# NUMERICALLY EXPLICIT ESTIMATES FOR THE DISTRIBUTION OF ROUGH NUMBERS

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ABSTRACT. For  $x \ge y > 1$  and  $u := \log x/\log y$ , let  $\Phi(x,y)$  denote the number of positive integers up to x free of prime divisors less than or equal to y. In 1950 de Bruijn [4] studied the approximation of  $\Phi(x,y)$  by the quantity

$$\mu_y(u)e^{\gamma}x\log y\prod_{p\leq y}\left(1-\frac{1}{p}\right),$$

where  $\gamma = 0.5772156...$  is Euler's constant and

$$\mu_y(u) := \int_1^u y^{t-u} \omega(t) \, dt.$$

He showed that the asymptotic formula

$$\Phi(x,y) = \mu_y(u)e^{\gamma}x\log y \prod_{p \le y} \left(1 - \frac{1}{p}\right) + O\left(\frac{xR(y)}{\log y}\right)$$

holds uniformly for all  $x \geq y \geq 2$ , where R(y) is a positive decreasing function related to the error estimates in the Prime Number Theorem. In this paper we obtain numerically explicit versions of de Bruijn's result.

#### 1. Introduction

Let  $x \ge y > 1$  be positive real numbers. Throughout the paper, we shall always write  $u := \log x/\log y$ , and the letters p and q will always denote primes. We say that a positive integer n is y-rough if all the prime divisors of n are greater than y. Let  $\Phi(x,y)$  denote the number of y-rough numbers up to x. Explicitly, we have

$$\Phi(x,y) = \sum_{\substack{n \le x \\ P^{-}(n) > y}} 1,$$

where  $P^-(n)$  denotes the least prime divisor of n, with the convention that  $P^-(1) = \infty$ . When  $1 \le u \le 2$ , or equivalently when  $\sqrt{x} \le y \le x$ , we simply have  $\Phi(x,y) = \pi(x) - \pi(y) + 1$ , where  $\pi(\cdot)$  is the prime-counting function. The function  $\Phi(x,y)$  is closely related to the sieve of Eratosthenes, one of the most ancient algorithms for finding primes, and it has been extensively studied by mathematicians. A simple application of the inclusion-exclusion principle enables us to write

$$\Phi(x,y) = \sum_{d|P(y)} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor, \tag{1.1}$$

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where  $\lfloor a \rfloor$  is the integer part of a for any  $a \in \mathbb{R}$ ,  $\mu$  is the Möbius function, and P(y) denotes the product of primes up to y. If y is relatively small in comparison with x, say  $y = x^{o(1)}$ , the above formula can be used to obtain  $\Phi(x,y) \sim e^{-\gamma} x/\log y$  as  $y \to \infty$ , where  $\gamma = 0.5772156...$  is Euler's constant. However, it turns out that this nice asymptotic formula does not hold uniformly, as already exemplified by the base case 1 < u < 2.

In 1937, Buchstab [3] showed that for any fixed u > 1, one has  $\Phi(x,y) \sim \omega(u)x/\log y$  as  $x \to \infty$ , where  $\omega(u)$  is defined to be the unique continuous solution to the delay differential equation  $(u\omega(u))' = \omega(u-1)$  for  $u \ge 2$ , subject to the initial value condition  $\omega(u) = 1/u$  for  $u \in [1,2]$ . Comparing this result with the asymptotic formula obtained from (1.1), one would expect that  $\omega(u) \to e^{-\gamma}$  as  $u \to \infty$ . Indeed, it can be shown [13, Corollary III.6.5] that  $\omega(u) = e^{-\gamma} + O(u^{-u/2})$  for  $u \ge 1$ . Moreover, it is known that  $\omega(u)$  oscillates above and below  $e^{-\gamma}$  infinitely often. It is convenient to extend the definition of  $\omega(u)$  by setting  $\omega(u) = 0$  for all u < 1, so that  $\omega(u)$  satisfies the same delay differential equation on  $\mathbb{R} \setminus \{1, 2\}$ . In the sequel, we shall write  $\omega'(1)$  and  $\omega'(2)$  for the right derivatives of  $\omega(u)$  at u = 1 and u = 2, respectively. With this convention, we have  $(u\omega(u))' = \omega(u-1)$  for all  $u \in \mathbb{R}$ .

Buchstab's asymptotic formula can be proved easily based on the following identity [13, Theorem III.6.3] named after him:

$$\Phi(x,y) = \Phi(x,z) + \sum_{y 1} \Phi(x/p^v, p)$$
(1.2)

for any  $z \in [y, x]$ . The Buchstab function  $\omega(u)$  then appears naturally in the iteration process, starting with  $\Phi(x, y) \sim x/(u \log y)$  in the range  $1 < u \le 2$ . Since  $1/2 \le \omega(u) \le 1$  for  $u \in [1, \infty)$ , Buchstab's asymptotic formula suggests that the relation  $\Phi(x, y) \approx x/\log y$  holds uniformly for  $x \ge y > 1$ . Thus, it is of interest to seek numerically explicit estimates for  $\Phi(x, y)$  that are applicable in wide ranges. Confirming a conjecture of Ford, the author [6] showed that  $\Phi(x, y) < x/\log y$  holds uniformly for  $x \ge y > 1$ , which is essentially best possible when  $x^{1-\epsilon} \le y \le \epsilon x$ , where  $\epsilon \in (0, 1)$  is fixed. On the other hand, the values of  $\omega(u)$  indicate that improvements should be expected in the narrower range  $2 \le y \le \sqrt{x}$ . In recent work jointly with Pomerance [7], the author proved that  $\Phi(x, y) < 0.6x/\log y$  holds uniformly for  $3 \le y \le \sqrt{x}$ . This inequality provides a fairly good upper bound for  $\Phi(x, y)$ , especially considering that the absolute maximum of  $\omega(u)$  over  $[2, \infty)$  is given by  $M_0 = 0.5671432...$ , attained at the unique critical point u = 2.7632228... of the function  $(\log(u - 1) + 1)u^{-1}$  on [2, 3]. With a bit more effort, one can show, using the Buchstab identity (1.2), that

$$\Phi(x,y) = \frac{x}{\log y} \left( \omega(u) + O\left(\frac{1}{\log y}\right) \right) \tag{1.3}$$

uniformly for  $2 \le y \le \sqrt{x}$  (see [13, Theorem III.6.4]). In Section 2, we shall derive a numerically explicit lower bound of this type that suits our needs. Our method can also be modified with ease to obtain a numerically explicit upper bound of the same type.

In [4] de Bruijn provided a more precise approximation for  $\Phi(x,y)$  than  $\omega(u)x/\log y$ . Let us fix some  $y_0 \geq 2$  for the moment. Suppose that there exist a positive constant  $C_0(y_0)$  and a positive decreasing function R(z) defined on  $[y_0,\infty)$ , such that  $R(z) \gg z^{-1}$ , that  $R(z) \downarrow 0$  as  $z \to \infty$  and that for all  $z \geq y_0$  we have

$$|\pi(z) - \operatorname{li}(z)| \le \frac{z}{\log z} R(z) \tag{1.4}$$

and

$$\int_{z}^{\infty} \frac{|\pi(t) - \operatorname{li}(t)|}{t^{2}} dt \le C_{0}(y_{0})R(z), \tag{1.5}$$

where li(z) is the logarithmic integral defined by

$$\operatorname{li}(z) := \int_0^z \frac{dt}{\log t}.$$

The classical version of the Prime Number Theorem allows us to take  $R(z) = \exp(-c\sqrt{\log z})$  for some suitable constant c > 0. Using the zero-free region of Korobov and Vinogradov for the Riemann zeta-function, we obtain  $R(z) = \exp(-c'(\log z)^{3/5}(\log\log z)^{-1/5})$  for some absolute constant c' > 0. If the Riemann Hypothesis holds, then one can take  $R(z) = c''z^{-1/2}\log^2 z$ , where c'' > 0 is an absolute constant.

To state de Bruijn's result, we define

$$\mu_y(u) := \int_1^u y^{t-u} \omega(t) \, dt.$$

It is easy to see that  $0 \le \mu_y(u) \log y \le 1 - y^{1-u}$  and that for every fixed  $u \ge 1$ , we have  $\mu_y(u) \log y \to \omega(u)$  as  $y \to \infty$ . Precise expansions for  $\mu_y(u)$  in terms of the powers of  $\log y$  can be found in [13, Theorem III.6.18]. When  $1 \le u \le 2$ , the change of variable  $t = \log v/\log y$  shows that

$$\mu_y(u)x = \int_1^u t^{-1}y^t dt = \int_y^x \frac{dv}{\log v} = \mathrm{li}(x) - \mathrm{li}(y).$$

Since  $\Phi(x,y) = \pi(x) - \pi(y) + 1$  when  $1 \le u \le 2$ , (1.4) clearly implies that

$$\Phi(x,y) = \mu_y(u)x + (\pi(x) - \text{li}(x)) - (\pi(y) - \text{li}(y)) + 1 = \mu_y(u)x + O\left(\frac{xR(y)}{\log y}\right).$$

It can be shown using (1.4) and (1.5) that

$$\prod_{p \le y} \left( 1 - \frac{1}{p} \right) = \frac{e^{-\gamma}}{\log y} \left( 1 + O(R(y)) \right).$$

Thus we have, equivalently,

$$\Phi(x,y) = \mu_y(u)e^{\gamma}x\log y \prod_{p \le y} \left(1 - \frac{1}{p}\right) + O\left(\frac{xR(y)}{\log y}\right). \tag{1.6}$$

Essentially, de Bruijn [4] showed that this formula holds uniformly for  $x \geq y \geq y_0$ . In Section 3 we shall derive an explicit version of (1.6), which will be applied in Section 4 to obtain numerically explicit estimates with suitable  $y_0$  and R(y). Our main results are summarized in the following theorem.

