The localization of orthogonal calculus with respect to homology

NIALL TAGGART

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Niall Taggart

For a set of maps of based spaces S we construct a version of Weiss's orthogonal calculus which depends only on the S–local homotopy type of the functor involved. We show that S–local homogeneous functors of degree n are equivalent to levelwise S–local spectra with an action of the orthogonal group $O(n)$ via a zigzag of Quillen equivalences between appropriate model categories. Our theory specialises to homological localizations and nullifications at a based space. We give a variety of applications including a reformulation of the telescope conjecture in terms of our local orthogonal calculus and a calculus version of Postnikov sections. Our results also apply when considering the orthogonal calculus for functors which take values in spectra.

## 1 Introduction

1.1 Motivation

Weiss's orthogonal calculus [1995] studies functors from the category of real inner product spaces and isometries to the category of based spaces or spectra. The motivation for such a version of functor calculus comes from a desire to study geometric and differential topology through a homotopy-theoretic lens. For example, Arone, Lambrechts and Volić [Arone et al. 2007] and Arone [2009] utilised Weiss's calculus to provide a comprehensive study of the (stable) homotopy type of spaces of embeddings $\mathrm{Emb}(M, N \times \mathbb{R}^k)$ where M and N are fixed smooth manifolds. More recently, Krannich and Randal-Williams [2021] have studied the Weiss tower of the classifying space $\mathrm{BTOP}(\mathbb{R}^k)$ of the group of homeomorphisms of $\mathbb{R}^k$ to understand the homotopy type of the space of diffeomorphisms of discs. In all of these cases, the authors are only able to ascertain geometric information up to rational homotopy via ad-hoc means. These vastly varying approaches highlight the need for a comprehensive account of the interactions between orthogonal calculus and localizations.

The theory of localizations at homology theories are ubiquitous and have had wide applications; of particular note is *chromatic homotopy theory* which among other things gives a spectrum level interpretation for the periodic families appearing in the stable homotopy groups of spheres. An extensive amount of effort has been geared toward understanding how localization at homology theories — particularly the chromatic localizations — interact with Goodwillie's calculus of functors [Arone and Mahowald 1999;

Kuhn 2004; 2006a; 2006b]; see eg [Kuhn 2007] for a survey. Analogous questions remain in Weiss's orthogonal calculus, and we propose a noticeably different approach than those applied to the Goodwillie calculus.
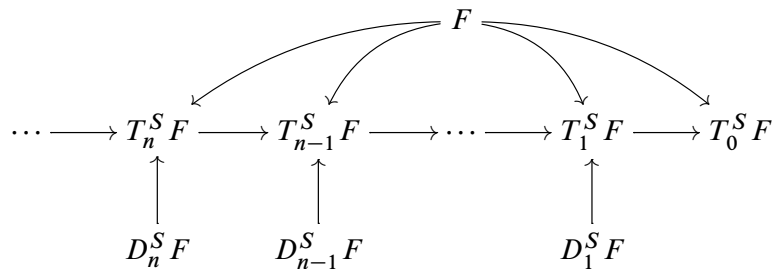
Overview

Given a functor F from the category of Euclidean spaces to the category of based spaces or spectra, the calculus assigns a tower of functors

$$\cdots \longrightarrow T_n F \longrightarrow T_{n-1} F \longrightarrow \cdots \longrightarrow T_1 F \longrightarrow T_0 F$$
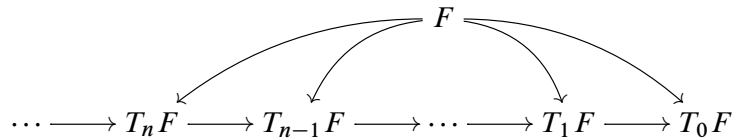
called the *Weiss tower* for F. The functor $T_n F$ is a categorification of the nth Taylor polynomial from differential calculus. The nth layer of the tower $D_n F$ is the homotopy fibre of the map $T_n F \to T_{n-1} F$, and is a categorification of homogeneous functions from differential calculus. Orthogonal calculus is synonymous with being the most computationally challenging flavour of functor calculus due to the interaction between the highly "geometric" nature of the objects of study and the highly homotopical constructions.

Let $\mathcal{C}$ denote either the category of based spaces or spectra. Given a set S of maps in $\mathcal{C}$ we produce an *S–local Weiss tower* of the form

$$\cdots \longrightarrow T_n^S F \longrightarrow T_{n-1}^S F \longrightarrow \cdots \longrightarrow T_1^S F \longrightarrow T_0^S F$$

$$D_n^S F \qquad D_{n-1}^S F \qquad D_1^S F$$

To understand the S–local Weiss tower, we utilise Bousfield's [1975; 1979] interpretation of localizations in terms of model structures on $\mathcal{C}$. We begin by constructing a model structure, denoted by $\mathrm{Poly}^{\leq n}(\mathcal{J}_0, L_S \mathcal{C})$, which captures the homotopy theory of functors which are S–locally polynomial of degree less than or equal n. Under some assumptions on the set of localizing objects the composite $T_n L_S$ is a fibrant replacement functor, hence satisfying the necessary universal property.

We further construct a model structure, denoted by $\mathrm{Homog}^n(\mathcal{J}_0, L_S \mathcal{C})$, which captures the homotopy theory of functors which are S–locally homogeneous of degree n. Through a zigzag of Quillen equivalences we characterise the S–local n–homogeneous functors in terms of appropriately S–local spectra with an action of $O(n)$.

Theorem (Corollary 5.16) *Let S be a set of maps of based spaces and $n \geq 0$. There is a zigzag of Quillen equivalences*

$$\mathsf{Homog}^n(\mathcal{J}_0, L_S\mathsf{Top}_*) \simeq_Q \mathsf{Sp}(L_S\mathsf{Top}_*)[O(n)],$$

where $\mathsf{Sp}(L_S\mathsf{Top}_*)[O(n)]$ is the category of levelwise S–local spectra with an action of $O(n)$.

Theorem (Corollary 5.18) *Let S be a set of maps of spectra and $n \geq 0$. There is a zigzag of Quillen equivalences*

$$\mathsf{Homog}^n(\mathcal{J}_0, L_S\mathsf{Sp}) \simeq_Q L_S\mathsf{Sp}[O(n)],$$

where $L_S\mathsf{Sp}[O(n)]$ is the category of S–local spectra with an action of $O(n)$.

In particular, an S–local n–homogeneous functor F is determined by and determines an appropriately S–local spectrum with an $O(n)$–action, denoted by $\partial_n^S F$. On the derived level, we obtain a computationally accessible classification theorem for S–local homogeneous of degree n functors.

Theorem (Theorem 5.20) *Let S be a set of maps of in $\mathcal{C}$ and $n \geq 1$.*

(1) *A Top_*–valued S–local n–homogeneous functor F is objectwise weakly equivalent to the functor*

$$V \mapsto \Omega^\infty[(S^{\mathbb{R}^n \otimes V} \wedge \partial_n^S F)_{hO(n)}],$$

    and any functor of the above form is objectwise S–local and n–homogeneous.

(2) *An Sp–valued S–local n–homogeneous functor F is objectwise weakly equivalent to the functor*

$$V \mapsto (S^{\mathbb{R}^n \otimes V} \wedge \partial_n^S F)_{hO(n)},$$

    and any functor of the above form is objectwise S–local and n–homogeneous.

Applications

We envision that the applications of this local version of orthogonal calculus are vast. For example, extending the rational computations of [Arone 2009; Arone et al. 2007; Krannich and Randal-Williams 2021] to higher chromatic height or another perspective on the full understanding of the Weiss tower of BO$(-)$ in v_n–periodic homotopy theory achieved by Arone [2002] using computations of Arone and Mahowald [1999].

Very little of our results use the fact that the target category is based spaces or spectra. The largest hurdle to having a theory of localizations of orthogonal calculus with target any (simplicial cofibrantly generated) model category is the development of orthogonal calculus in this realm. We hope that our exposé of orthogonal calculus with target space a localization of spaces or spectra will motivate the construction of orthogonal calculus based on more general homotopy theories such as arbitrary model categories or ∞–categories.

In the last part of this paper, we give several initial applications, which we survey here.

Bousfield classes   Bousfield [1979] introduced an equivalence relation on the stable homotopy category that has turned out to be of extreme importance. Define the Bousfield class $\langle E \rangle$ of a spectrum E to be the collection of E–acyclic spectra, and say that E and E' are Bousfield equivalent if and only if $\langle E \rangle = \langle E' \rangle$. These Bousfield classes assemble into a lattice, the understanding of which has been a major task in stable homotopy theory. For example, the nilpotence theorem of Devanitz, Hopkins and Smith [Devinatz et al. 1988; Hopkins and Smith 1998] is equivalent to a classification of the Bousfield classes for *finite spectra*. The Bousfield lattice has many interesting interactions with homological localizations of orthogonal calculus.

Theorem   (Example 6.6)   *Let E and E' be spectra. The E–local orthogonal calculus is equivalent to the E'–local orthogonal calculus if and only if E and E' are Bousfield equivalent.*

Fix a prime p. Ravenel's height n telescope conjecture [1984, Conjecture 10.5] is the statement that the height n Morava K–theory, $K(n)$, is Bousfield equivalent to $T(n)$, the telescope of any v_n–self map on a finite type n complex. The telescope conjecture is trivial at height $n = 0$, has been verified at height $n = 1$ and at all primes by Bousfield [1979], Mahowald [1981] and Miller [1981], but in general, is widely believed to be false.

Theorem   (Corollary 6.9)   *The height n telescope conjecture holds if and only if the $K(n)$–local orthogonal calculus and the $T(n)$–local orthogonal calculus are equivalent.*

The Weiss tower of a functor F produces a spectral sequence as it is a tower of fibrations. We call this spectral sequence the *Weiss spectral sequence*. From a computational perspective we obtain the following relation between the telescope conjecture and the local Weiss spectral sequences.

Theorem   (Lemma 6.10)   *If the height n telescope conjecture holds, then for all $r \geq 0$, the rth page of the $T(n)$–local Weiss spectral sequence is isomorphic to the rth page of the $K(n)$–local Weiss spectral sequence.*

Nullifications   For functors from the category of Euclidean spaces to the category of based spaces we also consider localization at a based space W, which is sometimes referred to as *nullification*. In this setting W–local objects are also called W–periodic, following Bousfield [1994] and Dror Farjoun [1996].

We give alternative constructions for the n–polynomial and n–homogenous model structures when the localization is a nullification. These alternative constructions yield an identical n–polynomial model structure but sheds new light on some of the formal properties of the model structure, and yield an n–homogeneous model structure which is Quillen equivalent to the original W–local model structure via the identity functor. These alternative descriptions are particularly useful when considering Postnikov sections of orthogonal calculus.

The results obtained for nullifications do not hold for more general localizations as the techniques employed rely crucially on a right properness condition on the model categories. We show in Proposition 7.2 that the right proper condition is satisfied if and only if the localization is a nullification. This is an extension of a remark of Bousfield [2001].

Postnikov sections Considering nullifications with respect to the spheres produces a theory of Postnikov sections in orthogonal calculus. We prove that our S^{k+1}–local projective model structure on the category of functors from Euclidean spaces to based spaces is identical to the model structure of k–types in the category of functors from Euclidean spaces to based spaces in the sense of k–types in an arbitrary model category developed by Gutiérrez and Roitzheim [2017, Section 4].

Theorem (Proposition 8.2) *Let $k \geq 0$. The model structure of k–types in orthogonal functors is identical to the S^{k+1}–local model structure*; *that is, there is an equality of model structures,*

$$P_k \operatorname{Fun}(\mathcal{J}_0, \operatorname{Top}_*) := L_{W_k} \operatorname{Fun}(\mathcal{J}_0, \operatorname{Top}_*) = \operatorname{Fun}(\mathcal{J}_0, L_{S^{k+1}} \operatorname{Top}_*).$$

As an application we produce a tower of model categories

$$\cdots \to \operatorname{Homog}^n(\mathcal{J}_0, P_k \operatorname{Top}_*) \to \cdots \to \operatorname{Homog}^n(\mathcal{J}_0, P_0 \operatorname{Top}_*),$$

where $P_k \operatorname{Top}_*$ denotes the S^{k+1}–local model structure on based spaces. By applying the theory of homotopy limits of model categories, we show that the n–homogeneous model structure of Barnes and Oman [2013, Proposition 6.9] is the homotopy limit of this tower, in the following sense.

Theorem (Corollary 8.13) *There is a Quillen equivalence*

$$\operatorname{Homog}^n(\mathcal{J}_0, \operatorname{Top}_*) \simeq_Q \operatorname*{holim}_k \operatorname{Homog}^n(\mathcal{J}_0, P_k \operatorname{Top}_*).$$

Relation to other work

This work is intimately related to the rational orthogonal calculus developed by Barnes [2017]; by replacing our generalised homology theory E_* with rational homology one recovers Barnes' theory.

Unstable chromatic homotopy theory can be described algebraically, via Heuts's [2021] *algebraic model* for v_n–periodic spaces via an equivalence (of ∞–categories) with Lie algebras in $T(n)$–local spectra. This model indicated that there is likely a relationship between v_n–periodic orthogonal calculus and orthogonal calculus of Heuts's Lie algebra models. Such an equivalence at chromatic height zero suggests a relationship between rational orthogonal calculus and the algebraic models for rational homotopy theory of Sullivan [1977] and Quillen [1969]. This together with Barnes' [2017] model for rational n–homogeneous functors using the classification of rational spectra with an $O(n)$–action as torsion modules over the rational cohomology ring of $BSO(n)$ of Greenlees and Shipley [2014] suggests the existence of *algebraic model calculi*. We plan to return to this in future work.

This work also forms part of an extensive program to go "beyond orthogonal calculus" which was initiated in the PhD thesis of the author [Taggart 2020], together with a series of articles exploring extensions of the orthogonal calculus and the relations between these [Taggart 2021; 2022a; 2022b; 2023]. This extensive project hopes to illuminate our understanding of orthogonal calculus which (at least relative to Goodwillie calculus) remains largely unexplored.

The future applications of the homological localization of orthogonal calculus are abounding. For example in the recent work of Beaudry, Bobkova, Pham and Xu [Beaudry et al. 2022], the authors compute the tmf–homology of $\mathbb{R}P^2$, where tmf denotes the connective spectrum of topological modular forms. Their computation for $\mathbb{R}P^2$ and the tmf–local Weiss tower for the functor $V \mapsto \mathbb{R}P(V)$ should yield a calculation of the tmf–homology of $\mathbb{R}P^k$ for all k. Such a connection would, for example, feed into a chromatic understanding of block structures; see eg [Macko 2007].

Conventions

We work extensively with model categories and refer the reader to the survey article [Dwyer and Spaliński 1995] and the textbooks [Hovey 1999; Hirschhorn 2003] for a detailed account of the theory. We further assume the reader has familiarity with orthogonal calculus, references for which include [Barnes and Oman 2013; Weiss 1995].

The category Top_* will always denote the category of based compactly generated weak Hausdorff spaces, and we will, for brevity, call the objects of this category "based spaces". The category of based spaces will always be equipped with the Quillen model structure unless specified otherwise. The weak equivalences are the weak homotopy equivalences and fibrations are Serre fibrations. This is a cellular, proper and topological model category with sets of generating cofibrations and acyclic cofibrations denoted by I and J, respectively.

Unless otherwise stated the word "spectra" is synonymous with the phrase "orthogonal spectra", details of which can be found in [Mandell et al. 2001] in the nonequivariant case, and [Mandell and May 2002] in the equivariant situation.

We will denote by $\mathcal{C}$ either the category of based spaces or of orthogonal spectra.

