Fixed-Parameter Debordering of Waring Rank

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Abstract

Border complexity measures are defined via limits (or topological closures), so that any function which can approximated arbitrarily closely by low complexity functions itself has low border complexity. Debordering is the task of proving an upper bound on some non-border complexity measure in terms of a border complexity measure, thus getting rid of limits.

Debordering is at the heart of understanding the difference between Valiant's determinant vs permanent conjecture, and Mulmuley and Sohoni's variation which uses border determinantal complexity. The debordering of matrix multiplication tensors by Bini played a pivotal role in the development of efficient matrix multiplication algorithms. Consequently, debordering finds applications in both establishing computational complexity lower bounds and facilitating algorithm design. Currently, very few debordering results are known.

In this work, we study the question of debordering the border Waring rank of polynomials. Waring and border Waring rank are very well studied measures in the context of invariant theory, algebraic geometry, and matrix multiplication algorithms. For the first time, we obtain a Waring rank upper bound that is exponential in the border Waring rank and only *linear* in the degree. All previous known results were exponential in the degree. For polynomials with constant border Waring rank, our results imply an upper bound on the Waring rank linear in degree, which previously was only known for polynomials with border Waring rank at most 5.

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1 Introduction

Given a homogeneous polynomial f of degree d over $\mathbb{C}$, its *Waring rank* $\mathsf{WR}(f)$ is defined as the smallest number r such that there exist homogeneous linear forms $\ell_1, \ldots, \ell_r$ with

$$f = \sum_{i=1}^{r} \ell_i^d.$$

Equivalently, $\mathsf{WR}(f)$ is the minimal top fanin of a homogeneous $\Sigma \wedge \Sigma$ circuit computing f. In the case of quadratic forms (polynomials of degree 2), this notion is equivalent to the rank of the symmetric matrix associated with a quadratic form; hence Waring rank can be regarded as a generalization of the rank of a symmetric matrix. Unlike the case of matrices, when $d \geq 3$, Waring rank is in general not lower semicontinuous [1], that is, a limit of a family of polynomials with low Waring rank can have higher Waring rank. The simplest example is given by the polynomial $x^{d-1}y$, which has Waring rank d (this is a classical result [47]), but can be presented as a limit

$$x^{d-1}y = \lim_{\epsilon \to 0} \frac{1}{d\epsilon} \left[(x + \epsilon y)^d - x^d \right]$$

of a family of Waring rank 2 polynomials (note that we work over $\mathbb{C}$, so this expression can be rearranged into a sum of two powers by moving constants inside the parentheses). The *border Waring rank* is a semicontinuous variation of Waring rank defined as follows: the border Waring rank of f, denoted $\underline{\mathsf{WR}}(f)$, is the smallest r such that f can be written as a limit of a sequence of polynomials of Waring rank at most r. We have $\underline{\mathsf{WR}}(x^{d-1}y) = 2$.

Waring rank was studied already in the eighteenth century [23, 50, 25] in the context of invariant theory, with the aim to determine normal forms for homogeneous polynomials. We mention the famous Sylvester Pentahedral Theorem, stating that a generic cubic form in four variables can be written uniquely as a sum of five cubes. At the beginning of the twentieth century, the early work on secant varieties in classical algebraic geometry [48, 51] implicitly commenced the study of border Waring rank. The notion of border rank for tensors was introduced in [11] to construct faster-than-Strassen matrix multiplication algorithms. In [10], Bini proved that tensor border rank and tensor rank define the same matrix multiplication exponent. Today this theory is deeply related to the study of Gorenstein algebras [35, 15], the Hilbert scheme of points [38], and deformation theory [18, 39].

In the context of algebraic complexity theory, Waring rank defines a model of computation known as the homogeneous diagonal depth 3 circuits or homogeneous $\Sigma \wedge \Sigma$ circuits, see e.g. [49]. This is a very weak computational model (determinants have provably exponential Waring rank [33]). Nevertheless, it is important as one of the simplest nontrivial computational models and has many unresolved open problems associated with it. The Waring rank of a generic homogeneous polynomial of degree $d \geq 3$ in n variables is $\lceil \frac{1}{n} \cdot \binom{n+d-1}{d} \rceil = \Omega(\frac{n^{d-1}}{d!})$ with finite number of exceptional values of (n, d) [2], but the best lower bounds obtained are of order $2n^{\lfloor d/2 \rfloor}$ (from tensor rank lower bounds in [3]). For a large class of lower bound methods, so called rank methods, there are barrier results showing that cannot give bounds significantly larger than $n^{\lfloor d/2 \rfloor}$ [28, 31, 30]. Waring rank can be useful when the degree of the polynomials considered is constant. For example, the results of [24] guarantee that the

[1] A function f is lower semicontinuous at a if $\liminf_{x \to a} f(x) \geq f(a)$.

matrix multiplication exponent is controlled by the Waring rank or border Waring rank of the polynomial $\mathrm{Tr}(X^3)$ with $X \in \mathbb{C}^{n\times n}$, which is a symmetrized version of the matrix multiplication tensor.

The lack of semicontinuity is a common phenomenon in algebraic complexity not specific to Waring rank. Most complexity measures defined in terms of discrete structures (such as circuits or formulas) or in terms of decompositions (such as Waring rank or tensor rank) are not lower semicontinuous. To any algebraic complexity measure one can define the corresponding *border complexity measure* in the same way as border Waring rank arises from Waring rank: the border complexity of f is the smallest number s such that f can be approximated arbitrarily closely by polynomials of complexity at most s. Border tensor rank appears in the study of the computational complexity of matrix multiplication [11, 10], border complexity for algebraic circuits was first discussed in [45] and [20].