**Theorem 1.1.** Uniformly for  $x \geq y \geq 2$ , we have

$$\left| \Phi(x,y) - \mu_y(u)e^{\gamma}x \log y \prod_{p \le y} \left( 1 - \frac{1}{p} \right) \right| < 4.403611 \frac{x}{(\log y)^{3/4}} \exp\left( -\sqrt{\frac{\log y}{6.315}} \right).$$

Conditionally on the Riemann Hypothesis, we have

$$\left| \Phi(x,y) - \mu_y(u)e^{\gamma}x \log y \prod_{p \le y} \left(1 - \frac{1}{p}\right) \right| < 0.449774 \frac{x \log y}{\sqrt{y}}$$

uniformly for  $x \ge y \ge 11$ .

The following consequence of Theorem 1.1 is sometimes more convenient to use.

Corollary 1.2. Uniformly for  $x \geq y \geq 2$ , we have

$$|\Phi(x,y) - \mu_y(u)x| < 4.434084 \frac{x}{(\log y)^{3/4}} \exp\left(-\sqrt{\frac{\log y}{6.315}}\right).$$

Conditionally on the Riemann Hypothesis, we have

$$|\Phi(x,y) - \mu_y(u)x| < 0.460680 \frac{x \log y}{\sqrt{y}}$$

uniformly for  $x \ge y \ge 11$ .

## 2. Lower Bounds for $\Phi(x,y)$

Before moving on to the derivation of Theorem 1.1, we prove a clean lower bound for  $\Phi(x,y)$  which is applicable in a wide range. This lower bound, which is interesting in itself, will be used in the proof of Theorem 1.1 and Corollary 1.2 in Section 4. We start by proving the following result, which provides a numerically explicit lower bound for the implicit constant in the error term in (1.3). As we already mentioned, our method can easily be adapted to yield a numerically explicit upper bound as well, though it will not be needed in the present paper.

**Proposition 2.1.** Define  $\Delta(x,y)$  by

$$\Phi(x,y) = \frac{x}{\log y} \left( \omega(u) + \frac{\Delta(x,y)}{\log y} \right)$$

for  $2 \le y \le \sqrt{x}$ . Let  $y_0 = 602$ . For every positive integer  $k \ge 3$ , we define

$$\Delta_k^- = \Delta_k^-(y_0) := \inf \left\{ \min(\Delta(x, y), 0) : y \ge y_0 \text{ and } 2 \le u < k \right\}.$$

Then  $\Delta_3^- > -0.563528$ ,  $\Delta_4^- > -0.887161$ , and  $\Delta_k^- > -0.955421$  for all  $k \geq 5$ .

*Proof.* Let  $y_1 := 2,278,383$ . Suppose first that  $y \ge y_1$  and set

$$G(v) := \sum_{x^{1/v}$$

for  $2 \le v \le u$ . By [5, Theorem 5.6]<sup>1</sup>, we have

$$\left| G(v) - \log \frac{v}{2} \right| \le \frac{c_1}{\log^2 y} \tag{2.1}$$

for all  $y \ge y_1$ , where  $c_1 = 0.4/\log y_1$ . We shall also make use of the following inequality [5, Corollary 5.2]<sup>2</sup>:

$$\frac{z}{\log z} \left( 1 + \frac{c_3}{\log z} \right) \le \pi(z) \le \frac{z}{\log z} \left( 1 + \frac{c_2}{\log z} \right), \tag{2.2}$$

where  $c_2 = 1 + 2.53816/\log y_1$  and  $c_3 = 1 + 2/\log y_1$ . We start with the range  $2 \le u \le 3$ . In this range, we have

$$\Phi(x,y) = \#\{n \le x: P^-(n) > y \text{ and } \Omega(n) \le 2\}$$

$$= \pi(x) - \pi(y) + 1 + \sum_{y 
$$= \pi(x) - \pi(y) + 1 + \sum_{y$$$$

where  $\Omega(n)$  denotes the total number of prime factors of n, with multiplicity counted. Since

$$\sum_{y$$

we see that

$$\pi(x) - \pi(y) + 1 - \sum_{y \le p \le \sqrt{x}} (\pi(p) - 1) > \pi(x) - \frac{\pi(\sqrt{x})^2}{2} + \frac{\pi(\sqrt{x})}{2}.$$

It follows from (2.2) that

$$\Phi(x,y) > \frac{x}{\log x} \left( 1 + \frac{c_3}{\log x} \right) - \frac{x}{2\log^2 \sqrt{x}} \left( 1 + \frac{c_2}{\log \sqrt{x}} \right)^2 + \frac{\sqrt{x}}{2\log \sqrt{x}} + \sum_{y$$

To handle the sum in (2.3), we appeal to (2.2) again to arrive at

$$\sum_{y \le p \le \sqrt{x}} \pi(x/p) \ge \sum_{y \le p \le \sqrt{x}} \left( \frac{x}{p \log(x/p)} + \frac{c_3 x}{p \log^2(x/p)} \right).$$

<sup>&</sup>lt;sup>1</sup>In [2] it is claimed that the proof of [5, Theorem 4.2] is incorrect due to the application of an incorrect zero density estimate of Ramare [10, Theorem 1.1]. In a footnote on p. 2299 of the same paper, the authors state that the bounds asserted in [5] are likely affected for this reason. However, since they also give a correct proof of [5, Theorem 4.2] (see [2, Corollary 11.2]), one verifies easily that the proof of [5, Theorem 5.6], which relies only on [5, Theorem 4.2], partial summation, and numerical computation, remains valid.

<sup>&</sup>lt;sup>2</sup>For the same reason mentioned above, it is reasonable to suspect that the bounds given in [5, Corollary 5.2] are also affected. However, one can verify these bounds without much difficulty. Indeed, (5.2) of [5, Corollary 5.2] is superseded by [11, Corollary 1], while (5.3) and (5.4) of [5, Corollary 5.2] follow from [1, Lemmas 3.2–3.4] and direct calculations.

By partial summation we see that

$$\sum_{y$$

From (2.1) it follows that

$$\frac{G(u)}{u-1} \ge \frac{1}{u-1} \left( \log \frac{u}{2} - \frac{c_1}{\log^2 y} \right),$$

and

$$\begin{split} \int_{2^{-}}^{u} \frac{G(v)}{(v-1)^{2}} \, dv &\geq \int_{2}^{u} \frac{1}{(v-1)^{2}} \left( \log \frac{v}{2} - \frac{c_{1}}{\log^{2} y} \right) \, dv \\ &= -\frac{1}{u-1} \log \frac{u}{2} + \int_{2}^{u} \frac{1}{v(v-1)} \, dv - \frac{c_{1}}{\log^{2} y} \left( 1 - \frac{1}{u-1} \right) \\ &= -\frac{u}{u-1} \log \frac{u}{2} + \log(u-1) - \frac{c_{1}}{\log^{2} y} \left( 1 - \frac{1}{u-1} \right). \end{split}$$

Hence

$$\sum_{y 
$$= \frac{x}{\log y} \left( \omega(u) - \frac{2c_1}{u \log^2 y} \right) - \frac{x}{\log x}.$$
(2.4)$$

Similarly, we have

$$\sum_{u < v \le \sqrt{x}} \frac{1}{p \log^2(x/p)} = \frac{1}{\log^2 x} \int_{2^-}^u \left(\frac{v}{v-1}\right)^2 dG(v) = \frac{1}{\log^2 x} \left(\frac{G(u)u^2}{(u-1)^2} + 2\int_{2^-}^u \frac{vG(v)}{(v-1)^3} dv\right).$$

By (2.1) we have

$$\frac{G(u)u^2}{(u-1)^2} \ge \frac{u^2}{(u-1)^2} \left( \log \frac{u}{2} - \frac{c_1}{\log^2 y} \right),$$

and

$$\int_{2^{-}}^{u} \frac{vG(v)}{(v-1)^3} dv \ge \int_{2^{-}}^{u} \frac{v}{(v-1)^3} \left( \log \frac{v}{2} - \frac{c_1}{\log^2 y} \right) dv.$$

Since

$$\begin{split} \int_2^u \frac{v}{(v-1)^3} \log \frac{v}{2} \, dv &= -\left(\frac{1}{u-1} + \frac{1}{2(u-1)^2}\right) \log \frac{u}{2} + \int_2^u \left(\frac{1}{v-1} + \frac{1}{2(v-1)^2}\right) \frac{dv}{v} \\ &= -\frac{2u-1}{2(u-1)^2} \log \frac{u}{2} + \frac{1}{2} \int_2^u \left(\frac{1}{(v-1)^2} + \frac{1}{v(v-1)}\right) \, dv \\ &= -\frac{u^2}{2(u-1)^2} \log \frac{u}{2} + \frac{1}{2} \left(\log(u-1) + 1 - \frac{1}{u-1}\right) \end{split}$$

and

$$\int_{2}^{u} \frac{v}{(v-1)^{3}} dv = -\frac{2u-1}{2(u-1)^{2}} + \frac{3}{2},$$

we have

$$\sum_{u \le p \le \sqrt{x}} \frac{x}{p \log^2(x/p)} \ge \frac{x}{\log^2 x} \left( \log(u - 1) + \frac{u - 2}{u - 1} - \frac{4c_1}{\log^2 y} \right). \tag{2.5}$$

Inserting (2.4) and (2.5) into (2.3) yields

$$\Delta(x,y) \ge g(u) - \frac{2c_1}{u\log y} + \frac{\log y}{uy^{3/2}} - \frac{1}{u^2} \left( 2 - c_3 + \frac{4c_1c_3}{\log^2 y} + \frac{8c_2}{u\log y} + \frac{8c_2^2}{u^2\log^2 y} \right),$$

where

$$g(u) := \frac{c_3}{u^2} \left( \log(u-1) + \frac{u-2}{u-1} \right).$$

Using Mathematica we find that  $\Delta_3^- > -0.301223$  when  $y \ge y_1$ .