Acknowledgements

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Part I Local orthogonal calculus

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2 Orthogonal functors

Denote by $\mathcal{C}$ the category Top_* of based topological spaces or the category Sp of (orthogonal) spectra. Define $\mathcal{J}$ to be the category with finite-dimensional inner product subspaces of $\mathbb{R}^\infty$ as objects and with the linear isometries as morphisms. Define $\mathcal{J}_0$ to be the category with the same objects and $\mathcal{J}_0(U, V) = \mathcal{J}(U, V)_+$. The morphism set $\mathcal{J}(U, V)$ may be topologised as the Stiefel manifold of $\dim(U)$–frames in V. As such, $\mathcal{J}$ is a topologically enriched category, and $\mathcal{J}_0$ is enriched in based spaces. Since the functor

$$\Sigma^\infty \colon \mathsf{Top}_* \to \mathsf{Sp}$$

is symmetric monoidal — see eg [Mandell and May 2002, Lemma II.4.8] — we may enhance the topological enrichment of $\mathcal{J}_0$ to a spectral enrichment, resulting in a category $\mathcal{J}_0^{\mathsf{Sp}}$, whose class of objects agrees with the class of objects in $\mathcal{J}_0$, and morphism spectrum

$$\mathcal{J}_0^{\mathsf{Sp}}(V, W) = \Sigma^\infty \mathcal{J}_0(V, W).$$

We will omit the superscript "Sp" when confusion is unlikely to occur.

The category $\mathsf{Fun}(\mathcal{J}_0, \mathcal{C})$ of $\mathcal{C}$–enriched functors from $\mathcal{J}_0$ to $\mathcal{C}$ is the category of input functors for orthogonal calculus. We will refer to such functors as $\mathcal{C}$–*valued orthogonal functors* or simply *orthogonal functors* when confusion is unlikely. Examples of orthogonal functors are abound in geometry, topology and homotopy theory, and examples of Top_*–valued orthogonal functors include

(1) the one-point compactification functor $\mathbb{S} \colon V \mapsto S^V$;

(2) the functor $\mathsf{BO}(-) \colon V \mapsto \mathsf{BO}(V)$ which sends an inner product space to the classifying space of its orthogonal group;

(3) the functor $\mathsf{BTOP}(-) \colon V \mapsto \mathsf{BTOP}(V)$, which sends an inner product space V to $\mathsf{BTOP}(V)$, the classifying space of the space of self-homeomorphisms of V;

(4) the functor $\mathsf{BDiff}^b(M \times -) \colon V \mapsto \mathsf{BDiff}^b(M \times V)$, which for a fixed smooth and compact manifold M sends an inner product space V to the classifying space of the group of bounded diffeomorphisms from $M \times V$ to $M \times V$ which are the identity on $\partial M \times V$; and

(5) the restriction of an endofunctor on based spaces to evaluation on spheres.[1]

The category of orthogonal functors may be equipped with a projective model structure.

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[1]Endofunctors of based spaces are particularly interesting from a homotopy-theoretic point of view when you restrict to the values on spheres; see eg [Arone 2002; Arone and Mahowald 1999; Behrens 2012]. In particular for F the identity functor, the Weiss tower of $F \circ \mathbb{S} = \mathbb{S}$ and the Goodwillie tower for F agree up to weak equivalence [Barnes and Eldred 2016]; hence orthogonal calculus is intimately related to understanding the (stable) homotopy groups of spheres.

Proposition 2.1 *There is a model category structure on the category of orthogonal functors* $\mathsf{Fun}(\mathcal{J}_0, \mathcal{C})$ *with weak equivalences and fibrations defined objectwise. This model structure is cellular, proper and topological, and in the case of* Sp*–valued orthogonal functors, this model structure is spectral and stable.*

2.1 Local input functors

The "base" model structure for the S–local orthogonal calculus will be the S–local model structure on the category of orthogonal functors.

Proposition 2.2 *Let S be a set of maps in $\mathcal{C}$. There is model structure on the category of orthogonal functors such that a map is a weak equivalence or fibration if it is an objectwise S–local equivalence or an objectwise S–local fibration in $\mathcal{C}$, respectively. This model structure is cellular, left proper and topological, and in the case of* Sp*–valued orthogonal functors this model structure is spectral. We call this model structure the S–local projective model structure and denote it by* $\mathsf{Fun}(\mathcal{J}_0, L_S\mathcal{C})$.

Proof This model structure is an instance of a projective model structure on a category of functors; see eg [Hirschhorn 2003, Theorem 11.6.1]. □

Example 2.3 For E_* a generalised homology theory, the model structure of Proposition 2.2 has weak equivalences the objectwise E_*–isomorphisms, and fibrant objects objectwise E_*–local objects. This follows since the E_*–localization of spaces and spectra exist by work of Bousfield [1975; 1979].

3 Polynomial functors

3.1 Polynomial functors

Polynomial functors behave in many ways like polynomial functions from classical calculus; eg a functor which is polynomial of degree less than or equal to n is polynomial of degree less than or equal to $n+1$. We give only the necessary details here and refer the reader to [Weiss 1995] or [Barnes and Oman 2013] for more details on polynomial functors in orthogonal calculus.

Definition 3.1 An orthogonal functor F is *polynomial of degree less than or equal n* if F is objectwise fibrant and for each $U \in \mathcal{J}_0$, the canonical map

$$F(U) \to \underset{0 \neq V \subseteq \mathbb{R}^{n+1}}{\mathrm{holim}} F(U \oplus V) =: \tau_n F(U)$$

is a weak homotopy equivalence. Functors which are polynomial of degree less than or equal to n will sometimes be referred to as *n–polynomial* functors.

Remark 3.2 Given an orthogonal functor F and an inner product space U we can restrict the orthogonal functor $F(U \oplus -)$ to a functor

$$F(U \oplus -) \colon \mathcal{P}(\mathbb{R}^{n+1}) \to \mathsf{Top}_*,$$

where $\mathcal{P}(\mathbb{R}^{n+1})$ is the poset of finite-dimensional inner product subspaces of $\mathbb{R}^{n+1}$. Such functors are deserving of the name $\mathbb{R}^{n+1}$–*cubes* by analogy with cubical homotopy theory. The orthogonal functor F being n–polynomial is equivalent to asking that for each U this restricted functor is homotopy cartesian. Informally speaking, orthogonal calculus can be thought of as calculus built from $\mathbb{R}^n$–*cubical homotopy theory* in a similar way to how Goodwillie calculus is built from cubical homotopy theory.

There is a functorial assignment of a universal (up to homotopy) n–polynomial functor to any orthogonal functor F. It is the n–*polynomial approximation* of F, and is defined as

$$T_n F(U) = \operatorname{hocolim}(F(U) \to \tau_n F(U) \to \cdots \to \tau_n^k F(U) \to \cdots).$$

Barnes and Oman [2013, Propositions 6.5 and 6.6] construct a localization of the projective model structure on the category of orthogonal functors which captures the homotopy theory of n–polynomial functors, in particular the n–polynomial approximation functor is a fibrant replacement. There are two equivalent ways to consider this model structure; as the Bousfield–Friedlander localization of $\mathsf{Fun}(\mathcal{J}_0, \mathcal{C})$ at the n–polynomial approximation endofunctor

$$T_n \colon \mathsf{Fun}(\mathcal{J}_0, \mathcal{C}) \to \mathsf{Fun}(\mathcal{J}_0, \mathcal{C}),$$

or as the left Bousfield localization at the set

$$\mathcal{S}_n = \{ S\gamma_{n+1}(U, V)_+ \to \mathcal{J}_0(U, V) \mid U, V \in \mathcal{J}_0 \}$$

for Top_*–valued orthogonal functors, or the set $\Sigma^\infty \mathcal{S}_n = \{ \Sigma^\infty f \mid f \in \mathcal{S}_n \}$ for Sp–valued orthogonal functors, where $S\gamma_{n+1}(V, W)$ is the sphere bundle of the $(n+1)$–fold Whitney sum of the orthogonal complement bundle over the space of linear isometries $\mathcal{J}(V, W)$.

Proposition 3.3 [Barnes and Oman 2013, Proposition 6.5] *There is a model category structure on the category of orthogonal functors with weak equivalences the T_n–equivalences[2] and fibrations those objectwise fibrations $f \colon X \to Y$ such that the square*

$$
\begin{array}{ccc}
X & \longrightarrow & T_n X \\
\downarrow & & \downarrow \\
Y & \longrightarrow & T_n Y
\end{array}
$$

is a homotopy pullback in the projective model structure. This model structure is cellular, proper and topological, and in the case of Sp–valued orthogonal functors this model structure is spectral. We call this the n–polynomial model structure and denote it by $\mathsf{Poly}^{\leq n}(\mathcal{J}_0, \mathcal{C})$.

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[2] A map $f \colon X \to Y$ is a T_n–equivalence if $T_n(f) \colon T_n X \to T_n Y$ is an objectwise weak equivalence.

3.2   Local polynomial functors

The definition of an S–locally n–polynomial functor is the analogous definition of an n–polynomial functor when the base model category is $L_S\mathcal{C}$, ie an objectwise fibrant functor which satisfies a cartesian $\mathbb{R}^{n+1}$–cube condition.

Definition 3.4   Let S be a set of maps in $\mathcal{C}$. An orthogonal functor is S–*locally* n–*polynomial* if it is objectwise S–local and n–polynomial.

The S–locally n–polynomial model structure is an iterated left Bousfield localization involving the set $\mathcal{S}_n$ and the set
$$J_S = \{\mathcal{J}_0(U,-) \wedge j \mid U \in \mathcal{J}, \ j \in J_{L_S\mathcal{C}}\},$$
as this iterative localization will necessarily have the S–locally n–polynomial functors as fibrant objects. This model structure was first constructed by Barnes [2017] for the rationalization of Top_*–valued orthogonal functors.

Proposition 3.5   *Let S be a set of maps in $\mathcal{C}$. There is model category structure on the category of orthogonal functors with cofibrations the projective cofibrations, and fibrant objects the S–locally n–polynomial functors. This model structure is cellular, left proper, topological, and in the case of Sp–valued orthogonal functors this model structure is spectral. We call this model structure the S–local n–polynomial model structure and denote it by $\mathrm{Poly}^{\leq n}(\mathcal{J}_0, L_S\mathcal{C})$.*

Proof   The process of left Bousfield localizations may be iterated and it follows that the J_S–localization of the n–polynomial model structure and the $\mathcal{S}_n$–localization of the S–local projective model structure are identical, and have as cofibrations the projective cofibrations.

For the fibrant objects, notice that the model structure is equivalently described as the left Bousfield localization of the projective model structure with respect to the set of maps $\mathcal{S}_n \cup J_S$. By definition an object X is $\mathcal{S}_n \cup J_S$–local if and only if it is both $\mathcal{S}_n$–local and J_S–local, and hence the fibrant objects are precise those S–locally n–polynomial functors. $\qquad\square$

The S–local n–polynomial model structure behaves precisely like a left Bousfield localization of the n–polynomial model structure in the following sense.

Lemma 3.6   *Let S be a set of maps in $\mathcal{C}$. The adjoint pair*
$$\mathbb{1}\colon \mathrm{Poly}^{\leq n}(\mathcal{J}_0, \mathcal{C}) \rightleftarrows \mathrm{Poly}^{\leq n}(\mathcal{J}_0, L_S\mathcal{C})\colon \mathbb{1}$$
is a Quillen adjunction.

Proof   The left adjoint preserves cofibrations since the classes of cofibrations are identical. The right adjoint is right Quillen since it preserves fibrant objects as every S–locally n–polynomial functor is necessarily n–polynomial. $\qquad\square$

The composite $T_n L_S$ need not be a fibrant replacement functor in the S–local n–polynomial model structure since the class of S–local objects need not be closed under filtered homotopy colimits. Imposing a condition on the set S which forces $T_n L_S$ to be S–local in turn forces $T_n L_S$ to be a functorial fibrant replacement.

Proposition 3.7   *Let S be a set of maps in $\mathcal{C}$. If the class of S–local objects is closed under sequential homotopy colimits, then the weak equivalences of the S–local n–polynomial model structure are those maps $f\colon X \to Y$ such that the induced map*

$$T_n L_S f\colon T_n L_S X \to T_n L_S Y$$

is an S–local equivalence. In particular, The composite $T_n L_S$ is a functorial fibrant replacement in the S–local n–polynomial model structure.

Proof   We apply [Barnes 2017, Lemma 5.5], which shows that a map $f\colon X \to Y$ is weak equivalence in the iterated left Bousfield localization if and only if

$$L_S f\colon L_S X \to L_S Y$$

is an $\mathcal{S}_n$–local equivalence. This last is equivalent to $L_S f\colon L_S X \to L_S Y$ being a T_n–equivalence, ie $T_n L_S f\colon T_n L_S X \to T_n L_S Y$ being an objectwise weak equivalence. Since both the domain and codomain of this map are S–local, checking this map is an objectwise weak equivalence is equivalent to checking that it is an S–local equivalence by the S–local Whitehead theorem.   □

Remark 3.8   Let S be a set of maps in $\mathcal{C}$. To ease notation, we will denote the composite $T_n L_S$ by T_n^S. In particular, for E a spectrum we denote the composite functor $T_n L_E$ by T_n^E. In general, T_n^S need not be S–local, but will be when the class of S–local objects is closed under sequential homotopy colimits.

Example 3.9   (1)   For a finite cell complex W, $T_n^W F$ is W–local (or W–periodic) for all Top_*–valued orthogonal functors F.

 (2)   For localization at the Eilenberg–Mac Lane spectrum associated to a subring R of the rationals, $T_n^{HR} F$ is HR–local for all orthogonal functors F.

 (3)   For E a spectrum such that the associated localization of spectra is smashing, $T_n^E F$ is E–local for all Sp–valued orthogonal functors F.

4   Differentiation

The analogy between orthogonal calculus and differential calculus (Taylor's version) indicated the existence of an inductive "formula" for the n–polynomial approximation. The building blocks of such a "formula" are the derivatives of the functor under consideration.

4.1 The derivatives

The orthogonal complement of the pullback of the tautological bundle to the Stiefel manifold $\mathcal{J}_0(V, W)$ is a vector bundle $\gamma_1(V, W)$ with fibre over an isometry f given by $f(V)^\perp$. For $n \geq 0$, we denote the n–fold Whitney sum of $\gamma_1(V, W)$ by $\gamma_n(V, W)$. Define $\mathcal{J}_n$ to be the category with the same objects as $\mathcal{J}$ and morphism space $\mathcal{J}_n(U, V)$ given as the Thom space of $\gamma_n(U, V)$. Define $\mathcal{J}_n^{\mathsf{Sp}}$ to be the spectral enriched version of $\mathcal{J}_n$, ie the category with the same objects but morphism spectrum given by

$$\mathcal{J}^{\mathsf{Sp}}(V, W) = \Sigma^\infty \mathcal{J}_n(V, W).$$

The standard action of $O(n)$ on $\mathbb{R}^n$ via the regular representation induces an action on the vector bundles that is compatible with the composition; hence $\mathcal{J}_n$ is naturally enriched over based spaces with an $O(n)$–action.

