A CHROMATIC VANISHING RESULT FOR TR

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Abstract. In this note, we establish a vanishing result for telescopically localized TR. More precisely, we prove that $T(k)$-local TR vanishes on connective $L_n^{p,f}$-acyclic $\mathbb{E}_1$-rings for every $1 \le k \le n$ and deduce consequences for connective Morava K-theory and the Thom spectra $y(n)$. The proof relies on the relationship between TR and the spectrum of curves on K-theory together with fact that algebraic K-theory preserves infinite products of additive ∞-categories which was recently established by Córdova Fedeli.

1. Introduction

In this note, we study the telescopic localizations of TR inspired by the work of Land–Mathew–Meier–Tamme [24] and Mathew [28]. Our starting point is the following result which follows from the main result of [24]: If R is an $\mathbb{E}_1$-ring with $L_n^{p,f} R \simeq 0$, then

$$L_{T(k)} \mathrm{K}(R) \simeq 0$$

for every $1 \le k \le n$. For instance, if $R = \mathbb{Z}/p^n$ for some integer $n \ge 1$, then $L_{T(1)} \mathrm{K}(\mathbb{Z}/p^n) \simeq 0$. We consider this result as an extension of Quillen's fundamental calculation that $\mathrm{K}(\mathbb{F}_p)_p^\wedge \simeq H\mathbb{Z}_p$ which in particular yields that $L_{T(1)} \mathrm{K}(\mathbb{F}_p) \simeq 0$. This particular consequence was also obtained by Bhatt–Clausen–Mathew [4] by means of a calculation in prismatic cohomology. Additionally, the vanishing result above for $T(k)$-local K-theory can be applied to the Morava K-theories $K(n)$ and to the Thom spectra $y(n)$ considered by Mahowald–Ravenel–Shick in [27].

1.1. **Results.** We will be interested in similar vanishing results for $T(k)$-local TR[1]. The invariant TR plays an instrumental role in the classical construction of topological cyclic homology in [7, 19, 6], where TC is obtained as the fixedpoints of a Frobenius operator on TR. In §3, we briefly review the construction of TR following [29] which produces TR together with its Frobenius operator entirely in the Borel–equivariant formalism of Nikolaus–Scholze [30]. Even though TR does not feature prominently in the construction of TC given in [30], TR remains an important invariant by virtue of its close relationship to the Witt vectors and the de Rham–Witt complex [18, 19, 20, 21]. In [28], Mathew proves that $T(1)$-local TR is truncating on connective $H\mathbb{Z}$-algebras which means that if R is a connective $H\mathbb{Z}$-algebra, then the canonical map of spectra

$$L_{T(1)} \mathrm{TR}(R) \to L_{T(1)} \mathrm{TR}(\pi_0 R)$$

is an equivalence. This property was verified for $T(1)$-local K-theory and $T(1)$-local TC in [4, 24]. Our main result is a version of this at higher chromatic heights:

[1] Note that $L_{T(k)} \mathrm{TR}(R) \simeq L_{T(k)} \mathrm{TR}(R,p)$, where $\mathrm{TR}(R,p)$ denotes the p-typical version of TR. Indeed, the canonical map $\mathrm{TR}(R) \to \mathrm{TR}(R,p)$ is a p-adic equivalence and $T(n)$-localization is insensitive to p-completion. Therefore, we will not distinguish between p-typical TR and integral TR in this note.

Theorem A. *Let $n \geq 1$. If R is a connective $\mathbb{E}_1$-ring such that $L_n^{p,f} R \simeq 0$, then*

$$L_{T(k)} \operatorname{TR}(R) \simeq 0$$

for every $1 \leq k \leq n$.

We remark that Theorem A is a consequence of the work of [24] in the case where R admits a more refined multiplicative structure; If R admits an $\mathbb{E}_m$-ring structure for $m \geq 2$, then the refined cyclotomic trace $\mathrm{K}(R) \to \operatorname{TR}(R)$ is a map of $\mathbb{E}_1$-rings. Consequently, the spectrum $L_{T(k)} \operatorname{TR}(R)$ admits the structure of a $L_{T(k)} \mathrm{K}(R)$-module and $L_{T(k)} \mathrm{K}(R) \simeq 0$ by [24, Theorem 3.8]. A similar sort of reasoning has recently been employed with great success to study redshift phenomena for algebraic K-theory in [9, 11, 15, 31]. We deduce the following results from Theorem A:

Corollary B. *Let $n \geq 1$. Then $L_{T(k)} \operatorname{TR}(\mathbb{Z}/p^n) \simeq 0$ for every $k \geq 1$.*

We stress that Corollary B is a consequence of the work of [4, 24] by the reasoning above. For $n = 1$, Corollary B can also be deduced from the work of Mathew [28]. Since $T(1)$-local TR is truncating on connective $H\mathbb{Z}$-algebra it is in particular nilinvariant by [25], so

$$L_{T(1)} \operatorname{TR}(\mathbb{Z}/p^n) \simeq L_{T(1)} \operatorname{TR}(\mathbb{F}_p) \simeq 0,$$

where the final equivalence follows since $\operatorname{TR}(\mathbb{F}_p, p) \simeq H\mathbb{Z}_p$ by Hesselholt–Madsen [19]. As a consquence of Theorem A we deduce a new chromatic vanishing result for the connective Morava K-theories, which we denote by $k(n)$. While $k(n)$ admits the structure of an $\mathbb{E}_1$-ring, it does not admit the structure of an $\mathbb{E}_2$-ring so we cannot argue using the refined cyclotomic trace above.

Corollary C. *Let $n \geq 2$. Then $L_{T(k)} \operatorname{TR}(k(n)) \simeq 0$ for every $1 \leq k \leq n-1$.*

Similarly, we obtain a chromatic vanishing result for the Thom spectra $y(n)$ considered in [27].

1.2. **Methods.** We end by explaining the strategy of our proof of Theorem A. They key input is the close relationship between TR and the spectrum of curves on K-theory as studied in [3, 5, 18, 29]. For every $\mathbb{E}_1$-ring R, the spectrum of curves on K-theory is defined by

$$\mathrm{C}(R) = \varprojlim_i \Omega \tilde{\mathrm{K}}(R[t]/t^i),$$

where $\tilde{\mathrm{K}}(R[t]/t^i)$ denotes the fiber of the map $\mathrm{K}(R[t]/t^i) \to \mathrm{K}(R)$ induced by the augmentation. If we assume that R is connective, then $\operatorname{TR}(R) \simeq \mathrm{C}(R)$ by [29, Corollary 4.2.5]. This result was preceded by Hesselholt [18] and Betley–Schlichtkrull [3] who established the result for associative rings after profinite completion. Combining the theorem of the weighted heart (cf. [13, 17, 16]) with the recent result of Córdova Fedeli [12, Corollary 2.11.1] which asserts that algebraic K-theory preserves arbitrary products of additive ∞-categories, we reduce to proving that

$$L_{T(k)} \mathrm{K}^{\oplus}\Big(\prod_{i \geq 1} \operatorname{Proj}^{\omega}_{R[t]/t^i}\Big) \simeq 0$$

provided that $L_n^{p,f} R \simeq 0$, where $\operatorname{Proj}^{\omega}_{R[t]/t^i}$ denotes the additive ∞-category of finitely generated projective $R[t]/t^i$-modules and $\mathrm{K}^{\oplus}$ denotes additive algebraic K-theory. This claim can be verified explicitly by using [24, Proposition 3.6].

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2. Preliminaries on weight structures and K-theory

The main technical apparatus for deducing our chromatic vanishing result for TR is the notion of a weight structure on a stable ∞-category in conjunction with the closely related theorem of the weighted heart (cf. [13, 16]). This will help us reduce to studying additive algebraic K-theory of additive ∞-categories.

Definition 2.1. A weight structure on a stable ∞-category $\mathcal{C}$ consists of a pair of full subcategories $\mathcal{C}_{[0,\infty]}$ and $\mathcal{C}_{[-\infty,0]}$ of $\mathcal{C}$ such that the following conditions are satisfied:

(1) The full subcategories $\mathcal{C}_{[0,\infty]}$ and $\mathcal{C}_{[-\infty,0]}$ are closed under retracts in $\mathcal{C}$.

(2) For $X \in \mathcal{C}_{[-\infty,0]}$ and $Y \in \mathcal{C}_{[0,\infty]}$, the mapping spectrum $\mathrm{map}_{\mathcal{C}}(X,Y)$ is connective.

(3) For every $X \in \mathcal{C}$, there is a fiber sequence

$$X' \to X \to X''$$

with $X' \in \mathcal{C}_{[-\infty,0]}$ and $X''[-1] \in \mathcal{C}_{[0,\infty]}$.