Now we proceed to bound  $\Delta_k^-$  for  $k \geq 4$  recursively when  $y \geq y_1$ . Let  $k \geq 3$  be arbitrary. It is easily seen that the following variant of Buchstab's identity (1.2) holds for any  $z \in [y, x]$ :

$$\Phi(x,y) = \Phi(x,z) + \sum_{y$$

where  $p^- < p$  is any real number sufficiently close to p. For  $3 \le k \le u < k+1$  and  $y \ge y_1$ , we obtain by taking  $z = x^{1/3}$  that

$$\Phi(x,y) = \Phi(x,x^{1/3}) + \sum_{y 
(2.7)$$

We have already shown that

$$\Phi\left(x, x^{1/3}\right) \ge \frac{x}{\log x^{1/3}} \left(\omega\left(\frac{\log x}{\log x^{1/3}}\right) + \frac{\Delta_3^-}{\log x^{1/3}}\right) = \frac{3x}{\log y} \left(\frac{\omega(3)}{u} + \frac{3\Delta_3^-}{u^2 \log y}\right). \tag{2.8}$$

Note that  $2 < \log(x/p)/\log(p^{-}) < k$ . Thus, we have

$$\Phi(x/p, p^{-}) \ge \frac{x}{p \log(p^{-})} \left( \omega \left( \frac{\log(x/p)}{\log(p^{-})} \right) + \frac{\Delta_k^{-}}{\log(p^{-})} \right).$$

Since  $\omega(u)$  is continuous on  $[1, \infty)$ , it follows from (2.7) and (2.8) that

$$\Phi(x,y) \ge \frac{3x}{\log y} \left( \frac{\omega(3)}{u} + \frac{3\Delta_3^-}{u^2 \log y} \right) + \sum_{y$$

By partial summation we see that

$$\sum_{y \le n \le x^{1/3}} \frac{1}{p \log^2 p} < \int_y^\infty \frac{1}{t \log^2 t} d\pi(t) = -\frac{\pi(y)}{y \log^2 y} + \int_y^\infty \frac{\log t + 2}{t^2 \log^3 t} \pi(t) dt,$$

which, by (2.2), is

$$< -\frac{1}{\log^3 y} \left( 1 + \frac{c_3}{\log y} \right) + \int_y^\infty \frac{\log t + 2}{t \log^4 t} \left( 1 + \frac{c_2}{\log t} \right) dt$$

$$= -\frac{1}{\log^3 y} \left( 1 + \frac{c_3}{\log y} \right) + \frac{1}{2 \log^2 y} + \frac{c_2 + 2}{3 \log^3 y} + \frac{c_2}{2 \log^4 y}$$

$$= \frac{1}{\log^2 y} \left( \frac{1}{2} + \left( \frac{c_2}{3} - 1 \right) \frac{1}{\log y} + \left( \frac{c_2}{2} - c_3 \right) \frac{1}{\log^2 y} \right)$$

$$< \frac{1}{\log^2 y} \left( \frac{1}{2} + \left( \frac{c_2}{3} - 1 \right) \frac{1}{\log y} \right).$$

Hence

$$\sum_{y$$

On the other hand, we have

$$\sum_{y 
$$= \frac{1}{\log x} \left( \int_3^u \omega(v - 1) dv + \int_{3^-}^u v \omega(v - 1) d \left( G(v) - \log \frac{v}{2} \right) \right).$$$$

Observe that

$$\int_{3}^{u} \omega(v-1) dv = u\omega(u) - 3\omega(3)$$

and that

$$\int_{3^{-}}^{u} v\omega(v-1) d\left(G(v) - \log\frac{v}{2}\right) = u\omega(u-1) \left(G(v) - \log\frac{v}{2}\right) - 3\omega(2) \left(G(3) - \log\frac{3}{2}\right)$$
$$-\int_{3^{-}}^{u} \left(G(v) - \log\frac{v}{2}\right) d(v\omega(v-1)).$$

By [13, (6.23), p. 562] and [13, Theorems III.5.7 & III.6.6], we have, for all  $v \ge 3$ , that

$$\frac{d}{dv}(v\omega(v-1)) = \omega(v-2) + \omega'(v-1) \ge \frac{1}{2} - \rho(v-1) \ge \frac{1}{2} - \rho(2) = \log 2 - \frac{1}{2},$$
$$\frac{d}{dv}(v\omega(v-1)) \le 1 + \rho(v-1) \le 1 + \rho(2) = 2 - \log 2,$$

where  $\rho$  is the Dickman-de Bruijn function defined to be the unique continuous solution to the delay differential equation  $t\rho'(t) + \rho(t-1) = 0$  for  $t \ge 1$ , subject to the initial value condition  $\rho(t) = 1$  for  $0 \le t \le 1$ . Moreover, we have

$$\lim_{v \to 3^{-}} \frac{d}{dv} (v\omega(v-1)) = \lim_{v \to 3^{-}} (\omega(v-2) + \omega'(v-1)) = -\frac{1}{4}.$$

It follows by (2.1) that

$$\int_{3^{-}}^{u} \left( G(v) - \log \frac{v}{2} \right) d(v\omega(v-1)) \le \frac{c_1}{\log^2 y} \left( u\omega(u-1) - 3\omega(2) \right).$$

Thus we have

$$\int_{3^{-}}^{u} v\omega(v-1) d\left(G(v) - \log \frac{v}{2}\right) \ge -\frac{2c_1 u\omega(u-1)}{\log^2 y} \ge -\frac{2c_1 M_0 u}{\log^2 y},$$

where  $M_0 = 0.5671432...$  Hence we have shown that

$$\sum_{y$$

Combining (2.9), (2.10) and (2.11), we deduce that

$$\Delta(x,y) \ge \frac{9\Delta_3^-}{u^2} + \frac{\Delta_k^-}{2} - \frac{1}{\log y} \left( 2c_1 M_0 - \left(\frac{c_2}{3} - 1\right) \Delta_k^- \right)$$

for  $k \leq u < k+1$ . Therefore,  $\Delta_{k+1}^- \geq \min(\Delta_k^-, a_k^-)$  for all  $k \geq 3$ , where

$$a_k^- := \frac{9\Delta_3^-}{k^2} + \frac{\Delta_k^-}{2} - \frac{1}{\log y_1} \cdot \max\left(2c_1M_0 - \left(\frac{c_2}{3} - 1\right)\Delta_k^-, 0\right).$$

Consequently, we have  $\Delta_4^- > -0.451835$  and  $\Delta_k^- > -0.480075$  for all  $k \ge 5$ . Suppose now that  $602 \le y \le y_1$ . By [11, Theorem 20] we can replace (2.1) with

$$\left| G(v) - \log \frac{v}{2} \right| \le \frac{d_1}{\sqrt{y} \log y},$$

where  $d_1 = 2$ . Moreover, (2.2) remains true if we replace  $c_2$  and  $c_3$  by  $d_2 = 1.2762$  and  $d_3 = 1$ , respectively, according to [5, Corollary 5.2]. With these changes, we run the same argument used to handle the case  $y \ge y_1$  and get

$$\Delta(x,y) > g(u) - \frac{2d_1}{u\sqrt{y}} + \frac{\log y}{uy^{3/2}} - \frac{1}{u^2} \left( 2 - d_3 + \frac{4d_1d_3}{\sqrt{y}\log y} + \frac{8d_2}{u\log y} + \frac{8d_2^2}{u^2\log^2 y} \right).$$

when  $2 \le u \le 3$  and

$$\Delta(x,y) \ge \frac{9\Delta_3^-}{u^2} + \frac{\Delta_k^-}{2} - \frac{1}{\log y} \left( \frac{2d_1 M_0 \log y}{\sqrt{y}} - \left( \frac{d_2}{3} - 1 \right) \Delta_k^- \right)$$

when  $3 \le k \le u < k+1$ , so that we can take

$$a_k^- = \frac{9\Delta_3^-}{k^2} + \frac{\Delta_k^-}{2} - \frac{1}{\log y_0} \cdot \max\left(\frac{2d_1 M_0 \log y_0}{\sqrt{y_0}} - \left(\frac{d_2}{3} - 1\right)\Delta_k^-, 0\right).$$

As a consequence, we have  $\Delta_3^- > -0.563528$ ,  $\Delta_4^- > -0.887161$  and  $\Delta_k^- > -0.955421$  for all  $k \ge 5$ . This completes the proof of the proposition.

The next result provides a numerical lower bound for  $\omega(u)$  on  $[3, \infty)$ .

