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RESEARCH ARTICLE

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Bilinear sums with $GL(2)$ coefficients and the exponent of distribution of d_3

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Abstract

We obtain the exponent of distribution $1/2 + 1/30$ for the ternary divisor function d_3 to square-free and prime power moduli, improving the previous results of Fouvry–Kowalski–Michel, Heath-Brown and Friedlander–Iwaniec. The key input is certain estimates on bilinear sums with $GL(2)$ coefficients obtained using the delta symbol approach.

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1 | INTRODUCTION

Given an arithmetically interesting function $f : \mathbb{N} \to \mathbb{C}$ and q of reasonable size, we expect that

$$\sum_{\substack{n \leqslant X \\ n = a(q)}} f(n) \sim \frac{1}{\phi(q)} \sum_{\substack{n \leqslant X \\ (n,q)=1}} f(n), \tag{1.1}$$

for each $(a, q) = 1$. It is a fundamental problem in number theory to show that the above asymptotic holds for q as large as possible. To this end, we call a positive number δ an *exponent of distribution for f restricted to a set Q of moduli*, if for any $q \in Q$ with $q \leqslant X^{\delta-\epsilon}$ and any residue class $a \pmod q$ with $(a, q) = 1$, the asymptotic formula

$$\sum_{\substack{n \leqslant X \\ n = a(q)}} f(n) = \frac{1}{\phi(q)} \sum_{\substack{n \leqslant X \\ (n,q)=1}} f(n) + O\left(\frac{X}{q(\log X)^A}\right)$$

holds for any $A > 0$ and $X \geqslant 2$.

For the very important Von Mangoldt function $\Lambda(n)$, the classical Siegel–Walfisz theorem implies that the above asymptotics hold for $q \leqslant (\log X)^{B(A)}$, where $B(A) > 0$ depends on A, whereas the GRH predicts $q \leqslant X^{1/2-\epsilon}$. The celebrated Bombieri–Vinogradov theorem confirms this prediction on an average over the moduli.

Another important class of examples comes from the *k-fold* divisor function

$$d_k(n) = \sum_{n_1 n_2 \cdots n_k = n} 1.$$

It is widely believed that $\delta = 1$ is a exponent of distribution for all $k \geqslant 2$. This has deep consequences for our understanding of primes which goes far beyond the direct reach of the Generalized Riemann hypothesis. For $k = 2$, the best known exponent of distribution is $\delta = 2/3$ due to Selberg (unpublished), Hooley [15] and Heath-Brown [13]. Several authors have achieved improvement to $\delta = 2/3$ in special cases. See [3, 4, 6, 7].

The only other case known for surpassing the 'Bombieri–Vinogradov range' $\delta = 1/2$ is when $k = 3$. Let us briefly take a look at the previous approaches. After an application of the $GL(3)$ Voronoi summation formula to the left-hand side of (1.1) (or equivalently, a three-fold application of the Poisson summation formula), one observes that to beat the $\delta_3 = 1/2$ barrier, one needs non-trivial estimates for

$$\sum_{m \sim q} d_3(m) \mathrm{Kl}_3(am, q), \tag{1.2}$$

where $\mathrm{Kl}_3(\cdots)$ is the hyper-Kloosterman sum. Opening the divisor function

$d_3(m) = \sum_{m_1 m_2 m_3 = m} 1$ and dividing the m_i-sum into dyadic blocks $m_i \sim Y_i$ with $Y_1 \leqslant Y_2 \leqslant Y_3$, it suffices to obtain non-trivial estimates for

$$\sum_{m_1 \sim Y} \sum_{m_2 \sim q/Y} d(m_2) \mathrm{Kl}_3(am_1 m_2, q) \tag{1.3}$$

for each $Y \leqslant q^{1/3}$. When Y is not too small, good estimates can be obtained by applying Cauchy–Schwarz inequality to (1.3) keeping the m_2 variable outside the absolute value square followed by a Poisson summation in the m_2-sum. Therefore, the main effort lies in obtaining good estimates for (1.3) when Y is small. Alternatively, by applying the $GL(2)$ Voronoi summation formula to the m_2-sum in (1.3), one can also consider

$$\sum_{m_1 \sim Y} \sum_{m_2 \sim qY} d(m_2) e(am_1 \overline{m_2}/q). \tag{1.4}$$

