Essential implications of similarities in non-Hermitian systems

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In this paper, we show that three different generalized similarities enclose all unitary and anti-unitary symmetries that induce exceptional points in lower-dimensional non-Hermitian systems. We prove that the generalized similarity conditions result in a larger class of systems than any class defined by a unitary or anti-unitary symmetry. Further we highlight that the similarities enforce spectral symmetry on the Hamiltonian resulting in a reduction of the codimension of exceptional points. As a consequence we show that the similarities drive the emergence of exceptional points in lower dimensions without the more restrictive need for a unitary and/or anti-unitary symmetry.

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I. INTRODUCTION

In recent years non-Hermitian (NH) Hamiltonians have attracted increasing attention, and one active branch of research focusses on the role of symmetries in NH systems.[1] A complete classification in terms of 38 symmetry classes was derived by Kawabata *et al*,[2] and the topological features of these classes as well as the connection between some of them is studied in the literature.[2,3] Further it is generally recognized that certain unitary and anti-unitary symmetries lower the codimension of exceptional points (EPs), which are degeneracies where the eigenvalues and the corresponding eigenvectors coalesce.[2,4–7] The generic appearance of EPns, where n is the order of the EP set by the number of coalescing eigenvectors, is determined by the codimension of the EP. As such unitary and anti-unitary symmetries, which are local in parameter space—namely, parity-time ($\mathcal{PT}$) and particle-hole ($\mathcal{CP}$) symmetry, pseudo-Hermitian symmetry, as well as sublattice symmetry, chiral symmetry and pseudo-chiral symmetry[8]—inflict symmetries on the spectrum, and therefore reduce the codimension of the EPs.[8–12] The (anti-)unitary symmetries can be grouped into three pairs based on the type of constraint they enforce on the set of eigenenergies.[8,13] In this work, we show that the spectral symmetries already come about in the presence of similarity relations and not just in the presence of more restrictive symmetries. These similarity relations, namely pseudo Hermiticity, chirality and skew self-similarity, naturally pair the anti-unitary and unitary symmetries, cf. Fig. 1, and enforce the spectral symmetry. The symmetries appear as special cases of these three EP-inducing generalized similarities.

The relation of $\mathcal{PT}$ symmetry and pseudo Hermiticity, which denotes the similarity of H and its adjoint $H^\dagger$, is well established. Quantum mechanics formulated on the basis of $\mathcal{PT}$-symmetric operators was investigated by Bender *et al.*,[14–16] and has been related to pseudo-Hermitian operators. For diagonalizable $\mathcal{PT}$-symmetric operators Mostafazadeh proofed pseudo Hermiticity by explicitly showing the similarity between H and $H^\dagger$.[17–20] Later this was extended to any finite $\mathcal{PT}$-symmetric Hamiltonian by Zhang *et al.*.[21] In Section II we summarize and expand upon their results by showing a further connection to Hermitian and pseudo-Hermitian symmetric matrices, where we note the subtle difference between pseudo Hermiticity and pseudo-Hermitian symmetry. Pseudo Hermiticity alone already enforces symmetries on the spectrum, and thus lowers the codimension of EPs, while $\mathcal{PT}$ symmetry and pseudo-Hermitian symmetry constitute two special cases. Further we include Hermitian