**Lemma 2.2.** We have  $\omega(u) > 0.549307$  for all  $u \ge 3$ .

*Proof.* Consider first the case  $u \in [3,4]$ . Since  $(t\omega(t))' = \omega(t-1)$  for  $t \geq 2$  and  $\omega(t) = (\log(t-1)+1)/t$  for  $t \in [2,3]$ , we have

$$\omega(u) = \frac{1}{u} \left( \log 2 + 1 + \int_3^u \frac{\log(t-2) + 1}{t-1} dt \right)$$

for  $u \in [3, 4]$ . Note that  $u\omega'(u) = \omega(u - 1) - \omega(u) = S(u)/u$ , where

$$S(u) := \frac{u(\log(u-2)+1)}{u-1} - \log 2 - 1 - \int_3^u \frac{\log(t-2)+1}{t-1} dt.$$

Since

$$S'(u) = \frac{1}{u-1} \left( \log(u-2) + 1 + \frac{u}{u-2} - \frac{u(\log(u-2)+1)}{u-1} - (\log(u-2)+1) \right)$$
$$= \frac{u(1-(u-2)\log(u-2))}{(u-2)(u-1)^2},$$

we know that S(u) is strictly increasing on  $[3, u_1]$  and strictly decreasing on  $[u_1, 4]$ , where  $u_1 = 3.7632228...$  is the unique solution to the equation  $(u - 2) \log(u - 2) = 1$ . But  $S(3) = 1/2 - \log 2 < 0$  and

$$S(4) = \frac{\log 2 + 1}{3} - \int_{3}^{4} \frac{\log(t - 2) + 1}{t - 1} dt > 0.$$

It follows that S(u) has a unique zero  $u_2 \in [3,4]$ . The numerical value of  $u_2$  is given by  $u_2 = 3.4697488...$ , according to Mathematica. Hence S(u) < 0 for  $u \in [3, u_2)$  and S(u) > 0 for  $u \in (u_2, 4]$ . The same is true for  $\omega'(u)$ , which implies that  $\omega(u)$  is strictly decreasing on  $[3, u_2]$  and strictly increasing on  $[u_2, 4]$ . Thus,  $\omega(u) \ge \omega(u_2) = 0.5608228...$  for  $u \in [3, 4]$ .

Consider now the case  $u \in [4, \infty)$ . It is known [8] that  $\omega(t)$  satisfies

$$|\omega(t) - e^{-\gamma}| \le \frac{\rho(t-1)}{t}$$

for all  $t \ge 1$ . Since  $\rho(t)$  is strictly decreasing on  $[4, \infty)$ , we have  $\omega(u) \ge e^{-\gamma} - \rho(3)/4$  for all  $u \ge 4$ . To find the value of  $\rho(3)$ , we use  $t\rho'(t) + \rho(t-1) = 0$  for  $t \ge 1$  and  $\rho(t) = 1 - \log t$  for  $t \in [1, 2]$  to obtain

$$\rho(u) = 1 - \log 2 - \int_2^u \frac{1 - \log(t - 1)}{t} dt$$

for  $u \in [2,3]$ . It follows that

$$\omega(u) \ge e^{-\gamma} - \frac{1}{4} \left( 1 - \log 2 - \int_2^3 \frac{1 - \log(t - 1)}{t} dt \right) = 0.5493073...$$

for all  $u \ge 4$ . We have therefore shown that  $\omega(u) > 0.549307$  for all  $u \ge 3$ .

We are now ready to prove the following clean lower bound for  $\Phi(x,y)$  that we alluded to.

**Theorem 2.3.** We have  $\Phi(x,y) > 0.4x/\log y$  uniformly for all  $7 \le y \le x^{2/3}$ .

*Proof.* In the range  $\max(7, x^{2/5}) \le y \le x^{2/3}$ , we have trivially  $\Phi(x, y) \ge \pi(x) - \pi(y) + 1$ . By [5, Corollary 5.2] we have

$$\pi(x) - \pi(y) \ge \frac{x}{\log x} \left( 1 + \frac{1}{\log x} \right) - \frac{y}{\log y} \left( 1 + \frac{1.2762}{\log y} \right)$$

$$= \left( \frac{1}{u} \left( 1 + \frac{1}{\log x} \right) - \frac{y}{x} \left( 1 + \frac{1.2762u}{\log x} \right) \right) \frac{x}{\log y}$$

$$> \left( \frac{2}{5} \left( 1 + \frac{1}{\log x} \right) - \frac{1}{x^{1/3}} \left( 1 + \frac{3.1905}{\log x} \right) \right) \frac{x}{\log y} > 0.4 \frac{x}{\log y}$$

whenever  $x \ge 41,217$ . Furthermore, we have verified  $\Phi(x,y) > 0.4x/\log y$  for  $\max(7,x^{2/5}) \le y \le x^{2/3}$  with  $x \le 41,217$  using Mathematica. Hence,  $\Phi(x,y) > 0.4x/\log y$  holds in the range  $\max(7,x^{2/5}) \le y \le x^{2/3}$ .

Consider now the case  $\max(x^{1/3},7) \leq y \leq x^{2/5}$ . Following the proof of Proposition 2.1, we have

$$\Phi(x,y) = \pi(x) - \pi(y) + 1 + \sum_{y 
$$= \pi(x) - M(x,y) + \sum_{y$$$$

where

$$M(x,y) := \frac{1}{2}\pi \left(\sqrt{x}\right)^2 - \frac{1}{2}\pi \left(\sqrt{x}\right) - \frac{1}{2}\pi(y)^2 + \frac{3}{2}\pi(y) - 1.$$

To handle the sum in (2.12), we appeal to Theorem 5 and its corollary from [11] to arrive at

$$G(v) - \log \frac{v}{2} > -\frac{1}{2\log^2 \sqrt{x}} - \frac{1}{\log^2 y} \ge -\frac{33}{25\log^2 y}$$

in the range  $\max(x^{1/3},7) \leq y \leq x^{2/5}$ . By [11, Corollary 1] we have

$$\sum_{y x \sum_{y$$

provided that  $x \ge 289$ . The right-hand side of the above can be estimated in the same way as in the proof of Proposition 2.1, so we obtain

$$\sum_{u \frac{x}{\log y} \left( \omega(u) - \frac{66}{25u \log^2 y} \right) - \frac{x}{\log x}.$$

On the other hand, we see by [5, Corollary 5.2] and [11, Corollary 2] that

$$\pi(x) - M(x, y) > \pi(x) - \frac{1}{2}\pi \left(\sqrt{x}\right)^2 \ge \frac{x}{\log x} \left(1 + \frac{1}{\log x}\right) - \frac{25x}{8\log^2 x} = \frac{x}{\log x} - \frac{17x}{8\log^2 x}.$$

for  $x \ge 114^2$ . Collecting the estimates above and using the inequality  $\omega(u) \ge \omega(5/2) = 2(\ln(3/2) + 1)/5$  for  $u \in [5/2, 3]$ , we find that

$$\Phi(x,y) > \frac{\omega(5/2)x}{\log y} - \frac{17x}{8\log^2 x} - \frac{66x}{25u\log^3 y} \ge \frac{\omega(5/2)x}{\log y} - \frac{17x}{50\log^2 y} - \frac{132x}{125\log^3 y} > 0.4 \frac{x}{\log y}$$

for all  $\max(46, x^{1/3}) \le y \le x^{2/5}$ . For  $x^{1/3} \le y \le x^{2/5}$  with  $7 \le y \le 46$ , we have verified the inequality  $\Phi(x, y) > 0.4x/\log y$  directly through numerical computation.

Next, we consider the range  $7 \le y < x^{1/3}$ . By Proposition 2.1 and Lemma 2.2 we have

$$\Phi(x,y) > \frac{x}{\log y} \left( 0.549307 - \frac{0.955421}{\log y} \right) > 0.4 \frac{x}{\log y},$$

provided that  $y \ge 602$ . To deal with the range  $7 \le y \le \min(x^{1/3}, 602)$ , we follow the inclusion-exclusion technique used in [7, Section 3]. For any integer  $n \ge 1$ , let  $\nu(n)$  denote the number of distinct prime factors of n. We start by "pre-sieving" with the primes 2, 3, and 5: for any  $x \ge 1$  the number of integers  $n \le x$  with  $\gcd(n, 30) = 1$  is  $(4/15)x + r_x$ ,

where  $|r_x| \le 14/15$ . Let  $P_5(y)$  be the product of the primes in (5, y]. Then we have by the Bonferroni inequalities that

$$\Phi(x,y) \ge \sum_{\substack{d \mid P_5(y) \\ \nu(d) \le 3}} \mu(d) \left( \frac{4}{15} \cdot \frac{x}{d} + r_{x/d} \right) \ge a(y)x - b(y),$$

where

$$a(y) := \frac{4}{15} \sum_{\substack{d \mid P_5(y) \\ \nu(d) \le 3}} \frac{\mu(d)}{d} = \frac{4}{15} \sum_{j=0}^{3} (-1)^j \sum_{\substack{d \mid P_5(y) \\ \nu(d) = j}} \frac{1}{d},$$

$$b(y) := \frac{14}{15} \sum_{i=0}^{3} {\pi(y) - 3 \choose j}.$$

By Newton's identities, the inner sum in the definition of a(y) can be represented in terms of the power sums of 1/p over all primes  $5 . Thus, we have <math>\Phi(x,y) > 0.4x/\log y$  whenever  $a(y) > 0.4/\log y$  and  $x > b(y)/(a(y) - 0.4/\log y)$ . Using Mathematica, we find that the inequality  $\Phi(x,y) > 0.4x/\log y$  holds for  $7 \le y \le 602$  and  $x \ge 13,160,748$ . Finally, we have verified the inequality  $\Phi(x,y) > 0.4x/\log y$  directly for  $7 \le y \le x^{1/3}$  with  $x \le 13,160,748$  by numerical calculations, completing the proof of our theorem.

Remark 2.1. Note that for  $y \in [5,7)$  we have

$$\Phi(x,y) \ge \frac{4}{15}x - \frac{14}{15} > 0.4 \frac{x}{\log 5} \ge 0.4 \frac{x}{\log y},$$

provided that  $x \ge 52$ . Combined with Theorem 2.3 and numerical examination of the case  $11 \le x \le 52$ , this implies that the inequality  $\Phi(x, y) > 0.4x/\log y$  holds in the slightly larger range  $5 \le y \le x^{2/3}$  if one assumes  $x \ge 41$ .