Replacing a complexity measure by its border measure in a complexity class, we obtain the *closure* of this class. For example, $\overline{\mathsf{VP}}$ is the class of all polynomial sequences with polynomially bounded degree and border circuit size, and $\overline{\mathsf{VF}}$ is defined analogously using formula size. Formally, the closure $\overline{\mathcal{C}}$ of a complexity class $\mathcal{C}$ consists of all polynomial sequences $(f_n)_{n\in\mathbb{N}}$ such that there exists a bivariate sequence $(g_{n,m})_{n,m\in\mathbb{N}}$ with the property that $(g_{n,m})_{n\in\mathbb{N}}$ lies in $\mathcal{C}$ for every fixed m, and $f_n = \lim_{m\to\infty} g_{n,m}$. The operation of going to the closure is indeed a closure operator in the sense of topology, see [36].

The relationship between border and non-border complexity is far from straightforward. In some contexts taking a limit can be a very strong operation, which sometimes turns non-universal computational models into universal ones. For example, there are polynomials which cannot be computed by width 2 algebraic branching programs [4], but the corresponding border measure is related to border formula size [14], so every polynomial is a limit of width 2 ABPs. Kumar [42] gives an even easier example: every polynomial can be presented as a limit of a sum of 2 products of affine linear forms. On the other hand, there are examples of complexity measures which are lower semicontinuous, so that there is no difference between border and non-border complexity measures. A simplest example is the number of monomials in a polynomial (equivalently, top fanin of a $\Sigma\Pi$ circuit). Other examples are noncommutative ABP width (implicit in [46]) and read-once ABP width [26].

Semicontinuous complexity measures and closed complexity classes are easier to work with using geometric methods. Because of this, the geometric complexity theory program [45] proposes to study conjectures $\mathsf{VNP} \not\subseteq \overline{\mathsf{VBP}}$ and $\mathsf{VNP} \not\subseteq \overline{\mathsf{VP}}$ instead of Valiant's conjectures $\mathsf{VNP} \neq \mathsf{VBP}$ and $\mathsf{VNP} \neq \mathsf{VP}$. The $\mathsf{VNP} \not\subseteq \overline{\mathsf{VP}}$ conjecture was also proposed in [20]. These border variants of Valiant's conjecture are now usually referred to as the Mulmuley–Sohoni conjectures. Mulmuley–Sohoni conjectures are stronger that Valiant's conjectures, but it is not clear how much stronger, as most questions about the relations between complexity classes and their closures are wide open. It is unknown even whether or not $\overline{\mathsf{VF}} \subseteq \mathsf{VNP}$. Theorems of the form $\overline{\mathcal{C}} \subseteq \mathcal{D}$ for algebraic complexity classes $\mathcal{C}$ and $\mathcal{D}$ are called *debordering* results. These kind of results can also be proven directly on the complexity measures, by giving an upper bound on a non-border complexity in terms of border complexity. For example, $\mathsf{abpw}(f) \leq \underline{\mathrm{WR}}(f)$, where $\mathsf{abpw}(f)$ is the algebraic branching program width of f. This is proven using semicontinuity of noncommutative ABP width, see [12, Thm 4.2] and [29]. In terms of complexity classes, this means $\overline{\mathsf{VWaring}} \subseteq \mathsf{VBP}$, where $\mathsf{VWaring}$ is the class of p-families that have polynomially bounded Waring rank.

Forbes [52] conjectures that $\overline{\mathsf{VWaring}} = \mathsf{VWaring}$. Since this puts $\overline{\mathsf{VWaring}}$ in VF, a proof of this conjecture will also improve the results of Dutta, Dwivedi and Saxena [26] from $\overline{\Sigma^{[r]}\Pi\Sigma} \subset \mathsf{VBP}$ to $\overline{\Sigma^{[r]}\Pi\Sigma} \subset \mathsf{VF}$. Ballico and Bernardi [7] propose an even stronger conjecture stating that $\mathsf{WR}(f) \leq (\underline{\mathsf{WR}}(f) - 1) \cdot \deg f$. This was proven by case analysis for small values of border Waring rank: for $\underline{\mathsf{WR}}(f) \leq 3$ in [43], for $\underline{\mathsf{WR}}(f) = 4$ in [6], and for $\underline{\mathsf{WR}}(f) = 5$ and $\deg f \geq 9$ in [5].

Main result

We prove the following improved debordering theorem for border Waring rank.

▶ **Theorem 1** (Fixed-parameter debordering). *Let f be a homogeneous polynomial with $\deg f = d$ and $\underline{\mathsf{WR}}(f) = r$. Then $\mathsf{WR}(f) \leq 4^r \cdot d$.*

Note that the example of the polynomial $x^{d-1}y$ with $\underline{\mathsf{WR}}(x^{d-1}y) = 2$ and $\mathsf{WR}(x^{d-1}y) = d$ shows that any debordering bound must necessarily depend on both border Waring rank r and the degree d. We call our result a *fixed-parameter debordering* because the bound is polynomial (in this case even linear) in d, but exponential in the complexity parameter r. In the case of a fixed border Waring rank this gives a bound linear in the degree. This was previously known only for $\underline{\mathsf{WR}}(f) \leq 5$. Even for $r = O(\log d)$ we obtain an upper bound polynomial in d.