In their groundbreaking work, Friedlander and Iwaniec [11] successfully obtained non-trivial estimates for (1.4) which led to the exponent $\delta_3 = 1/2 + 1/230$. More precisely, their main input was non-trivial estimates for the short exponential sums

$$\sum_{h \sim H} \sum_{m \sim M} \sum_{n \sim N} e(h\overline{mn}/q), \tag{1.5}$$

which is a further decomposition of (1.4), using the 'shifting by ab' technique. Heath-Brown [14] improved the exponent to $\delta_3 = 1/2 + 1/82$ by utilising a more elementary treatment of (1.5) based on the methods of Heath-Brown [12] and Balasubramanian, Conrey and Heath-Brown [2]. Since both of these approaches were based on decomposing the sum (1.3) into multiple exponential sums (1.5), they were far from optimal.

With a more structural approach by viewing the divisor function $d(m_2)$ in (1.3) as the Fourier coefficients of Eisenstein series, Fouvry, Kowalski and Michel [10] were able to produce the exponent $\delta_3 = 1/2 + 1/46$ for prime moduli improving the previous results. Their key input was the estimates for short sums of $GL(2)$ coefficients

$$\sum_{m\sim M} \lambda(m)K(m) \tag{1.6}$$

twisted by general trace functions $K(\cdots)$ of prime modulus, which they obtained in [9] using the $GL(2)$ spectral theory. Note that the relevant estimate for (1.6) (when $\lambda(m) = d(m)$) was obtained in the separate paper [8], which required additional arguments to isolate its contribution from the continuous spectrum. They further improved their exponent to $\delta_3 = 1/2 + 1/34$ on an average over the moduli by combining their results with the estimates for sums of Kloosterman sums pioneered by Deshouillers and Iwaniec. P. Xi [22] obtained the exponent $\delta_3 = 1/2 + 1/34$ for moduli with special factorisation using the q-analogue of the van der Corput method.

In this paper, we go even further and utilise the complete bilinear structure in (1.3), which results in an improvement over all the above exponents. We use the delta symbol approach to obtain non-trivial estimates for bilinear sums (1.3) involving $GL(2)$ coefficients. The method provides a uniform treatment for the holomorphic/Maass and Eisenstein cases and essentially covers all moduli.

Theorem 1.1. *Let $\epsilon > 0$ and a be a non-zero integer. For every square-free $q \geqslant 1$ and every odd prime power $q = p^\gamma, \gamma \geqslant 28$ with $(a, q) = 1$ and satisfying*

$$q \leqslant X^{1/2+1/30-\epsilon},$$

we have

$$\sum_{\substack{n\leqslant X\\ n=a(q)}} d_3(n) = \frac{1}{\phi(q)} \sum_{\substack{n\leqslant X\\ (n,q)=1}} d_3(n) + O(X^{1-\epsilon}/q),$$

where the implied constant depends only on ϵ.

Remarks.

- We have considered the complementary cases of square-free and prime power, but it is possible to merge the arguments to cover all natural numbers and, in particular, to close the gap from $\gamma \geqslant 28$ in Theorem 1.1 to $\gamma \geqslant 2$. All one needs is a version of Lemma 2.4 with restriction $u \leqslant 4\gamma/5$ lifted to $u \leqslant \gamma - 1$ for $\gamma \geqslant 2$. The current estimation of the character sum leads to complicated counting arguments, which we decided to avoid for the exposition's simplicity.
- For $q = p^\gamma$, the methods of this paper can produce a better exponent $\delta > 1/2 + 1/30$ by using the p-adic stationary phase analysis followed by an exponent pair estimate (see [20, Theorem 2]) to bound certain average of the product of two Kloosterman sums non-trivially. See the remarks just before Lemma 5.2 and Remark 3.

- The exponents can be further improved by combining our estimates with the Kloostermania techniques when averaging over the moduli.

The key input is the following estimate for the bilinear sums with $GL(2)$ coefficients obtained using the separation of oscillation technique.