Recall that $\mathcal{C}$ denotes the category of based spaces or spectra. We denote by $\mathcal{C}[O(n)]$ the category of $O(n)$–objects in $\mathcal{C}$. For $\mathcal{C} = \mathsf{Top}_*$, this recovers the category of $O(n)$–spaces, and for $\mathcal{C} = \mathsf{Sp}$, this is the category of spectra with an $O(n)$–action. Let $0 \leq m \leq n$. The inclusion $i_m^n \colon \mathbb{R}^m \to \mathbb{R}^n$ induces a functor $i_m^n \colon \mathcal{J}_m \to \mathcal{J}_n$. Postcomposition with i_m^n induces a topological functor

$$\mathsf{res}_m^n \colon \mathsf{Fun}(\mathcal{J}_n, \mathcal{C}) \to \mathsf{Fun}(\mathcal{J}_m, \mathcal{C}),$$

which by [Weiss 1995, Proposition 2.1] has a right adjoint

$$\mathsf{ind}_m^n \colon \mathsf{Fun}(\mathcal{J}_m, \mathcal{C}) \to \mathsf{Fun}(\mathcal{J}_n, \mathcal{C}),$$

the right Kan extension along i_m^n, and is given by

$$\mathsf{ind}_m^n F(U) = \mathsf{nat}_m(\mathcal{J}_n(U, -), F),$$

where $\mathsf{nat}_m(-, -)$ denotes the space of natural transformations in $\mathsf{Fun}(\mathcal{J}_m, \mathcal{C})$ and $\mathcal{J}_n(U, -)$ is considered as an object of $\mathsf{Fun}(\mathcal{J}_m, \mathcal{C})$ by restriction. Combining the restriction and induction functors with change of group adjunctions from [Mandell and May 2002], we obtain an adjoint pair

$$\mathsf{res}_m^n / O(n-m) \colon \mathsf{Fun}_{O(n)}(\mathcal{J}_n, \mathcal{C}[O(n)]) \rightleftarrows \mathsf{Fun}_{O(m)}(\mathcal{J}_m, \mathcal{C}[O(m)]) \colon \mathsf{ind}_m^n \mathsf{Cl}$$

(see [Barnes and Oman 2013, Section 4]), where $\mathsf{Fun}_{O(n)}(\mathcal{J}_n, \mathcal{C}[O(n)])$ is the category of $\mathcal{C}[O(n)]$–enriched functors from $\mathcal{J}_n$ to $\mathcal{C}[O(n)]$. We refer to this category as the nth *intermediate category* on the point of its role as an intermediate in the classification of n–homogeneous functors; see Section 5.

Definition 4.1 Let F be an orthogonal functor. For $n \geq 0$, the nth *derivative of* F is given by $\mathsf{ind}_0^n \mathsf{Cl} F$. For this we write $\mathsf{ind}_0^n \varepsilon^* F$ or $F^{(n)}$.

Restricted evaluation in the nth intermediate category induces structure maps of the form

$$X(V) \wedge S^{\mathbb{R}^n \otimes W} \to X(V \oplus W)$$

for $X \in \mathsf{Fun}_{O(n)}(\mathcal{J}_n, \mathcal{C}[O(n)])$ and $V, W \in \mathcal{J}_n$; see eg [Barnes and Oman 2013, Section 7]. It is thus reasonable to think of the objects of the nth intermediate category as *spectra of multiplicity n*; see eg

[Weiss 1995, Section 9]. This idea leads to an object Z in the nth intermediate category being called an $n\Omega$–*spectrum* if the adjoint structure maps

$$Z(V) \to \Omega^{\mathbb{R}^n \otimes W} Z(V \oplus W)$$

are weak equivalences in $\mathcal{C}$, and a map $f \colon X \to Y$ in the nth intermediate category being called an n–*stable equivalence* if the induced map

$$f^* \colon [Y, Z] \to [X, Z]$$

on objectwise homotopy classes of maps is an isomorphism for all $n\Omega$–spectra Z. With these definitions we get an n–stable model structure on the nth intermediate category analogous to the stable model structure on spectra; see eg [Barnes and Oman 2013, Section 7].

Proposition 4.2 [Barnes and Oman 2013, Proposition 7.14] *There is a model category structure on the nth intermediate category with weak equivalences the n–stable equivalences and fibrations the objectwise fibrations $X \to Y$ such that the square*

$$
\begin{array}{ccc}
X(U) & \longrightarrow & \Omega^{\mathbb{R}^n \otimes V} X(U \oplus V) \\
\downarrow & & \downarrow \\
Y(U) & \longrightarrow & \Omega^{\mathbb{R}^n \otimes V} Y(U \oplus V)
\end{array}
$$

is a homotopy pullback in $\mathcal{C}$ for all $U, V \in \mathcal{J}_n$. The fibrant objects are the $n\Omega$–spectra. This model structure is cellular, proper, stable and topological, and in the case of Sp–valued orthogonal functors, this model structure is spectral. We call this the n–stable model structure and denote it by $\mathsf{Fun}_{O(n)}(\mathcal{J}_n, \mathcal{C}[O(n)])$.

4.2 The local n–stable model structure

We now equip the nth intermediate category with an S–local model structure which will be intermediate in our classification of S–local n–homogeneous functors as appropriately[3] S–local spectra with an action of $O(n)$. This model structure was first defined by Barnes [2017] for the rationalization of Top_*–valued orthogonal functors.

Proposition 4.3 *Let S be a set of maps in $\mathcal{C}$. There is a model category structure on the nth intermediate category with cofibrations the cofibrations of the n–stable model structure and fibrant objects the $n\Omega$–spectra which are objectwise S–local. This model structure is cellular, left proper and topological, and in the case of Sp–valued orthogonal functors, this model structure is spectral. We call this the S–local n–stable model structure and denote it by $L_S \mathsf{Fun}_{O(n)}(\mathcal{J}_n, \mathcal{C}[O(n)])$.*

Proof This model structure is the left Bousfield localization of the n–stable model structure at the set

$$\mathcal{Q}_n = \{O(n)_+ \wedge \mathcal{J}_n(U, -) \wedge j \mid U \in \mathcal{J}, j \in J_{L_S \mathcal{C}}\}. \qquad \square$$

---

[3]Here "appropriately" means levelwise S–local spectra for Top_*–valued orthogonal functors and S–local spectra for Sp–valued orthogonal functors.

We record the following fact which will prove useful later.

Lemma 4.4 *Let S be a set of maps in $\mathcal{C}$. If F is an S–local functor, then $F^{(n)} = \operatorname{ind}_0^n F$ is S–local.*

Proof The objectwise smash product

$$(-) \wedge (-) \colon \mathsf{Fun}(\mathcal{J}_n, L_S\mathcal{C}) \times L_S\mathcal{C} \to \mathsf{Fun}(\mathcal{J}_n, L_S\mathcal{C})$$

is a Quillen bifunctor, and the result follows from the definition of $\operatorname{ind}_0^n F$. $\qquad\qquad\square$

4.3 The derivatives as spectra

The nth derivative ($n \geq 0$) is naturally an object of the nth intermediate category, ie is a spectrum of multiplicity n. This multiplicity may be reduced to $n = 1$ through a Quillen equivalence

$$(\alpha_n)_! \colon \mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\mathsf{Top}_*) \rightleftarrows \mathsf{Sp}[O(n)] \colon (\alpha_n)^*$$

in the topological case (see eg [Barnes and Oman 2013, Section 8]) and by a series of Quillen equivalences

$$\mathsf{Fun}(\mathcal{J}_n, \mathsf{Sp}[O(n)]) \underset{(\alpha_n)^*}{\overset{(\alpha_n)_!}{\rightleftarrows}} \mathsf{Sp}(\mathsf{Sp}[O(n)]) \underset{\mathrm{Ev}_0}{\overset{F_0}{\rightleftarrows}} \mathsf{Sp}[O(n)]$$

in the spectral case (see eg [Barnes and Oman 2013, Section 11]). Here $\mathsf{Sp}(\mathsf{Sp}[O(n)])$ denotes the category of spectrum objects in spectra with an $O(n)$–action or equivalently, orthogonal bispectra with an $O(n)$–action, and is Quillen equivalent to orthogonal spectra by arguments similar to [Hovey 2001, Theorem 5.1] or [Schwede and Shipley 2003, Theorem 3.8.2].

Example 4.5 The (spectrum representing the) n^{th} derivative of the Top_*–valued orthogonal functor $\mathsf{BO}(-)$ have been completely calculated by Arone [2002]. Weiss [1995] calculated the first few examples by hand, for instance the first derivative is the sphere spectrum with trivial $O(1)$–action, the second derivative is the shifted sphere spectrum $\mathbb{S}^{-1}$ with trivial action, and the third derivative is the 2–fold loops on the mod-3 Moore spectrum $\Omega^2(\mathbb{S}/3)$. Higher derivatives have a striking resemblance with the Goodwillie derivatives of the identity functors on based spaces.

We now prove that this result holds S–locally for any set S of maps in our category $\mathcal{C}$. Since the adjunctions are slightly different, we prove each separately.

Theorem 4.6 *Let S be a set of maps of based spaces. The adjoint pair*

$$(\alpha_n)_! \colon L_S \mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\mathsf{Top}_*) \rightleftarrows \mathsf{Sp}(L_S\mathsf{Top}_*)[O(n)] \colon (\alpha_n)^*$$

is a Quillen equivalence between the S–local model structures.

Proof For the Quillen adjunction apply [Hirschhorn 2003, Theorem 3.3.20(1)], noting that there is an isomorphism

$$(\alpha_n)_!(O(n)_+ \wedge \mathcal{J}_n(U, -) \wedge j) \cong O(n)_+ \wedge \mathcal{J}_1(\mathbb{R}^n \otimes U, -) \wedge j$$

for j a generating acyclic cofibration for the S–local model structure on based spaces.

By [Barnes and Oman 2013, Proposition 8.3], the adjoint pair

$$(\alpha_n)_! \colon \mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\mathsf{Top}_*) \rightleftarrows \mathsf{Sp}[O(n)] \colon (\alpha_n)^*$$

is a Quillen equivalence. To show that the adjunction between the S–local model structures is a Quillen equivalence, it suffices by [Hovey 2001, Proposition 2.3] to show that if Y is fibrant in $\mathsf{Sp}[O(n)]$ such that $(\alpha_n)^* Y$ is fibrant in the S–local n–stable model structure, then Y is fibrant in the S–local model structure on $\mathsf{Sp}[O(n)]$. This follows readily from the definitions of fibrant objects in both model structures. □

The category of orthogonal bispectra with an $O(n)$–action, or equivalently the category of (orthogonal) spectrum objects in spectra with an $O(n)$–action may be equipped with an L_S–local model structure, similar to Proposition 4.3. For S a set of maps of spectra, the S–local model structure $L_S \mathsf{Sp}(\mathsf{Sp}[O(n)])$ is the left Bousfield localization of the stable model structure at the set

$$\{\mathcal{J}_1(V, -) \wedge j \mid V \in \mathcal{J}, \ j \in J_{L_S \mathsf{Sp}[O(n)]}\}$$

since the category $\mathsf{Sp}(\mathsf{Sp}[O(n)])$ may also be described as the category of $O(n)$–objects in $\mathsf{Fun}(\mathcal{J}_1, \mathsf{Sp})$. In particular, the fibrant objects of the S–local model structure on $\mathsf{Sp}(\mathsf{Sp}[O(n)])$ are $O(n)$–objects $X \in \mathsf{Fun}(\mathcal{J}_1, \mathsf{Sp})$ such that $X(V)$ is an S–local spectrum for each $V \in \mathcal{J}_1$.

Theorem 4.7 *Let S be a set of maps of spectra. The adjoint pairs*

$$L_S \mathsf{Fun}(\mathcal{J}_n, \mathsf{Sp}[O(n)]) \underset{(\alpha_n)^*}{\overset{(\alpha_n)_!}{\rightleftarrows}} L_S \mathsf{Sp}(\mathsf{Sp}[O(n)]) \underset{\mathsf{Ev}_0}{\overset{F_0}{\rightleftarrows}} L_S \mathsf{Sp}[O(n)]$$

are Quillen equivalences between the S–local model structures.

Proof Identifying the category of spectrum objects in spectra with an $O(n)$–action with the category of $O(n)$–objects in $\mathsf{Fun}(\mathcal{J}_1, \mathsf{Sp})$, the proof that the adjunction

$$(\alpha_n)_! \colon L_S \mathsf{Fun}(\mathcal{J}_n, \mathsf{Sp}[O(n)]) \rightleftarrows L_S \mathsf{Sp}(\mathsf{Sp}[O(n)]) \colon (\alpha_n)^*,$$

is a Quillen equivalence follows analogously to Theorem 4.6.

For the adjunction

$$F_0 \colon L_S \mathsf{Sp}[O(n)] \rightleftarrows L_S \mathsf{Sp}(\mathsf{Sp}[O(n)]) \colon \mathsf{Ev}_0,$$

note that the composite functor

$$\mathsf{Sp}[O(n)] \xrightarrow{F_0} \mathsf{Sp}(\mathsf{Sp}[O(n)]) \xrightarrow{\mathbb{1}} L_S \mathsf{Sp}(\mathsf{Sp}[O(n)])$$

is left Quillen, and to extend to a left Quillen functor from $L_S \mathsf{Sp}[O(n)]$, it suffices by [Hirschhorn 2003, Proposition 3.3.18(1) and Theorem 3.1.6(1)] to exhibit that the right adjoint preserves S–local objects, which follows immediately from the definition of S–local objects in the respective model structures.

To see that the adjunction is a Quillen equivalence, we apply [Hovey 2001, Proposition 2.3], which reduces the problem to showing that if Y is an Ω–spectrum object in $\mathsf{Sp}[O(n)]$ (ie fibrant in $\mathsf{Sp}(\mathsf{Sp}[O(n)])$) such that $\mathsf{Ev}_0(Y)$ is S–local, then Y is S–local. This follows from the Ω–spectrum structure and the interaction of homotopy function complexes with the suspension-loops adjunction. □

5 Homogeneous functors and their classification

5.1 Homogeneous functors

The layers of the Weiss tower associated to an orthogonal functor F are the homotopy fibres of maps $T_nF \to T_{n-1}F$ and have two interesting properties: first, they are polynomial of degree less than or equal to n; and second, their $(n-1)$–polynomial approximation is trivial. We denote the nth layer of the Weiss tower of F by D_nF.

Definition 5.1 For $n \geq 0$, an orthogonal functor F is said to be *n–reduced* if its $(n-1)$–polynomial approximation is objectwise weakly equivalent to the terminal object. An orthogonal functor F is said to be *homogeneous of degree n* if it is both polynomial of degree less than or equal n and n–reduced. We will sometimes refer to a functor which is homogeneous of degree n as being *n–homogeneous*.

There is a model structure on the category of orthogonal functors which contains the n–homogeneous functors as the bifibrant objects. This model structure is a right Bousfield localization of the n–polynomial model structure.

Proposition 5.2 [Barnes and Oman 2013, Proposition 6.9] *There is a model category structure on the category of orthogonal functors with weak equivalences the D_n–equivalences and fibrations the fibrations of the n–polynomial model structure. The cofibrant objects are the n–reduced projectively cofibrant objects and the fibrant objects are the n–polynomial functors. In particular, cofibrant-fibrant objects of this model structure are the projectively cofibrant n–homogeneous functors. This model structure is cellular, proper, stable and topological, and in the case of Sp–valued orthogonal functors this model structure is spectral. We call this the n–homogeneous model structure and denote it by $\mathsf{Homog}^n(\mathcal{J}_0, \mathcal{C})$.*

Remark 5.3 The model structure of [Barnes and Oman 2013, Proposition 6.9] has as weak equivalences those maps which induce objectwise weak equivalences on the nth derivatives of their n–polynomial approximations. We showed in [Taggart 2022a, Proposition 8.2] that the class of such equivalences is precisely the class of D_n–equivalences. The proof of [Taggart 2022a, Proposition 8.2] is valid for Sp–valued orthogonal functors since Sp–valued n–homogeneous functors admit and analogous classification in terms of spectral with an $O(n)$–action; see eg [Barnes and Oman 2013, Section 11].

The n–homogeneous model structure is (zigzag) Quillen equivalent to spectra with an action of $O(n)$.

Proposition 5.4 [Barnes and Oman 2013, Proposition 8.3, Theorems 10.1 and 11.3, and Corollary 11.4] *Let $n \geq 0$. There is a zigzag of Quillen equivalences*

$$\mathsf{Homog}^n(\mathcal{J}_0, \mathcal{C}) \simeq_Q \mathsf{Sp}[O(n)].$$

On the homotopy category level, the Barnes–Oman zigzag of Quillen equivalences recovers Weiss's characterisation of homogeneous functors of degree n.

