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RESEARCH ARTICLE

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Categorical Torelli theorems for Gushel–Mukai threefolds

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Abstract

We show that a general ordinary Gushel–Mukai (GM) threefold X can be reconstructed from its Kuznetsov component $\mathcal{K}u(X)$ together with an extra piece of data coming from tautological subbundle of the Grassmannian $\mathrm{Gr}(2, 5)$. We also prove that $\mathcal{K}u(X)$ determines the birational isomorphism class of X, while $\mathcal{K}u(X')$ determines the isomorphism class of a special GM threefold X' if it is general. As an application, we prove a conjecture of Kuznetsov–Perry in dimension 3 under a mild assumption. Finally, we use $\mathcal{K}u(X)$ to restate a conjecture of Debarre–Iliev–Manivel regarding fibers of the period map for ordinary GM threefolds.

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Contents

1 | INTRODUCTION

In recent times, derived categories have played an important role in algebraic geometry; in many cases, much of the geometric information of a variety/scheme X is encoded by its bounded derived category of coherent sheaves $\mathrm{D}^b(X)$. In this setting, one of the most fundamental questions that can be asked is whether $\mathrm{D}^b(X)$ recovers X up to isomorphism, in other words, whether a *derived Torelli theorem* holds for X. For varieties with ample or antiample canonical bundle (which include Fano varieties and varieties of general type), this question was answered affirmatively by Bondal–Orlov in [10].

1.1 | Kuznetsov components and categorical Torelli theorems

Therefore, for the class of varieties above, it is natural to ask whether they are also determined up to isomorphism by *less* information than the whole derived category $\mathrm{D}^b(X)$. A natural candidate for this is a subcategory $\mathcal{K}u(X)$ of $\mathrm{D}^b(X)$ called the *Kuznetsov component*. This subcategory has been studied extensively by Kuznetsov and others (e.g., [28, 29, 33]) for many Fano varieties, including Gushel–Mukai (GM) varieties.

The question of whether $\mathcal{K}u(X)$ determines X up to isomorphism has been studied for certain cases in the setting of Fano threefolds. In [8], the authors show that the Kuznetsov component completely determines cubic threefolds up to isomorphism, in other words, a *categorical Torelli theorem* holds for cubic threefolds Y. The same result was also verified in [51]. On the other hand, for many Fano varieties, the Kuznetsov component $\mathcal{K}u(X)$ does not determine the isomorphism class, but only the birational isomorphism class of X. This is known as a *birational categorical Torelli theorem*. For instance, Kuznetsov components determine the birational isomorphism class of every index 1 prime Fano threefolds of even genus $g \geqslant 8$. For GM threefolds — the focus of our paper — by [34], it is known that there are birational GM threefolds with equivalent Kuznetsov components. So, there are two natural questions to ask in this setting.

Question 1.1.

(1) Does $\mathcal{K}u(X)$ determine the birational equivalence class of X?
(2) What extra data along with $\mathcal{K}u(X)$ do we need to identify a particular GM threefold X from its birational equivalence class?

1.2 | Main results

1.2.1 | (Refined) categorical Torelli for Gushel–Mukai threefolds

In the present paper, we deal with the case of index 1 prime Fano threefolds of degree 10 and genus 6, also known as Gushel–Mukai threefolds (GM threefolds for short), which are split into two types: ordinary GM threefolds that arise as a quadric section of a linear section of the Grassmannian Gr(2, 5), and special GM threefolds that arise as double covers of a codimension three linear section of Gr(2, 5), branched over a degree 10 K3 surface. By [33], we have a semiorthogonal decomposition

$$\mathrm{D}^b(X) = \langle \mathcal{K}u(X), \mathcal{E}, \mathcal{O}_X \rangle,$$

where $\mathcal{E}$ is the pull-back of the tautological subbundle on Gr(2, 5) along the natural map $X \to \mathrm{Gr}(2,5)$.

Our first main theorem is concerned with ordinary GM threefolds and answers Question 1.1 (2).

Theorem 1.2 (Theorem 9.2). *Let X be a general ordinary GM threefold and $\pi : \mathrm{D}^b(X) \to \mathcal{K}u(X)$ be the right adjoint to the inclusion $\mathcal{K}u(X) \subset \mathrm{D}^b(X)$. Then the data of $\mathcal{K}u(X)$ along with the object $\pi(\mathcal{E})$ are enough to determine X up to isomorphism.*

On the other hand, for special GM threefolds that are general ("general special" for short), we show that a categorical Torelli theorem holds.

Theorem 1.3 (Theorem 9.9). *Let X and X' be general special GM threefolds, and assume that there is an equivalence of categories $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$. Then X and X' are isomorphic.*

1.2.2 | Birational categorical Torelli for Gushel–Mukai threefolds

Next, returning to the setting of ordinary GM threefolds, we show that a birational categorical Torelli theorem holds for general ordinary GM threefolds, which answers Question 1.1 (1).

Theorem 1.4 (Theorem 9.3). *Let X and X' be general ordinary GM threefolds, and suppose that there is an equivalence of categories $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$. Then X is birationally equivalent to X'.*

In [34], the authors studied GM varieties of arbitrary dimension and proved the Duality Conjecture [33, Conjecture 3.7] for them, that is, they showed that the period partner or period dual of a GM variety X shares the same Kuznetsov component $\mathcal{K}u(X)$ as X. Combining earlier results [14, Theorem 4.20] on the birational equivalence of these varieties, this gives strong evidence for the following conjecture.

Conjecture 1.5 [34, Conjecture 1.7]. *If X and X' are GM varieties of the same dimension such that there is an equivalence $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$, then X and X' are birationally equivalent.*

Thus, our result Theorem 1.4 actually proves Conjecture 1.5 under the assumption that X and X' are both of dimension 3, ordinary and general.

Moreover, by a careful study of Bridgeland moduli spaces of stable objects in the Kuznetsov components $\mathcal{A}_X$ for not only smooth ordinary GM threefolds but also special GM threefolds X, we can prove that the Kuznetsov component of a general ordinary GM threefold cannot be equivalent to the one of a general special GM threefold. Therefore, combined with Theorems 1.4 and 1.3, we have the following improved version of Theorem 1.4, which allows threefolds to be either ordinary or special.

Theorem 1.6 (Theorem 9.7 and Corollary 9.8). *If X and X' are general ordinary or general special GM threefolds such that there is an equivalence $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$, then X and X' are birationally equivalent.*

1.2.3 | The Debarre–Iliev–Manivel conjecture

In [13], the authors conjecture that the general fiber of the classical period map from the moduli space of ordinary GM threefolds to the moduli space of 10-dimensional principally polarized abelian varieties is birational to the disjoint union of the minimal model $\mathcal{C}_m(X)$ of the Fano surface of conics and a moduli space of stable sheaves $M_G^X(2,1,5)$, both quotiented by involutions, which we call the *Debarre–Iliev–Manivel conjecture* (cf. Conjecture 10.1). Within the moduli space of smooth ordinary GM threefolds, we define the fiber of the "categorical period map" through $[X]$ as the isomorphism classes of all ordinary GM threefolds X' whose Kuznetsov components satisfy $\mathcal{K}u(X') \simeq \mathcal{K}u(X)$. Then the following categorical analog of the *Debarre–Iliev–Manivel conjecture* follows from Theorem 1.4 and results on Bridgeland moduli spaces with respect to the two (-1)-classes in the numerical Grothendieck group of $\mathcal{A}_X$.

Theorem 1.7 (Theorem 10.3). *A general fiber of the "categorical period map" through an ordinary GM threefold X is the union of $\mathcal{C}_m(X)/\iota$ and $M_G^X(2,1,5)/\iota'$ where ι, ι' are geometrically meaningful involutions.*

As an application, the *Debarre–Iliev–Manivel* Conjecture 10.1 can be restated in an equivalent form as follows.

Conjecture 1.8. *Let X be a general ordinary GM threefold. The intermediate Jacobian $J(X)$ determines the Kuznetsov component $\mathcal{K}u(X)$.*

Remark 1.9. In [13], the authors actually conjecture that a general fiber of the period map is *birational* to the disjoint union of two surfaces, parametrizing conic transforms and conic transforms of a line transform of X, which is birational to the disjoint union of $\mathcal{C}_m(X)$ and $M_G^X(2,1,5)$, both quotiented by involutions. In Corollary 9.5, we show that this birational equivalence is indeed an *isomorphism*.

1.2.4 | Uniqueness of Serre-invariant stability conditions

One of the key steps when we identify Bridgeland moduli spaces via an equivalence of Kuznetsov components in the proofs of Theorems 1.2 and 1.4. A stability condition σ on the Kuznetsov component $\mathcal{K}u(X)$ of a prime Fano threefold X is *Serre-invariant* if $S_{\mathcal{K}u(X)} \cdot \sigma = \sigma \cdot g$ for some $g \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ (see Section 4.4). Serre-invariance is one of the fundamental tools in studying relationship of classical Gieseker moduli spaces and Bridgeland moduli spaces for Kuznetsov components (cf. [1, 17, 40, 51, 54]). A natural question is whether any two Serre-invariant stability conditions are in the same $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-orbit. In the present paper, we answer this question affirmatively.

Theorem 1.10 (Theorem A.10). *Let X be a prime Fano threefold of index 1 of genus $g \geqslant 6$, or a del Pezzo threefold of degree $d \geqslant 2$. Then all Serre-invariant stability conditions on $\mathcal{K}u(X)$ are in the same $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-orbit.*

1.3 | Methods

For convenience, we work with the alternative Kuznetsov component $\mathcal{A}_X$, defined by the semiorthogonal decomposition $\mathrm{D}^b(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{E}^\vee \rangle$ and there is an equivalence $\Xi : \mathcal{K}u(X) \simeq \mathcal{A}_X$. We prove the above Theorems 1.2, 1.4, 1.6, and 1.7 by considering the moduli spaces of Bridgeland stable objects in the alternative Kuznetsov component $\mathcal{A}_X$ with respect to (-1)-classes in the numerical Grothendieck group of $\mathcal{A}_X$, that is, a vector v with $\chi(v,v) = -1$ where χ is the Euler form. Up to sign, there are two (-1)-classes in the numerical Grothendieck group of $\mathcal{A}_X$, call them $-x$ and $y - 2x$.

First, we show that the moduli space with the class $-x$ is isomorphic to the minimal model $\mathcal{C}_m(X)$ of the Fano surface of conics (Theorem 7.12). Indeed, we first show that the unique exceptional curve contracted in $\mathcal{C}(X)$ is the rational curve of conics whose ideal sheaf I_C is not in $\mathcal{A}_X$ and that the image is the smooth point represented by $\pi(\mathcal{E})$ (Proposition 7.1), so $\mathcal{C}_m(X)$ forms an irreducible component of the moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ of stable objects in $\mathcal{A}_X$ with respect to $-x$.

Then, we show that this component actually occupies the whole moduli space (Proposition 7.11), which is the most difficult and technical part of the article, and we only briefly sketch the argument here. We start with a stable object $F \in \mathcal{A}_X$ of the class $-x$. It suffices to show that F is isomorphic to the projection of ideal sheaf I_C of a conic $C \subset X$. First, we assume that F is semistable in the double tilted heart $\mathrm{Coh}^0_{\alpha,\beta}(X)$ (cf. Section 4.4). Then, by a wall-crossing argument, we prove that $F[-1]$ is a slope-semistable sheaf of rank one. Since its class is $[F] = -[I_C]$, we get $F \cong I_C[1]$. Next, we assume that F is not semistable in the double-tilted heart $\mathrm{Coh}^0_{\alpha,\beta}(X)$. Our main tools are inequalities in [36] and Theorem 4.7, which allow us to bound the rank and first two Chern characters $\mathrm{ch}_1, \mathrm{ch}_2$ of the destabilizing objects and their cohomology objects. Since $F \in \mathcal{A}_X$, by using the Euler characteristics $\chi(\mathcal{O}_X, -)$ and $\chi(\mathcal{E}^\vee, -)$, we can obtain a bound on ch_3. Then, we deduce that the Harder–Narasimhan factors of F are the expected ones (Proposition 7.10). As a result, $\mathcal{M}_\sigma(\mathcal{A}_X, -x) \cong \mathcal{C}_m(X)$. Similarly, we identify the moduli space $M^X_G(2,1,5)$ of Gieseker semistable sheaves of rank 2, $c_1 = 1, c_2 = 5$, and $c_3 = 0$ on X with the Bridgeland moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, y - 2x)$ in Theorem 8.9.

As we have seen, $\mathcal{C}(X)$ is exactly the blow-up of $\mathcal{C}_m(X) \cong \mathcal{M}_\sigma(\mathcal{A}_X, -x)$ at the point $\Xi(\pi(\mathcal{E}))$; hence, the data $(\mathcal{K}u(X), \pi(\mathcal{E}))$ determine $\mathcal{C}(X)$. A classical result of Logachev [41] states that X can be determined up to isomorphism from $\mathcal{C}(X)$. Thus, Theorem 1.2 is proved.

We prove Theorem 1.3 via another method. By considering the equivariant Kuznetsov components $\mathcal{K}u(X)^{\mu_2}$, first discussed in [32], and exploiting the fact that X is the double cover of a degree 5 index 2 prime Fano threefold Y, branched over a quadric hypersurface $\mathcal{B} \subset Y$. In this case, the equivariant Kuznetsov component is equivalent to $\mathrm{D}^b(\mathcal{B})$ where $\mathcal{B}$ is a K3 surface. Therefore, a number of results concerning the Fourier–Mukai partners of K3 surfaces can be used to deduce that $\mathcal{K}u(X)^{\mu_2} \simeq \mathcal{K}u(X')^{\mu_2}$ implies $\mathcal{B} \cong \mathcal{B}'$. Then, the fact that the del Pezzo threefold Y of degree 5 is rigid can be used to deduce that indeed, $X \cong X'$.

To prove Theorem 1.4, we invoke a few more results from [13]. More precisely, an equivalence of categories $\Phi : \mathcal{A}_X \simeq \mathcal{A}_{X'}$ identifies the moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ with either $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ or $\mathcal{M}_\sigma(\mathcal{A}_{X'}, y-2x)$. The former case gives an isomorphism of minimal surfaces $\mathcal{C}_m(X) \cong \mathcal{C}_m(X')$. Blowing $\mathcal{C}_m(X)$ up at the smooth point associated to $\pi(\mathcal{E})$ gives $\mathcal{C}(X)$, and blowing up $\mathcal{C}_m(X')$ at the image of $\pi(\mathcal{E})$ under Φ gives $\mathcal{C}(X'_C)$, where X'_C is certain birational transformation of X', associated with a conic $C \subset X'$. Then by Logachev's reconstruction theorem for $\mathcal{C}(X)$, X is isomorphic to X'_C that is birational to X'. For the latter case, we start with the isomorphism $\mathcal{C}_m(X) \cong M_G^{X'}(2,1,5)$. In fact, $M_G^{X'}(2,1,5)$ is birational to $\mathcal{C}(X'_L)$, where X'_L is another birational transformation of X', associated with a line $L \subset X'$. Since $\mathcal{C}(X'_L)$ is a surface of general type, we get $\mathcal{C}_m(X) \cong \mathcal{C}_m(X'_L)$. Then, by the same argument as in the previous case, X is isomorphic to some birational transformation of X'.

Finally, the proof of Theorem 1.6 is similar to that of Theorem 1.4. First, we identify the Bridgeland moduli spaces $\mathcal{M}_\sigma(\mathcal{A}_{X'}, -x)$ and $\mathcal{M}_\sigma(\mathcal{A}_{X'}, y-2x)$ on a special GM threefold X' with $\mathcal{C}_m(X')$ and $M_G^{X'}(2,1,5)$, respectively (Theorem 7.12 and Theorem 8.9), where $\mathcal{C}_m(X')$ is the contraction of the Fano surface $\mathcal{C}(X')$ of conics on X' along one of the components to a singular point. Then, if X is ordinary, the equivalence $\Phi : \mathcal{A}_X \simeq \mathcal{A}_{X'}$ would identify those moduli spaces on a general ordinary GM threefold X with those on a special GM threefold X'; we show that this is impossible by analyzing their singularities. Then, Theorem 1.6 reduces to Theorem 1.4 and Theorem 1.3.

1.4 | Related work

1.4.1 | Categorical Torelli theorems

There is a very nice survey article [50] on recent results and remaining open questions on this topic. In [8] and [51], the authors prove categorical Torelli theorems for cubic threefolds. In [1] and [9], the authors prove categorical Torelli theorems for general quartic double solids. In [38] and [39], the authors prove a refined categorical Torelli theorem for Enriques surfaces. In [26], the authors generalize Theorem 9.2 to all prime Fano threefolds of genus $g \geqslant 6$. In [18], the authors prove a birational categorical Torelli theorem for general non-Hodge-special Gushel–Mukai fourfolds.

1.4.2 | Identifying classical moduli spaces as Bridgeland moduli spaces for Kuznetsov components

In the present article, we realize the Fano surface of conics and a certain Gieseker moduli space of semistable sheaves as Bridgeland moduli spaces of stable objects in Kuznetsov components of GM threefolds. In [51], the authors realize the Fano surface of lines $\Sigma(Y_d)$ (for $d \geqslant 2$) as a Bridgeland moduli space of stable objects in the Kuznetsov component $\mathcal{K}u(Y_d)$. In [40], the authors realize the moduli space of rank two instanton sheaves on a del Pezzo threefold Y_d (for $d \geqslant 3$) and the compactification of the moduli space of ACM sheaves on X_{4d+2} (for $d \geqslant 3$) as Bridgeland moduli spaces of stable objects in $\mathcal{K}u(Y_d)$ and $\mathcal{K}u(X_{4d+2})$, respectively. In [17], the authors realize the moduli space of Ulrich bundles of arbitrary rank on a cubic threefold Y_3 as an open locus of a Bridgeland moduli space of stable objects in $\mathcal{K}u(Y_3)$.

1.4.3 | Serre-invariant stability conditions

In [48] and [51], the authors prove that stability conditions on Kuznetsov components of every del Pezzo threefold Y_d of degree $d \geqslant 1$ and every index 1 prime Fano threefold of genus $g \geqslant 6$ are Serre-invariant. In [17], the authors prove the uniqueness of Serre-invariant stability conditions for a general triangulated category satisfying a list of very natural assumptions, which include Kuznetsov components of a series of prime Fano threefolds.

1.5 | Notation and conventions

- We work over the field $k = \mathbb{C}$. All triangulated categories and abelian categories are assumed to be k-linear.
- We use hom and ext^i to represent the dimension of the vector spaces Hom and Ext^i.
- The numerical K group of a triangulated category $\mathcal{D}$ is denoted by $\mathcal{N}(\mathcal{D})$, which is the Grothendieck group $K_0(\mathcal{D})$ modulo the kernel of the Euler form $\chi(E,F) = \sum_i (-1)^i \mathrm{ext}^i(E,F)$.
- We denote the bounded derived category of a smooth projective variety X by $\mathrm{D}^b(X)$. The derived dual functor is denoted by $\mathbb{D} := \mathrm{R}\mathcal{H}om_X(-, \mathcal{O}_X)$.
- We denote the phase and slope with respect to a weak stability condition σ by ϕ_σ and μ_σ, respectively. The maximal and minimal slopes (phases) of the Harder–Narasimhan factors of a given object F will be denoted by $\mu_\sigma^+(F)$ ($\phi_\sigma^+(F)$) and $\mu_\sigma^-(F)$ ($\phi_\sigma^-(F)$), respectively.
- $\mathcal{H}^i_{\mathcal{A}}$ means the ith cohomology with respect to the heart $\mathcal{A}$. When the $\mathcal{A}$ subscript is dropped, we take the heart to be $\mathrm{Coh}(X)$.
- The symbol $\simeq$ denotes an equivalence of categories and a birational equivalence of varieties. The symbol $\cong$ denotes an isomorphism of varieties.
- Let X be a GM threefold. Then a conic means a closed subscheme $C \subset X$ with Hilbert polynomial $p_C(t) = 1 + 2t$, and a line means a closed subscheme $L \subset X$ with Hilbert polynomial $p_L(t) = 1 + t$.

1.6 | Organization of the paper

In Section 2, we collect basic facts about semiorthogonal decompositions. In Section 3, we introduce Gushel–Mukai threefolds and their Kuznetsov components. In Section 4, we introduce the definition of weak stability conditions on $\mathrm{D}^b(X)$, and the induced stability conditions on the alternative Kuznetsov components $\mathcal{A}_X$ of GM threefolds. In Section 5, we introduce a distinguished object $\pi(\mathcal{E}) \in \mathcal{K}u(X)$ and its alternative Kuznetsov component analog $\Xi(\pi(\mathcal{E})) \in \mathcal{A}_X$ and prove its stability. In Section 6, we discuss the geometry of the Fano surface of conics of a GM threefold. In Section 7, we construct the Bridgeland moduli space of σ-stable objects with class $-x$ in $\mathcal{A}_X$. In Section 8, we construct the Bridgeland moduli space of σ-stable objects with respect to the other (-1)-class $y - 2x$ in $\mathcal{A}_X$. In Section 9, we prove several birational/refined categorical Torelli theorems (Theorems 1.2, 1.3, and 1.4) and Conjecture 1.5 in dimension 3 with mild assumptions. In Section 10, we describe the general fiber of the "categorical period map" for ordinary GM threefolds 1.7, and restate the *Debarre–Iliev–Manivel conjecture* in terms of Conjecture 10.6. Finally, we study Serre-invariant stability conditions on Kuznetsov components and show that they are contained in one $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ orbit in the Appendix.

2 | SEMIORTHOGONAL DECOMPOSITIONS

In this section, we collect some useful facts about semiorthogonal decompositions. Background on triangulated categories and derived categories of coherent sheaves can be found in [21], for example. From now on, let $\mathrm{D}^b(X)$ denote the bounded derived category of coherent sheaves on a smooth projective variety X, and for $E, F \in \mathrm{D}^b(X)$, define

$$\mathrm{RHom}^\bullet(E,F) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}(E, F[i])[-i].$$

2.1 | Exceptional collections and semiorthogonal decompositions

Definition 2.1. Let $\mathcal{D}$ be a triangulated category and $E \in \mathcal{D}$. We say that E is an *exceptional object* if $\mathrm{RHom}^\bullet(E,E) = k$. Now let $(E_1, \dots, E_m)$ be a collection of exceptional objects in $\mathcal{D}$. We say that it is an *exceptional collection* if $\mathrm{RHom}^\bullet(E_i, E_j) = 0$ for $i > j$.

Definition 2.2. Let $\mathcal{D}$ be a triangulated category and $\mathcal{C}$ be a triangulated subcategory of $\mathcal{D}$. We define the *right orthogonal complement* of $\mathcal{C}$ in $\mathcal{D}$ as the full triangulated subcategory

$$\mathcal{C}^\perp = \{X \in \mathcal{D} \mid \mathrm{Hom}(Y,X) = 0 \text{ for all } Y \in \mathcal{C}\}.$$

The *left orthogonal complement* is defined similarly, as

$$^\perp\mathcal{C} = \{X \in \mathcal{D} \mid \mathrm{Hom}(X,Y) = 0 \text{ for all } Y \in \mathcal{C}\}.$$

Definition 2.3. Let $\mathcal{D}$ be a triangulated category. We say that a triangulated subcategory $\mathcal{C} \subset \mathcal{D}$ is *admissible* if the inclusion functor $i : \mathcal{C} \hookrightarrow \mathcal{D}$ has left adjoint i^* and right adjoint $i^!$.

Definition 2.4. Let $\mathcal{D}$ be a triangulated category, and $(\mathcal{C}_1, \dots, \mathcal{C}_m)$ be a collection of full admissible subcategories of $\mathcal{D}$. We say that $\mathcal{D} = \langle \mathcal{C}_1, \dots, \mathcal{C}_m \rangle$ is a *semiorthogonal decomposition* of $\mathcal{D}$ if $\mathcal{C}_j \subset \mathcal{C}_i^\perp$ for all $i > j$, and the subcategories $(\mathcal{C}_1, \dots, \mathcal{C}_m)$ generate $\mathcal{D}$, that is, the category resulting from taking all shifts and cones of objects in the categories $(\mathcal{C}_1, \dots, \mathcal{C}_m)$ is equivalent to $\mathcal{D}$.

Let $S_{\mathcal{D}}$ be the Serre functor of $\mathcal{D}$, then we have the following standard result, see, for example, [4, Section 3]:

Proposition 2.5 [4, Section 3]. *If $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ is a semiorthogonal decomposition, then $\mathcal{D} = \langle S_{\mathcal{D}}(\mathcal{D}_2), \mathcal{D}_1 \rangle = \langle \mathcal{D}_2, S_{\mathcal{D}}^{-1}(\mathcal{D}_1) \rangle$ are also semiorthogonal decompositions.*

2.2 | Mutations

Let $\mathcal{C} \subset \mathcal{D}$ be an admissible triangulated subcategory. Then the *left mutation functor* $\mathbf{L}_{\mathcal{C}}$ through $\mathcal{C}$ is defined as the functor lying in the canonical functorial exact triangle

$$ii^! \to \mathrm{id} \to \mathbf{L}_{\mathcal{C}},$$

and the *right mutation functor* $\mathbf{R}_{\mathcal{C}}$ through $\mathcal{C}$ is defined similarly, by the triangle

$$\mathbf{R}_{\mathcal{C}} \to \mathrm{id} \to ii^*.$$

When $E \in \mathrm{D}^b(X)$ is an exceptional object, and $F \in \mathrm{D}^b(X)$ is any object, the left mutation $\mathbf{L}_E F$ fits into the triangle

$$E \otimes \mathrm{RHom}^\bullet(E, F) \to F \to \mathbf{L}_E F,$$

and the right mutation $\mathbf{R}_E F$ fits into the triangle

$$\mathbf{R}_E F \to F \to E \otimes \mathrm{RHom}^\bullet(F, E)^\vee.$$

Proposition 2.6 [31, Lemma 2.6]. *Let $\mathcal{D} = \langle \mathcal{A}, \mathcal{B} \rangle$ be a semiorthogonal decomposition. Then*

$$S_{\mathcal{B}} = \mathbf{R}_{\mathcal{A}} \circ S_{\mathcal{D}} \quad \textit{and} \quad S_{\mathcal{A}}^{-1} = \mathbf{L}_{\mathcal{B}} \circ S_{\mathcal{D}}^{-1}.$$

Lemma 2.7 [30, Lemma 2.7]. *Let $\mathcal{D} = \langle \mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n \rangle$ be a semiorthogonal decomposition with all components being admissible. Then, for each $1 \leqslant k \leqslant n-1$, there is a semiorthogonal decomposition*

$$\mathcal{D} = \langle \mathcal{C}_1, \dots, \mathcal{C}_{k-1}, \mathbf{L}_{\mathcal{C}_k} \mathcal{C}_{k+1}, \mathcal{C}_k, \mathcal{C}_{k+2} \dots, \mathcal{C}_n \rangle$$

and for each $2 \leqslant k \leqslant n$, there is a semiorthogonal decomposition

$$\mathcal{D} = \langle \mathcal{C}_1, \dots, \mathcal{C}_{k-2}, \mathcal{C}_k, \mathbf{L}_{\mathcal{C}_k} \mathcal{C}_{k-1}, \mathcal{C}_{k+1} \dots, \mathcal{C}_n \rangle.$$

3 | GUSHEL–MUKAI THREEFOLDS AND THEIR DERIVED CATEGORIES

Let X be a prime Fano threefold of index 1 and degree $H^3 = 10$, where H is the ample generator of $\mathrm{CaCl}(X)$. Then, X is either a quadric section of a linear section of codimension 2 of the Grassmannian $\mathrm{Gr}(2,5)$, in which case it is called an ordinary Gushel–Mukai (GM) threefold, or X is a double cover of a degree 5 and index 2 Fano threefold Y ramified in a quadric hypersurface, in which case it is called a special GM threefold. In the latter case, it has a natural involution $\tau : X \to X$ induced by the double cover $\pi : X \to Y$. By [5, 43], there exists a stable vector bundle $\mathcal{E}$ of rank 2 with $c_1(\mathcal{E}) = -H, c_2(\mathcal{E}) = 4L$, and $c_3(\mathcal{E}) = 0$, where L is the class of a line on X. In addition, $\mathcal{E}$ is exceptional and $H^\bullet(X, \mathcal{E}) = 0$. In fact, $\mathcal{E}$ is the pullback of the tautological bundle on the Grassmannian Gr(2, 5). By [13, Proposition 4.1], $\mathcal{E}$ is the unique stable sheaf with $c_1(\mathcal{E}) = -H, c_2(\mathcal{E}) = 4L$ and $c_3(\mathcal{E}) = 0$.

