0 & \cdots & \beta_p\end{smallmatrix}\right];\left[\begin{smallmatrix}\tilde\alpha_0 & \cdots & \tilde\alpha_q \\ \tilde\beta_0 & \cdots & \tilde\beta_q\end{smallmatrix}\right];i_0,\dots,i_l} : E^{(k,l)}_{\left[\begin{smallmatrix}\alpha_0 & \cdots & \alpha_p \\ \beta_0 & \cdots & \beta_p\end{smallmatrix}\right];\left[\begin{smallmatrix}\tilde\alpha_q \\ \tilde\beta_q\end{smallmatrix}\right];i_0,\dots,i_l} \to E^{(k,l)}_{\left[\begin{smallmatrix}\alpha_0 & \cdots & \alpha_p \\ \beta_0 & \cdots & \beta_p\end{smallmatrix}\right];\left[\begin{smallmatrix}\tilde\alpha_0 \\ \tilde\beta_0\end{smallmatrix}\right];i_0,\dots,i_l}\Big); \end{aligned}$$

here the g's are associated to any subsequence $\left[\begin{smallmatrix}\tilde\alpha_0 & \cdots & \tilde\alpha_q \\ \tilde\beta_0 & \cdots & \tilde\beta_q\end{smallmatrix}\right]$ for $q \geq 0$, of the indices from $\left[\begin{smallmatrix}\alpha_0 & \cdots & \alpha_p \\ \beta_0 & \cdots & \beta_p\end{smallmatrix}\right]$. Moreover, these g's satisfy the relations from (2-5). Since the simplices of $x^{(k,l)}$ fit together via the simplicial set relations, the above data (4-5) does not depend on the chosen p–cell determined by $\left[\begin{smallmatrix}\alpha_0 & \cdots & \alpha_p \\ \beta_0 & \cdots & \beta_p\end{smallmatrix}\right]$, and thus $x^{(k,l)}$ is given by the data

$$x^{(k,l)} = \Big(E^{(k,l)}_{\left[\begin{smallmatrix}\alpha \\ \beta\end{smallmatrix}\right];i_0,\dots,i_l} \to U_{i_0,\dots,i_l}, \nabla^{(k,l)}_{\left[\begin{smallmatrix}\alpha \\ \beta\end{smallmatrix}\right];i_0,\dots,i_l}, g^{(k,l)}_{\left[\begin{smallmatrix}\tilde\alpha_0 \cdots \tilde\alpha_q \\ \tilde\beta_0 \cdots \tilde\beta_q\end{smallmatrix}\right];i_0,\dots,i_l} : E^{(k,l)}_{\left[\begin{smallmatrix}\tilde\alpha_q \\ \tilde\beta_q\end{smallmatrix}\right];i_0,\dots,i_l} \to E^{(k,l)}_{\left[\begin{smallmatrix}\tilde\alpha_0 \\ \tilde\beta_0\end{smallmatrix}\right];i_0,\dots,i_l}\Big). \tag{4-6}$$

For example, for $k = 2$ and $l = 0, 1$, some of this data is visualized below, where both the nablas and the open set indices $i_0, \dots, i_l$ are suppressed for better readability:

on U_{i_0} for each fixed i_0 on U_{i_0,i_1} for each fixed i_0 and i_1



Now, by the compatibility relations (B-7) in $\mathrm{Tot}(K)$, the data given by the right-hand side of (4-6) is determined by the lowest l for which a given set of indices $\left[\begin{smallmatrix}\tilde{\alpha}_0 & \cdots & \tilde{\alpha}_q\\ \tilde{\beta}_0 & \cdots & \tilde{\beta}_q\end{smallmatrix}\right]$ can be obtained via a face map. For example,

$$E^{(k,l+1)}_{\left[\begin{smallmatrix}\alpha\\ \delta_j(\beta)\end{smallmatrix}\right];i_0,\dots,i_{l+1}} \overset{\text{(B-7)}}{=} (\text{component of } d^j(x^{(k,l)})) = E^{(k,l)}_{\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right];i_0,\dots,\hat{i}_j,\dots,i_{l+1}}|_{U_{i_0,\dots,i_{l+1}}},$$

where d^j acts by pulling back a bundle to a subset (by Definitions 2.5 and 2.1), ie by restricting the vector bundle to this subset. In particular,

$$E^{(k,l)}_{\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right];i_0,\dots,i_l} = E^{(k,0)}_{\left[\begin{smallmatrix}\alpha\\ 0\end{smallmatrix}\right];i_\beta}|_{U_{i_0,\dots,i_l}},$$

and similar statements apply to the g's.

Thus, the data of a k–simplex in $\mathrm{Tot}(K)$ is given by (suppressing the tildes)

(1) chain complexes of holomorphic vector bundles $E_{\alpha;i} := E^{(k,0)}_{\left[\begin{smallmatrix}\alpha\\ 0\end{smallmatrix}\right];i} \to U_i$ with differential $g_{\left[\begin{smallmatrix}\alpha\\ 0\end{smallmatrix}\right];i} = g^{(k,0)}_{\left[\begin{smallmatrix}\alpha\\ 0\end{smallmatrix}\right];i}$ for any index $\left[\begin{smallmatrix}\alpha\\ 0\end{smallmatrix}\right]$ on the $(k+1)\times(0+1)$ grid;

(2) connections $\nabla_{\alpha;i} := \nabla^{(k,0)}_{\left[\begin{smallmatrix}\alpha\\ 0\end{smallmatrix}\right];i}$ on $E_{\alpha;i}$;

(3) maps

$$g_{\left[\begin{smallmatrix}\alpha_0 & \cdots & \alpha_q\\ \beta_0 & \cdots & \beta_q\end{smallmatrix}\right];i_0,\dots,i_l} := g^{(k,l)}_{\left[\begin{smallmatrix}\alpha_0 & \cdots & \alpha_q\\ \beta_0 & \cdots & \beta_q\end{smallmatrix}\right];i_0,\dots,i_l} : E_{\alpha_q;i_{\beta_q}}|_{U_{i_0,\dots,i_l}} \to E_{\alpha_0;i_{\beta_0}}|_{U_{i_0,\dots,i_l}}$$

for $l \geq 1$ and for any β's which include all the indices from 0 to l, ie for $\{\beta_0, \ldots, \beta_q\} = \{0, \ldots, l\}$; this is because if there was a $j \in \{0, \ldots, l\}$ with $j \notin \{\beta_0, \ldots, \beta_q\}$, then the map $g^{(k,l)}_{\left[\begin{smallmatrix}\alpha_0 & \cdots & \alpha_q \\ \beta_0 & \cdots & \beta_q\end{smallmatrix}\right]; i_0, \ldots, i_j, \ldots, i_l}$ would, according to (B-7), just be the restriction

$$g^{(k,l-1)}_{\left[\begin{smallmatrix}\alpha_0 & \cdots & \alpha_q \\ \gamma_0 & \cdots & \gamma_q\end{smallmatrix}\right]; i_0, \ldots, \hat{i}_j, \ldots, i_l}\Big|_{U_{i_0, \ldots, i_j, \ldots, i_l}},$$

where $\beta_i = \delta_j(\gamma_i)$ for all i, and so the data could be recovered from the map $g^{(k,l-1)}_{\left[\begin{smallmatrix}\alpha_0 & \cdots & \alpha_q \\ \gamma_0 & \cdots & \gamma_q\end{smallmatrix}\right]; i_0, \ldots, \hat{i}_j, \ldots, i_l}$ via restriction.

Of course, as before, the sequence of indices $\left[\begin{smallmatrix}\alpha_0 & \cdots & \alpha_q \\ \beta_0 & \cdots & \beta_q\end{smallmatrix}\right]$ has to come from a nondecreasing set of indices on a $(k+1) \times (l+1)$ grid (see Section B.3). Sometimes we simply write $g_{\left[\begin{smallmatrix}\alpha_0 & \cdots & \alpha_q \\ \beta_0 & \cdots & \beta_q\end{smallmatrix}\right]}$ when the context of the open set $U_{i_0, \ldots, i_l}$ is clear.

In particular note that:

- Using the fact that we land in the maximal Kan subcomplex $\mathrm{dg}\mathcal{N}(\mathrm{Perf}(U))^\circ$ of $\mathrm{dg}\mathcal{N}(\mathrm{Perf}(U))$, for $q = 1$, the maps on 1–cells $g_{\left[\begin{smallmatrix}\alpha_0 & \alpha_1 \\ \beta_0 & \beta_1\end{smallmatrix}\right]; i_0, i_1}$ are all quasi-isomorphisms.
- Finally, these maps satisfy the relations from (2-5) on $U_{i_0, \ldots, i_l}$:

$$\begin{aligned} g_{\left[\begin{smallmatrix}\alpha_0 & \cdots & \alpha_q \\ \beta_0 & \cdots & \beta_q\end{smallmatrix}\right]} \circ g_{\left[\begin{smallmatrix}\alpha_q \\ \beta_q\end{smallmatrix}\right]} + (-1)^q \cdot g_{\left[\begin{smallmatrix}\alpha_0 \\ \beta_0\end{smallmatrix}\right]} \circ g_{\left[\begin{smallmatrix}\alpha_0 & \cdots & \alpha_q \\ \beta_0 & \cdots & \beta_q\end{smallmatrix}\right]} \\ = \sum_{j=1}^{q-1} (-1)^{j-1} g_{\left[\begin{smallmatrix}\alpha_0 & \cdots & \hat{\alpha}_j & \cdots & \alpha_q \\ \beta_0 & \cdots & \hat{\beta}_j & \cdots & \beta_q\end{smallmatrix}\right]} + \sum_{j=1}^{q-1} (-1)^{q(j-1)+1} g_{\left[\begin{smallmatrix}\alpha_0 & \cdots & \alpha_j \\ \beta_0 & \cdots & \beta_j\end{smallmatrix}\right]} \circ g_{\left[\begin{smallmatrix}\alpha_j & \cdots & \alpha_q \\ \beta_j & \cdots & \beta_q\end{smallmatrix}\right]}. \end{aligned} \tag{4-7}$$

The above note is applied below to the case of 0–simplices, in order to relate them to twisting cochains defined by O'Brian, Toledo and Tong [1981c, Definition 1.3], which we now briefly review.

Note 4.6 Let $(U_i \to X)_{i \in I}$ be a given cover, and let $E_i^\bullet \to U_i$ be graded holomorphic vector bundles over U_i. Then, according to [loc. cit.], a is a *twisting cochain* if $a = \sum_{j \geq 0} a^{j,1-j}$ with $a^{j,1-j} \in C^j(\mathcal{U}, \mathrm{Hom}^{1-j}(E, E))$, which is given by a collection of bundle morphisms on intersections of open sets, $a^{j,1-j} = \{a_{i_0, \ldots, i_j} : E_{i_j}|_{U_{i_0, \ldots, i_j}} \to E_{i_0}|_{U_{i_0, \ldots, i_j}}\}_{i_0, \ldots, i_j \in I}$, satisfying conditions [loc. cit., (1.5)] on each $U_{i_0, \ldots, i_q}$

$$\sum_{j=1}^{q-1} (-1)^j a_{i_0, \ldots, \hat{i}_j, \ldots, i_q} + \sum_{j=0}^{q} (-1)^{(1-j)(q-j)} a_{i_0, \ldots, i_j} \circ a_{i_j, \ldots, i_q} = 0. \tag{4-8}$$

Note that, compared to the data of a k–simplex in $\check{\mathbf{IVB}}^\dagger(X)$ (see Note 4.5(1)–(3)), there is a priori no chosen connection. A version of $\check{\mathbf{IVB}}^\dagger(X)$ is also provided then without connection. Recall from (2-7) that $\mathbf{IVB}(U)_n = \mathrm{sSet}\big(\hat{\Delta}^n, \mathrm{dg}\mathcal{N}(\mathrm{Perf}(U))^\circ\big)$.

Definition 4.7 Define $\widetilde{\mathrm{Perf}} \colon \mathbb{C}\mathrm{Man}^{\mathrm{op}} \to \mathrm{dgCat}$ by setting $\widetilde{\mathrm{Perf}}(U)$ to be the dg-category of finite chain complexes of holomorphic vector bundles, just as in Definition 2.1, but with the difference that we do not choose any connection on $E_\bullet$. Analogously to $\mathbf{IVB}$ from Definition 2.11, define $\widetilde{\mathbf{IVB}}\colon \mathbb{C}\mathrm{Man}^{\mathrm{op}} \to \mathrm{sSet}$ by setting $\widetilde{\mathbf{IVB}}(U)_n := \mathrm{sSet}\big(\widehat{\Delta}^n, \mathrm{dg}\mathcal{N}(\widetilde{\mathrm{Perf}}(U))^\circ\big)$.

For a Čech cover $(U_\bullet \to X)$, Note 4.5 can be repeated to obtain an explicit description of $\mathrm{Tot}(\widetilde{\mathbf{IVB}}(\check{N}\mathcal{U}_\bullet))$. Indeed the data of a k–simplex of $\mathrm{Tot}(\widetilde{\mathbf{IVB}}(\check{N}\mathcal{U}_\bullet))$ is given by the data of chain complexes of holomorphic vector bundles $E_{\alpha;i}$ as in (1) together with maps $g_{\left[\begin{smallmatrix}\alpha_0 \\ \beta_0\end{smallmatrix}\middle|\begin{smallmatrix}\cdots \\ \cdots\end{smallmatrix}\middle|\begin{smallmatrix}\alpha_q \\ \beta_q\end{smallmatrix}\right];i_0,\dots,i_l}$ as in (3), but *without* any connections as stated in (2).

The following lemma relates the above definition to the one with connections:

Lemma 4.8 *The dg-functor* $\mathrm{Perf} \to \widetilde{\mathrm{Perf}}$ *that forgets the connection induces a map of simplicial presheaves* $\mathbf{IVB} \to \widetilde{\mathbf{IVB}}$, *which after applying the Čech sheafification* (*Definition 4.1*) *yields an isomorphism of simplicial sets* $\mathbf{IVB}^{\check{\dagger}}(X) \xrightarrow{\cong} \widetilde{\mathbf{IVB}}^{\check{\dagger}}(X)$.

Proof For a fixed cover $(U_\bullet \to X)$, the forgetful map $\mathrm{Tot}(\mathbf{IVB}(\check{N}\mathcal{U}_\bullet)) \to \mathrm{Tot}(\widetilde{\mathbf{IVB}}(\check{N}\mathcal{U}_\bullet))$ forgets the information of the connections as stated in (2) in Note 4.5. Taking colimit over covers, this descends to a well-defined map $\mathbf{IVB}^{\check{\dagger}}(X) \to \widetilde{\mathbf{IVB}}^{\check{\dagger}}(X)$ which is surjective, since every complex manifold has a (Stein) open cover such that, for every open set of the cover, there exists a connection on the corresponding bundles.

