MAX PLANCK INSTITUTE
FOR RESEARCH ON
COLLECTIVE GOODS

Martin F. Hellwig

Discussion Paper
2024/8

DYNAMIC EFFICIENCY AND INEFFICIENCY IN A CLASS OF OVERLAPPING-GENERATIONS ECONOMIES WITH MULTIPLE ASSETS

Dynamic Efficiency and Inefficiency in a Class of Overlapping-Generations Economies with Multiple Assets*

Martin F. Hellwig
Max Planck Institute for Research on Collective Goods
Kurt-Schumacher-Str. 10, D-53113 Bonn, Germany
hellwig@coll.mpg.de

February 20, 2024

Abstract

For overlapping-generations models with multiple assets and without labour, welfare assessments of equilibrium allocations depend on whether the certainty equivalents of the one-period-ahead marginal rates of return on assets that are held are larger or smaller than the population growth rate. Conditional on the period and the history up to that period, the equilibrium values of these certainty equivalents are the same for all assets held and equal to the riskless rate if a riskless asset is held. If population growth is uncertain, the standard of comparison is the certainty equivalent of the population growth rate when interpreted as the marginal rate of return on an additional asset.

Key Words: Dynamic Inefficiency, overlapping-generations models, First Welfare Theorem, certainty-equivalents criterion.

JEL: D15, D61, E21, E22, E62, H30.

*Revision of Hellwig (2023a). Whereas this paper considers the implications of asset multiplicity on efficiency assessments in overlapping-generations models, Hellwig (2024), considers the implications of productivity shocks affecting uncertain wage incomes. Both papers are based on material in Hellwig (2021). I thank Andrew Abel, Gaetano Bloise, Subir Chattopadhyay, Christian Hellwig, Greg Mankiw, Pietro Reichlin, Larry Summers, Christian von Weizsäcker, and Richard Zeckhauser for helpful exchanges and advice. The usual caveat applies.

1 Introduction

In overlapping-generations models with infinite time horizons, equilibrium allocations under *laissez-faire* need not be Pareto efficient.[1] Such "dynamic inefficiency" is often tied to the question whether the real rate of return r on capital is smaller or larger than the real growth rate g of the economy. If r is less than g, efficiency can be improved by reducing capital investments in all periods and using the resources saved to provide for the consumption of old participants. The growth rate g comes in as the "rate of return" from participating in a scheme under which agents make contributions to older participants' consumption when young and receive contributions from the next generation of young participants when old. If $r < g$, participation in such a scheme is advantageous because the rate of return on contributions to the scheme exceeds the rate of return on capital investment.

As stated, the argument presumes that assets are riskless so that in equilibrium they all bear the same rate of return. The argument is no longer clear, however, if some assets, or even all, are risky so that their rates of return are given by random variables, rather than real numbers. What are we to conclude if the equilibrium rate of return on safe assets is smaller than the growth rate of the economy and the expected rates of return on risky assets are larger than the growth rate of the economy?

For a particular class of overlapping-generations models, this paper shows that the relevant variable for comparison with the growth rate is given by the certainty equivalent of the uncertain marginal rate of return on any risky asset that is actually held. By standard portfolio choice considerations, the equilibrium value of this certainty equivalent is the same for all assets that are held in positive amounts. If this equilibrium value of the certainty equivalent of the uncertain marginal rates of return on risky assets that are held falls short of the population growth rate, the equilibrium allocation is not Pareto efficient; if it exceeds the population growth rate, the equilibrium allocation is Pareto efficient.[2]

Thus, with uncertainty about asset returns, the r versus g comparison is as relevant as in the certainty case. The only change is that r must be thought of as the common certainty equivalent of the uncertain marginal

[1] The argument goes back to Allais (1947, Appendix 2), Samuelson (1958), and Diamond (1965). Blanchard (2019), as well as von Weizsäcker (2014) and Weizsäcker and Krämer (2019/2022), have provided the discussion with a new impetus.

[2] For a particular model with one riskless and one risky asset, a special case of this finding is already contained in Hellwig (2022). The present paper distills the general principle.

rates of return on assets that are held. This criterion also coincides with the so-called dominant-root criterion of Peled and Aiyagari (1991).[3]

If one of the assets is riskless and positive amounts of this asset are held, the common value of the certainty equivalent of the marginal rates of return on risky assets must be equal to the marginal rate of return on this riskless asset. In this case, the r versus g comparison can rely on the marginal rate of return on the riskless asset.

I also consider the case where the population growth rate is uncertain. Under the assumption that population growth rates from one period to the next are given by a sequence of independent and identically distributed random variables, I show that, for the class of models under consideration, the assessment of dynamic efficiency of an equilibrium allocation hinges on whether the common certainty equivalent of the uncertain marginal rates of return on assets that are held is larger or smaller than the certainty equivalent of the marginal rate of return on a fictitious asset whose uncertain rate of return is equal to the population growth rate.

The results of this paper contradict a claim of Abel et al. (1989, pp. 13f.) that certainty equivalents of marginal rates of return on assets being smaller than growth rates is not a sufficient condition for dynamic inefficiency. These authors, however, do not prove their claim. They merely support it with an example involving an infinitely-lived representative consumer and a single risky asset. Such an example cannot tell us anything about overlapping-generations economies.[4] Nor can it tell us anything about equilibria in which riskless assets are held in positive amounts.[5]

Abel et al. (1989) also have a theorem on overlapping-generations models. This theorem gives sufficient conditions for dynamic efficiency and for dynamic inefficiency in terms of the sign of net payment flows between the consumer sector and the producer sector of the economy, without any explicit reference to rates of return on assets. However, these conditions are far from necessary. For the class of models considered here, they are much stronger than the sufficient conditions I give in terms of the r versus g com-

[3] See also Manuelli (1990), Chattopadhyay and Gottardi (1999), Demange and Laroque (1999, 2000), Chattopadhyay (2001), and Bloise and Reichlin (forthcoming).

[4] Despite this lack of a serious foundation, the claim of Abel et al. (1989) in presuming that assessments of dynamic inefficiency must consider aggregates of returns on all assets, rather than merely the riskless rate, has been very influential. See, e.g., Homburg (2014), Geerolf (2018), Yared (2019), Acharya and Droga (2020), Reis (2020), Bloise and Reichlin (forthcoming).

[5] In the example of Abel et al. (1989), as in the model of Bloise and Reichlin (forthcoming), riskless assets could be constructed synthetically, as packages of contingent claims, but equilibrium holdings of these synthetic assets are zero.

parison. In fact, the "gap" between my sufficient conditions for dynamic inefficiency and my sufficient conditions for dynamic efficiency concerns only the case $r = g$.[6]

The class of models I consider is special in that there is no labour and therefore no market for labour. At any date, output is produced with capital that belongs to members of the old generation. This output makes up the old generation's real income at that date. In contrast to most other papers with this setup, e.g. Bloise and Reichlin (forthcoming), I assume that there are different kinds of real capital, with different return risks. Members of the young generation have a commodity endowment that they can use for immediate consumption and for investments in the different kinds of capital.[7] Because the different kinds of capital involve different return risks, the young face a nontrivial problem of portfolio choice.

In Hellwig (2024), I also consider the model of Demange and Laroque (2000) and Blanchard (2019), in which there are active labour markets because production in any period relies on a combination of the young people's labour with the old people's assets. In this model, the r versus g comparison at any date t depends on the wage rate at this date, which in turn depends on the productivity shock at this date. The uncertainty about productivity at date t affects not only the returns on investments at date $t - 1$ but also the wage incomes of the young at date t, and these wage incomes in turn affect consumption and investment of the young at date t. High wage rates at t allow the young to make large investments. If investments are large, marginal rates of return on investments are likely to be low and so is the equilibrium value of the certainty equivalent of these rate of return. More generally, the certainty equivalents of the uncertain marginal rates of return on investments, and therefore the r vs. g comparison, and any date t depends on the wage rate and, indirectly, the productivity shock at this date.

The companion paper studies the implications of this dependence in an economy with a single asset and gives a more general formulation of the r vs. g criterion under uncertainty. This more general criterion compares the

[6] The literature on the dominant-root criterion fills this gap by showing that, if preferences are *strictly* quasi-concave, with Gaussian curvature bounded away from zero, laissez-faire allocations with $r = g$ are efficient. See, e.g., Chattopadhyay and Gottardi (1999). The need for *strict* quasi-concavity indicates that the economics of the argument in this case is slightly more complicated. For simplicity, I only analyse the cases $r < g$ and $r > g$.

[7] A generalization giving the young generation a labour endowment that they can use for their own production of current consumption and investments would be trivial but in this generalization there also would be no market for labour.

conditional certainty equivalents of the uncertain marginal rates of return on assets that are held to the conditional certainty equivalents of the uncertain implicit marginal rates of return on social security contributions. The analysis there shows that the "g" in the analysis of this paper, or in other versions of the r vs. g criterion is merely a stand-in for the implicit rate of return on social security contributions.