The heart of the weight structure is the subcategory $\mathcal{C}^{\mathrm{ht}} = \mathcal{C}_{[0,0]}$, where $\mathcal{C}_{[a,b]} = \mathcal{C}_{[a,\infty]} \cap \mathcal{C}_{[-\infty,b]}$. The weight structure is said to be exhaustive if every object is bounded, in the sense that

$$\mathcal{C} = \bigcup_{n\in\mathbb{Z}} \mathcal{C}_{[-n,n]}.$$

A weighted ∞-category is a stable ∞-category equipped with a weight structure.

Remark 2.2. The heart of a weighted ∞-category is an additive ∞-category ([16, Lemma 3.1.2]).

We recall the following terminology which will play an important role throughout this note. For every connective $\mathbb{E}_1$-ring R, let Proj_R^{ω} denote the full subcategory of the ∞-category $\mathrm{LMod}_R^{\geq 0}$ spanned by those connective left R-modules which are finitely generated and projective. Recall that an object of Proj_R^{ω} can be written as a retract of a finitely generated free R-module (cf. [26, Proposition 7.2.2.7]). For any not necessarily connective $\mathbb{E}_1$-ring, let Perf_R denote the ∞-category of perfect R-modules defined as the smallest stable subcategory of LMod_R which contains R and is closed under retracts. The following is our main example of interest:

Example 2.3. For a connective $\mathbb{E}_1$-ring R, let $\mathrm{Perf}_{R,\geq 0}$ be the full subcategory of Perf_R spanned by those perfect R-modules which are connective, and let $\mathrm{Perf}_{R,\leq 0}$ denote the full subcategory of Perf_R spanned by those perfect R-modules M which have projective amplitude ≤ 0. This means that every R-linear map $M \to N$ is nullhomotopic provided that N is 1-connective. The pair $(\mathrm{Perf}_{R,\geq 0}, \mathrm{Perf}_{R,\leq 0})$ defines an exhaustive weight structure on Perf_R whose heart is equivalent to the additive ∞-category Proj_R^{ω} of finitely generated projective R-modules (cf. [17, 1.38 & 1.39]);

while the proofs therein are stated for connective $\mathbb{E}_\infty$-rings, the same arguments work in the $\mathbb{E}_1$ case.

The algebraic K-theory of a weighted ∞-category is often determined by the additive algebraic K-theory of its heart by virtue of the theorem of the weighted heart first established by Fontes [13] but we also refer the reader to [16, Corollary 8.1.3, Remark 8.1.4]. Let $\mathcal{A}$ denote an additive ∞-category regarded as a symmetric monoidal ∞-category with the cocartesian symmetric monoidal structure, so that the core $\mathcal{A}^{\simeq}$ inherits the structure of an $\mathbb{E}_\infty$-monoid. Recall that the additive algebraic K-theory of $\mathcal{A}$ is defined by

$$\mathrm{K}^{\oplus}(\mathcal{A}) = (\mathcal{A}^{\simeq})^{\mathrm{grp}},$$

where $(\mathcal{A}^{\simeq})^{\mathrm{grp}}$ denotes the group completion of the $\mathbb{E}_\infty$-monoid $\mathcal{A}^{\simeq}$. We have the following result which will play an instrumental role below (cf. [13, Theorem 5.1] and [16, Corollary 8.1.3]):

Theorem 2.4. *The canonical map of spectra*

$$\mathrm{K}^{\oplus}(\mathcal{C}^{\mathrm{ht}}) \to \mathrm{K}(\mathcal{C})$$

is an equivalence for every stable ∞-category $\mathcal{C}$ equipped with an exhaustive weight structure.

3. Chromatic vanishing results

The main goal of this section is to prove Theorem A from §1 and discuss various consequences. As explained, our proof of this result relies on the close relationship between TR and the spectrum of curves in K-theory (cf. [3, 18, 29]). We will regard TR as a functor $\mathrm{TR} : \mathrm{Alg}^{\mathrm{cn}}_{\mathbb{E}_1} \to \mathrm{Sp}$ given by

$$\mathrm{TR}(R) \simeq \mathrm{map}_{\mathrm{CycSp}}(\widetilde{\mathrm{THH}}(\mathbf{S}[t]), \mathrm{THH}(R))$$