For $m, q \geqslant 1$, let $\tilde{\mathrm{Kl}}_3(m, q)$ denote the normalised hyper-Kloosterman sum

$$\tilde{\mathrm{Kl}}_3(m,q) = \frac{1}{q} \sum_{x,y\ (q)}^{*} e\left(\frac{mx + y + \overline{xy}}{q}\right). \tag{1.7}$$

Let $\lambda(n)$ denote the nth Fourier coefficient of an $SL(2,\mathbb{Z})$ holomorphic cusp form or Maass cusp form or the Eisenstein series $E(z, 1/2 + w)$ for a complex number

$$w \ll q^{\epsilon}.$$

Note that in the case of Eisenstein series, $\lambda(n) = \sigma_{-2w}(n)$, which will be the relevant case for the application to Theorem 1.1. We fix a smooth function $V(x)$ compactly supported in $\mathbb{R}_{>0}$ and satisfying $V^{(j)}(x) \ll_{j,\epsilon} q^{j\epsilon}$, $j \geqslant 0$.

Theorem 1.2. *Let $q \geqslant 1$ be square-free, $b \in \mathbb{Z}$ co-prime to q and $\lambda(n)$, $\tilde{Kl}_3(\cdots)$ be as above. Let $\mathcal{N} \subset \mathbb{Z}$ be a set of N consecutive integers and let $\{\alpha_n\}_{n\in\mathcal{N}}$ be a sequence of complex numbers with $|\alpha_n| \ll 1$. Suppose $M \geqslant 1$ is such that $N \leqslant q^{1/2}(1 + M/q)^{-2}$, then*

$$\sum_{n\in\mathcal{N}} \sum_{m\geqslant 1} \alpha_n \lambda(m) \tilde{Kl}_3(mnb, q) V(m/M) \ll_\epsilon q^{3/8+\epsilon} M^{1/2} N^{3/4} (1 + M/q)^{1/2} + q^{-1/4+\epsilon} M N^{3/2} (1 + M/q) + N q^{3/4+\epsilon} (1 + M/q)^{1/2}.$$

The flexibility of the method allows us to obtain stronger estimates for moduli with special factorisation.

Theorem 1.3. *Let $q = p^\gamma$, $\gamma \geqslant 2$ and $p > 2$. With the notations of Theorem 1.2, suppose that $N \leqslant q^{1/5}(1 + M/q)^{-2}$. Then*

$$\sum_{n\in\mathcal{N}} \sum_{m\geqslant 1} \alpha_n \lambda(m) \tilde{Kl}_3(mnb, q) V(m/M) \ll_\epsilon p^{7/12} q^{1/3+\epsilon} M^{1/2} N^{5/6} (1 + M/q)^{2/3} + q^{13/20+\epsilon} N.$$

Remarks.

- Each $(1 + M/q)$ factor that appears in last two theorems can be eliminated by first dualising the $GL(2)$ sum when $M \gg q$ and proceeding same as below. The restriction $N \leqslant q^{1/5}(1 + M/q)^{-2}$ in Theorem 1.3 is for the sake of technical simplicity and can be easily extended to $N \leqslant q^{1/2}(1 + M/p)^{-2}$ with additional computations.
- The choice of the trace function $\mathrm{Kl}_3(\cdots)$ in Theorems 1.2 and 1.3 is made for its application towards Theorem 1.1, but the results should hold for more general trace functions (see [9] for examples).

- Choosing $\mathcal{N} = \{1\}$ and $\alpha_1 = 1$, the two theorems give

$$\sum_{m\sim M} \lambda(m)\tilde{\mathrm{Kl}}_3(mb,q) \ll_\epsilon q^\epsilon\left(M^{1/2}q^{3/8} + M/q^{1/4} + q^{3/4}\right), \tag{1.8}$$

  and

$$\sum_{m\sim M} \lambda(m)\tilde{\mathrm{Kl}}_3(mb,q) \ll_\epsilon q^\epsilon(M^{1/2}q^{1/3} + q^{13/20}).$$

  when $M \leqslant q$. These are non-trivial as long as $M \gg q^{3/4+\epsilon}$, which is the 'Burgess range', and $M \gg q^{2/3+\epsilon}$, which is the 'Weyl range', respectively. Hence, with the additional cancellation in the n-sum, Theorem 1.3 is, on average, of sub-Weyl strength (with twists by trace functions). In the square-free case, the '$Nq^{3/4}$' term in Theorem 1.1, which pops out as an additional term from a certain zero-frequency, prevents us from going beyond Burgess. It would be of interest to get an improvement over this term.
- For composite moduli $q = p_1p_2$, with $p_1 \asymp q^\alpha$, $\alpha > 0$ not too large, estimates somewhere between the Weyl and the Burgess range can be obtained using a similar approach.