Furthermore, there is a standard short exact sequence

$$0 \to \mathcal{E} \to \mathcal{O}_X^{\oplus 5} \to \mathcal{Q} \to 0, \tag{1}$$

where $\mathcal{Q}$ is the pull-back of the tautological quotient bundle on Gr(2, 5) along the natural map $X \to \mathrm{Gr}(2,5)$. Since $\mathrm{rk}(\mathcal{E}) = 2$, we have $\mathcal{E}(H) \cong \mathcal{E}^\vee$.

Definition 3.1. Let X be a GM threefold.

- The *Kuznetsov component* of X is defined as $\mathcal{K}u(X) := \langle \mathcal{E}, \mathcal{O}_X\rangle^\perp$. In particular, it fits into the semiorthogonal decomposition $\mathrm{D}^b(X) = \langle \mathcal{K}u(X), \mathcal{E}, \mathcal{O}_X\rangle$.
- The *alternative Kuznetsov component* of X is defined as $\mathcal{A}_X := \langle \mathcal{O}_X, \mathcal{E}^\vee\rangle^\perp$. In particular, it fits into the semiorthogonal decomposition $\mathrm{D}^b(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{E}^\vee\rangle$.

Remark 3.2. By [33, Proposition 2.6], there is a natural involutive autoequivalence functor $\tau_\mathcal{A} := S_{\mathcal{A}_X}[-2]$ of $\mathcal{A}_X$. When X is special, it is induced by the natural involution τ on X as $\tau_\mathcal{A} = \tau^*|_{\mathcal{A}_X}$.

Definition 3.3. The left adjoint to the inclusion $\mathcal{A}_X \hookrightarrow \mathrm{D}^b(X)$ is given by $\mathrm{pr} := \mathbf{L}_{\mathcal{O}_X}\mathbf{L}_{\mathcal{E}^\vee} : \mathrm{D}^b(X) \to \mathcal{A}_X$. We call this the *projection functor*.

The analogous natural projection functor can be defined for $\mathcal{K}u(X)$, and we denote it by $\mathrm{pr}' := \mathbf{L}_\mathcal{E}\mathbf{L}_{\mathcal{O}_X}$.

3.1 | Kuznetsov components

Let $K_0(\mathcal{D})$ denote the Grothendieck group of a triangulated category $\mathcal{D}$. We have the bilinear Euler form

$$\chi(E,F) = \sum_{i\in\mathbb{Z}} (-1)^i \operatorname{ext}^i(E,F)$$

for $[E],[F] \in K_0(\mathcal{D})$. By the Hirzebruch–Riemann–Roch formula, it takes the following form on GM threefolds. We have [29, p. 5] $\chi(u,v) = \chi_0(u^* \cap v)$ where $u \mapsto u^*$ is an involution of $\oplus_{i=0}^3 H^i(X,\mathbb{Q})$ given by multiplication with $(-1)^i$ on $H^{2i}(X,\mathbb{Q})$, and χ_0 is given by

$$\chi_0(x + yH + zL + wP) = x + \frac{17}{6}y + \frac{1}{2}z + w,$$

where L is the class of lines and P is the class of points. The *numerical Grothendieck group* of $\mathcal{D}$ is $\mathcal{N}(\mathcal{D}) = K_0(\mathcal{D})/\ker\chi$.

Lemma 3.4 [29, p. 5]. *The numerical Grothendieck group $\mathcal{N}(\mathcal{K}u(X))$ of the Kuznetsov component is a rank 2 integral lattice generated by the basis elements $v = 1 - 3L + \frac{1}{2}P$ and $w = H - 6L + \frac{1}{6}P$.*

Using this basis, χ is given by the matrix

$$\begin{pmatrix} -2 & -3 \\ -3 & -5 \end{pmatrix}.$$

3.2 | Alternative Kuznetsov components

As in [29, Proposition 3.9], the following lemma follows from a straightforward computation.

Lemma 3.5. *The numerical Grothendieck group of $\mathcal{A}_X$ is a rank 2 integral lattice with basis vectors $x = 1 - 2L$ and $y = H - 4L - \frac{5}{6}P$, and the Euler form with respect to the basis is*

$$\begin{pmatrix} -1 & -2 \\ -2 & -5 \end{pmatrix}.$$

Remark 3.6. It is straightforward to check that the (-1)-classes of $\mathcal{N}(\mathcal{A}_X)$ are $x = 1 - 2L$ and $2x - y = 2 - H + \frac{5}{6}P$, up to sign.

Indeed, the Kuznetsov components from Subsection 3.1 and the alternative Kuznetsov components from this section are equivalent.

Lemma 3.7. *The original and alternative Kuznetsov components are equivalent. More precisely, there is an equivalence of categories $\Xi : \mathcal{K}u(X) \xrightarrow{\sim} \mathcal{A}_X$ given by $E \mapsto \mathbf{L}_{\mathcal{O}_X}(E \otimes \mathcal{O}_X(H))$, with inverse given by $F \mapsto (\mathbf{R}_{\mathcal{O}_X} F) \otimes \mathcal{O}_X(-H)$.*

Proof. Using Lemma 2.7 and noting that $\mathcal{E} \otimes \mathcal{O}_X(H) \cong \mathcal{E}^\vee$, we manipulate the semiorthogonal decomposition as follows:

$$\begin{aligned} \mathrm{D}^b(X) &= \langle \mathcal{K}u(X), \mathcal{E}, \mathcal{O}_X \rangle \\ &\simeq \langle \mathcal{K}u(X) \otimes \mathcal{O}_X(H), \mathcal{E}^\vee, \mathcal{O}_X(H) \rangle \\ &\simeq \langle \mathcal{O}_X, \mathcal{K}u(X) \otimes \mathcal{O}_X(H), \mathcal{E}^\vee \rangle \\ &\simeq \langle \mathbf{L}_{\mathcal{O}_X}(\mathcal{K}u(X) \otimes \mathcal{O}_X(H)), \mathcal{O}_X, \mathcal{E}^\vee \rangle. \end{aligned}$$

Now comparing with the definition of $\mathcal{A}_X$, we get $\mathcal{A}_X \simeq \mathbf{L}_{\mathcal{O}_X}(\mathcal{K}u(X) \otimes \mathcal{O}_X(H))$ and the desired result follows. The reverse direction is similar. □

4 | BRIDGELAND STABILITY CONDITIONS

In this section, we recall (weak) Bridgeland stability conditions on $\mathrm{D}^b(X)$, and the notions of tilt stability, double-tilt stability, and stability conditions induced on Kuznetsov components from weak stability conditions on $\mathrm{D}^b(X)$. We follow [4, § 2].

4.1 | Weak stability conditions

Let $\mathcal{D}$ be a triangulated category, and $K_0(\mathcal{D})$ its Grothendieck group. Fix a surjective morphism $v: K_0(\mathcal{D}) \to \Lambda$ to a finite rank lattice.

Definition 4.1. A *stability condition (resp. weak stability condition)* on $\mathcal{D}$ is a pair $\sigma = (\mathcal{A}, Z)$ where $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}$, and $Z: \Lambda \to \mathbb{C}$ is a group homomorphism such that the following conditions hold.

(1) The composition $Z \circ v: K_0(\mathcal{A}) \cong K_0(\mathcal{D}) \to \mathbb{C}$ satisfies: for any $E \neq 0 \in \mathcal{D}$, we have $\operatorname{Im} Z(E) \geqslant 0$ and if $\operatorname{Im} Z(E) = 0$, then $\operatorname{Re} Z(E) < 0$ (resp. $\operatorname{Re} Z(E) \leqslant 0$). From now on, we write $Z(E)$ rather than $Z(v(E))$.

We define a *slope function* μ_σ for σ using Z. For any $E \in \mathcal{A}$, set

$$\mu_\sigma(E) := \begin{cases} -\frac{\operatorname{Re} Z(E)}{\operatorname{Im} Z(E)}, & \operatorname{Im} Z(E) > 0 \\ +\infty, & \text{else.} \end{cases}$$

We say that an object $0 \neq E \in \mathcal{A}$ is σ-(semi)stable if $\mu_\sigma(F) < \mu_\sigma(E/F)$ (respectively, $\mu_\sigma(F) \leqslant \mu_\sigma(E/F)$) for all proper subobjects $F \subset E$.

(2) Any object $E \in \mathcal{A}$ has a Harder–Narasimhan filtration in terms of σ-semistability defined above.
(3) There exists a quadratic form Q on $\Lambda \otimes \mathbb{R}$ such that $Q|_{\ker Z}$ is negative definite, and $Q(E) \geqslant 0$ for all σ-semistable objects $E \in \mathcal{A}$. This is known as the *support property*.

Definition 4.2. Let $\sigma = (\mathcal{A}, Z)$ be a stability condition on $\mathcal{D}$. The *phase* of a σ-semistable object $E \in \mathcal{A}$ is

$$\phi(E) := \frac{1}{\pi}\arg(Z(E)) \in (0, 1].$$

Specially, if $Z(E) = 0$, then $\phi(E) = 1$. If $F = E[n]$, then we define

$$\phi(F) := \phi(E) + n.$$

A *slicing* $\mathcal{P}$ of $\mathcal{D}$ consists of full additive subcategories $\mathcal{P}(\phi) \subset \mathcal{D}$ for each $\phi \in \mathbb{R}$ satisfying

(1) for $\phi \in (0, 1]$, the subcategory $\mathcal{P}(\phi)$ is given by the zero object and all σ-semistable objects whose phase is ϕ;
(2) for $\phi + n$ with $\phi \in (0, 1]$ and $n \in \mathbb{Z}$, we set $\mathcal{P}(\phi + n) := \mathcal{P}(\phi)[n]$.

We will use both notations $\sigma = (\mathcal{A}, Z)$ and $\sigma = (\mathcal{P}, Z)$ for a stability condition σ with heart $\mathcal{A} = \mathcal{P}((0, 1])$ where $\mathcal{P}$ is the slicing of σ.

We say that σ is a *numerical stability condition* on $\mathcal{D}$ if the surjective morphism $v: K_0(\mathcal{D}) \to \Lambda$ factors through the natural surjection $K_0(\mathcal{D}) \twoheadrightarrow \mathcal{N}(\mathcal{D})$ (assuming that $\mathcal{N}(\mathcal{D})$ is well defined).

Next, we recall two natural group actions on the set of stability conditions $\operatorname{Stab}(\mathcal{D})$.

(1) An element $\tilde{g} = (g, G)$ in the universal covering $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ of the group $\mathrm{GL}^+(2, \mathbb{R})$ consists of an increasing function $g : \mathbb{R} \to \mathbb{R}$ such that $g(\phi + 1) = g(\phi) + 1$ and a matrix $G \in \mathrm{GL}^+(2, \mathbb{R})$ with $\det(G) > 0$. It acts on the right on the stability manifold by $\sigma \cdot \tilde{g} := (G^{-1} \circ Z, \mathcal{P}(g(\phi)))$ for any $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}(\mathcal{D})$ (see [12, Lemma 8.2]).
(2) Let $\mathrm{Aut}_\Lambda(\mathcal{D})$ be the group of exact autoequivalences of $\mathcal{D}$, whose action Φ_* on $K_0(\mathcal{D})$ is compatible with $v : K_0(\mathcal{D}) \to \Lambda$. For $\Phi \in \mathrm{Aut}_\Lambda(\mathcal{D})$ and $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}(\mathcal{D})$, we define a left action of the group of linear exact autoequivalences $\mathrm{Aut}_\Lambda(\mathcal{D})$ by $\Phi \cdot \sigma = (\Phi(\mathcal{P}), Z \circ \Phi_*^{-1})$, where Φ_* is the automorphism of $K_0(\mathcal{D})$ induced by Φ.

4.2 | Tilt-stability

Let (X, H) be a polarized smooth projective variety of dimension n and $\sigma_H = (\mathrm{Coh}(X), Z_H)$ be the standard weak stability condition on $\mathrm{Coh}(X)$ defined as

$$Z_H(E) := -H^{n-1}\mathrm{ch}_1(E) + \mathfrak{i}H^n \,\mathrm{rk}(E).$$

Its σ_H-stability coincides with classical μ_H-stability (slope stability). Now for a fixed real number β, consider the following subcategories[†] of $\mathrm{Coh}(X)$:

$$\mathcal{T}^\beta = \langle E \in \mathrm{Coh}(X) \mid E \text{ is } \sigma_H\text{-semistable with } \mu_{\sigma_H}(E) > \beta \rangle,$$

$$\mathcal{F}^\beta = \langle E \in \mathrm{Coh}(X) \mid E \text{ is } \sigma_H\text{-semistable with } \mu_{\sigma_H}(E) \leqslant \beta \rangle.$$

Then it is a result of [19] that the tilted heart $\mathrm{Coh}^\beta(X) := \langle \mathcal{T}^\beta, \mathcal{F}^\beta[1] \rangle$ is the heart of a bounded t-structure on $\mathrm{Coh}(X)$.

Proposition 4.3 [6, 7]. *Let $\alpha > 0$ and $\beta \in \mathbb{R}$. Then, the pair $\sigma_{\alpha,\beta} = (\mathrm{Coh}^\beta(X), Z_{\alpha,\beta})$ defines a weak stability condition on $\mathrm{D}^b(X)$, where*

$$Z_{\alpha,\beta}(E) = \frac{1}{2}\alpha^2 H^n \mathrm{ch}_0^\beta(E) - H^{n-2}\mathrm{ch}_2^\beta(E) + \mathfrak{i}H^{n-1}\mathrm{ch}_1^\beta(E).$$

The quadratic form Q is given by the discriminant

$$\Delta_H(E) = (H^{n-1}\mathrm{ch}_1(E))^2 - 2H^n\mathrm{ch}_0(E)H^{n-2}\mathrm{ch}_2(E).$$

We denote the slope function by $\mu_{\alpha,\beta} := \mu_{\sigma_{\alpha,\beta}}$.

The weak stability conditions $\sigma_{\alpha,\beta}$ constructed above are also known as *tilt-stability* and the heart $\mathrm{Coh}^\beta(X)$ are called the *tilted heart*.

Now pick a weak stability condition $\sigma_{\alpha,\beta}$. We define

$$\mathcal{T}^0_{\alpha,\beta} = \langle E \in \mathrm{Coh}^\beta(X) \mid E \text{ is } \sigma_{\alpha,\beta}\text{-semistable with } \mu_{\alpha,\beta}(E) > 0 \rangle,$$

$$\mathcal{F}^0_{\alpha,\beta} = \langle E \in \mathrm{Coh}^\beta(X) \mid E \text{ is } \sigma_{\alpha,\beta}\text{-semistable with } \mu_{\alpha,\beta}(E) \leqslant 0 \rangle.$$

[†] The angle brackets here mean extension closure.

Moreover, we "rotate" the stability function $Z_{\alpha,\beta}$ by setting

$$Z^0_{\alpha,\beta} := \frac{1}{\mathfrak{i}} Z_{\alpha,\beta}.$$

Then, we have the following result.

Proposition 4.4 [4, Proposition 2.15]. *The pair $\sigma^0_{\alpha,\beta} = (\mathrm{Coh}^0_{\alpha,\beta}(X) = \langle \mathcal{T}^0_{\alpha,\beta}, \mathcal{F}^0_{\alpha,\beta}[1]\rangle, Z^0_{\alpha,\beta})$ defines a weak stability condition on $\mathrm{D}^b(X)$. We denote the slope function by $\mu^0_{\alpha,\beta} := \mu_{\sigma^0_{\alpha,\beta}}$.*

We now state a useful lemma that relates 2-Gieseker-stability (see [2, Definition 4.3]) and tilt-stability.

Lemma 4.5 [6, Lemma 2.7], [2, Proposition 4.8, 4.9]. *Let $E \in \mathrm{D}^b(X)$.*

(1) *Let $\beta < \mu(E)$. Then $E \in \mathrm{Coh}^\beta(X)$ is $\sigma_{\alpha,\beta}$-(semi)stable for $\alpha \gg 0$ if and only if $E \in \mathrm{Coh}(X)$ and E is 2-Gieseker-(semi)stable.*
(2) *If $E \in \mathrm{Coh}^\beta(X)$ is $\sigma_{\alpha,\beta}$-semistable for $\beta \geqslant \mu(E)$ and $\alpha \gg 0$, then $\mathcal{H}^{-1}(E)$ is a torsion-free μ-semistable sheaf and $\mathcal{H}^0(E)$ is supported in dimension not greater than one. If $\beta > \mu(E)$ and $\alpha > 0$, then $\mathcal{H}^{-1}(E)$ is also reflexive.*

4.3 | Stronger BG inequalities

In this subsection, we state stronger Bogomolov–Gieseker (BG) style inequalities, which hold for tilt-semistable objects. These will be useful later on for ruling out potential walls for tilt-stability of objects in $\mathrm{D}^b(X)$. The first is a stronger version of Proposition 4.3, which was proved by Chunyi Li in [36, Proposition 3.2] for Fano threefolds of Picard number one.

Lemma 4.6 (Stronger BG I). *Let X be an index 1 prime Fano threefold with degree d, and $E \in \mathrm{D}^b(X)$ a $\sigma_{\alpha,\beta}$-stable object where $\alpha > 0$. Let $k := \lfloor \mu(E) \rfloor$. Then we have:*

$$\frac{H \cdot \mathrm{ch}_2(E)}{H^3 \cdot \mathrm{ch}_0(E)} \leqslant \max\left\{ k\mu_H(E) - \frac{k^2}{2}, \frac{1}{2}\mu_H(E)^2 - \frac{3}{4d}, (k+1)\mu_H(E) - \frac{(k+1)^2}{2} \right\}.$$

Moreover, if the equality holds, then E has rank one or two.

The second is due to Naoki Koseki and Chunyi Li. It is based on [27, Lemma 4.2, Theorem 4.3]. Chunyi Li also sent us a similar inequality from his upcoming paper [37].

Theorem 4.7 (Stronger BG II). *Let X be an index 1 Fano threefold of degree d, and $E \in \mathrm{Coh}^0(X)$ be a $\sigma_{\alpha,0}$-semistable object for some $\alpha > 0$ with $|\mu_H(E)| \in [0,1]$ and $\mathrm{rk}(E) \geqslant 2$. Then*

$$\frac{H \cdot \mathrm{ch}_2(E)}{H^3 \cdot \mathrm{ch}_0(E)} \leqslant \max\left\{ \frac{1}{2}\mu_H(E)^2 - \frac{3}{4d}, \mu_H(E)^2 - \frac{1}{2}|\mu_H(E)| \right\}.$$

Before we prove Theorem 4.7, we first state an easy lemma.

Lemma 4.8. *Let S be a K3 surface of degree d and H_S the ample polarization. Let E be a μ_{H_S}-semistable sheaf in $\mathrm{D}^b(S)$ with $\mathrm{rk}(E) \geqslant 2$. Then*

$$\frac{\mathrm{ch}_2(E)}{H_S^2 \cdot \mathrm{rk}(E)} \leqslant \frac{1}{2}\mu_{H_S}(E)^2 - \frac{3}{4d}.$$

Proof. Let $v(E)$ be the Mukai vector of E. We have

$$\begin{aligned} v(E)^2 &= H_S^2 \cdot \mathrm{ch}_1(E)^2 - 2\,\mathrm{rk}(E)^2 - 2\,\mathrm{rk}(E) \cdot \mathrm{ch}_2(E) \\ &\geqslant -2 \geqslant -\frac{1}{2}\,\mathrm{rk}(E)^2. \end{aligned}$$

Dividing through by $\mathrm{rk}(E)^2$ and rearranging, we get

$$\frac{\mathrm{ch}_2(E)}{\mathrm{rk}(E)} \leqslant \frac{1}{2}\mu_{H_S}(E)^2 H_S^2 - \frac{3}{4}$$

as required. □

Proof of Theorem 4.7. Let $f : [0,1] \to \mathbb{R}$ be defined as

$$f(t) := \max\left\{\frac{1}{2}t^2 - \frac{3}{4d}, t^2 - \frac{1}{2}t\right\}.$$

Note that f is star-shaped [27, Definition 3.2] and satisfies $f(0) = 0$ and $f(1) = 1/2$ as well as

$$t^2 - \frac{1}{2}t \leqslant f(t) \leqslant \frac{1}{2}t^2$$

for all $t \in [0,1]$. We now follow the strategy of proof in [27, Theorem 4.3]. Assume for a contradiction that there is an $E \in \mathrm{D}^b(X)$ such that the inequality in the statement of Theorem 4.7 is not true. Then conditions (a) and (b) in [27, Lemma 3.3] are satisfied for f. Then by [27], the restriction $E|_{S_d}$ where S_d is a general hyperplane section of X_d is $\mu_{H_{S_d}}$-semistable. Also note that $\mu_{H_{S_d}}(E|_{S_d}) = \mu_H(E)$ and S_d is a smooth K3 surface. But then by assumption,

$$\frac{\mathrm{ch}_2(E|_{S_d})}{H_{S_d}^2 \cdot \mathrm{rk}(E|_{S_d})} > \frac{1}{2}\mu_{H_{S_d}}(E)^2 - \frac{3}{4d},$$

which contradicts Proposition 4.8, so the assumption is false and the result follows. □

4.4 | Stability conditions on the Kuznetsov component of a GM threefold

Proposition 5.1 in [4] gives a criterion for checking when weak stability conditions on a triangulated category can be used to induce stability conditions on a subcategory. Each of the criteria of this proposition can be checked for $\mathcal{A}_X \subset \mathrm{D}^b(X)$ to give stability conditions on $\mathcal{A}_X$.

More precisely, let $\mathcal{A}(\alpha,\beta) = \mathrm{Coh}^0_{\alpha,\beta}(X) \cap \mathcal{A}_X$ and $Z(\alpha,\beta) = Z^0_{\alpha,\beta}|_{\mathcal{A}_X}$. Furthermore, if we take suitable (α,β), by [4, Theorem 6.9] and [48, Proposition 3.2], we have the following.

Theorem 4.9. *Let X be a GM threefold. Then $\sigma(\alpha,\beta)$ is a stability condition on $\mathcal{A}_X$ for all $(\alpha,\beta) \in V$, where*

$$V := \{(\alpha,\beta) : -\frac{1}{10} < \beta < 0, 0 < \alpha < -\beta\}.$$

Now we introduce a special class of stability condition, which will play a central role in our paper.

Definition 4.10. Let σ be a stability condition on a triangulated category $\mathcal{D}$. It is called *Serre-invariant* if $S_{\mathcal{D}} \cdot \sigma = \sigma \cdot g$ for some $g \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$, where $S_{\mathcal{D}}$ is the Serre functor of $\mathcal{D}$.

We recall a recent result proved in [48].

Theorem 4.11. *Let X be a GM threefold and σ (resp. σ') be a stability condition on $\mathcal{K}u(X)$ (resp. $\mathcal{A}_X$) defined by [4]. Then σ (resp. σ') is Serre-invariant.*

Proposition 4.12. *Let X be a GM threefold and E a nonzero object in $\mathcal{A}_X$ such that $\mathrm{ext}^1(E,E) \leqslant 3$ and $-\chi(E,E)$ is not a perfect square. Then E is σ-stable for every Serre-invariant stability condition σ on $\mathcal{A}_X$.*

Proof. The proof is the same as in [54, Lemma 9.12]. We omit the details. □

5 | PROJECTION OF $\mathcal{E}$ INTO $\mathcal{K}u(X)$

In this section, we consider the object that results from projecting the vector bundle $\mathcal{E}$ into $\mathcal{K}u(X)$, and its stability in $\mathcal{K}u(X)$. We start with a lemma.

Lemma 5.1. *Let X be a GM threefold.*

(1) $\mathrm{RHom}^\bullet(\mathcal{Q}(-H),\mathcal{E}) = \mathrm{RHom}^\bullet(\mathcal{E},\mathcal{Q}^\vee) = k^2$ *when X is ordinary.*
(2) $\mathrm{RHom}^\bullet(\mathcal{Q}(-H),\mathcal{E}) = \mathrm{RHom}^\bullet(\mathcal{E},\mathcal{Q}^\vee) = k^3 \oplus k[-1]$ *when X is special.*
(3) $\mathrm{RHom}^\bullet(\mathcal{E},\mathcal{Q}(-H)) = k[-2]$.
(4) $\mathrm{RHom}^\bullet(\mathcal{Q}^\vee,\mathcal{E}) = \mathrm{RHom}^\bullet(\mathcal{E}^\vee,\mathcal{Q}) = k[-2]$.