Proposition 5.5 [Weiss 1995, Theorem 7.3; Barnes and Oman 2013, Theorem 11.5] *Let $n \geq 1$.*

(1) *An n–homogeneous functor F is determined by and determines a spectrum $\partial_n F$ with an $O(n)$–action.*

(2) *A Top_*–valued n–homogeneous functor F is objectwise weak homotopy equivalent to the functor*

$$V \mapsto \Omega^\infty[(S^{\mathbb{R}^n \otimes V} \wedge \partial_n F)_{hO(n)}],$$

and any functor of the above form is homogeneous of degree n.

(3) *An Sp–valued n–homogeneous functor F is objectwise weak homotopy equivalent to the functor*

$$V \mapsto (S^{\mathbb{R}^n \otimes V} \wedge \partial_n F)_{hO(n)},$$

and any functor of the above form is homogeneous of degree n.

5.2 Local homogeneous functors

Definition 5.6 Let S be a set of maps in $\mathcal{C}$. An orthogonal functor F is S–*locally homogeneous of degree n* if it is objectwise S–local and n–homogeneous.

Lemma 5.7 *Let S be a set of maps of in $\mathcal{C}$, and F and orthogonal functor. For $n \geq 1$, there is a homotopy fibre sequence*

$$D_n^S F \to T_n^S F \to T_{n-1}^S F$$

in which $D_n^S(F)$ is

(1) *homogeneous of degree n; and*

(2) *S–locally n–homogeneous if, in addition, the class of S–local objects is closed under sequential homotopy colimits.*

Proof By [Weiss 1995, Lemma 5.5] the homotopy fibre of a map between n–polynomial functors is n–polynomial; hence $D_n^S F$ is n–polynomial. Applying T_{n-1} to the homotopy fibre sequence, yields that the $(n-1)$–polynomial approximation of $D_n^S F$ is objectwise weakly contractible, proving (1).

For (2), observe that the homotopy fibre of a map between S–local objects is S–local and when the class of S–local objects is closed under sequential homotopy colimits, $T_n L_S F$ is S–local for all n. □

Example 5.8   (1)   For homological localization at the Eilenberg–Mac Lane spectrum associated to a subring R of the rationals, $D_n^{HR} F$ is HR–locally n–homogeneous for any orthogonal functor F.

(2) For nullification at a based finite cell complex W, $D_n^W F$ is W–locally n–homogeneous for any Top_*–valued orthogonal functor F.

(3) For a spectrum E whose associated localization of spectra is smashing, $D_n^E F$ is E–locally n–homogeneous for any Sp–valued orthogonal functor.

Proposition 5.9 *Let S be a set of maps in $\mathcal{C}$. There is model category structure on the category of orthogonal functors with cofibrations the cofibrations of the n–homogeneous model structure and fibrant objects the n–polynomial functors whose n^{th} derivative is objectwise S–local in the nth intermediate category. This model structure is cellular, left proper and topological, and in the case of Sp–valued orthogonal functors this model structure is spectral. We call this the S–local n–homogeneous model structure and denote it by $\mathsf{Homog}^n(\mathcal{J}_0, L_S\mathcal{C})$.*

Proof We left Bousfield localize the n–homogeneous model structure at the set of maps

$$\mathcal{K}_n = \{\mathcal{J}_n(U, -) \wedge j \mid U \in \mathcal{J}, j \in J_{L_S\mathcal{C}}\}.$$

This left Bousfield localization exists since the n–homogeneous model structure is cellular and left proper by [Barnes 2017, Lemma 6.1]. The description of the cofibrations follows immediately.

The fibrant objects are the $\mathcal{K}_n$–local objects which are also fibrant in the n–homogeneous model structure, ie those n–polynomial functors Z for which the induced map

$$[\mathcal{J}_n(U, -) \wedge B, Z] \to [\mathcal{J}_n(U, -) \wedge A, Z]$$

is an isomorphism for all maps $\mathcal{J}_n(U, -) \wedge A \to \mathcal{J}_n(U, -) \wedge B$ in $\mathcal{K}_n$. A straightforward adjunction argument and the definition of the nth derivative of an orthogonal functor yield the required characterisation of the fibrant objects. $\square$

Corollary 5.10 *Let S be a set of maps in $\mathcal{C}$. The cofibrant objects of the S–local n–homogeneous model structure are the projectively cofibrant functors which are n–reduced.*

Proof The S–local n–homogeneous model structure is a particular left Bousfield localization of the n–homogeneous model structure, hence has the same cofibrant objects. The result follows by the orthogonal calculus version of [Taggart 2022a, Corollary 8.6]. $\square$

The S–local n–homogeneous model structure behaves like a right Bousfield localization of the S–local n–polynomial model structure in the following sense.

Lemma 5.11 *Let S be a set of maps in $\mathcal{C}$. The adjoint pair*

$$\mathbb{1}\colon \mathsf{Homog}^n(\mathcal{J}_0, L_S\mathcal{C}) \rightleftarrows \mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_S\mathcal{C}) \colon \mathbb{1}$$

is a Quillen adjunction.

Proof The cofibrations of the S–local n–homogeneous model structure are the cofibrations of the n–homogeneous model structure, which are contained in the cofibrations of the n–polynomial model structure, which in turn are precisely the cofibrations of the S–local n–polynomial model structure, hence

$$\mathbb{1}\colon \mathsf{Homog}^n(\mathcal{J}_0, L_S\mathcal{C}) \to \mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_S\mathcal{C})$$

preserves cofibrations.

On the other hand, to show that the right adjoint is right Quillen it suffices to show that the identity functor sends fibrant objects in the S–local n–polynomial model structure to fibrant objects in the S–local n–homogeneous model structure. This follows from Lemma 4.4 since the fibrant objects in the S–local n–polynomial model structure are the S–locally n–polynomial functors by Proposition 3.5 and the fibrant objects of the S–local n–homogeneous model structure are the n–polynomial functors with S–local n^{th} derivative by Proposition 5.9. $\square$

5.3 Characterisations for stable localizations

We obtain a characterisation of the fibrations of the S–local n–homogeneous model structure when the localizing set S is stable in the sense of [Barnes and Roitzheim 2014, Definition 4.2], ie when the class of S–local spaces is closed under suspension. For the statement of the following result recall the definition of the nth derivative of an orthogonal functor from Definition 4.1.

Proposition 5.12 *If S is a set of maps in $\mathcal{C}$ which is stable, then the fibrations of the S–local n–homogeneous model structure are those maps $f\colon X \to Y$ which are fibrations in the n–polynomial model structure such that*

$$X^{(n)} \to Y^{(n)}$$

is an objectwise fibration in $L_S\mathcal{C}$.

Proof We first given an explicit characterisation of the acyclic cofibrations since the fibrations are characterised by the right lifting property against these maps. The maps in $\mathcal{K}_n$ are cofibrations between cofibrant objects since $\mathcal{J}_n(U, -)$ is cofibrant in $\mathsf{Homog}^n(\mathcal{J}_0, \mathcal{C})$ and the maps in $J_{L_S\mathcal{C}}$ are cofibrations of the S–local model structure on $\mathcal{C}$. Moreover, since the localizing set S is stable, it follows the set of generating acyclic cofibrations $J_{L_S\mathcal{C}}$ is stable and in turn that the set $\mathcal{K}_n$ is stable. Hence by [Barnes and Roitzheim 2014, Theorem 4.11], the generating acyclic cofibrations are given by the set $J_{\mathsf{Homog}^n} \cup \Lambda(\mathcal{K}_n)$, where J_{Homog^n} is the set of the generating acyclic cofibrations of the n–homogeneous model structure and $\Lambda(\mathcal{K}_n)$ the set of horns on $\mathcal{K}_n$ in the sense of [Hirschhorn 2003, Definition 4.2.1]. As horns in topological model categories are given by pushouts and $\mathcal{K}_n$ is a set of cofibrations between cofibrant objects it suffices to use the set $J_{\mathsf{Homog}^n} \cup \mathcal{K}_n$ as the generating acyclic cofibrations of the S–local n–homogeneous model structure.

If $f\colon X \to Y$ is a map with the right lifting property with respect to $J_{\mathsf{Homog}^n} \cup \mathcal{K}_n$, then f has the right lifting property with respect to J_{Homog^n} and the right lifting property with respect to $\mathcal{K}_n$ independently. Having the right lifting property with respect to J_{Homog^n} is equivalent to being a fibration in the n–

polynomial model structure. On the other hand, a map in $\mathcal{K}_n$ is of the form $\mathcal{J}_n(U,-) \wedge A \to \mathcal{J}_n(U,-) \wedge B$ for $A \to B$ a generating acyclic cofibration of the S–local model structure on $\mathcal{C}$. A lift in the diagram

$$\begin{array}{ccc} \mathcal{J}_n(U,-) \wedge A & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \mathcal{J}_n(U,-) \wedge B & \longrightarrow & Y \end{array}$$

(indicated by the dotted arrow) exists if and only if the lift in the diagram

$$\begin{array}{ccc} A & \longrightarrow & \mathsf{nat}_0(\mathcal{J}_n(U,-), X) \\ \downarrow & \nearrow & \downarrow \\ B & \longrightarrow & \mathsf{nat}_0(\mathcal{J}_n(U,-), Y) \end{array}$$

exists, which is equivalent to the statement that $X^{(n)} \to Y^{(n)}$ is an objectwise fibration of S–local objects in $\mathcal{C}$; see Section 4.1. $\qquad\square$

This specialises to homological localizations.

Corollary 5.13 *Let E be a spectrum. The fibrations of the E–local n–homogeneous model structure are those maps $f \colon X \to Y$ which are fibrations in the n–polynomial model structure such that*

$$X^{(n)} \to Y^{(n)}$$

is an objectwise fibration in $L_E \mathcal{C}$.

Proof Combine Proposition 5.12 with [Barnes and Roitzheim 2014, Example 4.3]. $\qquad\square$

Corollary 5.14 *Let E be a spectrum. An orthogonal functor F is fibrant in the E–local n–homogeneous model structure if and only if F is n–polynomial and $F^{(n)}$ is objectwise E–local. In particular, the bifibrant objects are the projectively cofibrant n–homogeneous functors with E–local n^{th} derivative.*

Proof Apply Corollary 5.13 to the map $F \to *$. $\qquad\square$

5.4 Differentiation as a Quillen functor

The nth derivative is a right Quillen functor as part of a Quillen equivalence between the n–homogeneous model structure and the nth intermediate category; the adjunction

$$\mathsf{res}_0^n / O(n) \colon \mathsf{Fun}_{O(n)}(\mathcal{J}_n, \mathcal{C}[O(n)]) \rightleftarrows \mathsf{Homog}^n(\mathcal{J}_0, \mathcal{C}) \colon \mathsf{ind}_0^n \varepsilon^*$$

is a Quillen equivalence [Barnes and Oman 2013, Theorem 10.1]. We now show that this extends to the S–local situation.

Theorem 5.15 *Let S be a set of maps in $\mathcal{C}$. The adjoint pair*

$$\mathsf{res}_0^n / O(n) \colon L_S \mathsf{Fun}_{O(n)}(\mathcal{J}_n, \mathcal{C}[O(n)]) \rightleftarrows \mathsf{Homog}^n(\mathcal{J}_0, L_S \mathcal{C}) \colon \mathsf{ind}_0^n \varepsilon^*$$

is a Quillen equivalence between the S–local model structures.

Proof The left adjoint applied to the localizing set of the S–local n–stable model structure is precisely the localization set of the S–local n–homogeneous model structure, hence the result follows from [Hirschhorn 2003, Theorem 3.3.20(1)]. □

Corollary 5.16 *Let S be a set of maps of based spaces, and $n \geq 0$. There is a zigzag of Quillen equivalences*

$$\mathsf{Homog}^n(\mathcal{J}_0, L_S\mathsf{Top}_*) \simeq_Q \mathsf{Sp}(L_S\mathsf{Top}_*)[O(n)].$$

Example 5.17 Let R be a subring of the rationals. Then there is a zigzag of Quillen equivalences

$$\mathsf{Homog}^n(\mathcal{J}_0, L_{HR}\mathsf{Top}_*) \simeq_Q \mathsf{Sp}_{HR}[O(n)]$$

between HR–local n–homogeneous functors and HR–local[4] spectra with an action of $O(n)$.

Corollary 5.18 *Let S be a set of maps of spectra, and $n \geq 0$. There is a zigzag of Quillen equivalences*

$$\mathsf{Homog}^n(\mathcal{J}_0, L_S\mathsf{Sp}) \simeq_Q L_S\mathsf{Sp}[O(n)].$$

Example 5.19 Let E be a spectrum. Then there is a zigzag of Quillen equivalences

$$\mathsf{Homog}^n(\mathcal{J}_0, L_E\mathsf{Sp}) \simeq_Q \mathsf{Sp}_E[O(n)]$$

between E–local n–homogeneous functors and E–local spectra with an action of $O(n)$.

5.5 The classification

As in the classical theory, any S–locally n–homogeneous functor may be expressed concretely in terms of a levelwise S–local spectrum with an action of $O(n)$. The proof of which follows as in the classical setting [Weiss 1995, Theorem 7.3] and can be realised through the derived equivalence of homotopy categories provided by our zigzag of Quillen equivalences.

Theorem 5.20 *Let S be a set of maps of in $\mathcal{C}$ and $n \geq 1$.*

(1) *An S–local n–homogeneous functor F is determined by and determines an appropriately S–local spectrum with an $O(n)$–action, denoted by $\partial_n^S F$.*

(2) *A Top_*–valued S–local n–homogeneous functor F is objectwise weakly equivalent to the functor*

$$V \mapsto \Omega^\infty[(S^{\mathbb{R}^n \otimes V} \wedge \partial_n^S F)_{hO(n)}],$$

and any functor of the above form is objectwise S–local and n–homogeneous.

(3) *An Sp–valued S–local n–homogeneous functor F is objectwise weakly equivalent to the functor*

$$V \mapsto (S^{\mathbb{R}^n \otimes V} \wedge \partial_n^S F)_{hO(n)},$$

and any functor of the above form is objectwise S–local and n–homogeneous.

---

[4] In particular, the HR–local model structure on spectra is identical to the levelwise HR–local model structure since a spectrum is HR–local if and only if it is levelwise HR–local; see eg [Barnes and Roitzheim 2011, Lemma 8.6].

Part II Applications

6 Bousfield classes

6.1 Bousfield classes

For a spectrum E, the *Bousfield class* of E, denoted by $\langle E \rangle$, is the equivalence class of E under the relation: $E \sim E'$ if for any spectrum X, $E \wedge X = 0$ if and only if $E' \wedge X = 0$. If $\langle E \rangle = \langle E' \rangle$, then the classes of E_*–isomorphisms and E'_*–isomorphisms agree and hence the localization functors (on spaces or spectra) agree. The collection of all Bousfield classes forms a lattice, with partial ordering $\langle E \rangle \leq \langle E' \rangle$ given by reverse containment, ie if and only if the class of E'–acyclic spectra is contained in the class of E–acyclic spectra, in particular, the partial ordering induces a natural transformation $L_{E'} \to L_E$. Bousfield classes have been studied at length; see eg [Bousfield 1979; Ravenel 1984].

A similar story remains true unstably. Given a based space W the *unstable Bousfield class* of W, or the *nullity class* of W, is the equivalence class $\langle W \rangle$ of all spaces W' such that the class of W–periodic[5] spaces agrees with the class of W'–periodic spaces. There is a partial ordering $\langle W \rangle \leq \langle W' \rangle$ given by reverse containment, ie if and only if every W'–periodic space is W–periodic. In particular, the relation $\langle W \rangle \leq \langle W' \rangle$ implies that every W–local equivalence is a W'–local equivalence and there is a natural transformation $P_W \to P_{W'}$, which is a W'–localization. Nullity classes have also been studied at length; see eg [Bousfield 1994; Dror Farjoun 1996].