Remark 1 (Notation). In this paper, the notation $\alpha \ll A$ will mean that for any $\epsilon > 0$, there is a constant c such that $|\alpha| \leqslant cAX^\epsilon$. The dependence of the constant on ϵ, when occurring, will be ignored. We will follow the usual ϵ-convention: the letter ϵ denotes sufficiently small positive quantity that may change from line to line. We will also use the phrase "negligible error" by which we mean an error term $O_B(X^{-B})$ for an arbitrary $B > 0$. The notation $x = y(q)$ will mean $x \equiv y \bmod q$ throughout the paper.

2 | PRELIMINARIES

2.1 | Voronoi summation formula for $d_3(n)$

We use the version due to X. Li [19]. Let

$$\sigma_{0,0}(k,l) = \sum_{d_1|l}\ \sum_{\substack{d_2|\frac{l}{d_1}\\(d_2,k)=1}} 1 = \sum_{a|(k,l)} \mu(a)d_3(l/a). \tag{2.1}$$

For $\phi(y) \in C_c(0,\infty)$, $k = 0,1$ and $\sigma > -1-2k$, set

$$\Phi_k(y) = \frac{1}{2\pi i}\int_{(\sigma)} (\pi^3 y)^{-s}\cdot\frac{\Gamma\left(\frac{1+s+2k}{2}\right)^3}{\Gamma\left(\frac{-s}{2}\right)^3}\tilde{\phi}(-s-k)\,ds, \tag{2.2}$$

where $\tilde{\phi}$ is the Mellin transform of ϕ, and

$$\Phi_\pm(y) = \Phi_0(y) \pm \frac{1}{i\pi^3 y}\Phi_1(y). \tag{2.3}$$

Lemma 2.1 (X. Li). *For integers $a, q \geqslant 1$ with $(a, q) = 1$, with the above notation, we have*

$$\sum_{n\geqslant 1} d_3(n)e(an/q)\phi(n)$$
$$= \frac{1}{q}\int_0^\infty P(\log y, q)\phi(y)dy$$
$$+ \frac{q}{2\pi^{3/2}}\sum_{\pm}\sum_{r|q}\sum_{m\geqslant 1}\frac{1}{rm}\sum_{r_1|r}\sum_{r_2|\frac{r}{r_1}}\sigma_{0,0}(r/(r_1r_2), m)S(\pm m, \overline{a}; q/r)\Phi_\pm(mr^2/q),$$

where $P(y, q) = A_0(q) + A_1(q)y + A_2(q)y^2$ is a quadratic polynomial whose coefficients depend only on q.

When ϕ is a nice weight function, the corresponding transform $\Phi_\pm$ also behaves nicely as conveyed by the following lemma.

Lemma 2.2. *Suppose that the smooth function $\phi(y)$ is supported in $[X, 2X], X \geqslant 1$ and satisfies $y^j\phi^{(j)}(y) \ll_j 1, j \geqslant 0$. Then, $\Phi_\pm(y) \ll_B X^{-B}$ unless $yX \ll X^\epsilon$ in which case*

$$y^j\Phi_\pm^{(j)}(y) \ll_j \min\{yX, 1\}.$$

Proof. From (2.3), it is enough to prove the lemma for $\Phi_0(y)$. From the definition (2.2), we have for $j \geqslant 0$,

$$y^j\Phi_0^{(j)}(y) = (-1)^j\frac{1}{2\pi i}\int_{(\sigma)} s(s+1)\cdots(s+j)(\pi^3y)^{-s}\cdot\frac{\Gamma\left(\frac{1+s}{2}\right)^3}{\Gamma\left(\frac{-s}{2}\right)^3}\tilde{\phi}(-s)\,ds.$$