It remains to check injectivity. Assume that two k–simplices $x, x' \in \mathbf{IVB}^{\check{\dagger}}(X)_k$ are mapped, respectively, to $\tilde{x}, \tilde{x}' \in \widetilde{\mathbf{IVB}}^{\check{\dagger}}(X)_k$ by forgetting the connections, and that these are equal, ie $\tilde{x} = \tilde{x}'$. This means that there is a zigzag of refinements and extensions with respect to the colimit over covers which connects $\tilde{x}$ and $\tilde{x}'$ in $\widetilde{\mathbf{IVB}}^{\check{\dagger}}(X)_k$. Since every k–simplex in $\widetilde{\mathbf{IVB}}^{\check{\dagger}}(X)$ has a refinement which is in the image of $\mathbf{IVB}^{\check{\dagger}}(X)$ under the forgetful functor, (ie it has a choice of connections on the bundles for each open set,) it is enough to consider the case where $\tilde{x}$ and $\tilde{x}'$ are both refinements of $\tilde{y} \in \widetilde{\mathbf{IVB}}^{\check{\dagger}}(X)_k$, where $\tilde{y}$ may *not* be in the image of the forgetful functor. In order to prove injectivity, it is enough to show that there exists a $\tilde{z}$ for which both $\tilde{x}$ and $\tilde{x}'$ are refinements, and which is in the image of the forgetful functor, so that taking a preimage z of $\tilde{z}$ shows that x and x' are equal in $\mathbf{IVB}^{\check{\dagger}}(X)_k$. To this end, note that, if x and x' are represented on fixed covers $U_\bullet$ and $U'_\bullet$, respectively. Then we define $\tilde{z}$ represented on the cover $U_\bullet \sqcup U'_\bullet$ as follows. To define the bundle data (1) for $\tilde{z}$, if V is an open set in the cover $U_\bullet$ or $U'_\bullet$ pick the bundle for that open set from $\tilde{x}$ or $\tilde{x}'$, respectively, which we note to be equal to bundles from $\tilde{y}$ appropriately restricted. To define the maps g from (3) for $\tilde{z}$, if $V_1, \dots, V_l$ are open sets from $U_\bullet \sqcup U'_\bullet$, we have bundles over V_i coming from the data $\tilde{y}$, and so we take the maps of bundles as provided by $\tilde{y}$. Note that $\tilde{x}$ and $\tilde{x}'$ both extend $\tilde{z}$, and, moreover, $\tilde{z}$ is in the image of the forgetful functor by the extension z of x and x', since there are connections on each of the bundles coming from the data (2) provided by x and x'. □

With this definition, the main theorem of this section is stated below.

Theorem 4.9 *The equivalence classes of O'Brian, Toledo and Tong* [1981c] *of twisting cochains inject into the vertices of* $\mathbf{IVB}^{\check{\dagger}}(X)$.

Proof By Lemma 4.8, we may forget about the connections, and simply inject twisting cochains into vertices of $\widetilde{\mathbf{IVB}}^{\check{\dagger}}(X)$. By Note 4.6, a twisting cochain on a cover $(U_i \to X)_{i\in I}$ with holomorphic vector bundles $E_i^\bullet \to U_i$ is given by a collection $a = \{a_{i_0,\dots,i_j}\}_{i_0,\dots,i_j\in I, j\geq 0}$ satisfying (4-8). To this, we assign the data of a 0–simplex in $\mathbf{IVB}^{\check{\dagger}}(X)$ as stated in (1) and (3) from page 4962 as follows. First, the $E_{0;i} \to U_i$ from (1) are just the given E_i. As for the g's in (1) and (3), define

$$g^{(k,l)}_{\left[\begin{smallmatrix}0 & | \cdots | & 0\\ \beta_0 & | \cdots | & \beta_q\end{smallmatrix}\right];i_0,\dots,i_l} := a_{i_{\beta_0},\dots,i_{\beta_q}}. \tag{4-9}$$

Note that the twisting cochain equations (4-8) imply (4-7). Moreover, the equivalence of twisting cochains is generated by refinements and extensions (see [loc. cit., page 232, above Proposition 1.10]), which identifies the corresponding infinity vector bundles (due to the colimit in (4-3)).

To check injectivity, we give a map in the opposite direction, which is a left-inverse to the above map. Explicitly, for a 0–simplex in $\widetilde{\mathbf{IVB}}^{\check{\dagger}}(X)$ represented by a cover $(U_i \to X)_i$ and bundles $E_i^\bullet$ with maps g as in (1) and (3), we define the twisting cochain

$$a_{i_0,\dots,i_j} := g^{(k,l)}_{\left[\begin{smallmatrix}0 & | & 0 & | \cdots | & 0\\ 0 & | & 1 & | \cdots | & j\end{smallmatrix}\right];i_0,\dots,i_j}, \tag{4-10}$$

which preserves the twisting cochain equations (4-8) due to (4-7). The colimit construction implies equivalence of twisting cochains. The composition of these two constructions, which maps twisting cochains to $\widetilde{\mathbf{IVB}}^{\check{\dagger}}(X)_0$ via (4-9) and then back to twisting cochains via (4-10), is the identity on twisting cochains.

As a final remark, we note that there are different (nonequivalent) choices for a left-inverse other than (4-10). In fact, equation (4-9) assigns the *same* homotopy $a_{j_0,\dots,j_q}$ to any

$$g_{\left[\begin{smallmatrix}\alpha_0 & | \cdots | & \alpha_q\\ \beta_0 & | \cdots | & \beta_q\end{smallmatrix}\right];i_0,\dots,i_l} \quad \text{with } i_{\beta_0} = j_0, \dots, i_{\beta_q} = j_q, \tag{4-11}$$

while in $\widetilde{\mathbf{IVB}}^{\check{\dagger}}(X)_0$ these maps (4-11) may generally be different. Therefore, any choice (consistent within the Maurer–Cartan equation (4-7)) may thus be used as a left-inverse for (4-9). □

To end this subsection, consider the restriction of the simplicial presheaf **IVB** to the one which only utilizes chain complexes of vector bundles whose homology is concentrated in degree zero. Below we show that the associated simplicial presheaf contains (after sheafification) all of the data of isomorphism classes of coherent sheaves in its vertices.

Note 4.10 For the reader's convenience, we review here a construction from [Toledo and Tong 1978a, Section 2]. Let $X \in \mathbb{C}\mathrm{Man}$ and $a_\bullet$ be a twisting cochain for a cover $(U_\bullet \to X)$ with holomorphic vector bundles $E_{\bullet}^{\bullet}$ (see O'Brian, Toledo and Tong [1981c] or Note 4.6 above). Consider the locally defined sheaf of $\mathcal{O}_X$–modules, $\mathcal{H}_i := H_{\bullet}(\Gamma(E_i), a_i)$, given by the homology of sections of $E_i^{\bullet}$ with differential a_i over U_i. Since each $a_{i,j}$ gives a quasi-isomorphism on the level of complexes, there is an induced isomorphism of sheaves on homology $a_{i,j}: U_{i,j}|_{\mathcal{H}_j} \xrightarrow{\sim} U_{i,j}|_{\mathcal{H}_i}$. Taking the colimit[2] of the $\mathcal{H}_i$ over the diagram induced by these a_{ij} produces a sheaf on X which we will call *the homology sheaf* and denote by $\mathcal{H}$. This construction further produces a map[3] of simplicial presheaves

$$\check{\mathbf{IVB}}^{\dagger} \xrightarrow{\mathcal{H}} \mathcal{N}(\mathrm{Sh}\mathcal{O}^{\bullet}), \tag{4-12}$$

where $\mathcal{N}$ denotes the nerve, and $\mathrm{Sh}\mathcal{O}^{\bullet}$ is the category of sheaves of graded $\mathcal{O}_X$–modules (without differential) with morphisms given by isomorphisms. The relevance of this construction to coherent sheaves is recorded in the following definition and proposition.

Definition 4.11 The simplicial presheaf $\mathbf{CohSh} \hookrightarrow \mathbf{IVB}$ is the subsimplicial presheaf defined by considering the full subpresheaf of dg-categories, $\mathrm{Perf}_{\mathrm{coh}} \hookrightarrow \mathrm{Perf}$ utilizing only chain complexes of bundles whose homology is concentrated in degree zero and then taking $\mathbf{CohSh}(X)_n := \mathrm{sSet}\big(\widehat{\Delta}^n, \mathrm{dg}\,\mathcal{N}(\mathrm{Perf}_{\mathrm{coh}}(U))^{\circ}\big)$.

Lemma 4.12 *Given a manifold M and a coherent sheaf $\mathcal{F}$, there exists an open cover by relatively compact Stein open submanifolds on which $\mathcal{F}$ is locally resolved by a chain complex of vector bundles.*

Proof M admits a cover $\{U_i\}_{i\in I}$ by Stein open subsets. For each Stein submanifold U_i, it admits an open cover by relatively compact open sets $\{V_{i,j}\}_{i\in I, j\in J_i}$. Now, for each relatively compact open submanifold $V_{i,j}$, we cover it one final step further by open Stein sets $W_{i,j,k}$. As each $W_{i,j,k}$ is a subset of a relatively compact open Stein manifold U_i, then, by [Field 1982, Theorem 7.2.6], $\mathcal{F}$ admits a resolution by vector bundles on $W_{i,j,k}$. □

Proposition 4.13 *The set of isomorphism classes of coherent sheaves on X is in bijective correspondence with the connected components of $\check{\mathbf{CohSh}}^{\dagger}(X)$.*

Proof Recall the map $\mathcal{H}: \check{\mathbf{IVB}}^{\dagger}(X) \to \mathcal{N}(\mathrm{Sh}\,\mathcal{O}_X^{\bullet})$ from Note 4.10. But, since $\mathbf{CohSh}$ requires the local chain complex's homology to be concentrated in degree zero, the map's image lands in $\mathcal{N}(\mathrm{Sh}\,\mathcal{O}_X) \hookrightarrow \mathcal{N}(\mathrm{Sh}\,\mathcal{O}_X^{\bullet})$, where $\mathcal{N}(\mathrm{Sh}\,\mathcal{O}_X)$ is the nerve of the category of sheaves of $\mathcal{O}_X$–modules (concentrated in degree 0). Since the image of our map is precisely an $\mathcal{O}_X$ which satisfies the properties of a coherent sheaf, then the map factors through the nerve of the groupoid of coherent sheaves with isomorphisms, $\mathcal{H}: \check{\mathbf{CohSh}}^{\dagger}(X) \to \mathcal{N}(\mathrm{CohSh}\,\mathcal{O}_X) \hookrightarrow \mathcal{N}(\mathrm{Sh}\,\mathcal{O}_X)$ which in turn is well defined as a map which sends connected components of $\check{\mathbf{CohSh}}^{\dagger}$ to connected components of $\mathcal{N}(\mathrm{CohSh}\,\mathcal{O}_X)$, ie precisely the isomorphism classes of $\mathrm{CohSh}\,\mathcal{O}_X$.

[2]Here we mean the concrete set-theoretic colimit given by a coproduct of $\mathcal{H}_i$ and then mod out by the equivalence generated by $a_{i,j}$ on $U_{i,j}$.

[3]Which, importantly, is *not* coming from a map of complexes or even graded modules.

To observe injectivity, we consider the image of two vertices $x, y \in \mathbf{CohSh}^{\check{\dagger}}(X)_0$, represented by cocycle data on some common refinement by a Stein cover, $(U_\bullet \to X)$, whose images $\mathcal{H}(x), \mathcal{H}(y) \in \mathcal{N}(\mathrm{CohSh}\,\mathcal{O}_X)$ are connected by an edge. In particular, this means that the global homology sheaves for x and y are isomorphic as $\mathcal{O}$–modules. In order to construct an edge $z \in \mathbf{CohSh}^{\check{\dagger}}(X)_1$ connecting x and y, we first need local quasi-isomorphisms connecting the local resolutions for the chain complexes of bundles x and y, respectively. These maps are given by recalling that these complexes over a Stein space are projective resolutions [Forstnerič 2011, Corollary 2.4.5] and so maps on homology induce chain maps between the complexes [Hilton and Stammbach 1971, Theorem 4.1]. So far, these quasi-isomorphisms produce the edge data for z on U_i, and the 1–skeleton of the edge data for z on higher intersections. To move up to the 2–skeleton, say on $U_{i,j}$, we see that we now have two quasi-isomorphisms between the complexes for x and y: one restricted from the quasi-isomorphism over U_i and the other from U_j. Again appealing to [Hilton and Stammbach 1971, Theorem 4.1] we now know these two quasi-isomorphisms are chain-homotopic and this provides all of the data for z on U_i's, $U_{i,j}$'s, and the 2–skeleton of the data on higher intersections. Now, by O'Brian, Toledo and Tong [1981c, Lemma 1.6] and the ensuing discussion there, one uses an inductive argument for how our higher homotopies of z would be constructed to satisfy the Maurer–Cartan equation and since their constructions include into ours (see our proof of Theorem 4.9), one indeed can construct an edge z connecting x and y to prove injectivity.

For surjectivity, applying Lemma 4.12 and then following [Toledo and Tong 1978a, Propsoition 2.4], for a coherent sheaf $\mathcal{F}$ there exists a Stein open cover $(U_i \hookrightarrow X)_{i \in I}$, so we can choose a twisting cochain class in $\mathbf{CohSh}^{\check{\dagger}}(X)_0$ by locally/projectively resolving the coherent sheaf by a complex of vector bundles, coherent on intersections $U_{i,j}$ up to quasi-isomorphisms, and further coherent on $U_{i_0,\dots,i_p}$ by higher homotopies which again exist by virtue of Lemma 1.6 of O'Brian, Toledo and Tong [1981c] and the discussion which follows it. It follows that the map $\mathcal{H}$ is surjective on connected components since in the proof of Theorem 4.9, we show how their constructions include into ours. □

4.2 Čech sheafification of the Chern map Ch

This section continues the study of the Čech sheafified Chern character map $\mathbf{Ch}^{\check{\dagger}} : \mathbf{IVB}^{\check{\dagger}} \to \mathbf{\Omega}^{\check{\dagger}}$ (where $\boldsymbol{F}^{\check{\dagger}}(X) = \mathrm{colim}_{(U_\bullet \to X) \in \breve{\mathcal{S}}} \mathrm{Tot}(\boldsymbol{F}(\check{N} U_\bullet))$ was defined in (4-2)). In Theorem 4.9 twisting cochains à la [loc. cit.] were already interpreted as 0–simplices of $\mathbf{IVB}^{\check{\dagger}}$. Next, in Note 4.16, $\mathbf{\Omega}^{\check{\dagger}}$ is explicitly described as well as the map $\mathbf{Ch}^{\check{\dagger}}$ for the case of 0–simplices. Comparing the formulas for the Čech sheafified Chern character map $\mathbf{Ch}^{\check{\dagger}}$ with the Chern character map from [loc. cit.] for a coherent sheaf (which is reviewed in 4.17), shows, that these are given by precisely the same formulas. This result is stated in Theorem 4.18.

The following note reviews $\mathrm{Tot}(\mathbf{\Omega}(\check{N} U_\bullet))$:

Note 4.14 Fix a Čech cover $(U_\bullet \to X)$. Then $\mathrm{Tot}(\mathbf{\Omega}(\check{N} U_\bullet))$ is the totalization of the cosimplicial simplicial set $\mathbf{\Omega}(\check{N} U_\bullet) = \underline{\mathrm{DK}}(\Omega^\bullet_{\mathrm{hol}}(\check{N} U)[u]^{\bullet \le 0})$. Recall from Note 2.16 that the n–simplices of Dold and Kan applied to the chain complex $\Omega^\bullet_{\mathrm{hol}}(V)[u]^{\bullet\le 0}$ for some open set V, are decorations of the standard n–simplex, ie they assign to each l–simplex, polynomials $a \in \Omega^\bullet_{\mathrm{hol}}(V)[u]^{\bullet\le 0}$ of total degree $-l$,

$$a = \begin{cases} \sum_{j=0}^\infty a^{2j} \cdot u^{l/2+j} & \text{when } l \text{ is even,} \\ \sum_{j=0}^\infty a^{2j+1} \cdot u^{l+1/2+j} & \text{when } l \text{ is odd,} \end{cases} \tag{4-13}$$

where $a^p \in \Omega^p_{\mathrm{hol}}(V)$. The condition (2-9) imposed for these decorations is that the alternating sum of the faces of a l–simplex agrees with applying the chain complex's differential to the data of the l–simplex:

$$0 = d_C(a) = \sum_{j=0}^{l} (-1)^j d_j(a),$$

where C is the complex $C = \Omega^\bullet_{\mathrm{hol}}(V)[u]^{\bullet\le 0}$ with zero differential $d_C = 0$, (see Definition 2.15).