#### 3. An Explicit Version of de Bruijn's Estimate

To prove Theorem 1.1, we shall first develop an explicit version of (1.6) with a general R(y), following [4], where R(y) is a positive decreasing function satisfying the same conditions described in the introduction. Suppose that  $y_0 \ge 3$ . For each  $z \ge 2$ , put

$$Q(z) := \prod_{p \le z} \left( 1 - \frac{1}{p} \right).$$

We start by estimating Q(y) for  $y \geq y_0$ . Using a Stieltjes integral, we may write

$$\log \frac{Q(z)}{Q(y)} = \int_{y}^{z} \log (1 - t^{-1}) d \operatorname{li}(y) + \int_{y}^{z} \log (1 - t^{-1}) d(\pi(y) - \operatorname{li}(t)), \tag{3.1}$$

where  $z \geq y \geq y_0$ . The first integral on the right-hand side of the above is equal to

$$\int_{y}^{z} \log (1 - t^{-1}) \frac{dt}{\log t} = -\log \frac{\log z}{\log y} + \int_{y}^{z} (t^{-1} + \log (1 - t^{-1})) \frac{dt}{\log t}.$$

Since

$$-\frac{1}{2t(t-1)} < t^{-1} + \log(1 - t^{-1}) < 0$$

for all  $t \geq y_0$ , we have

$$-\frac{1}{2} \int_{y}^{\infty} \frac{dt}{t(t-1)\log t} < \int_{y}^{z} \left(t^{-1} + \log\left(1 - t^{-1}\right)\right) \frac{dt}{\log t} < 0.$$

But a change of variable shows that

$$\int_{y}^{\infty} \frac{dt}{t(t-1)\log t} = \int_{1}^{\infty} \frac{dt}{t(y^{t}-1)} \le \frac{1}{y-1} \int_{1}^{\infty} \frac{dt}{t^{2}} = \frac{1}{y-1},$$

where we have used the inequality  $y^t - 1 \ge (y - 1)t$  for  $t \ge 1$  and  $y \ge y_0$ . It follows that

$$-\frac{1}{2(y-1)} \le \int_{y}^{z} \log(1-t^{-1}) d \operatorname{li}(y) + \log \frac{\log z}{\log y} < 0.$$
 (3.2)

Now we estimate the second integral on the right-hand side of (3.1). By (1.4) and partial integration we have

$$\left| \int_{y}^{z} \log \left( 1 - t^{-1} \right) \, d(\pi(y) - \operatorname{li}(t)) \right| \le \log \left( 1 - y^{-1} \right)^{-1} \frac{y}{\log y} R(y) + \log \left( 1 - z^{-1} \right)^{-1} \frac{z}{\log z} R(z) + \int_{y}^{z} \frac{|\pi(t) - \operatorname{li}(t)|}{t(t-1)} \, dt.$$

Using (1.5) we see that

$$\int_{y}^{z} \frac{|\pi(t) - \operatorname{li}(t)|}{t(t-1)} dt \le \frac{C_{0}(y_{0})y_{0}}{y_{0} - 1} R(y).$$

It is clear that the function

$$\log (1 - t^{-1})^{-1} \frac{t}{\log t} = \frac{1}{\log t} \sum_{n=0}^{\infty} \frac{t^{-n}}{n+1}$$

is strictly decreasing for  $t \in (1, \infty)$ . Since R(t) is decreasing on  $[y_0, \infty)$ , we find that

$$\left| \int_{y}^{z} \log \left( 1 - t^{-1} \right) \, d(\pi(y) - \operatorname{li}(t)) \right| \le \left( 2 \log \left( 1 - y_{0}^{-1} \right)^{-1} \frac{y_{0}}{\log y_{0}} + \frac{C_{0}(y_{0})y_{0}}{y_{0} - 1} \right) R(y).$$

Combining this inequality with (3.1) and (3.2) yields

$$-C_2(y_0)R(y) \le \log \frac{Q(z)}{Q(y)} + \log \frac{\log z}{\log y} \le C_1(y_0)R(y)$$
(3.3)

for  $z \geq y \geq y_0$ , where

$$C_1(y_0) = 2\log\left(1 - y_0^{-1}\right)^{-1} \frac{y_0}{\log y_0} + \frac{C_0(y_0)y_0}{y_0 - 1},$$

$$C_2(y_0) = C_1(y_0) + \sup_{t \ge y_0} \frac{1}{2(t - 1)R(t)}.$$

Exponentiating (3.3) we obtain

$$-C_4(y_0)R(y) \le \frac{Q(z)\log z}{Q(y)\log y} - 1 \le C_3(y_0)R(y)$$
(3.4)

for  $z \geq y \geq y_0$ , where

$$C_3(y_0) = \sup_{t \ge y_0} \frac{\exp(C_1(y_0)R(t)) - 1}{R(t)} = \frac{\exp(C_1(y_0)R(y_0)) - 1}{R(y_0)},$$

$$C_4(y_0) = \sup_{t \ge y_0} \frac{1 - \exp(-C_2(y_0)R(t))}{R(t)} = C_2(y_0).$$

As a consequence, we have by letting  $z \to \infty$  in (3.4) and using the fact that  $Q(z) \log z \to e^{-\gamma}$  as  $z \to \infty$ , that

$$e^{\gamma} \log y (1 - C_4(y_0)R(y)) \le \frac{1}{Q(y)} \le e^{\gamma} \log y (1 + C_3(y_0)R(y)).$$
 (3.5)

Similarly, we derive from (3.3) that

$$\frac{e^{-\gamma}}{\log y}(1 - C_6(y_0)R(y)) \le Q(y) \le \frac{e^{-\gamma}}{\log y}(1 + C_5(y_0)R(y))$$
(3.6)

for  $y \geq y_0$ , where

$$C_5(y_0) = \sup_{t \ge y_0} \frac{\exp(C_2(y_0)R(t)) - 1}{R(t)} = \frac{\exp(C_2(y_0)R(y_0)) - 1}{R(y_0)},$$

$$C_6(y_0) = \sup_{t \ge y_0} \frac{1 - \exp(-C_1(y_0)R(t))}{R(t)} = C_1(y_0).$$

For  $x \geq y \geq 2$ , we define

$$\psi(x,y) := \frac{\Phi(x,y)}{xQ(y)}.$$

We then need to estimate  $\eta(x,y) = \psi(x,y) - \lambda(x,y)$ , where  $\lambda(x,y) := e^{\gamma} \mu_y(u) \log y$ . For  $1 \le u \le 2$  this can be done straightforward. Indeed, we have  $\Phi(x,y) = \pi(x) - \pi(y) + 1$  and  $\omega(u) = 1/u$  when  $1 \le u \le 2$ , so that

$$\eta(x,y) = \frac{\pi(x) - \pi(y) + 1}{xQ(y)} - e^{\gamma} \log y \int_{1}^{u} t^{-1} y^{t-u} dt.$$

Note that

$$\left| \pi(x) - \pi(y) - x \int_1^u t^{-1} y^{t-u} dt \right| = \left| \pi(x) - \pi(y) - \int_y^x \frac{dt}{\log t} \right| \le \left( \frac{x}{\log x} + \frac{y}{\log y} \right) R(y).$$

From (3.5) it follows that  $|\eta(x,y)| \le e^{\gamma} \alpha_y(u) R(y)$  for  $y \ge y_0$  and  $u \in [1,2]$ , where

$$\alpha_y(u) := \frac{\log y}{y^u R(y)} + C_3(y_0) \left( \frac{\log y}{y^u} + \log y \int_1^u t^{-1} y^{t-u} dt \right) + \left( 1 + C_3(y_0) R(y) \right) \left( \frac{1}{u} + y^{1-u} \right).$$

Integration by parts enables us to write

$$\log y \int_{1}^{u} t^{-1} y^{t-u} dt = \frac{1}{u} - y^{1-u} + \int_{1}^{u} t^{-2} y^{t-u} dt$$

for  $y \geq y_0$ . Hence  $|\eta(x,y)| \leq e^{\gamma} \eta_1(y) R(y)$  for  $y \geq y_0$  and  $u \in [1,2]$ , where

$$\eta_1(y) := \sup_{t \ge y} \frac{\log t}{tR(t)} + \max_{u \in [1,2]} \left( C_3(y_0) I_y(u) + (1 + C_3(y_0) R(y)) \left( \frac{1}{u} + y^{1-u} \right) \right) \tag{3.7}$$

with

$$I_y(u) := \frac{1}{u} + \int_1^u t^{-2} y^{t-u} dt.$$

We remark that  $I_y(u)$  is strictly decreasing on [1, 2] and hence satisfies  $I_y(u) < 1$  for  $u \in (1, 2]$ , since its derivative is

$$I_y'(u) = -\int_1^u t^{-2} y^{t-u} \log y \, dt < 0.$$

Thus, (3.7) simplifies to

$$\eta_1(y) = \sup_{t \ge y} \frac{\log t}{tR(t)} + C_3(y_0) + 2(1 + C_3(y_0)R(y)). \tag{3.8}$$

Suppose now that  $y \geq y_0$  and  $u \geq 2$ . From (2.6) it follows that

$$\psi(x,y) = \psi(x,z) \frac{Q(z)}{Q(y)} + \sum_{y$$

where  $z \ge y \ge y_0$ . Put  $h := \log z/\log y \ge 1$  and

$$H_y(v) := \sum_{y (3.10)$$

for  $v \ge 1$ . Then we have  $H_y(v) = 1 - Q(y^v)/Q(y)$ . By partial summation, we see that (3.9) becomes

$$\psi(x,y) = \psi(y^u, y^h)(1 - H_y(h)) + \int_1^h \psi(y^{u-v}, (y^v)^-) dH_y(v).$$
(3.11)