following [29] and this agrees with the classical construction of TR by [29, Theorem 3.3.12]. By virtue of our assumption that R is connective, there is an equivalence of spectra

$$\mathrm{TR}(R) \simeq \varprojlim \Omega \tilde{\mathrm{K}}(R[t]/t^i),$$

where $\tilde{\mathrm{K}}(R[t]/t^i)$ denotes the fiber of the map $\mathrm{K}(R[t]/t^i) \to \mathrm{K}(R)$ induced by the augmentation. In this generality, the result was obtained by the second author in [29] preceded by Hesselholt [18] and Betley–Schlichtkrull [3] who proved the result for associative rings after profinite completion. With this equivalence at our disposal, we prove the following result:

Theorem 3.1. *Let $n \geq 1$. If R is a connective $\mathbb{E}_1$-ring such that $L_n^{p,f} R \simeq 0$, then*

$$L_{T(k)} \mathrm{TR}(R) \simeq 0$$

for every $1 \leq k \leq n$.

The limit in the definition of the spectrum of curves on K-theory above does not commute with $T(k)$-localization. Instead, the proof of Theorem 3.1 relies on the following result, which is proved by combining the theorem of the weighted heart and a recent result which asserts that additive algebraic K-theory preserves infinite products of additive ∞-categories, due to Córdova Fedeli [12].

Proposition 3.2. *Let R be a connective $\mathbb{E}_1$-ring which vanishes after $L_n^{p,f}$-localization. If $\{S_i\}_{i\in I}$ is collection of connective $\mathbb{E}_1$-rings with a map of $\mathbb{E}_1$-rings $R \to S_i$ for every $i \in I$, then*

$$L_{T(k)}\big(\prod_{i\in I} \mathrm{K}(S_i)\big) \simeq 0$$

for every $1 \le k \le n$.

Proof. For $i \in I$, the stable ∞-category Perf_{S_i} admits an exhaustive weight structure whose heart is equivalent to the additive ∞-category $\mathrm{Proj}^{\omega}_{S_i}$ by Example 2.3. The canonical composite

$$\mathrm{K}^{\oplus}\Big(\prod_{i\in I} \mathrm{Proj}^{\omega}_{S_i}\Big) \to \prod_{i\in I} \mathrm{K}^{\oplus}(\mathrm{Proj}^{\omega}_{S_i}) \to \prod_{i\in I} \mathrm{K}(\mathrm{Perf}_{S_i})$$

is an equivalence by [12, Corollary 2.11.1] and Theorem 2.4, so we have reduced to proving that

$$L_{T(k)}\,\mathrm{K}^{\oplus}\Big(\prod_{i\in I} \mathrm{Proj}^{\omega}_{S_i}\Big) \simeq 0$$

for $1 \le k \le n$. By [24, Proposition 3.6], it suffices to prove that the endomorphism $\mathbb{E}_1$-rings of

$$\mathcal{A} = \prod_{i\in I} \mathrm{Proj}^{\omega}_{S_i}$$

vanish after $L_n^{p,f}$-localization. If $P \in \mathcal{A}$, then the endomorphism $\mathbb{E}_1$-ring of P is given by

$$\mathrm{End}_{\mathcal{A}}(P) \simeq \prod_{i\in I} \mathrm{map}_{S_i}(P_i, P_i),$$

where $\mathrm{map}_{S_i}(P_i, P_i)$ denotes the mapping spectrum in LMod_{S_i}. For each $i \in I$, we may choose a positive integer $n_i \ge 1$ such that P_i is a retract of $S_i^{\oplus n_i}$ by virtue of our assumption that P_i is a finitely generated projective S_i-module. Consequently, we obtain a retract diagram of spectra

$$\mathrm{End}_{\mathcal{A}}(P) \to \prod_{i\in I} S_i^{\oplus n_i^2} \to \mathrm{End}_{\mathcal{A}}(P)$$

which proves the desired statement since the middle term is a left R-module, hence vanishes after $L_n^{p,f}$-localization by virtue of our assumption that R is $L_n^{p,f}$-acyclic. □

Remark 3.3. In general, E-acyclic spectra are not closed under infinite products; for each $n \ge 0$, the nth Postnikov truncation $\tau_{\le n}\mathbb{S}$ is $K(1)$-acyclic, whereas $\prod_{n\ge 0} \tau_{\le n}\mathbb{S}$ is not, else $L_{K(1)}\mathbb{S} \simeq 0$. The assumptions of Proposition 3.2 should be viewed as a uniformity condition on the spectra $\mathrm{K}(S_i)$, forcing their product to become acyclic.