The plan of the paper is as follows: Section 2 introduces the basic model. Given that labour plays no role, there is no direct trade between the two generations that are alive in any period t. I define and characterize the autarky allocation and show that it can be generated as an equilibrium allocation in a sequence of markets such that, in each period, there is a complete system of one-period-ahead contingent-claims markets.

Section 3 contains the main result on the efficiency of this equilibrium allocation. Following the literature, I use a concept of *interim* Pareto efficiency where each generation t assesses a change of allocation from an *interim* perspective, knowing the history of productivity shocks up to and including t.[8] This information assumption eliminates the possibility of Pareto improvements from having people born in period t take over some of the return risks of people born in period $t-1$. The equilibrium allocation is shown to be interim Pareto efficient if the common certainty equivalent of the uncertain marginal rates of return on assets that are held exceeds the population growth rate; the equilibrium allocation is interim Pareto dominated if this certainty equivalent is smaller than the population growth rate.

Section 4 generalizes the analysis to allow for growth rates given by independent and identically distributed random variables. In this case, the role of the growth rate in the efficiency criterion is taken by the certainty equivalent of the uncertain growth rate when interpreted as a rate of return on an asset.

Section 5 provides further perspectives on the main result. Section 5.1 shows that this result implies the theorem of Abel et al. (1989) that was mentioned above. Section 5.2 shows that the result can be interpreted as a failure of the First Welfare Theorem for competitive equilibria in a complete market system *ex ante when* agents' identities include the histories up to and including their births. In such a system, the autarky allocation is a competitive equilibrium allocation and is efficient if the value of aggregate

[8] This is the terminology of the literature on incentive mechanisms, as well as Demange and Laroque (1999, 2000). Peled and Aiyagari (1991), Chattopadhyay and Gottardi (1999) and Chattopadhyay (2001, 2008) refer to "conditional" Pareto efficiency. Demange and Laroque (1999) refer to conditional Pareto efficiency when potentially improving allocations are required to be stationary.

consumption at equilibrium prices is finite and inefficient if the value of aggregate consumption at equilibrium prices is unbounded. This criterion is equivalent to the criterion in the main result of this paper..

Formal proofs are given in the appendix.

2 An Overlapping-Generations Model with Multiple Assets and no Labour

Consider an economy in periods $t = 1, 2, ...$ In each period t, there is a single produceable good. This good serves for consumption and investments. There are I types of investments. For $i = 1, ..., I$, an investment k_i^t of type i in period t generates an output $f_i(A_{t+1}, k_i^t)$ in period $t+1$, where A_{t+1} is the realization of a nondegenerate random variable $\tilde{A}_{t+1}$ with values in a finite set $\mathcal{A} = \{a_1, ..., a_S\}$. This realization only becomes known in period $t+1$. After production, investments of all types are fully depreciated. For any $a \in \mathcal{A}$, the return functions $f_i(a, \cdot)$, $i = 1, ..., I$, are continuously differentiable, nondecreasing, and concave, with $f_i(a, 0) = 0$. Moreover, for any $a \in \mathcal{A}$, $f_i'(a, 0) > 0$ for at least one $i \in \{1, ..., I\}$.

In each period t, a new generation of N_t people is born and lives for two periods. There are also $N_0 = N$ old people in period 1. I assume that the population grows at a constant rate n, so $N_t = (1+n)^t N_0$ for all t.

For simplicity, I assume that, except for the old people in period 1, all people have the same characteristics. A person born in period $t \geq 1$ has an initial endowment $E > 0$ of the period t good and no endowment of the period t' good for $t' \neq t$. Moreover, this person is interested in the utility

$$u(c_1^t) + v(c_2^t) \tag{2.1}$$

that is obtained from consuming c_1^t in period t and c_2^t in period $t+1$. The utility functions $u(\cdot)$ and $v(\cdot)$ are assumed to be twice continuously differentiable, increasing and concave, with $u'(0) = \infty$ and $v'(0) = \infty$. An old person in period 1 has past investments $k_1^0, ..., k_I^0$ and is interested in the utility $v(c_2^0)$.

In the absence of trade, a person born in period $t \geq 1$ chooses a first-period consumption level c_1^t, and investment levels k_i^t, $i = 1, ..., I$ under the constraint

$$c_1^t + \sum_{i=1}^{I} k_i^t = E. \tag{2.2}$$

The person also chooses a plan $c_2^t(\cdot)$ for second period consumption subject to the constraint that

$$c_2^t(a_s) = \sum_{i=1}^{I} f_i(a_s, k_i^t) \tag{2.3}$$

for $s = 1, ..., S$. An old person in period 1 just has the consumption

$$c_2^0(a_s) = \sum_{i=1}^{I} f_i(a_s, k_i^0)$$

for $s = 1, ..., S$. I assume that $\sum_{i=1}^{I} f_i(a_s, k_i^0) > 0$ for all s.

The parameters $\tilde{A}_1, \tilde{A}_2, ...$ are assumed to be independent and identically distributed, with strictly positive probabilities $p_1, ..., p_S$. for the outcomes $a_1, ..., a_S$. A person born in period $t \geq 1$ thus gets the expected utility

$$u(c_1^t) + \sum_{s=1}^{S} p_s \cdot v(c_2^t(a_s)) \tag{2.4}$$

from the plan $(c_1^t, k_1^t, ..., k_I^t, c_2^t(\cdot))$.

An *autarky allocation* is an array of plans $(c_1^t, k_1^t, ..., k_I^t, c_2^t(\cdot))$ for $t = 1, 2, ...$ such that, for each t, the plan $(c_1^t, k_1^t, ..., k_I^t, c_2^t(\cdot))$ maximizes (2.4) subject to the constraints (2.2) and (2.3). Given the assumptions imposed on utility functions and return functions, the following lemma is immediate.

Lemma 2.1 *There is a unique autarky allocation. For each generation $t \geq 1$, the autarky allocation involves the unique plan $(c_1^a, k_1^a, ..., k_I^a, c_2^a(\cdot))$ that satisfies the first-order conditions*

$$u'(c_1^a) \leq \sum_{s=1}^{S} p_s \cdot f_i'(a_s, k_i^a) \cdot v'(c_2^a(a_s) \tag{2.5}$$

for $i = 1, ...I$, as well as the constraints (2.2) and (2.3), where, for any i, (2.5) holds as an equation unless $k_i^a = 0$. This plan satisfies $c_1^a > 0$ and $c_2^a(a_s) > 0$ for all s.

The autarky allocation can be implemented as an equilibrium allocation in a sequence of complete one-period-ahead market systems. For suppose that, in period t, there is a market system in which consumers can buy state-contingent claims for period $t + 1$ consumption at prices

$$\psi(a_s) := \frac{p_s \cdot v'(c_2^a(a_s))}{u'(c_1^a)}, s = 1, ..., S, \tag{2.6}$$

and they can sell the period t good to firms at a price $q_t = 1$. These firms acquire the period t good at the price $q_t = 1$ in order to make investments, and they dispose of the state-dependent outputs from these investments by selling state-contingent claims for the period $t+1$ good at the prices $\psi(a_s)$, $s = 1, ..., S$. The profits of these firms are distributed to people of generation t.

Lemma 2.2 *For any t, the autarky consumption plan $(c_1^t, c_2^t(\cdot)) = (c_1^a, c_2^a(\cdot))$ maximizes the expected utility (2.4) of a person born in period t subject to the budget constraint*

$$c_1^t + \sum_{s=1}^{S} \psi(a_s) c_2^t(a_s) = E + \Pi^t, \tag{2.7}$$

where

$$\Pi^t = \max_{k_1^t, ..., k_I^t} \left[\sum_{s=1}^{S} \psi(a_s) \sum_{i=1}^{I} f_i(a_s, k_i^t) - \sum_{i=1}^{I} k_i^t \right] \tag{2.8}$$

and, moreover, the maximum in (2.8) is attained at the autarky investment plan $(k_1^t, ..., k_I^t) = (k_1^a, ..., k_I^a)$.

In any period, old agents play no active role because they do not trade. They merely consume the returns on the contingent claims they acquired in the preceding period. From Lemma 2.2, one therefore obtains the following result.

Proposition 2.3 *Suppose that, in each period t, there is a market system of the sort considered in Lemma 2.2. A sequence $\{q_t\}_{t=1}^{\infty}$ of price vectors satisfying*

$$q_t = (1, \psi(a_1), ..., \psi(a_S)) \tag{2.9}$$

for all t and all histories $(A_1, ..., A_t)$ up to t, supports the autarky allocation as a rational-expectations equilibrium allocation.

The sequence of markets in this proposition is *not* equivalent to a complete market system *ex ante* in which claims on all contingencies can be traded. In a complete market system *ex ante*, there would be active trading of contingent claims on the period t goods that allows people born in period $t-1$ to share some of their return risk with people born in period t. Such risk sharing cannot take place if people born in period t know the realization of $\tilde{A}_t$ when they enter the market.