Proof. When X is ordinary, (1) and (2) follow from the Koszul resolution of $X \subset \mathrm{Gr}(2,5)$ and the Borel–Weil–Bott Theorem. When X is special with the double cover $\pi : X \to Y$, note that $\pi_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{O}_Y(-1)$. Then the (1) and (2) follow from the projection formula and [53, Lemma 2.14, Proposition 2.15]. And applying $\mathrm{Hom}(-,\mathcal{E})$ to (1) and using Serre duality, we get $\mathrm{RHom}^\bullet(\mathcal{E},\mathcal{Q}(-H)) = \mathrm{RHom}^\bullet(\mathcal{Q},\mathcal{E})^\vee[-3] = k[-2]$, which proves (3). Finally, (4) follows from applying $\mathrm{Hom}(-,\mathcal{E})$ to (1) and using Serre duality and $\mathrm{RHom}^\bullet(\mathcal{E},\mathcal{E}) = k$. □

5.1 | The projection of $\mathcal{E}$ into $\mathcal{K}u(X)$

Let $\pi := \mathbf{R}_{\mathcal{O}_X(-H)}\mathbf{R}_{\mathcal{E}(-H)} : \mathrm{D}^b(X) \to \mathcal{K}u(X)$ be the right adjoint to the inclusion $\mathcal{K}u(X) \hookrightarrow \mathrm{D}^b(X)$. Here, $\mathcal{K}u(X) = \langle \mathcal{E}, \mathcal{O}_X\rangle^{\perp}$ is the original Kuznetsov component.

Lemma 5.2. *The projection object $\pi(\mathcal{E})$ is the unique object that fits into a nontrivial exact triangle*

$$\mathcal{Q}(-H)[1] \to \pi(\mathcal{E}) \to \mathcal{E}. \tag{2}$$

Proof. By Serre duality, we have $\mathrm{RHom}^\bullet(\mathcal{E}, \mathcal{E}(-H)) = \mathrm{RHom}^\bullet(\mathcal{E}, \mathcal{E})^\vee[-3] = k[-3]$. Then, we have an exact triangle $\mathbf{R}_{\mathcal{E}(-H)}\mathcal{E} \to \mathcal{E} \to \mathcal{E}(-H)[3]$. And by (1), we see $\mathbf{R}_{\mathcal{O}_X(-H)}\mathcal{E}(-H) = \mathcal{Q}(-H)[-1]$. Thus, from $\mathbf{R}_{\mathcal{O}_X(-H)}\mathcal{E} = \mathcal{E}$, we obtain the triangle (2). It is nontrivial since $\pi(\mathcal{E}) \in \mathcal{K}u(X)$, so $\mathcal{E}$ cannot be a direct summand of $\pi(\mathcal{E})$. Finally, the uniqueness follows from Lemma 5.1 (4). □

Lemma 5.3. *Let X be a GM threefold. Then, we have*

- $\mathrm{RHom}^\bullet(\pi(\mathcal{E}), \pi(\mathcal{E})) = k \oplus k^2[-1]$ *when X is ordinary.*
- $\mathrm{RHom}^\bullet(\pi(\mathcal{E}), \pi(\mathcal{E})) = k \oplus k^3[-1] \oplus k[-2]$ *when X is special.*

Hence, $\pi(\mathcal{E})$ is stable with respect to every Serre-invariant stability condition on $\mathcal{K}u(X)$.

Proof. The first statement follows from applying $\mathrm{Hom}(-, \mathcal{E})$ to triangle (2) and Lemma 5.1, and also the fact that $\mathrm{RHom}^\bullet(\pi(\mathcal{E}), \pi(\mathcal{E})) = \mathrm{RHom}^\bullet(\pi(\mathcal{E}), \mathcal{E})$ which is by adjunction. The last statement follows from Lemma 4.12. □

5.2 | The analogous projection object for $\mathcal{A}_X$

In this subsection, we state and prove the analogous results as in Subsection 5.1, except for $\mathcal{A}_X$ instead of $\mathcal{K}u(X)$. Let $\pi' := \mathbf{R}_{\mathcal{E}}\mathbf{R}_{\mathcal{O}_X(-H)} : \mathrm{D}^b(X) \to \mathcal{A}_X$ be the right adjoint to the inclusion $\mathcal{A}_X \hookrightarrow \mathrm{D}^b(X)$.

Lemma 5.4. *The projection object $\pi'(\mathcal{Q}^\vee)$ is the unique object fits into a nontrivial exact triangle*

$$\mathcal{E}[1] \to \pi'(\mathcal{Q}^\vee) \to \mathcal{Q}^\vee. \tag{3}$$

Proof. The proof is completely analogous to the proof of Lemma 5.2. By Serre duality, we have the vanishing $\mathrm{RHom}^\bullet(\mathcal{Q}^\vee, \mathcal{O}_X(-H)) = \mathrm{RHom}^\bullet(\mathcal{O}_X, \mathcal{Q}^\vee)^\vee = 0$. Thus, $\pi'(\mathcal{Q}^\vee) = \mathbf{R}_{\mathcal{E}}\mathcal{Q}^\vee$. Then, the result follows from Lemma 5.1 (4). Finally, the uniqueness also follows from Lemma 5.1 (4) and (3) is nontrivial since $\mathrm{RHom}^\bullet(\mathcal{Q}^\vee, \pi'(\mathcal{Q}^\vee)) = 0$. □

Remark 5.5. Later, in Section 7, we will see that we have $\pi'(\mathcal{Q}^\vee) \cong \mathrm{pr}(I_C)[1]$ where $C \subset X$ is a conic such that $I_C \notin \mathcal{A}_X$.

Lemma 5.6. *Let X be a GM threefold. Then*

- $\mathrm{RHom}^\bullet(\pi'(\mathcal{Q}^\vee), \pi'(\mathcal{Q}^\vee)) = k \oplus k^2[-1]$ *when X is ordinary.*
- $\mathrm{RHom}^\bullet(\pi'(\mathcal{Q}^\vee), \pi'(\mathcal{Q}^\vee)) = k \oplus k^3[-1] \oplus k[-2]$ *when X is special.*

Hence, $\pi'(\mathcal{Q}^\vee)$ is stable with respect to every Serre-invariant stability condition on $\mathcal{A}_X$.

Proof. It is not hard to check that $\Xi(\pi(\mathcal{E})) \cong \pi'(\mathcal{Q}^\vee)[1]$, where Ξ is the equivalence $\mathcal{K}u(X) \simeq \mathcal{A}_X$ in Lemma 3.7. Then the result follows from Lemma 5.3. $\square$

6 | CONICS ON GM THREEFOLDS

In this section, we collect some useful results regarding the birational geometry of GM threefolds and their Hilbert schemes of conics. The results in this section are all from [13, 41], and [22].

Recall that a conic means a closed subscheme $C \subset X$ with Hilbert polynomial $p_C(t) = 1 + 2t$, and a line means a closed subscheme $L \subset X$ with Hilbert polynomial $p_L(t) = 1 + t$. Denote their Hilbert schemes by $\mathcal{C}(X)$ and $\Gamma(X)$, respectively.

6.1 | Conics on ordinary GM threefolds

Let X be an ordinary GM threefold. Recall that it is a quadric section of a linear section of codimension 2 of the Grassmannian $\mathrm{Gr}(2,5) = \mathrm{Gr}(2, V_5)$, where V_5 is a five-dimensional complex vector space. Let V_i be an i-dimensional vector subspace of V_5. There are two types of 2-planes in $\mathrm{Gr}(2,5)$; σ*-planes* are given set-theoretically as $\{[V_2] \mid V_1 \subset V_2 \subset V_4\}$, and ρ*-planes* are given by $\{[V_2] \mid V_2 \subset V_3\}$.

Remark 6.1. In [13, Section 3.1], the σ-planes and ρ-planes are called α-planes and β-planes, respectively.

By [13, Section 3.1] and [24, Section 3.1], we have the following classification of conics on X.

Definition 6.2 [13, p. 5].

- A conic $C \subset X$ is called a τ*-conic* if the 2-plane $\langle C \rangle$ is not contained in $\mathrm{Gr}(2, V_5)$, there is a unique $V_4 \subset V_5$ such that $C \subset \mathrm{Gr}(2, V_4)$, the conic C is reduced and if it is smooth, the union of corresponding lines in $\mathbb{P}(V_5)$ is a smooth quadric surface in $\mathbb{P}(V_4)$.
- A conic $C \subset X$ is called a σ*-conic* if the 2-plane $\langle C \rangle$ spanned by C is an σ-plane, and if there is a unique hyperplane $V_4 \subset V_5$ such that $C \subset \mathrm{Gr}(2, V_4)$ and the union of the corresponding lines in $\mathbb{P}(V_5)$ is a quadric cone in $\mathbb{P}(V_4)$.
- A conic $C \subset X$ is called a ρ*-conic* if the 2-plane $\langle C \rangle$ spanned by C is a ρ-plane, and the union of corresponding lines in $\mathbb{P}(V_5)$ is this 2-plane.

The following lemma is very useful for computations:

Lemma 6.3. *Let X be an ordinary GM threefold and C be a conic on X.*

(1) *If C is a τ-conic, then we have* $\mathrm{RHom}^\bullet(\mathcal{E}, I_C) = k$ *and* $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, I_C) = 0$.
(2) *If C is a ρ-conic, then we have* $\mathrm{RHom}^\bullet(\mathcal{E}, I_C) = k^2 \oplus k[-1]$ *and* $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, I_C) = 0$.
(3) *If C is a σ-conic, then we have* $\mathrm{RHom}^\bullet(\mathcal{E}, I_C) = k$ *and* $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, I_C) = k[-1] \oplus k[-2]$.

Proof. Note that if $\mathrm{Hom}(\mathcal{E}, I_C) = k^a$, then $C \subset \mathrm{Gr}(2, 5-a) \cap X$. Since for any conic C, there is some V_4 such that $C \subset \mathrm{Gr}(2, V_4)$, then we have $\hom(\mathcal{E}, I_C) \geqslant 1$ for any conic C. Now if

$\hom(\mathcal{E}, I_C) \geqslant 2$, we know that C is contained in a ρ-plane $\mathrm{Gr}(2,3)$. Since $\langle C\rangle$ is not in $\mathrm{Gr}(2,5)$ for a τ-conic C, and $\langle C\rangle$ is a σ-plane $\{V_2 | V_1 \subset V_2 \subset V_4\}$ for a σ-conic, for these two types of conics, we have $\mathrm{Hom}(\mathcal{E}, I_C) = k$. Also, for a ρ-conic C, since $\langle C\rangle = \mathrm{Gr}(2,3)$, we have $\hom(\mathcal{E}, I_C) \geqslant 2$. But if $\hom(\mathcal{E}, I_C) \geqslant 3$, we know that $C \subset \mathrm{Gr}(2,2)$ that is impossible. Hence, for a ρ-conic C, we have $\mathrm{Hom}(\mathcal{E}, I_C) = k^2$. Now the result for Ext groups follows from applying $\mathrm{Hom}(\mathcal{E}, -)$ to the short exact sequence $0 \to I_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0$ and $\chi(\mathcal{E}, I_C) = 1$.

First, by stability and Serre duality, we have $\mathrm{Hom}(\mathcal{E}^\vee, I_C) = \mathrm{Ext}^3(\mathcal{E}^\vee, I_C) = 0$. From $\chi(\mathcal{E}^\vee, I_C) = 0$, we only need to compute $\mathrm{Ext}^1(\mathcal{E}^\vee, I_C)$. Since $\mathrm{RHom}^\bullet(\mathcal{O}_X, I_C) = 0$, applying $\mathrm{Hom}(-, I_C)$ to the tautological sequence, we have $\mathrm{Hom}(\mathcal{Q}^\vee, I_C) = \mathrm{Ext}^1(\mathcal{E}^\vee, I_C)$. Note that if $\mathrm{Hom}(\mathcal{Q}^\vee, I_C) = k^a$, then $C \subset \mathrm{Gr}(2-a, 5-a) \cap X$. Thus, we have $\hom(\mathcal{Q}^\vee, I_C) \leqslant 1$ for any conic C. And since $\hom(\mathcal{Q}^\vee, I_C) = 1$ if and only if C is contained in the zero locus of a global section of $\mathcal{Q}$, which is a σ-3-plane in $\mathrm{Gr}(2,5)$, we know that $\mathrm{Hom}(\mathcal{Q}^\vee, I_C) = 0$ for C of type τ or ρ, and $\mathrm{Hom}(\mathcal{Q}^\vee, I_C) = k$ for a σ-conic. Then the result follows. □

Now we recall some properties of the Fano surface of conics $C(X)$.

Theorem 6.4 [13, 41]. *Let X be an ordinary GM threefold. Then $C(X)$ is an irreducible projective surface. If X is furthermore general, then $C(X)$ is smooth.*

It is a fact that there is a unique ρ-conic on X, and there is a curve $L_\sigma \subset C(X)$ parameterize all σ-conics on X (cf. [13, Section 5.1]), and we denote it by c_X. Furthermore, we have the following result that is a corollary of Logachev's tangent bundle theorem [41, Section 4].

Lemma 6.5 [13, p. 16]. *The only rational curve in $C(X)$ is L_σ. Furthermore, there exists a surface $C_m(X)$ and a map $C(X) \to C_m(X)$ that contracts L_σ to a point $[\pi]$. If X is general, then $C_m(X)$ is the minimal surface of $C(X)$.*

Theorem 6.6 [13, Section 5.2]. *Let X be a general ordinary GM threefold. Then there is a natural involution ι on $C_m(X)$, switching the points $[c_X]$ and $[\pi]$.*

Another important result that we require is Logachev's reconstruction theorem. This was originally proved in [41, Theorem 7.7], and then reproved later in [13, Theorem 9.1].

Theorem 6.7 (Logachev's reconstruction theorem). *Let X and X' be general ordinary GM threefolds. If $C(X) \cong C(X')$, then $X \cong X'$.*

6.2 | Conic and line transforms

For this section, we follow [13, Section 6.1]. Let X be a general ordinary GM threefold, and let C be a conic. Then, in [13, § 6.1, Theorem 6.4], the authors construct a new GM threefold X_C and a birational map $\psi_C : X \dashrightarrow X_C$, called the *conic transform*. Similarly, for any line $L \subset X$, a new GM threefold X_L and a birational morphism $\psi_L : X \to X_L$ are constructed in [13, Section 6.2], called the *line transform*.

Note that in [14], such an X_C is called the *period partner* of X, and the line transforms are called the *period duals*. We now list some important results about conic and line transforms below.

Theorem 6.8 [13, Theorem 6.4]. *Let X be a general ordinary GM threefold, and let $C \subset X$ be a conic. Then $\mathcal{C}(X_C)$ is isomorphic to $\mathcal{C}_m(X)$ blown up at the point $[C] \in \mathcal{C}_m(X)$, where $\mathcal{C}_m(X)$ is the minimal surface of $\mathcal{C}(X)$.*

Proposition 6.9 [13, Theorem 6.4, Remark 7.2]. *Let X be a general ordinary GM threefold. Then the isomorphism classes of conic transforms of X are parametrized by the surface $\mathcal{C}_m(X)/\iota$.*

Theorem 6.10 [34, Theorem 1.6]. *Let X be a general ordinary GM threefold. Then, the Kuznetsov components of all conic transforms and line transforms of X are equivalent to $\mathcal{A}_X$.*

6.3 | Conics on special GM threefolds

Let X be a special GM threefold. Recall that X is a double cover $X \to Y$ of a degree five del Pezzo threefold Y with branch locus a quadric hypersurface $\mathcal{B} \subset Y$. When X is general, $\mathcal{B}$ is a smooth K3 surface of Picard number 1 and degree 10. Recall that Y is a codimension 3 linear section of $\mathrm{Gr}(2,5)$. Let $\mathcal{V}$ be the tautological quotient bundle on Y. We recall some properties of $\mathcal{C}(X)$ from [22].

Theorem 6.11 [22]. *Let X be a special GM threefold. Then $\mathcal{C}(X)$ has two components $\mathcal{C}_1$ and $\mathcal{C}_2$. One of the components $\mathcal{C}_2 \cong \Sigma(Y) \cong \mathbb{P}^2$ parameterizes the preimage of lines on Y. Moreover, when X is general, $\mathcal{C}(X)$ is smooth away from $\mathcal{C}_1 \cap \mathcal{C}_2$.*

The following lemma will be useful in computations; it is similar to Lemma 6.3.

Lemma 6.12. *Let X be a special GM threefold and C a conic on X. Then $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, I_C) \neq 0$ if and only if C is the preimage of a line on Y. In this case, $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, I_C) = k[-1] \oplus k[-2]$, and such a family of conics is parametrized by the Hilbert scheme of lines $\Sigma(Y) \cong \mathbb{P}^2$ on Y.*

Proof. The proof is almost the same as the second part of the proof of Lemma 6.3. The same argument shows that $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, I_C) \neq 0$ if and only if $\mathrm{Hom}(\mathcal{Q}^\vee, I_C) \neq 0$. The image of a nontrivial map $\mathcal{Q}^\vee \to I_C$ is the ideal sheaf of the zero locus of a section s of $\mathcal{Q}$, which is the preimage of the zero locus of a section of $\mathcal{V}$. By [53, Lemma 2.18], the zero locus of a section of $\mathcal{V}$ is either a line or a point. Thus, the zero locus of a section of $\mathcal{Q}$ is either the preimage of a line on Y that is a conic on X, or a zero-dimensional closed subscheme of length two. But this zero locus contains a conic $C \subset X$, so $C = Z(s)$ is the preimage of a line on Y and the map $\mathcal{Q}^\vee \to I_C$ is surjective. In particular, such conics are exactly the preimages of lines on Y and are parametrized by $\Sigma(Y) \cong \mathbb{P}^2$. □

7 | CONICS AND BRIDGELAND MODULI SPACES

In this section, we study the moduli space of σ-stable objects of the (-1)-class $-x$ in the alternative Kuznetsov component $\mathcal{A}_X$ of a GM threefold X and its relation to $\mathcal{C}(X)$. Our main result in this section is Theorem 7.12, which realizes the Bridgeland moduli space as a contraction of $\mathcal{C}(X)$.

First, we study those conics C such that $I_C \notin \mathcal{A}_X$.

Proposition 7.1. *Let $C \subset X$ be a conic on a GM threefold X. Then, $I_C \notin \mathcal{A}_X$ if and only if*

(1) *C is a σ-conic when X is ordinary. In particular, such a family of conics is parametrized by the line L_σ.*
(2) *C is the preimage of a line on Y when X is special. In particular, such a family of conics is parametrized by the Hilbert scheme of lines $\Sigma(Y) \cong \mathbb{P}^2$ on Y.*

Moreover, we have an exact sequence

$$0 \to \mathcal{E} \to \mathcal{Q}^\vee \to I_C \to 0.$$

Proof. Note that $I_C \notin \mathcal{A}_X$ if and only if $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, I_C) \neq 0$. When X is ordinary, (1) follows from Lemma 6.3. When X is special, we deduce (2) from Lemma 6.12. Note that since $I_C \notin \mathcal{A}_X$, we have $\mathrm{Hom}(\mathcal{Q}^\vee, I_C) \neq 0$. The nontrivial map $\mathcal{Q}^\vee \to I_C$ is surjective by the arguments in Lemma 6.3 and 6.12. Note that by the stability of $\mathcal{Q}^\vee$, the kernel of $\mathcal{Q}^\vee \twoheadrightarrow I_C$ is μ-stable with the same Chern character as $\mathcal{E}$, hence we have $\ker(\mathcal{Q}^\vee \twoheadrightarrow I_C) \cong \mathcal{E}$ by [13, Proposition 4.1]. □

Proposition 7.2. *Let X be a GM threefold and $C \subset X$ a conic on X. If $I_C \notin \mathcal{A}_X$, then we have the exact triangle*

$$\mathcal{E}[1] \to \mathrm{pr}(I_C) \to \mathcal{Q}^\vee$$

and $\mathrm{pr}(I_C) \cong \pi'(\mathcal{Q}^\vee)$

Proof. By Proposition 7.1, I_C fits into the short exact sequence

$$0 \to \mathcal{E} \to \mathcal{Q}^\vee \to I_C \to 0.$$

Applying the projection functor to this exact sequence, and note that applying the functor pr to the dual exact sequence of (1) gives $\mathrm{pr}(\mathcal{Q}^\vee) = 0$. Then, we have $\mathrm{pr}(I_C) \cong \mathrm{pr}(\mathcal{E})[1]$. Now we compute the projection $\mathrm{pr}(\mathcal{E})$. Since $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, \mathcal{E}) \cong k[-3]$, we get an exact triangle $\mathcal{E}^\vee[-3] \to \mathcal{E} \to \mathbf{L}_{\mathcal{E}^\vee}\mathcal{E}$. Now applying $\mathbf{L}_{\mathcal{O}_X}$ to this triangle and using $\mathbf{L}_{\mathcal{O}_X}\mathcal{E}^\vee = \mathcal{Q}^\vee[1]$, we get

$$\mathcal{Q}^\vee[-2] \to \mathcal{E} \to \mathrm{pr}(\mathcal{E}).$$

Therefore, we obtain the triangle

$$\mathcal{E}[1] \to \mathrm{pr}(\mathcal{E})[1] \to \mathcal{Q}^\vee,$$

and the desired result follows from Lemma 5.4. □

Now the following two results follow from Proposition 7.2 and Lemma 5.6.

Lemma 7.3. *Let X be a GM threefold. If $C \subset X$ is a conic such that $I_C \notin \mathcal{A}_X$, then*

- $\mathrm{RHom}^\bullet(\mathrm{pr}(I_C), \mathrm{pr}(I_C)) = k \oplus k^2[-1]$ *when X is ordinary.*
- $\mathrm{RHom}^\bullet(\mathrm{pr}(I_C), \mathrm{pr}(I_C)) = k \oplus k^3[-1] \oplus k[-2]$ *when X is special.*

Lemma 7.4. *Let X be a GM threefold. If $I_C \notin \mathcal{A}_X$, the projection* $\mathrm{pr}(I_C)[1]$ *is stable with respect to every Serre-invariant stability condition on $\mathcal{A}_X$.*

When $I_C \in \mathcal{A}_X$, we cannot use Proposition 4.12 to prove the Bridgeland stability of I_C, since $C(X)$ can be singular and $\mathrm{Ext}^1(I_C, I_C)$ may have large dimension. Instead, we use a wall-crossing argument and the uniqueness of Serre-invariant stability conditions (Theorem A.10).

Lemma 7.5. *Let X be a GM threefold. Let F be an object with* $\mathrm{ch}_{\leqslant 2}(F) = (1, 0, -2L)$. *Then there are no walls for F in the range $-\frac{1}{2} \leqslant \beta < 0$ and $\alpha > 0$.*

Proof. Recall that by [2, Theorem 4.13], $\beta = 0$ is the unique vertical wall of F. Any other wall is a semicircle centered along the β-axis, and its apex lies on the hyperbola $\mu_{\alpha,\beta}(F) = 0$. Moreover, no two walls intersect.

Note that when $\mu_{\alpha,\beta}(F) = 0$ holds, we have $\beta < -\sqrt{\frac{2}{5}} < -\frac{1}{2}$; thus, we know that there is no semicircular wall centered in the interval $-\frac{1}{2} \leqslant \beta < 0$. Therefore, any semicircular wall in the range $-\frac{1}{2} \leqslant \beta < 0$ will intersect $\beta = -\frac{1}{2}$. To prove the statement, we only need to show that there are no walls when $\beta = -\frac{1}{2}$. This follows from the fact that $\mathrm{ch}_1^{-\frac{1}{2}}(F)$ is minimal. □

Lemma 7.6. *Let $C \subset X$ be a conic on a GM threefold X such that $I_C \in \mathcal{A}_X$. Then $I_C[1]$ is stable with respect to every Serre-invariant stability condition on $\mathcal{A}_X$.*

Proof. By Lemma 4.5 and Lemma 7.5, we know that I_C is $\sigma_{\alpha,\beta}$-semistable for every $(\alpha, \beta) \in V$. Since I_C is torsion-free, $I_C[1] \in \mathrm{Coh}^0_{\alpha,\beta}(X)$ is $\sigma^0_{\alpha,\beta}$-semistable. Thus, $I_C[1] \in \mathcal{A}(\alpha, \beta)$ is $\sigma(\alpha, \beta)$-semistable. Then, stability with respect to every Serre-invariant stability condition follows from Theorem 4.11 and Theorem A.10. □

7.1 | The Bridgeland moduli space of class $-x$

In this subsection, we are going to describe the Bridgeland moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ in Theorem 7.12.

The proofs in this section seem technical. However, the only results in this section that will be used in other sections are Proposition 7.11 in the proof of Theorem 7.12, so there is no harm for readers in skipping this whole section and assuming Proposition 7.11 and Theorem 7.12.

We start with two lemmas.

Lemma 7.7. *Let X be a GM threefold and E a μ-semistable sheaf on X with truncated Chern character* $\mathrm{ch}_{\leqslant 2}(E) = (2, -H, aL)$. *If $a \geqslant 1$ and $c_3(E) \geqslant 0$, then we have $E \cong \mathcal{E}$.*

Proof. By Lemma 4.6, we have $a \leqslant 1$ which means $a = 1$ by our assumption. Then $c_1(E) = -1$ and $c_2(E) = 4$. Since $c_3(E) \geqslant 0$, by [11, Proposition 3.5], we have $\chi(E) = 0$. This implies $c_3(E) = 0$. Moreover, $E^{\vee\vee}$ also satisfies the assumptions above. Hence, by the previous argument, we have $c_1(E^{\vee\vee}) = -1, c_2(E^{\vee\vee}) = 4$ and $c_3(E^{\vee\vee}) = 0$ as well. In other words, $E = E^{\vee\vee}$. Since $c_3(E) = 0$ and E is reflexive of rank 2, it is a vector bundle. Moreover, E is a globally generated bundle by [11, Proposition 3.5]. Thus, $E \cong \mathcal{E}$ by [13, Proposition 4.1]. □

Lemma 7.8. *Let X be a GM threefold and E a μ-semistable sheaf on X with* $\mathrm{ch}(E) = \mathrm{ch}(\mathcal{Q})$. *Then we have $E \cong \mathcal{Q}$.*

Proof. First we show that $h^2(E) = 0$; then from $\chi(E) = 5$, we have $h^0(E) \geqslant 5$. Indeed, if $h^2(E) \neq 0$, then $\mathrm{Hom}(E, \mathcal{O}_X(-H)[1]) \neq 0$ by Serre duality. Therefore, we have a nontrivial extension

$$0 \to \mathcal{O}_X(-H) \to F \to E \to 0.$$

If F is not μ-semistable, then by the stability of $\mathcal{O}_X(-H)$ and E, the minimal destabilizing quotient sheaf F' of F has $\mathrm{ch}_{\leqslant 1}(F') = (1, -H)$. Thus, $F'^{\vee\vee} \cong \mathcal{O}_X(-H)$. But if we apply $\mathrm{Hom}(-, \mathcal{O}_X(-H))$ to the exact sequence above, we obtain $\mathrm{Hom}(F, \mathcal{O}_X(-H)) = 0$ since this extension is nontrivial, which gives a contradiction. Then, F is μ-semistable with $\mathrm{ch}_{\leqslant 2}(F) = (4, 0, 4L)$, which is impossible since $\Delta(F) < 0$.