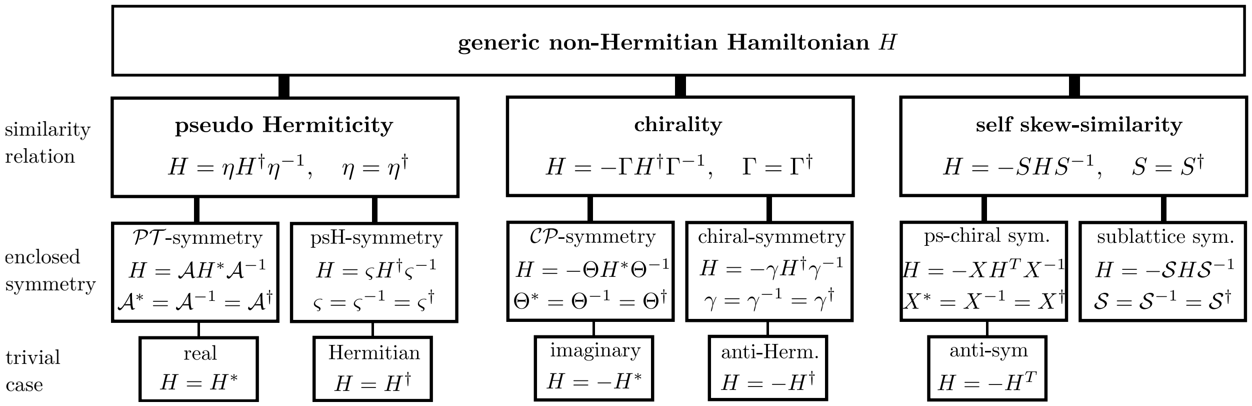

FIG. 1. Pairing of all EP inducing symmetries as special cases of generalized similarities. The three generalized similarity relations that lower the codimension of EPs are given. The different (anti-)unitary symmetries enclosed by the generalized similarities are shown. Trivial cases of the symmetries, where the generator is the identity, included in the overview.

Hamiltonians as a special case of pseudo-Hermitian symmetric systems, which has additional spectral symmetry that naturally prevents EPs from emerging. We also comment on real Hamiltonian as a special case of $\mathcal{PT}$-symmetry.

We find a similar structure for $\mathcal{CP}$-symmetric and chiral-symmetric systems. Both symmetries enforce chirality on the systems, which we define in Section III. We prove that all $\mathcal{CP}$-symmetric and chiral-symmetric systems are chiral. We show the spectral constraint follows from chirality, and relate anti-Hermiticity to chirality. We compare pseudo Hermiticity and chirality, which have a resembling effect on the spectrum, and point out the similarities and differences between them.

In Section IV we follow the same approach for pseudo-chiral symmetry and sublattice symmetry, where we find that in both cases the Hamiltonian exhibits self skew-similarity. This skew self-similarity is the origin of the spectral symmetry. We note that self skew-similarity behaves differently from pseudo-Hermiticity and chirality, because it does not relate the Hamiltonian to its adjoint, but instead is a property of the Hamiltonian itself. This results in differences in the treatment of self skew-similarity.

We provide a conclusion in Section V.

II. PSEUDO-HERMITIAN SYSTEMS

We start from the definition of pseudo-Hermiticity. A Hamiltonian is called pseudo-Hermitian if there exists an invertible Hermitian matrix η such that

$$H = \eta H^{\dagger} \eta^{-1}, \tag{1}$$

where $H^{\dagger}$ denotes the conjugate transpose of H.

Theorem 1.1. *For a matrix $H \in \mathbb{C}^{n\times n}$, H is pseudo-Hermitian if and only if there exist a nonsingular Hermitian matrix η and a Hermitian matrix A such that $H = \eta A$.*

Proof. Insert $H = \eta A$ in the definition of pseudo-Hermiticity using η as the similarity matrix.

We give this theorem here to provide a general decomposition of pseudo-Hermitian matrices, and to highlight a method of generating generic pseudo-Hermitian matrices.

For a pseudo-Hermitian matrix H with the eigenstate $|\psi\rangle$, defined by $H|\psi\rangle = \epsilon|\psi\rangle$, it follows from Eq. (1) that

$$\eta^{-1} H |\psi\rangle = \epsilon \left(\eta^{-1}|\psi\rangle\right) = H^{\dagger}\left(\eta^{-1}|\psi\rangle\right) , \tag{2}$$

thus $\eta^{-1}|\psi\rangle$ is an eigenstate of $H^{\dagger}$ with the eigenvalue ϵ, and the eigenvalues of H are either real or appear in complex conjugate pairs, i.e., $\{\epsilon\} = \{\epsilon^{*}\}$.