Now, from Sections B.1 and B.2, 0–simplices of the totalization $\mathrm{Tot}(\mathbf{\Omega}(\check{N}\mathcal{U}_\bullet))_0$ consist of coherent decorations of the standard n–simplex by data coming from $\mathbf{\Omega}(\check{N}\mathcal{U}_n)$:

- on each U_i, a 0–simplex in $\underline{\mathrm{DK}}(\Omega^\bullet_{\mathrm{hol}}(U_i)[u]^{\bullet\le 0})$, ie a polynomial a_i as in (4-13) with $l = 0$: $a_i = \sum_{j=0}^\infty a_i^{2j} \cdot u^j$,
- on each U_{i_0,i_1}, a 1–simplex in $\underline{\mathrm{DK}}(\Omega^\bullet_{\mathrm{hol}}(U_{i_0,i_1})[u]^{\bullet\le 0})$, ie a polynomial a_{i_0,i_1} as in (4-13) with $l = 1$: $a_{i_0,i_1} = \sum_{j=0}^\infty a_{i_0,i_1}^{2j+1} \cdot u^{j+1}$,
- on each $U_{i_0,\dots,i_l}$, an l–simplex in $\underline{\mathrm{DK}}(\Omega^\bullet_{\mathrm{hol}}(U_{i_0,\dots,i_l})[u]^{\bullet\le 0})$, ie a polynomial $a_{i_0,\dots,i_l}$ as in (4-13).

These polynomials satisfy the conditions

$$0 = \sum_{j=0}^{l} (-1)^j d_j(a_{i_0,\dots,i_l}) = \sum_{j=0}^{l} U_{i_0,\dots,i_l}|_{a_{i_0,\dots,\hat{\imath}_j,\dots,i_l}},$$

where the last equality follows from (B-5) and Example B.1.

Recall from [Grothendieck 1966] that the *Hodge cohomology* $\bigoplus_{p,q} H^p(X, \Omega^q)$ is given by a sum over the p^{th} sheaf cohomology of the sheaf of holomorphic q forms (see also "Hodge theory" or "Hodge decomposition" [Frölicher 1955]). O'Brian, Toledo and Tong [1981c, Section 4] defined the Chern character as an element in $\bigoplus_k H^k(X, \Omega^k)$. Below we see how our $\mathbf{\Omega}^{\check{\dagger}}$ relates to the Hodge cohomology.

Proposition 4.15 *The set of connected components of $\mathbf{\Omega}^{\check{\dagger}}(X)$ forms a ring which is isomorphic to the even part of the Hodge cohomology ring,*

$$\pi_0(\mathbf{\Omega}^{\check{\dagger}}(X)) \simeq \bigoplus_{\substack{p,q \\ p+q \text{ even}}} H^p(X, \Omega^q).$$

Proof The proof follows first from a direct observation that the vertices of $\mathrm{Tot}(\mathbf{\Omega}(\check{N}U_\bullet))$ are precisely (since the differentials are all zero) a direct sum of Čech l–cocycles of holomorphic forms (even degree forms for l even and odd degree forms for l odd), and then from the observation that edges in $\mathrm{Tot}(\mathbf{\Omega}(\check{N}U_\bullet))$ correspond to Čech coboundaries. □

We next illustrate our sheafified Chern map $\mathbf{Ch}^{\check{\dagger}}$.

Note 4.16 Consider a Čech cover $(U_\bullet \to X)$, and a vertex in $\mathrm{Tot}(\mathbf{IVB}(\check{N}\mathcal{U}))_0$ as provided by Note 4.5, ie the data of holomorphic bundles $E_{0;i}$ with

- differentials $d = g_{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right];i}$ from (1);
- connections $\nabla_{0;i}$ from (2); and
- maps $g_{\left[\begin{smallmatrix}0 & \cdots & 0\\ \beta_0 & \cdots & \beta_q\end{smallmatrix}\right];i_0,\dots,i_l}$ from (3).

Then our sheafified Chern character map $\mathbf{IVB}^{\check{\dagger}} \xrightarrow{\mathbf{Ch}^{\check{\dagger}}} \mathbf{\Omega}^{\check{\dagger}}$ simply applies the Chern character $\mathbf{Ch}\colon \mathbf{IVB} \to \mathbf{\Omega}$ from Definition 3.13 locally to the data in our vertex by allowing the indices from that definition to be given by the indices of the open cover. To clarify, the vertex above gets mapped to the following vertex in $\mathrm{Tot}(\mathbf{\Omega}(\check{N}\mathcal{U}))_0$:

- On each U_i, assign the Euler characteristic of $E_{0;i}$, denoted by $\chi(E_{0;i}) \cdot u^0 \in \Omega^0_{\mathrm{hol}}(U_i)[u]^{\bullet \le 0}$.
- On each U_{i_0,i_1}, using $g = \{g_{\left[\begin{smallmatrix}0 & \cdots & 0\\ \beta_0 & \cdots & \beta_q\end{smallmatrix}\right];i_0,i_1}\}_{(\beta_0,\dots,\beta_q)\in\widehat{\Delta}^1}$, assign the monomial
$$\mathrm{Tr}_g(\nabla(d+g))_{(0,1)} \cdot u \in \Omega^1_{\mathrm{hol}}(U_{i_0,i_1})[u]^{\bullet\le 0},$$
and restrict the Euler characteristic above on the vertices (see Definition 3.13):
$$\underset{\bullet}{U_{i_0,i_1}|_{\chi(E_{0;i_0})}} \xleftarrow{\quad \mathrm{Tr}_g(\nabla(d+g))_{(0,1)} \cdot u \quad} \underset{\bullet}{U_{i_0,i_1}|_{\chi(E_{0;i_1})}}$$
- For each $U_{i_0,i_1,\dots,i_l}$, using $g = \{g_{\left[\begin{smallmatrix}0 & \cdots & 0\\ \beta_0 & \cdots & \beta_q\end{smallmatrix}\right];i_0,i_1,\dots,i_l}\}_{(\beta_0,\dots,\beta_q)\in\widehat{\Delta}^l}$, assign the monomial
$$\mathrm{Tr}_g\big((\nabla(d+g))^l\big)_{(0,1,\dots,l)} \cdot \frac{u^l}{l!} \in \Omega^l_{\mathrm{hol}}(U_{i_0,i_1,\dots,i_l})[u]^{\bullet\le 0} \tag{4-14}$$
to the top cell and to each face assign appropriate restrictions of the monomials defined for lower intersections.

The above formula is now compared to the one provided by O'Brian, Toledo and Tong for the Chern character map of a coherent sheaf.

Note 4.17 O'Brian, Toledo and Tong [1981c] construct characteristic classes for coherent sheaves via the following four steps:

(i) Given a coherent sheaf, a twisting cochain a is constructed using [loc. cit., below Lemma 1.6]. This construction is well defined with respect to equivalences of twisting cochains; see [loc. cit., Proposition 1.10].

(ii) Connection data is chosen for a so that we obtain a twisting cochain with holomorphic connection data; see [loc. cit., above Proposition 4.4].

(iii) The Atiyah class is represented by the class ∇a in [loc. cit., Proposition 4.4].

(iv) The Chern character is defined [loc. cit., above Proposition 4.5] using the trace map τ_a to be given by

$$\mathrm{ch} := \sum_{k\geq 0} \mathrm{ch}_k := \sum_{k\geq 0} \frac{1}{k!}\tau_a((\nabla a)^k). \tag{4-15}$$

Note that the trace map τ_a from [loc. cit., above Proposition 3.2] is defined in the same way as our trace map Tr_g in Definition 3.7.

Comparing the formulas (4-14) and (4-15) for the Chern character, these involve the same trace terms, and so we obtain the following theorem:

Theorem 4.18 *For a given coherent sheaf, the formula for the Chern character* (4-15) *from* [loc. cit.] *is given by the terms in the formula* (4-14) *of the Chern character map*

$$\{\text{isomorphism classes of coherent sheaves}\} \simeq \pi_0(\mathbf{CohSh}^{\check{\dagger}}) \xrightarrow{\pi_0(\mathbf{Ch}^{\check{\dagger}})} \pi_0(\mathbf{\Omega}^{\check{\dagger}}) \simeq \bigoplus_{\substack{p,q \\ p+q \text{ even}}} H^p(\Omega^q) \tag{4-16}$$

applied to the corresponding twisting cochain interpreted (*by Theorem 4.9*) *as a* 0*–simplex in* $\mathbf{CohSh}^{\check{\dagger}}$.

Proof A twisting cochain a defines the Maurer–Cartan element via (4-9). With this, the terms in the traces in (4-14) and (4-15) coincide. (We note that the additional factor u^l in (4-15) does not add any extra information, as the power l is precisely the "Čech degree" given by the number of intersections in $U_{i_0,\dots,i_l}$.). Finally, the left and right isomorphisms in (4-16) are given by Propositions 4.13 and 4.15, respectively. □

Note, in particular, that our sheafified $\mathbf{Ch}^{\check{\dagger}}$ provides not only a Chern character to resolutions of coherent sheaves but also provides invariants for morphisms and higher homotopies between these resolutions.

Remark 4.19 A version of the Chern–Simons invariant for the straight line path between connections is computed by $\pi_1(\mathbf{Ch}^{\check{\dagger}})$ as we outline here. In the case where $\mathbf{Vect} \hookrightarrow \mathbf{CohSh}$ is the full subcategory of vector bundles, a loop representing a class in $\pi_1(\mathbf{Vect}^{\check{\dagger}})$ is given by a vector bundle $E \to X$ and locally chosen connections $\{(E_i \to U_i, \nabla_i)\}$, along with a bundle automorphism $f\colon E \to E$. Then $\mathbf{Ch}^{\check{\dagger}}$ sends the vertex of this loop to the Chern character $\mathbf{Ch}^{\check{\dagger}}\big(\{(E_i, \nabla_i, g_{ij})\}\big)$ and the edge induced by f is sent to an odd Čech–Hodge form, which we denote by $\mathbf{Ch}^{\check{\dagger}}(f)$, whose differential is the difference between $\mathbf{Ch}^{\check{\dagger}}\big(\{(E_i, \nabla_i, g_{ij})\}\big)$ and $\mathbf{Ch}^{\check{\dagger}}\big(\{(E_i, f^*\nabla_i, g_{ij})\}\big)$. Since $\mathbf{Ch}^{\check{\dagger}}\big(\{(E_i, \nabla_i, g_{ij})\}\big) = \mathbf{Ch}^{\check{\dagger}}\big(\{(E_i, f^*\nabla_i, g_{ij})\}\big)$, f is sent to a closed odd form in the Čech–Hodge complex. Moreover, if two loops f and f' in $\mathbf{Vect}^{\check{\dagger}}$ are homotopic, then the difference between $\mathbf{Ch}^{\check{\dagger}}(f)$ and $\mathbf{Ch}^{\check{\dagger}}(f')$ is exact, and so $\pi_1(\mathbf{Ch}^{\check{\dagger}})$ indeed computes a higher invariant.

5 The induced map on classifying stacks

In this section we show that the previously considered Čech sheafified Chern character map (Definition 4.3) is a map of simplicial sheaves when we restrict $\mathbf{IVB}^{\check{\dagger}}$ (see Definition 4.1 and Note 4.5) to the subsimplicial sheaf $\mathbf{IVB}^{\check{\dagger}}_{\leq n}$ which considers complexes of vector bundles of a fixed length, n (see Definition 5.4). Moreover, each of these simplicial presheaves contains a subsimplicial presheaf which considers complexes, $\mathbf{CohSh}^{\check{\dagger}}$ and $\mathbf{CohSh}^{\check{\dagger}}_{\leq n}$ respectively (see Definition 4.11), whose homology is concentrated in degree zero, yielding the commutative diagram

$$\begin{tikzcd}
\mathbf{CohSh}_{\leq n} \arrow[rrr, hook] \arrow[dr] \arrow[ddd, hook] & & & \mathbf{CohSh} \arrow[dl] \arrow[ddd, hook] \\
& \mathbf{CohSh}^{\check{\dagger}}_{\leq n} \arrow[r, hook] \arrow[d, hook] & \mathbf{CohSh}^{\check{\dagger}} \arrow[d, hook] & \\
& \mathbf{IVB}^{\check{\dagger}}_{\leq n} \arrow[r, hook] & \mathbf{IVB}^{\check{\dagger}} & \\
\mathbf{IVB}_{\leq n} \arrow[rrr, hook] \arrow[ur] & & & \mathbf{IVB} \arrow[ul]
\end{tikzcd}$$

(see Proposition 4.13 for a justification of our notation **CohSh**). As such we offer in Theorem 5.13 an upgrade on the statement of Theorem 4.18 to a statement about sheaves.

5.1 Sheaves in the local projective model structure

This section's main goal is to sort out which of the (maps of) presheaves in this paper are in fact (maps of) sheaves.

Given the Verdier site à la Dugger, Hollander and Isaksen [2004, Section 9] of complex manifolds and holomorphic maps, $\mathbb{C}\mathrm{Man}$, the category of simplicial presheaves $\mathrm{sPre}(\mathbb{C}\mathrm{Man})$ has multiples model structures. One particular choice is the (global) projective model structure whose weak equivalences are objectwise weak equivalences of simplicial sets and whose fibrations are objectwise fibrations of simplicial sets [Blander 2001, Theorem 1.5]. Further this model structure forms a (proper simplicial cellular) simplicial model category when we use the simplicial mapping space $\underline{\mathrm{sPre}}(X, Y)_n := \mathrm{sPre}(X \otimes \Delta^n, Y)$. After localizing this simplicial category over the class of maps induced by hypercovers, we further obtain the local projective (proper simplicial cellular) model structure $\mathrm{sPre}(\mathbb{C}\mathrm{Man})_{\mathrm{proj,loc}}$ [loc. cit., Theorem 1.6]. The relevant criteria in this structure for us is that an object in $\mathrm{sPre}(\mathbb{C}\mathrm{Man})_{\mathrm{proj,loc}}$ is fibrant if it is fibrant in the projective model structure and satisfies descent with respect to any hypercover thanks to Dugger, Hollander and Isaksen [2004]. Such an object is referred to below as a (hyper)sheaf.

In presenting a classifying stack (ie classifying hypersheaf) for coherent sheaves, one could produce a simplicial presheaf, $\boldsymbol{F} \in \mathrm{sPre}(\mathbb{C}\mathrm{Man})$, and prove (at the very least) that for any manifold $X \in \mathbb{C}\mathrm{Man}$, the set of equivalence classes of coherent sheaves coincides with the connected components of the derived mapping space, $\mathbb{R}\mathrm{Hom}(X, \boldsymbol{F})$. Since we are working with the local projective simplicial model category of simplicial presheaves this mapping space can be computed by cofibrantly approximating X with $\widetilde{X} \to X$ (which in this case is the identity since X is representable and thus cofibrant), fibrantly approximating $\boldsymbol{F}$ by $\boldsymbol{F} \to \widehat{\boldsymbol{F}}$, and defining the right derived mapping space (ie the homotopy function complex from [Hirschhorn 2003, Section 17]) as the simplicial mapping space on the replacements:

$$\mathbb{R}\mathrm{Hom}(X, \boldsymbol{F}) := \underline{\mathrm{sPre}}(\widetilde{X}, \widehat{\boldsymbol{F}}) = \underline{\mathrm{sPre}}(X, \widehat{\boldsymbol{F}}). \tag{5-1}$$

Thus $\widehat{\boldsymbol{F}}$ would provide a more concrete description of this classifying stack and any map of simplicial presheaves $\boldsymbol{F} \to \boldsymbol{\Omega}$ provides cohomological invariants by inducing a map between fibrant replacements $\widehat{\boldsymbol{F}} \to \widehat{\boldsymbol{\Omega}}$; offering more explicit, cocycle-level cohomological invariants.

It is not immediate that our Čech sheafification computes the fibrant replacement. Below we first show that if $\boldsymbol{F}$ is already a hypersheaf then $\boldsymbol{F}^{\check{\dagger}}$ is again a hypersheaf, even though this result is not used in this paper.