3 Welfare Assessments

For welfare assessments, I take an *interim* perspective where each generation t assesses a change of allocation on the basis of the information that it has, assuming that it knows the history $A_1, ..., A_t$ of productivity parameters up to t. From this perspective, an allocation is *interim* Pareto-preferred to another if, conditioning on the information that is available to agents when they take their decisions and regardless of the value that information may take, no participant is worse off and some participants are strictly better off under the first allocation than under the second allocation. The *interim* perspective avoids a trivial finding of inefficiency due to the absence of risk sharing between generations.

To assess the *interim* Pareto efficiency of the autarky allocation, I consider the welfare impact of reducing the first-period consumption of agents born in period t by $\Delta > 0$ and increasing second-period consumption of these agents by $(1+n)\Delta$ while leaving everything else unchanged. With a population growth factor $1+n$, this change is obviously feasible. For a person born in period t expected utility shifts from $u(c_1^a) + \sum_{s=1}^{S} p_s \cdot v(c_2^a(a_s))$ to $u(c_1^a - \Delta) + \sum_{s=1}^{S} p_s \cdot v(c_2^a(a_s) + (1+n)\Delta)$. For small Δ, the change in expected utility is approximately equal to

$$\left[-u'(c_1^a) + \sum_{s=1}^{S} p_s \cdot (1+n) \cdot v'(c_2^a(a_s)) \right] \cdot \Delta = -u'(c_1^a) \left[1 - (1+n) \sum_{s=1}^{S} \psi(a_s) \right] \cdot \Delta, \tag{3.1}$$

where $\psi(a_s)$ is given by (2.6). If the term in brackets is positive, the intervention considered lowers welfare; if this term is negative, the intervention raises welfare. In the latter case, the new allocation Pareto dominates the autarky allocation, in the former case, it does not dominate the autarky allocation.

Proposition 3.1 *If $(1+n) \sum_{s=1}^{S} \psi(a_s) < 1$, the autarky allocation is interim Pareto efficient. If $(1+n) \sum_{s=1}^{S} \psi(a_s) > 1$, the autarky allocation fails to be interim Pareto efficient.*

The second part of Proposition 3.1 follows from the argument just given. That argument also shows that, if $(1+n) \sum_{s=1}^{S} \psi(a_s) < 1$, the specified intervention, with a fixed Δ, does not provide a Pareto improvement. A more general argument is needed, however, in order to show that in this case no intervention at all provides for a Pareto improvement, not even a time-dependent or state-dependent intervention that provides for the sharing of risks from the random variable $\tilde{A}_{t+1}$ between generations t and $t+1$.

The *interim* efficiency or inefficiency of the autarky allocation thus depends on whether the sum $\sum_{s=1}^{S} \psi(a_s)$ is less than or greater than $\frac{1}{1+n}$. To understand what this comparison is about, it is useful to recall that, for any t and any s, $\psi(a_s)$ is the period t price of a claim on the period $t+1$ good contingent on the event $\tilde{A}_{t+1} = a_s$ expressed in units of the period t good. The sum $\sum_{s=1}^{S} \psi(a_s)$ is therefore the period t price of a non-contingent claim on the period $t+1$ good expressed in units of the period t good. The proposition asserts that the interim efficiency or inefficiency of the allocation depends on whether this price is less than or greater than $\frac{1}{1+n}$.

The condition in Proposition 3.1 can be interpreted as a version of the so-called *dominant-root or unit root criterion* of Aiyagari and Peled (1991), Chattopadhyay and Gottardi (1999), Demange and Laroque (1999, 2000), Chattopadhyay (2001), and Bloise and Reichlin (forthcoming).[9] To see the relation, consider the strictly positive $S \times S$ matrix $\Psi = (\psi_{s,s^+})$, where, for any s and s^+ in $\{1, ..., S\}$,

$$\psi_{s,s^+} := (1+n) \cdot \psi(a_{s^+}),$$

regardless of s. One easily verifies that this matrix, whose rows are all equal, has eigenvalues

$$\lambda^*(\Psi) = (1+n) \cdot \sum_{s^+=1}^{S} \psi(a_{s^+}),$$

with eigenvector $(1, ..., 1)$, and 0, with an $S-1$-dimensional space of eigenvectors that equals the null space of Ψ. By Proposition 3.1 therefore, the efficiency of inefficiency of the autarky allocation depends on whether the maximal eigenvalue of Ψ is less than or greater than one. This is exactly the dominant-root criterion.[10]

Proposition 3.1 makes no reference to assets or asset returns. Rates of return enter implicitly because the equilibrium price system depends on the allocation and the allocation in turn reflects the available investment opportunities. Using (2.6) and Lemma 2.1, one finds that, for any asset i

[9] See also Manuelli (1990), who uses a somewhat different formulation.

[10] Under the additional assumption that the participants' indifference curves in (c_1, c_2)-space have non-zero, bounded Gaussian curvature, the cited papers also establish interim efficiency if $\lambda^*(\Psi) = 1$.

satisfying $k_i^a > 0$, one has

$$\frac{1}{\sum_{s=1}^{S} \psi(a_s)} = \frac{u'(c_1^{a)}}{\sum_{s=1}^{S} p_s \cdot v'(c_2^a(a_s))} = \frac{\sum_{s=1}^{S} p_s \cdot f_i'(a_s, k_i^a) \cdot v'(c_2^a(a_s))}{\sum_{s=1}^{S} p_s \cdot v'(c_2^a(a_s))}. \tag{3.2}$$

Upon combining this finding with Proposition 3.1, one obtains:

Proposition 3.2 *The autarky allocation fails to be interim Pareto efficient if*

$$\frac{\sum_{s=1}^{S} p_s \cdot f_i'(a_s, k_i^a) \cdot v'(c_2^a(A_s))}{\sum_{s=1}^{S} p_s \cdot v'(c_2^a(a_s))} < 1 + n \tag{3.3}$$

for all i*. The autarky allocation is interim Pareto efficient if*

$$\frac{\sum_{s=1}^{S} p_s \cdot f_i'(a_s, k_i^a) \cdot v'(c_2^a(a_s))}{\sum_{s=1}^{S} p_s \cdot v'(c_2^a(a_s))} > 1 + n \tag{3.4}$$

for all i *satisfying* $k_i^a > 0$.

The term on the left-hand side of (3.3) and (3.4) is a marginal-utility-weighted expectation of the marginal return random variable $f_i'(\tilde{A}_{t+1}, k_i^a)$ for asset i. This marginal-utility-weighted expectation is the same for all assets that are actually held. It can be interpreted as the certainty-equivalent of the marginal return $f_i'(\tilde{A}_{t+1}, k_i^a)$, i.e., as that value of the marginal return on a (possibly fictitious) riskless asset at which the investor would be indifferent between a marginal investment in asset i and in the riskless asset.

The term $1+n$ on the right-hand side of (3.3) and (3.4) can be interpreted as a rate of return that is implicit in participants's paying Δ in the first period of their lives and receiving $(1+n)\Delta$ in the second period of their lives. Proposition 3.2 asserts that, if this implicit rate of return exceeds the common value of the certainty equivalents of the marginal returns on assets, the autarky allocation is Pareto dominated; if this implicit rate of return is smaller than than the common value of the certainty equivalents of the marginal returns on assets, the autarky allocation is Pareto efficient.

For an asset that satisfies

$$f_i'(a_s, k_i^a) = \hat{f}_i'(k_i^a) \tag{3.5}$$

for some function $\hat{f}_i$ and all s, the left-hand side of (3.3) and (3.4) is simply equal to $\hat{f}_i'(k_i^a)$.

Corollary 3.3 *Assume that the autarky allocation satisfies $k_i^a > 0$ for some asset i that is riskless, i.e., that satisfies (3.5) for all s. Then this allocation is interim Pareto efficient if $\hat{f}_i'(k_i^a) > 1 + n$ and interim Pareto-dominated if $\hat{f}_i'(k_i^a) < 1 + n$.*

Corollary 3.3 restates the old result that the efficiency or inefficiency of a competitive-equilibrium allocation in an overlapping-generations economy depends on whether the marginal rate of return on a riskless asset that is held in positive amounts exceeds the growth rate of the economy or falls short of it. Contrary to a claim in Abel et al. (1989), the criterion for efficiency and inefficieny is specified only in terms of the marginal rate of return on the safe asset, seemingly without regard to the rates of return on risky assets. The marginal rates of return on risky assets come in implicitly because, by portfolio choice considerations, the certainty equivalents of marginal rates of return must be the same for all assets that are held in positive amounts. In particular, they must be equal to the riskless rate if there is a riskless asset that is held. If this rate lower than the growth rate and yet the riskless asset is held, the equilibrium allocation is inefficient even though the expected returns on risky assets may be very large.[11]

The follows remark shows that that there exist constellations in which the assumption $k_s^a > 0$ is satisfied so Corollary 3.3 is not vacuous.