Note that $\tilde{\phi}(-s) \ll X^{-\Re(s)}$. Shifting the contour above to the right $\sigma = A > 0$ and trivially estimating, we obtain

$$y^j\Phi_0^{(j)}(y) \ll_{A,j} (yX)^{-A}. \tag{2.4}$$

Since A is arbitrary, the first part of lemma follows. On the other hand, shifting the contour to the left $\sigma = -3/2$ while picking up the residue at $\sigma = -1$, we obtain

$$y^j\Phi_0^{(j)}(y) \ll_j yX + (yX)^{3/2}. \tag{2.5}$$

The second part of the lemma from (2.4) and (2.5). □

2.2 | Voronoi summation formula for $GL(2)$

See appendix A.4 of [18] and appendix of [17] for details.

Lemma 2.3. *Let $\lambda(n)$ be either the nth Fourier coefficient of a Maass cusp form with Laplacian eigenvalue $1/4+\nu^2$, $\nu \geqslant 0$ or $\lambda(n)=\sigma_w(n)$, $w \in \mathbb{C}$. For integers $a, q \geqslant 1$ with $(a,q)=1$, $h(x) \in C_c(0,\infty)$, we have*

$$\sum_{n=1}^{\infty} \lambda(n) e\left(\frac{an}{q}\right) h(n) = \frac{1}{q}\int_0^{\infty} g(q,x)h(x)\,dx + \frac{1}{q}\sum_{\pm}\sum_{n\geqslant 1}\lambda(n)e\left(\frac{\pm\bar{a}n}{q}\right)H^{\pm}\left(\frac{n}{q^2}\right),$$

where

- *if $\lambda(n)$ corresponds to Maass form, then $g(q,x)=0$ and*

$$H^-(\alpha) = \frac{-\pi}{\sin(\pi i\nu)}\int_0^{\infty} h(y)\{J_{2i\nu} - J_{-2i\nu}\}(4\pi\sqrt{y\alpha})dy\,,$$
$$H^+(\alpha) = 4\varepsilon_f \cosh(\pi\nu)\int_0^{\infty} h(y)K_{2i\nu}(4\pi\sqrt{y\alpha})dy\,,$$

  for $\nu > 0$, and

$$H^-(\alpha) = -2\pi\int_0^{\infty} h(y)Y_0(4\pi\sqrt{y\alpha})dy, \quad \text{and} \quad H^+(\alpha) = 4\varepsilon_f\int_0^{\infty} h(y)K_0(4\pi\sqrt{y\alpha})dy\,,$$

  for $\nu = 0$.
- *If $\lambda(n) = \sigma_0(n) = d(n)$, then $g(q,x) = \log(\sqrt{x}/q) + \gamma$ and*

$$H^-(\alpha) = -2\pi\int_0^{\infty} h(y)Y_0\left(4\pi\sqrt{y\alpha}\right)dy\,,$$
$$H^+(\alpha) = 4\int_0^{\infty} h(y)K_0(4\pi\sqrt{y\alpha})dy\,.$$

- *If $\lambda(n) = \sigma_w(n)$, $w \neq 0$, then $g(q,x) = \zeta(1+w)(x/q)^w + \zeta(1-w)q^w$ and*

$$H^-(\alpha) = \int_0^{\infty} h(y)\tilde{Y}_w\left(4\pi\sqrt{y\alpha}\right)dy\,,$$
$$H^+(\alpha) = \int_0^{\infty} h(y)\tilde{K}_w(4\pi\sqrt{y\alpha})dy\,,$$

  where $\tilde{Y}_w, \tilde{K}_w$ are closely related to Y_w, K_w, and have the integral representations

$$\tilde{Y}_w(x) = \frac{1}{2\pi i}\int_{(2)}(x/2)^{-s}\Gamma(s-w)\Gamma(s+w)\cos(\pi s)ds,$$
$$\tilde{K}_w(x) = \frac{\cosh(\pi|w|)}{2\pi i}\int_{(2)}(x/2)^{-s}\Gamma(s-w)\Gamma(s+w)ds.$$