Proof of Theorem 3.1. Since R is a connective $\mathbb{E}_1$-ring, there is an equivalence of spectra $\mathrm{TR}(R) \simeq \mathrm{C}(R)$ by [29, Corollary 4.2.5]. Thus, the spectrum $\Sigma\,\mathrm{TR}(R)$ is the fiber of a suitable map

$$\prod_{i\ge 1} \widetilde{\mathrm{K}}(R[t]/t^i) \to \prod_{i\ge 1} \widetilde{\mathrm{K}}(R[t]/t^i)$$

which proves the desired statement as these products vanish after $T(k)$-localization for $1 \le k \le n$ by virtue of Theorem 3.2. □

Remark 3.4. As remarked above, we have used work by Córdova Fedeli [12] in a crucial way. This result on K-theory of additive ∞-categories is part of a long tradition of examining the interaction of algebraic K-theory and infinite products of categories. One of the first results of this kind is due to Carlsson, who showed that K-theory preserves infinite products of exact 1-categories with a cylinder functor [10]. In [23], Kasprowski–Winges proved that K-theory preserves infinite products of additive categories. Furthermore, Kasprowski–Winges [22] used a characterization of Grayson [14] to prove that non-connective algebraic K-theory preserves infinite products of stable ∞-categories and this was used in [8] with Bunke to prove the analogous statement of prestable ∞-categories.

Remark 3.5. Another attempt to prove Proposition 3.2 proceeds by invoking the recent result of Kasprowski–Winges [22], which asserts that the canonical map of spectra

$$\mathrm{K}\Big(\prod_{i\in I}\mathrm{Perf}(S_i)\Big)\to\prod_{i\in I}\mathrm{K}(S_i)$$

is an equivalence (cf. Remark 3.5). Proceeding as in the proof of Proposition 3.2, it suffices to prove that the endomorphism $\mathbb{E}_1$-rings of the product of the stable ∞-categories $\mathrm{Perf}(S_i)$ vanish after $L_n^{p,f}$-localization. This is closely related to the following assertion:

($\ast$) Let E denote the endomorphism $\mathbb{E}_1$-ring of a finite spectrum V of type n. If $v:\Sigma^k E\to E$ is the associated v_n self-map of E, then there is a canonical lift of v to a map of E-E-bimodules.

By the description of the $\mathbb{E}_1$-center as Hochschild cohomology, the statement ($\ast$) is equivalent to asking for a lift of the class $v\in\pi_\ast(E)$ to a class $\tilde{v}\in\pi_\ast\mathcal{Z}_{\mathbb{E}_1}(E)$ along the $\mathbb{E}_1$-map $\mathcal{Z}_{\mathbb{E}_1}(E)\to E$. Classes which do lift in this way can be viewed as "homotopically central" elements of E, and we remark that such lifts exist for all $\mathbb{E}_2$-rings, by the universal property of the $\mathbb{E}_1$-center.

However, the assertion ($\ast$) is false as we learned from Maxime Ramzi, and we thank him for help with the following argument. If such a lift exists, then we obtain an equivalence of $L_{K(n)}$-$L_{K(n)}$-bimodules

$$\varphi:\Sigma^k L_{K(n)}E\to L_{K(n)}E,$$

and there is an equivalence of $\mathbb{E}_1$-rings $\mathrm{End}_{K(n)}(L_{K(n)}V)\simeq L_{K(n)}E$ since V is a finite spectrum. The ∞-category of $K(n)$-local spectra is equivalent to the ∞-category $\mathrm{Mod}_{L_{K(n)}E}(\mathrm{Sp}_{K(n)})$ since $L_{K(n)}V$ is a compact generator of $\mathrm{Sp}_{K(n)}$. As a consequence, for every $K(n)$-local spectrum X, we obtain an equivalence $\Sigma^k X\to X$ by base-changing along φ. This is a contradiction since the homotopy groups of a $K(n)$-local spectrum are not periodic. We indicate an example of this at every height $n\geq 1$. Let k be a perfect field of characteristic p, let $\mathbb{G}$ be a 1-dimensional formal group of height n, and let E_n denote the associated Lubin–Tate theory which canonically carries the structure of an $\mathbb{E}_\infty$-ring. For every topological generator g of $\mathbb{Z}_p^\times$, there is a map of $\mathbb{E}_\infty$-rings $\psi_g:E_n\to E_n$, and we let F_n denote the fiber of the map

$$E_n\xrightarrow{1-\psi_g}E_n.$$

A calculation reveals that the homotopy groups of F_n are not periodic. For instance, if $n=1$, then $F_1\simeq L_{K(1)}\mathbb{S}$ since the map ψ_g is induced by Adams operations on $E_1\simeq\mathrm{KU}_p^\wedge$.