To the best of our knowledge, this is the first fixed-parameter debordering result. Previous methods applied to border Waring rank only allow upper bounds of the order d^r or r^d. To get $\mathsf{WR}(f) \leq O(d^r)$, note that a polynomial with border Waring rank r can be transformed into a polynomial in only r variables using a linear change of variables (see Lemma 4), and then take the maximal possible Waring rank of an r-variate polynomial of degree d as an upper bound. Alternatively, an upper bound $\mathsf{WR}(f) \leq 2^{d-1}r^d$ can be obtained by using the previously mentioned debordering into an ABP ($\mathsf{abpw}(f) \leq \underline{\mathsf{WR}}(f)$) and writing the ABP as a sum of at most r^d products, one for each path. Other known debordering techniques, such as the interpolation technique using the bound on the degree of ϵ in the approximation from the work of Lehmkuhl and Lickteig [44] (which is exponential in the degree of the polynomial), or the DiDIL technique from [26] can be applied in the border Waring rank setting, but do not improve over the simpler results discussed above.

Proof ideas

The main ideas for the proof come from *apolarity theory* and the study of 0-dimensional schemes in projective space (we discuss these ideas in Appendix A of the extended version of the paper [27]). We managed to simplify the proof so that it is elementary and *does not use* the language of algebraic geometry and is based on partial derivative techniques (see Section 2.3).

To prove the debordering, we transform a border Waring rank decomposition for f into a *generalized additive decomposition* [34, 8, 9] of the form $f = \sum_{k=1}^{m} \ell_k^{d-r_k+1} g_k$, where ℓ_k are linear forms, and g_k are homogeneous polynomials of degrees $r_k - 1$. We then obtain an upper bound on the Waring rank, by first decomposing each g_k with respect to a basis consisting of powers of linear forms, and then using the classical fact (see also [19]) that $\mathsf{WR}(\ell_1^a \ell_2^b) \leq \max(a+1, b+1)$.

To construct a generalized additive decomposition, we divide the summands of a border rank decomposition into several parts such that cancellations happen *only* between summands belonging to the same part; see Lemma 10. The key insight is that if the degree of polynomials involved is high enough, namely when $\deg f \geq \underline{\mathsf{WR}}(f) - 1$, then all parts of the decomposition are "local" in the sense that the lowest order term in each summand is a multiple of the same linear form. Each local part gives one term of the form $\ell^{d-r+1}g$, where r is the number of rank one summands in the part and ℓ is the common lowest order linear form; see Lemma 7.

For example, consider the family of polynomials $f_d = x_0^{d-1}y_0 + x_1^{d-1}y_1 + 2(x_0+x_1)^{d-1}y_2$, adapted from [16]. If $d > 3$, then the border Waring rank of f is at most 6, as evidenced by the decomposition

$$f_d = \lim_{\epsilon\to 0} \frac{1}{d\epsilon}\left[(x_0+\epsilon y_0)^d - x_0^d + (x_1+\epsilon y_1)^d - x_1^d + 2(x_0+x_1+\epsilon y_2)^d - 2(x_0+x_1)^d\right], \quad (1)$$

and a matching lower bound is obtained by considering the dimension of the space of second order partial derivatives. The summands of the decomposition (1) can be divided into three pairs. The lowest order term of the first pair is x_0^d, the one of the second pair is x_1^d and the one of the third pair is $(x_0+x_1)^d$. For each pair, the sum of the two powers individually converges to a limit as $\epsilon \to 0$; these three limits are, respectively, $x_0^{d-1}y_0$, $x_1^{d-1}y_1$, and $2(x_0+x_1)^{d-1}y_2$, which are the summands of a generalized additive decomposition for f_d.

When $d = 3$, the polynomial f_d is an example of a "wild form" [16]. It has border Waring rank 5 given for example by the decomposition

$$f_3 = \lim_{\epsilon\to 0} \frac{1}{9\epsilon}\big[3(x_0+\epsilon y_0)^3 + 3(x_1+\epsilon y_1)^3 + \\ 6(x_0+x_1+\epsilon y_2)^3 - (x_0+2x_1)^3 - (2x_0+x_3)^3\big]. \quad (2)$$

Unlike the previous decomposition, this one cannot be divided into parts that have limits individually, and is not local – all summands have different lowest order terms. This is only possible if the degree is low.

The condition on the degree is related to algebro-geometric questions about regularity of 0-dimensional schemes [35, Thm. 1.69], but for the schemes arising from border rank decompositions, this is ultimately a consequence of the fact that r distinct linear forms have linearly independent d-th powers when $d \geq r-1$.

2 Debordering border Waring rank

The goal of this section is to prove Theorem 1. Given a homogeneous degree d polynomial f, we provide upper bounds for $\mathsf{WR}(f)$ in terms of $\underline{\mathsf{WR}}(f)$ and d.

2.1 Definitions

In this section we introduce some notation and give a formal definition of Waring rank and border Waring rank. We work over the field $\mathbb{C}$ of complex numbers. The space of homogeneous polynomials of degree d in variables $\boldsymbol{x} = (x_1,\dots,x_n)$ is denoted by $\mathbb{C}[\boldsymbol{x}]_d$. We write $f \simeq g$ for $f, g \in \mathbb{C}(\epsilon)[\boldsymbol{x}]$ if $\lim_{\epsilon\to 0} f = \lim_{\epsilon\to 0} g$ (in particular, both limits must exist). Recall that the projective space $\mathbb{P}V$ is defined as the set of lines through the origin in V, that is, for each nonzero $v \in V$ we have a corresponding line $[v] \in \mathbb{P}V$, and $[v] = [w]$ if and only if $v = \alpha w$ for some α.