Remark 3.4 *Suppose that asset 1 is riskless, so that $f_1(a_s, \cdot) = \hat{f}_1(\cdot)$ for some function $\hat{f}_1$ and all s. Then $k_1^a > 0$ if there exists a state in which the returns on all other assets are zero, i.e., if, for some s, $f_j(a_s, k_j^a) = 0$ for all $j \neq 1$. The condition $k_1^a > 0$ is also satisfied if $\lim_{k_j \to \infty} f_j'(a_s, k_j) = 0$ for all $j \neq 1$ and all s and the endowment E is very large.*

The first part of Remark 3.4 concerns constellations in which safe investments are needed as protection against the positive-probability event that risky investments may be completely lost. The second part concerns constellations in which endowments are so large that, without safe investments,

[11] Abel et al. (1989) overlook this point because they have only a single real asset and this asset is risky, so they do not consider the implications of a riskless asset's being held in positive amounts. Bloise and Reichlin (forthcoming) has the same shortcoming. The paper criticizes a previous version of Corollary **??** in Hellwig (2021) without however considering the implications of optimal portfolio choice for the assessment of dynamic inefficiency in the presence of a riskless asset that is held in positive amounts.

the marginal returns on risky investments would be so low (with probability one) that, at the margin, these investments would be dominated by safe investments.

If there is no riskless asset, one can still define a "shadow" safe rate of return

$$R^a := \frac{1}{\sum_{s=1}^{S} \psi(a_s)} = \frac{u'(c_1^{a)}}{\sum_{s=1}^{S} p_s \cdot v'(c_2^a(a_s))} \tag{3.6}$$

as that value of the rate of return on a fictitious safe asset at which agents would be exactly indifferent about a marginal investment in this asset. This number is given by the consumers' marginal rate of substitution between non-contingent changes in consumption in the first and second periods of their lives.

Corollary 3.5 *The autarky allocation is Pareto efficient if $R^a > 1+n$ and Pareto-dominated if $R^a < 1+n$.*

4 Uncertainty about Population Growth

The analysis so far has made extensive use of the assumption that the population growth rate is a known constant. There is an easy generalization, however, to the case where the population growth rate from period t to period $t+1$ is the realization of a random variable $\tilde{n}_{t+1}$ and the random variables $\tilde{n}_1, \tilde{n}_2, ...$ are independent and identically distributed.[12] Without loss of generality, one can write

$$\tilde{n}_t = \nu(\tilde{A}_t), \tag{4.1}$$

so that the state of the world in period t determines not only the returns on assets held from period $t-1$ but also the size of generation t relative to generation $t-1$. The autarky allocation is the same as before, but the transfer scheme considered in Section 3 now takes the form of a payment $\Delta > 0$ in period t by a person born in that period and a receipt $(1+\tilde{n}_{t+1})\Delta$ by that person in period $t+1$. Given this modification, for small Δ, the

[12] In a model with a single real asset, Demange and Laroque (1999, 2000) also allow for stochastic population growth rates, however, without considering the interpretation of growth rates as rates of return.

effect of such a scheme on the expected utility of a person born in period t now takes the form

$$\left[-u'(c_1^a) + \sum_{s=1}^{S} p_s \cdot (1+\nu(a_s)) \cdot v'(c_2^a(a_s))\right] \cdot \Delta, \tag{4.2}$$

which specializes to (3.1) if $\nu(a_s) = n$, regardless of a_s. Along the same lines as before, one obtains the following generalization of Proposition 3.2:

Proposition 4.1 *In the model with uncertain population growth given by 4.1), the autarky allocation fails to be interim Pareto efficient if*

$$\frac{\sum_{s=1}^{S} p_s \cdot f_i'(a_s, k_i^a) \cdot v'(c_2^a(a_s))}{\sum_{s=1}^{S} p_s \cdot v'(c_2^a(a_s))} < \frac{\sum_{s=1}^{S} p_s \cdot (1+\nu(a_s)) \cdot v'(c_2^a(a_s))}{\sum_{s=1}^{S} p_s \cdot v'(c_2^a(a_s))} \tag{4.3}$$

for all i. The autarky allocation is interim Pareto efficient if

$$\frac{\sum_{s=1}^{S} p_s \cdot f_i'(a_s, k_i^a) \cdot v'(c_2^a(a_s))}{\sum_{s=1}^{S} p_s \cdot v'(c_2^a(a_s))} > \frac{\sum_{s=1}^{S} p_s \cdot (1+\nu(a_s)) \cdot v'(c_2^a(a_s))}{\sum_{s=1}^{S} p_s \cdot v'(c_2^a(a_s))} \tag{4.4}$$

for all i satisfying $k_i^a > 0$.

To understand this result, consider a possibly fictitious asset whose rate of return from period t to period $t+1$ is equal to the population growth rate, so one unit of the good invested in this asset in period t yields $1+\tilde{n}_{t+1}$ in period $t+1$. The term on the right-hand sides of (4.3) and (4.4) can be interpreted as the certainty equivalent of the one-period rate of return on this asset. The proposition assets that the interim efficiency or inefficiency of the autarky allocation depends on how the certainty equivalent of marginal returns on assets that are held compare to the certainty equivalent of the marginal returns on this fictitious asset. The underlying rationale is the same as before: The transfer scheme considered in (4.2) can be interpreted in terms of an "investment" Δ in period t and a "return" $(1+\tilde{n}_{t+1})$ in period $t+1$.[13]

[13] The appearance of the growth rate in the return to social security contributions reflects the fact that individual contributions are constant. Demange and Laroque (2000) have an example with a Cobb-Douglas production function in which contributions are proportional to labour incomes and the population growth rate does not appear in the criterion for efficiency because the quantity effects of population growth on labour incomes are largely neutralized by a decline in wage rates. For a discussion, see Hellwig (2024).

5 Relation to the Net-Dividend Criterion of Abel et al. (1989)

Abel et al. (1989) have another criterion for dynamic efficiency and inefficiency, which on the face of it has nothing to do with rates of return. For any one period t, their net-dividend criterion compares the returns to investments that are payed out to consumers in that period to the payments for new investments that consumers make in that period. In the context of the model considered here, the comparison concerns the returns $N_{t-1} \cdot \tilde{d}_t := \sum_{i=1}^{I} f_i(\tilde{A}_t, k_i^{t-1})$ on past investments that go to the old generation in period t and the new investment $N_t \cdot \sum_{i=1}^{I} k_i^t$ that is made by the young generation in period t. According to Proposition 1 in Abel et al. (1989), under the assumption that production exhibits stochastic constant returns to scale, an equilibrium allocation is Pareto efficient if, for some $\varepsilon > 0$, $\tilde{d}_t \geq (1+\varepsilon)(1+n)\sum_{i=1}^{I} k_i^t$ for all t with probability one, and the allocation is Pareto dominated if, for some $\varepsilon > 0$, $\tilde{d}_t \leq (1-\varepsilon)(1+n)\sum_{i=1}^{I} k_i^t$ for all t with probability one. For the autarky allocation in the present analysis, these conclusions are actually a special case of Corollary 3.5. This is shown by the following result.

Proposition 5.1 *Assume that production exhibits stochastic constant returns to scale, i.e., that, for some functions $\rho_1(\cdot), ..., \rho_I(\cdot)$ from $\mathcal{A}$ to $\mathbb{R}_+$,*

$$f_i(a_s, k_i) = \rho_i(a_s) \cdot k_i \tag{5.1}$$

for all s and all $k_i > 0$. Then the autarky allocation satisfies $R^a > 1+n$ if, for some $\varepsilon > 0$,

$$\sum_{i=1}^{I} f_i(a_s, k_i^a) \geq (1+\varepsilon)(1+n)\sum_{i=1}^{I} k_i^a \tag{5.2}$$

for all s. It satisfies $R^a < 1+n$ if, for some $\varepsilon > 0$,

$$\sum_{i=1}^{I} f_i(a_s, k_i^a) \leq (1-\varepsilon)(1+n)\sum_{i=1}^{I} k_i^a \tag{5.3}$$

for all s.

The proof of Proposition 5.1 makes essential use of the stationarity of the autarky allocation. The fact that $k_i^t = k_i^a$ for all t makes it possible to translate the net-dividend criterion into a rate-of-return criterion: Given (5.1), (5.2) takes the form

$$\sum_{i=1}^{I} \rho_i(a_s) \cdot k_i^a \geq (1+\varepsilon)(1+n) \cdot \sum_{i=1}^{I} k_i^a \tag{5.4}$$

for all s, implying that, in all possible states of nature, the overall rate of return on the portfolio $(k_1^a, ..., k_I^a)$ is at least $(1+\varepsilon)(1+n)$. From the optimization conditions (2.5), one has

$$u'(c_1^a) \cdot \sum_{i=1}^{I} k_i^a = \sum_{i=1}^{I} \sum_{s=1}^{S} [p_s \cdot v'(\tilde{c}_2^a) \cdot \rho_i(a_s) \cdot k_i^a],$$

so the net-dividend condition (5.4) implies

$$u'(c_1^a) \cdot \sum_{i=1}^{I} k_i^a \geq \sum_{s=1}^{S} p_s \cdot v'(\tilde{c}_2^a) \cdot (1+\varepsilon)(1+n) \cdot \sum_{i=1}^{I} k_i^a$$

and, therefore,

$$1 \geq \sum_{s=1}^{S} \psi(a_s) \cdot (1+\varepsilon)(1+n)$$

or

$$R^a \geq (1+\varepsilon)(1+n),$$

which implies interim Pareto efficiency. Similarly, (5.3) implies that in all possible states of nature the overall rate of return on the portfolio $(k_1^a, ..., k_I^a)$ is at most $(1-\varepsilon)(1+n) > 1+n$, so that the optimization conditions (2.5) yield

$$u'(c_1^a) \cdot \sum_{i=1}^{I} k_i^a \leq \sum_{s=1}^{S} p_s \cdot v'(\tilde{c}_2^a) \cdot (1-\varepsilon)(1+n) \cdot \sum_{i=1}^{I} k_i^a$$

and, therefore,

$$R^a \leq (1-\varepsilon)(1+n),$$

implying a failure of interim Pareto efficiency.