Now we can take five linearly independent elements in $H^0(E)$ and obtain a map $t : \mathcal{O}_X^{\oplus 5} \to E$. From the stability of $\mathcal{O}_X$ and E, we have $\mu(\mathrm{Im}(t)) = 0$ or $\mu(\mathrm{Im}(t)) = \frac{1}{3}$. But the first case cannot happen, since then $\mathrm{Im}(t)$ is the direct sum of a number of copies of $\mathcal{O}_X$, and this contradicts the construction of t. Thus, $\mu(\mathrm{Im}(t)) = \frac{1}{3}$ and $\mathrm{ch}_{\leqslant 1}(\mathrm{Im}(t)) = (3, H)$. Also $\mathrm{ch}_{\leqslant 2}(\ker(t)) = (2, -H, xL)$, where $x \geqslant 1$. Note that $\ker(t)$ is reflexive, thus we have $c_3(\ker(t)) \geqslant 0$ since $\ker(t)$ has rank 2. Then by stability of $\mathcal{O}_X$ and $\mathrm{Hom}(\mathcal{O}_X, \ker(t)) = 0$, it is not hard to see that $\ker(t)$ is μ-semistable. Thus, by Lemma 7.7, we have $\ker(t) \cong \mathcal{E}$. Therefore, $\mathrm{ch}(\mathrm{Im}(t)) = \mathrm{ch}(E)$ and thus t is surjective.

Now applying $\mathrm{Hom}(\mathcal{Q}, -)$ to the exact sequence

$$0 \to \mathcal{E} \to \mathcal{O}_X^{\oplus 5} \to E \to 0,$$

from $\mathrm{RHom}^\bullet(\mathcal{Q}, \mathcal{O}_X) = 0$ and $\mathrm{Ext}^1(\mathcal{Q}, \mathcal{E}) = k$, we have $\mathrm{Hom}(\mathcal{Q}, E) = k$. Thus, from the stability of E and $\mathcal{Q}$, we have $E \cong \mathcal{Q}$ and the result follows. □

Now we introduce some notations. Let $\alpha > 0$ and $\beta < 0$. For an object $E \in \mathrm{D}^b(X)$, the limit central charge $Z^0_{0,0}(E)$ is defined as the limit of $Z^0_{\alpha,\beta}(E)$ when $(\alpha, \beta) \to (0, 0)$. Note that $Z^0_{\alpha,\beta}(E)$ is given by $\mathbb{Q}$-linear combinations of $\alpha, \beta, \alpha^2, \beta^2$, thus such a limit $Z^0_{0,0}(E)$ always exists. For $Z^0_{0,0}(E) \neq 0$, we can also define the limit slope $\mu^0_{0,0}(E)$ as follows:

- If $\mathrm{Im}(Z^0_{0,0}(E)) \neq 0$, then we define $\mu^0_{0,0}(E) := -\frac{\mathrm{Re}(Z^0_{0,0}(E))}{\mathrm{Im}(Z^0_{0,0}(E))}$.
- If $\mathrm{Im}(Z^0_{0,0}(E)) = 0$ and $\mathrm{Re}(Z^0_{0,0}(E)) > 0$, then we define $\mu^0_{0,0}(E) := -\infty$.
- If $\mathrm{Im}(Z^0_{0,0}(E)) = 0$ and $\mathrm{Re}(Z^0_{0,0}(E)) < 0$, then we define $\mu^0_{0,0}(E) := +\infty$.

Note that $Z^0_{0,0}(E) = 0$ if and only if $\mathrm{ch}_{\leqslant 2}(E)$ is a multiple of $\mathrm{ch}_{\leqslant 2}(\mathcal{O}_X)$.

Let $E \in \mathrm{Coh}^0_{\alpha,\beta}(X)$. By continuity, we can find a neighborhood U_E of the origin such that for any $(\alpha, \beta) \in U_E$, the slopes $\mu^0_{\alpha,\beta}(E)$ and $\mu^0_{0,0}(E)$ are both negative or positive. Let $F \in \mathrm{Coh}^0_{\alpha,\beta}(X)$ be another object such that E, F are both $\sigma^0_{\alpha,\beta}$-semistable in a neighborhood $U_{E,F}$ of the origin. If $\mu^0_{0,0}(E) > \mu^0_{0,0}(F)$, then by continuity, we can find a smaller neighborhood $U'_{E,F}$ such that $\mu^0_{\alpha,\beta}(E) > \mu^0_{\alpha,\beta}(F)$ holds for every $(\alpha, \beta) \in U'_{E,F}$. Thus, we have $\mathrm{Hom}(E, F) = 0$. We will use these two elementary facts repeatedly.

Proposition 7.9. *If $F \in \mathcal{A}(\alpha,\beta)$ is $\sigma(\alpha,\beta)$-stable such that $[F] = -x$ and F is $\sigma^0_{\alpha,\beta}$-semistable for some $(\alpha,\beta) \in V$, then $F \cong I_C[1]$ for a conic C on X.*

Proof. Since F is $\sigma^0_{\alpha,\beta}$-semistable and $\mu^0_{\alpha,\beta}(F) > 0$, as in [51, Proposition 4.6], there is a triangle

$$F_1[1] \to F \to F_2,$$

where $F_1 \in \mathrm{Coh}^\beta(X)$ with $\mu^+_{\alpha,\beta}(F_1) < 0$ and F_2 is supported on points. Thus, $\mathrm{ch}(F_1) = (1, 0, -2L, mP)$, where m is the length of F_2. By Lemmas 7.5 and 4.5, F_1 is a rank one torsion-free sheaf; hence, it is the ideal sheaf of a closed subscheme. Thus, by [53, Corollary 1.38], we have $m \leqslant 0$, which means $F_2 = 0$ and $F_1 \cong F[-1]$. Thus, by Lemma 7.5 again, $F[-1]$ is a μ-semistable torsion free sheaf, which is of the form $F[-1] \cong I_C$ for a conic C on X since $\mathrm{Pic}(X) = \mathbb{Z}\cdot H$. □

When F is not $\sigma^0_{\alpha,\beta}$-semistable for $(\alpha,\beta) \in V$, the argument is more complicated. Our main tools are the inequalities in [49, 51, Proposition 4.1], Lemma 4.6, and Theorem 4.7, which allow us to bound the rank and first two Chern characters $\mathrm{ch}_1, \mathrm{ch}_2$ of the destabilizing objects and their cohomology objects. Since $F \in \mathcal{A}_X$, by using the Euler characteristics $\chi(\mathcal{O}_X, -)$ and $\chi(\mathcal{E}^\vee, -)$, we can obtain a bound on ch_3. Finally, via a similar argument as in Lemma 7.7, we deduce that the Harder–Narasimhan factors of F are the ones we expect.

Proposition 7.10. *If $F \in \mathcal{A}(\alpha,\beta)$ is $\sigma(\alpha,\beta)$-stable such that $[F] = -x$ and F is not $\sigma^0_{\alpha,\beta}$-semistable for every $(\alpha,\beta) \in V$, then F fits into a triangle*

$$\mathcal{E}[2] \to F \to \mathcal{Q}^\vee[1].$$

Proof. Since there are no walls for F tangent to the wall $\beta = 0$, by the local finiteness of walls and [7, Proposition 2.2.2], we can find an open neighborhood U' of the origin such that the Harder–Narasimhan filtration with respect to $\sigma^0_{\alpha,\beta}$ is constant for every $(\alpha,\beta) \in U := U' \cap V$. In the following, we will only consider $\sigma^0_{\alpha,\beta}$ for $(\alpha,\beta) \in U$.

Let B be the minimal destabilizing quotient object of F and $0 \to A \to F \to B \to 0$ be the destabilizing short exact sequence of F in $\mathrm{Coh}^0_{\alpha,\beta}(X)$. Hence, we know that $A, B \in \mathrm{Coh}^0_{\alpha,\beta}(X)$ and B is $\sigma^0_{\alpha,\beta}$-semistable with $\mu^{0,-}_{\alpha,\beta}(A) > \mu^0_{\alpha,\beta}(F) > \mu^0_{\alpha,\beta}(B)$ for all $(\alpha,\beta) \in U$. By [4, Remark 5.12], we have $\mu^0_{\alpha,\beta}(B) \geqslant \min\{\mu^0_{\alpha,\beta}(F), \mu^0_{\alpha,\beta}(\mathcal{O}_X), \mu^0_{\alpha,\beta}(\mathcal{E}^\vee)\}$. Hence, the following relations hold for all $(\alpha,\beta) \in U$:

(a) $\mu^0_{\alpha,\beta}(A) > \mu^0_{\alpha,\beta}(F) > \mu^0_{\alpha,\beta}(B)$,
(b) $\mathrm{Im}(Z^0_{\alpha,\beta}(A)) \geqslant 0$, $\mathrm{Im}(Z^0_{\alpha,\beta}(B)) > 0$,
(c) $\mu^0_{\alpha,\beta}(B) \geqslant \min\{\mu^0_{\alpha,\beta}(F), \mu^0_{\alpha,\beta}(\mathcal{O}_X), \mu^0_{\alpha,\beta}(\mathcal{E}^\vee)\}$,
(d) $\Delta(B) \geqslant 0$.

By continuity and taking $(\alpha,\beta) \to (0,0)$, we have:

(1) $\mu^0_{0,0}(A) \geqslant \mu^0_{0,0}(F) = 0 \geqslant \mu^0_{0,0}(B)$,
(2) $\mathrm{Im}(Z^0_{0,0}(A)) \geqslant 0$, $\mathrm{Im}(Z^0_{0,0}(B)) \geqslant 0$,
(3) $\mu^0_{0,0}(B) \geqslant \min\{\mu^0_{0,0}(F), \mu^0_{0,0}(\mathcal{O}_X), \mu^0_{0,0}(\mathcal{E}^\vee)\}$,
(4) $\Delta(B) \geqslant 0$.

Assume that $[A] = a[\mathcal{O}_X] + b[\mathcal{O}_H] + c[\mathcal{O}_L] + d[\mathcal{O}_P]$ for integers $a, b, c, d \in \mathbb{Z}$. Then, we have $[B] = (-1-a)[\mathcal{O}_X] - b[\mathcal{O}_H] + (2-c)[\mathcal{O}_L] - (1+d)[\mathcal{O}_P]$. Then we see

- $\mathrm{ch}(A) = (a, bH, \frac{c-5b}{10}H^2, \frac{\frac{5}{3}b+\frac{c}{2}+d}{10}H^3)$,
- $Z^0_{0,0}(A) = bH^3 + (\frac{c-5b}{10}H^3)\mathfrak{i}$, $Z^0_{0,0}(B) = -bH^3 + (\frac{2-c+5b}{10}H^3)\mathfrak{i}$,
- $\mu^0_{0,0}(A) = \frac{10b}{5b-c}$, $\mu^0_{0,0}(B) = \frac{-10b}{c-5b-2}$.

Note that $[F] = -[\mathcal{O}_X] + 2[\mathcal{O}_L] - [\mathcal{O}_P]$. From (2), we know $c - 5b = 0$, 1 or 2. But when $c - 5b = 2$, it is not hard to see that (c) fails near the origin. Thus, $c - 5b = 0$ or 1.

We begin with two claims.

Claim 1. We have $\mathrm{RHom}^\bullet(\mathcal{O}_X, B) = \mathrm{Hom}(\mathcal{O}_X, B)$ and $\mathrm{RHom}^\bullet(\mathcal{O}_X, A) = \mathrm{Ext}^1(\mathcal{O}_X, A)[-1]$.

Since $F \in \mathcal{A}_X$, we only need to prove that $\mathrm{Ext}^i(\mathcal{O}_X, A) = 0$ for $i \neq 1$. Indeed, since $\mathcal{O}_X \in \mathrm{Coh}^0_{\alpha,\beta}(X)$ and $F \in \mathcal{A}_X$, we have $\mathrm{Ext}^i(\mathcal{O}_X, A) = 0$ for all $i \leqslant 0$. Also, by Serre duality, we have $\mathrm{Ext}^i(\mathcal{O}_X, A) = \mathrm{Hom}(A, \mathcal{O}_X(-H)[3-i])$. Thus, from the fact that $\mathcal{O}_X(-H) \in \mathrm{Coh}^0_{\alpha,\beta}(X)$, we obtain $\mathrm{Hom}(A, \mathcal{O}_X(-H)[3-i]) = 0$ for $i \geqslant 2$. Therefore, we have $\mathrm{Ext}^i(\mathcal{O}_X, A) = 0$ for $i \neq 1$.

Claim 2. We have $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, B) = \mathrm{Hom}(\mathcal{E}^\vee, B)$ and $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, A) = \mathrm{Ext}^1(\mathcal{E}^\vee, A)[-1]$.

Since $\mathcal{E}^\vee$ and $\mathcal{E}[2] \in \mathrm{Coh}^0_{\alpha,\beta}(X)$, the argument is the same as Claim 1.

Now we deal with the cases $c - 5b = 0$ and $c - 5b = 1$ separately.

Case 1 ($c - 5b = 0$):

First, we assume that $c - 5b = 0$. By 7.1, we have:

(1) $-2 \leqslant b \leqslant 0$,
(2) $b^2 + \frac{2a+2}{5} \geqslant 0$.

Case 1.1 ($b = 0$): If $b = 0$, then $c = 0$ and $a \geqslant -1$. In this case, we have $\mathrm{ch}_{\leqslant 2}(B) = (-1-a, 0, 2L)$. If $a = -1$, then $\mathrm{ch}_{\leqslant 2}(A) = \mathrm{ch}_{\leqslant 2}(\mathcal{O}_X[1]) = (-1, 0, 0)$, which is impossible since $\mathrm{Im}(Z^0_{\alpha,\beta}(A)) < 0$ for $(\alpha, \beta) \in V$. Thus, $a \geqslant 0$, and $a \neq 0$ otherwise $\mu^0_{\alpha,\beta}(F) = \mu^0_{\alpha,\beta}(B)$ for any $(\alpha, \beta) \in V$. But then we have $\mu^0_{\alpha,\beta}(F) < \mu^0_{\alpha,\beta}(B)$ when $(\alpha, \beta) \in U$ is sufficiently close to the origin. This contradicts our assumption on B.

Case 1.2 ($b = -1$): If $b = -1$, we have $c = -5$. In this case, $\mathrm{ch}_{\leqslant 2}(A) = (a, -H, 0)$. Since $A \in \mathrm{Coh}^0_{\alpha,\beta}(X)$, we have $\mathrm{Im}(Z^0_{\alpha,\beta}(A)) \geqslant 0$ for every $(\alpha, \beta) \in U$. Note that $\mathrm{Im}(Z^0_{\alpha,\beta}(A)) = (\beta + \frac{a(\beta^2-\alpha^2)}{2})H^3$ and $0 < \alpha < -\beta$, and we have $a \geqslant \frac{-2\beta}{\beta^2-\alpha^2}$. But note that when $\alpha = \frac{-\beta}{2}$ and $\beta \to 0$, we have $\frac{-2\beta}{\beta^2-\alpha^2} \to +\infty$, thus we get a contradiction since a is a finite number.

Case 1.3 ($b = -2$): If $b = -2$, we have $c = -10$. In this case, we have $\mathrm{ch}_{\leqslant 2}(A) = (a, -2H, 0)$. Similarly to case 1.2, we have $\mathrm{Im}(Z^0_{\alpha,\beta}(A)) \geqslant 0$ for every $(\alpha, \beta) \in U$. Note that $\mathrm{Im}(Z^0_{\alpha,\beta}(A)) = (2\beta + \frac{a(\beta^2-\alpha^2)}{2})H^3$ and $\alpha < -\beta$, and we have $a \geqslant \frac{-4\beta}{\beta^2-\alpha^2}$. Then as in case 1.2, we get a contradiction.

Case 2 ($c - 5b = 1$): Now we assume that $c - 5b = 1$. Then by 7.1, we have:

(1) $-1 \leqslant b \leqslant 0$,
(2) $b^2 + \frac{a+1}{5} \geqslant 0$.

Case 2.1 ($b = 0$): If $b = 0$, then $c = 1$. Therefore, $-1 \leqslant a$. If $a = -1$, since B is $\sigma^0_{\alpha,\beta}$-semistable, we know that $\mathcal{H}^0_{\mathrm{Coh}^\beta(X)}(B)$ is either 0 or supported on points. Thus, $\mathrm{ch}_{\leqslant 2}(\mathcal{H}^{-1}_{\mathrm{Coh}^\beta(X)}(B)) = (0, 0, -L)$. But $\mathrm{Re}(Z_{\alpha,\beta}(\mathcal{H}^{-1}_{\mathrm{Coh}^\beta(X)}(B))) > 0$ which is impossible since $\mathcal{H}^{-1}_{\mathrm{Coh}^\beta(X)}(B) \in \mathrm{Coh}^\beta(X)$ with $\mathrm{Im}(Z_{\alpha,\beta}(\mathcal{H}^{-1}_{\mathrm{Coh}^\beta(X)}(B))) = 0$.

Therefore, we have $a \geqslant 0$. Hence, $\mathrm{ch}_{\leqslant 2}(B) = -(a+1, 0, -L)$, where $a + 1 \geqslant 1$. This is also impossible since when $(\alpha, \beta) \in U$ is sufficiently close to the origin, we have $\mu^0_{\alpha,\beta}(B) > \mu^0_{\alpha,\beta}(F)$.

Case 2.2 ($b = -1$): We have $b = -1$ and $c = -4$. Hence, $-6 \leqslant a$. In this case, $\mathrm{ch}_{\leqslant 2}(B) = (-1 - a, H, L)$ and we have $\mu^0_{\alpha,\beta}(B) < 0$ and $\mathrm{ch}^\beta_1(B) > 0$ for when $(\alpha, \beta) \in U$ is sufficiently close to the origin. Thus, $B \in \mathrm{Coh}^\beta(X)$ is $\sigma_{\alpha,\beta}$-semistable. Applying Lemma 4.6 to B, we have $a \geqslant -3$.

We first prove a claim.

Claim 3. In the situation of case 2.2, we have A is $\sigma^0_{\alpha,\beta}$-semistable. Hence, $\mathrm{RHom}^\bullet(\mathcal{O}_X, A) = 0$, $\mathrm{ch}(A) = (a, -H, L, (\frac{7}{3} - a)P)$ and $\chi(\mathcal{E}^\vee, A) = 3 - 2a$.

Assume that A is not $\sigma^0_{\alpha,\beta}$-semistable for some $(\alpha, \beta) \in U$. Then, we can take a neighborhood U'_A of the origin such that A has constant Harder–Narasimhan factors for any $(\alpha, \beta) \in U_A := U \cap U'_A \cap V$. Let C be the minimal destabilizing quotient object of A with respect to $\sigma^0_{\alpha,\beta}$ for $(\alpha, \beta) \in U_A$. In this case, we have $\mathrm{ch}_{\leqslant 2}(A) = (a, -H, L)$. Since $\mathrm{Im}(Z^0_{0,0}(A)) = \frac{1}{10}H^3$, we know that $\mathrm{Im}(Z^0_{0,0}(C)) = 0$ or $\frac{1}{10}H^3$. If $\mathrm{Im}(Z^0_{0,0}(C)) = 0$, then $\mu^0_{0,0}(C) = +\infty$ or $-\infty$. But the previous case contradicts $\mu^0_{\alpha,\beta}(A) > \mu^0_{\alpha,\beta}(C)$ and the latter case contradicts $\mu^0_{\alpha,\beta}(C) > \mu^0_{\alpha,\beta}(F)$. Therefore, we have $\mathrm{Im}(Z^0_{0,0}(C)) = \frac{1}{10}H^3$ and we can assume that $\mathrm{ch}_{\leqslant 2}(C) = (e, fH, L)$ where $e, f \in \mathbb{Z}$. Since $\mu^0_{0,0}(A) \geqslant \mu^0_{0,0}(C) \geqslant \mu^0_{0,0}(F) = 0$, we have $10 \geqslant -10f \geqslant 0$. If $f = 0$, then $\mathrm{ch}_{\leqslant 2}(C) = (e, 0, L)$ and $\mathrm{ch}_{\leqslant 2}(D) = (a - e, -H, 0)$, where $D = \mathrm{cone}(A \to C)[-1]$. Then, $\mu^{0-}_{\alpha,\beta}(D) > \mu^0_{\alpha,\beta}(A)$ for any $(\alpha, \beta) \in U_A$. Hence,

$$\mu^0_{\alpha,\beta}(D) = \frac{1 + (a-e)\beta}{\beta + \frac{a-e}{2}(\beta^2 - \alpha^2)} > \mu^0_{\alpha,\beta}(A) > \mu^0_{\alpha,\beta}(F).$$

But note that $\mu^0_{\alpha,\beta}(D) < 0$ for $(\alpha, \beta) \in U_A$ that sufficiently closed to the origin, which gives a contradiction since $\mu^0_{\alpha,\beta}(D) > \mu^0_{\alpha_0,\beta_0}(F)$ holds for any $(\alpha, \beta) \in U_A$.

Therefore, the only possible case is $f = -1$, and hence $\mu^0_{0,0}(C) = 10$. Since $\mu^0_{\alpha,\beta}(A) > \mu^0_{\alpha,\beta}(C)$ for $(\alpha, \beta) \in U_A$, we have $\mathrm{rk}\, C > a$. But this is impossible since $D, \mathcal{O}_X \in \mathrm{Coh}^0_{\alpha,\beta}(X)$ but $\mathrm{ch}_{\leqslant 2}(D) = (s, 0, 0) = s \cdot \mathrm{ch}_{\leqslant 2}(\mathcal{O}_X)$ where $s = a - \mathrm{rk}\, C < 0$. Now for the last statement, note that $\mathcal{O}_X(-H)[2] \in \mathrm{Coh}^0_{\alpha,\beta}(X)$ is $\sigma^0_{\alpha,\beta}$-semistable with $\mu^0_{0,0}(\mathcal{O}_X(-H)[2]) = 2$, hence we have $\mathrm{Hom}(A, \mathcal{O}_X(-H)[2]) = \mathrm{Hom}(\mathcal{O}_X, A[1]) = 0$. Now combined with Claim 1, this proves our claim.

Now we deal with the three cases $a = -3$, $-2 \leqslant a \leqslant 1$ and $a \geqslant 2$ separately.

When $a = -3$, we have $\mathrm{ch}_{\leqslant 2}(B) = \mathrm{ch}_{\leqslant 2}(\mathcal{E}^\vee)$. Then since $\mathrm{ch}_{\leqslant 2}(B)$ is on the boundary of Lemma 4.6, by a standard argument, we know that B is $\sigma_{\alpha,\beta}$-semistable for every $\alpha > 0$ and $\beta < 0$, as explained in [48, Proposition 3.2]. Thus, by Lemma 4.5, B is a μ-semistable sheaf. From Claim 3, we have $\chi(\mathcal{O}_X, B) = 0$, hence $\mathrm{ch}(B) = \mathrm{ch}(\mathcal{E}^\vee)$, and by Lemma 7.7, we have $B \cong \mathcal{E}^\vee$. But this implies $\mathrm{Hom}(\mathcal{O}_X, A[1]) = k^5$ since $F \in \mathcal{A}_X$, which contradicts Claim 3.

When $-2 \leqslant a \leqslant 1$, we have $\mu^0_{\alpha,\beta}(A) > \mu^0_{\alpha,\beta}(\mathcal{E}[2])$. Since A is $\sigma^0_{\alpha,\beta}$-semistable, we have $\mathrm{Hom}(A, \mathcal{E}[2]) = \mathrm{Hom}(\mathcal{E}^\vee, A[1]) = 0$. Thus, $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, A) = 0$ by Claim 2. But this contradicts Claim 3 since $\chi(\mathcal{E}^\vee, A) = 3 - 2a$.

When $a \geqslant 2$, applying Theorem 4.7 to B, we have $a = 2$. Thus, $\mathrm{ch}_{\leqslant 2}(B) = \mathrm{ch}_{\leqslant 2}(\mathcal{Q}^\vee[1])$. By Claim 3, we know that $\mathrm{RHom}^\bullet(\mathcal{O}_X, B) = 0$ and we get $\mathrm{ch}(B) = \mathrm{ch}(\mathcal{Q}^\vee[1])$. Thus, $\chi(\mathcal{E}^\vee, B) = \hom(\mathcal{E}^\vee, B) > 0$. Therefore, if we apply $\mathrm{Hom}(-, B)$ to the exact sequence $0 \to \mathcal{Q}^\vee \to \mathcal{O}_X^{\oplus 5} \to \mathcal{E}^\vee \to 0$, we obtain $\hom(\mathcal{Q}^\vee[1], B) > 0$. Now by stability, we have $B \cong \mathcal{Q}^\vee[1]$. Now $\mathrm{ch}(A) = \mathrm{ch}(\mathcal{E}[2])$. By Claim 2 and Claim 3, we have $\mathrm{ext}^1(\mathcal{E}^\vee, A) = \hom(A, \mathcal{E}[2]) = 1$. Since A is $\sigma^0_{\alpha,\beta}$-semistable and $\mathcal{E}[2]$ is $\sigma^0_{\alpha,\beta}$-stable, we have $A \cong \mathcal{E}[2]$. □

Proposition 7.11. *Let X be a GM threefold. Then every object in the moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ is of form $\mathrm{pr}(I_C)[1]$ for a conic $C \subset X$.*

Proof. Note that $\hom(\mathcal{Q}^\vee[1], \mathcal{E}[2]) = 1$. Then the result follows from Proposition 7.9 and Proposition 7.10. □

Now we are ready to realize the Bridgeland moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ as the contraction $C_m(X)$ of the Fano surface $C(X)$:

Theorem 7.12. *Let X be a GM threefold and σ a Serre-invariant stability condition on $\mathcal{A}_X$. The projection functor $\mathrm{pr} : \mathrm{D}^b(X) \to \mathcal{A}_X$ induces a surjective morphism $p : C(X) \to \mathcal{M}_\sigma(\mathcal{A}_X, -x)$, where p is*

- *a blow-down morphism to a smooth point when X is ordinary;*
- *a contraction of the component $\mathbb{P}^2$ to a singular point when X is special.*

In particular, when X is general and ordinary, $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ is isomorphic to the minimal model $C_m(X)$ of the Fano surface of conics on X. When X is general and special, the moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ has only one singular point.