▶ **Definition 2.** *A* Waring rank decomposition *of a homogeneous polynomial $f \in \mathbb{C}[\boldsymbol{x}]_d$ is a decomposition of the form*

$$f = \sum_{k=1}^{r} \ell_k^d$$

for some linear forms $\ell_1,\dots,\ell_r \in \mathbb{C}[\boldsymbol{x}]_1$. The minimal number of summands in a Waring rank decomposition is called the Waring rank *of f and is denoted by* $\mathsf{WR}(f)$.

It is known that every homogeneous polynomial over $\mathbb{C}$ has finite Waring rank [47].

▶ **Definition 3.** *A* border Waring rank decomposition *of a homogeneous polynomial $f \in \mathbb{C}[\boldsymbol{x}]_d$ is an expression of the form*

$$f = \lim_{\epsilon \to 0} \sum_{k=1}^{r} \ell_k^d$$

where $\ell_1, \dots, \ell_r \in \mathbb{C}(\epsilon)[\boldsymbol{x}]_1$, that is, ℓ_i are linear forms in $\boldsymbol{x}$ with coefficients rationally dependent on ϵ. The border Waring rank $\underline{\mathsf{WR}}(f)$ *is the minimal number of summands in a border Waring rank decomposition.*

Equivalently, the border Waring rank of $f \in \mathbb{C}[\boldsymbol{x}]_d$ can be defined as the minimal number r such that f lies in the closure of the set $W_{d,r} = \{g \in \mathbb{C}[\boldsymbol{x}]_d \mid \mathsf{WR}(g) \le r\}$ of all polynomials with Waring rank at most r. The set $W_{d,r}$ is constructible, so its Zariski and Euclidean closures coincide, see e.g. [41, Anh.I.7.2 Folgerung]. The equivalence to the definition given above was established by Alder [1] (cited by [21, Ch.20]) for a similar notion of tensor rank, the proof remains essentially the same for Waring rank of polynomials.

2.2 Orbit closure and essential variables

The number of essential variables of a homogeneous polynomial f is the minimum integer m such that there is a linear change of coordinates after which f can be written as a polynomial in m variables. Denote the number of essential variables of f by $\mathrm{N_{ess}}(f)$. It is a classical fact, which already appears in [50], that the number of essential variables of f equals the dimension of the linear span of its first order partial derivatives, or equivalently the rank of the first partial derivative map. In particular $\mathrm{N_{ess}}(-)$ is a lower semicontinuous function. We refer to [22] and [40, Lemma B.1] for modern proofs of this result.

An immediate consequence of the semicontinuity of the number of essential variables is the following result.

▶ **Lemma 4.** *For a homogeneous polynomial $f \in \mathbb{C}[\boldsymbol{x}]_d$ we have $\mathrm{N_{ess}}(f) \le \underline{\mathsf{WR}}(f)$.*

Proof. We first prove $\mathrm{N_{ess}}(f) \le \mathsf{WR}(f)$. Let p be the dimension of the linear space spanned by the linear forms ℓ_k in the decomposition $f = \sum_{k=1}^{r} \ell_k^d$. Without loss of generality the linear forms $\ell_1, \dots, \ell_p$ are linearly independent, and $\ell_{p+1}, \dots, \ell_r$ are linear combinations of $\ell_1, \dots, \ell_p$. After applying a change of variables such that $y_k = \ell_k(\boldsymbol{x})$ for $k = 1, \dots, p$ we see that $\mathrm{N_{ess}}(f) \le p \le r$.

The inequality $\mathrm{N_{ess}}(f) \le \underline{\mathsf{WR}}(f)$ now follows from the semicontinuity of $\mathrm{N_{ess}}$: if

$$f = \lim_{\epsilon \to 0} \sum_{k=1}^{r} \ell_k^d \ ,$$

with $\ell_k \in \mathbb{C}(\epsilon)[\boldsymbol{x}]$, then $\mathrm{N_{ess}}(f) \le \lim_{\epsilon \to 0} \mathrm{N_{ess}}(\sum_{k=1}^{r} \ell_k^d(\epsilon)) \le r$. ◀

2.3 Fixed-parameter debordering

The proof of Theorem 1 is based on generalized additive decompositions of polynomials, in the sense of [34]. These decompositions were studied in algebraic geometry, usually in connection to 0-dimensional schemes and the notion of cactus rank. We defer the discussion of connections to algebraic geometry to the next section. Here we provide elementary proofs of some statements on generalized additive decompositions based on partial derivatives techniques, without using the language of 0-dimensional schemes. We bring from geometry a key insight: a border rank decomposition can be separated into *local* parts if the degree of the polynomial is large enough.

To define formally what it means for a border rank decomposition to be local, note that a rational family of linear forms $\ell \in \mathbb{C}(\epsilon)[\boldsymbol{x}]_1$ always has a limit when viewed projectively. Specifically, expanding $\ell(\epsilon)$ as a Laurent series $\ell(\epsilon) = \sum_{i=q}^{\infty} \epsilon^i \ell_i$ with $\ell_q \neq 0$, we have $\lim_{\epsilon\to 0}[\ell(\epsilon)] = \lim_{\epsilon\to 0}[\sum_{i=0}^{\infty} \epsilon^i \ell_{q+i}] = [\ell_q]$. A border Waring rank decomposition is called local if for all summands in the decomposition this limit is the same. More precisely, we give the following definition.