Without some element of stationarity, the status of the result of Abel et al. (1989) is unclear. Their conditions compare payouts of returns from investments of period $t-1$ with new investments of period t. Chattopadhyay (2008) has examples where the technology involves technical regress, so investments decline over time. In these examples, the condition $\tilde{d}_t \geq (1+\varepsilon)(1+n) \sum_{i=1}^{I} k_i^t$ holds for all t with probability one, and yet the competitive-equilibrium allocation is Pareto-dominated.

6 Relation to the First Welfare Theorem

"Dynamic inefficiency" has little to do with dynamics. "Dynamic inefficiency" reflects a breakdown of the First Welfare Theorem in certain economies with an infinity of goods and an infinity of consumers.[14] The First Welfare Theorem asserts that, in the absence of external effects, public goods, and the like, under quite general assumptions on preferences and technologies, competitive equilibrium allocations are Pareto efficient. For a breakdown of this theorem, having a "large-square" economy with a large number of agents (at least two for every good) as well as a large-number of goods is crucial. The scope for a breakdown depends on the structure of the equilibrium price system, which in turn depends on the interplay of consumer preferences and investment opportunities. A breakdown of the First Welfare Theorem can occur even if there is no investment; in this case, only consumer preferences matter.[15]

In the present context, these observations are relevant even though the sequence of markets in Proposition 2.3 is *not* equivalent to a complete market system *ex ante* in which claims on all contingencies can be traded. The reason is that, for a slightly modified economy, the sequence of markets in Proposition 2.3 is equivalent to a complete market system *ex ante* in which claims on all contingencies can be traded. In this modified economy, the failure of interim Pareto efficiency when $(1+n)\sum_{s=1}^{S}\psi(a_s) > 1$ is in fact a failure of the First Welfare Theorem.

The modified economy is identical to the one studied so far *except* that the set of agents is expanded by treating agents born in period t as different agents if the histories $(A^1, ..., A^t)$ up to period t are different.[16] Thus an agent born in period t is treated as S^t different agents, who differ from each other according to the histories $(A^1, ..., A^t) \in \mathcal{A}^t$. Conditional on any one history up to t, the set of agents in the economy at t is the same as in the original model. However, from an *ex ante* perspective, this construction eliminates the scope for using active trading of contingent claims on the period t goods to allow people born in period $t-1$ to share some of their return risk with people born in period t. Such risk sharing cannot take place

[14] See, e.g., Balasko and Shell (1980), Mas-Colell et al. (1995), Ch. 20.H.

[15] For a discussion, see Shell (1971) and Hellwig (2023).

[16] Although the total number of participants in this *ex ante* market system is countably infinite, the definition and analysis of competitive equilibrium do not raise any technical or conceptual problems. Because of the underlying overlapping-generations specification of preferences, technologies and endowments, the number of participants interested in any one contingent claim is finite, so the aggregate excess demand for that claim is well-defined as a finite sum of excess demands of the interested participants.

if the plans of people born in period t condition on the histories up to and including t that determine their identities.[17]

Proposition 6.1 *The autarky allocation is a competitive equilibrium allocation in a complete system of contingent-claims markets ex ante in which agents born in period t are distinguished by the histories* $(A^1, ..., A^t)$ *up to t as well as their names. The equilibrium involves a sequence* $\{q_t(\cdot)\}$ *of time-and-history-contingent prices for the consumption that satisfies the equations*

$$q_1(A_1) = 1 \tag{6.1}$$

and, for any $t > 1$ *and history* $(A_1, ...A_t)$ *up to t,*

$$q_t(A_1, ..., A_t) = \psi(A_t) \cdot q_{t-1}(A_1, ..., A_{t-1}). \tag{6.2}$$

For the modified economy in which agents born in period t are distinguished by the histories $(A^1, ..., A^t)$ up to t as well as their names, the concepts of *interim* Pareto efficency and *ex ante* Pareto efficiency coincide because, in this economy, each agent naturally conditions on the history up to and including the date of his or her birth. Proposition 3.1 thus becomes a result about *ex ante* Pareto efficiency. The comparison of $\sum_{s=1}^{S} \psi(a_s)$ and $\frac{1}{1+n}$, which is crucial for the distinction between the efficiency and inefficiency pats of Proposition 3.1 can now be translated into a condition on the equilibrium price system $\{q_t(\cdot)\}$.

Proposition 6.2 *If* $\sum_{s=1}^{S} \psi(a_s) < \frac{1}{1+n}$*, the value of aggregate consumption at the equilibrium prices in Proposition 6.1 is finite, and the autarky allocation is ex ante Pareto efficient for an economy in which agents born in period t are distinguished by the histories* $(A^1, ..., A^t)$ *up to t as well as their names. If* $\sum_{s=1}^{S} \psi(a_s) > \frac{1}{1+n}$*, the value of aggregate consumption at the equilibrium prices in Proposition 6.1 is unbounded, and the autarky allocation fails to be ex ante Pareto efficient for an economy in which agents born in period t are distinguished by the histories* $(A^1, ..., A^t)$ *up to t as well as their names.*

The standard proof of the First Welfare Theorem begins by observing that, if an alternative allocation provides each participant with greater utility than the competitive equilibrium allocation, then for each participant

[17] This procedure is the same as the procedure for constructing the agent normal form of an extensive-form game, treating the same agent at two different information sets at two different agents. See Selten (1975).

the consumption plan under the new allocation must be unaffordable at the equilibrium prices. Upon adding this inequality over all consumers, one finds that the value at equilibrium prices of aggregate consumption under the alternative allocation must exceed the value of aggregate consumption under the competitive equilibrium allocation and therefore the value of the aggregate available resources. This leads to the conclusion that the alternative allocation cannot be feasible: For at least one good, the alternative allocation must stipulate consumption in excess of the resources available for providing this good.

In the present model, with infinitely many agents and infinitely many goods, one for each period and history up to that period, this argument goes through if the value of aggregate consumption at the equilibrium prices in Proposition 6.1 is finite, as it is if $\sum_{s=1}^{S}\psi(a_s) < \frac{1}{1+n}$, and it breaks down if this value is unbounded, as it is if $\sum_{s=1}^{S}\psi(a_s) > \frac{1}{1+n}$. Proposition 6.2 thus links the classification of cases in Proposition 3.1 to the applicability or breakdown of the standard proof of the First Welfare Theorem in a "large-square" economy,which has an infinity of people as well as an infinity of goods.

A Proofs

The first-order conditions in Lemma 2.1 as well as Lemma 2.2 and Proposition 2.3 follow by standard arguments, so their proofs are left to the reader. Positivity of c_1^a and $c_2^a(A)$ for all $A \in \mathcal{A}$ follows from the first-order conditions in Lemma 2.1 and the assumptions that $u'(0) = \infty$, $v'(0) = \infty$, $p_s > 0$ for all $s \in \{1, ..., S\}$ and that, for all $s \in \{1, ..., S\}$, there exists some i such that $f_i'(a_s, 0) > 0$.

As for the proof of Proposition 3.1, the argument in the text shows that the autarky allocation is *interim* Pareto-dominated if $\sum_{s=1}^{S}\psi(a_s) > \frac{1}{1+n}$. It remains to be proved that the autarky allocation is *interim* Pareto efficient if $\sum_{s=1}^{S}\psi(a_s) < \frac{1}{1+n}$. I follow the same strategy as Abel et al. (1989). The idea is to show that the equilibrium allocation maximizes the welfare of the old generation in period 1 over the set of feasible allocations subject to the constraint that no other generation be made worse off, using a Lagrangian approach to deal with the constraints. The approach requires some care in order to ensure that the duality conditions underlying the Lagrangian approach are satisfied.[18]

[18] Abel et al. (1989) take it for granted that the Paretian maximization problem is

If $\mathcal{A} = \{a_1, ..., a_S\}$ is the set of possible values of the productivity parameters in any one period, then $\mathcal{A}^t$ is the set of possible histories of the productivity parameter up to t, i.e., the set of contingencies on which choices at t may be conditioned. The space

$$\mathcal{E} := \bigcup_{t=1}^{\infty} \left[\{t\} \times \mathcal{A}^t\right], \tag{A.1}$$

corresponds to the union of these sets of contingencies over all t. Notice that $\mathcal{E}$ is a countable union of finite sets and is therefore a countable set.