Proof. By Lemmas 7.4 and 7.6, $\mathrm{pr}(I_C)[1]$ is σ-stable for any conic $C \subset X$. Then we obtain a morphism $p : C(X) \to \mathcal{M}_\sigma(\mathcal{A}_X, -x)$. Moreover, Proposition 7.11 implies that p is surjective.

Now according to Proposition 7.1, the family of conics $C \subset X$ with the property that $I_C \notin \mathcal{A}_X$ is parametrized by the line L_σ when X is ordinary, and the component $\mathbb{P}^2$ when X is special. Since $\mathrm{pr}(I_C)[1] \cong \pi'(\mathcal{Q}^\vee)[1]$ for $I_C \notin \mathcal{A}_X$ by Proposition 7.2, we know that $p(L_\sigma) = [\pi'(\mathcal{Q}^\vee)[1]]$ when X is ordinary, and $p(\mathbb{P}^2) = [\pi'(\mathcal{Q}^\vee)[1]]$ when X is special, where $[\pi'(\mathcal{Q}^\vee)[1]] \in \mathcal{M}_\sigma(\mathcal{A}_X, -x)$ is the point represented by the object $\pi'(\mathcal{Q}^\vee)[1]$. Thus, p is a blow-down morphism to a smooth point when X is ordinary and a contraction of the component $\mathbb{P}^2$ to a singular point when X is special by Lemma 5.6.

When X is general and ordinary, the Fano surface $C(X)$ is smooth by Theorem 6.4. Thus, $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ is a smooth surface obtained by blowing down a smooth rational curve L_σ on the smooth irreducible projective surface $C(X)$. This implies that $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ is also a smooth irreducible projective surface. On the other hand, it is known that there is a unique rational curve $L_\sigma \subset C(X)$ and it is the unique exceptional curve by Lemma 6.5. Thus, $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ is isomorphic to the minimal model $C_m(X)$ of Fano surface of conics on X.

When X is general and special, the last statement follows from Theorem 6.11 and Lemma 7.3. □

7.2 Involutions on $\mathcal{C}_m(X)$

In this section, we are going to describe the involution ι on $\mathcal{C}_m(X)$ in Theorem 6.6, described in [13, Section 5.2] using the involution on $\mathcal{A}_X$. Recall that there is a natural involutive autoequivalence functor of $\mathcal{A}_X$, denoted by $\tau_{\mathcal{A}}$ (cf. Remark 3.2). When X is special, it is induced by the natural involution τ on X, which comes from the double cover $X \to Y$. In this case, it is easy to see that $\tau_{\mathcal{A}}(\mathrm{pr}(I_C)) \cong \mathrm{pr}(I_{\tau(C)})$.

When X is ordinary, the situation is more subtle. In the following, we describe the action of $\tau_{\mathcal{A}}$ on the projection into $\mathcal{A}_X$ of an ideal sheaf of a conic $\mathrm{pr}(I_C)$ in this case, and compare with the involution ι on $\mathcal{C}_m(X)$ described in [13, Section 5.2].

Proposition 7.13. *Let X be an ordinary GM threefold and C a conic on X.*

(1) *If $I_C \in \mathcal{A}_X$, then $\iint_{\mathcal{A}}(I_C)$ is either*
  (a) *$I_{C'}$ such that $C \cup C' = Z(s)$ for $s \in H^0(\mathcal{E}^\vee)$, where $Z(s)$ is the zero locus of the section s;*
  (b) *or $\pi'(\mathcal{Q}^\vee)$, and in this case C is the ρ-conic*

(2) *If $I_C \notin \mathcal{A}_X$, then $\tau_{\mathcal{A}}(\mathrm{pr}(I_C)) \cong I_{C''}$ for the ρ-conic $C'' \subset X$.*

Therefore, the involution induced by $\tau_{\mathcal{A}}$ on $\mathcal{C}_m(X)$ is the same as ι in Theorem 6.6.

Remark 7.14. We can define a birational involution on $\mathcal{C}(X)$ for any GM threefold X as in Proposition 7.13 (1)(a), which is regular on the locus of conics C with $\hom(\mathcal{E}, I_C) = 1$.

We first state some lemmas which we require for the proof of the proposition above.

Lemma 7.15. *Let X be an ordinary GM threefold and C be the ρ-conic on X. Then the natural morphism $s' : \mathcal{E}^{\oplus 2} \to I_C$ is surjective and there is a short exact sequence*

$$0 \to \mathcal{Q}(-H) \to \mathcal{E}^{\oplus 2} \xrightarrow{s'} I_C \to 0.$$

Proof. By Lemma 6.3, we have $\mathrm{Hom}(\mathcal{E}, I_C) = k^2$. Thus, taking two linearly independent elements in $\mathrm{Hom}(\mathcal{E}, I_C)$, we have a natural map $s' : \mathcal{E}^{\oplus 2} \to I_C$. Moreover, since $\langle C \rangle = \mathrm{Gr}(2,3)$ and $\langle C \rangle \cap X = C$, we know that s' is surjective. Let $K := \ker(s')$. Then, it is not hard to see that $\mathrm{ch}(K) = \mathrm{ch}(\mathcal{Q}(-H))$. Note that $\mathrm{Hom}(\mathcal{E}, K) = 0$ and K is reflexive.

We claim that K is μ-semistable. Indeed, suppose K is not μ-semistable and let K' be its maximal destabilizing subsheaf. Then K' is also reflexive. Since $\mathrm{Hom}(\mathcal{E}, K) = 0$, we have $K' \neq \mathcal{E}$. By the stability of $\mathcal{E}$ and the fact that $K \subset \mathcal{E}^{\oplus 2}$, we know that $\mu(K') = -\frac{1}{2}$. Since $\mathrm{Hom}(K', \mathcal{E}) \neq 0$, by the stability of K' and $\mathcal{E}$, we have $K' \subset \mathcal{E}$. Thus, from $\mathrm{ch}_{\leqslant 1}(K') = \mathrm{ch}_{\leqslant 1}(\mathcal{E})$, we know that $\mathcal{E}/K'$ is supported in codimension $\geqslant 2$, which gives a contradiction since $\mathcal{E}$ and K' are both reflexive.

Now the result follows from Lemma 7.8, since $K(H)$ is μ-semistable with $\mathrm{ch}(K(H)) = \mathrm{ch}(\mathcal{Q})$. □

Lemma 7.16. *Let X be an ordinary GM threefold. Let C be a conic on X. Then*

$$\mathbf{L}_{\mathcal{E}}(I_C) = \begin{cases} \mathbb{D}(I_{C'}) \otimes \mathcal{O}_X(-H)[1], & \mathrm{RHom}^\bullet(\mathcal{E}, I_C) = k \\ \pi(\mathcal{E}), & \mathrm{RHom}^\bullet(\mathcal{E}, I_C) = k^2 \oplus k[-1] \end{cases}$$

such that $C \cup C' = Z(s)$ for $s \in H^0(\mathcal{E}^\vee)$, where $Z(s)$ is the zero locus of the section s

Proof. By Lemma 6.3, we have that $\mathrm{RHom}^\bullet(\mathcal{E}, I_C)$ is either k or $k^2 \oplus k[-1]$. If $\mathrm{RHom}^\bullet(\mathcal{E}, I_C) = k$, then we have the triangle

$$\mathcal{E} \to I_C \to \mathbf{L}_{\mathcal{E}}(I_C).$$

Taking cohomology with respect to the standard heart, we get

$$0 \to \mathcal{H}^{-1}(\mathbf{L}_{\mathcal{E}}(I_C)) \to \mathcal{E} \xrightarrow{s} I_C \to \mathcal{H}^0(\mathbf{L}_{\mathcal{E}}(I_C)) \to 0.$$

The image of the map s is the ideal sheaf of an elliptic quartic $D = Z(s)$ for $s \in H^0(\mathcal{E}^\vee)$; thus, we have following two short exact sequences: $0 \to \mathcal{H}^{-1}(\mathbf{L}_{\mathcal{E}}(I_C)) \to \mathcal{E} \to I_D \to 0$ and $0 \to I_D \to I_C \to \mathcal{H}^0(\mathbf{L}_{\mathcal{E}}(I_C)) \to 0$. Then $\mathcal{H}^{-1}(\mathbf{L}_{\mathcal{E}}(I_C))$ is a torsion-free sheaf of rank 1 with the same Chern character as $\mathcal{O}_X(-H)$. It is easy to show that it must be $\mathcal{O}_X(-H)$. On the other hand, $\mathcal{H}^0(\mathbf{L}_{\mathcal{E}}(I_C))$ is supported on the residual curve C' of C in D and $\mathcal{H}^0(\mathbf{L}_{\mathcal{E}}(I_C)) \cong \mathcal{O}_{C'}(-H)$. Thus, we have the triangle

$$\mathcal{O}_X(-H)[1] \to \mathbf{L}_{\mathcal{E}}(I_C) \to \mathcal{O}_{C'}(-H),$$

and we observe that $\mathbf{L}_{\mathcal{E}}(I_C)$ is exactly the twisted derived dual of the ideal sheaf $I_{C'}$ of a conic $C' \subset X$, that is, $\mathbf{L}_{\mathcal{E}}(I_C) \cong \mathbb{D}(I_{C'}) \otimes \mathcal{O}_X(-H)[1]$.

If $\mathrm{RHom}^\bullet(\mathcal{E}, I_C) = k^2 \oplus k[-1]$, then we have the triangle

$$\mathcal{E}^2 \oplus \mathcal{E}[-1] \to I_C \to \mathbf{L}_{\mathcal{E}}(I_C).$$

Taking the long exact sequence in cohomology with respect to the standard heart, we get

$$0 \to \mathcal{H}^{-1}(\mathbf{L}_{\mathcal{E}}(I_C)) \to \mathcal{E}^2 \xrightarrow{s'} I_C \to \mathcal{H}^0(\mathbf{L}_{\mathcal{E}}(I_C)) \to \mathcal{E} \to 0.$$

Now by Lemma 7.15, s' is surjective and the cohomology objects are given by $\mathcal{H}^{-1}(\mathbf{L}_{\mathcal{E}}(I_C)) \cong \mathcal{Q}(-H)$ and $\mathcal{H}^0(\mathbf{L}_{\mathcal{E}}(I_C)) \cong \mathcal{E}$, which implies that $\mathbf{L}_{\mathcal{E}}(I_C) \cong \pi(\mathcal{E})$. □

Proof of Proposition 7.13. Since $\tau_{\mathcal{A}} \circ \tau_{\mathcal{A}} \cong \mathrm{id}$, we have $\tau_{\mathcal{A}} \cong \tau_{\mathcal{A}}^{-1}$. By Proposition 2.6, we have $\tau_{\mathcal{A}} \cong \tau_{\mathcal{A}}^{-1} \cong \mathbf{L}_{\mathcal{O}_X} \circ \mathbf{L}_{\mathcal{E}^\vee}(- \otimes \mathcal{O}_X(H))[-1]$. Then

$$\begin{aligned}\tau_{\mathcal{A}}(I_C) &\cong \mathbf{L}_{\mathcal{O}_X} \circ \mathbf{L}_{\mathcal{E}^\vee}(I_C \otimes \mathcal{O}_X(H))[-1] \\ &\cong \mathbf{L}_{\mathcal{O}_X}(\mathbf{L}_{\mathcal{E}}(I_C) \otimes \mathcal{O}_X(H))[-1].\end{aligned}$$

The left mutation $\mathbf{L}_{\mathcal{E}}(I_C)$ is given by

$$\mathrm{RHom}^\bullet(\mathcal{E}, I_C) \otimes \mathcal{E} \to I_C \to \mathbf{L}_{\mathcal{E}}(I_C).$$

Note that by Lemma 6.3, $\mathrm{RHom}^\bullet(\mathcal{E}, I_C)$ is either k or $k^2 \oplus k[-1]$, and in the latter case, C is the unique ρ-conic. Then, by Lemma 7.16,

$$\mathbf{L}_{\mathcal{E}}(I_C) = \begin{cases} \mathbb{D}(I_{C'}) \otimes \mathcal{O}_X(-H)[1], & \mathrm{RHom}^\bullet(\mathcal{E}, I_C) = k \\ \pi(\mathcal{E}), & \mathrm{RHom}^\bullet(\mathcal{E}, I_C) = k^2 \oplus k[-1]. \end{cases}$$

If $\mathrm{RHom}^\bullet(\mathcal{E}, I_C) = k$, then $\tau_{\mathcal{A}}(I_C) \cong \mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'}))$. We have the triangle

$$\mathrm{RHom}^\bullet(\mathcal{O}_X, \mathbb{D}(I_{C'})) \otimes \mathcal{O}_X \to \mathbb{D}(I_{C'}) \to \mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'})).$$

Note that $\mathrm{RHom}^\bullet(\mathcal{O}_X, \mathbb{D}(I_{C'})) \cong \mathrm{RHom}^\bullet(I_{C'}, \mathcal{O}_X) = k \oplus k[-1]$. Then we have the triangle

$$\mathcal{O}_X \oplus \mathcal{O}_X[-1] \to \mathbb{D}(I_{C'}) \to \mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'})). \tag{4}$$

The derived dual $\mathbb{D}(I_{C'})$ is given by the triangle $\mathcal{O}_X \to \mathbb{D}(I_{C'}) \to \mathcal{O}_{C'}[-1]$. Then taking cohomology with respect to the standard heart of triangle (4), we have the long exact sequence

$$0 = \mathcal{H}^{-1}(\mathbb{D}(I_{C'})) \to \mathcal{H}^{-1}(\mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'}))) \to \mathcal{O}_X \to \mathcal{O}_X$$

$$\to \mathcal{H}^{0}(\mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'}))) \to \mathcal{O}_X \to \mathcal{O}_{C'} \to \mathcal{H}^{1}(\mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'}))) \to 0.$$

Thus, we have $\mathcal{H}^{-1}(\mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'}))) = 0$, $\mathcal{H}^{1}(\mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'}))) = 0$ and $\mathcal{H}^{0}(\mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'}))) \cong I_{C'}$. Hence, $\tau_{\mathcal{A}}(I_C) \cong \mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'})) \cong I_{C'}$.

If $\mathrm{RHom}^\bullet(\mathcal{E}, I_C) = k^2 \oplus k[-1]$, then $\tau_{\mathcal{A}}(I_C) \cong \mathbf{L}_{\mathcal{O}_X} \circ \mathbf{L}_{\mathcal{E}^\vee}(I_C \otimes \mathcal{O}_X(H))[-1] \cong \mathbf{L}_{\mathcal{O}_X}(\pi(\mathcal{E}) \otimes \mathcal{O}_X(H)[-1])$ by Lemma 7.16. Then using the triangle (2), we have $\tau_{\mathcal{A}}(I_C) \cong \pi'(\mathcal{Q}^\vee)$. Then, (2) follows from $\tau_{\mathcal{A}} \cong \tau_{\mathcal{A}}^{-1}$.

Now since $\tau_{\mathcal{A}} = S_{\mathcal{A}_X}[-2]$ and $\tau_{\mathcal{A}}$ acts trivially on $\mathcal{N}(\mathcal{A}_X)$, it induces an involution on the Bridgeland moduli space of any class with respect to any Serre-invariant stability condition. In particular, $\tau_{\mathcal{A}}$ induces an involution on $\mathcal{C}_m(X) \cong \mathcal{M}_\sigma(\mathcal{A}_X, -x)$ by Theorem 7.12. By (1) and (2), this induced involution coincides with ι in Theorem 6.6, described in [13, Section 5.2]. □

Remark 7.17. Smooth τ-conics form an open subscheme U of $\mathcal{C}(X)$. Therefore, the open subscheme $U \cap \iota(U)$ parameterizes smooth τ-conics C such that their involutive conics in Proposition 7.13 are smooth as well. The same also works for special GM threefolds, but replace τ-conics with conics with $\hom(\mathcal{E}, I_C) = 1$ and $I_C \in \mathcal{A}_X$, which are parametrized by $\mathcal{C}(X) \setminus \mathbb{P}^2$. In other words, for any GM threefold X, there is a two-dimensional open subscheme $\mathcal{C}_1 \subset \mathcal{C}(X)$ parameterizing smooth conics C with $\hom(\mathcal{E}, I_C) = 1$ such that their involutive conics are smooth.

8 | THE MODULI SPACE $M_G(2, 1, 5)$ FOR GM THREEFOLDS

In this section, we investigate the moduli space of rank 2 Gieseker-semistable sheaves on a GM threefold X with Chern classes $c_1 = H, c_2 = 5L$, and $c_3 = 0$, denoted as $M_G^X(2, 1, 5)$. We drop X from the notation when it is clear from context on which threefold we work. Note that if $F \in M_G(2, 1, 5)$, then

$$\mathrm{ch}(F) = \left(2, H, 0, -\frac{5}{6}P\right).$$

We are interested in $M_G(2, 1, 5)$ since it naturally appears in the description of the period fiber in [13]. Our main theorem in Section is Theorem 8.9, which realizes $M_G(2, 1, 5)$ as the Bridgeland moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, y - 2x)$.

First, we prove a classification result of sheaves in $M_G(2, 1, 5)$.

Proposition 8.1. *Let X be a GM threefold and $F \in M_G(2,1,5)$. Then, we have* $\mathrm{RHom}^\bullet(\mathcal{O}_X, F) = k^4$ *and* $\mathrm{RHom}^\bullet(\mathcal{O}_X, F(-H)) = 0$. *Moreover, F is either a*

(1) *globally generated bundle that fits into a short exact sequence*

$$0 \to \mathcal{O}_X \to F \to I_Z(H) \to 0$$

where Z is a projective normal smooth elliptic quintic curve;

(2) *nonlocally free sheaf with a short exact sequence*

$$0 \to F \to \mathcal{E}^\vee \to \mathcal{O}_L \to 0$$

where L is a line on X. Moreover, F is uniquely determined by L.

Remark 8.2. In [13, Section 8], they also did computations for nonglobally generated bundles in $M_G^X(2,1,5)$. However, in the following proof, we will show such sheaves do not exist.

Proof. The first statement follows from [11, Proposition 3.5 (1)] and the fact $\chi(F) = 4$. (1) and (2) also follow from [11, Proposition 3.5] or the argument in [13, Section 8]. So, we only need to prove the nonexistence of nonglobally generated bundles in $M_G^X(2,1,5)$. If $F \in M_G^X(2,1,5)$ is a nonglobally generated bundle, then as showed in [13, Section 8], we have an exact sequence

$$0 \to F^\vee \to \mathcal{O}_X^{\oplus 4} \to \mathcal{E}^\vee \xrightarrow{a} \mathcal{O}_L \to 0.$$

By (2), we know that $E := \ker(a)$ is a nonlocally free stable sheaf and $E \in M_G^X(2,1,5)$. Thus, we have an exact sequence $0 \to F^\vee \to \mathcal{O}_X^{\oplus 4} \to E \to 0$. In particular, E is generated by global sections.

However, we also have the following commutative diagram of exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X^{\oplus 4} \arrow[r] \arrow[d, "\mathrm{ev}"'] & \mathcal{O}_X^{\oplus 5} \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E \arrow[r] & \mathcal{E}^\vee \arrow[r] & \mathcal{O}_L \arrow[r] & 0
\end{tikzcd}$$

where $\mathrm{ev} : \mathcal{O}_X^{\oplus 4} \to E$ is the evaluation map. Then, using Snake Lemma, we have an exact sequence

$$0 \to \ker(\mathrm{ev}) \to \mathcal{Q}^\vee \xrightarrow{s} I_L \to \mathrm{cok}(\mathrm{ev}) \to 0.$$

As shown in Lemma 6.3 and Proposition 7.1, the image of s is the zero locus of a nonzero section of $\mathcal{Q}$. It is a σ-conic when X is ordinary, and a preimage of a line on Y when X is special. Hence, in both cases, $\mathrm{im}(s)$ is an ideal sheaf of a conic, and s is not surjective. Therefore, ev is not surjective as well and we get a contradiction. □

A natural question to ask is what Bridgeland moduli space we get after projecting a sheaf in $M_G(2,1,5)$ into the Kuznetsov component. Since it is easier in this setting, we will work with the alternative Kuznetsov component $\mathcal{A}_X$ in this section. Our analysis of the projections of objects in $M_G(2,1,5)$ is based on the three cases listed in Proposition 8.1. We begin with a Hom-vanishing result.

Lemma 8.3. *Let X be a GM threefold and $F \in M_G^X(2,1,5)$. Then, we have* $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, F) = 0$.

Proof. By Serre duality and the stability of $\mathcal{E}^\vee$ and F, we have $\mathrm{Hom}(\mathcal{E}^\vee, F) = \mathrm{Ext}^3(\mathcal{E}^\vee, F) = 0$. Since $\chi(\mathcal{E}^\vee, F) = 0$, we only need to show that $\mathrm{Ext}^2(\mathcal{E}^\vee, F) = 0$ or $\mathrm{Ext}^2(\mathcal{E}^\vee, F) = 0$. By Serre duality, we have $\mathrm{Ext}^2(\mathcal{E}^\vee, F) = \mathrm{Hom}(F, \mathcal{E}[1])$. Since $\mathrm{ch}_1^0(F) = \mathrm{ch}_1^0(\mathcal{E}[1]) = 1$, by Lemma 4.5, we know that F and $\mathcal{E}[1]$ are both $\sigma_{\alpha,0}$-stable for any $\alpha > 0$. Then $\mathrm{Hom}(F, \mathcal{E}[1]) = 0$ since $\mu_{\alpha,0}(F) > \mu_{\alpha,0}(\mathcal{E}[1])$ when $0 < \alpha$ is sufficiently small. □

We are now ready to give an explicit description of $\mathrm{pr}(F)$, for all objects $F \in M_G(2,1,5)$. Recall that for any line $L \subset X$, we have $\mathcal{Q}|_L \cong \mathcal{O}_L^{\oplus 2} \oplus \mathcal{O}_L(1)$. Hence, L is contained in a unique σ-conic C. We define *the residue line of L* to be the support of $\mathcal{O}_C \twoheadrightarrow \mathcal{O}_L$. Note that when C is a double line, we have $L' = L$.

Lemma 8.4. *Let X be a GM threefold and $F \in M_G(2,1,5)$.*

- *If F is a globally generated bundle, then*

$$\mathrm{pr}(F) \cong \ker(\mathrm{ev})[1],$$

  where $\mathrm{ev} : \mathcal{O}_X^{\oplus 4} \twoheadrightarrow F$ *is the evaluation map.*
- *If F is a nonlocally free sheaf determined by a line $L \subset X$, then* $\mathrm{pr}(F)$ *is the unique object fits into a nontrivial exact triangle*

$$\mathcal{E}[1] \to \mathrm{pr}(F) \to \mathcal{O}_{L'}(-1),$$

  where L' is the residue line of L.

Proof. As a result of Lemma 8.3, $\mathbf{L}_{\mathcal{E}^\vee} F = F$, so $\mathrm{pr}(F) = \mathbf{L}_{\mathcal{O}_X} F$. By Proposition 8.1, we have $\mathrm{RHom}^\bullet(\mathcal{O}_X, F) = k^4$, and the triangle defining the left mutation is

$$\mathcal{O}_X^{\oplus 4} \xrightarrow{\mathrm{ev}} F \to \mathrm{pr}(F). \tag{5}$$

In the cases where F is globally generated, the evaluation map ev is surjective, so $\mathrm{pr}(F) = \ker(\mathrm{ev})[1]$.

When F is nonlocally free, as in Proposition 8.1, we have an exact sequence

$$0 \to \ker(\mathrm{ev}) \to \mathcal{Q}^\vee \xrightarrow{s} I_L \to \mathrm{cok}(\mathrm{ev}) \to 0.$$

As shown in Lemma 6.3 and Proposition 7.1, the image of s is the zero locus of a nonzero section of $\mathcal{Q}$, which is a σ-conic. Hence, by Proposition 7.1, we obtain $\ker(\mathrm{ev}) = \mathcal{E}$ and $\mathrm{cok}(\mathrm{ev}) = \mathcal{O}_{L'}(-1)$. Since $\mathrm{pr}(F) \in \mathcal{A}_X$, by Serre duality, we have $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, \mathrm{pr}(F)) = \mathrm{RHom}^\bullet(\mathrm{pr}(F), \mathcal{E})^\vee[-3] = 0$, which implies that such triangle is nontrivial. And the uniqueness follows from $\mathrm{Ext}^2(\mathcal{O}_{L'}(-1), \mathcal{E}) = H^1(\mathcal{E}(-1)|_L) = k$. □

8.1 | Stability of projection objects

In the following, we prove the stability of $\mathrm{pr}(F)$ for any $F \in M_G^X(2,1,5)$.

Lemma 8.5. *The functor* $\mathrm{pr}: \mathrm{D}^b(X) \to \mathcal{A}_X$ *induces isomorphisms of* $\mathrm{Ext}^k(\mathrm{pr}(F_1), \mathrm{pr}(F_2))$ *and* $\mathrm{Ext}^k(F_1, F_2)$ *for all k and for all* $F_1, F_2 \in M_G(2,1,5)$.

Proof. We apply $\mathrm{Hom}(F_1, -)$ to the exact triangle $\mathcal{O}_X^{\oplus 4} \to F_2 \to \mathrm{pr}(F_2)$. By adjunction of pr and the inclusion $\mathcal{A}_X \hookrightarrow \mathrm{D}^b(X)$, we have $\mathrm{Ext}^k(F_1, \mathrm{pr}(F_2)) = \mathrm{Ext}^k(\mathrm{pr}(F_1), \mathrm{pr}(F_2))$ for all k. Thus, we get a long exact sequence

$$\cdots \to \mathrm{Ext}^k(F_1, \mathcal{O}_X)^{\oplus 4} \to \mathrm{Ext}^k(F_1, F_2) \to \mathrm{Ext}^k(\mathrm{pr}(F_1), \mathrm{pr}(F_2)) \to \mathrm{Ext}^{k+1}(F_1, \mathcal{O}_X)^{\oplus 4} \to \cdots.$$

Note that $\mathrm{Ext}^k(F_1, \mathcal{O}_X) = \mathrm{Ext}^{3-k}(\mathcal{O}_X, F_1(-H)) = 0$ for all k by Proposition 8.1. Thus, the desired result follows. □

Before we show the stability of projection objects, let us recall a classical result:

Proposition 8.6. *Let X be an ordinary GM threefold and* $L \subset X$ *be a line. Then* $\mathrm{RHom}^\bullet(\mathcal{O}_L, \mathcal{O}_L) = k \oplus k[-1]$ *or* $k \oplus k^2[-1] \oplus k[-2]$. *Moreover, when X is general, we always have* $\mathrm{RHom}^\bullet(\mathcal{O}_L, \mathcal{O}_L) = k \oplus k[-1]$.