▶ **Definition 5.** *Let $f \in \mathbb{C}[\boldsymbol{x}]_d$ be a homogeneous polynomial. A border Waring rank decomposition*

$$f = \lim_{\epsilon\to 0} \sum_{k=1}^{r} \ell_k^d ,$$

with $\ell_k \in \mathbb{C}(\epsilon)[\boldsymbol{x}]_1$ is called a local border decomposition *if there exists a linear form $\ell \in \mathbb{C}[\boldsymbol{x}]_1$ such that $\lim_{\epsilon\to 0}[\ell_k(\epsilon)] = [\ell]$ for all $k \in \{1, \dots, r\}$. We call the point $[\ell] \in \mathbb{P}\mathbb{C}[\boldsymbol{x}]_1$ the* base *of the decomposition. A local decomposition is called* standard *if $\ell_1 = \epsilon^q \gamma \ell$ for some $q \in \mathbb{Z}$ and $\gamma \in \mathbb{C}$.*

▶ **Lemma 6.** *If f has a local border decomposition, then it has a standard local border decomposition with the same base and the same number of summands.*

Proof. After applying a linear change of variables, we may assume that the base of the local decomposition for f is $[x_1]$. This means

$$f = \lim_{\epsilon\to 0} \sum_{k=1}^{r} \ell_k^d$$

with $\ell_k = \epsilon^{q_k} \cdot \gamma_k x_1 + \sum_{j=q_k+1}^{\infty} \epsilon^j \ell_{k,j}$.

Write $\ell_1 = \epsilon^{q_1} (\sum_{i=1}^{n} \alpha_i x_i)$ where $\alpha_i \in \mathbb{C}(\epsilon)$. Let $\widehat{x}_1 = \frac{\gamma_1}{\alpha_1} x_1 - \sum_{i=2}^{n} \frac{\alpha_n}{\alpha_1} x_i$. Note that $\alpha_1 \simeq \gamma_1$ and $\alpha_i \simeq 0$ for $i > 1$, hence $\widehat{x}_1 \simeq x_1$ and

$$f \simeq f(\widehat{x}_1, \dots, x_n) \simeq \ell_1(\widehat{x}_1, x_2, \dots, x_n)^d + \sum_{k=2}^{r} \ell_k(\widehat{x}_1, x_2, \dots, x_n)^d = (\epsilon^{q_1}\gamma_1 x_1)^d + \sum_{k=2}^{r} \widehat{\ell}_k^d.$$

where $\widehat{\ell}_k(x_1, \dots, x_n) = \ell_k(\widehat{x}_1, x_2, \dots, x_n)$. This defines a new border rank decomposition of f. Moreover, notice that $\lim_{\epsilon\to 0}[\widehat{\ell}_k] = [x_1]$ for every k, so the new decomposition is again local with base $[x_1]$. Since the first summand is $\epsilon^{q_1}\gamma_1 x_1$, this is the desired standard local border decomposition. ◀

▶ **Lemma 7.** *Suppose $f \in \mathbb{C}[\boldsymbol{x}]_d$ has a local border decomposition with r summands based at $[\ell]$. If $d \geq r-1$, then $f = \ell^{d-r+1} g$ for some homogeneous polynomial g of degree $r-1$.*

Proof. After applying a linear change of variables we may assume $\ell = x_1$. We prove the statement by induction on r and the difference $d - (r-1)$.

The cases $r = 1$ and $d = r - 1$ are trivial.

If $d > r-1$, then by the previous Lemma there exists a standard local border decomposition

$$f = \lim_{\epsilon\to 0} \sum_{k=1}^{r} \ell_k(\epsilon)^d.$$

Write $\ell_k = \sum_{i=1}^{n} \alpha_{ki} x_i$ for some $\alpha_{ki} \in \mathbb{C}(\epsilon)$. Since the decomposition is standard, $\alpha_{1i} = 0$ for $i > 1$. For the derivatives of f we have the following border decompositions

$$\frac{\partial f}{\partial x_1} = \lim_{\epsilon\to 0} \sum_{k=1}^{r} d \cdot \alpha_{k1}(\epsilon)\ell_k(\epsilon)^{d-1},$$

and

$$\frac{\partial f}{\partial x_i} = \lim_{\epsilon \to 0} \sum_{k=2}^{r} d \cdot \alpha_{ki}(\epsilon) \ell_k(\epsilon)^{d-1}.$$

for $i \neq 1$. These decompositions involve the same linear forms ℓ_k with multiplicative coefficients, so they are local with the same base $[x_1]$. By inductive hypothesis $\frac{\partial f}{\partial x_1} = x_1^{d-r} g_1$ and $\frac{\partial f}{\partial x_i} = x_1^{d-r+1} g_i$ for some homogeneous polynomials $g_1, \ldots, g_n$ of appropriate degrees. To get an analogous expression for f, combine these expressions using Euler's formula for homogeneous polynomials as follows

$$f = \frac{1}{d} \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = \frac{1}{d} \left(x_1 \cdot x_1^{d-r} g_1 + \sum_{i=2}^{n} x_i x_1^{d-r+1} g_i \right) = \frac{1}{d} x_1^{d-r+1} \left(g_1 + \sum_{i=2}^{n} x_i g_i \right) . \quad ◀$$

We will now extend this result to non-local border Waring rank decompositions. As long as the degree of the approximated polynomial is high enough, every border rank decomposition can be divided into local parts and transformed into a sum of terms of the form $\ell^{d-r+1} g$.