Theorem 1.2. *For a matrix $H \in \mathbb{C}^{n\times n}$, H is pseudo-Hermitian if and only if it is similar to its complex conjugate H^{*}.*

The proof of Theorem 1.2 can be found in Ref. 21. They show that the necessity follows from the definition of pseudo Hermiticity and the similarity of every matrix to its transpose. The proof of sufficiency is shown explicitly. The similarity of H and H^{*} results in real eigenvalues or pairs of complex conjugate eigenvalues. By ordering the Jordan canonical form of H in real Jordan blocks and block structures of complex conjugate Jordan blocks, Zhang *et al.* are able to construct the Hermitian similarity transformation η for any Hamiltonian that is similar to its complex conjugate. Theorem 1.1 and 1.2 are equivalent criteria for pseudo Hermiticity.

We establish a connection between pseudo Hermiticity and (anti-)unitary symmetries in non-Hermitian systems. $\mathcal{PT}$ symmetry is defined by

$$H = \mathcal{A} H^{*} \mathcal{A}^{-1} , \tag{3}$$

with $\mathcal{A}^{-1} = \mathcal{A}^\dagger$ and $\mathcal{A}\mathcal{A}^* = 1$. A different symmetry of non-Hermitian systems is pseudo-Hermitian symmetry defined by

$$H = \varsigma H^\dagger \varsigma^{-1}\,, \tag{4}$$

where $\varsigma^{-1} = \varsigma^\dagger$ and $\varsigma^2 = 1$. We emphasize the subtle difference between pseudo Hermiticity and pseudo-Hermitian symmetry.

Theorem 1.3. *For finite-dimensional systems, a $\mathcal{PT}$-symmetric or pseudo-Hermitian symmetric Hamiltonian H is necessarily pseudo-Hermitian.*

Proof. By the definition of pseudo-Hermiticity and pseudo-Hermitian symmetry this is clear for the later statement. For a $\mathcal{PT}$-symmetric system Eq. (3) shows that H is similar to H^*. Therefore, according to Theorem 1.2 the Hamiltonian is pseudo Hermitian. This was already realized by Zhang *et al.* in Ref. 21.

Theorem 1.4. *For any $H \in \mathbb{C}^{2\times 2}$, if H is pseudo-Hermitian it is necessarily $\mathcal{PT}$-symmetric and pseudo-Hermitian symmetric. For finite-dimensional systems with dimension $n > 2$, pseudo-Hermiticity does not imply either symmetry of the Hamiltonian.*

Proof. H has a certain symmetry if and only if there exists a unitary matrix U fulfilling Eq. (3) or (4) with the symmetry specific additional condition on U. Note that the case $n = 1$ is trivial due to the fact that H reduces to a real number. We show first whether a unitary similarity U between H and H^* or $H^\dagger$ exists in general and then investigate the properties of U. For the proof we make use of Specht's criterion.[22]

Specht's criterion *The matrices $A, B \in \mathbb{C}^{n\times n}$ are unitarily similar, i.e., $A = UBU^\dagger$ with U unitary, if and only if*

$$\mathrm{tr}\left[w\left(A, A^\dagger\right)\right] = \mathrm{tr}\left[w\left(B, B^\dagger\right)\right] \tag{5}$$

for ever finite word w in two letters.