By (3.4) we have

$$|H_y(v) - 1 + v^{-1}| \le C_7(y_0)R(y)$$

where  $C_7(y_0) := \max(C_3(y_0), C_4(y_0))$ . Thus, one can think of  $1 - v^{-1}$  as a smooth approximation to  $H_y(v)$ . Since we also expect  $\lambda(x, y)$  to be a smooth approximation to  $\psi(x, y)$ , in view of (3.11) it is reasonable to expect

$$E_1(h; y, u) := \lambda(y^u, y) - \lambda(y^u, y^h)h^{-1} - \int_1^h \lambda(y^{u-v}, y^v)v^{-2} dv$$

to be small in size as a function of y. This can be easily verified when  $1 \le h \le u/2$ . Following de Bruijn [4], we have

$$\frac{\partial}{\partial h}E_1(h;y,u) = -h^{-1} \cdot \frac{\partial}{\partial h}\lambda(y^u,y^h) + h^{-2}\lambda(y^u,y) - h^{-2}\lambda(y^{u-h},y^h). \tag{3.12}$$

Since

$$\frac{\lambda(y^u, y^h)}{e^{\gamma} \log u} = h \int_1^{u/h} y^{ht-u} \omega(t) dt,$$

we find

$$\frac{\partial}{\partial h} \left( \frac{\lambda(y^u, y^h)}{e^{\gamma} \log y} \right) = \int_1^{u/h} y^{ht-u} \omega(t) dt + h \left( \log y \int_1^{u/h} y^{ht-u} (t\omega(t)) dt - uh^{-2} \omega(uh^{-1}) \right).$$

Recall that  $(t\omega(t))' = \omega(t-1)$  for  $t \in \mathbb{R}$  with the obvious extension  $\omega(t) = 0$  for t < 1. It follows that

$$\log y \int_{1}^{u/h} y^{ht-u}(t\omega(t)) dt = h^{-1}y^{ht-u}(t\omega(t)) \Big|_{1}^{u/h} - h^{-1} \int_{1}^{u/h} y^{ht-u}\omega(t-1) dt$$

$$= uh^{-2}\omega(uh^{-1}) - h^{-1}y^{h-u} - h^{-1}y^{h} \int_{1}^{u/h-1} y^{ht-u}\omega(t) dt$$

$$= uh^{-2}\omega(uh^{-1}) - h^{-1}y^{h-u} - \left(h^{2}e^{\gamma}\log y\right)^{-1} \lambda(y^{u-h}, y^{h}).$$

Hence we have

$$\frac{\partial}{\partial h}\lambda(y^u, y^h) = e^{\gamma}\log y \left( \int_1^{u/h} y^{ht-u}\omega(t) dt - y^{h-u} \right) - h^{-1}\lambda(y^{u-h}, y^h)$$
$$= h^{-1}\lambda(y^u, y^h) - e^{\gamma}y^{h-u}\log y - h^{-1}\lambda(y^{u-h}, y^h).$$

Inserting this in (3.12) yields

$$\frac{\partial}{\partial h} E_1(h; y, u) = h^{-1} e^{\gamma} y^{h-u} \log y.$$

Integrating both sides with respect to h and using the initial value condition  $E_1(1; y, u) = 0$ , we obtain

$$E_1(h; y, u) = e^{\gamma} \log y \int_1^h t^{-1} y^{t-u} dt < e^{\gamma} y^{h-u}.$$
 (3.13)

In what follows, we shall always suppose that  $1 \le h \le u/2$ . Following de Bruijn [4], we proceed to show that

$$E_3(h; y, u) := \lambda(y^u, y) - \lambda(y^u, y^h)(1 - H(h)) - \int_1^h \lambda(y^{u-v}, y^v) dH(h)$$

is small in size as a function of y. This is intuitive, because

$$\lambda(y^u, y^h)h^{-1} - \int_1^h \lambda(y^{u-v}, y^v)v^{-2} dv,$$

which is a good approximation to  $\lambda(y^u, y)$  as we have already demonstrated, can be thought of as a smooth approximation to

$$\lambda(y^{u}, y^{h})(1 - H(h)) - \int_{1}^{h} \lambda(y^{u-v}, y^{v}) dH(h).$$

Moreover, we have by (3.11) that

$$\eta(x,y) = \eta(y^u, y^h)(1 - H_y(h)) + \int_1^h \eta(y^{u-v}, (y^v)^-) dH_y(v) - E_3(h; y, u), \tag{3.14}$$

which will later be used to estimate  $\eta(x,y)$ . To estimate  $E_3(h;y,u)$ , let us write  $E_3(h;y,u) = E_1(h;y,u) + E_2(h;y,u)$ , where

$$E_2(h; y, u) := -\int_1^h \lambda(y^{u-v}, y^v) d\left(H(v) - 1 + v^{-1}\right) + (H(h) - 1 + h^{-1})\lambda(y^u, y^h).$$

Then we expect  $E_2(h; y, u)$  to be small in size as a function of y. Using (3.10) and the observation that H(1) = 0, we have

$$|E_2(h;y,u)| \le \left( \left| \lambda(y^u, y^h) - \lambda(y^{u-h}, y^h) \right| + \int_1^h \left| \frac{\partial}{\partial v} \lambda(y^{u-v}, y^v) \right| dv \right) C_7(y_0) R(y). \tag{3.15}$$

Note that

$$\begin{split} \frac{\lambda(y^u, y^h) - \lambda(y^{u-h}, y^h)}{he^{\gamma} \log y} &= \int_1^{u/h} y^{ht-u} \omega(t) \, dt - \int_2^{u/h} y^{ht-u} \omega(t-1) \, dt \\ &= \int_1^2 y^{ht-u} \omega(t) \, dt + \int_2^{u/h} y^{ht-u} (\omega(t) - \omega(t-1)) \, dt \\ &= \int_1^2 t^{-1} y^{ht-u} \, dt - \int_2^{u/h} y^{ht-u} t \omega'(t) \, dt. \end{split}$$

By Theorems III.5.7 and III.6.6 in [13] we have

$$|\omega'(t)| \le \rho(t) \le \frac{1}{\Gamma(t+1)} \tag{3.16}$$

for all  $t \geq 1$ . It follows that

$$\left| \lambda(y^u, y^h) - \lambda(y^{u-h}, y^h) \right| \le he^{\gamma} \log y \left( \int_1^2 t^{-1} y^{ht-u} dt + \int_2^{u/h} y^{ht-u} t \rho(t) dt \right).$$
 (3.17)

This inequality will later be used in conjunction with the formulas

$$h\log y \int_{1}^{2} t^{-1} y^{ht-u} dt = \frac{y^{2h-u}}{2} - y^{h-u} + \int_{1}^{2} t^{-2} y^{ht-u} dt$$
 (3.18)

and

$$h \log y \int_{2}^{u/h} y^{ht-u} t \rho(t) dt = uh^{-1} \rho(uh^{-1}) - 2\rho(2)y^{2h-u} - \int_{2}^{u/h} y^{ht-u} (t\rho(t))' dt$$

$$\leq uh^{-1} \rho(uh^{-1}) - 2\rho(2)y^{2h-u} + \int_{2}^{u/h} y^{ht-u} \rho(t-1) dt.$$
 (3.19)

On the other hand, we have

$$\frac{\lambda(y^{u-v}, y^v)}{e^{\gamma} \log y} = v \int_2^{u/v} y^{vt-u} \omega(t-1) dt,$$

which implies that

$$\frac{\partial}{\partial v} \left( \frac{\lambda(y^{u-v}, y^v)}{e^{\gamma} \log y} \right) = \int_2^{u/v} y^{vt-u} (1 + tv \log y) \omega(t-1) dt - uv^{-1} \omega(uv^{-1} - 1).$$

By partial integration, the right side of the above is easily seen to be

$$-2y^{2v-u} - \int_{2}^{u/v} y^{vt-u} t\omega'(t-1) dt.$$

Hence, we arrive at

$$\int_1^h \left| \frac{\partial}{\partial v} \lambda(y^{u-v}, y^v) \right| dv \le e^{\gamma} \log y \left( 2 \int_1^h y^{2v-u} dv + \int_1^h \int_2^{u/v} y^{vt-u} t |\omega'(t-1)| dt dv \right).$$

Furthermore, we have by Fubini's theorem that

$$\int_1^h \int_2^{u/v} y^{vt-u} t |\omega'(t-1)| \, dt dv = \int_2^{u/h} \int_1^h y^{vt-u} t |\omega'(t-1)| \, dv \, dt + \int_{u/h}^u \int_1^{u/t} y^{vt-u} t |\omega'(t-1)| \, dv \, dt,$$

the right side of which is easily seen to be

$$\frac{1}{\log y} \left( \int_{2}^{u/h} y^{ht-u} |\omega'(t-1)| \, dt + \int_{u/h}^{u} |\omega'(t-1)| \, dt - \int_{2}^{u} y^{t-u} |\omega'(t-1)| \, dt \right).$$

It follows that

$$\int_{1}^{h} \left| \frac{\partial}{\partial v} \lambda(y^{u-v}, y^{v}) \right| dv < e^{\gamma} \left( y^{2h-u} + \int_{2}^{u/h} y^{ht-u} |\omega'(t-1)| dt + \int_{u/h}^{u} |\omega'(t-1)| dt \right). \tag{3.20}$$

This estimate together with (3.17) will lead us to a good estimate for  $E_2(h; y, u)$ .