▶ **Definition 8.** *A* generalized additive decomposition *of f is a decomposition of the form*

$$f = \sum_{k=1}^{m} \ell_k^{d-r_k+1} g_k \ ,$$

where ℓ_k are linear forms such that ℓ_i is not proportional to ℓ_j when $i \neq j$, and g_k are homogeneous polynomials of degrees $\deg g_k = r_k - 1$.

To show that there are no cancellations between different local parts, we need the following lemma, which in the case of 2 variables goes back to Jordan [35, Lem. 1.35]. This lemma can be seen as a generalization of a well-known fact that m pairwise non-proportional linear forms $\ell_1, \ldots, \ell_m$ have linearly independent powers $\ell_1^d, \ldots, \ell_m^d$ for $d \geq m - 1$.

▶ **Lemma 9.** *Let $\ell_1, \ldots, \ell_m \in \mathbb{C}[\boldsymbol{x}]_1$ be linear forms such that ℓ_i is not proportional to ℓ_j when $i \neq j$. Let $g_1, \ldots, g_m$ be homogeneous polynomials of degrees $r_1 - 1, \ldots, r_m - 1$ respectively. If*

$$\sum_{k=1}^{m} \ell_k^{d-r_k+1} g_k \ = \ 0 \ ,$$

and $d \geq \sum_{k=1}^{m} r_i - 1$, then all g_k are zero.

Proof. We first prove the statement for polynomials in 2 variables y_1, y_2 by induction on the number of summands m; this part of the proof closely follows [32, Appx.III].

The case $m = 1$ with one summand is clear. Consider the case $m > 2$. We can assume $\ell_1 = y_1$ by applying a linear change of variables if required. Note two simple facts about partial derivatives. First, for a homogeneous polynomial $f \in \mathbb{C}[y_1, y_2]_d$ we have $\partial_2^r f = 0$ if and only if $f = y_1^{d-r+1} g$ (here $\partial_2 := \frac{\partial}{\partial y_2}$). Second, differentiating r times a homogeneous polynomial of the form $\ell^{d-s+1} g$, we obtain a polynomial of the form $\ell^{d-r-s+1} h$.

Suppose

$$y_1^{d-r_1+1} g_1 + \sum_{k=2}^{m} \ell_k^{d-r_k+1} g_k = 0.$$

Differentiating r_1 times with respect to y_2, we obtain

$$\sum_{k=2}^{m} \ell_k^{d-r_1-r_k+1} h_k = 0,$$

where $\ell_k^{d-r_1-r_k+1} h_k = \partial_2^{r_1}(\ell_k^{d-r_k+1} g_k)$. The degree condition $d - r_1 \geq \sum_{k=2}^{m} r_k - 1$ holds for this new expression. Therefore, by induction hypothesis we have $h_k = 0$ and thus $\partial_2^{r_1}(\ell_k^{d-r_k+1} g_k) = 0$. It follows that $\ell_k^{d-r_k+1} g_k = y_1^{d-r_1+1} \widehat{g}_k$ for some homogeneous polynomial $\widehat{g}_k$. This implies that $y_1^{d-r_1+1}$ divides g_k, which is impossible since $d - r_1 + 1 \geq \sum_{k=2}^{m} r_k \geq r_k > \deg g_k$.

Consider now the general case where the number of variables $n \geq 2$ (the case $n = 1$ is trivial). Suppose $\sum_{k=1}^{m} \ell_k^{d-r_k+1} g_k = 0$. The set of linear maps $A\colon (y_1, y_2) \mapsto (x_1, \ldots, x_n)$ such that $\ell_i \circ A$ and $\ell_j \circ A$ are not proportional to each other is a nonempty Zariski open set given by the condition $\operatorname{rank}(\ell_i \circ A, \ell_j \circ A) > 1$. Hence for a nonempty Zariski open (and therefore dense) set of linear maps A the linear forms $\ell_k \circ A$ are pairwise non-proportional. From the binary case above we have $g_k \circ A = 0$ if A lies in this open set. By continuity this implies $g_k \circ A = 0$ for all A. Since every point lies in the image of some linear map A we have $g_k = 0$. ◀

▶ **Lemma 10.** *Let $f \in \mathbb{C}[\boldsymbol{x}]_d$ be such that $\underline{\mathsf{WR}}(f) = r$. If $d \geq r - 1$, then there exists a partition $r = r_1 + \cdots + r_m$ such that f has a generalized additive decomposition*

$$f = \sum_{k=1}^{m} \ell_k^{d-r_k+1} g_k,$$

and moreover $\underline{\mathsf{WR}}(\ell_k^{d-r_k+1} g_k) \leq r_k$.

Proof. Consider a border Waring rank decomposition

$$f = \lim_{\epsilon \to 0} \sum_{k=1}^{r} \ell_k^d$$

Divide the summands between several local decompositions as follows. Define an equivalence relation $\sim$ on the set of indices $\{1, 2, \ldots, r\}$ as $i \sim j \Leftrightarrow \lim_{\epsilon \to 0}[\ell_i] = \lim_{\epsilon \to 0}[\ell_j]$ and let $I_1, \ldots, I_m$ be the equivalence classes with respect to this relation. Further, let $r_k = |I_k|$ and let $[L_k] = \lim_{\epsilon \to 0}[\ell_i]$ for $i \in I_k$.