Remark 6.1 It is worth noting that in both cases there is a choice of ordering of the equivalence classes, and our choices have been made to align with the predominant references on the subject, which unfortunately means the "stable" and "unstable" directions are dual. The choice of ordering used by Bousfield and that of Dror Farjoun also differ, adding further confusion to the literature on these matters.

Theorem 6.2 *Let S and S' be sets of maps in $\mathcal{C}$. The class of S–local objects agrees with the class of S'–local objects if and only if for every orthogonal functor F, the S–local Weiss tower of F is objectwise weakly equivalent to the S'–local Weiss tower of F.*

Proof If the class of S–local objects agrees with the class of S'–local objects, then the localization functors L_S and $L_{S'}$ agree on $\mathcal{C}$ and hence on the level of orthogonal functors. In particular, for every orthogonal functor F, the canonical map[6]

$$L_S F \to L_{S'} F$$

---

[5] W–periodic spaces are precisely W–local spaces. This change in terminology is classical; see eg [Bousfield 1994; Dror Farjoun 1996].

[6] This map is induced from the S–local objects being contained in the S'–local objects. We could also use the canonical $L_{S'} F \to L_S F$ since the S–local objects also contained the S'–local objects.

is an objectwise weak equivalence. Now, consider the commutative diagram

$$
\begin{array}{ccccc}
D_n^S F & \longrightarrow & T_n^S F & \longrightarrow & T_{n-1}^S F \\
\downarrow & & \downarrow & & \downarrow \\
D_n^{S'} F & \longrightarrow & T_n^{S'} F & \longrightarrow & T_{n-1}^{S'} F
\end{array}
$$

in which the rows are homotopy fibre sequences. For each $n \geq 0$, the map

$$
T_n^S F \to T_n^{S'} F
$$

is an objectwise weak equivalence since polynomial approximation preserves objectwise weak equivalences. It follows that the leftmost vertical arrow is also an objectwise weak equivalence and that the S–local Weiss tower is objectwise weakly equivalent to the S'–local Weiss tower.

The converse is immediate from specialising for every object $C \in \mathcal{C}$ to the constant functor at C. □

Example 6.3 (1) Let E and E' be spectra. For every orthogonal functor F the E–local Weiss tower of F and the E'–local Weiss tower of F agree if and only if E and E' are Bousfield equivalent.

(2) Let W and W' be based spaces. For every Top_*–valued orthogonal functor F the W–local Weiss tower of F and the W'–local Weiss tower of F agree if and only if W and W' have the same nullity class.

6.2 Bousfield classes and model categories for orthogonal calculus

On the model category level, we have the following.

Theorem 6.4 *Let S and S' be sets of maps of maps in $\mathcal{C}$. The class of S–local objects in $\mathcal{C}$ agrees with the class of S'–local objects in $\mathcal{C}$ if and only if there are equalities of model structures making the diagram*

$$
\begin{array}{ccccc}
\mathsf{Fun}(\mathcal{J}_0, L_S\mathcal{C}) & \underset{\mathbb{1}}{\overset{\mathbb{1}}{\rightleftarrows}} & \mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_S\mathcal{C}) & \underset{\mathbb{1}}{\overset{\mathbb{1}}{\rightleftarrows}} & \mathsf{Homog}^n(\mathcal{J}_0, L_S\mathcal{C}) \\
\| & & \| & & \| \\
\mathsf{Fun}(\mathcal{J}_0, L_{S'}\mathcal{C}) & \underset{\mathbb{1}}{\overset{\mathbb{1}}{\rightleftarrows}} & \mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_{S'}\mathcal{C}) & \underset{\mathbb{1}}{\overset{\mathbb{1}}{\rightleftarrows}} & \mathsf{Homog}^n(\mathcal{J}_0, L_{S'}\mathcal{C})
\end{array}
$$

commute.

Proof For one direction assume that the class of S–local objects agrees with the class of S'–local objects. Then the S–local model structure and the S'–local model structure on $\mathcal{C}$ agree as they have the same cofibrations and fibrant objects. This equality lifts to the local projective model structures on the category of orthogonal functors. As left Bousfield localization does not alter the cofibrations, the cofibrations of the S–local n–polynomial model structure agree with the cofibrations of the S'–local n–polynomial model structure. These model structures also have the same fibrant objects since a functor is S–locally n–polynomial if and only if it is S'–local n–polynomial under our assumption.

For the local n–homogeneous model structures, recall that these are certain left Bousfield localizations of the n–homogeneous model structure (see Proposition 5.9), hence have the same cofibrations. As before, these model structures have the same fibrant objects since our assumption together with Lemma 4.4 implies that the nth derivative of a functor is S–local if and only if it is S'–local, and the fibrant objects are the n–polynomial functors with local derivatives; see Proposition 5.9.

For the converse note that since the S–local model structure on the category of orthogonal functors agrees with the S'–local model structure, the objectwise S–local equivalences are precise the objectwise S–local equivalences. It follows that the local model structures on $\mathcal{C}$ must agree. $\square$

6.3   The partial ordering of Bousfield classes

Lemma 6.5   *Let S and S' be sets of maps in $\mathcal{C}$ and F an orthogonal functor. If the class of S'–local objects of $\mathcal{C}$ is contained in the class of S–local objects, then*

(1)   *there is an S'–local equivalence $D_n^S F \to D_n^{S'} F$; and*

(2)   *if F is reduced, then the S–local Weiss tower of F is S'–locally equivalent to the S'–local Weiss tower of F.*

Proof   For (1), note that the map on derivatives $\partial_n^S F \to \partial_n^{S'} F$ induced by the natural transformation $L_S \to L_{S'}$ is an S'–local equivalence; hence the n–homogeneous functors which correspond to these spectra are S'–locally equivalent, ie the map $D_n^S F \to D_n^{S'} F$ is an S'–local equivalence. For (2), since F is reduced, [Weiss 1995, Corollary 8.3] implies that there is a commutative diagram

$$
\begin{array}{ccccc}
T_n^S F & \longrightarrow & T_{n-1}^S F & \longrightarrow & R_n^S F \\
\downarrow & & \downarrow & & \downarrow \\
T_n^{S'} F & \longrightarrow & T_{n-1}^{S'} F & \longrightarrow & R_n^{S'} F
\end{array}
$$

in which both rows are homotopy fibre sequences. The map $R_n^S F \to R_n^{S'} F$ is an S'–local equivalence by part (1), and the map $T_0^S F \to T_0^{S'} F$ is also an S'–local equivalence since F is reduced. An induction argument on the degree of polynomials yields the result. $\square$

Example 6.6   (1)   Let E and E' be spectra and F an orthogonal functor. If $\langle E \rangle \le \langle E' \rangle$, then

(a)   there is an E–local equivalence $D_n^{E'} F \to D_n^E F$; and

(b)   if F is reduced, then the E'–local Weiss tower of F is E–locally equivalent to the E–local Weiss tower of F.

(2)   Let W and W' be based spaces and F a Top_*–valued orthogonal functor. If $\langle W \rangle \le \langle W' \rangle$, then

(a)   there is an W'–local equivalence $D_n^W F \to D_n^{W'} F$; and

(b)   if F is reduced, then the W–local Weiss tower of F is W'–locally equivalent to the W'–local Weiss tower of F.

6.4 The telescope conjecture

The height n telescope conjecture of Ravenel [1984, Conjecture 10.5] asserts that the $T(n)$–localization and $K(n)$–localization of spectra agree. There are numerous equivalent formalisations of the conjecture — see eg [Barthel 2020, Proposition 3.6] — and we choose the following as it best suits any possible interaction with the calculus.

Conjecture 6.7 (the height n telescope conjecture) *Let $n \geq 0$. The Bousfield class of $T(n)$ agrees with the Bousfield class of $K(n)$.*

Corollary 6.8 *Let $n \geq 0$. The validity of the height n telescope conjecture implies equality of model structures*

$$
\begin{array}{ccc}
\mathsf{Fun}(\mathcal{J}_0, L_{T(n)}\mathcal{C}) \underset{\mathbb{1}}{\overset{\mathbb{1}}{\rightleftarrows}} \mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_{T(n)}\mathcal{C}) \overset{\mathbb{1}}{\underset{\mathbb{1}}{\rightleftarrows}} \mathsf{Homog}^n(\mathcal{J}_0, L_{T(n)}\mathcal{C}) \\
\| \qquad\qquad\qquad \| \qquad\qquad\qquad \| \\
\mathsf{Fun}(\mathcal{J}_0, L_{K(n)}\mathcal{C}) \underset{\mathbb{1}}{\overset{\mathbb{1}}{\rightleftarrows}} \mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_{K(n)}\mathcal{C}) \overset{\mathbb{1}}{\underset{\mathbb{1}}{\rightleftarrows}} \mathsf{Homog}^n(\mathcal{J}_0, L_{K(n)}\mathcal{C})
\end{array}
$$

Proof The telescope conjecture implies that the Bousfield class of $T(n)$ and the Bousfield class of $K(n)$ agree, hence the result follows by Theorem 6.4. □

The following is an immediate corollary to Theorem 6.2.

Corollary 6.9 *Let $n \geq 0$. The height n telescope conjecture holds if and only if for every orthogonal functor F the $K(n)$–local Weiss tower of F and the $T(n)$–local Weiss tower of F agree.*

This provides new insight into the height n telescope conjecture. For example, to find a counterexample it now suffices to find an orthogonal functor such that one corresponding term in the $K(n)$–local and $T(n)$–local Weiss towers disagree. This can also be seen through the spectral sequences associated to the local Weiss towers. The $K(n)$–local and $T(n)$–local Weiss towers of an orthogonal functor F produce two spectral sequences,

$$
\pi_{t-s} D_s^{K(n)} F(V) \cong \pi_{t-s}((S^{\mathbb{R}^s \otimes V} \wedge \partial_s^{K(n)} F)_{hO(n)}) \Rightarrow \pi_* \operatorname*{holim}_d T_d^{K(n)} F(V),
$$

$$
\pi_{t-s} D_s^{T(n)} F(V) \cong \pi_{t-s}((S^{\mathbb{R}^s \otimes V} \wedge \partial_s^{T(n)} F)_{hO(n)}) \Rightarrow \pi_* \operatorname*{holim}_d T_d^{T(n)} F(V).
$$

These are closely related to the telescope conjecture as follows.

Lemma 6.10 *Let F be an orthogonal functor. If the height n telescope conjecture holds, then for all $r \geq 1$, the E_r–page of the $T(n)$–local Weiss spectral sequence is isomorphic to the E_r–page of the $K(n)$–local Weiss spectral sequence.*

Proof It suffices to prove the claim for $r = 1$. The validity of the height n telescope conjecture implies that there is a natural transformation $L_{K(n)} \to L_{T(n)}$. This natural transformation induces a map $D_d^{K(n)} F \to D_d^{T(n)} F$, which by Corollary 6.9 is an objectwise weak equivalence. It hence suffices to show that the natural map $D_d^{K(n)} F \to D_d^{T(n)} F$ induces a map on the E_1–pages of the spectral sequences, that is, we have to show that the induced diagram

$$
\begin{array}{ccc}
\pi_{t-s} D_s^{K(n)} F(V) & \xrightarrow{d_1^{K(n)}} & \pi_{t-s+1} D_{s+1}^{K(n)} F(V) \\
\downarrow & & \downarrow \\
\pi_{t-s} D_s^{T(n)} F(V) & \xrightarrow[d_1^{T(n)}]{} & \pi_{t-s+1} D_{s+1}^{T(n)} F(V)
\end{array}
$$

commutes for all s and t. This follows from the commutativity of the induced diagram of long exact sequences induced by the diagram of homotopy fibre sequences

$$
\begin{array}{ccc}
D_s^{K(n)} F(V) \longrightarrow T_s^{K(n)} F(V) \longrightarrow T_{s-1}^{K(n)} F(V) \\
\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \\
D_s^{T(n)} F(V) \longrightarrow T_s^{T(n)} F(V) \longrightarrow T_{s-1}^{T(n)} F(V)
\end{array}
$$

and the construction of the d_1–differential in the homotopy spectral sequence associated to a tower of fibrations. $\square$

7 The calculus for nullifications

7.1 Nullifications of orthogonal functors

Bousfield, Dror Farjoun and others — see eg [Bousfield 1994; 1996; Casacuberta 1994; Dror Farjoun 1996] — have extensively studied the nullification of the category of based spaces at a based space W. This nullification is functorial, giving a functor

$$P_W \colon \mathsf{Top}_* \to \mathsf{Top}_*,$$

and the Bousfield–Friedlander localization of Top_* at the endofunctor P_W defines a model structure which we call the W–periodic model structure, and denote by $P_W \mathsf{Top}_*$. This model structure is precisely the left Bousfield localization at the set $S = \{* \to W\}$, ie the W–periodic and W–local model structures agree.

The endofunctor $P_W \colon \mathsf{Top}_* \to \mathsf{Top}_*$ extends objectwise to a functor

$$P_W \colon \mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*) \to \mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*),$$

and the W–periodic model structure on spaces — see eg [Bousfield 2001, Section 9.8] — extends in a canonical way to give the Bousfield–Friedlander localization of the category of orthogonal functors at

the functor P_W, which we denote by $\mathsf{Fun}(\mathcal{J}_0, P_W \mathsf{Top}_*)$, and call the W–periodic model structure. This model structure agrees with the S–local model structure on orthogonal functors for $S = \{* \to W\}$.

In this section we give an alternative construction of the model structures for W–local orthogonal calculus. The key to this is that the W–periodic model structure on based spaces is right proper.

Remark 7.1 The process of left Bousfield localization can interfere with other model categorical properties, for instance left Bousfield localization need not preserve right properness. For example if $E = H\mathbb{Q}$, then the $H\mathbb{Q}$–local model structure on based spaces is not right proper since there is a pullback square

$$\begin{array}{ccc} K(\mathbb{Q}/\mathbb{Z}, 0) & \longrightarrow & P \\ \downarrow & & \downarrow \\ K(\mathbb{Z}, 1) & \xrightarrow{\simeq H\mathbb{Q}} & K(\mathbb{Q}, 1) \end{array}$$

in which the right-hand vertical map is a fibration, P is contractible and the lower horizontal map is a $H\mathbb{Q}$–equivalence but the left-hand vertical map is not. Another example is provided by Quillen [1969, Remark 2.9].

The property of being right proper has many advantages including the ability to right Bousfield localize. As such we investigate when the S–local model structure is right proper. It suffices to examine when the f–local model structure is right proper for some map $f \colon X \to Y$ of based spaces.

The following has motivation in [Bousfield 2001, Remark 9.11], which notes that the f–local model structure cannot be right proper unless the localization functor L_f is equivalent to a nullification. We extend Bousfield's remark by showing that his nullification condition is both necessary and sufficient in a stronger sense than originally proposed by Bousfield. This result depends on two constructions also due to Bousfield: the first is the construction of a based space $A(f)$ associated to a map $f \colon X \to Y$ of based spaces (see [Bousfield 1997, Theorem 4.4]); the second is the nullification functor $P_W \colon \mathsf{Top}_* \to \mathsf{Top}_*$ associated to any based space W (see [Bousfield 1994, Theorem 2.10]). This nullification functor has two key properties which we would also like to highlight: first, when W is connected, P_W preserves disjoint unions (see eg [Bousfield 2001, Theorem 9.9]); and second, P_W is contractible when W is not connected (see eg [Bousfield 1994, Example 2.3]). For example, if f is the map which induces localization with respect to integral homology, then $P_{A(f)}$ is Quillen's plus construction; see eg [Dror Farjoun 1996, 1.E.5].