This criterion is useful, because an upper bound on the length of the words w was introduced by Pearcy in Ref. 23 and was later refined.[24–29] For small n the sets of non-redundant words one has to check is given in Ref. 30. We make use of $n = 2$ and $n = 3$, where the non-redundant words $w(X, X^\dagger)$ are given by

$$n = 2: \quad X,\, X^2,\, XX^\dagger\,, \tag{6}$$

$$n = 3: \quad X,\, X^2,\, XX^\dagger,\, X^3,\, X^2X^\dagger,\, X^2\left(X^\dagger\right)^2,\, X^2\left(X^\dagger\right)^2,\, XX^\dagger\,. \tag{7}$$

For $n = 2$ we use that the traces of the three non-redundant words of H, H^* and $H^\dagger$ are equal due to the pseudo-Hermiticity constraint. Thus H and H^* as well as H and $H^\dagger$ are unitarily similar. The first condition from Eq. (3) for $\mathcal{PT}$-symmetry and from Eq. (4) for pseudo-Hermitian symmetry is therefore fulfilled. The special properties of the unitary similarity matrices $\mathcal{A}$ and η can be shown by

$$H = \mathcal{A} H^* \mathcal{A}^{-1} \implies H^* = \mathcal{A}^* H \left(\mathcal{A}^{-1}\right)^* \implies \mathcal{A}\mathcal{A}^* = 1\,, \tag{8}$$

$$H = \varsigma H^\dagger \varsigma^{-1} \implies H^\dagger = \varsigma H \varsigma^{-1} \implies \varsigma^2 = 1\,. \tag{9}$$

Therefore pseudo-Hermiticity implies $\mathcal{PT}$-symmetry and pseudo-Hermitian symmetry for any $H \in \mathbb{C}^{2\times 2}$.

For any $n \geq 3$ to find unitary similarity it is necessary that the traces of the words for $n = 3$ have to be equal, while there are more non-redundant words for $n > 3$. However, for non-normal H, i.e., $[H, H^*] \neq 0$, equality of the word traces of H and H^* as well as of H and $H^\dagger$ does not follow from pseudo-Hermiticity. Thus for $n \geq 3$ pseudo-Hermiticity is more general and not equivalent to $\mathcal{PT}$-symmetry or pseudo-Hermitian symmetry.

We note that normality of H restores the equivalence of similarity and symmetry, which can be shown from the diagonalisability of H and the pseudo-Hermitian spectral properties. For any normal H pseudo Hermiticity is equivalent to both $\mathcal{PT}$-symmetry and pseudo-Hermitian symmetry. However, normality prohibits the emergence of EPs altogether, because the Hamiltonian must be diagonalizable in the whole parameter space. Therefore, we consider non-normal Hamiltonians in the following for which the pseudo Hermiticity is more general then any of the two symmetries.

Theorem 1.5. *For any $H \in \mathbb{C}^{n\times n}$, if H is pseudo-Hermitian the codimension of an EPn is reduced to $n - 1$.*

Proof. It has been shown in Ref. 8 that the $2(n-1)$ real constraints for the emergence of an EPn can be cast as $\det[H] = \prod_i \epsilon_i = 0$ and $\mathrm{tr}\left[H^k\right] = \sum_i \epsilon_i^k = 0$ for $2 \leq k < n$ with the eigenvalues ϵ_i of H. The determinant and the traces are in general complex. Pseudo-Hermiticity implies the spectral symmetry $\{\epsilon\} = \{\epsilon^*\}$, which results in $\left\{\det[H], \mathrm{tr}\left[H^k\right]\right\} \in \mathbb{R}$. This reduces the codimension of the EPn to $n - 1$.

For $\mathcal{PT}$-symmetry and pseudo-Hermitian symmetry this was shown in Ref. 8, but the symmetries are special cases of pseudo-Hermiticity according to Theorem 1.3. From Theorem 1.4 we know that pseudo-Hermiticity is more general than the two symmetries, and it already induces the EPs in lower dimension without the need of symmetry. Thus we have shown that not symmetry but similarity drives the emergence of exceptional points in lower dimensions. Further the spectral structure surrounding the similarity-induced EPs is fully determined by the similarity even in the presence of the more restrictive $\mathcal{PT}$-symmetry or pseudo-Hermitian symmetry. This spectral structure is discussed in detail in previous papers on symmetry-induced EPs. Symmetry-protected EP2 rings were found,[1] and the rich spectral features surrounding symmetry-induced EP3s, EP4s and EP5s in two dimensions have also been analyzed.[13,31]