Proof. The first statement follows from [46, Lemma 4.2.1] and the second one follows from [46, Theorem 4.2.7]. □

Now we are ready to prove the stability of $\mathrm{pr}(F)$.

Proposition 8.7. *Let X be a GM threefold and* $F \in M_G^X(2,1,5)$. *Then, we have* $\mathrm{RHom}^\bullet(F,F) = k \oplus k^2[-1]$ *or* $\mathrm{RHom}^\bullet(F,F) = k \oplus k^3[-1] \oplus k[-2]$. *Hence,* $\mathrm{pr}(F)$ *is stable with respect to every Serre-invariant stability condition on* $\mathcal{A}_X$.

Moreover,

(1) *when X is ordinary, if* $\mathrm{RHom}^\bullet(F,F) = k \oplus k^3[-1] \oplus k[-2]$, *then F is a nonglobally generated bundle or a nonlocally free sheaf determined by a line L, and* $[L] \in \Gamma(X)$ *is a singular point. In particular, we always have* $\mathrm{RHom}^\bullet(F,F) = k \oplus k^2[-1]$ *when X is general;*
(2) *when X is special,* $\mathrm{RHom}^\bullet(F,F) = k \oplus k^3[-1] \oplus k[-2]$ *if and only if* $\tau^* F \cong F$, *where* τ *is the natural involution on X*.

Proof. First, we assume that X is ordinary. We have $\hom(F,F) = 1$ and $\mathrm{ext}^3(F,F) = 0$ by Serre duality and the stability of F. Since $\chi(F,F) = -1$, we need to prove $\mathrm{ext}^2(F,F) = 0$ or 1.

When F is a globally generated bundle, by the proof of [13, Theorem 8.2], we have $\mathrm{ext}^1(F,F) = 2$ and $\mathrm{ext}^2(F,F) = 0$. When F is nonlocally free, there is a mistake made in the proof of [13, Theorem 8.2] and we fix it here. From Proposition 8.1, we have an exact sequence $0 \to F \to \mathcal{E}^\vee \to \mathcal{O}_L \to 0$. Since $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, F) = 0$ by Lemma 8.3, applying $\mathrm{Hom}(-, F)$ to this exact sequence, we get $\mathrm{Ext}^k(F,F) = \mathrm{Ext}^{k+1}(\mathcal{O}_L, F)$ for any k. Now applying $\mathrm{Hom}(\mathcal{O}_L, -)$ to this exact sequence, we get a long exact sequence

$$\cdots \to \mathrm{Ext}^2(\mathcal{O}_L, \mathcal{O}_L) \to \mathrm{Ext}^3(\mathcal{O}_L, F) \to \mathrm{Ext}^3(\mathcal{O}_L, \mathcal{E}^\vee) \to 0.$$

By Serre duality, we have $\mathrm{Ext}^3(\mathcal{O}_L, \mathcal{E}^\vee) = H^0(\mathcal{E}(-1)|_L) = 0$. Then, from Proposition 8.6, we have $\mathrm{ext}^2(F,F) = \mathrm{ext}^3(\mathcal{O}_L, F) \leqslant \mathrm{ext}^2(\mathcal{O}_L, \mathcal{O}_L) \leqslant 1$. Moreover, if $\mathrm{ext}^2(F,F) = 1$, then $\mathrm{ext}^2(\mathcal{O}_L, \mathcal{O}_L) = 1$. In other words, $[L] \in \Sigma(X)$ is a singular point. This proves (1).

Now we assume that X is special. Then, by Lemma 8.5 and Serre duality in $\mathcal{K}u(X)$, we have

$$\begin{aligned}\mathrm{Ext}^2(F,F) &\cong \mathrm{Ext}^2(\mathrm{pr}(F),\mathrm{pr}(F))\\ &\cong \mathrm{Hom}(\mathrm{pr}(F),\tau_{\mathcal{A}}(\mathrm{pr}(F)))\\ &\cong \mathrm{Hom}(\mathrm{pr}(F),\mathrm{pr}(\tau^*F)) \cong \mathrm{Hom}(F,\tau^*F),\end{aligned}$$

where τ is the involution on X induced by the double cover. Thus, when $F \cong \tau^*F$, we have $\mathrm{Ext}^2(F,F) = k$, and $\mathrm{Ext}^2(F,F) = 0$ otherwise. Since $\mathrm{Ext}^3(F,F) = 0$ and $\mathrm{Hom}(F,F) = k$, the result follows from $\chi(F,F) = -1$.

Finally, the stability of $\mathrm{pr}(F)$ follows from Lemma 8.5 and Proposition 4.12. □

8.2 | Involutions on $M_G(2,1,5)$

In this subsection, we briefly recall the involutions that exist on $M_G(2,1,5)$ and compare it with the one induced by $\tau_{\mathcal{A}}$. Let F be a globally generated vector bundle, and consider the short exact sequence

$$0 \to \ker(\mathrm{ev}) \to H^0(X,F)\otimes \mathcal{O}_X \xrightarrow{\mathrm{ev}} F \to 0.$$

Note that $\ker(\mathrm{ev})$ is a rank 2 vector bundle with $c_1 = -H$ and $c_2 = 5L$ and no global sections, hence $\ker(\mathrm{ev})^\vee \in M_G(2,1,5)$. Define $\iota F := \ker(\mathrm{ev})^\vee$. This bundle ιF is globally generated, and we have $H^0(X,\iota F) \cong H^0(X,F)^\vee$ [13, p. 29]. This defines a birational involution on $M_G^X(2,1,5)$.

Note that there is no nonglobally generated bundle in $M_G^X(2,1,5)$ by Proposition 8.1, then the definition of ι on the nonlocally free locus in [13, Theorem 8.2] does not work. However, we can fix this issue as follows: for any nonlocally free $F \in M_G^X(2,1,5)$ determined by a line L, we define $\iota(F) := F'$, where $F' := \ker(\mathcal{E}^\vee \twoheadrightarrow \mathcal{O}_{L'})$ is a nonlocally free stable sheaf determined by the residue line L' of L. This extends ι to be a regular involution on $M_G^X(2,1,5)$.

Note that for a special GM threefold, there is another involution on $M_G(2,1,5)$ induced by the involution τ on X,

$$\tau^* : M_G(2,1,5) \to M_G(2,1,5),\ F \mapsto \tau^*F.$$

And it is clear that $\tau_{\mathcal{A}}(\mathrm{pr}(F)) \cong \mathrm{pr}(\tau^*F)$.

Now let X be an ordinary GM threefold, $\tau_{\mathcal{A}}$ be the involution of $\mathcal{A}_X$, and ι be the geometric involution of $M_G(2,1,5)$ defined above. Then $\tau_{\mathcal{A}}$ induces involutions of the Bridgeland moduli spaces of σ-stable objects $\mathcal{M}_\sigma(\mathcal{A}_X,-x)$ and $\mathcal{M}_\sigma(\mathcal{A}_X,y-2x)$. In Proposition 7.13, we already showed that the action of $\tau_{\mathcal{A}}$ on $\mathcal{M}_\sigma(\mathcal{A}_X,-x)$ induces a geometric involution on $\mathcal{C}_m(X)$. In this section, we show that the involution induced by $\tau_{\mathcal{A}}$ is also compatible with ι on $M_G(2,1,5)$.

Proposition 8.8. *Let X be an ordinary GM threefold and $F \in M_G^X(2,1,5)$. Then $\tau_{\mathcal{A}}\mathrm{pr}(F) \cong \mathrm{pr}(\iota(F))$.*

Proof.

(1) If F is a nonlocally free sheaf determined by a line L, then by Lemma 8.4, we have the triangle

$$\mathcal{E}[1] \to \mathrm{pr}(F) \to \mathcal{O}_{L'}(-1).$$

Then since $\tau_{\mathcal{A}} \cong \tau_{\mathcal{A}}^{-1} \cong \mathbf{L}_{\mathcal{O}_X} \circ \mathbf{L}_{\mathcal{E}^\vee}(- \otimes \mathcal{O}_X(H))[-1]$, $\tau_{\mathcal{A}}(\mathrm{pr}(F))$ is given by a triangle

$$\mathbf{L}_{\mathcal{O}_X}\mathbf{L}_{\mathcal{E}^\vee}(\mathcal{E}^\vee) \to \tau_{\mathcal{A}}(\mathrm{pr}(F)) \to \mathbf{L}_{\mathcal{O}_X}\mathbf{L}_{\mathcal{E}^\vee}(\mathcal{O}_{L'})[-1].$$

Note that $\mathbf{L}_{\mathcal{E}^\vee}(\mathcal{E}^\vee) = 0$, hence $\tau_{\mathcal{A}}(\mathrm{pr}(F)) \cong \mathbf{L}_{\mathcal{O}_X}\mathbf{L}_{\mathcal{E}^\vee}(\mathcal{O}_{L'})[-1]$. It is easy to see $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, \mathcal{O}_{L'}) = k$; therefore, we have $\mathcal{E}^\vee \to \mathcal{O}_{L'} \to \mathbf{L}_{\mathcal{E}^\vee}\mathcal{O}_{L'}$. Also, since $\mathcal{E}^\vee \to \mathcal{O}_{L'}$ is surjective, we have $\mathbf{L}_{\mathcal{E}^\vee}\mathcal{O}_{L'} \cong F'[1]$, where $F' := \ker(\mathcal{E}^\vee \to \mathcal{O}_{L'})$ is a nonlocally free sheaf in $M_G(2,1,5)$ determined by L' as in Proposition 8.1. Thus, $\tau_{\mathcal{A}}(\mathrm{pr}(F)) \cong \mathbf{L}_{\mathcal{O}_X}F' \cong \mathrm{pr}(F') = \mathrm{pr}(\iota(F))$.

(2) If F is a globally generated vector bundle, consider the standard short exact sequence

$$0 \to \ker(\mathrm{ev}) \to H^0(X, F) \otimes \mathcal{O}_X \xrightarrow{\mathrm{ev}} F \to 0.$$

Dualizing the sequence and applying pr, we get the triangle

$$\mathrm{pr}(F^\vee) \to \mathrm{pr}(\mathcal{O}_X^{\oplus 4}) \to \mathrm{pr}(\ker(\mathrm{ev})^\vee) \cong \mathrm{pr}(\iota F).$$

Note that $F^\vee \in \mathcal{A}_X$ and $\mathrm{pr}(\mathcal{O}_X) = 0$, thus we get $\mathrm{pr}(\iota F) \cong F^\vee[1]$. Since $F \in M_G(2,1,5)$ is a globally generated vector bundle, we have $F \cong \iota E$ for some globally generated vector bundle E. Then $\mathrm{pr}(F) = \mathrm{pr}(\iota E) \cong E^\vee[1] \cong E \otimes \mathcal{O}_X(-H)[1]$, hence $\tau_{\mathcal{A}}(\mathrm{pr}(F)) \cong \tau_{\mathcal{A}}(E \otimes \mathcal{O}_X(-H))[1] \cong \mathrm{pr}(E) \cong \mathrm{pr}(\iota F)$.

□

8.3 | The Bridgeland moduli space of class $y - 2x$

In this subsection, we show that $M_G(2,1,5) \cong \mathcal{M}_\sigma(\mathcal{A}_X, y - 2x)$.

Theorem 8.9. *Let X be a GM threefold and σ be a Serre-invariant stability condition on $\mathcal{A}_X$. Then the projection functor $\mathrm{pr} : \mathrm{D}^b(X) \to \mathcal{A}_X$ induces an isomorphism $M_G^X(2,1,5) \cong \mathcal{M}_\sigma(\mathcal{A}_X, y - 2x)$.*

We split the proof of this theorem into a series of lemmas and propositions. Recall that in 4.9, we defined

$$V := \{(\alpha, \beta) : -\frac{1}{10} < \beta < 0, 0 < \alpha < -\beta\}.$$

Proposition 8.10. *Let $F \in \mathcal{A}(\alpha, \beta)$ be a $\sigma(\alpha, \beta)$-stable object with numerical class $y - 2x$ for every $(\alpha, \beta) \in V$. Then, $F = \mathrm{pr}(E)$ for some $E \in M_G(2,1,5)$.*

Proof. First, we argue as in [51, Proposition 4.6]. When $(\alpha_0, \beta_0) = (0,0)$, we have $\mu^0_{\alpha_0,\beta_0}(F) = -\infty$. Since there are no walls intersecting with $\beta = 0$ as in [51, Proposition 4.6], we know that F is

$\sigma^0_{\alpha,0}$-semistable for all $\alpha > 0$. By the definition of the double-tilted heart, we have a triangle

$$A[1] \to F \to B$$

such that A (respectively, B) is in $\mathrm{Coh}^0(X)$ with its $\sigma_{\alpha,0}$-semistable factors having slope $\mu_{\alpha,0} \leqslant 0$ (respectively, $\mu_{\alpha,0} > 0$). Since F is $\sigma^0_{\alpha,0}$-semistable and $\mu^0_{\alpha,0}(F) < 0$, we have that $A[1] = 0$ and $B \cong F$. Since $\mathrm{ch}^0_1(F)$ is minimal, there are no walls on $\beta = 0$, and we know that F is $\sigma_{\alpha,0}$-semistable for every $\alpha > 0$. Thus, by Lemma 4.5, $\mathcal{H}^{-1}(F)$ is a μ-semistable reflexive sheaf and $\mathcal{H}^0(F)$ is 0 or supported in dimension $\leqslant 1$.

If $\mathcal{H}^0(F)$ is supported in dimension 0, then $\mathrm{ch}(\mathcal{H}^0(F)) = bP$ for $b \geqslant 1$. But this is impossible since then $c_3(\mathcal{H}^{-1}(F)) > 0$ and by [11, Proposition 3.5], we have $\chi(\mathcal{H}^{-1}(F)) = 0$, which implies $b = 0$.

If $\mathcal{H}^0(F)$ is supported in dimension 1, we can assume $\mathrm{ch}(\mathcal{H}^0(F)) = aL + \frac{b}{2}P$ where $a \geqslant 1$ and b are integers. Thus, $\mathrm{ch}(\mathcal{H}^{-1}(F)) = 2 - H + aL + (\frac{5}{6} + \frac{b}{2})P$. Now from Lemma 7.7, we know $\mathcal{H}^{-1}(F) \cong \mathcal{E}$ and $\mathrm{ch}(\mathcal{H}^0(F)) = L - \frac{P}{2}$. Thus, $\mathcal{H}^0(F) \cong \mathcal{O}_L(-1)$ for some line L on X. Therefore, we have a triangle

$$\mathcal{E}[1] \to F \to \mathcal{O}_L(-1).$$

In this case, we have $\mathrm{Hom}(\mathcal{O}_L(-1), \mathcal{E}[2]) = \mathrm{Hom}(\mathcal{E}^\vee(1), \mathcal{O}_L[1]) = H^1(L, \mathcal{E}(-1)|_L) = H^1(L, \mathcal{O}_L(-1) \oplus \mathcal{O}_L(-2)) = k$. Hence, by Lemma 8.4, $F \cong \mathrm{pr}(E)$ for some $E \in M_G(2,1,5)$ such that E is locally free but not globally generated.

If $\mathcal{H}^0(F) = 0$, we have $F[-1] \cong \mathcal{H}^{-1}(F)$. Then $F[-1]$ is a μ-semistable sheaf. Since $F[-1]$ is reflexive and $c_3(F[-1]) = 0$, $F[-1] \in M_G(2,-1,5)$ is a stable vector bundle. Thus, by Lemma 8.4, we know $F[-1] = \mathrm{pr}(E)$ for some $E \in M_G(2,1,5)$ such that E is a globally generated vector bundle. □

Lemma 8.11. *The functor* $\mathrm{pr}\colon \mathrm{D}^b(X) \to \mathcal{A}_X$ *is injective on all objects in* $M_G(2,1,5)$*, that is, if* $\mathrm{pr}(F_1) \cong \mathrm{pr}(F_2)$*, then* $F_1 \cong F_2$.

Proof. For the case of globally generated vector bundles, by Corollary 8.4, $\mathrm{pr}(F_1) \cong \mathrm{pr}(F_2)$ implies that

$$(\iota F_1)^\vee \cong (\iota F_2)^\vee.$$

Note that $(\iota F_i)^\vee \cong \iota F_i \otimes \mathcal{O}_X(-H)$ for $i = 1, 2$. Then we get $\iota F_1 \cong \iota F_2$. Finally, we apply ι to both sides. Since it is an involution $\iota^2 = \mathrm{id}$, so $F_1 \cong F_2$ as required.

For the case of nonlocally free sheaves F, recall that from Lemma 8.4, we have $\mathcal{H}^{-1}(\mathrm{pr}(F)) = \mathcal{E}$ and $\mathcal{H}^0(\mathrm{pr}(F)) = \mathcal{O}_L(-H)$. Since F is uniquely determined by the line L, and $\mathrm{Hom}(\mathcal{O}_L(-H), \mathcal{E}[2]) = k$, the object $\mathrm{pr}(F)$ is also uniquely determined by the line L. Thus, $\mathrm{pr}(F_1) \cong \mathrm{pr}(F_2)$ implies $F_1 \cong F_2$, as required. □

Proof of Theorem 8.9. Using Proposition 8.7, we know that the projection functor pr induces a morphism

$$p\colon M_G(2,1,5) \to \mathcal{M}_\sigma(\mathcal{A}_X, y - 2x),$$

which is bijective on points by Proposition 8.10 and Lemma 8.11, and bijective on tangent spaces by Lemma 8.5. Hence, it is an isomorphism. □

9 | REFINED AND BIRATIONAL CATEGORICAL TORELLI THEOREMS FOR GM THREEFOLDS

In this section, we will prove several refined/birational categorical Torelli theorems for GM threefolds, using results from the previous sections.

9.1 | The universal family for $\mathcal{C}_m(X)$

In this subsection, we show that $\mathcal{C}_m(X) \cong \mathcal{M}_\sigma(\mathcal{A}_X, -x)$ admits a universal family, which thus gives a fine moduli space. Let $\mathcal{I}$ be the universal ideal sheaf of conics on $X \times \mathcal{C}(X)$ and $\mathcal{I}_{L_\sigma}$ be the universal ideal sheaf of conics restricted to $X \times L_\sigma$. Let $q : X \times \mathcal{C}(X) \to X$ and $\pi : X \times \mathcal{C}(X) \to \mathcal{C}(X)$ be the projection maps on the first and second factors, respectively. Let $\mathcal{G}' := \mathrm{pr}(\mathcal{I}_{L_\sigma})$ be the projected family in $\mathcal{A}_{X \times L_\sigma}$. Let $t \in L_\sigma \cong \mathbb{P}^1$ be any point. Then $j_t^* \mathrm{pr}(\mathcal{I}_{L_\sigma}) \cong A$, where $j_t : X_t \to X_t \times L_\sigma$ and $A \in \mathcal{A}_X$ is $A \cong \mathrm{pr}(I_C)$ for $I_C \notin \mathcal{A}_X$ by Proposition 7.2. Then $\mathcal{G}' \cong q^*(A) \otimes \pi^* \mathcal{O}_{L_\sigma}(k)$ for some $k \in \mathbb{Z}$. Now let $\mathcal{G} := \mathrm{pr}(\mathcal{I}) \otimes \pi^* \mathcal{O}_{\mathcal{C}(X)}(kE)$, where $E \cong L_\sigma \cong \mathbb{P}^1$ is the unique exceptional curve on $\mathcal{C}(X)$.

Proposition 9.1. *The object $(p_X)_* \mathcal{G}$ is the universal family of $\mathcal{C}_m(X)$, where $p_X = \mathrm{id}_X \times p : X \times \mathcal{C}(X) \to X \times \mathcal{C}_m(X)$.*

Proof.

(1) If $s = [A] = \pi \in \mathcal{C}_m(X)$, s is contracted from the unique rational curve $L_\sigma \cong \mathbb{P}^1 \subset \mathcal{C}(X)$. Note that in this case, $p_X|_{L_\sigma} = q$. Then

$$\begin{aligned}
i_s^*(p_X)_* \mathcal{G} &\cong i_s^*(p_X)_*(\mathcal{G}' \otimes \pi^* \mathcal{O}_{\mathcal{C}(X)}(kE)) \\
&\cong i_s^* q_*(q^*(A) \otimes \pi^* \mathcal{O}_{L_\sigma}(k) \otimes \pi^* \mathcal{O}_{\mathcal{C}(X)}(kE)) \\
&\cong i_s^* q_*(q^*(A) \otimes (\pi^* \mathcal{O}_{L_\sigma}(k) \otimes \mathcal{O}_{L_\sigma}(kE))) \\
&\cong i_s^* q_*(q^*(A) \otimes \pi^*(\mathcal{O}_{L_\sigma}(k) \otimes \mathcal{O}_{L_\sigma}(-k))) \\
&\cong i_s^* q_*(q^*(A)) \cong i_s^*(A) \cong A.
\end{aligned}$$

(2) If $s = [I_C]$, then $\mathcal{C}_m(X)$ and $\mathcal{C}(X)$ are isomorphic outside L_σ. Note that p restricts to id on $\mathcal{C}(X) \setminus L_\sigma$. Then

$$\begin{aligned}
i_s^*(p_X)_* \mathcal{G} &\cong i_s^*(p_X)_*(\mathrm{pr}(\mathcal{I}) \otimes \pi^* \mathcal{O}_{\mathcal{C}(X)}(kE)) \\
&\cong j_s^*(\mathrm{pr}(\mathcal{I})) \otimes j_s^* \pi^* \mathcal{O}_{\mathcal{C}(X)}(kE) \\
&\cong I_C \otimes (\pi \circ j_s)^* \mathcal{O}_{\mathcal{C}(X)}(kE) \\
&\cong I_C \otimes (i_s \circ \pi_s)^* \mathcal{O}_{\mathcal{C}(X)}(kE) \cong I_C.
\end{aligned}$$

See below for the commutative diagrams that summarize the maps in the proof:

$$\begin{tikzcd}
X_s \arrow[r, "\cong"] \arrow[d, hook, "j_s"] & X_s \arrow[r, "\pi_s"] \arrow[d, hook, "i_s"] & \{s\} \arrow[d] \\
X \times \mathcal{C}(X) \arrow[r, "p_X"'] & X \times \mathcal{C}_m(X) \arrow[r] & \mathcal{C}_m(X)
\end{tikzcd}$$

$$\begin{tikzcd}
X_s \arrow[r, "j_s"] \arrow[d, "\pi_s"] & X \times \mathcal{C}(X) \arrow[d, "\pi"] \\
\{s\} \arrow[r, "i_s"'] & \mathcal{C}(X)
\end{tikzcd}$$

□

9.2 | A refined categorical Torelli theorem for ordinary GM threefolds

We now prove a refined categorical Torelli theorem for ordinary GM threefolds.

Theorem 9.2. *Let X and X' be general ordinary GM threefolds such that $\Phi : \mathcal{K}u(X) \simeq \mathcal{K}u(X')$ is an equivalence and $\Phi(\pi(\mathcal{E})) \cong \pi(\mathcal{E}')$. Then $X \cong X'$.*

Proof. Since Φ commutes with Serre functors, it preserves the stability of an object with respect to any Serre-invariant stability condition. Then, the existence of the universal family on $\mathcal{C}_m(X) \cong \mathcal{M}_\sigma(\mathcal{A}_X, -x)$ guarantees a morphism from $\mathcal{C}_m(X)$ to $\mathcal{C}_m(X')$, denoted by ψ, which is induced by Ψ (for more details on the construction of the morphism ψ, see [1, 8]). Since Φ is an equivalence, ψ is an isomorphism. On the other hand, we have $\psi([\pi_X]) = [\pi_{X'}]$ by the assumption, where $\pi_X := \pi'(\mathcal{Q}^\vee)$ and $\pi_{X'} := \pi'(\mathcal{Q}^\vee)$. Then ψ induces an isomorphism $\phi : \mathcal{C}(X) \cong \mathcal{C}(X')$ by blowing up $[\pi_X] \in \mathcal{C}_m(X)$ and $[\pi_{X'}] \in \mathcal{C}_m(X')$, respectively. Then we have $X \cong X'$ by Logachev's Reconstruction Theorem 6.7. □

9.3 | Birational categorical Torelli theorem for ordinary GM threefolds

In this subsection, we show a birational categorical Torelli theorem for ordinary GM threefolds, that is, assuming the Kuznetsov components are equivalent leads to a birational equivalence of the ordinary GM threefolds.

Theorem 9.3. *Let X and X' be general ordinary GM threefolds such that $\mathcal{A}_X \simeq \mathcal{A}_{X'}$. Then X' is a conic transform or a conic transform of a line transform of X. In particular, we have $X \simeq X'$.*

Proof. The equivalence $\Phi : \mathcal{A}_X \xrightarrow{\sim} \mathcal{A}_{X'}$ sends $-x$ to either itself or $y - 2x$ in $\mathcal{N}(\mathcal{A}_{X'})$ up to sign since they are only (-1)-class. By the same argument as in Theorem 9.2 and [1, 8], we thus get two possible induced isomorphisms between Bridgeland moduli spaces

$$\begin{tikzcd}
\mathcal{C}_m(X) \cong \mathcal{M}_\sigma(\mathcal{A}_X, -x) \arrow[r, "\gamma"] \arrow[dr, "\gamma'"] & \mathcal{C}_m(X') \cong \mathcal{M}_{\Phi(\sigma)}(\mathcal{A}_{X'}, -x) \\
& M_G^{X'}(2,1,5) \cong \mathcal{M}_{\Phi(\sigma)}(\mathcal{A}_{X'}, y - 2x)
\end{tikzcd}$$

If we have the isomorphism γ, then we blow up $\mathcal{C}_m(X)$ at the distinguished point $[\pi_X] := [\Xi(\pi(\mathcal{E}))]$, and blow up $\mathcal{C}_m(X')$ at the point $[C] := [\Phi(\pi_X)] = \gamma([\pi_X])$. We have

$$\mathcal{C}(X) \cong \mathrm{Bl}_{[\pi_X]}\mathcal{C}_m(X) \cong \mathrm{Bl}_{[C]}\mathcal{C}_m(X'),$$

and $\mathrm{Bl}_{[C]}\mathcal{C}_m(X') \cong \mathcal{C}(X'_C)$ by Theorem 6.8, so $\mathcal{C}(X) \cong \mathcal{C}(X'_C)$. Therefore, by Logachev's Reconstruction Theorem 6.7, we have $X \cong X'_C$.