Consider the sum of all summands with indices in I_k. Let q_k be the power of ϵ in the lowest order term, that is,

$$\sum_{i \in I_k} \ell_i^d \; = \; \epsilon^{q_k} f_k + \sum_{j=q_k+1}^{\infty} \epsilon^j f_{k,j} \; ,$$

with $f_k \in \mathbb{C}[\mathbf{x}]_d$ nonzero. This expression can be transformed into a local border decomposition

$$f_k = \lim_{\epsilon \to 0} \sum_{i \in I_k} \left(\frac{\ell_i(\epsilon^d)}{\epsilon^{q_k}} \right)^d .$$

based at $[L_k]$. By Lemma 7 we have $f_k = L_k^{d-r_k+1} g_k$ for some homogeneous polynomial g_k of degree $r_k - 1$. The decomposition also gives $\underline{\mathsf{WR}}(f_k) \leq r_k$.

Note that $q_k \leq 0$ since otherwise the summands ℓ_i with $i \in I_k$ can be removed from the original border rank decomposition of f without changing the limit. Let $q = \min\{q_1, \ldots, q_m\}$. Note that if $q < 0$, then, comparing the terms before ϵ^q in the left and right hand sides of the equality

$$f + O(\epsilon) = \sum_{k=1}^{m} \sum_{i \in I_k} \ell_i^d$$

we get

$$0 = \sum_{k\colon q_k = q} f_k = \sum_{k\colon q_k = q} L_k^{d-r_k+1} g_k.$$

From Lemma 9 we obtain $g_k = 0$ and $f_k = 0$, in contradiction with the definition of f_k.

We conclude that $q = 0$ and

$$f = \sum_{k=1}^{m} f_k = \sum_{k=1}^{m} L_k^{d-r_k+1} g_k,$$

obtaining the required generalized additive decomposition. ◀

We will now take a brief detour to define a function $M(r)$ which we use to upper bound the Waring rank of generalized additive decomposition.

▶ **Definition 11.** *Let* $\mathsf{maxR}(n, d)$ *denote the* maximum Waring rank *of a degree d homogeneous polynomial in n variables, that is* $\mathsf{maxR}(n, d) = \max\{\mathsf{WR}(f) \mid f \in \mathbb{C}[x_1, \ldots, x_n]_d\}$. *Define the* partition-maxrank function *as*

$$M(r) = \max_{r_1 + \cdots + r_m = r} \sum_{k=1}^{m} \mathsf{maxR}(r_k, r_k - 1).$$

Since every homogeneous polynomial has finite Waring rank, the space $\mathbb{C}[x_1, \ldots, x_n]_d$ is spanned by powers of linear forms. This implies a trivial upper bound on the maximum Waring rank: $\mathsf{maxR}(n, d) \leq \dim \mathbb{C}[x_1, \ldots, x_n]_d = \binom{n+d-1}{d}$. Improved upper bounds were proven in [13, 37].

▶ **Proposition 12.** $\mathsf{maxR}(n, d_1) \leq \mathsf{maxR}(n, d_2)$ *when* $d_1 \leq d_2$.

Proof. Every form f of degree d_1 can be represented as a partial derivative of some form g of degree d_2. By differentiating a Waring rank decomposition of g we obtain a Waring rank decomposition of f, thus $\mathsf{WR}(f) \leq \mathsf{WR}(g) \leq \mathsf{maxR}(n, d_2)$. Since f is arbitrary, $\mathsf{maxR}(n, d_1) \leq \mathsf{maxR}(n, d_2)$. ◀

We are now ready to prove a debordering theorem for Waring rank.

▶ **Theorem 13.** *Let* $f \in \mathbb{C}[\boldsymbol{x}]_d$ *be such that* $\underline{\mathsf{WR}}(f) = r$. *Then*

$$\mathsf{WR}(f) \leq M(r) \cdot d.$$

Proof. We consider two cases depending on relation of degree d and border Waring rank r.

Case $d < r - 1$. Since $\underline{\mathsf{WR}}(f) = r$, the number of essential variables of f is at most r. Taking the maximum Waring rank as an upper bound, we obtain

$$\mathsf{WR}(f) \leq \mathsf{maxR}(r, d) < \mathsf{maxR}(r, r-1) \leq M(r) \leq M(r) \cdot d.$$

Case $d \geq r-1$. By Lemma 10 f has a generalized additive decomposition

$$f = \sum_{k=1}^{m} \ell_k^{d-r_k+1} g_k$$

with $r_1 + \cdots + r_m = r$, $\deg g_k = r_k - 1$ and $\underline{\mathsf{WR}}(\ell_k^{d-r_k+1} g_k) \leq r_k$. Since $\underline{\mathsf{WR}}(\ell_k^{d-r_k+1} g_k) \leq r_k$, the number of essential variables $\mathrm{N}_{\mathrm{ess}}(g_k) \leq r_k$. If $r_k = 1$, then

$$\mathsf{WR}(\ell_k^{d-r_k+1} g_k) = \mathsf{WR}(\ell_k^d) = 1 \leq d.$$

If $r_k \geq 2$, then we upper bound $\mathsf{WR}(g_k)$ by $\mathsf{maxR}(\mathrm{N}_{\mathrm{ess}}(g_k), \deg g_k) = \mathsf{maxR}(r_k, r_k - 1)$. Taking a Waring rank decomposition $g_k = \sum_{i=1}^{\mathsf{WR}(g_k)} L_i^{r_k-1}$ and multiplying it by $\ell_k^{d-r_k+1}$, we obtain a decomposition

$$\ell_k^{d-r_k+1} g_k = \sum_{i=1}^{\mathsf{WR}(g_k)} \ell_k^{d-r_k+1} \cdot L_i^{r_k-1}.$$

It is known that $\mathsf{WR}(y_1^a y_2^b) = \max\{a,b\} + 1$ (this is a classical fact known at least to Oldenburger [47], see also [19])[2]. It follows that

$$\mathsf{WR}(\ell_k^{d-r_k+1} L_i^{r_k-1}) \leq \mathsf{WR}(y_1^{d-r_k+1} y_2^{r_k-1}) = \max\{d - r_k + 2, r_k\} \leq d.$$

Hence we have $\mathsf{WR}(\ell_k^{d-r_k+1} g_k) \leq d \cdot \mathsf{WR}(g_k) \leq d \cdot \mathsf{maxR}(r_k - 1, r_k)$.