In addition to $\mathcal{PT}$-symmetry and pseudo-Hermitian symmetry, pseudo-Hermiticity encloses two more special cases, namely Hermitian and real matrices. We note that a Hermitian matrix is a special case of a pseudo-Hermitian symmetric systems, and a real Hamiltonian a special case of $\mathcal{PT}$-symmetric systems, with the symmetry generator being the identity operation in both cases. Our results concerning EPs are applicable to real matrices, while Hermiticity does not allow for EPs.

III. CHIRAL SYSTEMS

To define chirality we first define skew-similarity. Two matrices $A, B \in \mathbb{C}^{n\times n}$ are said to be skew-similar to each other if there exists an invertible matrix S such that

$$A = -SBS^{-1}\,. \tag{10}$$

We define chirality as Hermitian skew-similarity between the Hamiltonian H and its adjoint $H^\dagger$. A Hamiltonian H is called chiral if there exists an invertible Hermitian matrix Γ such that

$$H = -\Gamma H^\dagger \Gamma^{-1}\,. \tag{11}$$

Theorem 2.1. *For a matrix $H \in \mathbb{C}^{n\times n}$, H is chiral if and only if there exists a nonsingular Hermitian matrix Γ and a Hermitian matrix C such that $H = i\Gamma C$.*

Proof. To prove this Theorem insert $H = i\Gamma C$ in the definition of chirality Eq. (11) and take Γ as the similarity matrix.

With Theorem 2.1 we show a decomposition of chiral matrices, which can be implemented to obtain generic chiral Hamiltonians.

For any chiral Hamiltonian, we find constraints on the spectrum. Consider a chiral Hamiltonian H and an eigenstate $|\phi\rangle$ to the eigenvalue ϵ. From Eq. (11) follows

$$-\Gamma^{-1}H\,|\phi\rangle = -\epsilon\left(\Gamma^{-1}\,|\phi\rangle\right) = H^{\dagger}\left(\Gamma^{-1}\,|\phi\rangle\right). \tag{12}$$

Therefore the spectrum of H is mirrored across the real axis, and the eigenenergies fulfill $\{\epsilon\} = \{-\epsilon^*\}$.

Theorem 2.2. *For a matrix $H \in \mathbb{C}^{n\times n}$, H is chiral if and only if it is skew-similar to its complex conjugate H^*.*

Proof. The necessity in the statement is easy to prove. If a Hamiltonian H is chiral it is by definition skew-similar to its adjoint $H^{\dagger}$. It is well known that any matrix is similar to its transpose and therefore $H^{\dagger} = PH^TP^{-1}$ for some nonsingular matrix P. With $S = \Gamma P$ this proves the skew-similarity of H and H^*.

We explicitly construct the Hermitian matrix Γ to prove the sufficiency of the theorem. Using its Jordan canonical form J the Hamiltonian H can be written by

$$H = QJQ^{-1}\,, \tag{13}$$

where Q is a reversible matrix. The Jordan canonical form consists of Jordan blocks that have the form

$$J(\epsilon) = \begin{pmatrix} \epsilon & 1 & & \\ & \ddots & \ddots & \\ & & \epsilon & 1 \\ & & & \epsilon \end{pmatrix}_{m\times m}\,, \tag{14}$$

where m is the size of the Jordan block and $m = 1$ reduces to ϵ. Because H is skew-similar to H^* the eigenvalues are symmetric with respect to the imaginary axis. Thus the eigenvalues are either purely imaginary $\epsilon = ib$ or come in pairs of the form $\epsilon = a + ib$ and $-\epsilon^* = -a + ib$, where $a, b \in \mathbb{R}$. We define two kinds of matrix blocks

$$K_1 = J(ib)_{m\times m}\,, \tag{15}$$

$$K_2 = \begin{pmatrix} J(a+ib) & 0 \\ 0 & J(-a+ib) \end{pmatrix}_{2l\times 2l}\,, \tag{16}$$

and express the Jordan canonical form as the block diagonal matrix $J = \mathrm{diag}(M_1, ..., M_k)$ with each block M_J being either of the form K_1 or K_2. We can prove explicitly that the