For the second case, we get $\mathcal{C}_m(X) \cong M_G^{X'}(2,1,5)$. And by [13, Proposition 8.1], we have a birational equivalence $M_G^{X'}(2,1,5) \simeq \mathcal{C}(X'_L)$ of surfaces, where $L \subset X'$ is a line. Then we see $\mathcal{C}_m(X)$ is birationally equivalent to $\mathcal{C}(X'_L)$. Let $\mathcal{C}_m(X'_L)$ be the minimal surface of $\mathcal{C}(X'_L)$. Note that the surfaces here are all smooth surfaces of general type. By the uniqueness of minimal models of surfaces of general type, we get $\mathcal{C}_m(X) \cong \mathcal{C}_m(X'_L)$, which implies $X \cong (X'_L)_C \simeq X'$ for a conic $C \subset X'_L$ as in the first case. $\square$

Remark 9.4. Theorem 9.3 proves a conjecture [34, Conjecture 1.7] of Kuznetsov–Perry for *general ordinary* GM varieties of dimension 3.

In [13], the authors proved that $\mathcal{C}_m(X_L)$ is birational to $M_G^X(2,1,5)$. The following corollary shows that they are indeed isomorphic.

Corollary 9.5. *Let X be a general ordinary GM threefold, and X_L be a line transform of X. Then, we have $\mathcal{C}_m(X_L) \cong M_G^X(2,1,5)$. Moreover, this isomorphism commutes with involutions ι and ι' on both sides, thus giving an isomorphism $\mathcal{C}_m(X_L)/\iota \cong M_G^X(2,1,5)/\iota'$.*

Proof. By the same argument as in the proof of Theorem 9.3, we have $\mathcal{C}_m(X_L) \cong \mathcal{C}_m(X)$ or $\mathcal{C}_m(X_L) \cong M_G^X(2,1,5)$. Note that $\mathcal{C}_m(X_L) \cong \mathcal{C}_m(X)$ implies that $X_L \cong X_C$ for some conic $C \subset X$ as in Theorem 9.3. But this is impossible by [13, Remark 7.3]. Thus, we always have $\mathcal{C}_m(X_L) \cong M_G^X(2,1,5)$. The last statement follows from the fact that any equivalence between Kuznetsov components commutes with Serre functors, and the involutions on $\mathcal{C}_m(X_L)$ and $M_G^X(2,1,5)$ can be induced by Serre functors up to shift by Propositions 7.13 and 8.8. $\square$

Since the intermediate Jacobian $J(X)$ is invariant under conic and line transforms, as a corollary we have the following.

Corollary 9.6. *Let X and X' be general ordinary GM threefolds. If $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$, then we have $J(X) \cong J(X')$.*

In fact, we can relax the assumptions on X by looking at the singularities of Bridgeland moduli spaces.

Theorem 9.7. *Let X and X' be general GM threefolds (they can be either general ordinary or general special), and suppose that their Kuznetsov components $\mathcal{A}_X \simeq \mathcal{A}_{X'}$ are equivalent. Then X is birationally equivalent to X'.*

Proof. First, we claim that if X and X' are general GM threefolds such that $\Phi : \mathcal{A}_X \simeq \mathcal{A}_{X'}$, then both X and X' are ordinary or special simultaneously. Indeed, we may assume that X' is ordinary and X is special. Then the equivalence would identify the moduli space $\mathcal{C}_m(X) \cong \mathcal{M}_\sigma(\mathcal{A}_X, -x)$ of stable objects of class $-x$ in $\mathcal{A}_X$ with either the moduli space $\mathcal{C}_m(X') \cong \mathcal{M}_{\sigma'}(\mathcal{A}_{X'}, -x)$ or $M_G^{X'}(2,1,5) \cong \mathcal{M}_{\sigma'}(\mathcal{A}_{X'}, y-2x)$. But $\mathcal{C}_m(X)$ has a unique singular point by Theorem 7.12, and both $\mathcal{C}_m(X')$ and $M_G^{X'}(2,1,5)$ are smooth for X' general by Theorems 7.12 and 8.9. This means that neither identification is possible, so the claim follows.

Now X and X' are both general ordinary or general special, hence the result follows from Theorem 9.3 and 9.4. □

Corollary 9.8. *Let X and X' be general GM threefolds such that one of them is ordinary and their Kuznetsov components $\mathcal{A}_X \simeq \mathcal{A}_{X'}$ are equivalent. Then they are both general ordinary and X is birationally equivalent to X'.*

9.4 | A categorical Torelli theorem for special GM threefolds

In this subsection, we show that the Kuznetsov component of a general special GM threefold X determines the isomorphism class of X.

Recall from Section 3 that every special GM threefold X is a double cover of a degree 5 index 2 prime Fano threefold Y branched over a quadric hypersurface $\mathcal{B}$ in Y. Since X is smooth and general, $(\mathcal{B}, h)$ is a smooth degree $h^2 = 10$ K3 surface with Picard number 1. There is a natural geometric involution τ on X induced by the double cover. The Serre functor on $\mathcal{K}u(X)$ is given by $S_{\mathcal{K}u(X)} = \tau \circ [2]$.

Theorem 9.9. *Let X and X' be general special GM threefolds with $\Phi : \mathcal{K}u(X) \simeq \mathcal{K}u(X')$. Then $X \cong X'$.*

Proof. By [32, Theorem 1.1, Section 8.2], the equivariant triangulated category $\mathcal{K}u(X)^{\mu_2}$ is equivalent to $\mathrm{D}^b(\mathcal{B})$, where μ_2 is the group of square roots of 1 generated by the involution τ acting on $\mathcal{K}u(X)$. Assume that there is an equivalence $\Phi : \mathcal{K}u(X) \simeq \mathcal{K}u(X')$. Since $S_{\mathcal{K}u(X)} \cong \tau[2]$ and $S_{\mathcal{K}u(X')} \cong \tau'[2]$, Φ commutes with the involutions τ and τ' on $\mathcal{K}u(X)$ and $\mathcal{K}u(X')$, respectively. Then we get an induced equivalence

$$\Psi : \mathcal{K}u(X)^{\mu_2} \simeq \mathcal{K}u(X')^{\mu_2'},$$

where $\mu_2 = \langle \tau \rangle$, $\mu_2' = \langle \Phi \circ \tau \circ \Phi^{-1} = \tau' \rangle$ and $\mu_2 \cong \mu_2'$. Thus, we have $\Psi : \mathrm{D}^b(\mathcal{B}) \simeq \mathrm{D}^b(\mathcal{B}')$. We know that $\mathcal{B}$ and $\mathcal{B}'$ are smooth projective surfaces with polarizations h and h', respectively, so Ψ is a Fourier–Mukai functor by Orlov's representability theorem [45, Theorem 2.2]. Moreover, $(\mathcal{B}, h)$ and $(\mathcal{B}', h')$ are both Picard number 1 smooth projective K3 surfaces of degree $h^2 = h'^2 = 10 = 2 \times 5$. Then by [44, Theorem 1.10] and [20, Corollary 1.7], there is an isomorphism $\phi : \mathcal{B} \cong \mathcal{B}'$. Since they both have Picard number one, we obtain $\phi^*(h') = h$. On the other hand, Y_5 is rigid [29, § 4.1], which implies $X \cong X'$. □

Remark 9.10. Theorem 9.9 can also be proved via Bridgeland moduli spaces with respect to the Kuznetsov component $\mathcal{A}_X$. The details are contained in another paper of authors [26].

10 | THE DEBARRE–ILIEV–MANIVEL CONJECTURE

Let $\mathcal{X}_{10}$ be the moduli space of smooth ordinary GM threefolds and $\mathcal{A}_{10}$ be the moduli space of 10-dimensional principal polarized abelian varieties. In [13, pp. 3–4], the authors make the following conjecture regarding the general fiber of the period map.

Conjecture 10.1 [13, pp. 3-4]. *A general fiber $\mathcal{P}^{-1}([J(X)])$ of the period map $\mathcal{P}: \mathcal{X}_{10} \to \mathcal{A}_{10}$ at the intermediate Jacobian $J(X)$ of an ordinary GM threefold X is the union of $C_m(X)/\iota$ and a surface birationally equivalent to $M_G^X(2,1,5)/\iota'$, where ι, ι' are geometrically meaningful involutions.*

Remark 10.2. Note that by Corollary 9.5, the surface birationally equivalent to $M_G(2,1,5)/\iota'$ in [13], parametrizing conic transforms of a line transform of X, is actually isomorphic to $M_G(2,1,5)/\iota'$. Thus, this conjecture predicts that a general fiber $\mathcal{P}^{-1}([J(X)])$ is actually the disjoint union of $C_m(X)/\iota$ and $M_G(2,1,5)/\iota'$.

We will prove a categorical analog of this conjecture. Consider the "categorical period map"

$$\mathcal{P}_{\mathrm{cat}}: \mathcal{X}_{10} \to \{\mathcal{A}_X\}/\simeq, \; X \mapsto \mathcal{A}_X,$$

where $\mathcal{X}_{10}$ is the moduli space of isomorphism classes of GM threefolds and $\{\mathcal{A}_X\}/\simeq$ is the set of equivalence classes of Kuznetsov components of GM threefolds. Note that a global description of a "moduli of Kuznetsov components" $\{\mathcal{A}_X\}/\simeq$ is not known; however, local deformations are controlled by the second Hochschild cohomology $\mathrm{HH}^2(\mathcal{A}_X)$. The fiber of the "categorical period map" $\mathcal{P}_{\mathrm{cat}}$ over $\mathcal{A}_X$ for an ordinary GM threefold is defined as the isomorphism classes of all ordinary GM threefolds X' such that $\mathcal{A}_{X'} \simeq \mathcal{A}_X$.

Theorem 10.3. *The general fiber $\mathcal{P}_{\mathrm{cat}}^{-1}([\mathcal{A}_X])$ of the categorical period map over the alternative Kuznetsov component of an ordinary GM threefold X is the union of $C_m(X)/\iota$ and $M_G^X(2,1,5)/\iota'$ where ι, ι' are geometrically meaningful involutions.*

Proof. The general fiber $\mathcal{P}_{\mathrm{cat}}^{-1}([\mathcal{A}_X])$ of the categorical period map consists of GM threefolds X' such that there is an equivalence of Kuznetsov components $\mathcal{A}_{X'} \simeq \mathcal{A}_X$. Then, by Theorem 9.7, X' is also a general ordinary GM threefold. Thus, by Theorems 9.3 and 6.10, we know that $\mathcal{A}_{X'} \simeq \mathcal{A}_X$ if and only if X' is a conic transform of X, or a conic transform of a line transform of X. Then, the result follows from Proposition 6.9 and Corollary 9.5. □

The Kuznetsov components of prime Fano threefolds of index 1 and 2 are often regarded as categorical analogues of the intermediate Jacobians of these threefolds, and it is known that if there is a Fourier–Mukai-type equivalence $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$ (or $\mathcal{A}_X \simeq \mathcal{A}_{X'}$), then $J(X) \cong J(X')$ by [47]. For the converse, we have the following result.

Theorem 10.4. *For smooth prime Fano threefolds X, if X is one of the following:*

- Y_d, $\quad 2 \leqslant d \leqslant 5$
- X_{2g-2}, $\quad g = 5, 7, 8, 9, 10, 12$,

then the intermediate Jacobian $J(X)$ uniquely determines the Kuznetsov component $\mathcal{K}u(X)$, that is, for another prime Fano threefold X' of the same degree, if $J(X) \cong J(X')$, then $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$.

Proof. If X is an index 2 prime Fano threefold Y_d of degree $2 \leqslant d \leqslant 5$, then the statement follows from the Torelli theorems for Y_d. Now let X_d be a degree d index one prime Fano threefold. If $X = X_8$, the statement follows from its Torelli theorem. If $X = X_{12}, X_{18}, X_{16}$, their intermediate Jacobians are Jacobians of curves: $J(X_{12}) \cong J(C_7)$, $J(X_{16}) \cong J(C_3)$, and $J(X_{18}) \cong J(C_2)$. But $\mathcal{K}u(X_{12}) \simeq \mathrm{D}^b(C_7)$, $\mathcal{K}u(X_{16}) \simeq \mathrm{D}^b(C_3)$, and $\mathcal{K}u(X_{18}) \simeq \mathrm{D}^b(C_2)$. Thus, the statement follows from the classical Torelli theorem for curves. If $X = X_{14}$, the statement follows from the Kuznetsov conjecture for the pair (Y_3, X_{14}) [28] and the Torelli theorem for cubic threefolds. If $X = X_{22}$, the statement is trivial since $\mathcal{K}u(X_{22}) \cong \mathcal{K}u(Y_5)$ ([35]) and Y_5 is rigid, so $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$ is always true. □

Therefore, it is natural to make the following conjecture.

Conjecture 10.5. *Let X be a prime Fano threefold of index one or two. Then the intermediate Jacobian $J(X)$ uniquely determines the Kuznetsov component $\mathcal{K}u(X)$, that is, for another prime Fano threefold X' of the same degree, if $J(X) \cong J(X')$, then $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$.*

Surprisingly, in the case of general ordinary GM threefolds, we can restate the Debarre–Iliev–Manivel Conjecture 10.1 as Conjecture 10.5.

Proposition 10.6. *The Debarre–Iliev–Manivel Conjecture 10.1 is equivalent to Conjecture 10.5 for general ordinary GM threefolds.*

Proof. First, we assume that Conjecture 10.5 holds. Then by Corollary 9.6 and Theorem 10.3, the Debarre–Iliev–Manivel Conjecture 10.1 holds.

On the other hand, we assume that the Debarre–Iliev–Manivel Conjecture 10.1 holds. Then for any X and X' such that $J(X) \cong J(X')$, X is either a conic transform of X', or X is a conic transform of a line transform of X'. In both cases, we have $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$ by the Duality Conjecture Theorem 6.10. Thus, Conjecture 10.5 holds. □

APPENDIX: UNIQUENESS OF SERRE-INVARIANT STABILITY CONDITIONS

In this appendix, we aim to prove the uniqueness of Serre-invariant stability conditions on $\mathcal{K}u(X)$ for several prime Fano threefolds X (Theorem A.10). We start with a general criterion for when two numerical stability conditions with the same central charge are equal. We always assume that any triangulated category is k-linear and of finite type, that is, $\sum_i \mathrm{ext}^i(A, B) < +\infty$ for any two objects A, B. Therefore, the Euler form and the numerical Grothendieck group are well defined.

Theorem A.1. *Let $\mathcal{D}$ be a k-linear triangulated category of finite type. Assume that*

(A) *$\chi(x, x) \leqslant 1 - n$ for a positive integer n and any nonzero $x \in \mathcal{N}(\mathcal{D})$,*
(B) *there exists an object D satisfies*

$$n \leqslant \mathrm{ext}^1(D, D) < 2n,$$

Let $\sigma_1 = (\mathcal{A}_1, Z)$ be a numerical stability condition on $\mathcal{D}$ and $D_1, D_2 \in \mathcal{A}_1$ be two σ_1-stable objects satisfying:

(C) *for any two objects $A, B \in \mathcal{D}$, if $\phi^+_{\sigma_1}(B) < \phi^-_{\sigma_1}(A)$, then $\mathrm{Hom}(B, A[2]) = 0$,*

(D) *if E is a σ_1-semistable object with $\chi(E, D_1) \geqslant 0$ and $\chi(E, D_2) \geqslant 0$, then there exist $k \in \{1, 2\}$ such that $\chi(E, D_k) > 0$ and $\mu_{\sigma_1}(E) < \mu_{\sigma_1}(D_k)$, and*

(E) *if E is a σ_1-semistable object with $\chi(D_1, E) \geqslant 0$ and $\chi(D_2, E) \geqslant 0$, then there exist $k \in \{1, 2\}$ such that $\chi(D_k, E) > 0$ and $\mu_{\sigma_1}(E) > \mu_{\sigma_1}(D_k)$.*

If $\sigma_2 = (\mathcal{A}_2, Z)$ is a numerical stability condition on $\mathcal{D}$ that satisfies (C), (D), and (E) such that D_1 and D_2 are σ_2-stable with $\phi_{\sigma_2}(D_1) = \phi_{\sigma_1}(D_1)$ and $\phi_{\sigma_2}(D_2) = \phi_{\sigma_1}(D_2)$, then $\sigma_1 = \sigma_2$.

We first prove several lemmas. By the same proof as in [3, Lemma 2.5], we have the following generalized version of Weak Mukai lemma.

Lemma A.2. *Let $\mathcal{D}$ be a k-linear triangulated category with finite-dimensional* Hom*-space. Then for any exact triangle $A \to E \to B$ with* $\mathrm{Hom}(A, B) = \mathrm{Hom}(B, A[2]) = 0$, *we have*

$$\mathrm{ext}^1(A, A) + \mathrm{ext}^1(B, B) \leqslant \mathrm{ext}^1(E, E).$$

Lemma A.3. *Let $\mathcal{D}$ be a k-linear triangulated category of finite type that satisfies (A). Assume that there is a stability condition $\sigma = (\mathcal{A}, Z)$ on $\mathcal{D}$ satisfies (C).*

(1) *The homological dimension of $\mathcal{A}$ is at most 2.*
(2) *For any exact triangle $A \to E \to B$ with $\phi_\sigma^-(A) > \phi_\sigma^+(B)$, we have*

$$\mathrm{ext}^1(A, A) + \mathrm{ext}^1(B, B) \leqslant \mathrm{ext}^1(E, E).$$

(3) *For any nonzero object $A \in \mathcal{D}$, we have* $\mathrm{ext}^1(A, A) \geqslant n$.
(4) *If a nonzero object E is not σ-semistable, then any Harder–Narasimhan factor A of E satisfies*

$$\mathrm{ext}^1(A, A) < \mathrm{ext}^1(E, E).$$

(5) *Any object E with*

$$n \leqslant \mathrm{ext}^1(E, E) < 2n$$

is σ-semistable.

Proof. Let $A, B \in \mathcal{A}$. Then, we have $\phi_\sigma^+(A) \leqslant 1 < \phi_\sigma^-(B[k])$ for any $k \geqslant 1$. Therefore, by (C), we get $\mathrm{Hom}(A, B[k+2]) = \mathrm{Hom}(A, B[k][2]) = 0$ for any $k \geqslant 0$. This proves (1).

Now for (2), note that $\mathrm{Hom}(A, B) = 0$ and by (C), we have $\mathrm{Hom}(B, A[2])$. Then the result follows from Lemma A.2.

Next, we prove (3). If $A \neq 0 \in \mathcal{A}$, then from (1), we get $\chi(A, A) = \hom(A, A) - \mathrm{ext}^1(A, A) + \mathrm{ext}^2(A, A)$. Since $\chi(A, A) \leqslant 1 - n$ by (A), we know that $\mathrm{ext}^1(A, A) \geqslant n$ in this case. Now for a general nonzero object $A \in \mathcal{D}$, if A is σ-semistable, then it is in $\mathcal{A}$ up to shift and the result follows from the previous argument. So, we assume that A is not σ-semistable. Let A' be the first Harder–Narasimhan factor of A with respect to σ, and $A'' := \mathrm{cone}(A' \to A)$. We have $\phi_\sigma(A') > \phi_\sigma^+(A'')$. Using (2) and σ-semistability of A', we obtain $n \leqslant \mathrm{ext}^1(A', A') + \mathrm{ext}^1(A'', A'') \leqslant \mathrm{ext}^1(A, A)$, and hence, (3) follows. And (4) follows from the induction on the number of Harder–Narasimhan factors of E and using (2) and (3).

Finally, if such E in (5) is not σ-semistable, then by the existence of Harder–Narasimhan filtration, we can find a triangle $A \to E \to B$ with $\phi_\sigma^-(A) > \phi_\sigma^+(B)$. By (2) and (3), this contradicts our assumption on $\mathrm{ext}^1(E, E)$. Thus, E is σ-semistable. □

Now we are ready to prove our criterion.

Proof of Theorem A.1. Since σ_1 and σ_2 have the same central charge, it remains to show $\mathcal{A}_1 = \mathcal{A}_2$. By our assumptions, $D_1, D_2 \in \mathcal{A}_1 \cap \mathcal{A}_2$ are both σ_1-stable and σ_2-stable with phases in (0,1].

Step 1. First, we show that if E is a σ_i-semistable object that is also σ_j-semistable, then $\phi_{\sigma_1}(E) = \phi_{\sigma_2}(E)$, where $\{i, j\} = \{1, 2\}$. Since σ_1 and σ_2 satisfy the same assumptions, in the following, we will take $i = 2$ and $j = 1$. The other case can be deduced from the same argument but exchanges the role of σ_1 and σ_2.

Up to shift, we can assume that $E \in \mathcal{A}_2$. Since σ_1 and σ_2 have the same central charge, we have $\phi_{\sigma_1}(E) - \phi_{\sigma_2}(E) = 2m$ for an integer m. Then, we see

$$2m < \phi_{\sigma_1}(E) \leqslant 2m + 1. \tag{A.1}$$

- Assume that there exist $k, l \in \{1, 2\}$ such that $\chi(D_k, E) < 0$ and $\chi(E, D_l) < 0$. Then by Lemma A.3 (1) and the fact that $E, D_1, D_2 \in \mathcal{A}_2$, we have

  $$\mathrm{ext}^1(D_k, E) \neq 0, \quad \mathrm{ext}^1(E, D_l) \neq 0,$$

  which imply

  $$-1 < \phi_{\sigma_1}(D_k) - 1 \leqslant \phi_{\sigma_1}(E) \leqslant \phi_{\sigma_1}(D_l) + 1 \leqslant 2.$$

  Hence, by (A.1), we get $2m < 2$ and $-1 < 2m + 1$, which means $m = 0$ and we obtain $\phi_{\sigma_1}(E) = \phi_{\sigma_2}(E)$ as required.
- Assume that $\chi(E, D_1) \geqslant 0$ and $\chi(E, D_2) \geqslant 0$. By (D), there is an integer $s \in \{1, 2\}$ such that $\chi(E, D_s) > 0$ and $\mu_{\sigma_1}(E) < \mu_{\sigma_1}(D_s)$. Thus, we have $\mu_{\sigma_2}(E) < \mu_{\sigma_2}(D_s)$ as well. Since $E, D_s \in \mathcal{A}_2$, we get $\phi_{\sigma_2}(E) < \phi_{\sigma_2}(D_s)$, which implies $\mathrm{Hom}(E, D_s[2]) = 0$ by (C). Then from $\chi(E, D_s) > 0$ and Lemma A.3 (1), we obtain $\mathrm{Hom}(E, D_s) \neq 0$, and hence,

  $$\phi_{\sigma_1}(E) \leqslant \phi_{\sigma_1}(D_s) \leqslant 1.$$

  Now if one of $\chi(D_1, E)$ and $\chi(D_2, E)$ is negative, the same argument as in the first case shows that $-1 < \phi_{\sigma_1}(E)$.

  If $\chi(D_1, E) \geqslant 0$ and $\chi(D_2, E) \geqslant 0$, then by (E), there is an integer $t \in \{1, 2\}$ such that $\chi(D_t, E) > 0$ and $\mu_{\sigma_1}(E) = \mu_{\sigma_2}(E) > \mu_{\sigma_1}(D_t) = \mu_{\sigma_2}(D_t)$. Since $E, D_t \in \mathcal{A}_2$, we get $\phi_{\sigma_2}(E) > \phi_{\sigma_2}(D_t)$, which by (C) implies $\mathrm{Hom}(D_t, E[2]) = 0$. Then together with $\chi(D_t, E) > 0$ and Lemma A.3 (1), we see $\mathrm{Hom}(D_t, E) \neq 0$. Therefore, we have $0 < \phi_{\sigma_1}(D_t) \leqslant \phi_{\sigma_1}(E)$. In both cases, we always have $\phi_{\sigma_1}(E) \in (-1, 2]$. By (A.1), we get $m = 0$ and $\phi_{\sigma_1}(E) = \phi_{\sigma_2}(E)$ as required.
- Assume that $\chi(D_1, E) \geqslant 0$ and $\chi(D_2, E) \geqslant 0$. Then using (E), by a similar argument as the second case, we obtain $\phi_{\sigma_1}(E) = \phi_{\sigma_2}(E)$. This completes the first step.

Step 2. Next, we prove that an object E is σ_1-semistable if and only if σ_2-semistable. We show this by induction on $\mathrm{ext}^1(E, E)$. If $\mathrm{ext}^1(E, E) < 2n$, then from (B), we know that such E exists. By Lemma A.3 (5), E is both σ_1-semistable and σ_2-semistable.

Now assume that the statement holds for any object F with $\mathrm{ext}^1(F, F) < N$ for an integer $N > 2n$. Let E be an object with $\mathrm{ext}^1(E, E) = N$. If E is σ_i-semistable but not σ_j-semistable for $\{i, j\} = \{1, 2\}$, let A be the first Harder–Narasimhan factor of E with respect to σ_j and B be the last one. Therefore, we see

$$\phi_{\sigma_j}(A) > \phi_{\sigma_j}(B). \tag{A.2}$$

And from Lemma A.3 (4), we have

$$\mathrm{ext}^1(A,A) < \mathrm{ext}^1(E,E), \quad \mathrm{ext}^1(B,B) < \mathrm{ext}^1(E,E).$$

Therefore, by the induction hypothesis, A and B are σ_i-semistable as well and the first step implies

$$\phi_{\sigma_1}(A) = \phi_{\sigma_2}(A), \quad \phi_{\sigma_1}(B) = \phi_{\sigma_2}(B). \tag{A.3}$$

But then from $\mathrm{Hom}(A,E) \neq 0$ and $\mathrm{Hom}(E,B) \neq 0$, we have $\phi_{\sigma_i}(A) \leqslant \phi_{\sigma_i}(E) \leqslant \phi_{\sigma_i}(B)$, which contradicts (A.2) and (A.3). Hence, E is σ_j-semistable. This completes our induction argument.

Step 3. Finally, by the previous two steps, we know that an object E is σ_1-semistable if and only if σ_2-semistable with $\phi_{\sigma_1}(E) = \phi_{\sigma_2}(E)$. Since every nonzero object in the heart is obtained by extensions of semistable objects with phases in (0,1], we know that $\mathcal{A}_1 = \mathcal{A}_2$. This ends the proof of our theorem. □

A.1 | Applications to Kuznetsov components of Fano threefolds

Let Y_d be smooth index 2 degree $d \geqslant 2$ prime Fano threefold and X_{4d+2} an index 1 degree $4d+2$ prime Fano threefold. In this section, we apply Theorem A.1 to show that all Serre-invariant stability conditions on $\mathcal{K}u(Y_d)$ and $\mathcal{K}u(X_{4d+2})$ (or $\mathcal{A}_{X_{4d+2}}$) are in the same $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-orbit for each $d \geqslant 2$ (Theorem A.10).