An allocation is a mapping from $\mathcal{E}$ to $\mathbb{R}_+^{2+I}$ that assigns to each date t and each history $(A_1, ..., A_t)$ up to t a vector

$$(c_2^{t-1}(A_1, ..., A_t), c_1^t(A_1, ..., A_t), k_1^t(A_1, ..., A_t), ..., k_I^t(A_1, ..., A_t))$$

of actions in period t following the history $(A_1, ..., A_t)$, second-period consumption of generation $t-1$ and first-period consumption and investments of generation t. An allocation is *feasible* if it satisfies the constraints

$$\begin{aligned} c_2^{t-1}(A_1, ..., A_t) + (1+n)\left[c_1^t(A_1, ..., A_t) + \sum_{i=1}^{I} k_i^t(A_1, ..., A_t)\right] \\ \leq (1+n)E + \sum_{i=1}^{I} f_i(A_t, k_i^{t-1})), \end{aligned} \tag{A.2}$$

for all $[t, (A_1, ..., A_t)] \in \mathcal{E}$, where, for $i = 1, ..., I$, $k_i^{t-1} = k_i^0$ for $t = 1$ and $k_i^{t-1} = k_i^{t-1}(A_1, ..., A_{t-1})$ for $t > 1$. An allocation is interim Pareto-preferred to the autarky allocation if it satisfies the inequalities

$$v\left(c_2^0(A_1)\right) \geq v\left(\sum_{i=1}^{I} f_i(A_1, k_i^0)\right) \tag{A.3}$$

for all $A_1 \in \mathcal{A}$ and

$$u(c_1^t(A_1, ..., A_t)) + \sum_{s=1}^{S} p_s v(c_2^t(A_1, ..., A_t, a_s)) \geq u(c_1^a) + \sum_{s=1}^{S} p_s v(c_2^a) \tag{A.4}$$

equivalent to the Lagrangian maximization problem. Moreover, they do not verify that the Lagrangian has a maximum, rather than a supremum. At the level of generality of their formulation, which allows for technical regress and shrinkage of the economy, their result is actually invalid. Chattopadhyay (2008) gives counterexamples, in which the Lagrangian is unbounded and fails to have a maximum.

for all $[t,(A_1,...,A_t)] \in \mathcal{E}$ with $t>1$.

The proof strategy is to show that, for each $A_1 \in \mathcal{A}$, the autarky allocation is a solution to the problem of maximizing $v\left(c_2^0(A_1)\right)$ subject to the feasibility constraints (A.2) and the Pareto constraints (A.4). For this purpose, I consider introduce a suitably specified Lagrangian and show that, if $\sum_{s=1}^{S} \psi(a_s) < \frac{1}{1+n}$, the autarky allocation maximizes this Lagrangian and, moreover, any solution to the problem of maximizing the Lagrangian is also a solution to the Paretian problem of maximizing $v\left(c_2^0(A_1)\right)$ subject to (A.2) and (A.4).

To specify the Lagrange multipliers $\lambda_t(A_1,...,A_t)$ for the feasibility constraints (A.2), I set

$$\lambda_1(A_1) = v'\left(\sum_{i=1}^{I} f_i(A_1,k_i^0)\right) \tag{A.5}$$

for $t=1$ and $A_1 \in \mathcal{A}$ and

$$\lambda_t(A_1,...,A_t) = (1+n)\cdot\psi(A_t)\cdot\lambda_{t-1}(A_1,...,A_{t-1}) \tag{A.6}$$

for $t>1$ and $(A_1,...,A_t) \in \mathcal{A}^t$. For the Pareto constraints, I specify Lagrange multipliers $\mu_t(A_1,...,A_t)$ such that

$$\mu_t(A_1,...,A_t) = \frac{1+n}{u'(c_1^a)}\cdot\lambda_t(A_1,...,A_t) \tag{A.7}$$

for all $[t,(A_1,...,A_t)] \in \mathcal{E}$.

Lemma A.1 *If $\sum_{s=1}^{S} \psi(a_s) < \frac{1}{1+n}$, the Lagrange multipliers $\lambda_t(A_1,...,A_t)$, $\mu_t(A_1,...,A_t)$, $[t,(A_1,...,A_t)] \in \mathcal{E}$, define a pair $\lambda^\infty, \mu^\infty$ of bounded additive set functions on $\mathcal{E}$.*

Proof. Given the Lagrange multipliers $\lambda_t(A_1,...,A_t)$, $\mu_t(A_1,...,A_t)$, the formulae

$$\lambda^\infty(\{t\}\times\{(A_1,...,A_t)\}) := \lambda_t(A_1,...,A_t) \tag{A.8}$$

and

$$\mu^\infty(\{t\}\times\{(A_1,...,A_t)\}) := \mu_t(A_1,...,A_t) \tag{A.9}$$

define a pair of set functions on the singletons in $\mathcal{E}$. One easily sees that these set functions can be extended to additive measures $\lambda^\infty, \mu^\infty$ on the algebra of all finite subsets of $\mathcal{E}$.

I will show that these measures are uniformly bounded if $\sum_{s=1}^{S}\psi(a_s) < \frac{1}{1+n}$. I begin with λ^∞ and note that, for any $t > 1$, one has

$$\begin{aligned}
\lambda^\infty(\{t\}\times\mathcal{A}^{t-1}) &= \sum_{(A^1,...,A^t)\in\mathcal{A}^t} \lambda_t(A_1,...,A_t) \\
&= \sum_{(A_1,...,A_t)\in\mathcal{A}^t} [(1+n)^{t-1}\cdot\psi(A_t)\cdot ...\cdot\psi(A_2)\cdot\lambda_1(A^1)] \\
&= (1+n)^{t-1}\cdot\sum_{A_t\in\mathcal{A}}\psi(A_t)\cdot ...\cdot\sum_{A_2\in\mathcal{A}}\psi(A_2)\cdot\sum_{A_1\in\mathcal{A}}\lambda_1(A_1) \\
&= \left[(1+n)\cdot\sum_{s=1}^{S}\psi(a_s)\right]^{t-1}\cdot\sum_{A_1\in\mathcal{A}}\lambda_1(A_1).
\end{aligned}$$

If $(1+n)\cdot\sum_{s=1}^{S}\psi(A_s) < 1$, (A.9) implies that the infinite series $\sum_{t=1}^{\infty}\lambda^\infty(\{t\}\times\mathcal{A}^t)$ is well defined and satisfies

$$\sum_{t=1}^{\infty}\lambda^\infty(\{t\}\times\mathcal{A}^t) = \frac{1}{1-(1+n)\cdot\sum_{s=1}^{S}\psi(a_s)}\cdot\sum_{A_1\in\mathcal{A}}\lambda_1(A_1).$$

The additive measure λ^∞ on the algebra of all finite subsets of $\mathcal{E}$ is therefore σ-finite and has a unique extension λ^∞ to the algebra of all subsets of the countable set $\mathcal{E}$.[19] This yields

$$\begin{aligned}
\lambda^\infty(\mathcal{E}) &= \frac{1}{1-(1+n)\cdot\sum_{s=1}^{S}\psi(a_s)}\cdot v'\left(\sum_{i=1}^{I} f_i(A_1,k_i^0)\right) \\
&\leq \frac{1}{1-(1+n)\cdot\sum_{s=1}^{S}\psi(a_s)}\cdot v'\left(\min_{s'}\sum_{i=1}^{I} f_i(a_{s'},k_i^0)\right). \qquad (A.10)
\end{aligned}$$

By the assumption that $\sum_{i=1}^{I} f_i(a_s,k_i^0) > 0$ for all s, (A.10) provides a finite upper bound on $\lambda^\infty(\mathcal{E})$.

The corresponding claim for μ^∞ follows upon observing that, by construction, $\mu^\infty(\cdot) = \frac{1}{u'(c_1^a)}\cdot\lambda^\infty(\cdot)$. ■

Lemma A.2 *If $\sum_{s=1}^{S}\psi(a_s) < \frac{1}{1+n}$, then, for any $A_1 \in \mathcal{A}$, the autarky allocation with $c_2^0(A_1) = \sum_{i=1}^{I} f_i(A_1,k_i^0)$ and, for $t = 1,2,...$ and $A_1,...,A_{t+1}$,*

$$\begin{aligned}
c_1^t(A_1,...,A_t) &= c_1^a, \\
c_2^t(A_1,...,A_{t+1}) &= c_2^a(A_{t+1}),
\end{aligned}$$

[19] See Theorem A, p. 54, in Halmos (1950).

and, for $i = 1, ..., I$,

$$k_i^t(A_1, ..., A_t) = k_i^a.$$

maximizes the value of the Lagrangian

$$\mathcal{L}(\{c_2^{t-1}(\cdot), c_1^t(\cdot), k_1^t(\cdot), ..., k_I^t(\cdot)\}_{t=1}^{\infty}|A_1) = v\left(c_2^0(A_1)\right) \tag{A.11}$$

$$+ \sum_{[t,(A_1,...,A_t)]\in\mathcal{E}} \mu_t(A_1, ..., A_t)\left[u(c_1^t(A_1, ..., A_t)) + \sum_{s=1}^{S} p_s v(c_2^t(A_1, ..., A_t, a_s)) - W^a\right]$$

$$+ \sum_{[t,(A_1,...,A_t)]\in\mathcal{E}} \lambda_t(A_1, ..., A_t)[(1+n)E + \sum_{i=1}^{I} f_i(A_t, k_i^{t-1}(A_1, ..., A_{t-1}))]$$

$$- \sum_{[t,(A_1,...,A_t)]\in\mathcal{E}} \lambda_t(A_1, ..., A_t)\left\{c_2^{t-1}(A_1, ..., A_t) + (1+n)\left[c_1^t(A_1, ..., A_t) + \sum_{i=1}^{I} k_i^t(A_1, ..., A_t)\right]\right\},$$

over the set of bounded allocations, where

$$W^a := u(c_1^a) + \sum_{s=1}^{S} p_s u(\tilde{c}_2^a(a_s)) \tag{A.12}$$

is the expected utility obtained by a member of generation $t > 0$. *The value of the Lagrangian at the autarky allocation is* $v\left(\sum_{i=1}^{I} f_i(A_1, k_i^0)\right)$.