Jordan form constructed in this ordered form is chiral. Both types of matrix blocks satisfy $M_j = -G_j M_j^\dagger G_j^{-1}$ with the Hermitian matrix

$$G_j = \begin{pmatrix} 0 & \dots & 0 & 1 \\ \vdots & \iddots & 1 & 0 \\ 0 & \iddots & \iddots & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix} . \tag{17}$$

From these G_j we construct the Hermitian matrix $G = \mathrm{diag}(G_1, ..., G_k)$, and this satisfies

$$J = -GJ^\dagger G^{-1} , \tag{18}$$

which shows that the Jordan canonical form is chiral if H is skew-similar to H^*. We insert this in Eq. (13) and we obtain

$$\begin{aligned} H &= QJQ^{-1} \\ &= -QGJ^\dagger G^{-1} Q^{-1} \\ &= -QG \left(Q^{-1}HQ\right)^\dagger G^{-1}Q^{-1} \\ &= -\left(QGQ^\dagger\right) H^\dagger \left(QGQ^\dagger\right)^{-1} . \end{aligned} \tag{19}$$

Because G is Hermitian, $\Gamma = QGQ^\dagger$ is Hermitian as well. This completes the proof that H is chiral.

We now show the connection between chirality and $\mathcal{CP}$-symmetry and chiral symmetry. We define $\mathcal{CP}$-symmetry by

$$H = -\Theta H^* \Theta^{-1} , \tag{20}$$

where $\Theta^{-1} = \Theta^\dagger$ and $\Theta\Theta^* = 1$. We note that $\mathcal{CP}$-symmetry is sometimes called anti-$\mathcal{PT}$-symmetry due to the structure of the symmetry constraint. Chiral symmetry is defined as

$$H = -\gamma H^\dagger \gamma^{-1} , \tag{21}$$

with $\gamma^{-1} = \gamma^\dagger$ and $\gamma^2 = 1$. The subtle difference in the properties of the similarity matrix between chirality and chiral symmetry is emphasized here.

Theorem 2.3. *For finite-dimensional systems, a $\mathcal{CP}$-symmetric or chiral-symmetric Hamiltonian H is necessarily chiral.*

Proof. A $\mathcal{CP}$-symmetric Hamiltonian H is by the definition in Eq. (20) skew-similar to its complex conjugate H^*. Chirality follows directly from Theorem 2.2. By the definition of chiral symmetry it is clear that chiral symmetry entails chirality.

Theorem 2.4. *For any $H \in \mathbb{C}^{n\times n}$, if $n = 2$ and H is chiral it is necessarily $\mathcal{CP}$-symmetric and chiral symmetric. For finite-dimensional systems with dimension $n > 2$, chirality does not imply either symmetry of the Hamiltonian.*

Proof. For $n = 1$ the statement is obvious, because H is restricted to be an imaginary number. For $H \in \mathbb{C}^{2\times 2}$ the traces of the three non-redundant words from Specht's refined criterion are equal for H, $-H^*$ and $-H^\dagger$. This follows from the spectral symmetry of the chiral Hamiltonian. Hence, there is a unitary similarity transformation between H and $-H^*$, and H and $-H^\dagger$, respectively. To prove that this unitary similarity is in fact a symmetry of the system, we show the properties of the different similarity matrices by

$$H = -\Theta H^* \Theta^{-1} \implies -H^* = \Theta^* H \left(\Theta^{-1}\right)^* \implies \Theta\Theta^* = 1\,, \tag{22}$$

$$H = -\gamma H^\dagger \gamma^{-1} \implies -H^\dagger = \gamma H \gamma^{-1} \implies \gamma^2 = 1\,. \tag{23}$$

Thus chirality implies both $\mathcal{CP}$ symmetry and chiral symmetry for any $H \in \mathbb{C}^{2\times 2}$.