Proposition 5.1 *If $\boldsymbol{F}$ is a hypersheaf, then $\boldsymbol{F}^{\check{\dagger}}$ is a hypersheaf and the natural map $\boldsymbol{F} \to \boldsymbol{F}^{\check{\dagger}}$ is an objectwise weak equivalence.*

Proof By construction, we have already shown in the proof of Proposition 5.2 that Čech sheafification preserves objectwise fibrancy without any assumptions on the homotopy type of $\boldsymbol{F}$. To see that there is an objectwise weak equivalence, we compute

$$\boldsymbol{F}^{\check{\dagger}}(X) = \operatorname*{colim}_{(\mathcal{W}\to X)\in S} \underline{\mathrm{sPre}}(\mathcal{W}, \boldsymbol{F}),$$

where S is a full subcategory of the overcategory $\mathbb{C}\mathrm{Man}/X$, whose objects are hypercovers $\mathcal{W} \to X$. Since $\boldsymbol{F}$ already satisfies descent, ie $\underline{\mathrm{sPre}}(\mathcal{W}, \boldsymbol{F}) \xleftarrow{\sim} \underline{\mathrm{sPre}}(X, \boldsymbol{F})$,

$$\boldsymbol{F}^{\check{\dagger}}(X) \xleftarrow{\sim} \operatorname*{colim}_{(\mathcal{W}\to X)\in S} \underline{\mathrm{sPre}}(X, \boldsymbol{F}) \xleftarrow{\sim} \underline{\mathrm{sPre}}(X, \boldsymbol{F}) = \boldsymbol{F}(X).$$

Now, to show that the Čech sheafification preserves hyperdescent, we choose a hypercover $\mathcal{U} \to X$ and argue that the natural map $\underline{\mathrm{sPre}}(X, \boldsymbol{F}^{\check{\dagger}}) \to \underline{\mathrm{sPre}}(\mathcal{U}, \boldsymbol{F}^{\check{\dagger}})$ is a weak equivalence of simplicial sets. On the one hand, we have

$$\underline{\mathrm{sPre}}(X, \boldsymbol{F}^{\check{\dagger}}) = \boldsymbol{F}^{\check{\dagger}}(X) \xleftarrow{\sim} \boldsymbol{F}(X),$$

while, on the other hand, we have

$$\begin{aligned}
\underline{\mathrm{sPre}}(\mathcal{U}, \boldsymbol{F}^{\check{\dagger}}) \xrightarrow{\sim} \underline{\mathrm{sPre}}\Bigl(\operatorname*{hocolim}_{i\in\Delta} \coprod_{i,\alpha_i} U_{i,\alpha_i}, \boldsymbol{F}^{\check{\dagger}}\Bigr)
&= \operatorname*{holim}_{i\in\Delta} \prod_{i,\alpha_i} \underline{\mathrm{sPre}}(U_{i,\alpha_i}, \boldsymbol{F}^{\check{\dagger}}) \\
&= \operatorname*{holim}_{i\in\Delta} \prod_{i,\alpha_i} \boldsymbol{F}^{\check{\dagger}}(U_{i,\alpha_i}) \xleftarrow{\sim} \operatorname*{holim}_{i\in\Delta} \prod_{i,\alpha_i} \boldsymbol{F}(U_{i,\alpha_i}) = \underline{\mathrm{sPre}}(\mathcal{U}, \boldsymbol{F}) \xleftarrow{\sim} \boldsymbol{F}(X),
\end{aligned}$$

where the last weak equivalence follows from $\boldsymbol{F}$ already satisfying descent. After repeated application of the two-out-of-three property for weak equivalences, we see that $\boldsymbol{F}^{\check{\dagger}}$ satisfies descent as well. $\square$

Under a modest boundedness condition on a simplicial presheaf $\boldsymbol{F}$ which takes values in Kan complexes, its Čech sheafification (Definition 4.1) is a sheaf; this result is key to the rest of this paper.

Proposition 5.2 *Let $\boldsymbol{F} \in \mathrm{sPre}(\mathbb{C}\mathrm{Man})$ be a projectively fibrant simplicial presheaf whose homotopy groups are all trivial above level n. Then $\boldsymbol{F}^{\check{\dagger}}$ is a fibrant approximation of $\boldsymbol{F}$ in the local projective model structure of simplicial presheaves on complex manifolds.*

Proof Given a projectively fibrant simplicial presheaf $\boldsymbol{F} \in \mathrm{sPre}(\mathbb{C}\mathrm{Man})$ we can consider its fibrant replacement in the local projective model structure $\boldsymbol{F} \xrightarrow{\sim} \boldsymbol{F}' \in \mathrm{sPre}(\mathbb{C}\mathrm{Man})_{\mathrm{loc}}$. By [Lurie 2017, Remark 6.2.2.12], we see that in general we can compute this fibrant replacement on a test manifold $X \in \mathbb{C}\mathrm{Man}$ with the *hypersheafification* of $\boldsymbol{F}$, written $\boldsymbol{F}^{\dagger}$, by taking a homotopy colimit of the simplicial mapping space $\underline{\mathrm{sPre}}(\mathcal{U}, \boldsymbol{F})$ over all hypercovers $(\mathcal{U} \to X)$. Below, as is standard, we identify the manifold X with its representable simplicial presheaf, ie with the functor $Y \mapsto \mathbb{C}\mathrm{Man}(Y, X)$, postcomposed by the functor which sends sets to simplicially constant simplicial sets. Thus, if S denotes the category of all hypercovers,

$$\boldsymbol{F}^{\dagger}(X) := \operatorname*{hocolim}_{(\mathcal{U}\to X)\in S} \underline{\mathrm{sPre}}(\mathcal{U}, \boldsymbol{F}).$$

More formal references for this fact include [Anel and Subramaniam 2020, Example 3.4.9; Low 2015, Proposition 6.6]. We can now follow a series of steps to rewrite the above sheafification up to weak equivalence: Starting with

$$\boldsymbol{F}^{\dagger}(X) := \operatorname*{hocolim}_{(\mathcal{U}\to X)\in S} \underline{\mathrm{sPre}}(\mathcal{U}, \boldsymbol{F}) = \operatorname*{hocolim}_{(\mathcal{U}\to X)\in S} \underline{\mathrm{sPre}}\Big(\operatorname*{hocolim}_{i\in\boldsymbol{\Delta}} \mathcal{U}_i, \boldsymbol{F}\Big), \tag{5-2}$$

pulling the homotopy colimit out as a homotopy limit, and then using the fact that $\boldsymbol{F}$ is of bounded homotopy type so $\boldsymbol{F} \xrightarrow{\sim} \mathbf{cosk}_n \boldsymbol{F}$ with both of these projectively fibrant,

$$\boldsymbol{F}^{\dagger}(X) = \operatorname*{hocolim}_{(\mathcal{U}\to X)\in S} \operatorname*{holim}_{i\in\boldsymbol{\Delta}} \underline{\mathrm{sPre}}(\mathcal{U}_i, \boldsymbol{F}) \xrightarrow{\sim} \operatorname*{hocolim}_{(\mathcal{U}\to X)\in S} \operatorname*{holim}_{i\in\boldsymbol{\Delta}} \underline{\mathrm{sPre}}(\mathcal{U}_i, \mathbf{cosk}_n \boldsymbol{F}).$$

Now, using the skeleton–coskeleton adjunction and then that we can change the indexing set of hypercovers to also be n–skeletal,

$$\boldsymbol{F}^{\dagger}(X) \xrightarrow{\sim} \operatorname*{hocolim}_{(\mathcal{U}\to X)\in S} \operatorname*{holim}_{i\in\boldsymbol{\Delta}} \underline{\mathrm{sPre}}(\mathbf{sk}_n \mathcal{U}_i, \boldsymbol{F}) = \operatorname*{hocolim}_{(\mathcal{U}\to X)\in S_{\le n}} \operatorname*{holim}_{i\in\boldsymbol{\Delta}} \underline{\mathrm{sPre}}(\mathbf{sk}_n \mathcal{U}_i, \boldsymbol{F}).$$

Now, since Čech covers are cofinal in bounded hypercovers on a paracompact manifold [Schreiber 2013, Proposition 3.6.63], denoting by $\check{S}$ the category of Čech covers,

$$\begin{aligned}\operatorname*{hocolim}_{(\mathcal{U}\to X)\in S_{\le n}} \operatorname*{holim}_{i\in\boldsymbol{\Delta}} \underline{\mathrm{sPre}}(\mathbf{sk}_n \mathcal{U}_i, \boldsymbol{F}) &\xleftarrow{\sim} \operatorname*{hocolim}_{(\check{N}U_\bullet\to X)\in \check{S}} \operatorname*{holim}_{i\in\boldsymbol{\Delta}} \underline{\mathrm{sPre}}(\mathbf{sk}_n \check{N}U_i, \boldsymbol{F}) \\ &= \operatorname*{hocolim}_{(\check{N}U_\bullet\to X)\in \check{S}} \operatorname*{holim}_{i\in\boldsymbol{\Delta}} \underline{\mathrm{sPre}}(\check{N}U_i, \mathbf{cosk}_n \boldsymbol{F}) \xleftarrow{\sim} \operatorname*{hocolim}_{(\check{N}U_\bullet\to X)\in \check{S}} \operatorname*{holim}_{i\in\boldsymbol{\Delta}} \underline{\mathrm{sPre}}(\check{N}U_i, \boldsymbol{F}).\end{aligned}$$

Next we apply a simplicial Yoneda lemma and then use the fact that Tot computes holim when the cosimplicial simplicial set is Reedy fibrant [Hirschhorn 2003, Theorem 18.7.4] to obtain

$$\begin{aligned}\operatorname*{hocolim}_{(\check{N}U_\bullet\to X)\in\check{S}}\ \operatorname*{holim}_{i\in\mathbf{\Delta}} \underline{\mathrm{sPre}}(\check{N}U_i,\boldsymbol{F}) = \operatorname*{hocolim}_{(\check{N}U_\bullet\to X)\in\check{S}}\ \operatorname*{holim}_{i} \prod_{\alpha_0,\dots,\alpha_i} \boldsymbol{F}(U_{\alpha_0,\dots,\alpha_i})& \\ \xrightarrow{\sim} \operatorname*{hocolim}_{(\check{N}U_\bullet\to X)\in\check{S}} \mathrm{Tot}(\boldsymbol{F}(\check{N}U_\bullet)),&\end{aligned}$$

and finally we use the fact that the colimit over Čech covers is a filtered colimit to compute hocolim with a colim to obtain

$$\operatorname*{hocolim}_{(\check{N}U_\bullet\to X)\in\check{S}} \mathrm{Tot}(\boldsymbol{F}(\check{N}U_\bullet)) \xrightarrow{\sim} \operatorname*{colim}_{(\check{N}U_\bullet\to X)\in\check{S}} \mathrm{Tot}(\boldsymbol{F}(\check{N}U_\bullet)) = \boldsymbol{F}^{\check{\dagger}}(X).$$

By Proposition 4.2, $\boldsymbol{F}^{\check{\dagger}}$ is already globally projectively fibrant (ie takes values in Kan complexes). Now it remains to show that $\boldsymbol{F}^{\check{\dagger}}$ satisfies hyperdescent. Given a hypercover, $\mathcal{U}\to X$, we use the commutative square

$$\begin{tikzcd} \underline{\mathrm{sPre}}(X,\boldsymbol{F}^{\dagger}) = \boldsymbol{F}^{\dagger}(X) \arrow[r] \arrow[d] & \underline{\mathrm{sPre}}(\mathcal{U},\boldsymbol{F}^{\dagger}) \arrow[d] \\ \underline{\mathrm{sPre}}(X,\boldsymbol{F}^{\check{\dagger}}) = \boldsymbol{F}^{\check{\dagger}}(X) \arrow[r] & \underline{\mathrm{sPre}}(\mathcal{U},\boldsymbol{F}^{\check{\dagger}}) \end{tikzcd} \tag{5-3}$$

where the equalities are given by Yoneda. Since $\boldsymbol{F}^{\dagger}$ satisfies descent, the top horizontal map is a weak equivalence by definition of descent. The left vertical map was proven to be an equivalence above. With $\mathcal{U}$ projectively cofibrant it follows that the simplicial mapping spaces preserve the weak equivalence $\boldsymbol{F}^{\dagger}\xrightarrow{\sim}\boldsymbol{F}^{\check{\dagger}}$ between projectively fibrant objects and so the right vertical map is a weak equivalence. Thus, by the two-out-of-three property afforded to our model category, we have shown that the bottom horizontal map is a weak equivalence. Since we have shown that $\boldsymbol{F}^{\check{\dagger}}$ is projectively fibrant, satisfies hyperdescent, and that $\boldsymbol{F}\xrightarrow{\sim}\boldsymbol{F}^{\check{\dagger}}$, then $\boldsymbol{F}^{\check{\dagger}}$ is a fibrant replacement of $\boldsymbol{F}$ in the local projective model structure. □

Lemma 5.3 *Let* $\mathrm{Ch}^{\le 0}(\mathcal{A})$ *be the* dg*–category of nonpositively graded chain complexes over some additive category* $\mathcal{A}$*, where the hom-complex* $\mathrm{Ch}^{\bullet}(E,E')$ *consists of chain maps and* (*higher*) *chain homotopies from* E *to* E'*, and let* $\mathcal{Q}\hookrightarrow\mathrm{Ch}^{\le 0}(\mathcal{A})$ *be a full subcategory which only considers complexes of height at most* m *for some fixed* $m\in\mathbb{N}$*. Then the simplicial set* $\mathrm{dg}\mathcal{N}(\mathcal{Q})\simeq\mathbf{cosk}_{m+1}\,\mathrm{dg}\mathcal{N}(\mathcal{Q})$ *is* $(m+1)$*–coskeletal.*

Proof For any two objects in $\mathcal{Q}$ and for an integer $k>m+1$, we have $\mathcal{Q}^k(E,E')=0$ due to the restricted height of all complexes in our dg-category. Thus the only way to decorate a k–simplex with $k>m+1$ is to have the boundary data all satisfy the condition $\hat{\delta}g+g\cdot g=0$ and then uniquely assign a 0–homotopy to the $(m+1)$–simplex. But recall that, whenever each decorated boundary simplex has a unique filler, this means the simplicial set is isomorphic to its coskeleton, so in our case we have $\mathrm{dg}\mathcal{N}(\mathcal{Q})\simeq\mathbf{cosk}_{m+1}\,\mathrm{dg}\mathcal{N}(\mathcal{Q})$, as required. □

Definition 5.4 Define $\mathrm{Perf}_{\leq n} : \mathbb{C}\mathrm{Man}^{\mathrm{op}} \to \mathrm{dgCat}$ by setting $\mathrm{Perf}_{\leq n}(U)$ to be the dg-category of finite chain complexes of holomorphic vector bundles just as in Definition 2.1, but with the difference that we require the complexes to be trivial above level n. Analogously to **IVB** from Definition 2.11, we then define $\mathbf{IVB}_{\leq n} : \mathbb{C}\mathrm{Man}^{\mathrm{op}} \to \mathrm{sSet}$ by setting $\mathbf{IVB}_{\leq n}(U)_n := \mathrm{sSet}\big(\widehat{\Delta}^n, \mathrm{dg}\mathcal{N}(\mathrm{Perf}_{\leq n}(U))^{\circ}\big)$.

Corollary 5.5 *The fibrant replacement of $\mathbf{IVB}_{\leq n}$ in the local projective model structure can be computed by its Čech sheafification*, $\mathbf{IVB}_{\leq n} \xrightarrow{\sim} \check{\mathbf{IVB}}^{\dagger}_{\leq n}$.