Now we derive estimates for  $E_3(h; y, u)$  that suit our needs. Suppose that  $k \le u < k + 1$  and take  $h = h_k = u/k$ , where  $k \ge 2$  is a positive integer. We first consider the case k = 2. In view of (3.18), we see that (3.17) simplifies to

$$\left|\lambda\left(y^{u},y^{h_{2}}\right)-\lambda\left(y^{u-h_{2}},y^{h_{2}}\right)\right| < e^{\gamma}\left(\frac{1}{2}+\int_{1}^{2}t^{-2}y_{0}^{t-2}\,dt\right) = e^{\gamma}I_{y_{0}}(2)$$

for  $y \geq y_0$ . By (3.20) we have

$$\int_{1}^{h_2} \left| \frac{\partial}{\partial v} \lambda(y^{u-v}, y^v) \right| dv \le e^{\gamma} \left( 1 + \int_{2}^{3} |\omega'(t-1)| dt \right) = \frac{3e^{\gamma}}{2},$$

since  $\omega'(t) = -1/t^2$  for  $t \in [1,2)$ . Combining these estimates with (3.13) and (3.15), we obtain  $E_3(h_2; y, u) \le e^{\gamma} \xi_2(y_0) R(y)$  for  $y \ge y_0$  and  $z \le u < 3$ , where

$$\xi_2(y_0) := \max_{t \ge y_0} \frac{1}{tR(t)} + C_7(y_0) \left( I_{y_0}(2) + \frac{3}{2} \right).$$

Now we handle the case  $k \geq 3$ . From (3.16)–(3.19) it follows that

$$\left| \lambda \left( y^{u}, y^{h_{k}} \right) - \lambda \left( y^{u - h_{k}}, y^{h_{k}} \right) \right| < e^{\gamma} \left( \frac{1}{\Gamma(k)} + \left( 2 \log 2 - \frac{3}{2} \right) y^{2 - k} + \int_{1}^{2} t^{-2} y^{t - k} dt + \int_{2}^{3} y^{t - k} (1 - \log(t - 1)) dt + \int_{3}^{k} y^{t - k} \frac{dt}{\Gamma(t)} \right),$$

where we have used the fact that  $\rho(t) = 1 - \log t$  for  $t \in [1, 2]$ . By (3.16) and (3.20) we have

$$\int_1^{h_k} \left| \frac{\partial}{\partial v} \lambda(y^{u-v}, y^v) \right| dv \le e^{\gamma} \left( y^{2-k} + \int_2^3 y^{t-k} \frac{dt}{(t-1)^2} + \int_3^k y^{t-k} \frac{dt}{\Gamma(t)} + \int_k^{k+1} \frac{dt}{\Gamma(t)} \right).$$

Together with (3.13) and (3.15), these inequalities imply that  $E_3(h_k; y, u) \leq e^{\gamma} \xi_k(y_0) R(y)$  for  $y \geq y_0$  and  $3 \leq k \leq u < k+1$ , where

$$\xi_k(y_0) := \left(\max_{t \ge y_0} \frac{1}{tR(t)}\right) y_0^{2-k} + C_7(y_0) \left(\frac{1}{(k-1)!} + \int_k^{k+1} \frac{dt}{\Gamma(t)} + \left(2\log 2 - \frac{1}{2}\right) y_0^{2-k} + \int_1^2 t^{-2} y_0^{t-k} dt + \int_2^3 y_0^{t-k} \left(1 - \log(t-1) + \frac{1}{(t-1)^2}\right) dt + 2\int_3^k y_0^{t-k} \frac{dt}{\Gamma(t)}\right).$$

As a direct corollary, we obtain

$$\sum_{k=2}^{\infty} \xi_k(y_0) = \frac{y_0}{y_0 - 1} \max_{t \ge y_0} \frac{1}{tR(t)} + C_7(y_0) \left( e - \frac{1}{2} + \int_3^{\infty} \frac{dt}{\Gamma(t)} + \frac{1}{y_0 - 1} \left( 2\log 2 - 1 + y_0 I_{y_0}(2) + \int_2^3 y_0^{t-2} \left( 1 - \log(t - 1) + \frac{1}{(t - 1)^2} \right) dt + 2 \int_3^{\infty} y_0^{t} \frac{dt}{\Gamma(t)} \right) \right),$$

where we have applied partial summation to derive

$$\begin{split} \sum_{k=3}^{\infty} \int_{3}^{k} y_{0}^{t-k} \frac{dt}{\Gamma(t)} &= \left(\sum_{k=3}^{\infty} y_{0}^{-k}\right) \int_{3}^{\infty} y_{0}^{t} \frac{dt}{\Gamma(t)} - \int_{3}^{\infty} \left(\sum_{3 \leq k \leq t} y_{0}^{-k}\right) y_{0}^{t} \frac{dt}{\Gamma(t)} \\ &= \frac{y_{0}^{-3}}{1 - y_{0}^{-1}} \int_{3}^{\infty} y_{0}^{t} \frac{dt}{\Gamma(t)} - \int_{3}^{\infty} \frac{y_{0}^{t-3} (1 - y_{0}^{-\lfloor t \rfloor + 2})}{1 - y_{0}^{-1}} \cdot \frac{dt}{\Gamma(t)} \\ &= \frac{1}{y_{0} - 1} \int_{3}^{\infty} y_{0}^{\{t\}} \frac{dt}{\Gamma(t)}. \end{split}$$

For computational purposes, we can transform the last integral above by observing that

$$\int_{3}^{\infty} y_0^{\{t\}} \frac{dt}{\Gamma(t)} = \int_{0}^{1} \left( \sum_{n=0}^{\infty} \frac{1}{(t+2)\cdots(t+2+n)} \right) y_0^t \frac{dt}{\Gamma(t+2)}.$$

Let

$$\gamma(s,z) := \int_0^z v^{s-1} e^{-v} dv$$

be the lower incomplete gamma function, where  $s \in \mathbb{C}$  with  $\Re(s) > 0$  and  $z \geq 0$ . It is well known that

$$\gamma(s,z) = z^s e^{-z} \sum_{n=0}^{\infty} \frac{z^n}{s(s+1)\cdots(s+n)},$$

from which it follows that

$$\sum_{n=0}^{\infty} \frac{1}{(t+2)\cdots(t+2+n)} = \gamma(t+2,1)e.$$

Thus we obtain

$$\sum_{k=2}^{\infty} \xi_k(y_0) = \frac{y_0}{y_0 - 1} \max_{t \ge y_0} \frac{1}{tR(t)} + C_7(y_0) \left( e - \frac{1}{2} + \int_3^{\infty} \frac{dt}{\Gamma(t)} + \frac{1}{y_0 - 1} \left( 2\log 2 - 1 + y_0 I_{y_0}(2) + \int_2^3 y_0^{t-2} \left( 1 - \log(t - 1) + \frac{1}{(t - 1)^2} \right) dt + 2e \int_0^1 y_0^t \frac{\gamma(t + 2, 1)}{\Gamma(t + 2)} dt \right) \right).$$
(3.21)

In Mathematica, the function  $\gamma(t+2,1)$  can be evaluated by "Gamma[t+2,0,1]". Finally, we are ready to estimate  $\eta(x,y)$ . Let

$$\eta_k(y) := \frac{1}{e^{\gamma} R(y)} \sup_{\substack{u \in [k, k+1) \\ t > y}} |\eta(t^u, t)|$$

for  $k \ge 1$  and  $y \ge y_0$ , where the value of  $\eta_1(y)$  is provided by (3.8). Using (3.14) and the estimates for  $E_3(h_k; y, u)$  with  $y \ge y_0$  and  $2 \le k \le u < k + 1$ , we find

$$\eta_k(y) \le \eta_{k-1}(y) + \xi_k(y_0)$$

for all  $k \geq 2$  and  $y \geq y_0$ , from which we derive

$$\eta_k(y) \le \eta_1(y) + \sum_{\ell=2}^k \xi_{\ell}(y_0)$$

for all  $k \geq 1$  and  $y \geq y_0$ . Since  $\eta_1(y)$  is decreasing on  $[y_0, \infty)$ , we have therefore shown that

$$|\eta(x,y)| \le e^{\gamma} \left( \eta_1(y_0) + \sum_{k=2}^{\infty} \xi_k(y_0) \right) R(y)$$
 (3.22)

for all  $y \ge y_0$ , where the infinite sum can be evaluated using (3.21). To derive an explicit version of de Bruijn's result (1.6), we observe that (3.6), (3.22) and [11, Theorem 23] imply that  $Q(y)|\eta(x,y)| \le C_8(y_0)R(y)/\log y$  for all  $y \ge y_0$ , where

$$C_8(y_0) := \beta(y_0) \left( \eta_1(y_0) + \sum_{k=2}^{\infty} \xi_k(y_0) \right)$$

with

$$\beta(y_0) := \begin{cases} 1, & \text{if } 3 \le y_0 < 10^8, \\ \exp(C_2(y_0)R(y_0)), & \text{if } y_0 \ge 10^8. \end{cases}$$

Hence, it follows that

$$\left| \Phi(x,y) - \mu_y(u)e^{\gamma}x \log y \prod_{p \le y} \left( 1 - \frac{1}{p} \right) \right| < \frac{C_8(y_0)xR(y)}{\log y}$$
 (3.23)

for all  $y \geq y_0$ .