Combining all parts of the decomposition together, we get

$$\mathsf{WR}(f) \leq d \sum_{k=1}^{m} \mathsf{maxR}(r - k - 1, r_k) \leq M(r) \cdot d. \qquad \blacktriangleleft$$

A more explicit upper bound is provided by the following immediate corollary.

▶ **Theorem 14.** *Let $f \in \mathbb{C}[x_1, \ldots, x_n]_d$ and let $\underline{\mathsf{WR}}(f) = r$. Then*

$$\mathsf{WR}(f) \leq \binom{2r-2}{r-1} \cdot d.$$

Proof. The space of homogeneous polynomials of degree $r-1$ in r variables has dimension $\binom{2r-2}{r-1}$ and is spanned by powers of linear forms. Therefore, $\mathsf{maxR}(r-1, r) \leq \binom{2r-2}{r-1}$. Note that if $r = p + q$ with $p, q \neq 0$, then the space $\mathbb{C}[x_1, \ldots, x_r]_{r-1}$ contains a direct sum of $x_1^q \cdot \mathbb{C}[x_1, \ldots, x_p]_{p-1}$ and $x_1^{p+1} \cdot \mathbb{C}[x_{p+1}, \ldots, x_r]_{q-1}$. Taking the dimensions of these spaces, we obtain $\binom{2r-2}{r-1} \geq \binom{2p-2}{p-1} + \binom{2q-2}{q-1}$. It follows that $M(r) \leq \binom{2r-2}{r-1}$. ◀

Using the Blekherman–Teitler bound on the maximum rank [13], we can get a slightly better bound. The proof is essentially the same as for the previous theorem.

▶ **Corollary 15.** *Let $f \in \mathbb{C}[x_1, \ldots, x_n]_d$ and let $\underline{\mathsf{WR}}(f) = r$. Then*

$$\mathsf{WR}(f) \leq 2 \left\lceil \frac{1}{r} \binom{2r-2}{r-1} \right\rceil \cdot d.$$

[2] it is easy to see that for $a \geq b$ the monomial $y_1^a y_2^b$ is proportional to $\sum_{k=0}^{a} \zeta^k (\zeta^k y_1 + y_2)^{a+b}$ where ζ is a primitive root of unity of order $a+1$.

2.4 Scheme-theoretic proof

In this section we give a proof of Lemma 10 based on the theory of 0-dimensional schemes and apolarity. This short section assumes familiarity with these topics, we review them in more details in Appendix A of the extended version of the paper [27].

▶ **Lemma 10.** *Let $f \in \mathbb{C}[\boldsymbol{x}]_d$ be such that $\underline{\mathsf{WR}}(f) = r$. If $d \geq r - 1$, then there exists a partition $r = r_1 + \cdots + r_m$ such that f has a generalized additive decomposition*

$$f = \sum_{k=1}^{m} \ell_k^{d-r_k+1} g_k,$$

and moreover $\underline{\mathsf{WR}}(\ell_k^{d-r_k+1} g_k) \leq r_k$.

Alternative proof. Denote by V the space of linear forms $\mathbb{C}[\boldsymbol{x}]_1$.

Since $d \geq r - 1$, the border Waring rank of f is equal to its *smoothable rank* $\mathsf{SR}(f)$ [16], that is, there exists a 0-dimensional scheme $Z \subset \mathbb{P}V$ of degree r which is smoothable (obtained as a flat limit of the family of r-point subsets of $\mathbb{P}V$) and f is apolar to Z. Let I be the ideal of Z and let $I = I^{(1)} \cap \cdots \cap I^{(m)}$ be the primary decomposition of this ideal. The primary ideals $I^{(j)}$ correspond to irreducible components Z_j of the scheme Z.

Since f is apolar to I, we have $f \in I_d^{\perp} = (I_d^{(1)})^{\perp} + \cdots + (I_d^{(m)})^{\perp}$. In particular, there exist $f_j \in (I_d^{(j)})^{\perp}$ such that $f = f_1 + \cdots + f_m$. Let r_j be the degree of Z_j. By the definition of degree, $r = r_1 + \cdots + r_m$. If Z_j is supported at the point $[\ell_j] \in \mathbb{P}V$, then for the ideal $I^{(j)}$ we have $(\ell_j^{\perp})^{r_j} \subset I^{(j)} \subset \ell_j^{\perp}$ and $(I_d^{(j)})^{\perp} \subset \ell_j^{d-r_j+1} \cdot \mathbb{C}[\boldsymbol{x}]_{r_j-1}$. Therefore the polynomials f_j have the form $\ell_j^{d-r_j+1} g_j$ for some g_j of degree $\deg g_j = r_j - 1$.

Additionally, all irreducible components of a smoothable scheme Z are smoothable [17, Thm. 1.1], and since f_j is apolar to Z_j, we have $\underline{\mathsf{WR}}(f_j) \leq \mathsf{SR}(f_j) \leq r_j$. ◀

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