Proposition 7.2 *Let $f \colon X \to Y$ be a map of based spaces. The f–local model structure on based spaces is right proper if and only if there exists a based space $A(f)$ and equality of model structures*

$$L_f \mathsf{Top}_* = P_{A(f)} \mathsf{Top}_*,$$

where $P_{A(f)} \mathsf{Top}_*$ is the Bousfield–Friedlander localization [Bousfield 2001, Theorem 9.3], *at the nullification endofunctor*

$$P_{A(f)} \colon \mathsf{Top}_* \to \mathsf{Top}_*.$$

Proof By [Bousfield 1997, Theorem 4.4], there exists a based space $A(f)$ such that the classes of $A(f)$–acyclic and f–acyclic spaces agree, and every $P_{A(f)}$–equivalence is an f–local equivalence.

Assume that the f–local model structure is right proper. For a connected based space X, the path fibration over $L_f X$ is an f–local fibration; hence the homotopy fibre of the map $X \to L_f X$ is f–acyclic, and hence $A(f)$–acyclic. It follows by [Bousfield 1994, Corollary 4.8(i)], the map $X \to L_f X$ is a $P_{A(f)}$–equivalence; hence every f–local equivalences of connected spaces is a $P_{A(f)}$–equivalence. Since the functor $P_{A(f)}$ on based spaces comes from a functor on unbased spaces which preserves disjoint unions when $A(f)$ is connected and which takes contractible values when $A(f)$ is not connected, every f–local equivalence must be a $P_{A(f)}$–equivalence. It follows that the class of f–local equivalences agrees with the class of $P_{A(f)}$–equivalences. The equality of the model structures follows immediately since both model structures have the same cofibrations inherited from the Quillen model structure on the category of based spaces.

For the converse, assume that the f–local model structure agrees with the $A(f)$–local model structure. The latter model structure is right proper by [Bousfield 2001, Theorem 9.9], and since both model structures have the same weak equivalences and fibrations, the f–local model structure must also be right proper. $\square$

Remark 7.3 The property of being right proper is completely determined by the weak equivalence class of the model structure; if two model structures have the same weak equivalences, then one is right proper if and only if the other is; see eg [Balchin 2021, Remark 2.5.6].

7.2 Nullifications and polynomial functors

Recall from Proposition 3.5 that we have minimal control over the W–local n–polynomial model structure, in particular, unless the localization is well-behaved with respect to sequential homotopy colimits, $T_n L_W$ is not a fibrant replacement functor. We construct a W–periodic n–polynomial model structure as the Bousfield–Friedlander localization at the composite

$$T_n \circ P_W \colon \mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*) \to \mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*)$$

and show that this model structure is precisely the W–local n–polynomial model structure.

We begin with a lemma which deals with fibrant objects in the Bousfield–Friedlander localization of orthogonal functors at the endofunctor P_W, which we call the W–periodic projective model structure.

Lemma 7.4 *For a finite cell complex W and an orthogonal functor F, the functor $T_n P_W F$ is fibrant in the Bousfield–Friedlander localization of the category of orthogonal functors at the functor P_W. In particular, the map*

$$\omega_{T_n P_W F} \colon T_n P_W F \to P_W T_n P_W F$$

is an objectwise weak homotopy equivalence.

Proof   The Bousfield–Friedlander localization of based spaces at the endofunctor P_W is identical to the left Bousfield localization of based spaces at the map $* \to W$, since both model structures have the same cofibrations and fibrant objects. It follows that the Bousfield–Friedlander localization of the category of orthogonal functors at the endofunctor P_W is identical to the W–local projective model structure. In particular, we see that $P_W F$ is fibrant and hence $\tau_n P_W F$ is also fibrant, since the class of W–local objects is closed under homotopy limits. The result follows since local objects for a nullification are closed under sequential homotopy colimits by [Dror Farjoun 1996, 1.D.6]. □

Proposition 7.5   *For a finite cell complex W the Bousfield–Friedlander localization of the category of orthogonal functors at the endofunctor*

$$T_n \circ P_W \colon \mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*) \to \mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*)$$

exists. This model structure is proper and topological. We call this the W–periodic n–polynomial model structure and denote it by $\mathsf{Poly}^{\leq n}(\mathcal{J}_0, P_W \mathsf{Top}_*)$.

Proof   We verify the axioms of [Bousfield 2001, Theorem 9.3]. First note that since P_W and T_n both preserve objectwise weak equivalences so does their composite, hence verifying [Bousfield 2001, Theorem 9.3(A1)].

The natural transformation from the identity to the composite $T_n \circ P_W$ is given in components as the composite

$$F \xrightarrow{\omega_F} P_W F \xrightarrow{\eta_{P_W F}} T_n P_W F,$$

where $\omega \colon \mathbb{1} \to P_W$ and $\eta \colon \mathbb{1} \to T_n$; hence at $T_n P_W F$, we obtain the composite

$$T_n P_W F \xrightarrow{\omega_{T_n P_W F}} P_W T_n P_W F \xrightarrow{\eta_{P_W T_n P_W F}} T_n P_W T_n P_W F.$$

Since the domain is fibrant in the W–periodic projective model structure the first map in the composite is an objectwise weak equivalence; see Lemma 7.4. The second map is also a weak equivalence. To see this, note that since $T_n P_W F$ is polynomial of degree less than or equal n, the functor $P_W T_n P_W F$ is also polynomial of degree less than or equal n by the commutativity of the diagram

$$
\begin{array}{ccc}
T_n P_W F & \longrightarrow & \tau_n T_n P_W F \\
\downarrow & & \downarrow \\
P_W T_n P_W F & \longrightarrow & \tau_n P_W T_n P_W F
\end{array}
$$

and the fact that homotopy limits preserve objectwise weak equivalences. It follows that the natural transformation $\eta \colon T_n P_W F \to T_n P_W T_n P_W F$ is an objectwise weak equivalence, as a composite of two objectwise weak equivalences.

The map $T_n P_W(\eta): T_n P_W F \to T_n P_W T_n P_W F$ is also an objectwise weak equivalence. To see this, note that there is a commutative diagram

$$
\begin{array}{ccccc}
F & \xrightarrow{\ \omega_F\ } & P_W F & \xrightarrow{\ \eta_{P_W F}\ } & T_n P_W F \\
{\scriptstyle \omega_F}\downarrow & {\scriptstyle (1)} & {\scriptstyle \omega_{P_W F}}\downarrow & {\scriptstyle (2)} & \downarrow{\scriptstyle \omega_{T_n P_W F}} \\
P_W F & \xrightarrow{\ P_W \omega_F\ } & P_W P_W F & \xrightarrow{\ P_W \eta_{P_W F}\ } & P_W T_n P_W F \\
{\scriptstyle \eta_{P_W F}}\downarrow & {\scriptstyle (3)} & {\scriptstyle \eta_{P_W P_W F}}\downarrow & {\scriptstyle (4)} & \downarrow{\scriptstyle \eta_{P_W T_n P_W F}} \\
T_n P_W F & \xrightarrow[T_n P_W \omega_F]{} & T_n P_W P_W F & \xrightarrow[T_n P_W \eta_{P_W F}]{} & T_n P_W T_n P_W F
\end{array}
$$

in which the required map is given by the lower horizontal composite. Since P_W is a homotopically idempotent functor, $P_W \omega_F$ is an objectwise weak equivalence. It follows that the bottom horizontal map

$$T_n P_W \omega_F: T_n P_W F \to T_n P_W P_W F$$

of (3) is a weak equivalence since T_n preserves weak equivalences.

Moreover, P_W being homotopically idempotent yields that the vertical map

$$\omega_{P_W F}: P_W F \to P_W P_W F$$

in (2) is an objectwise weak equivalence. The right-hand vertical map in this square is also an equivalence by Lemma 7.4. By [Weiss 1995, Theorem 6.3], the top right-hand horizontal map

$$\eta_{P_W F}: P_W F \to T_n P_W F$$

is an approximation of order n in the sense of [Weiss 1995, Definition 5.16]. By commutativity of (2), the lower horizontal map

$$P_W \eta_{P_W F}: P_W P_W F \to P_W T_n P_W F$$

is an approximation of order n. The proof of [Weiss 1995, Theorem 6.3] also demonstrates that the vertical maps in (4) are approximations of order n, and since three out of the four maps in the lower right square are approximations of order n, so too is the lower right-hand horizontal map

$$T_n P_W \eta_{P_W F}: T_n P_W P_W F \to T_n P_W T_n P_W F.$$

An application of [Weiss 1995, Theorem 5.15] yields that this map is an objectwise weak equivalence as both source and target are polynomial of degree less than or equal n. This concludes the proof that the map

$$T_n P_W(\eta): T_n P_W F \to T_n P_W T_n P_W F$$

is an objectwise weak equivalence, and verifies [Bousfield 2001, Theorem 9.3(A2)].

Finally we verify [Bousfield 2001, Theorem 9.3(A3)]. Let

$$\begin{array}{ccc} A & \xrightarrow{\ k\ } & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow{\ h\ } & D \end{array}$$

be a pullback square with f an objectwise fibration between W–local n–polynomial functors, and $T_n P_W h \colon T_n P_W C \to T_n P_W D$ an objectwise weak equivalence. By [Bousfield 2001, Theorem 9.9], we see that the fibre of k is P_W–acyclic, ie $P_W(\mathrm{fib}(k))$ is objectwise weakly contractible. Since T_n preserves objectwise weak equivalences, we see that $T_n P_W(\mathrm{fib}(k))$ is objectwise weakly contractible, and hence k is a $T_n P_W$–equivalence.

The fact that the resulting model structure is topological follows from [Bousfield 2001, Theorem 9.1]. □

This Bousfield–Friedlander localization results in an identical model structure to the W–local n–polynomial model structure of Proposition 3.5

Proposition 7.6 *For a finite cell complex W there is an equality of model structures*

$$\mathrm{Poly}^{\leq n}(\mathcal{J}_0, L_W \mathsf{Top}_*) = \mathrm{Poly}^{\leq n}(\mathcal{J}_0, P_W \mathsf{Top}_*),$$

that is, the W–local n–polynomial model structure and the W–periodic n–polynomial model structure agree. In particular, these model structures are cellular, proper and topological.

Proof Both model structures have the same cofibrations, namely the projective cofibrations. It suffices to show that they share the same fibrant objects. Working through the definition of a fibrant object in the Bousfield–Friedlander localization we see that an orthogonal functor F is fibrant if and only if the canonical map $F \to T_n P_W F$ is an objectwise weak equivalence. It follows that F must be W–local and n–polynomial, hence fibrant in the W–local n–polynomial model structure. Conversely, if F is fibrant in the W–local n–polynomial model structure, then the map $F \to P_W F$ is an objectwise weak equivalence and there is a commutative diagram

$$\begin{array}{ccc} F & \longrightarrow & P_W F \\ \downarrow & & \downarrow \\ T_n F & \longrightarrow & T_n P_W F \end{array}$$

in which three out of the four arrows are objectwise weak equivalences, hence so too is the right-hand vertical arrow. It follows that F is fibrant in the Bousfield–Friedlander localization. □

Remark 7.7 The nullification condition here is necessary. The above lemma does not hold in general. To see this, consider the (smashing) localization at the spectrum $E = H\mathbb{Q}$. The $H\mathbb{Q}$–local model structure is not right proper (see Remark 7.1), yet if this were expressible as a Bousfield–Friedlander localization it would necessarily be right proper [Bousfield 2001, Theorem 9.3].

Corollary 7.8 *For a finite cell complex W, a map $f : X \to Y$ is a fibration in the W–local n–polynomial model structure if and only if f is a fibration in the projective model structure and the square*

$$
\begin{array}{ccc}
X & \longrightarrow & T_n P_W X \\
\downarrow & & \downarrow \\
Y & \longrightarrow & T_n P_W Y
\end{array}
$$

is a homotopy pullback square in the projective model structure on $\mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*)$.

Remark 7.9 It is highly unlikely that this result holds in more general localizations than nullifications. Let $\mathcal{C}$ be a model category and S a set of maps in $\mathcal{C}$ such that the left Bousfield localization of $\mathcal{C}$ at S exists. By [Hirschhorn 2003, Proposition 3.4.8(1)] right properness of $\mathcal{C}$ and $L_S \mathcal{C}$ is sufficient for a map $f : X \to Y$ being a fibration in $L_S \mathcal{C}$ if and only if f is a fibration in $\mathcal{C}$ and the square

$$
\begin{array}{ccc}
X & \xrightarrow{\ j_X\ } & \widehat{X} \\
f \downarrow & & \downarrow \hat{f} \\
Y & \xrightarrow[\ j_Y\ ]{} & \widehat{Y}
\end{array}
$$

is a homotopy pullback square, where $\hat{f} : \widehat{X} \to \widehat{Y}$ is a S–localization of f in the sense of [Hirschhorn 2003, Definition 3.2.16]. In our situation, Proposition 7.2 guarantees that a homological localization is right proper if and only if it is a nullification. However, it is not clear in general if right properness of the base model category and the localized model category is a necessary condition for the above description of the fibrations in $L_S \mathcal{C}$.

7.3 Nullifications and homogeneous functors

In the case of a nullification, the W–local n–homogeneous model structure of Proposition 5.9 is not the only way of constructing a model structure with the correct homotopy category. Since the W–local model structure on based spaces is right proper, so too is the W–local n–polynomial model structure and hence we can also follow the more standard procedure and preform a right Bousfield localization at the set

$$
\mathcal{K}'_n = \{\mathcal{J}_n(U, -) \mid U \in \mathcal{J}\},
$$

to obtain a local n–homogeneous model category structure.

Proposition 7.10 *For a finite cell complex W there exists a model structure on the category of orthogonal functors with weak equivalences those maps $X \to Y$ such that*

$$
(T_n P_W X)^{(n)} \to (T_n P_W X)^{(n)}
$$

is an objectwise weak equivalence and with fibrations the fibrations of the W–local n–polynomial model structure. This model structure cellular, proper, stable and topological. We call this the W–periodic n–homogeneous model structure and denote it by $\mathsf{Homog}^n(\mathcal{J}_0, P_W \mathsf{Top}_*)$.

Proof This is the right Bousfield localization of the W–local n–polynomial model structure. The proof of which follows exactly as in [Barnes and Oman 2013, Proposition 6.9]. Note that this right Bousfield localization exists since the W–local n–polynomial model structure in right proper and cellular when the localization is a nullification; see Proposition 7.6. □

This right Bousfield localization behaves like a left Bousfield localization of the n–homogeneous model structure in the following sense.

Lemma 7.11 *For a finite cell complex W, the adjoint pair*

$$\mathbb{1} \colon \mathsf{Homog}^n(\mathcal{J}_0, \mathsf{Top}_*) \rightleftarrows \mathsf{Homog}^n(\mathcal{J}_0, P_W \mathsf{Top}_*) \colon \mathbb{1}$$

is a Quillen adjunction.

Proof Since the acyclic cofibrations of the n–homogeneous model structure are precisely the acyclic cofibrations of the n–polynomial model structure and similarly, the acyclic cofibrations of W–periodic n–homogeneous model structure are precisely the acyclic cofibrations of the W–local n–polynomial model structure, the identity functor preserves acyclic cofibrations by Lemma 3.6.

On the other hand, by [Hirschhorn 2003, Proposition 3.3.16(2)], cofibrations between cofibrant objects in a right Bousfield localization are cofibrations in the underlying model structure; hence Lemma 3.6 shows that the identity functor preserves cofibrations between cofibrant objects. The result follows by [Dugger 2001, Corollary A.2]. □

An analogous Quillen equivalence is obtained between the W–local intermediate category and the W–periodic n–homogeneous model structure of Proposition 7.10, which we recall is obtained as a right Bousfield localization of the W–local n–polynomial model structure. The proof is all but identical to [Barnes and Oman 2013, Theorem 10.1].