### 4. Deduction of Theorem 1.1 and Corollary 1.2

Now we apply (3.23) to obtain explicit estimates for  $\Phi(x, y)$  with specific choices of R(y). Unconditionally, it has been shown [9, Corollary 2] that

$$|\pi(z) - \operatorname{li}(z)| \le 0.2593 \frac{z}{(\log z)^{3/4}} \exp\left(-\sqrt{\frac{\log z}{6.315}}\right)$$

for all  $z \geq 229$ . With  $y_0 \geq 229$ , the function

$$R(z) = 0.2593(\log z)^{1/4} \exp\left(-\sqrt{\frac{\log z}{6.315}}\right)$$

is strictly decreasing on  $[y_0, \infty)$  and satisfies (1.4) and (1.5) with

$$C_0(y_0) = 2\sqrt{\frac{6.315}{\log y_0}},$$

since

$$\begin{split} \int_{z}^{\infty} \frac{1}{t (\log t)^{3/4}} \exp\left(-\sqrt{\frac{\log t}{6.315}}\right) \, dt &= 2 \int_{\sqrt{\log z}}^{\infty} \frac{1}{\sqrt{t}} \exp\left(-\frac{t}{\sqrt{6.315}}\right) \, dt \\ &< \frac{2}{(\log z)^{1/4}} \int_{\sqrt{\log z}}^{\infty} \exp\left(-\frac{t}{\sqrt{6.315}}\right) \, dt \\ &= \frac{2\sqrt{6.315}}{(\log z)^{1/4}} \exp\left(-\sqrt{\frac{\log z}{6.315}}\right) \end{split}$$

for  $z \geq y_0$ . Numerical computation using Mathematica allows us to conclude that

$$\left| \Phi(x,y) - \mu_y(u)e^{\gamma}x \log y \prod_{p \le y} \left( 1 - \frac{1}{p} \right) \right| < 4.403611 \frac{x}{(\log y)^{3/4}} \exp\left( -\sqrt{\frac{\log y}{6.315}} \right) \tag{4.1}$$

for all  $x \ge y \ge 229$ . Suppose now that  $2 \le y < 229$ . Using the inequalities  $\Phi(x,y) < x/\log y$  [6, Theorem],  $\prod_{p \le y} (1-1/p) < e^{-\gamma}/\log y$  [11, Theorem 23] and  $0 \le \mu_y(u) < 1/\log y$ , we have

$$\left| \Phi(x,y) - \mu_y(u)e^{\gamma}x \log y \prod_{p \le y} \left( 1 - \frac{1}{p} \right) \right| < \frac{2x}{\log y} < 4.403611 \frac{x}{(\log y)^{3/4}} \exp\left( -\sqrt{\frac{\log y}{6.315}} \right)$$

for all  $2 \le y < 229$ . Combining this with (4.1) proves the first half of Theorem 1.1. Under the assumption of the Riemann Hypothesis, it is known [12, Corollary 1] that

$$|\pi(z) - \operatorname{li}(z)| < \frac{1}{8\pi} \sqrt{z} \log z$$

for all  $z \ge 2,657$ . With  $y_0 = 2,657$  and

$$R(z) = \frac{\log^2 z}{8\pi\sqrt{z}},$$

we have

$$\int_{z}^{\infty} \frac{|\pi(t) - \operatorname{li}(t)|}{t^{2}} dt \le \frac{1}{8\pi} \int_{z}^{\infty} \frac{\log t}{t^{3/2}} dt = \frac{\log z + 2}{4\pi\sqrt{z}} \le C_{0}(y_{0})R(z)$$

for  $z \geq y_0$ , where

$$C_0(y_0) = \frac{2(\log y_0 + 2)}{\log^2 y_0}.$$

Therefore, we conclude by (3.23) and numerical calculations that

$$\left| \Phi(x,y) - \mu_y(u)e^{\gamma}x \log y \prod_{p < y} \left( 1 - \frac{1}{p} \right) \right| < 0.184563 \frac{x \log y}{\sqrt{y}}$$
 (4.2)

for all  $x \ge y \ge 2,657$ . The values of relevant constants are recorded in the table below.

Table:	Num	orical	Constants	7
Table.	$\pm NUIII$	енсаг	Constants	5

constants	unconditional estimates		conditional estimates	
$y_0$	229	$10^{8}$	2,657	$10^{8}$
$R(y_0)$	.156576	.097363	.047992	.001351
$C_0(y_0)$	2.156096	1.171019	.317985	.120362
$C_1(y_0)$	2.534430	1.279593	.571800	.228936
$C_2(y_0)$	2.548436	1.279593	.575723	.228940
$C_3(y_0)$	3.110976	1.362717	.579718	.228971
$C_4(y_0)$	2.548436	1.279593	.575723	.228940
$C_5(y_0)$	3.131827	1.362717	.583750	.228975
$C_6(y_0)$	2.534430	1.279593	.571800	.228936
$C_7(y_0)$	3.110976	1.362717	.579718	.228971
$C_8(y_0)$	16.982691	9.079975	4.638553	2.967998
$\eta_1(y_0)$	6.236726	3.628074	2.697198	2.229726
$\sum_{k=2}^{\infty} \xi_k(y_0)$	10.745960	4.388310	1.941356	.737355

To complete the proof of the second half of Theorem 1.1, it remains to deal with the case  $11 \le y \le 2,657$ . For simplicity of notation we set

$$D(x,y) := \Phi(x,y) - \mu_y(u)e^{\gamma}x\log y \prod_{p < y} \left(1 - \frac{1}{p}\right).$$

Using Mathematica we find that

$$\begin{split} M &:= \max_{11 \leq z \leq 2,657} \frac{\operatorname{li}(z) - \pi(z)}{\sqrt{z} \log z} < 0.259141, \\ m &:= \min_{11 \leq z \leq 2,657} e^{\gamma} \log z \prod_{p \leq z} \left(1 - \frac{1}{p}\right) > 0.876248. \end{split}$$

If  $\sqrt{x} \le y < x$ , then

$$\Phi(x,y) = \mu_y(u)x + (\pi(x) - \text{li}(x)) - (\pi(y) - \text{li}(y)) + 1.$$

Note that  $x \leq y^2 < 10^8$ . Since  $\pi(z) < \text{li}(z)$  for  $2 \leq z \leq 10^8$  by [11, Theorem 16] and

$$\prod_{p \le z} \left( 1 - \frac{1}{p} \right) < \frac{e^{-\gamma}}{\log z}$$

for  $0 < z \le 10^8$  by [11, Theorem 23], we have

$$|D(x,y)| < (1-m) (1-y^{-1}) \frac{x}{\log y} + M\sqrt{x} \log x + 1$$

$$\leq \left( (1-m) (1-y^{-1}) + M \frac{\log^2 y}{\sqrt{y}} + \frac{\log y}{y} \right) \frac{x}{\log y}, \tag{4.3}$$

where we have used the fact that  $\log x/\sqrt{x}$  is strictly decreasing on  $[e^2, \infty)$ . Numerical computation shows that the right side of (4.3) is  $< 0.449774x \log y/\sqrt{y}$  for  $11 \le y \le 2,657$ .

Suppose now that  $11 \le y \le \sqrt{x}$ . By [7, Theorem 1], Theorem 2.3 and [11, Theorem 23] we have, for  $11 \le y \le 2{,}657$ ,

$$D(x,y) \le \left(0.6 - \frac{m}{2} \left(1 - y^{-1}\right)\right) \frac{x}{\log y} < 0.449774 \frac{x \log y}{\sqrt{y}},$$
  
$$D(x,y) > (0.4 - M_0) \frac{x}{\log y} > -0.449774 \frac{x \log y}{\sqrt{y}}.$$

This settles the case  $11 \le y \le 2,657$  and completes the proof of Theorem 1.1.

The proof of Corollary 1.2 is similar, and we shall only sketch it. When  $y \ge y_0$ , where  $y_0 = 229$  for the unconditional estimate and  $y_0 = 2,657$  for the conditional estimate, we have by the triangle inequality that

$$|\Phi(x,y) - \mu_y(u)x| < |D(x,y)| + \left| 1 - e^{\gamma} \log y \prod_{p < y} \left( 1 - \frac{1}{p} \right) \right| \frac{x}{\log y}.$$

Then we bound |D(x,y)| using the values of  $C_8(y_0)$  listed in the table above. To estimate the second term, we use (3.6) when  $y \ge 10^8$  and the inequality

$$m(y) < e^{\gamma} \log y \prod_{p \le y} \left(1 - \frac{1}{p}\right) < 1$$

when  $y_0 \le y \le 10^8$ , where m(y) is given by

$$m(y) := \begin{cases} 0.983296, & \text{if } 229 \le y \le 2,657, \\ 0.996426, & \text{if } 2,657 \le y < 210,000, \\ 0.999643, & \text{if } 210,000 \le y \le 10^8, \end{cases}$$

according to [11, Theorem 23] and Mathematica. This leads to the asserted bounds for  $y \geq y_0$ . Suppose now that  $y \leq y_0$ . In this case, the proof of the unconditional bound is exactly the same as that of the unconditional bound in Theorem 1.1. As for the conditional bound, we argue in the same way as in the proof of Theorem 1.1 to get

$$|\Phi(x,y) - \mu_y(u)x| \le \left(M \frac{\log^2 y}{\sqrt{y}} + \frac{\log y}{y}\right) \frac{x}{\log y}$$

when  $\sqrt{x} \le y < x$  and

$$|\Phi(x,y) - \mu_y(u)x| \le \left(0.6 - \frac{1}{2} \left(1 - y^{-1}\right)\right) \frac{x}{\log y},$$
  
$$|\Phi(x,y) - \mu_y(u)x| > (0.4 - M_0) \frac{x}{\log y},$$

when  $11 \le y \le \sqrt{x}$ . Together, these inequalities yield the asserted conditional bound.

Remark 4.1. The bounds in Theorem 1.1 and its corrollary may be improved. For example, the numerical values of the sum  $\sum_{k=2}^{\infty} \xi_k(y_0)$  may be reduced by keeping  $\rho$  (or even  $|\omega'|$ ) in all of the relevant integrals, but of course the computational complexity is expected to increase as a cost. In addition, our method would allow an extension of the range  $x \geq y \geq 11$  in the second half of Theorem 1.1 to the entire range  $x \geq y \geq 2$  if we argue with  $y_0 = 2,657$  replaced by some smaller value and enlarge the constant 0.449774.

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