For a Hamiltonian $H \in \mathbb{C}^{n\times n}$ with $n \geq 3$ we find that the traces of the words in Specht's criterion do not vanish in general for non-normal H. Therefore, chirality does not imply unitary similarity between H, $-H^*$ and $H^\dagger$. For $n \geq 3$ chirality is not necessarily equivalent to $\mathcal{CP}$-symmetry or chiral symmetry.

The equivalence of chirality and $\mathcal{CP}$-symmetry as well as chiral symmetry can be restored by enforcing normality on the Hamiltonian H. However, this would disallow exceptional points to emerge in the systems as already mentioned before.

Theorem 2.5. *For any $H \in \mathbb{C}^{n\times n}$, if H is chiral the codimension of an EPn is reduced to $n - 1$.*

Proof. We consider the complex conditions $\det[H] = 0$ and $\mathrm{tr}\left[H^k\right] = 0$ for $2 \leq k < n$. The spectral symmetry $\{\epsilon\} = \{-\epsilon^*\}$, which is a consequence of the chirality, reduces the number of constraints, because the determinant of H and each of the traces of H^k is either real or purely imaginary. This reduces the codimension of the EP to $n-1$. For the two symmetries enclosed by chirality according to Theorem 2.3 this was shown in Ref. 8. However, from Theorem 2.4 it is clear that chirality is more general than either $\mathcal{CP}$-symmetry or chiral symmetry. Chirality already induced EPs by lowering their codimension without the need of symmetries of the Hamiltonian. Further, the spectral structure surrounding a chirality-induced exceptional point is fully determined by the skew-similarity of the Hamiltonian. For EP3s in two dimensions the spectral structure is equivalent to the structures described in Ref. 13 for $\mathcal{CP}$-symmetry and chiral symmetry.

Besides $\mathcal{CP}$-symmetric and chiral-symmetric systems there are two notable special cases of chiral systems. The first case is anti-Hermiticity of H, meaning $H = -H^\dagger$, which can be interpreted as chiral symmetry with the identity as generator. Because anti-Hermiticity implies normality no exceptional points can emerge in anti-Hermitian systems. The other case are imaginary matrices $H = -H^*$, which are $\mathcal{CP}$-symmetric with the identity as generator. Our results are thus also applicable for imaginary matrices.

IV. SELF SKEW-SIMILAR SYSTEMS

With skew-similarity defined in Eq. (10) any Hamiltonian H is self skew-similar if it fulfills

$$H = -SHS^{-1} \tag{24}$$

for an invertible hermitian matrix S.

Theorem 3.1. *A Hamiltonian H is self skew similar if and only if there exists an invertible Hermitian matrix S such that $\{H, S\} = 0$.*

Proof. This follows immediately from the definition of self skew-similarity.

The self skew-similarity constraints the spectrum to $\{\epsilon\} = \{-\epsilon\}$. This can be shown by considering an eigenstate $|\chi\rangle$ of the self skew-similar Hamiltonian H with eigenvalue ϵ. Applying the definition Eq. (24) yields

$$-S^{-1}H\,|\chi\rangle = -\epsilon\left(S^{-1}\,|\chi\rangle\right) = H\left(S^{-1}\,|\chi\rangle\right)\,, \tag{25}$$

and this shows the spectral symmetry.

The two symmetries that enforce the same spectral constraint on the system are sublattice symmetry and pseudo-chiral symmetry. Sublattice symmetry is defined by

$$H = -\mathcal{S}H\mathcal{S}^{-1}\,, \tag{26}$$

with $\mathcal{S}^{-1} = \mathcal{S}^\dagger$ and $\mathcal{S}^2 = 1$. We define pseudo-chiral symmetry as

$$H = -XH^TX^{-1}\,, \tag{27}$$

where $X^{-1} = X^{\dagger}$ and $XX^* = 1$.