Proof By construction, $\mathbf{IVB}_{\leq n}$ is still (globally) projectively fibrant, while combining Lemma 5.3 and Proposition A.1 gives us that $\mathbf{IVB}_{\leq n}$ is (globally) a homotopy-$(n+1)$ type. □

Lemma 5.6 *Let $Ch^{\leq 0}(\mathcal{A})$ be the dg–category of nonpositively graded chain complexes over some additive category $\mathcal{A}$, where the hom-complex $Ch^{\bullet}(E, E')$ consists of chain maps and* (*higher*) *chain homotopies from E to E', and let $\mathcal{Q} \hookrightarrow Ch^{\leq 0}(\mathcal{A})$ be a full subcategory which only considers complexes with homology concentrated in degree zero. Then the* (*Kan replacement of the*) *simplicial set $\mathrm{dg}\mathcal{N}(Q)$ is a 1–type.*

Proof If necessary, first replace $\mathrm{dg}\mathcal{N}(Q)$ with its maximal Kan subcomplex which only uses quasi-isomorphisms on edges. We will prove that $\pi_n(\mathrm{dg}\mathcal{N}(Q))$ is trivial for $n \geq 2$. A class in π_n consists of an n–simplex in $\mathrm{dg}\mathcal{N}(Q)$ whose entire boundary is in the image of a single vertex. Thus the vertices are given by the same chain complex, $E_0 = E, \dots, E_n = E$, the quasi-isomorphisms on the edges are the identity maps, and any homotopy decorating a $k < n$ face is the zero homotopy. By the definition of $\mathrm{dg}\mathcal{N}(Q)$, this data satisfies the condition $\hat{\delta}(g) + Dg + g \cdot g = 0$ using the notation of Definition 3.3. Since in this case $\hat{\delta}(g) + g \cdot g$ is an alternating sum of compositions of 0–homotopies and/or identity maps, one can show that the above condition reduces to $Dg = 0$. However, since E is a complex whose homology is concentrated in degree zero and $g \in Q^{1-n}(E, E)$ with $n \geq 2$, g is exact. From here we can fill this n–sphere with a higher homotopy and kill the class representing g in π_n. □

By a similar argument for Corollary 5.5 we can use the above lemma to see that **CohSh** is a 1–type and thus $\check{\mathbf{CohSh}}^{\dagger}$ is a sheaf, but without needing to further restrict the height of any chain complexes.

Corollary 5.7 *The simplicial presheaf* **CohSh** *is a 1–type and its fibrant replacement in the local projective model structure can be computed by its Čech sheafification*, $\mathbf{CohSh} \xrightarrow{\sim} \check{\mathbf{CohSh}}^{\dagger}$.

Remark 5.8 Now that under the right circumstances the Čech sheafification can act as a fibrant replacement functor, we can briefly present a different argument for Lemma 4.8 which makes use of equivalences being preserved under the various constructions we use to pass from the dgCat–valued presheaf Perf^{∇} to the simplicial presheaf $\check{\mathbf{IVB}}^{\dagger}$. The main idea used in the proof for Lemma 4.8 is that for a complex manifold X, and a point $x \in X$, there exists an (Stein) open subset $x \in U \subset X$ on which we have an equivalence of dg-categories, $\mathrm{Perf}^{\nabla}(U) \xrightarrow{\sim} \widetilde{\mathrm{Perf}}^{\nabla}(U)$, where the tilde again means we forget connection data. Since the dg-nerve construction preserves (weak) equivalences, we then obtain an equivalence of simplicial sets, $\mathbf{IVB}(U) \xrightarrow{\sim} \widetilde{\mathbf{IVB}}(U)$. We claim this then says that we have a weak equivalence for each stalk $\mathbf{IVB}_x \xrightarrow{\sim} \widetilde{\mathbf{IVB}}_x$ and thus a local weak equivalence of simplicial presheaves à la Jardine, $\mathbf{IVB} \xrightarrow{\sim} \widetilde{\mathbf{IVB}}$. The local weak equivalences for the local projective model structure happen to coincide with those of Jardine and thus we obtain a weak equivalence in the local projective model structure which is necessarily preserved under our (Čech) fibrant replacement functor if we restrict appropriately: $\mathbf{IVB}^{\check{\dagger}}_{\le n} \xrightarrow{\sim} \widetilde{\mathbf{IVB}}^{\check{\dagger}}_{\le n}$.

Remark 5.9 At this point, we'd like to take stock and summarize the relationships amongst some of the different constructions involving **IVB**. By the functoriality of our constructions, we obtain two commutative cubes of simplicial presheaves which actually fit together to form a commutative hypercube via the inclusion $\mathbf{CohSh} \hookrightarrow \mathbf{IVB}$:

$$
\begin{tikzcd}
\boxed{\mathbf{IVB}^{\check{\dagger}}_{\le n}} \arrow[rrr, hook, "\sim"] \arrow[ddd, hook] & & & \boxed{\widetilde{\mathbf{IVB}}^{\check{\dagger}}_{\le n}} \arrow[ddd, hook] \\
& \mathbf{IVB}_{\le n} \arrow[r, hook, "\sim"] \arrow[ul, "\sim_{\mathrm{loc}}"'] \arrow[d, hook] & \widetilde{\mathbf{IVB}}_{\le n} \arrow[ur, "\sim_{\mathrm{loc}}"] \arrow[d, hook] & \\
& \mathbf{IVB} \arrow[r, hook, "\sim"] \arrow[dl] & \widetilde{\mathbf{IVB}} \arrow[dr] & \\
\mathbf{IVB}^{\check{\dagger}} \arrow[rrr, hook, "\sim"] & & & \widetilde{\mathbf{IVB}}^{\check{\dagger}}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\boxed{\mathbf{CohSh}^{\check{\dagger}}_{\le n}} \arrow[rrr, hook, "\sim"] \arrow[ddd, hook] & & & \boxed{\widetilde{\mathbf{CohSh}}^{\check{\dagger}}_{\le n}} \arrow[ddd, hook] \\
& \mathbf{CohSh}_{\le n} \arrow[r, hook, "\sim"] \arrow[ul, "\sim_{\mathrm{loc}}"'] \arrow[d, hook] & \widetilde{\mathbf{CohSh}}_{\le n} \arrow[ur, "\sim_{\mathrm{loc}}"] \arrow[d, hook] & \\
& \mathbf{CohSh} \arrow[r, hook, "\sim"] \arrow[dl, "\sim_{\mathrm{loc}}"'] & \widetilde{\mathbf{CohSh}} \arrow[dr, "\sim_{\mathrm{loc}}"] & \\
\boxed{\mathbf{CohSh}^{\check{\dagger}}} \arrow[rrr, hook, "\sim"] & & & \boxed{\widetilde{\mathbf{CohSh}}^{\check{\dagger}}}
\end{tikzcd}
$$

where the hypersheaves are highlighted with boxes; we used $\sim$ to denote a global projective (ie objectwise) weak equivalence and $\sim_{\mathrm{loc}}$ to denote a local projective weak equivalence. Recall that the global weak equivalences are preserved in the local model structure.

Recall that in Proposition 4.13 we showed that $\mathbf{CohSh}^{\check{\dagger}}$ stands a chance of classifying coherent sheaves since the correspondence is bijective on connected components. We know, however, that $\mathcal{N}(\mathrm{Sh}\,\mathcal{O}_X^\bullet)$ is a 1–type and so, if we knew that $\mathbf{CohSh}^{\check{\dagger}}$ was also a 1–type, then it would only remain to prove the correspondence on π_1.

Lemma 5.10 *Given $\boldsymbol{F} \in \mathrm{sPre}(\mathbb{C}\mathrm{Man})$ which is objectwise an n–type (ie $\boldsymbol{F} \xrightarrow{\sim} \mathbf{cosk}_n\, F$ for some n), $\boldsymbol{F}^{\check{\dagger}}$ is again an n–type.*

Proof We begin by noting that, if $\boldsymbol{F} \xrightarrow{\sim} \mathbf{cosk}_n\, \boldsymbol{F}$, then

$$\begin{aligned}\boldsymbol{F}^{\check{\dagger}}(X) = \operatorname*{colim}_{(U_\bullet \to X)\in \check{S}} \underline{\mathrm{sPre}}(\check{N}U_\bullet, \boldsymbol{F}) &\xrightarrow{\sim} \operatorname*{colim}_{(U_\bullet \to X)\in \check{S}} \underline{\mathrm{sPre}}(\check{N}U_\bullet, \mathbf{cosk}_n\, \boldsymbol{F}) \\ &\xrightarrow{\sim} \operatorname*{colim}_{(U_\bullet \to X)\in \check{S}} \mathrm{Tot}(\mathbf{cosk}_n\, \boldsymbol{F}(\check{N}U_\bullet)) \xrightarrow{\sim} \operatorname*{colim}_{(U_\bullet \to X)\in \check{S}} \mathbf{cosk}_n\, \mathrm{Tot}(\boldsymbol{F}(\check{N}U_\bullet)),\end{aligned}$$

where we used that Tot computes the homotopy limit in this case and then we commuted the right adjoint $\mathbf{cosk}_n$ across this concrete limit, and now again using that Tot computes the holim,

$$\operatorname*{colim}_{(U_\bullet \to X)\in \check{S}} \mathbf{cosk}_n\, \mathrm{Tot}(\boldsymbol{F}(\check{N}U_\bullet)) \xleftarrow{\sim} \operatorname*{colim}_{(U_\bullet \to X)\in \check{S}} \mathbf{cosk}_n\, \underline{\mathrm{sPre}}(\check{N}U_\bullet, \boldsymbol{F}).$$

While we would love to commute this coskeleton across the colimit, we must proceed differently. Recall that filtered colimits commute with finite limits, and, since each homotopy group can be written as a finite limit, we have, for $m > n$,

$$\begin{aligned}\pi_m(\boldsymbol{F}^{\check{\dagger}}(X)) &\simeq \pi_m\Big(\operatorname*{colim}_{(U_\bullet \to X)\in \check{S}} \mathbf{cosk}_n\, \underline{\mathrm{sPre}}(\check{N}U_\bullet, \boldsymbol{F})\Big) \\ &\simeq \operatorname*{colim}_{(U_\bullet \to X)\in \check{S}} \pi_m(\mathbf{cosk}_n\, \underline{\mathrm{sPre}}(\check{N}U_\bullet, \boldsymbol{F})) = \operatorname*{colim}_{(U_\bullet \to X)\in \check{S}} 0 = 0.\end{aligned}$$

□

Theorem 5.11 *The simplicial presheaf* **CohSh** *is a classifying prestack for coherent sheaves.*

Proof Recall from [Hirschhorn 2003, Section 17] that the derived mapping space $\mathbb{R}\mathrm{Hom}(A, B)$ in a simplicial model category $\mathcal{C}$ can be computed by considering the simplicial mapping space $\underline{\mathcal{C}}(\widetilde{A}, B')$, where we use the cofibrant replacement $\widetilde{A} \xrightarrow{\sim}$ of A and the fibrant replacement $B \xrightarrow{\sim} B'$ of B. Then, since Corollary 5.7 tells us that **CohSh** is a 1–type whose (local projective) fibrant replacement is given by its Čech sheafification, we can compute the (local projective) derived mapping space from a manifold $X \in \mathbb{C}\mathrm{Man}$ (via its cofibrant representable presheaf) into **CohSh** as

$$\mathbb{R}\mathrm{Hom}(X, \mathbf{CohSh}) := \underline{\mathrm{sPre}}(\widetilde{X}, \mathbf{CohSh}') \simeq \underline{\mathrm{sPre}}(X, \mathbf{CohSh}^{\check{\dagger}}) = \mathbf{CohSh}^{\check{\dagger}}(X).$$

After combining Proposition 4.13 and Lemma 5.10, it remains to be shown that the map $\mathcal{H}\colon \mathbf{CohSh}^{\check{\dagger}}(X) \to \mathcal{N}(\mathrm{Sh}\,\mathcal{O}_X)$ is an isomorphism of fundamental groups. The ideas used to prove this fact are analogous to those of Proposition 4.13 but we will summarize them here for ease of reading. Given a vertex $\mathcal{E} = (U_\bullet, E_\bullet, g_\bullet) \in \mathbf{CohSh}^{\check{\dagger}}(X)_0$ and the coherent sheaf $\mathcal{F} := \mathcal{H}(\mathcal{E}) \in \mathcal{N}(\mathrm{CohSh}\,\mathcal{O}_X)_0$, we want to prove that there is an isomorphisms of based homotopy groups, $\pi_1(\mathbf{CohSh}^{\check{\dagger}}(X), \mathcal{E}) \xrightarrow{\pi_1(\mathcal{H})} \pi_1(\mathcal{N}(\mathrm{CohSh}\,\mathcal{O}_X), \mathcal{F})$.

To prove injectivity, if two loops in $\mathbf{CohSh}^{\check{\dagger}}(X)_1$, $a_\bullet, b_\bullet : \mathcal{E} \to \mathcal{E}$ have connected images in $\mathcal{N}(\mathrm{CohSh}\,\mathcal{O}_X)$, then by definition of the nerve of a groupoid, we have a commutative square of isomorphisms in $\mathrm{CohSh}\,\mathcal{O}_X$ where all four corners are the coherent sheaf $\mathcal{F}$. Lifting this commutative square to a homotopy in $\mathbf{CohSh}^{\check{\dagger}}(X)_1$ once again uses the fact that chain maps which induce the same map on homology are homotopic [Hilton and Stammbach 1971, Theorem 4.1] (and then the discussion of O'Brian, Toledo and Tong [1981c, near Lemma 1.6]). To prove surjectivity, a loop $f : \mathcal{F} = \mathcal{H}(\mathcal{E}) \to \mathcal{F} = \mathcal{H}(\mathcal{E})$ in $\mathcal{N}(\mathrm{CohSh}\,\mathcal{O}_X)_1$ is lifted to a loop in $\mathbf{CohSh}^{\check{\dagger}}(X)$ on $\mathcal{E}$ by using the fact that an isomorphism on homology lifts to a quasi-isomorphism of chain complexes [Hilton and Stammbach 1971, Theorem 4.1] (and then, again, the discussion of O'Brian, Toledo and Tong [1981c, near Lemma 1.6]). □

If we knew that $\mathbf{\Omega}$ somehow used complexes of bounded height, then our Čech sheafified Chern map from Definition 4.3 could be seen to restrict to a map of sheaves $\mathbf{Ch}^{\check{\dagger}} : \mathbf{IVB}^{\check{\dagger}}_{\leq n} \to \mathbf{\Omega}^{\check{\dagger}}$ out of infinity vector bundles of bounded complex height. One way to resolve this is by restricting our site as recorded below:

Proposition 5.12 *On the site $\mathbb{C}\mathrm{Man}_{\leq n}$ of complex manifolds of dimension at most n, the Čech sheafification of the restricted Chern map,*

$$\mathbf{Ch}^{\check{\dagger}} : \mathbf{IVB}^{\check{\dagger}}_{\leq n} \to \mathbf{\Omega}^{\check{\dagger}},$$

is a map of hypersheaves.

Proof By Corollary 5.5, $\mathbf{IVB}^{\check{\dagger}}_{\leq n}$ is already a sheaf. Now that we have restricted the site to $\mathbb{C}\mathrm{Man}_{\leq n}$, $\mathbf{\Omega}$ only makes use of chain complexes of length at most n and so it is coskeletal and, by Proposition 5.2, its sheafification is a hypersheaf. □

By different application of the same ideas above, we end with an upgrade on Theorem 4.18:

Theorem 5.13 *On the site $\mathbb{C}\mathrm{Man}_{\leq n}$ of complex manifolds of dimension at most n, the Čech sheafification of the Chern map restricted to coherent sheaves,*

$$\mathbf{Ch}^{\check{\dagger}} : \mathbf{CohSh}^{\check{\dagger}} \to \mathbf{\Omega}^{\check{\dagger}},$$

is a map of hypersheaves which restricts on π_0 to the Chern character (4-15) *from O'Brian, Toledo and Tong* [1981c].