2.3 | Character sum estimates

The endgame of the paper consists of getting square-root cancellations in certain character sums which we record here for convenience. Let p be a prime and $m \in \mathbb{Z}$. Suppose $s_j, t_j, \lambda_j, j = 1, 2$ are integers such that $(s_j, p) = (\lambda_j, p) = 1, j = 1, 2$. For $u \leqslant \gamma$, define

$$\mathfrak{C}_{\gamma,u} = \sum_{\substack{a_1,a_2(p^u) \\ \lambda_1\overline{a_1}-\lambda_2\overline{a_2}=m(p^u)}}^{*}\sum^{*} S(1, \overline{s_1 p^{\gamma-u} a_1 + t_1}, p^\gamma)\overline{S}(1, \overline{s_2 p^{\gamma-u} a_2 + t_2}, p^\gamma). \tag{2.6}$$

For $\gamma = 1$, such character sum has been studied in [5] using the l-adic techniques developed by Deligne and Katz, and in [9] in the broader framework of trace functions. When $\gamma > 1$, an estimate for $\mathfrak{C}_{\gamma,u}$ can be obtained in an elementary manner by reducing the sum to a set of congruence conditions. We begin with latter case.

Lemma 2.4. *Suppose $\gamma > 1, u \leqslant 4\gamma/5, m \neq 0$ and $(2t_j, p) = 1$. If $u/2 < \gamma - u$ or $\nu_p(m) < \gamma - u$, then*

$$\mathfrak{C}_{\gamma,u} \ll p^{\gamma+u/2+\epsilon(u)/2} \cdot p^{\nu_p(m)},$$

and if $u/2 \geqslant \gamma - u$ and $\nu_p(m) \geqslant \gamma - u$, then $\mathfrak{C}_{\gamma,u}$ vanishes unless $t_1^{-3/2}s_1\lambda_1 = t_2^{-3/2}s_2\lambda_2\ (p^{\gamma-u})$, in which case

$$\mathfrak{C}_{\gamma,u} \ll p^{\gamma+u}.$$

Here, $\epsilon(u) = 0$ or 1 depending on u is even or odd, respectively.

Proof. Without loss of generality, we can assume $\nu_p(m) < u/2$, since otherwise the claim follows after a trivial estimation of the Kloosterman sums. We perform some initial transformation. Firstly, suppose that u is even. Then, for $j = 1, 2$, we can write

$$a_j = p^{u/2}\alpha_j + \beta_j,\ \ 1 \leqslant \alpha_j, \beta_j \leqslant p^{u/2},\ \ (\beta_j, p) = 1. \tag{2.7}$$

From (2.7), we obtain

$$\overline{a_j} = \overline{\beta_j} - p^{u/2}\overline{\beta_j}^2\alpha_j\ (p^u). \tag{2.8}$$

Plugging (2.8), we see that the congruence

$$\lambda_1\overline{a_1} - \lambda_2\overline{a_2} = m\ (p^u),$$

is equivalent to

$$\begin{aligned} \overline{\beta_2} &= \lambda_1\overline{\lambda_2\beta_1} - \overline{\lambda_2}m\ (p^{u/2}), \\ \alpha_2 &= \lambda_1\overline{\lambda_2}\beta_2^2\overline{\beta_1}^2\alpha_1 - g(\beta_1)\ (p^{u/2}), \end{aligned} \tag{2.9}$$

where

$$g(\beta_1) = \overline{\lambda_2}\beta_2^2 \cdot \frac{(\lambda_1\overline{\beta_1} - \lambda_2\overline{\beta_2} - m)}{p^{u/2}}.$$

We proceed for the explicit evaluation of $\mathfrak{C}_{\gamma,u}$ (2.6) in terms of these decompositions. We use the following evaluation of the Kloosterman sums modulo prime powers, which can be found in [16, (12.39)]:

$$S(1,\beta,p^\gamma) = \begin{cases} 2\left(\frac{\ell}{p}\right)^\gamma p^{\gamma/2}\Re\, \varepsilon_{p^\gamma}\, e(2\ell/p^\gamma), & \left(\frac{\beta}{p}\right) = 1, \\ 0, & \left(\frac{\beta}{p}\right) = -1, \end{cases} \tag{2.10}$$

where $\ell^2 = \beta\ (p^\gamma)$, $\left(\frac{\cdot}{\cdot}\right)$ is the Legendre symbol, and ε_c equals 1 if $c \equiv 1 \bmod 4$ and i if $c \equiv 3 \bmod 4$.