Recall that the Kuznetsov component of an index two prime Fano threefold Y_d of degree d is defined by $\mathcal{K}u(Y_d) := \langle \mathcal{O}_{Y_d}, \mathcal{O}_{Y_d}(H)\rangle^{\perp}$. The numerical Grothendieck group $\mathcal{N}(\mathcal{K}u(Y_d))$ is a rank two lattice generated by two classes

$$v = 1 - \frac{1}{d}H^2, \quad w = H - \frac{1}{2}H^2 + \left(\frac{1}{6} - \frac{1}{d}\right)H^2.$$

Moreover, under this basis, the Euler form is given by the matrix

$$\begin{pmatrix} -1 & -1 \\ 1-d & -d \end{pmatrix}.$$

For index one cases, we assume that $d \geqslant 2$. Then the Kuznetsov component is defined by $\mathcal{K}u(X_{4d+2}) := \langle \mathcal{E}_{X_{4d+2}}, \mathcal{O}_{X_{4d+2}}\rangle^{\perp}$, where $\mathcal{E}_{X_{4d+2}}$ is a certain exceptional bundle pulled back from a Grassmannian (cf. [28]).

By [4], $\sigma(\alpha,\beta)$ is a stability condition on $\mathcal{K}u(Y_d)$ and $\mathcal{K}u(X_{4d+2})$ for suitable (α,β). Moreover, according to [48, 51], these stability conditions are all Serre-invariant.

Since for every index one prime Fano threefold X_{4d+2} with $d \geqslant 3$, there is an index two prime Fano threefold Y_d with $\mathcal{K}u(Y_d) \simeq \mathcal{K}u(X_{4d+2})$ by [28], hence we only need to consider Kuznetsov components of Y_d for $d \geqslant 2$ and X_{10}. Moreover, $\mathcal{K}u(Y_4)$ is equivalent to the derived category of a smooth curve, and $\mathcal{K}u(Y_5)$ is equivalent to the derived category of the 3-Kronecker quiver. In these two cases, the result is known by [42] and [15]. So, in the following, we mainly focus on $\mathcal{D} = \mathcal{K}u(Y_d)$ for $2 \leqslant d \leqslant 3$ or $\mathcal{K}u(X_{10})$. We first prove some properties of Serre-invariant stability conditions.

Lemma A.4. *Let $\mathcal{D} = \mathcal{K}u(Y_d)$ for $2 \leqslant d \leqslant 3$ or $\mathcal{K}u(X_{10})$ and $\sigma = (\mathcal{A}, Z)$ be a Serre-invariant stability condition on $\mathcal{D}$. Then $\mathcal{D}$ satisfies (A) and (B) in Theorem A.1 and σ satisfies (C). Moreover, for any σ-semistable object $E \in \mathcal{D}$, we have:*

(1) *if* $\mathcal{D} = \mathcal{K}u(Y_3)$*, then*

$$\phi_\sigma(E) + 1 \leqslant \phi_\sigma(S_\mathcal{D}(E)) < \phi_\sigma(E) + 2,$$

(2) *if* $\mathcal{D} = \mathcal{K}u(Y_2)$ *or* $\mathcal{K}u(X_{10})$*, then*

$$\phi_\sigma(S_\mathcal{D}(E)) = \phi_\sigma(E) + 2.$$

Proof. It is clear that $\mathcal{D}$ satisfies (A). And by [51, Lemma 5.16] and Proposition 8.7, $\mathcal{D}$ also satisfies (B). When $\mathcal{D} = \mathcal{K}u(Y_3)$, by [51, Lemma 5.9], we have $\phi_\sigma(S_\mathcal{D}(E)) < \phi_\sigma(E) + 2$. When $\mathcal{D} = \mathcal{K}u(Y_2)$ or $\mathcal{K}u(X_{10})$, recall that $S_\mathcal{D}^2 \cong [4]$. Then the same argument as in [51, Lemma 5.9] shows that $\phi_\sigma(S_\mathcal{D}(E)) \leqslant \phi_\sigma(E) + 2$. Then for any two objects $A, B \in \mathcal{D}$ with $\phi_\sigma^+(B) < \phi_\sigma^-(A)$, using (1) and (2), we see

$$\phi_\sigma^-(A[2]) = \phi_\sigma^-(A) + 2 > \phi_\sigma^+(B) + 2 \geqslant \phi_\sigma^+(S_\mathcal{D}(B)).$$

Hence, $\mathrm{Hom}(B, A[2]) = \mathrm{Hom}(A[2], S_\mathcal{D}(B)) = 0$ and the condition (C) is satisfied.

When $\mathcal{D} = \mathcal{K}u(Y_3)$, from [51, Lemma 5.11], we get $\mathrm{ext}^1(E, E) \neq 0$, which implies $\phi_\sigma(E) + 1 \leqslant \phi_\sigma(S_\mathcal{D}(E))$. This proves (1).

When $\mathcal{D} = \mathcal{K}u(Y_2)$ or $\mathcal{K}u(X_{10})$, since $\mathcal{D}$ satisfies (A) and (C), by Lemma A.3, we have $\mathrm{ext}^1(E, E) \neq 0$, which implies $\phi_\sigma(E) + 1 \leqslant \phi_\sigma(S_\mathcal{D}(E))$. Now since $[E] = [S_\mathcal{D}(E)] \in \mathcal{N}(\mathcal{D})$, we have $\phi_\sigma(E) - \phi_\sigma(S_\mathcal{D}(E)) \in 2\mathbb{Z}$. Hence, we get $\phi_\sigma(S_\mathcal{D}(E)) = \phi_\sigma(E) + 2$. □

Before verifying (D) and (E), we need several lemmas.

Lemma A.5. *Let X be a GM threefold.*

(1) $\mathrm{Hilb}_X^{3t+m} = \varnothing$ *for* $m < 1$*. Thus, for any conic* $C \subset X$ *and line* $L \subset X$*, we have* $\mathrm{Hom}(I_C, \mathcal{O}_L(-k)) = 0$ *for any* $k > 1$*.*
(2) *If a line L and a conic C satisfies $L \cap C \neq \varnothing$, then $L \cap C$ is of length one and $L \cup C$ is a twisted cubic.*
(3) *Let $\mathcal{I} \subset \Gamma(X) \times \mathcal{C}(X)$ be the incidence variety, that is,*

$$\mathcal{I} = \{(L, C) \, : \, L \cap C \neq \varnothing\}.$$

Then the projection maps $\mathcal{I} \to \mathcal{C}(X)$ and $\mathcal{I} \to \Gamma(X)$ are surjective.

Proof. By [53, Corollary 1.38], we have $\mathrm{Hilb}_X^{3t+m} = \varnothing$ for $m < 0$. Thus, to prove (1), we only need to show $\mathrm{Hilb}_X^{3t} = \varnothing$. From [53, Corollary 1.38], $\langle C \rangle \cong \mathbb{P}^2$ for any $[C] \in \mathrm{Hilb}_X^{3t}$. Since X is an intersection of quadrics, such a C cannot exist on X. Hence, $\mathrm{Hilb}_X^{3t} = \varnothing$. Now note that the kernel of any nonzero map $I_C \to \mathcal{O}_L(-k)$ is the ideal sheaf of a closed subscheme with the Hilbert polynomial $3t + m$ for $m \leqslant 2 - k$. Therefore, $\mathrm{Hom}(I_C, \mathcal{O}_L(-k)) = 0$ when $k > 1$. This proves (1). For (2), note that $\chi(\mathcal{O}_{L \cup C}) = 2 - \mathrm{length}(L \cap C)$, then the result follows from (1).

Finally, we prove (3). Since $\dim \Gamma(X) = 1$, all lines on X sweep out a surface S in X. By $\mathrm{Pic}(X) = \mathbb{Z}\mathcal{O}_X(H)$, we see $S \in |mH|$ for $m > 0$. Thus, $C.S \geqslant C.H > 0$ for any conic C. In other words, $C \cap S \neq \varnothing$, hence any conic on X intersects with a line. Thus, $\mathcal{I} \to \mathcal{C}(X)$ is surjective. Similarly, since X is covered by conics, any line intersects with a conic. Then, $\mathcal{I} \to \Gamma(X)$ is surjective. □

Lemma A.6. *Let X be a GM threefold. Then there exists a line L and twisted cubics C and D on X such that*

(1) $[L] \in \Gamma(X)$ *is a smooth point,*
(2) $I_C \notin \mathcal{K}u(X)$ *and* $L \cup C = Z(s)$ *for a section* $s \in H^0(\mathcal{E}^\vee)$,
(3) $L \subset D$, $I_D \in \mathcal{K}u(X)$ *and* $\mathrm{ext}^1(I_D, I_D) = 3$.

Proof. Let $\mathcal{I} \subset \Gamma(X) \times \mathcal{C}(X)$ be the incidence variety. We denote by $\mathcal{C}_1$ the sublocus of $\mathcal{C}(X)$ parametrizing smooth conics Z such that their involutive conics are also smooth and $\hom(\mathcal{E}, I_Z) = 1$. By Remark 7.17, $\mathcal{C}_1$ is an open subscheme of $\mathcal{C}(X)$. Let $\mathcal{I}_1 := \mathcal{I}|_{\Gamma(X)\times\mathcal{C}_1}$. From [25, Theorem 3.4 (iii)] and [23, Section 3.1], $\Gamma(X)$ is generic smooth. This implies that the image of $p : \mathcal{I}_1 \to \Gamma(X)$ contains a smooth point.

Let $L \subset X$ be a line such that $[L] \in \Gamma(X)$ is smooth and contained in the image of p. Then $p^{-1}([L])$ is nonempty and there is a conic $[Z] \in \mathcal{C}_1$ such that $L \cap Z \neq \emptyset$. We set $D := L \cup Z$. And since $\mathrm{Hom}(\mathcal{E}, I_L) \neq 0$, there is a section $s \in H^0(\mathcal{E})$ such that $L \subset Z(s)$. We define C to be the residue curve of L in $Z(s)$. It is clear that C and D are twisted cubics by Lemma A.5. Moreover, L and Z intersect transversely at a single point. Then, it remains to check $I_C \notin \mathcal{K}u(X)$, $\mathrm{ext}^1(I_D, I_D) = 3$ and $I_D \in \mathcal{K}u(X)$.

Since $C \subset Z(s)$, it is clear that $\mathrm{Hom}(\mathcal{E}, I_C) \neq 0$, that is, $I_C \notin \mathcal{K}u(X)$. Moreover, by the construction, we have an exact sequence

$$0 \to I_D \to I_Z \to \mathcal{O}_L(-1) \to 0. \tag{A.4}$$

Note that all conics are connected, hence the smoothness implies irreducibility. Since $\mathrm{Hom}(\mathcal{E}, I_Z) = k$ and the involutive conic Z' is smooth, we know that $Z \cup Z'$ only has two irreducible components that are both of degree 2, hence does not contain D. This means that the unique nonzero map in $\mathrm{Hom}(\mathcal{E}, I_Z) = k$ does not factor through I_D. Hence, the induced map $\mathrm{Hom}(\mathcal{E}, I_Z) \to \mathrm{Hom}(\mathcal{E}, \mathcal{O}_L(-1))$ is injective. By $\mathrm{RHom}^\bullet(\mathcal{E}, \mathcal{O}_L(-1)) = k$, this map is actually an isomorphism. Therefore, applying $\mathrm{Hom}(\mathcal{E}, -)$ to (A.4), we obtain $\mathrm{RHom}^\bullet(\mathcal{E}, I_D) = 0$, which implies $I_D \in \mathcal{K}u(X)$.

To show $\mathrm{ext}^1(I_D, I_D) = 3$, since $\hom(I_D, I_D) = 1$ and $\mathrm{ext}^3(I_D, I_D) = 0$, by $\chi(I_D, I_D) = -2$, we only need to prove $\mathrm{ext}^2(I_D, I_D) = 0$. From the construction above, we see $\mathrm{ext}^2(I_Z, I_Z) = 0$. Moreover, $\mathrm{ext}^2(\mathcal{O}_L(-1), \mathcal{O}_L(-1)) = 0$ since $[L] \in \Gamma(X)$ is a smooth point. And by the transversality of the intersection of L and Z, we see the derived restriction $\mathcal{O}_Z|_L \cong \mathcal{O}_{L\cap Z} \in \mathrm{D}^b(L)$. Hence, $\mathrm{ext}^2(\mathcal{O}_Z, \mathcal{O}_L(-1)) = \mathrm{ext}^1(I_Z, \mathcal{O}_L(-1)) = 0$. Finally, by Lemma A.5, we have $\hom(I_Z, \mathcal{O}_L(-2)) = \mathrm{ext}^3(\mathcal{O}_L(-1), I_Z) = 0$. Then $\mathrm{ext}^2(I_D, I_D) = 0$ follows from applying [52, Lemma 2.27] to (A.4). $\square$

Lemma A.7. *Let X be a GM threefold and L, C, D as in Lemma A.6. We define $F_1 := \mathrm{pr}'(I_C)$, $F_1' := I_D$, and $F_2 := \mathrm{pr}'(I_L)$. Then the objects F_1, F_1', F_2 are stable with respect to any Serre-invariant stability condition on $\mathcal{K}u(X)$. Moreover, F_1 and F_1' have the same phase.*

Proof. From the construction, we see $\mathrm{ext}^1(F_1', F_1') = 3$. By the same argument as in [54, Corollary 5.4], we have $\mathrm{ext}^1(F_1, F_1) = 3$. Finally, applying [52, Lemma 2.27] to $\mathcal{E}^{\oplus 2} \to I_L \to F_2$ and using $\mathrm{RHom}^\bullet(\mathcal{O}_L, \mathcal{O}_L) = k \oplus k[-1]$ implies $\mathrm{ext}^1(F_2, F_2) = 3$. Then, the stability of F_1, F_1', and F_2 follows from Proposition 4.12.

As $[F_1] = [F_1'] \in \mathcal{N}(\mathcal{K}u(X))$, we have $\chi(F_1, F_1') < 0$. Since $\mathrm{ext}^i(F_1, F_1') = \mathrm{ext}^i(I_C, I_D) = 0$ for $i \notin \{1, 2\}$, we get $\mathrm{Hom}(F_1, F_1'[1]) = \mathrm{Hom}(F_1'[1], S_{\mathcal{K}u(X)}(F_1)) \neq 0$. Using Lemma A.4, we obtain

$\phi_\sigma(F_1') - 1 < \phi_\sigma(F_1) < \phi_\sigma(F_1') + 1$ for any Serre-invariant stability condition σ. Thus, $\phi_\sigma(F_1) = \phi_\sigma(F_1')$ since $[F_1] = [F_1'] \in \mathcal{N}(\mathcal{K}u(X))$. □

Lemma A.8. *Let $\mathcal{D} = \mathcal{K}u(Y_d)$ for $2 \leqslant d \leqslant 3$ or $\mathcal{K}u(X_{10})$. Then there exist two objects $F_1, F_2 \in \mathcal{D}$ such that for any Serre-invariant stability condition σ on $\mathcal{D}$, F_1 and F_2 are σ-stable with*

$$\phi_\sigma(F_2) - 1 < \phi_\sigma(F_1) < \phi_\sigma(F_2).$$

In particular, the image of the central charge is not contained in a line for any Serre-invariant stability condition on $\mathcal{D}$.

Proof. When $\mathcal{D} = \mathcal{K}u(Y_d)$, we define $F_2 := R\mathcal{H}om(I_L, \mathcal{O}_{Y_d}(-H))[1]$ and $F_1 = I_L$, where $L \subset Y_d$ is a line. Then by [51, Lemma 5.13] and [51, Remark 4.8], F_1 and F_2 are σ-stable for any Serre-invariant stability condition σ on $\mathcal{D}$ with $\phi_\sigma(F_2) - 1 < \phi_\sigma(F_1) < \phi_\sigma(F_2)$.

Now assume that $\mathcal{D} = \mathcal{K}u(X_{10})$. We take F_1, F_1', and F_2 as in Lemma A.7. By [54, Proposition 3.3, 5.3], we have $\mathrm{pr}'(I_C) \cong \mathrm{pr}'(G)$, where G fits into an exact triangle

$$\mathcal{O}_X(-H)[1] \to G \to \mathcal{O}_L(-2)$$

and is the twisted derived dual of the line L.

First, we prove that $\mathrm{Hom}(F_2, F_1[1]) \neq 0$. By adjunction, we have $\mathrm{Hom}(F_2, F_1[1]) = \mathrm{Hom}(I_L, \mathrm{pr}'(G)[1])$. And by [54, Proposition 5.3], $\mathrm{pr}'(G)$ fits into an exact triangle

$$G \to \mathrm{pr}'(G) \to \mathcal{E}. \tag{A.5}$$

Since $\mathcal{E}|_L \cong \mathcal{O}_L \oplus \mathcal{O}_L(-1)$, it is easy to see $\mathrm{Ext}^i(I_L, \mathcal{E})$ for $i \neq 2$. So, applying $\mathrm{Hom}(I_L, -)$ to (A.5), we get $\mathrm{Hom}(I_L, \mathrm{pr}'(G)[1]) = \mathrm{Hom}(I_L, G[1]) = \mathrm{Hom}(I_L, \mathcal{O}_L(-2)[1])$. As the normal bundle $N_{L/X_{10}}$ is either $\mathcal{O}_L \oplus \mathcal{O}_L(-1)$ or $\mathcal{O}_L(1) \oplus \mathcal{O}_L(-2)$ by [46, Lemma 4.2.1], we see the derived restriction $I_L|_L \cong N^\vee_{L/X} \oplus \mathcal{O}_L(1)[1]$ from [21, Proposition 11.8]. Then $\mathrm{Hom}(I_L, \mathcal{O}_L(-2)[1]) \neq 0$ follows from a direct computation.

Next, we show that $\mathrm{Hom}(F_1', F_2) \neq 0$. By the definition of pr', $\mathrm{pr}'(I_L)$ fits into an exact triangle $\mathcal{E}^{\oplus 2} \to I_L \to \mathrm{pr}'(I_L)$. Then applying $\mathrm{Hom}(I_D, -)$ to this triangle, the result follows from $\mathrm{Hom}(I_D, \mathcal{E}) = 0$ and $\mathrm{Hom}(I_D, I_L) \neq 0$ since $L \subset D$.

By Lemma A.7, F_1, F_1', and F_2 are all stable with respect to any Serre-invariant stability condition on $\mathcal{D}$. Therefore, combined with above results, we get $\phi_\sigma(F_1) = \phi_\sigma(F_1') < \phi_\sigma(F_2) < \phi_\sigma(F_1) + 1$ as desired. □

Now we are ready to verify conditions (D) and (E) in Theorem A.1.

Lemma A.9. *Let $\mathcal{D} = \mathcal{K}u(Y_d)$ for $2 \leqslant d \leqslant 3$ or $\mathcal{K}u(X_{10})$. Then there exists a Serre-invariant stability condition σ_1 on $\mathcal{D}$ with two σ_1-stable objects D_1, D_2 satisfying (D) and (E). Moreover,*

- *we can assume that for any Serre-invariant stability condition σ on $\mathcal{D}$, D_1 and D_2 are σ-stable with*

$$\phi_\sigma(D_1) - 1 < \phi_\sigma(D_2) < \phi_\sigma(D_1), \text{ or } \phi_\sigma(D_1) < \phi_\sigma(D_2) < \phi_\sigma(D_1) + 1,$$

- *any Serre-invariant stability condition on $\mathcal{D}$ with the same central charge as σ_1 satisfies (D) and (E).*

Proof. When $\mathcal{D} = \mathcal{K}u(Y_d)$, we define $D_1 := R\mathcal{H}om(I_L, \mathcal{O}_{Y_d}(-H))[1]$ and $D_2 = I_L[1]$, where $L \subset Y_d$ is a line. Then by [51, Lemma 5.13] and [51, Remark 4.8], D_1 and D_2 are σ-stable for any Serre-invariant stability condition σ on $\mathcal{D}$ with $\phi_\sigma(D_1) < \phi_\sigma(D_2) < \phi_\sigma(D_1) + 1$. In this case, we take $\sigma_1 := \sigma(\alpha, -\frac{1}{2})$ for $\alpha > 0$ sufficiently small. Then, by [51, Section 4], $D_1, D_2 \in \mathcal{A}(\alpha, -\frac{1}{2})$. Now a direct computation shows that, for any object E with $[E] = av + bw$, we have

- $\chi(E, D_2) = a + (d-1)b$, $\chi(D_2, E) = a + b$; and $\mu^0_{\alpha,-\frac{1}{2}}(E) > \mu^0_{\alpha,-\frac{1}{2}}(D_2) \iff b < 0$
- $\chi(E, D_1) = -b$, $\chi(D_1, E) = -[(d-2)a + (d-1)b]$; and $\mu^0_{\alpha,-\frac{1}{2}}(E) > \mu^0_{\alpha,-\frac{1}{2}}(D_1) \iff a + b < 0$.

Then, it is straightforward to check (D) and (E) for σ_1.

When $\mathcal{D} = \mathcal{K}u(X_{10})$, we use the equivalence Ξ in Lemma 3.7 and prove every thing on $\mathcal{A}_{X_{10}}$. Let $\sigma_1 := \sigma(\alpha, \beta)$, where $\beta < 0$ and $\alpha > 0$ with $-\beta$ and α are sufficiently small. We set $D_1 = I_C[1]$ and $D_2 = \mathrm{pr}(F)[1]$, where $C \subset X$ is a smooth conic with $I_C \in \mathcal{A}_{X_{10}}$ and $F \in M_G(2, 1, 5)$ is non-locally free. It is clear that $D_1, D_2 \in \mathcal{A}(\alpha, \beta)$ and are stable with respect to any Serre-invariant stability condition on $\mathcal{A}_{X_{10}}$ by Lemma 7.5 and Proposition 8.7. As in the previous case, it is straightforward to check (D) and (E) for σ_1. Now we show that for any Serre-invariant stability condition σ on $\mathcal{A}_{X_{10}}$, we have $\phi_\sigma(D_1) - 1 < \phi_\sigma(D_2) < \phi_\sigma(D_1)$. Indeed, if $\sigma = \sigma_1$, then this follows from a direct computation of the slope function of σ_1. When $\sigma \neq \sigma_1$, by Lemma A.8, up to $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$-action, we can assume that σ and σ_1 have the same central charge and $\phi_\sigma(D_1) = \phi_{\sigma_1}(D_1)$. Thus, $\phi_\sigma(D_2) - \phi_{\sigma_1}(D_2) \in 2\mathbb{Z}$. We claim that $\phi_\sigma(D_1) - 2 < \phi_\sigma(D_2) < \phi_\sigma(D_1)$, which implies $\phi_\sigma(D_2) = \phi_{\sigma_1}(D_2)$ and the result follows. Indeed, by Proposition 8.1, we have an exact sequence $0 \to F \to \mathcal{E} \to \mathcal{O}_L(-1) \to 0$ for a line $L \subset X_{10}$. Hence, applying $\mathrm{Hom}(-, D_1)$ to this exact sequence and use $\mathrm{Hom}(\mathcal{E}, D_1[-1]) \neq 0$ (Lemma 6.3) and adjunction of pr, we have $\mathrm{Hom}(F, I_C) = \mathrm{Hom}(D_2, D_1) = \mathrm{Hom}(D_1, S_{\mathcal{A}_{X_{10}}}(D_2)) \neq 0$. Then, by Lemma A.4, we obtain $\phi_\sigma(D_1) - 2 < \phi_\sigma(D_2) < \phi_\sigma(D_1)$ as desired.

The final statement follows from the fact that (D) and (E) in this case only depend on the central charge and numerical classes $[D_1]$ and $[D_2]$, as we have seen above. □

Applying Theorem A.1, we obtain the uniqueness of Serre-invariant stability conditions.

Theorem A.10. *Let $\mathcal{D} = \mathcal{K}u(Y_d)$ for $2 \leqslant d \leqslant 3$ or $\mathcal{K}u(X_{10})$. Then all Serre-invariant stability conditions on $\mathcal{D}$ are in the same $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$-orbit.*

Proof. Let σ_1, D_1 and D_2 as in Lemma A.9. Let σ_2 be another Serre-invariant stability condition on $\mathcal{D}$. By Lemma A.8, up to $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$-action, we can assume that σ_1 and σ_2 have the same central charge. Moreover, up to shift we can assume that $\phi_{\sigma_1}(D_1) = \phi_{\sigma_2}(D_1)$. Thus, $\phi_{\sigma_1}(D_2) - \phi_{\sigma_2}(D_2) \in 2\mathbb{Z}$. And by Lemma A.9, σ_2 also satisfies (D) and (E).

Now from Lemma A.9, we have $\phi_{\sigma_k}(D_1) - 1 < \phi_{\sigma_k}(D_2) < \phi_{\sigma_k}(D_1)$ or $\phi_{\sigma_k}(D_1) < \phi_{\sigma_k}(D_2) < \phi_{\sigma_k}(D_1) + 1$ for any $k \in \{1, 2\}$. This implies $\phi_{\sigma_1}(D_2) = \phi_{\sigma_2}(D_2)$. Therefore, by Lemmas A.4 and A.9, we can apply Theorem A.1 and get $\sigma_1 = \sigma_2$. □

Remark A.11. The idea of the proof of Theorem A.10 was first explained to us by Arend Bayer. In [54, Proposition 4.21], one of the authors made an attempt to prove this statement but the argument is incomplete. Here, we fill the gaps and give a more general argument. Later, in [17, Theorem 3.1], the authors also prove the uniqueness of Serre-invariant stability conditions for a general triangulated category satisfying a list of assumptions and include Kuznetsov components of cubic threefolds and very general cubic fourfolds. The assumptions used in [17, Theorem 3.1] are (A) and (B), and the Serre functor of $\mathcal{D}$ satisfies $S_{\mathcal{D}}^r = [k]$ with $0 < k/r < 2$ or $r = 2$ and $k = 4$, while our Theorem A.1 also works for general triangulated categories that are not fractional Calabi–Yau but with extra assumptions (C), (D), and (E). Indeed, if we take $D_1 = D$ and $D_2 = S_{\mathcal{D}}(D)[-2]$ in (D) and (E) where D is an object in (B), then one can show that when $k/r < 2$, Theorem A.1 implies [17, Theorem 3.1]. Moreover, Theorem A.1 can be applied to the derived category of a smooth projective curve or a generalized Kronecker quiver as well.

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