Theorem 7.12 *For a finite cell complex W, the adjoint pair*

$$\mathsf{res}_0^n / O(n) \colon L_W \mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\mathsf{Top}_*) \rightleftarrows \mathsf{Homog}^n(\mathcal{J}_0, P_W \mathsf{Top}_*) \colon \mathsf{ind}_0^n \varepsilon^*$$

is a Quillen equivalence.

Propositions 5.9 and 7.10 provide two different model structures which both capture the homotopy theory of W–locally n–homogeneous functors. However, these model structures are not identical. For instance, the W–local model structure of Proposition 5.9 has fibrant objects the n–polynomial functors which have W–local n^{th} derivative, whereas the fibrant objects of the W–periodic n–homogeneous model structure (Proposition 7.10) are the W–local n–polynomial functors. However, they are Quillen equivalent via the identity functor.

Corollary 7.13 *For a finite cell complex W, the adjoint pair*

$$\mathbb{1}\colon \mathrm{Homog}^n(\mathcal{J}_0, L_W\,\mathrm{Top}_*) \rightleftarrows \mathrm{Homog}^n(\mathcal{J}_0, P_W\,\mathrm{Top}_*)\colon \mathbb{1}$$

is a Quillen equivalence.

Proof Since cofibrations between cofibrant objects in $\mathrm{Homog}^n(\mathcal{J}_0, L_W\,\mathrm{Top}_*)$ are projective cofibrations which are T_n–equivalences, and the cofibrations between cofibrant objects of $\mathrm{Homog}^n(\mathcal{J}_0, P_W\,\mathrm{Top}_*)$ are the projective cofibrations, it follows that the identity functor

$$\mathbb{1}\colon \mathrm{Homog}^n(\mathcal{J}_0, L_W\,\mathrm{Top}_*) \to \mathrm{Homog}^n(\mathcal{J}_0, P_W\,\mathrm{Top}_*)$$

necessarily preserves cofibrations between cofibrant objects. On the other hand, the identity functor

$$\mathbb{1}\colon \mathrm{Homog}^n(\mathcal{J}_0, P_W\,\mathrm{Top}_*) \to \mathrm{Homog}^n(\mathcal{J}_0, L_W\,\mathrm{Top}_*)$$

preserves fibrant objects since if X is objectwise W–local, $\mathrm{ind}_0^n X$ is objectwise W–local, by Lemma 4.4. It follows that the adjunction is a Quillen adjunction. To see that the adjunction is a Quillen equivalence, there is a commutative square

$$
\begin{array}{ccc}
L_W\,\mathrm{Fun}_{O(n)}(\mathcal{J}_n, O(n)\mathrm{Top}_*) & \underset{\mathrm{ind}_0^n\,\varepsilon^*}{\overset{\mathrm{res}_0^n\,/O(n)}{\rightleftarrows}} & \mathrm{Homog}^n(\mathcal{J}_0, L_W\,\mathrm{Top}_*) \\[2mm]
\mathbb{1}\Big\downarrow\Big\uparrow\mathbb{1} & & \mathbb{1}\Big\downarrow\Big\uparrow\mathbb{1} \\[2mm]
L_W\,\mathrm{Fun}_{O(n)}(\mathcal{J}_n, O(n)\mathrm{Top}_*) & \underset{\mathrm{ind}_0^n\,\varepsilon^*}{\overset{\mathrm{res}_0^n\,/O(n)}{\rightleftarrows}} & \mathrm{Homog}^n(\mathcal{J}_0, P_W\,\mathrm{Top}_*)
\end{array}
$$

of Quillen adjunctions, in which three out of four are Quillen equivalences by Theorems 5.15 and 7.12. Hence the remaining Quillen adjunction must also be a Quillen equivalence. $\qquad\square$

It follows that there is a zigzag of Quillen equivalences

$$\mathrm{Homog}^n(\mathcal{J}_0, P_W\,\mathrm{Top}_*) \simeq_Q \mathrm{Sp}(L_W\,\mathrm{Top}_*)[O(n)],$$

whenever both model structures exist.

8 Postnikov sections

Given a based space A, the kth Postnikov section of A is the nullification of A at S^{k+1}, ie $P_k A = P_{S^{k+1}} A$. Given a diagram of (simplicial, left proper, combinatorial) model categories, Barwick [2010, Section 5, Application 1] and Bergner [2012] develop a general machinery for producing a model structure which captures the homotopy theory of the homotopy limit of the diagram of model categories. Gutiérrez and Roitzheim [2016, Section 4] applied this to the study of Postnikov sections for model categories, which recovers the classical theory when $\mathcal{C}$ is the Kan–Quillen model structure on simplicial sets. We consider the relationship between Postnikov sections and orthogonal calculus via our local calculus.

8.1 A combinatorial model for calculus

The current theory of homotopy limits of model categories requires that the model categories in question be combinatorial, ie locally presentable and cofibrantly generated. Since the category of based compactly generated weak Hausdorff spaces is not locally presentable, the Quillen model structure is not combinatorial and hence none of our model categories for orthogonal functors are either. We invite the reader to take for granted that all of our cellular model categories may be replaced by combinatorial model categories by starting with a combinatorial model for the Quillen model structure on based spaces, and hence skip directly to Section 8.2.

We give the details of these combinatorial replacements here. We replace compactly generated weak Hausdorff spaces with Δ–*generated spaces*; a particular full subcategory of the category of topological spaces, which were developed by Vogt [1971] and unpublished work of Smith, and are surveyed by Dugger [2003]. The category of Δ–generated spaces may be equipped with a model structure analogous to the Quillen model structure on compactly generated weak Hausdorff spaces with weak equivalences the weak homotopy equivalences and fibrations the Serre fibrations. This model structure is combinatorial, proper and topological. The existence of the model structure follows from [Dugger 2003, Section 1.9]. The locally presentable (and hence combinatorial) property follows from [Fajstrup and Rosický 2008, Corollary 3.7]. The Quillen equivalence may be extracted from [Dugger 2003, Section 1.9].

This combinatorial model for spaces transfers to categories of functors and we obtain a projective model structure on the category of orthogonal functors which is Quillen equivalent to our original projective model structure but is now combinatorial. A left or right Bousfield localization of a combinatorial model category is again combinatorial; hence the n–polynomial, n–homogeneous and local versions of these model categories are all combinatorial when we begin with the combinatorial model for the projective model structure on orthogonal functors.

Hypothesis 8.1 *For the remainder of this section, we will assume that all our model structures are combinatorial, since they are all Quillen equivalent to combinatorial model categories using the combinatorial model for based spaces.*

8.2 The model structure of k–types in orthogonal functors

Denote by I the set of generating cofibrations of the projective model structure of orthogonal functors, and denote by W_k the set of maps of the form

$$B \wedge S^{k+1} \amalg_{A \wedge S^{k+1}} A \wedge D^{k+2} \to B \wedge D^{k+2},$$

where $A \to B$ is a map in I. The model category of k–types in $\mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*)$ is the left Bousfield localization of the projective model structure at $I \square \{S^{k+1} \to D^{k+2}\}$ used by Gutiérrez and Roitzheim [2016] to model Postnikov sections.

Proposition 8.2 *Let $k \geq 0$. Under Hypothesis 8.1, the model structure of k–types in the category of orthogonal functors is identical to the S^{k+1}–local model structure, that is, there is an equality of model structures,*

$$P_k \operatorname{Fun}(\mathcal{J}_0, \operatorname{Top}_*) := L_{W_k} \operatorname{Fun}(\mathcal{J}_0, \operatorname{Top}_*) = \operatorname{Fun}(\mathcal{J}_0, L_{S^{k+1}} \operatorname{Top}_*).$$

Proof It suffices to show that both model structures have the same fibrant objects since the cofibrations in both model structures are identical. To see this, note that by examining the pushout product we can rewrite the set W_k as

$$W_k = \{\mathcal{J}_0(U, -) \wedge S_+^{n+k+1} \to \mathcal{J}_0(U, -) \wedge D_+^{n+k+2} \mid n \geq 0, \, U \in \mathcal{J}_0\}.$$

It follows by an adjunction argument that an orthogonal functor Z is W_k–local if and only if $\pi_i Z(U)$ is trivial for all $i \geq k + 1$ and all $U \in \mathcal{J}_0$. This last condition is equivalent to being objectwise S^{k+1}–local. □

8.3 The model structure of k–types in spectra

Taking I_{Sp} to be the set of generating cofibrations of the stable model structure on Sp and denoting again by W_k the relevant pushout product maps, we obtain a similar characterisation of the category of k–types in spectra.

Proposition 8.3 *Let $k \geq 0$. Under Hypothesis 8.1, there is an equality of model structures between the model category of k–types in spectra, and the stabilisation of S^{k+1}–local spaces, that is,*

$$P_k \mathsf{Sp} := L_{W_k} \mathsf{Sp} = \mathsf{Sp}(L_{S^{k+1}} \operatorname{Top}_*).$$

Proof Both model structures can be described as particular left Bousfield localizations of the stable model structure on spectra, hence have the same cofibrations. The proof reduces to the fact that the model structures have the same fibrant objects. To see this, note that the fibrant objects of $P_k \mathsf{Sp}$ are the k–truncated Ω–spectra, and the fibrant objects of $\mathsf{Sp}(L_{S^{k+1}} \operatorname{Top}_*)$ are the levelwise k–truncated Ω–spectra. Since both fibrant objects are Ω–spectra a connectivity style argument yields that an Ω–spectrum is k–truncated if and only if it is levelwise k–truncated, and hence both model structures have the same fibrant objects. □

Remark 8.4 Given a compact Lie group G, a similar procedure shows that there is an equality of model structures

$$P_k \mathsf{Sp}[G] := L_{W_k} \mathsf{Sp}[G] = \mathsf{Sp}(L_{S^{k+1}} \operatorname{Top}_*)[G].$$

8.4 Postnikov reconstruction of orthogonal functors

The collection of S^{k+1}–local model structures on the category of orthogonal functors assembles into a tower of model categories[7]

$$\mathsf{P}_\bullet \colon \mathbb{N}^{\mathrm{op}} \to \mathsf{MCat}, \quad k \mapsto \operatorname{Fun}(\mathcal{J}_0, L_{S^{k+1}} \operatorname{Top}_*),$$

---

[7]A tower of model categories is a special instance of a left Quillen presheaf, that is a diagram of the form $F \colon \mathcal{J}^{\mathrm{op}} \to \mathsf{MCat}$ for some small indexing category $\mathcal{J}$.

where MCat denotes the category of model categories and left Quillen functors. The homotopy limit of this tower of model categories recovers the projective model structure on orthogonal functors. The existence of a model structure which captures the homotopy theory of the limit of these model categories follows from [Gutiérrez and Roitzheim 2016, Proposition 2.2]. In particular, the homotopy limit model structure is a model structure on the category of sections[8] of the diagram $\mathsf{P}_\bullet$ formed by right Bousfield localizing the injective model structure in which a map of sections is a weak equivalence or cofibration if it is an objectwise weak equivalence or cofibration respectively.

Lemma 8.5 [Gutiérrez and Roitzheim 2016, Theorem 1.3 and Proposition 2.2] *There is a combinatorial model structure on the category of sections of* $\mathsf{P}_\bullet$ *where a map* $f_\bullet\colon X_\bullet \to Y_\bullet$ *is a fibration if and only if* f_0 *is a fibration in* $\mathsf{Fun}(\mathcal{J}_0, L_{S^1}\mathsf{Top}_*)$ *and for every* $k \geq 1$ *the induced map*

$$
\begin{array}{ccc}
X_k & & \\
& Y_k \times_{Y_{k-1}} X_{k-1} \longrightarrow X_{k-1} & \\
& \downarrow \qquad\qquad\qquad \downarrow & \\
& Y_k \longrightarrow Y_{k-1} &
\end{array}
$$
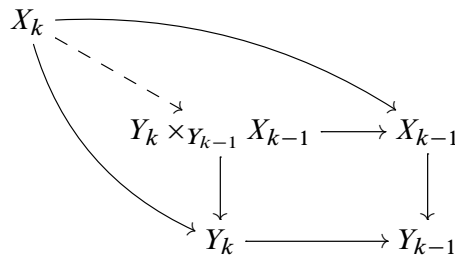
indicated by a dotted arrow in the above diagram is a fibration in $\mathsf{Fun}(\mathcal{J}_0, L_{S^{k+1}}\mathsf{Top}_*)$. *A section* $X_\bullet$ *is cofibrant if and only if* X_n *is cofibrant in* $\mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*)$ *and for every* $k \geq 0$, *the map* $X_{k+1} \to X_k$ *is a weak equivalence in* $\mathsf{Fun}(\mathcal{J}_0, L_{S^{k+1}}\mathsf{Top}_*)$. *A map of cofibrant sections is a weak equivalence if and only if the map is a weak equivalence in* $\mathsf{Fun}(\mathcal{J}_0, L_{S^{k+1}}\mathsf{Top}_*)$ *for each* $k \geq 0$. *We will refer to this model structure as the homotopy limit model structure and denote it by* $\mathrm{holim}\,\mathsf{P}_\bullet$.

Proposition 8.6 *Under Hypothesis 8.1 the adjoint pair*

$$\mathrm{const}\colon \mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*) \rightleftarrows \mathrm{holim}\,\mathsf{P}_\bullet \,:\mathrm{lim}$$

is a Quillen equivalence.

Proof The adjoint pair exists, and is a Quillen adjunction by [Gutiérrez and Roitzheim 2016, Lemma 2.4].

To see that the adjoint pair is a Quillen equivalence let $X_\bullet$ be a cofibrant and fibrant section in the homotopy limit model structure. Showing that

$$\mathrm{const}\,\lim X_\bullet \to X_\bullet$$

is a weak equivalence is equivalent to showing that the map $\lim X_\bullet \to X_k$ is a weak equivalence in $\mathsf{Fun}(\mathcal{J}_0, L_{S^{k+1}}\mathsf{Top}_*)$ for all $k \geq 0$. This is in turn, equivalent to the map $(\lim X_\bullet)(U) \to X_k(U)$ being a

---

[8]A section $X_\bullet$ of the tower $\mathsf{P}_\bullet$ is a sequence $\cdots \to X_k \to X_{k+1} \to \cdots \to X_0$ of orthogonal functors, and a morphism of sections $f\colon X_\bullet \to Y_\bullet$ is given by maps of orthogonal functors $f_k\colon X_k \to Y_k$ for all $k \geq 0$ subject to a commutative ladder condition.

weak equivalence in $L_{S^{k+1}}\mathsf{Top}_*$ for all $k \geq 0$. Since limits in functor categories are computed objectwise, the fact that the unit is a weak equivalence follows from [Gutiérrez and Roitzheim 2016, Theorem 2.5]. A similar argument shows that the counit is also a weak equivalence. $\square$

8.5 Postnikov reconstruction for spectra with an $O(n)$–action

The aim is to show that similar reconstruction theorems may be obtained for the n–homogeneous model structures. We first start by investigating analogous theorems for spectra and show that such reconstructions are compatible with the zigzag of Quillen equivalences between spectra with an $O(n)$–action and the n–homogeneous model structure. Proposition 8.3 and [Gutiérrez and Roitzheim 2016, Section 2.1] imply that the functor

$$\mathsf{P}_\bullet^{\mathsf{Sp}} : \mathbb{N}^{\mathrm{op}} \to \mathsf{MCat}, \quad k \mapsto \mathsf{Sp}(L_{S^{k+1}}\mathsf{Top}_*)$$

defines a left Quillen presheaf.[9] This left Quillen presheaf is "convergent" in the following sense.

Proposition 8.7 *Under Hypothesis 8.1 the adjoint pair*

$$\mathrm{const} \colon \mathsf{Sp} \rightleftarrows \mathrm{holim}\, \mathsf{P}_\bullet^{\mathsf{Sp}} \colon \mathrm{lim}$$

is a Quillen equivalence.