Proof By Theorem 5.11, $\mathbf{CohSh}^{\check{\dagger}}$ is already a sheaf. Now that we have restricted the site to $\mathbb{C}\mathrm{Man}_{\leq n}$, $\mathbf{\Omega}$ only makes use of chain complexes of length at most n and so it is coskeletal and, by Proposition 5.2, its sheafification is a hypersheaf. The fact that on π_0 it recovers the Chern map from O'Brian, Toledo and Tong [1981c] was already recorded in Theorem 4.18. □

Remark 5.14 For an arbitrary stack (ie hypersheaf) $\boldsymbol{F}$, recall as in (5-1) that the right derived mapping space

$$\mathbb{R}\mathrm{Hom}(\boldsymbol{F}, \boldsymbol{G}) := \underline{\mathrm{sPre}}(\widetilde{\boldsymbol{F}}, \widehat{\boldsymbol{G}})$$

for a simplicial model category can be computed by taking the simplicial mapping space between a cofibrant replacement of $\boldsymbol{F}$ and a fibrant replacement of $\boldsymbol{G}$. Letting $\boldsymbol{G}_1 = \mathbf{IVB}$ and $\boldsymbol{G}_2 = \boldsymbol{\Omega}$, Proposition 5.12 says that our presheafified Chern map $\mathbf{Ch}\colon \mathbf{IVB} \to \boldsymbol{\Omega}$ from Definition 3.13 induces a map of fibrant (ignoring the restrictions of sites and homotopy types for the moment) replacements $\check{\mathbf{Ch}}^{\dagger}\colon \check{\mathbf{IVB}}^{\dagger} \to \check{\boldsymbol{\Omega}}^{\dagger}$, and thus a map of right derived mapping spaces:

$$\mathbb{R}\mathrm{Hom}(\boldsymbol{F}, \mathbf{IVB}) = \underline{\mathrm{sPre}}(\widetilde{\boldsymbol{F}}, \check{\mathbf{IVB}}^{\dagger}) \xrightarrow{\check{\mathbf{Ch}}^{\dagger}} \underline{\mathrm{sPre}}(\widetilde{\boldsymbol{F}}, \check{\boldsymbol{\Omega}}^{\dagger}) =: \mathbb{R}\mathrm{Hom}(\boldsymbol{F}, \boldsymbol{\Omega}). \tag{5-4}$$

When $\boldsymbol{F} = X$ is the representable simplicial presheaf for a complex manifold, the above is explicitly calculated using Note 4.16. However, (5-4) suggests a reasonable definition for a *generalized Chern character map*. In a sequel to this paper, we will study this map for the case when a Lie group G acts on the complex manifold X and $\boldsymbol{F}_n = X \times G^{\times n}$ (see our previous paper [2022, Definition 5.1]), extending this paper to the equivariant setting.

Appendix A A weak equivalence $\mathrm{sSet}(\widehat{\boldsymbol{\Delta}}^{\bullet}, K) \to \mathrm{sSet}(\boldsymbol{\Delta}^{\bullet}, K)$

In this appendix, we prove Proposition A.1:

Proposition A.1 *If K is a Kan complex, then there exists a weak equivalence $F^{\sharp}\colon \mathrm{sSet}(\widehat{\Delta}^{\bullet}, K) \to \mathrm{sSet}(\Delta^{\bullet}, K)$.*

In order to define $F^{\sharp}$, we first establish some notation. Recall from Example 2.6 that Δ^n is the simplicial set whose k–simplices are nondecreasing sequences $(i_0 \le \cdots \le i_k)$ with $i_0, \dots, i_k \in \{0, \dots, n\}$, and recall from Example 2.8 that $\widehat{\Delta}^n$ is the simplicial set whose k–simplices are any sequences $(i_0, \dots, i_k)$ with $i_0, \dots, i_k \in \{0, \dots, n\}$. Both Δ^n and $\widehat{\Delta}^n$ have face maps d_j given by removing the j^{th} index i_j, and degeneracy maps s_j given by repeating the j^{th} index i_j. Furthermore both $\Delta^{\bullet}$ and $\widehat{\Delta}^{\bullet}$ are cosimplicial simplicial sets, so that for $\phi\colon [n] \to [m]$ in $\boldsymbol{\Delta}$ we get an induced map of $\phi_{\bullet}\colon \widetilde{\Delta}^n_{\bullet} \to \widetilde{\Delta}^m_{\bullet}$ via $\phi_k\colon \widetilde{\Delta}^n_k \to \widetilde{\Delta}^m_k$, $\phi_k(i_0, \dots, i_k) = (\phi(i_0), \dots, \phi(i_k))$, where $\widetilde{\Delta}^{\bullet}$ is either $\Delta^{\bullet}$ or $\widehat{\Delta}^{\bullet}$. Thus, there is an induced map of cosimplicial simplicial sets $F^{\bullet}\colon \Delta^{\bullet} \to \widehat{\Delta}^{\bullet}$, $(i_0 \le \cdots \le i_k) \mapsto (i_0, \dots, i_k)$. For any simplicial set X, both $X = \mathrm{sSet}(\Delta^{\bullet}, X)$ and $\widehat{X} := \mathrm{sSet}(\widehat{\Delta}^{\bullet}, X)$ are simplicial sets, and there is an induced map $F^{\sharp}\colon \widehat{X} \to X$ by precomposition with F.

Our first step towards proving Proposition A.1 is to show that $\widehat{K}$ is also a Kan complex:

Proposition A.2 *If K is a Kan complex, then $\widehat{K}$ is a Kan complex.*

To begin with, here is a useful lemma:

Lemma A.3 *A map $c\colon \Delta^n \to \widehat{K} = \mathrm{sSet}(\widehat{\Delta}^{\bullet}, K)$ is determined by the element $\bar{c} = c(0 \le \cdots \le n)\colon \widehat{\Delta}^n \to K$. Then $\delta_i(c) = c \circ \delta_i : \Delta^{n-1} \cong \delta_i(\Delta^{n-1}) \subset \Delta^n \xrightarrow{c} \widehat{K}$ is determined by $\delta_i(\bar{c}) = c \circ \delta_i : \widehat{\Delta}^{n-1} \cong \delta_i(\widehat{\Delta}^{n-1}) \subset \widehat{\Delta}^n \xrightarrow{c} K$.*

Proof Note that $\delta_i(\Delta^{n-1}) \subset \Delta^n$ are sequences that do not include i, which are generated by the $(n-1)$–simplex $(0 \leq \cdots \leq i-1 \leq i+1 \leq \cdots n) = d_i(0 \leq \cdots \leq n) \in \Delta^n_{n-1}$. Thus $\delta_i(c)$ is determined by the image of the simplex $d_i(0 \leq \cdots \leq n)$. Now $c(d_i(0 \leq \cdots \leq n)) = d_i(c(0 \leq \cdots \leq n)) = d_i(\bar{c}) = c \circ \delta_i$. □

Proof of Proposition A.2 Denote by $\Lambda^n_i := \bigcup_{j \neq i} \delta_j \Delta^{n-1}$ the i^{th} horn of Δ^n, which is a subsimplicial set of Δ^n. Similarly, denote by $\widehat{\Lambda}^n_i := \bigcup_{j \neq i} \delta_j \widehat{\Delta}^{n-1}$ the i^{th} horn of $\widehat{\Delta}^n$, which is a subsimplicial set of $\widehat{\Delta}^n$. As noted before, a simplicial set map $\Delta^n \to \widehat{K}$ is the same as an element $\widehat{K}_n$, ie a simplicial set map $\widehat{\Delta}^n \to K$. Similarly, a simplicial set map $\Lambda^n_i \to \widehat{K}$ is given by n maps $\delta_j \Delta^{n-1} \to \widehat{K}$, ie n maps $\widehat{\Delta}^{n-1} \to K$ (see Lemma A.3), which are compatible at their common boundary, ie whose induced common boundary maps $\widehat{\Delta}^{n-2} \to K$ coincide, and thus this is the same as a simplicial set map $\widehat{\Lambda}^n_i \to K$. Thus, the Kan condition for $\widehat{K}$ (left side of (A-1)) becomes equivalent to lifting a horn $\widehat{\Lambda}^n_i \to X$ to a map $\widehat{\Delta}^n \to X$ (right side of (A-1)):

$$
\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d] & \widehat{K} \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & *
\end{tikzcd}
\iff
\begin{tikzcd}
\widehat{\Lambda}^n_i \arrow[r] \arrow[d] & K \arrow[d] \\
\widehat{\Delta}^n \arrow[r] \arrow[ur, dashed] & *
\end{tikzcd}
\tag{A-1}
$$

Since K is a Kan complex, we have such a lift if $\widehat{\Lambda}^n_i \to \widehat{\Delta}^n$ is an trivial cofibration, ie if this map is injective and a weak equivalence. Clearly, $\widehat{\Lambda}^n_i \to \widehat{\Delta}^n$ is injective, and the weak equivalence follows since both $\widehat{\Lambda}^n_i$ and $\widehat{\Delta}^n$ are contractible, ie they have zero homotopy groups. First, it is well known that EG for any group G is contractible, since it has an extra degeneracy $s_{-1}(g_0, \ldots, g_k) = (e, g_0, \ldots, g_k)$; see for example [Goerss and Jardine 1999, Lemma III.5.1 and Example III.5.2]. Thus, $\widehat{\Delta}^n = E\mathbb{Z}_{n+1}$ is contractible, and, from the explicit extra degeneracy, we can see that it preserves $\widehat{\Lambda}^n_0$. Thus, $\widehat{\Lambda}^n_0$ is contractible as well. Now, there is a $\mathbb{Z}_{n+1}$–action on $E\mathbb{Z}_{n+1}$, which, in particular, can be used to map $\widehat{\Lambda}^n_0$ isomorphically to any other $\widehat{\Lambda}^n_i$, showing that indeed all $\widehat{\Lambda}^n_i$ are contractible. (Or, alternatively, one obtains that the extra degeneracy $s_{-1}(i_0, \ldots, i_k) = (i, i_0, \ldots, i_k)$ of $\widehat{\Delta}^n$ preserves $\widehat{\Lambda}^n_i$.) □

In order to prove Proposition A.1, we need one more ingredient. Denote by $\widehat{\Theta}^n := \big(\bigcup_{\text{all } j} \delta_j \widehat{\Delta}^{n-1}\big) \cup \Delta^n$ the subsimplicial set of $\widehat{\Delta}^n$ generated by *all* $\widehat{\Delta}^{n-1}$ boundary components, together with $\Delta^n \cong F^n(\Delta^n) \subset \widehat{\Delta}^n$.

Lemma A.4 $\widehat{\Theta}^n$ *is contractible.*

Proof For a subset $A \subset \{0, \ldots, n\}$, denote by $\widehat{\Upsilon}^n_A := \big(\bigcup_{j \in A} \delta_j \widehat{\Delta}^{n-1}\big) \cup \Delta^n$ the subsimplicial set $\widehat{\Upsilon}^n_A \subset \widehat{\Delta}^n$, given by Δ^n with "thickened" boundary components determined by A. In particular, $\widehat{\Upsilon}^n_{\{\}} = \Delta^n$ and $\widehat{\Upsilon}^n_{\{0,\ldots,n\}} = \widehat{\Theta}^n$. (Note that $\widehat{\Upsilon}^n_A$ may be explicitly described to have p–simplices given by sequences $(i_0, \ldots, i_p) \in \{0, \ldots, n\}^p$ such that either $i_0 \leq \cdots \leq i_p$, or there exists an element $i \in A$ such that $i_0 \neq i, \ldots, i_p \neq i$, or both.) We show that the $|\widehat{\Upsilon}^n_A|$ are contractible for all n and A. Since all $|\widehat{\Upsilon}^n_A|$ are CW–complexes, this is equivalent to showing that the $|\widehat{\Upsilon}^n_A|$ are connected and have zero homotopy groups.

We will repeatedly use the fact that, if X, Y, $X \cap Y$ and $X \cup Y$ are CW–complexes, and X, Y and $X \cap Y$ are contractible, then $X \cup Y$ is also contractible (which follows since $X \cup Y$ is certainly connected, has vanishing π_1 due to van Kampen, vanishing homology groups due to Mayer–Vietoris, and thus vanishing homotopy groups due to Hurewicz).

When $n = 1$, using that $\widehat{\Delta}^0 = \Delta^0$, we have for any $A \subset \{0,1\}$ that $\widehat{\Upsilon}^1_A = \Delta^1$, and $|\Delta^1|$ is contractible.

Now, for $n > 1$, assume by induction, that the $|\widehat{\Upsilon}^k_B|$ are contractible for all $k < n$ and all $B \subset \{0,\dots,k\}$. We perform a second induction on the number of elements of $A \subset \{0,\dots,n\}$. First, note that $\widehat{\Upsilon}^n_{\{\}} = \Delta^n$, and $|\Delta^n|$ is contractible. Thus, assume by induction that all $|\widehat{\Upsilon}^n_A|$ with $|A| < l$ are contractible. Now, let $A = \{i_1,\dots,i_l\} \subset \{0,\dots,n\}$ be an l–element set with, say, $i_1 < \cdots < i_l$. Writing

$$\widehat{\Upsilon}^n_{\{i_1,\dots,i_l\}} = \widehat{\Upsilon}^n_{\{i_1,\dots,i_{l-1}\}} \cup \delta_{i_l} \widehat{\Delta}^{n-1},$$

we know by induction that $|\widehat{\Upsilon}^n_{\{i_1,\dots,i_{l-1}\}}|$ is contractible, and also $|\delta_{i_l}\widehat{\Delta}^{n-1}| \approx |\widehat{\Delta}^{n-1}|$ is contractible (which was reviewed in the proof of Proposition A.2). Furthermore, $\widehat{\Upsilon}^n_{\{i_1,\dots,i_{l-1}\}} \cap \delta_{i_l}\widehat{\Delta}^{n-1} = \delta_{i_l}\widehat{\Upsilon}^{n-1}_{\{i_1,\dots,i_{l-1}\}} \cong \widehat{\Upsilon}^{n-1}_{\{i_1,\dots,i_{l-1}\}}$, and, by the first induction, $|\widehat{\Upsilon}^n_{\{i_1,\dots,i_{l-1}\}}| \cap |\delta_{i_l}\widehat{\Delta}^{n-1}| = |\widehat{\Upsilon}^{n-1}_{\{i_1,\dots,i_{l-1}\}}|$ is contractible as well. Thus, by the above fact, $|\widehat{\Upsilon}^n_{\{i_1,\dots,i_l\}}| = |\widehat{\Upsilon}^n_{\{i_1,\dots,i_{l-1}\}}| \cup |\delta_{i_l}\widehat{\Delta}^{n-1}|$ is also contractible. □

We are now ready to prove Proposition A.1.

Proof of Proposition A.1 Since both K and $\widehat{K}$ are Kan complexes, it suffices to show that $F^\sharp\colon \widehat{K} \to K$ induces isomorphisms on all simplicial homotopy groups (since these coincide with the homotopy groups of their geometric realizations; see [May 1967, Theorems 16.1 and 16.6]).

First, for $n = 0$, F induces a map $\pi_0(\widehat{K}) \to \pi_0(K)$ which is onto since $\widehat{\Delta}^0 = \Delta^0$ and thus $\widehat{K}_0 = K_0$. To see that the induced map $\pi_0(\widehat{K}) \to \pi_0(K)$ is one-to-one, assume $a, b \in K_0$ are equivalent $a \sim b$ in $\pi_0(K)$. Since K is a Kan complex, this means that (instead of a sequence of 1–simplices) there exists a single $c \in K_1$ such that $d_0(c) = a$ and $d_1(c) = b$. We need to check that $a \sim b$ in $\pi_0(\widehat{K})$, ie there exists a $\widehat{c} \in \widehat{K}_1$ with $d_0(\widehat{c}) = a$ and $d_1(\widehat{c}) = b$. Thus we need a simplicial set map $\Delta^1 \to \widehat{K}$, ie a map $\widehat{\Delta}^1 \to K$ making the following diagram commute:

$$\begin{tikzcd}
\widehat{\Theta}^1 = \Delta^1 \cup \delta_0\widehat{\Delta}^0 \cup \delta_1\widehat{\Delta}^0 \arrow[r, "c \cup a \cup b"] \arrow[d] & K \arrow[d] \\
\widehat{\Delta}^1 \arrow[r] \arrow[ur, dashed] & *
\end{tikzcd}$$

Note that the top arrow is well defined, and, since the left map is a trivial cofibration (ie injective and a weak equivalence) and K is a Kan complex, it follows that it lifts to a map $\widehat{\Delta}^1 \to K$, as needed.