Proof. By Theorem IV.8.16 in Dunford and Schwartz (1958), Lemma A.1 implies that the pair $(\lambda^\infty, \mu^\infty)$ of bounded additive set functions on the algebra of all finite subsets of $\mathcal{E}$ defines a continuous linear functional on the space $[\ell_\infty(\mathcal{E})]^2$ of violations of the feasibility and Pareto constraints (A.2) and (A.4). At any bounded allocation therefore the value of the Lagrangian (A.11) is well defined.

Because, by construction, the autarky allocation satisfies the constraints (A.4) and (A.2) with equality, the value of the Lagrangian at the autarky allocation is $v\left(\sum_{i=1}^{I} f_i(A_1, k_i^0)\right)$. By the concavity assumptions on utility and return functions, for any allocation, (A.11) yields

$$
\begin{aligned}
&\mathcal{L}(\{c_2^{t-1}(\cdot), c_1^t(\cdot), k_1^t(\cdot), ..., k_I^t(\cdot)\}_{t=1}^{\infty}|A_1) - v\left(\sum_{i=1}^{I} f_i(A_1, k_i^0)\right) \qquad \text{(A.13)} \\
&\leq \left(c_2^0(A_1) - \sum_{i=1}^{I} f_i(A_1, k_i^0)\right) \cdot v'\left(\sum_{i=1}^{I} f_i(A_1, k_i^0)\right) \\
&+ \sum_{[t,(A_1,...,A_t)]\in\mathcal{E}} \mu_t(A_1, ..., A_t) \cdot u'(c_1^a) \cdot [c_1^t(A_1, ..., A_t) - c_1^a] \\
&+ \sum_{[t,(A_1,...,A_t)]\in\mathcal{E}} \mu_t(A_1, ..., A_t) \cdot \sum_{s=1}^{S} p_s v'(c_2^a(a_s))[c_2^t(A_1, ..., A_t, a_s) - c_2^a(a_s)] \\
&+\lambda_1(A_1) \cdot \left(\left(\sum_{i=1}^{I} f_i(A_1, k_i^0) - c_2^0(A_1)\right) - (1+n)\left((c_1^1(A_1) - c_1^a) + \sum_{i=1}^{I}(k_i^1(A_1) - k_i^a)\right)\right) \\
&+ \sum_{[t,(A_1,...,A_t)]\in\mathcal{E}\backslash\{[1,A_1]\}} \lambda_t(A_1, ..., A_t) \cdot \sum_{i=1}^{I} f_i'(A_t, k_i^a)[k_i^{t-1}(A_1, ..., A_{t-1}) - k_i^a] \\
&- \sum_{[t,(A_1,...,A_t)]\in\mathcal{E}\backslash\{[1,A_1]\}} \lambda_t(A_1, ..., A_t) \cdot [c_2^{t-1}(A_1, ..., A_t) - c_2^a(A_t)] \\
&- \sum_{[t,(A_1,...,A_t)]\in\mathcal{E}\backslash\{[1,A_1]\}} \lambda_t(A_1, ..., A_t) \cdot (1+n) \cdot [c_1^t(A_1, ..., A_t) - c_1^a] \\
&- \sum_{[t,(A_1,...,A_t)]\in\mathcal{E}\backslash\{[1,A_1]\}} \lambda_t(A_1, ..., A_t) \cdot (1+n) \cdot \sum_{i=1}^{I}[k_i^t(A_1, ..., A_t) - k_i^a].
\end{aligned}
$$

I claim that the right-hand side of (A.13) is nonpositive. To prove this claim, I first note that the difference $(c_2^0(A_1) - \sum_{i=1}^{I} f_i(A_1, k_i^0))$ enters the right-hand side of (A.13) with a total weight $v'\left(\sum_{i=1}^{I} f_i(A_1, k_i^0)\right) - \lambda_1(A_1)$. By (A.5), this is equal to zero, so the terms involving this difference vanish. The difference $(c_1^t(A_1, ..., A_t) - c_1^a)$ enters with a total weight

$$\mu_t(A_1, ..., A_t) \cdot u'(c_1^a) - (1+n) \cdot \lambda_t(A_1, ..., A_t).$$

By (A.7), this is also zero, so, for any t and any $(A_1, ..., A_t)$, the terms involving the difference $(c_1^t(A_1, ..., A_t) - c_1^a)$ also vanish. The terms involving the difference $(c_2^t(A_{1,...,}A_t, a_s) - c_2^a(a_s))$ have the total weight

$$\mu_t(A_1, ..., A_t) \cdot p_s v'(c_2^a(a_s)) - \lambda_{t+1}(A_1, ..., A_t, a_s).$$

By (A.7), (A.6), and the definition of $\psi(a_s)$, this is also zero, so these terms also vanish. Finally, for any i, the difference $(k_i^t(A_1, ..., A_t) - k_i^a)$ enters the right-hand side of (A.13) with a total weight equal to

$$\sum_{A_{t+1}\in\mathcal{A}} \lambda_{t+1}(A_1, ..., A_{t+1}) \cdot f_i'(A_{t+1}, k_i^a) - (1+n) \cdot \lambda_t(A_1, ..., A_t).$$

By (A.6), this expression is equal to

$$\left(\sum_{A_{t+1}\in\mathcal{A}} \psi(A_{t+1}) f_i'(A_{t+1}, k_i^a) - 1\right) \cdot (1+n) \cdot \lambda_t(A_1, ..., A_t).$$

By the definition of $\psi(A_{t+1})$ and the first-order condition for k_i^a,

$$\sum_{A_{t+1}\in\mathcal{A}} \psi(A_{t+1}) f_i'(A_{t+1}, k_i^a) = \sum_{s=1}^{S} \frac{p_s v'(c_2^a(a_s)) f_i'(A_{t+1}, k_i^a)}{u'(c_1^a)} \leq 1,$$

and the inequality is strict only if $k_i^a = 0$. If $k_i^a > 0$, it follows that the weight with which the difference $(k_i^t(A_1, ..., A_t) - k_i^a)$ enters the right-hand side of (A.13) is equal to zero. If $k_i^a = 0$, the difference $(k_i^t(A_1, ..., A_t) - k_i^a)$ is nonnegative, and the contribution to the right-hand side of (A.13) of the terms that involve this difference is nonpositive. The lemma follows immediately. ■

Proposition A.3 *If $\sum_{s=1}^{S} \psi(a_s) < \frac{1}{1+n}$, then, for any $A_1 \in \mathcal{A}$, the autarky allocation is a solution to the problem of maximizing $v\left(c_2^0(A_1)\right)$ subject to the feasibility constraints (A.2) and the Pareto constraints (A.4).*

Proof. The proposition follows from Lemma A.2 above in combination with Theorem 1, p. 220, in Luenberger (1969). ■

Proposition 3.1 follows immediately. Proposition 3.2 and Corollaries 3.3 and 3.5 follow by the arguments sketched in the text.

Proof of Remark 3.4. The first statement follows from the first-order condition (2.5) in Lemma 2.1 and the observation, that for the critical s, $k_i^a = 0$ would imply $c_2^a(s) = 0$ and $v'(c_2^a(s)) = \infty$.

To prove the second statement, suppose that $\lim_{k_j\to\infty} f_j'(a_s, k_j) = 0$ for all $j \neq i$ and s and that $k_i^a = 0$ even if E is large.

I claim that, if E is very large, then, by the first-order condition (2.5), c_1^a is very large and $u'(c_1^a)$ is close to zero. Otherwise, $u'(c_1^a)$ would be bounded away from zero and, by (2.5), for every $j \neq i$, there would exist s such that $v'(c_2^a(s))f_j'(a_s, k_j^a)$ is also bounded away from zero. For the specified s the,n $c_2^a(s)$ is bounded and so is k_j. However, if $k_i^a = 0$ and c_1^a as well as $k_j^a, j \neq i$, are bounded, then, for large E, the constraint for generation t's first-period choices is not exhausted, contrary to the optimality of the autarky plan.