Proof The fact that the adjoint pair is a Quillen adjunction follows from [Gutiérrez and Roitzheim 2016, Lemma 2.4].

The left adjoint reflects weak equivalences between cofibrant objects. Indeed, if $X \to Y$ is a map between cofibrant spectra X and Y such that

$$\mathrm{const}(X) \to \mathrm{const}(Y)$$

is a weak equivalence in $\mathrm{holim}\, \mathsf{P}_\bullet^{\mathsf{Sp}}$, then

$$\mathrm{const}(X) \to \mathrm{const}(Y)$$

is a weak equivalence in $\mathsf{Sect}(\mathbb{N}, \mathsf{P}_\bullet^{\mathsf{Sp}})$ by the colocal Whitehead theorem and the fact that the left adjoint is left Quillen and thus preserves cofibrant objects. It follows that for each $k \in \mathbb{N}$, the induced map

$$\mathrm{const}(X)_k \to \mathrm{const}(Y)_k$$

is a weak equivalence in $\mathsf{Sp}(L_{S^{k+1}}\mathsf{Top}_*)$, that is, $X \to Y$ is a weak equivalence in $\mathsf{Sp}(L_{S^{k+1}}\mathsf{Top}_*)$ for all k. Unpacking the definition of a weak equivalence in $\mathsf{Sp}(L_{S^{k+1}}\mathsf{Top}_*)$ and using the fact that the right adjoint is a right Quillen functor and hence preserves weak equivalences between fibrant objects, we see that the induced map

$$\mathrm{lim}\, P_k X \to \mathrm{lim}\, P_k Y$$

is a weak equivalence in Sp, and hence, so is the map $X \to Y$.

---

[9] Alternatively, the adjunction $\mathbb{1} \colon \mathsf{Sp}(L_{S^{k+2}}\mathsf{Top}_*) \rightleftarrows \mathsf{Sp}(L_{S^{k+1}}\mathsf{Top}_*) \colon \mathbb{1}$, is a Quillen adjunction. This fact follows from the facts that both model structures have the same cofibrations and a S^{k+1}–local space is S^{k+2}–local as $\langle \Sigma W \rangle \leq \langle W \rangle$ for all based spaces W; see eg [Bousfield 1994, Section 9.9]. Hence $\mathsf{P}_\bullet^{\mathsf{Sp}}$ is a left Quillen presheaf.

It is left to show that the derived counit is an isomorphism. Let $Y_\bullet$ be bifibrant in holim $\mathsf{P}_\bullet^{\mathsf{Sp}}$. The condition that the counit applied to $Y_\bullet$ is a weak equivalence is equivalent to asking for the map

$$\lim_{\geq k} P_k Y_\bullet \to Y_k$$

to be a weak equivalence in $\mathsf{Sp}(L_{S^{k+1}}\mathsf{Top}_*)$ for all $k \in \mathbb{N}$. The structure maps of $Y_\bullet$ induce a map of towers

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & Y_j & \longrightarrow & \cdots & \longrightarrow & Y_{k+3} & \longrightarrow & Y_{k+2} & \longrightarrow & Y_{k+1} \\
& & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & Y_{k+1} & \longrightarrow & \cdots & \longrightarrow & Y_{k+1} & \longrightarrow & Y_{k+1} & \longrightarrow & Y_{k+1}
\end{array}
$$

in which each vertical arrow is a weak equivalence in $\mathsf{Sp}(L_{S^{k+1}}\mathsf{Top}_*)$. This map of towers induces a map

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & {\lim_{\geq k}}^1 \, \pi_{i+1}(Y_\bullet) & \longrightarrow & \pi_i(\lim_{\geq k} Y_\bullet) & \longrightarrow & \lim_{\geq k} \pi_i(Y_\bullet) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & {\lim_{\geq k}}^1 \, \pi_{i+1}(Y_{k+1}) & \longrightarrow & \pi_i(\lim_{\geq k} Y_{k+1}) & \longrightarrow & \lim_{\geq k} \pi_i(Y_{k+1}) & \longrightarrow & 0
\end{array}
$$

of short exact sequences. For $0 \leq i < n$ the left- and right-hand side maps are isomorphisms; hence the map

$$\lim_{\geq k} Y_\bullet \to Y_{k+1}$$

is a weak equivalence in $\mathsf{Sp}(L_{S^{k+1}}\mathsf{Top}_*)$ for all k, and it follows that the required map

$$\lim_{\geq k} Y_\bullet \to Y_{k+1} \to Y_k$$

is a weak equivalence in $\mathsf{Sp}(L_{S^{k+1}}\mathsf{Top}_*)$ for all k. $\qquad\square$

A similar justification to before provides a left Quillen presheaf

$$\mathsf{P}_\bullet^{\mathsf{Sp}[O(n)]} \colon \mathbb{N}^{\mathrm{op}} \to \mathsf{MCat}, \quad k \mapsto \mathsf{Sp}(L_{S^{k+1}}\mathsf{Top}_*)[O(n)],$$

where $\mathsf{Sp}(L_{S^{k+1}}\mathsf{Top}_*)[O(n)]$ is the category of $O(n)$–objects in the category of k–types in spectra. This is equivalent to the category of k–types in spectra with an $O(n)$–action. As a corollary to Proposition 8.7, we obtain that the induced left Quillen presheaf on spectra with an $O(n)$–action is also suitably convergent.

Corollary 8.8 *Under Hypothesis 8.1 the adjoint pair*

$$\mathrm{const} \colon \mathsf{Sp}[O(n)] \rightleftarrows \mathrm{holim}\, \mathsf{P}_\bullet^{\mathsf{Sp}[O(n)]} \colon \lim$$

is a Quillen equivalence.

8.6 Postnikov reconstruction for the intermediate categories

The functor

$$P_\bullet^{\mathcal{J}_n} : \mathbb{N}^{op} \to \mathsf{MCat}, \quad k \mapsto L_{S^{k+1}} \mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\mathsf{Top}_*)$$

defines a left Quillen presheaf, since there is an equality of model structures between the S^{k+1}–local n–stable model structure and the model structure of k–types in $\mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\mathsf{Top}_*)$. The proof of which is completely analogous to the case for spectra; see Proposition 8.3. Since the S^{k+1}–local n–stable model structure agrees with the model structure of k–types, we will denote both model structures by $P_k \mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\mathsf{Top}_*)$.

The homotopy limit of this left Quillen presheaf agrees with the homotopy limit of the left Quillen presheaf on spectra with an $O(n)$–action in the sense that the homotopy limit model categories are Quillen equivalent. In detail, the adjunction

$$(\alpha_n)_! : \mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\mathsf{Top}_*) \rightleftarrows \mathsf{Sp}[O(n)] : (\alpha_n)^*$$

of [Barnes and Oman 2013, Section 8] induces an adjunction

$$(\alpha_n)_!^{\mathbb{N}} : \mathsf{Fun}(\mathbb{N}, \mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\mathsf{Top}_*)) \rightleftarrows \mathsf{Fun}(\mathbb{N}, \mathsf{Sp}[O(n)]) : (\alpha_n^*)^{\mathbb{N}}$$

where $(\alpha_n^*)^{\mathbb{N}} = (\alpha_n)^* \circ (-)$. This adjunction in turn induces an adjunction

$$(\alpha_n)_!^{\mathbb{N}} : \mathrm{holim}\, P_\bullet^{\mathcal{J}_n} \rightleftarrows \mathrm{holim}\, P_\bullet^{\mathsf{Sp}[O(n)]} : (\alpha_n^*)^{\mathbb{N}}.$$

Proposition 8.9 *Under Hypothesis 8.1 the adjoint pair*

$$(\alpha_n)_!^{\mathbb{N}} : \mathrm{holim}\, P_\bullet^{\mathcal{J}_n} \rightleftarrows \mathrm{holim}\, P_\bullet^{\mathsf{Sp}[O(n)]} : (\alpha_n^*)^{\mathbb{N}}$$

is a Quillen equivalence.

Proof Fibrations of the homotopy limit model structure of $P_\bullet^{\mathsf{Sp}[O(n)]}$ are precisely the fibrations of the injective model structure on the category of sections of $P_\bullet^{\mathsf{Sp}[O(n)]}$ since the homotopy limit model structure is a right Bousfield localization of the injective model structure. A similar characterisation holds for the left Quillen presheaf $P_\bullet^{\mathcal{J}_n}$; hence to show that the right adjoint preserves fibrations it suffices to show that the left adjoint preserves acyclic cofibrations of the injective model structure on the categories of sections. To see this, note that the adjunction

$$(\alpha_n)_! : \mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\mathsf{Top}_*) \rightleftarrows \mathsf{Sp}[O(n)] : (\alpha_n)^*$$

is a Quillen adjunction, and hence so too is the induced adjunction on the injective model structures on the categories of sections.

To show that the left adjoint preserves cofibrations it suffices to show that cofibrations between cofibrant objects are preserved. As the homotopy limit model structures are right Bousfield localizations [Hirschhorn

2003, Proposition 3.3.16(2)] implies that cofibrations between cofibrant objects are cofibrations of the injective model structures on the categories of sections which by the analogous reasoning as above are preserved by the left adjoint. This yields that the adjunction in question is a Quillen adjunction.

To show that the adjunction is a Quillen equivalence notice that the right adjoint reflects weak equivalences between cofibrant objects by the colocal Whitehead theorem [Hirschhorn 2003, Theorem 3.2.13(2)], and the fact that the induced adjunction on the injective model structures on the categories of sections is a Quillen equivalence since for $B_\bullet \in \mathsf{Sect}(\mathbb{N}, \mathsf{P}_\bullet^{\mathcal{J}_n})$ and $X_\bullet \in \mathsf{Sect}(\mathbb{N}, \mathsf{P}_\bullet^{\mathsf{Sp}[O(n)]})$, a map $B_\bullet \to (\alpha_n^*)^{\mathbb{N}} X_\bullet$ is a weak equivalence if and only if for each $k \in \mathbb{N}$, the map $B_k \to (\alpha_n^*)^{\mathbb{N}} X_k$ is a weak equivalence of spectra, which in turn happens if and only if the adjoint map $(\alpha_n)_! B_k \to X_k$ is an n–stable equivalence, which is precisely the condition that the adjoint map $(\alpha_n)_!^{\mathbb{N}} B_\bullet \to X_\bullet$ is a weak equivalence.

It is left to show that the derived counit is an isomorphism. Let $Y_\bullet$ be bifibrant in the homotopy limit model structure of the left Quillen presheaf $\mathsf{P}_\bullet^{\mathsf{Sp}[O(n)]}$. Then the derived counit

$$(\alpha_n)_!^{\mathbb{N}} \hat{c}((\alpha_n^*)^{\mathbb{N}} Y_\bullet) \to Y_\bullet$$

is a map between cofibrant objects, hence a weak equivalence in the homotopy limit model structure if and only if a weak equivalence in the injective model structure on the category of sections, ie if and only if for each $k \in \mathbb{N}$, the induced map

$$(\alpha_n)_!(\alpha_n)^* Y_k \to Y_k$$

is a weak equivalence, which it always is by [Barnes and Oman 2013, Proposition 8.3]. $\qquad\square$

As a corollary, we see that the left Quillen presheaf $\mathsf{P}_\bullet^{\mathcal{J}_n}$ is convergent.

Corollary 8.10 *Under Hypothesis 8.1 the adjoint pair*

$$\mathrm{const}\colon \mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\mathsf{Top}_*) \rightleftarrows \mathrm{holim}\, \mathsf{P}_\bullet^{\mathcal{J}_n} \colon \mathrm{lim}$$

is a Quillen equivalence.

Proof Consider the commutative diagram

$$
\begin{array}{ccc}
\mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\mathsf{Top}_*) & \underset{(\alpha_n)^*}{\overset{(\alpha_n)_!}{\rightleftarrows}} & \mathsf{Sp}[O(n)] \\
\mathrm{const}\downarrow\uparrow\mathrm{lim} & & \mathrm{const}\downarrow\uparrow\mathrm{lim} \\
\mathrm{holim}\, \mathsf{P}_\bullet^{\mathcal{J}_n} & \underset{(\alpha_n^*)^{\mathbb{N}}}{\overset{(\alpha_n)_!^{\mathbb{N}}}{\rightleftarrows}} & \mathrm{holim}\, \mathsf{P}
\end{array}
$$

of Quillen adjunctions in which three out of the four adjoint pairs are Quillen equivalences by [Barnes and Oman 2013, Proposition 8.3], Corollary 8.8 and Proposition 8.9. It follows since Quillen equivalences satisfy the two-out-of-three property, that the remaining Quillen adjunction is a Quillen equivalence. $\qquad\square$

8.7 Postnikov reconstruction for homogeneous functors

The same approach as we have just employed for moving from spectra with an $O(n)$–action to the intermediate categories yields similar results for the homogeneous model structures. We choose to model S^{k+1}–local n–homogeneous functors by the S^{k+1}–periodic n–homogeneous model structures of Proposition 7.10.

Lemma 8.11 *The functor*

$$\mathsf{P}_\bullet^{\mathsf{Homog}^n} : \mathbb{N}^{\mathrm{op}} \to \mathsf{MCat}, \quad k \to \mathsf{Homog}^n(\mathcal{J}_0, P_{S^{k+1}}\mathsf{Top}_*)$$

defines a left Quillen presheaf.

Proof It suffices to show that the adjoint pair

$$\mathbb{1} : \mathsf{Homog}^n(\mathcal{J}_0, P_{S^{k+2}}\mathsf{Top}_*) \rightleftarrows \mathsf{Homog}^n(\mathcal{J}_0, P_{S^{k+1}}\mathsf{Top}_*) : \mathbb{1}$$

is a Quillen adjunction. The adjoint pair

$$\mathbb{1} : \mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_{S^{k+2}}\mathsf{Top}_*) \rightleftarrows \mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_{S^{k+1}}\mathsf{Top}_*) : \mathbb{1}$$

is a Quillen adjunction since the composite of Quillen adjunctions is a Quillen adjunction, so the adjunction

$$\mathbb{1} : \mathsf{Fun}(\mathcal{J}_0, L_{S^{k+2}}\mathsf{Top}_*) \rightleftarrows \mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_{S^{k+1}}\mathsf{Top}_*) : \mathbb{1}$$

is a Quillen adjunction, and by [Hirschhorn 2003, Proposition 3.3.18(1) and Theorem 3.1.6(1)], this composite Quillen adjunction extends to the S^{k+2}–local n–polynomial model structure since S^{k+1}–local n–polynomial functors are S^{k+2}–locally n–polynomial.

An application of [Hirschhorn 2003, Theorem 3.3.20(2)(a)] yields the desired result about the n–homogeneous model structures. □

Similar proofs to Proposition 8.9 and Corollary 8.10 yield the following results relating the n–homogeneous model structure to the homotopy limit of the tower of S^{k+1}–local n–homogeneous model structures.

Proposition 8.12 *Under Hypothesis 8.1 the adjunction*

$$(\mathrm{res}_0^n / O(n))^{\mathbb{N}} : \mathrm{holim}\, \mathsf{P}_\bullet^{\mathsf{Homog}^n} \rightleftarrows \mathrm{holim}\, \mathsf{P}_\bullet^{\mathcal{J}_n} : (\mathrm{ind}_0^n\, \varepsilon^*)^{\mathbb{N}}$$

is a Quillen equivalence.

Corollary 8.13 *Under Hypothesis 8.1 the adjunction*

$$\mathrm{const} : \mathsf{Homog}^n(\mathcal{J}_0, \mathsf{Top}_*) \rightleftarrows \mathrm{holim}\, \mathsf{P}_\bullet^{\mathsf{Homog}^n} : \mathrm{lim}$$

is a Quillen equivalence.

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Mathematical Institute, Utrecht University
Utrecht, Netherlands

`n.c.taggart@uu.nl`