Theorem 3.2. *For finite-dimensional systems, a sublattice-symmetric or pseudo-chiral symmetric Hamiltonian H is necessarily self skew-similar.*

Proof. By the definition of sublattice symmetry it is clear that a sublattice-symmetric Hamiltonian is self skew-similar. Because every matrix is similar to its transpose, pseudo-chiral symmetry entails self skew-similarity.

Theorem 3.3. *For any Hamiltonian $H \in \mathbb{C}^{n\times n}$, self skew-similarity does not imply either symmetry of the Hamiltonian for $n \geq 2$.*

Proof. For any $H \in \mathbb{C}^{n\times n}$ with $n \geq 3$ Specht's criterion is not fulfilled for a generic self skew-similar matrix. For $H \in \mathbb{C}^{2\times 2}$ Specht's criterion is always fulfilled, however, the additional properties enforced on the unitary operator to be a symmetry generator are not fulfilled for either sublattice symmetry or pseudo-chiral symmetry. Note that $n = 1$ is a special case, because the self skew-similarity implies $H = 0$, which has arbitrary unitary and anti-unitary symmetries.

Theorem 3.4. *For any $H \in \mathbb{C}^{n\times n}$, if H is self skew-similar the codimension of an EPn is reduced to n if n is even and to $n-1$ if n is odd.*

Proof. We consider the complex conditions $\det[H] = 0$ and $\mathrm{tr}\big[H^k\big] = 0$ for $2 \leq k < n$. For any odd k the trace $\mathrm{tr}\big[H^k\big]$ vanishes for any self skew-similar Hamiltonian due to the spectral constraint $\{\epsilon\} = \{-\epsilon\}$. For odd n the determinant always vanishes, because

$$\det[H] = \det[S]\det[-H]\det\big[S^{-1}\big] = (-1)^n \det[H] \overset{n\in\mathrm{odd}}{\Longrightarrow} \det[H] = 0\,. \tag{28}$$

This reduces the codimension in the case of odd n to $n-1$ and for even n the codimension of EPns is reduced only to n.

For the two symmetries that realize self skew-similar Hamiltonians this was shown in Ref. 8. From Theorem 3.3 it is clear that self skew-similarity is more general than either of the two symmetries. According to Theorem 3.4 self skew-similarity already induces EPns by lowering their codimension. Because the spectral symmetry is enforced by the similarity, symmetry of the Hamiltonian is not needed. Further the spectral structure accompanying the similarity-induced EPs is determined by the self skew-similarity, and also not affected by the additional constraints of pseudo-chiral symmetry or sublattice symmetry.

A special case of self skew-similarity are anti-symmetric Hamiltonians $H = -H^T$, which are pseudo-chiral symmetric with the identity as generator. All our results are applicable for anti-symmetric matrices.

V. CONCLUSIONS

It was previously shown that $\mathcal{PT}$-symmetry entails pseudo Hermiticity for finite dimensional systems. In this paper we show that this relation can be generalized to all unitary and anti-unitary symmetries, which lower the codimension of exceptional points. We proof that each of these symmetries is a special case of one of three generalized similarities, namely pseudo Hermiticity, chirality and self skew-similarity. Each similarity encompasses two symmetries, and in the case of pseudo Hermiticity and chirality for finite-dimensional systems of size $n > 2$ the similarities are more general than the respective symmetries. In the case of self skew-similar Hamiltonians this even holds for $n \geq 2$.

Overall we find that the spectral features of non-Hermitian systems and the emergence of stable EPs is linked to the relevant similarity, and not the symmetry of the system contrary to previous assumptions. The similarities are far less restrictive compared to unitary or anti-unitary symmetries. As such, the presence of similarities may lead to the robustness of symmetry-stabilized non-Hermitian features to symmetry-breaking perturbations.

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