Now, for $n \geq 1$, F induces a map $\pi_n(\widehat{K}, *) \to \pi_n(K, *)$ which is onto: if $c \in K_n$ with $d_i(c) = *$ for all i, represents an element of $\pi_n(K, *)$, then we want to produce a $\widehat{c} \in \widehat{K}_n$, ie $\widehat{c}\colon \widehat{\Delta}^n \to K$, with $d_i(\widehat{c}) = *$ for all i and which restricts to c under F. Thus, we need to find a lift making the following diagram commute:

$$\begin{tikzcd} \widehat{\Theta}^n = \Delta^n \cup \delta_0 \widehat{\Delta}^{n-1} \cup \cdots \cup \delta_n \widehat{\Delta}^{n-1} \arrow[r, "c \cup * \cup \cdots \cup *"] \arrow[d] & K \arrow[d] \\ \widehat{\Delta}^n \arrow[r] \arrow[ur, dashed] & * \end{tikzcd}$$

Again, the top arrow is well defined, since c restricts trivially to its boundaries. Just as before, we can find a lift, because $\widehat{\Theta}^n \to \widehat{\Delta}^n$ is a trivial cofibration and K is a Kan complex. Finally, we need to check that F induces a map $\pi_n(\widehat{K}, *) \to \pi_n(K, *)$, which is one-to-one. Since this map is a map of groups, it suffices to check that the kernel is trivial. More explicitly, we need to show that if $\widehat{c} \in K_n$ with $d_i(\widehat{c}) = *$ for all i represents a class of $\pi_n(\widehat{K}, *)$, which maps to $c = \widehat{c} \circ F^n \colon \Delta^n \xrightarrow{F^n} \widehat{\Delta}^n \xrightarrow{\widehat{c}} K$ which is trivial in $\pi_n(K, *)$, then $\widehat{c}$ is trivial in $\pi_n(\widehat{K}, *)$. For c to be trivial in $\pi_n(K, *)$ means that there is an $(n+1)$–simplex $q \in K_{n+1}$ such that $d_0(q) = c$ and $d_i(q) = *$ for all $i \geq 1$. We thus have the setup for the diagram

$$\begin{tikzcd} \widehat{\Theta}^{n+1} = \Delta^{n+1} \cup \delta_0 \widehat{\Delta}^n \cup \delta_1 \widehat{\Delta}^n \cup \cdots \cup \delta_n \widehat{\Delta}^n \arrow[r, "q \cup \widehat{c} \cup * \cup \cdots \cup *"] \arrow[d] & K \arrow[d] \\ \widehat{\Delta}^{n+1} \arrow[r] \arrow[ur, dashed] & * \end{tikzcd}$$

Since $\widehat{\Theta}^{n+1} \to \widehat{\Delta}^{n+1}$ is a trivial cofibration and K is a Kan complex, there exists a lift $\widehat{q} \in \widehat{K}_{n+1}$ with $d_0(\widehat{q}) = \widehat{c}$ and $d_i(\widehat{q}) = *$ for all $i \geq 1$. This shows that $\widehat{c}$ does indeed represent the trivial class in $\pi_n(\widehat{K}, *)$. □

Appendix B Explicit description of totalization

We now review the notion of totalization of a cosimplicial simplicial set.

B.1 Totalization

We recall from our previous work [2022, Definition D.1] and [Hirschhorn 2003, Definition 18.6.3]] the definition of totalization. Let $K^\bullet \colon \mathbf{\Delta} \to \mathrm{sSet}$ be a cosimplicial simplicial set, ie $K^l := K([l])$ is a simplicial set $K^l = K^l_\bullet$. Then, the totalization $\mathrm{Tot}(K^\bullet_\bullet)$ of K is defined as the simplicial set, which is the equalizer of the maps

$$\mathrm{Tot}(K^\bullet_\bullet) \to \prod_{[l] \in \mathbf{Obj}(\mathbf{\Delta})} (K^l)^{\Delta^l} \underset{\psi}{\overset{\phi}{\rightrightarrows}} \prod_{\rho \colon [n] \to [m]} (K^m)^{\Delta^n} \tag{B-1}$$

Here, by definition, $(K^p)^{\Delta^q}$ is the simplicial set whose n–simplices are simplicial set maps $((K^p)^{\Delta^q})_n = \mathrm{sSet}((\Delta^n \times \Delta^q)_\bullet, K^p_\bullet)$. Then a k–simplex in the totalization is given by some collection

$$\{x^{(k,l)}\}_{l \geq 0}, \quad \text{where } x^{(k,l)} \in \mathrm{sSet}(\Delta^k \times \Delta^l, K^l), \tag{B-2}$$

satisfying the coherence condition that they are in the above equalizer. Explicitly, for a fixed $j = 0, \dots, l+1$ the map $\delta_j : [l] \to [l+1]$ which skips j induces the maps

$$x^{(k,l+1)} \in \mathrm{sSet}(\Delta^k \times \Delta^{l+1}, K^{l+1}) \xrightarrow{d_j} \mathrm{sSet}(\Delta^k \times \Delta^l, K^{l+1}), \tag{B-3}$$

$$x^{(k,l)} \in \quad \mathrm{sSet}(\Delta^k \times \Delta^l, K^l) \quad \xrightarrow{d^j} \mathrm{sSet}(\Delta^k \times \Delta^l, K^{l+1}). \tag{B-4}$$

Then, for $x^{(k,l+1)}$ and $x^{(k,l)}$ as above,

$$d_j(x^{(k,l+1)}) = d^j(x^{(k,l)}). \tag{B-5}$$

Thus, a k–simplex, $\{x^{(k,l)}\}_{l=0,1,\dots}$, in the totalization of a cosimplicial simplicial set, $\mathrm{Tot}(K^\bullet_\bullet)$ is given by maps $x^{(k,l)} \in \mathrm{sSet}((\Delta^k \times \Delta^l)_\bullet, K^l_\bullet)$ for each $l = 0, 1, \dots,$ which can be thought of as a coherent "decoration" of the simplicial sets $\Delta^k \times \Delta^l$, for $l = 0, 1, \dots,$ by simplices in $K^l_\bullet$.

B.2 Simplices of $\Delta^k \times \Delta^l$

We now recall that there is a nice book-keeping device for the simplices of $\Delta^k \times \Delta^l$. In fact, the p–simplices of $\Delta^k \times \Delta^l$ can be described by nondecreasing paths with $p+1$ vertices in a $(k+1) \times (l+1)$ grid; we also call this a p–path. For example, the maximally nondegenerate (4+7)–simplices of $\Delta^4 \times \Delta^7$ can be labeled by paths[4] through a $(4+1) \times (7+1)$ grid, necessarily starting from $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$ and ending at $\left[\begin{smallmatrix}4\\7\end{smallmatrix}\right]$. For example, the following path of labels, which we denote by $\left[\begin{smallmatrix}0&1&1&1&2&3&4&4&4&4&4&4\\0&0&1&2&2&2&2&3&4&5&6&7\end{smallmatrix}\right]$, labels an element of $(\Delta^4 \times \Delta^7)_{11}$:

$$\begin{array}{cccccccc}
\boxed{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]} & \boxed{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} & \boxed{\left[\begin{smallmatrix}0\\2\end{smallmatrix}\right]} & \boxed{\left[\begin{smallmatrix}0\\3\end{smallmatrix}\right]} & \boxed{\left[\begin{smallmatrix}0\\4\end{smallmatrix}\right]} & \boxed{\left[\begin{smallmatrix}0\\5\end{smallmatrix}\right]} & \boxed{\left[\begin{smallmatrix}0\\6\end{smallmatrix}\right]} & \boxed{\left[\begin{smallmatrix}0\\7\end{smallmatrix}\right]} \\
\boxed{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} & \boxed{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]} & \boxed{\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]} & \boxed{\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]} & \boxed{\left[\begin{smallmatrix}1\\4\end{smallmatrix}\right]} & \boxed{\left[\begin{smallmatrix}1\\5\end{smallmatrix}\right]} & \boxed{\left[\begin{smallmatrix}1\\6\end{smallmatrix}\right]} & \boxed{\left[\begin{smallmatrix}1\\7\end{smallmatrix}\right]} \\
\boxed{\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]} & \boxed{\left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]} & \boxed{\left[\begin{smallmatrix}2\\2\end{smallmatrix}\right]} & \boxed{\left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]} & \boxed{\left[\begin{smallmatrix}2\\4\end{smallmatrix}\right]} & \boxed{\left[\begin{smallmatrix}2\\5\end{smallmatrix}\right]} & \boxed{\left[\begin{smallmatrix}2\\6\end{smallmatrix}\right]} & \boxed{\left[\begin{smallmatrix}2\\7\end{smallmatrix}\right]} \\
\boxed{\left[\begin{smallmatrix}3\\0\end{smallmatrix}\right]} & \boxed{\left[\begin{smallmatrix}3\\1\end{smallmatrix}\right]} & \boxed{\left[\begin{smallmatrix}3\\2\end{smallmatrix}\right]} & \boxed{\left[\begin{smallmatrix}3\\3\end{smallmatrix}\right]} & \boxed{\left[\begin{smallmatrix}3\\4\end{smallmatrix}\right]} & \boxed{\left[\begin{smallmatrix}3\\5\end{smallmatrix}\right]} & \boxed{\left[\begin{smallmatrix}3\\6\end{smallmatrix}\right]} & \boxed{\left[\begin{smallmatrix}3\\7\end{smallmatrix}\right]} \\
\boxed{\left[\begin{smallmatrix}4\\0\end{smallmatrix}\right]} & \boxed{\left[\begin{smallmatrix}4\\1\end{smallmatrix}\right]} & \boxed{\left[\begin{smallmatrix}4\\2\end{smallmatrix}\right]} & \boxed{\left[\begin{smallmatrix}4\\3\end{smallmatrix}\right]} & \boxed{\left[\begin{smallmatrix}4\\4\end{smallmatrix}\right]} & \boxed{\left[\begin{smallmatrix}4\\5\end{smallmatrix}\right]} & \boxed{\left[\begin{smallmatrix}4\\6\end{smallmatrix}\right]} & \boxed{\left[\begin{smallmatrix}4\\7\end{smallmatrix}\right]}
\end{array} \tag{B-6}$$

We can apply $x^{(4,7)} \in \mathrm{sSet}(\Delta^4 \times \Delta^7, K^7)$ to this path, which will give an element

$$x^{(4,7)}_{\left[\begin{smallmatrix}0&1&1&1&2&3&4&4&4&4&4&4\\0&0&1&2&2&2&2&3&4&5&6&7\end{smallmatrix}\right]} \in K^7_{11}$$

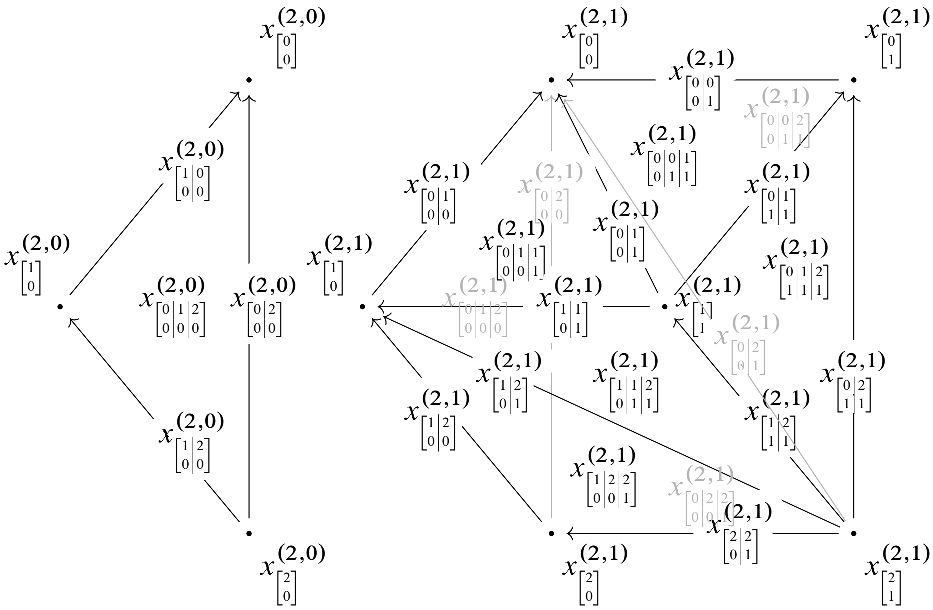
(note the simplicial degree 11 comes from the 11–path with 12 vertices). Note that, just as the simplices of the standard n–simplex have direction, these paths must be nondecreasing in both directions. Additionally, the faces of a p–simplex of $\Delta^k \times \Delta^l$ given by a path would consist of subsequences of that path, eg $\left[\begin{smallmatrix}1&1&2&4&4&4\\0&2&2&3&4&6\end{smallmatrix}\right]$ describes a 5–simplex in $(\Delta^4 \times \Delta^7)_5$ which is a lower face of the above 11–simplex. Degenerate simplices are described by paths where at least one of the indices is repeated, eg $\left[\begin{smallmatrix}1&2&4&4&4\\2&2&3&3&6\end{smallmatrix}\right]$.

[4]Informally, this path might be referred to as a "taxi-cab"path as it only moves in a rectangular fashion.

Using this notation, the coherence condition (B-5) can be stated more precisely as follows. Let $K_\bullet^\bullet$ be a cosimplicial simplicial set and let $\delta_j\colon [l]\to[l+1]$ be the map that skips j. We have the coface maps, $d^j\colon K_\bullet^l\to K_\bullet^{l+1}$, as well as the maps d_j in (B-3) given by precomposition with $\Delta_\bullet^l\to\Delta_\bullet^{l+1}$. Then we can explicitly describe the k–simplices of the totalization, $\mathrm{Tot}(K_\bullet^\bullet)_k$, as collections $\{x^{(k,l)}\in \mathrm{sSet}(\Delta^k\times\Delta^l, K_\bullet^l)\}_{l=0,1,\dots}$, which, applied to p–simplices of $\Delta^k\times\Delta^l$ labeled by the paths $\left[\begin{smallmatrix}\alpha_0 & \cdots & \alpha_p\\ \beta_0 & \cdots & \beta_p\end{smallmatrix}\right]$ with $0\le\alpha_0\le\cdots\le\alpha_p\le k$ and $0\le\beta_0\le\cdots\le\beta_p\le l$ as described above, assign elements $x^{(k,l)}_{\left[\begin{smallmatrix}\alpha_0 & \cdots & \alpha_p\\ \beta_0 & \cdots & \beta_p\end{smallmatrix}\right]}\in K_p^l$, satisfying

$$x^{(k,l+1)}_{\left[\begin{smallmatrix}\alpha_0 & \cdots & \alpha_p\\ \delta_j(\beta_0) & \cdots & \delta_j(\beta_p)\end{smallmatrix}\right]} = d^j\big(x^{(k,l)}_{\left[\begin{smallmatrix}\alpha_0 & \cdots & \alpha_p\\ \beta_0 & \cdots & \beta_p\end{smallmatrix}\right]}\big)\in K_p^{l+1}. \tag{B-7}$$

For example, for $k=2$, we have the assignments, for $l=0,1$,



As an example, for $\delta_0\colon[0]\to[1]$, equation (B-7) yields $x^{(2,1)}_{\left[\begin{smallmatrix}0&1&2\\1&1&1\end{smallmatrix}\right]} = d^0\big(x^{(2,0)}_{\left[\begin{smallmatrix}0&1&2\\0&0&0\end{smallmatrix}\right]}\big)$, which relates the cells for different l's.