Finally, we explore some immediate consequences of Theorem 3.1.

Corollary 3.6. *Let R be a connective $\mathbb{E}_1$-algebra over $\mathbb{Z}/p^j$. If $n\geq 1$, then $L_{T(n)}\,\mathrm{TR}(R)\simeq 0$.*

Proof. Note that $L_n^{p,f}R$ is a module over $L_n^{p,f}\mathbb{Z}/p^j\simeq 0$, so the assertion follows from Theorem 3.1. □

Recall that Corollary 3.6 above also follows from [4, 24, 28] as discussed in the introduction. We deduce some consequence for connective Morava K-theory. Let $k(n)$ denote the connective cover of the nth Morava K-theory $K(n)$. The spectrum $k(n)$ carries the structure of an $\mathbb{E}_1$-ring but not the structure of an $\mathbb{E}_2$-ring. We have the following:

Corollary 3.7. *If $n \geq 2$, then $L_{T(k)}\,\mathrm{TR}(k(n)) \simeq 0$ for every $1 \leq k \leq n-1$.*

Proof. For $n \geq 2$, the canonical map $k(n) \to \mathbb{F}_p$ is a $L^{p,f}_{n-1}$-local equivalence by [24, Lemma 2.2], so the assertion follows from Theorem 3.1. □

Remark 3.8. There is a fiber sequence of spectra

$$\mathrm{K}(\mathbb{F}_p) \to \mathrm{K}(k(n)) \to \mathrm{K}(K(n)),$$

by [1, Proposition 4.4] preceded by [2]. We consider this as an analogue of Quillen's dévissage theorem for algebraic K-theory of ring spectra. One might ask whether we can establish a similar fiber sequence for TR. In particular, this would allow us to deduce an analogue of Corollary 3.7 for the non-connective Morava K-theory.

Let $y(n)$ denote the Thom spectrum considered in [27, Section 3]. This is the Thom spectrum associated to the map of $\mathbb{E}_1$-spaces

$$\Omega J_{p^{n-1}} S^2 \hookrightarrow \Omega^2 S^3 \to \mathrm{BGL}_1(\mathbb{S}_p^\wedge)$$

where $J_{p^{n-1}}S^2$ is the $2(p^{n-1})$-skeleton of ΩS^3, which has a single cell in each even dimension. The map $\Omega^2 S^3 \to \mathrm{BGL}_1(\mathbb{S}_p^\wedge)$ is the spherical fibration constructed by Mahowald (for $p = 2$) and Hopkins (for p odd) whose Thom spectrum is $H\mathbb{F}_p$. We have the following:

Corollary 3.9. *If $n \geq 2$, then $L_{T(k)}\,\mathrm{TR}(y(n)) \simeq 0$ for every $1 \leq k \leq n-1$.*

Proof. This follows immediately by combining Theorem 3.1 with [24, Lemma 4.14]. □

Remark 3.10. If R is a connective $H\mathbb{Z}$-algebra, then the canonical map

$$L_{T(1)}\,\mathrm{K}(R) \to L_{T(1)}\,\mathrm{K}(R[1/p])$$

is an equivalence by [4, 24]. The analogue of this result does not hold for TC as explained in [24, Remark 4.27], which in particular means that the result also does not prolong to TR. However, at chromatic heights $n \geq 2$, TC does satisfy a version of chromatic purity (cf. [24, Corollary 4.5]). In particular, if $A \to B$ is an $L^{p,f}_n$-local equivalence of $\mathbb{E}_1$-rings, then the induced map

$$L_{T(n)}\,\mathrm{TC}(\tau_{\geq 0}A) \xrightarrow{\simeq} L_{T(n)}\,\mathrm{TC}(\tau_{\geq 0}B).$$

is an equivalence. One can wonder whether such a statement is true of $T(n)$-local TR, but our methods here do not seem to shed light on this problem.

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