Given that $u'(c_1^a)$ is close to zero if E is large, (2.5) implies that, for all j and all $s, v'(c_2^a(s))f_j'(a_s, k_j^a)$ is close to zero if E is large. Hence there exists j such that k_j^a is large if E is large. For this j, the first-order conditions (2.5) imply

$$\sum_{s=1}^{S} p_s v'(c_2^a(a_s)) f_j'(a_s, k_j^a) \geq \sum_{s=1}^{S} p_s v'(c_2^a(a_s)) \hat{f}_i'(0),$$

hence

$$\max_s f_j'(a_s, k_j^a) \geq \hat{f}_i'(0).$$

Given the assumption that $\lim_{k_j \to \infty} f_j'(a_s, k_j) = 0$ for all $j \neq i$ and s, it follows that k_j^a is bounded even if E is large. The assumption $\lim_{k_j \to \infty} f_j'(a_s, k_j) = 0$ for all $j \neq i$ and s and that $k_i^a = 0$ even if E is large has thus led to a contradiction and must be false. ■

The proof of Proposition 4.1 follows step by step the same line of argument as the proof of Proposition 3.1. If one replaces the term $(1 + n)$ in conditions (A.2), (A.6), (A.7), and (A.11) by $(1 + \nu(A_t))$, one finds that Lemma A.2 remains valid without change. The conclusion of Proposition A.3 then follows from the assumption that $\sum_{s=1}^{S} \psi(a_s)(1 + \nu(a_s)) < 1$, with a proof that is the same except for the replacement of $\sum_{s=1}^{S} \psi(a_s)(1+n)$ by $\sum_{s=1}^{S} \psi(a_s)(1 + \nu(a_s))$.

The proof of Proposition 5.1 follows from the argument sketched in the text. Proposition 6.1 follows from Proposition 2.3.

Proof of Proposition 6.2. For a participant who is to be born in period t, following the history $A_1, ...A_t$, the value of the autarky consumption vector $(c_1^a, c_2^a(a_1), ..., c_2^a(a_S))$ at the equilibrium prices in Proposition 6.1 is equal to the value $q_t(A_1, ..., A_S) \cdot (E + \Pi^a)$, where

$$\Pi^a = \sum_{s=1}^{S} \sum_{i=1}^{I} \psi(a_s) f_i(a_s, k_i^a) - \sum_{i=1}^{I} k_i^a$$

is the value of the maximum in (2.8), in Lemma 2.2. The aggregate of this value over all participants who are to born in period t at all is equal to

$$(1+n)^t \cdot N_0 \cdot \sum_{(A_1,...,A_S)\in\mathcal{A}^t} q_t(A_1,...,A_S)\cdot(E+\Pi^a).$$

By (6.2) and (6.1), this expression is equal to

$$(1+n)N_0 \cdot \left[(1+n)\sum_{s=1}^{S}\psi(a_s)\right]^{t-1} \cdot (E+\Pi^a).$$

By standard arguments, the infinite series that is obtained by adding over t converges if $(1+n)\sum_{s=1}^{S}\psi(a_s) < 1$ and diverges if $(1+n)\sum_{s=1}^{S}\psi(a_s) > 1$. In the case of convergence, *ex ante* Pareto efficiency follows by the usual argument for the First Welfare Theorem. In the case of divergence, the failure of *ex ante* Pareto efficiency follows by the argument used to prove Proposition 3.1. ■

References

[1] Abel, A.B., N.G. Mankiw, L.H. Summers, R.J. Zeckhauser (1989), Assessing Dynamic Efficiency, *Review of Economic Studies* 56 (1), 1-19.

[2] Acharya, S., and K. Droga (2021), The Side Effects of Safe Asset Creation, *Journal of the European Economic Association*, 20(2), 581–625.

[3] Aiyagari, S.R., and D. Peled (1991),. Dominant Root Characterization of Pareto Optimality and the Existence of Optimal Monetary Equilibria in Stochastic OLG models, *Journal of Economic Theory* 54, 69–83.

[4] Allais, M. (1947), *Économie et interêt*, Imprimerie Nationale, Paris.

[5] Balasko, Y., and K. Shell (1980), The Overlapping-Generations Model I: The Case of Pure Exchange without Money, *Journal of Economic Theory* 23, 281-306.

[6] Blanchard, O. (2019), Public Debt and Low Interest Rates, *American Economic Review* 109, 1197-1229.

[7] Bloise, G., and P. Reichlin (forthcoming), Low Safe Interest Rates: A Case for Dynamic Inefficiency?, *Review of Economic Dynamics*, forthcoming doi:https//doi.org/10.1016/j.red.2023.06.005.

[8] Chattopadhyay, S., and P. Gottardi (1999), Stochastic OLG Models, Market Structure, and Optimality, *Journal of Economic Theory* 89, 21–67.

[9] Chattopadhyay, S. (2001), The Unit Root Property and Optimality: A Simple Proof, *Journal of Mathematical Economics* 36, 151-159.

[10] Chattopadhyay, S. (2008), The Cass Criterion, the Net Dividend Criterion, and Optimality, *Journal of Economic Theory* 139, 335-352.

[11] Demange, G., and G. Laroque (1999), Social Security and Demographic Shocks, *Econometrica* 67, 527–542.

[12] Demange, G., and G. Laroque (2000), Social Security, Optimality, and Equilibria in a Stochastic Overlapping Generations Economy, *Journal of Public Economic Theory* 2, 1–23.

[13] Diamond, P.A. (1965), National Debt in a Neoclassical Growth Model, *American Economic Review* 55, 1126-1150.

[14] Geerolf, F. (2018), Reassessing Dynamic Inefficiency, mimeo, UCLA, September 2018, https://fgeerolf.com/r-g.pdf, accessed February 22, 2021.

[15] Hellwig, M.F. (2021), Safe Assets, Risky Assets, and Dynamic Inefficiency in Overlapping-Generations Economies, Discussion Paper 2021/10, Max Planck Institute for Research on Collective Goods, Bonn.

[16] Hellwig, M.F. (2022), Dynamic Inefficiency and Fiscal Interventions in an Economy with Land and Transaction Costs, *German Economic Review* 23(1), 21–60.

[17] Hellwig, M.F. (2023a), Overlapping-Generations Economies under Uncertainty: Dynamic Inefficiency/Efficiency with Multiple Assets and no Labour, Discussion Paper 04/2023, Max Planck Institute for Research on Collective Goods, Bonn.

[18] Hellwig, M.F. (2023b), The Economics of Hilbert's Hotel: An Expository Note, Discussion Paper 05/2023, Max Planck Institute for Research on Collective Goods, Bonn.

[19] Hellwig, M.F. (2024), The r vs. g Criterion for Pareto Efficiency in Blanchard's (2019) Overlapping-Generations Model under Uncertainty, Discussion Paper, Max Planck Institute for Research on Collective Goods, Bonn.

[20] Homburg, S. (2014), Overaccumulation, Public Debt and the Importance of Land, *German Economic Review* 15, 411-435.

[21] Manuelli, R. (1990), Existence and Optimality of Currency Equilibrium in Stochastic Overlapping Generations Models: The Pure Endowment Case, *Journal of Economic Theory* 51, 268-294.

[22] Mas-Colell, A., M.D.Whinston, and J.R.Green (1995), *Microeconomic Theory*, Oxford University Press, New York.

[23] Okuno, M., and I. Zilcha (1980), On the Efficiency of a Competitive Equilibrium in Infinite Horizon Monetary Economies, *Review of Economic Studies* 47 (4), 797-807.

[24] Rachel, L., and L.H. Summers (2019), On Falling Neutral Rates, Fiscal Policy, and the Risk of Secular Stagnation, *Brookings Papers on Economic Activity*, Spring 2019, 1-66.

[25] Reis, R. (2020), The constraint on Public Debt when $r < g$ but $g < m$, mimeo, London School of Economics.

[26] Samuelson, P.A. (1958), An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money, *Journal of Political Economy* 66, 467-482.

[27] Selten, R. (1975), Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games, *International Journal of Game Theory* 4, 25–55.

[28] Shell, K. (1971), Notes on the Economics of Infinity, *Journal of Political Economy* 79 (5), 1002-1011.

[29] Tirole, J. (1985), Asset Bubbles and Overlapping Generations, *Econometrica* 53, 1071-1100.

[30] von Weizsäcker, C.C. (2014), Public Debt and Price Stability, *German Economic Review* 15, 42-61.

[31] von Weizsäcker, C.C., and H. Krämer (2019/2021), *Sparen und Investieren im 21. Jahrhundert: Die große Divergenz,* SpringerGabler, Wiesbaden 2019. English version: *Saving and Investment in the 21st Century: The Great Divergence*, SpringerGabler, Wiesbaden 2021.

[32] Yared, P. (2019), Rising Government Debt: Causes and Solutions for a Decades-Old Trend, *Journal of Economic Perspectives* 33 (2), 115-140.