Note that, for a fixed k and l, the

$$x^{(k,l)}_{\left[\begin{smallmatrix}\alpha_0 & \cdots & \alpha_p\\ \beta_0 & \cdots & \beta_p\end{smallmatrix}\right]}\in K_p^l$$

are in fact determined by the maximal paths

$$x^{(k,l)}_{\left[\begin{smallmatrix}\alpha_0 & \cdots & \alpha_{k+l}\\ \beta_0 & \cdots & \beta_{k+l}\end{smallmatrix}\right]}\in K_{k+l}^l,$$

since each p–path is a subpath of a maximal path and so the p–cell is in the image of some face map $K_{k+l}^l\to K_p^l$ for some map $[p]\to[k+l]$.

Example B.1 For example, for a simplicial presheaf $\boldsymbol{F}:\mathbb{C}\mathrm{Man}^{\mathrm{op}}\to\mathrm{sSet}$, and an open cover $\mathcal{U}=\{U_i\}_{i\in\mathcal{I}}$ of $X\in\mathbb{C}\mathrm{Man}$, we take

$$K^l_p=\boldsymbol{F}_p(\check{N}\mathcal{U}_l)=\prod_{i_0,\dots,i_l\in\mathcal{I}}\boldsymbol{F}_p(U_{i_0,\dots,i_l}).$$

In this case a p–cell in $x\in K^l_p$ is given by $x=\{x_{i_0,\dots,i_l}\}$, where, for each $(l+1)$–fold intersection $U_{i_0,\dots,i_l}$, $x_{i_0,\dots,i_l}\in\boldsymbol{F}_p(U_{i_0,\dots,i_l})$ is a p–cell. Note that the map $d^j:K^l\to K^{l+1}$ in (B-4) and (B-7) is induced by the inclusions $\mathrm{incl}\colon U_{i_0,\dots,i_{l+1}}\hookrightarrow U_{i_0,\dots,\hat{i}_j,\dots,i_{l+1}}$ as $\boldsymbol{F}_p(\mathrm{incl})\colon\boldsymbol{F}_p(U_{i_0,\dots,\hat{i}_j,\dots,i_{l+1}})\to\boldsymbol{F}_p(U_{i_0,\dots,i_{l+1}})$. In particular, continuing the example from the figure above, $x^{(2,0)}_{\left[\begin{smallmatrix}0&1&2\\0&0&0\end{smallmatrix}\right]}$ and $x^{(2,1)}_{\left[\begin{smallmatrix}0&1&2\\1&1&1\end{smallmatrix}\right]}$ have components

$$x^{(2,0)}_{\left[\begin{smallmatrix}0&1&2\\0&0&0\end{smallmatrix}\right];i}\in\boldsymbol{F}_2(U_i),\qquad x^{(2,1)}_{\left[\begin{smallmatrix}0&1&2\\1&1&1\end{smallmatrix}\right];i_0i_1}\in\boldsymbol{F}_2(U_{i_0i_1}),$$

respectively and the compatibility of (B-7) now yields,

$$x^{(2,1)}_{\left[\begin{smallmatrix}0&1&2\\1&1&1\end{smallmatrix}\right];i_0i_1}=d^0\big(x^{(2,0)}_{\left[\begin{smallmatrix}0&1&2\\0&0&0\end{smallmatrix}\right]}\big)=U_{i_0i_1}|_{x^{(2,0)}_{\left[\begin{smallmatrix}0&1&2\\0&0&0\end{smallmatrix}\right];i_1}}.$$

B.3 Totalization for the case $K=\mathrm{sSet}(\widehat{\Delta},\widetilde{K})$

We are interested in the totalization of $K^\bullet_\bullet=\mathbf{Perf}^{\widehat{\Delta}}(\check{N}U)=\mathrm{sSet}(\widehat{\Delta},\mathbf{Perf}(\check{N}U))$. Thus, assume now that we have a cosimplicial simplicial set $K^\bullet_\bullet$, which is of the form $K^l_p:=\mathrm{sSet}(\widehat{\Delta}^p,\widetilde{K}^l)$ for some other cosimplicial simplicial set $\widetilde{K}^\bullet_\bullet$. By rewriting simplicial sets as colimits of their simplices, and using continuity of the hom-functor in the category sSet, we see that

$$\begin{aligned}\text{(B-8)}\qquad \mathrm{sSet}(\Delta^k\times\Delta^l,K^l)&=\mathrm{sSet}\big(\operatorname*{colim}_{\Delta^p\to\Delta^k\times\Delta^l}\Delta^p,K^l\big)=\lim_{\Delta^p\to\Delta^k\times\Delta^l}\mathrm{sSet}(\Delta^p,K^l)\\&=\lim_{\Delta^p\to\Delta^k\times\Delta^l}K^l_p=\lim_{\Delta^p\to\Delta^k\times\Delta^l}\mathrm{sSet}(\widehat{\Delta}^p,\widetilde{K}^l)\\&=\mathrm{sSet}\big(\operatorname*{colim}_{\Delta^p\to\Delta^k\times\Delta^l}\widehat{\Delta}^p,\widetilde{K}^l\big).\end{aligned}$$

We see from the above identification that decorations of simplicial sets $\Delta^k\times\Delta^l$ by simplices in $K^l_\bullet$ is equivalent to first gluing the simplicial sets $\widehat{\Delta}^n$ along the corresponding Δ^n sitting inside $\Delta^k\times\Delta^l$, and then decorating this colimit made of various $\widehat{\Delta}^n$ by simplices in $\widetilde{K}^l$. Using the description of $\widehat{\Delta}$ from Example 2.8, it now follows that the k–simplices of $\mathrm{Tot}(K^\bullet_\bullet)$ are in fact given by

$$x^{(k,l)}_{\left[\begin{smallmatrix}\alpha_0&\cdots&\alpha_p\\\beta_0&\cdots&\beta_p\end{smallmatrix}\right]}\in\widetilde{K}^l_p,$$

where this time the path described by $\left[\begin{smallmatrix}\alpha_0&\cdots&\alpha_p\\\beta_0&\cdots&\beta_p\end{smallmatrix}\right]$ is now permitted to move horizontally and vertically in each direction in the grid, ie possibly decreasing, but within the indices of a nondecreasing path. For example, in the $(2+1)\times(3+1)$ grid of vertices, take the 5–cell given by the map $\Delta^5\hookrightarrow\Delta^2\times\Delta^3$

whose nondecreasing path is $\left[\begin{array}{c|c|c|c|c|c} 0&0&0&1&1&2\\ 0&1&2&2&3&3 \end{array}\right]$. Then, for the corresponding $\widehat{\Delta}^5$, there is a nondegenerate 9–simplex

$$x^{(2,3)}_{\left[\begin{array}{c|c|c|c|c|c|c|c|c|c} 0&0&1&0&0&1&0&2&1&2\\ 0&1&3&1&2&2&1&3&3&3 \end{array}\right]} \in \widetilde{K}^3_9, \tag{B-9}$$

which is both increasing and decreasing using the indices of the 5–path $\left[\begin{array}{c|c|c|c|c|c} 0&0&0&1&1&2\\ 0&1&2&2&3&3 \end{array}\right]$ in $\Delta^2 \times \Delta^3$. Thus, in the totalization $\mathrm{Tot}(K)$, a 2–simplex $x = \{x^{(2,l)}\}$ needs to assign such an element in $\widetilde{K}^3_9$ to the 9–path from (B-9). However, note that there is no assignment to the path $\left[\begin{array}{c|c|c|c} 0&1&0&1\\ 0&0&1&1 \end{array}\right]$, because every map $\Delta^n \to \Delta^k \times \Delta^l$ is necessarily nondecreasing in both components and so one can never obtain both $\left[\begin{smallmatrix} 0\\ 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1\\ 0 \end{smallmatrix}\right]$ in the same path. To summarize, a cell in $\mathrm{Tot}(K)$ has to assign elements in $\widetilde{K}$ exactly to any path which uses the indices of a nondecreasing path.

Finally, note that the coherence condition on these simplices of the totalization is the same as expressed in (B-7).

Appendix C Totalization and fibrant objects

The purpose of this appendix is to prove Proposition C.1.

Proposition C.1 *If $\boldsymbol{F}$ is a projectively fibrant simplicial presheaf* (*such as $\boldsymbol{F} = \mathbf{IVB}$*) *then* $\mathrm{Tot}(\boldsymbol{F}(\check{N} U_\bullet))$ *is a Kan complex.*

We start with the following lemma:

Lemma C.2 *The totalization functor* (see Appendix B) $\mathrm{Tot}\colon (\mathrm{Set}^{\Delta^{\mathrm{op}}})^{\Delta} \to \mathrm{Set}^{\Delta^{\mathrm{op}}}$ *is a right adjoint.*

Proof We prove this directly by defining the left adjoint L. For any simplicial set $X^\bullet$, let $L(X^\bullet)$ be the cosimplicial simplicial set $n \mapsto X^\bullet \times \Delta^n$, where Δ^n is the standard n–simplex.

To show that these functors form an adjoint pair, let $X^\bullet$ be a simplicial set and $Y^\bullet_\bullet$ be a cosimplicial simplicial set. Since $\mathrm{Set}^{\Delta^{\mathrm{op}}}$ is a simplicial model category (under the usual Quillen structure), $\mathrm{Set}^{\Delta^{\mathrm{op}}}(X \times \Delta^n, Y^n_\bullet)$ is in bijection with $\mathrm{Set}^{\Delta^{\mathrm{op}}}(X, (Y^n_\bullet)^{\Delta^n})$. Since $\mathrm{Tot}(Y^\bullet_\bullet) = (Y^\bullet_\bullet)^{\Delta}$, we have our bijection. □

Lemma C.3 *The functors* (L, Tot) *form a Quillen adjunction between the Reedy model structure* [Hirschhorn 2003, Section 15] *of cosimplicial simplicial sets and the usual Quillen model structure on simplicial sets.*

Proof It is enough to show that L preserves cofibrations and trivial cofibrations. Suppose $f\colon X^\bullet \to Y^\bullet$ is a cofibration of simplicial sets, ie a levelwise monomorphism. By [Hirschhorn 2003, Theorem 15.9.9], to show that $L(f)$ is a Reedy cofibration, it is enough to show that $L(f)$ is a monomorphism that takes the maximal augmentation of $L(X^\bullet)$ isomorphically onto the maximal augmentation of $L(Y^\bullet)$.

Since $L(f) = f \times \mathrm{Id}$ and f is a levelwise monomorphism, $L(f)$ is a monomorphism. The maximal augmentation of $L(X^\bullet)$ and $L(Y^\bullet)$ are empty. So L preserves cofibrations.

Suppose $f : X^\bullet \to Y^\bullet$ is a trivial cofibration. We need to show that $L(f) \colon L(X^\bullet) \to L(Y^\bullet)$ is a Reedy weak equivalence. Since $L(f) = f \times \mathrm{Id}$, then $L_n F \colon X^\bullet \times \Delta^n \to Y^\bullet \times \Delta^n$ is a weak equivalence. □

Lemma C.4 *Let X be a Reedy fibrant cosimplicial simplicial set. Then* $\mathrm{Tot}(X)$ *is a Kan complex.*

Proof Since Tot is a right adjoint, it preserves fibrations and terminal objects. So Tot preserves fibrant objects. □

Lemma C.5 *Let V be a manifold and $U_\bullet$ be an open cover of V. Let $\boldsymbol{F}$ be a simplicial presheaf that takes values in Kan complexes. Then $\boldsymbol{F}(\check{N}U_\bullet) \colon \Delta \to \mathrm{Set}^{\Delta^{\mathrm{op}}}$* (see (4-1)) *is a Reedy fibrant cosimplicial simplicial set.*

Proof This proof uses some conventions from [Hirschhorn 2003, Section 15] for the Reedy model structure and is analogous to that of Block, Holstein and Wei [2017, Proposition 4.3]. We need to show that the matching map $\boldsymbol{F}(\check{N}U_n) \to M_n(\boldsymbol{F}(\check{N}U_\bullet))$ is a fibration for each n, where

$$\boldsymbol{F}(\check{N}U_n) := \underline{\mathrm{sPre}}\Big(\coprod_{i_0,\dots,i_n} yU_{i_0,\dots,i_n}, \boldsymbol{F}\Big) = \prod_{i_0,\dots,i_n} \boldsymbol{F}(U_{i_0,\dots,i_n}).$$

Write $\check{N}U_n$ as the coproduct

$$\check{N}U_n = \coprod_{\substack{i_0,\dots,i_n \\ i_j \neq i_{j+1}}} yU_{i_0,\dots,i_n} \amalg \Big(\coprod_{k=1}^{n} \coprod_{\substack{i_0,\dots,i_n \\ i_{j_1}=i_{j_1+1},\dots,i_{j_k}=i_{j_k+1}}} yU_{i_0,\dots,i_n}\Big)$$

and apply $\boldsymbol{F}$ to get

$$\prod_{\substack{i_0,\dots,i_n \\ i_j \neq i_{j+1}}} \boldsymbol{F}(U_{i_0,\dots,i_n}) \times \prod_{k=1}^{n} \Big(\prod_{\substack{i_0,\dots,i_n \\ i_{j_1}=i_{j_1+1},\dots,i_{j_k}=i_{j_k+1}}} \boldsymbol{F}(U_{i_0,\dots,i_n})\Big).$$

First note that the right side of this cartesian product is the matching object at n, $M_n\boldsymbol{F}(\check{N}U)$. This is seen directly by showing that this product is the terminal object in the category of cones under $\boldsymbol{F}(\check{N}U)$ restricted to the matching category $\partial([n] \downarrow \overleftarrow{\Delta})$ (see [Hirschhorn 2003, Definition 15.2.3.2]). The product

$$\begin{tikzcd}
 & & \prod_{k=1}^{n}\Big(\prod_{\substack{i_0,\dots,i_n \\ i_{j_1}=i_{j_1+1},\dots,i_{j_k}=i_{j_k+1}}} \boldsymbol{F}(U_{i_0,\dots,i_n})\Big) \arrow[dll] \arrow[dl] \arrow[dr] \arrow[drr] & & \\
\boldsymbol{F}(\check{N}U_{n-1}) \arrow[r] & \boldsymbol{F}(\check{N}U_{n-2}) \arrow[r] & \cdots \arrow[r] & \boldsymbol{F}(\check{N}U_1) \arrow[r] & \boldsymbol{F}(\check{N}U_0)
\end{tikzcd}$$

is a cone under $\boldsymbol{F}(\check{N}U)$, where $\boldsymbol{F}(\check{N}U_{n-j}) = \prod_{i_0,\dots,i_{n-j}} \boldsymbol{F}(U_{i_0,\dots,i_{n-j}})$ and the vertical maps are projections.

Now, suppose we have a cone under $\boldsymbol{F}(\check{N}U)$:

(C-1)
$$\begin{tikzcd}
 & & Y \arrow[dll, "f_1"'] \arrow[dl, "f_2"] \arrow[dr, "f_{n-1}"'] \arrow[drr, "f_n"] & & \\
\boldsymbol{F}(\check{N}U_{n-1}) \arrow[r] & \boldsymbol{F}(\check{N}_{n-2}) \arrow[r] & \cdots \arrow[r] & \boldsymbol{F}(\check{N}U_1) \arrow[r] & \boldsymbol{F}(\check{N}U_0)
\end{tikzcd}$$

Then, to define the map Y into the product, send y to $(f_1(y), f_2(y), \dots, f_n(y))$.

Finally, we see that the matching map $\boldsymbol{F}(\check{N}U_n) \times M_n(\check{N}U) \to M_n(\check{N}U)$ is the projection onto the second factor. Since $\boldsymbol{F}(\check{N}U_n)$ is a Kan complex, the projection is a fibration. □

Applying Lemma C.4 to Lemma C.5 proves Proposition C.1.

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