Hence, the Kloosterman sums in (2.6) vanish unless we have $\left(\frac{t_j}{p}\right) = 1$. From the formula (2.10), it follows

$$S(1, \overline{s_j p^{\gamma-u} a_j + t_j}, p^\gamma) = \sum_{\pm} p^{\gamma/2}\left(\frac{t_j^{1/2}}{p}\right)^\gamma e\left(\pm\frac{2\overline{(s_j p^{\gamma-u} a_j + t_j)^{1/2}}}{p^\gamma}\right). \tag{2.11}$$

Using the fact that $\gamma - u \geqslant 1$ and expanding $\overline{(s_j p^{\gamma-u} a_j + t_j)^{1/2}}$, we see that our character sum (2.6) can be written as sum of four terms of the form (up to constant factors)

$$\mathfrak{C} = p^\gamma\left(\frac{t_1^{1/2}}{p}\right)^\gamma\left(\frac{t_2^{1/2}}{p}\right)^\gamma \sideset{}{^*}\sum\sideset{}{^*}\sum_{\substack{a_1,a_2(p^u) \\ \lambda_1\overline{a_1}-\lambda_2\overline{a_2}=m(p^u)}} e\left(\frac{\sum_{i\geqslant 0} p^{i(\gamma-u)}\theta_i a_1^i - \sum_{i\geqslant 0} p^{i(\gamma-u)}\eta_i a_2^i}{p^\gamma}\right), \tag{2.12}$$

where

$$\theta_i = 2\binom{-1/2}{i} t_1^{-i-1/2} s_1^i \quad \text{and} \quad \eta_i = 2\binom{-1/2}{i} t_2^{-i-1/2} s_2^i.$$

Here,

$$\binom{-1/2}{i} = \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\cdots\left(-\frac{1}{2}-i+1\right)}{i!}$$

is the ith binomial coefficient, and in our context, $-1/2$ means $-\overline{2}(p^\gamma)$. Note that this way the numerator of this ith binomial coefficient is divisible by $i!$, and so, the expression makes sense modulo p^γ.

Using (2.8), modulo p^γ, the phase function above is

$$\sum_{i\geqslant 0} p^{i(\gamma-u)}\theta_i a_1^i - \sum_{i\geqslant 0} p^{i(\gamma-u)}\eta_i a_2^i = \sum_{i\geqslant 1} p^{i(\gamma-u)+u/2} i\theta_i \beta_1^{i-1}\alpha_1 - \sum_{i\geqslant 1} p^{i(\gamma-u)+u/2} i\eta_i\beta_2^{i-1}\alpha_2$$
$$+ \sum_{i\geqslant 0} p^{i(\gamma-u)}\theta_i\beta_1^i - \sum_{i\geqslant 0} p^{i(\gamma-u)}\eta_i\beta_2^i$$
$$= \sum_{i=1,2} i\left(p^{i(\gamma-u)+u/2} i\theta_i\beta_1^{i-1}\alpha_1 - p^{i(\gamma-u)+u/2} i\eta_i\beta_2^{i-1}\alpha_2\right)$$
$$+ \sum_{i\geqslant 0} p^{i(\gamma-u)}\theta_i\beta_1^i - \sum_{i\geqslant 0} p^{i(\gamma-u)}\eta_i\beta_2^i.$$

We have truncated the last sum up to $i \leqslant 2$ since $3(\gamma-u)+u/2 \geqslant \gamma$ by our assumption. Substituting α_2 from (2.9), the right-hand side of the last display becomes

$$\alpha_1 \sum_{i=1,2} p^{i(\gamma-u)+u/2} i\left(\theta_i - \eta_i\lambda_1\overline{\lambda_2}(\overline{\beta_1}\beta_2)^{i+1}\right)\beta_1^{i-1} - \sum_{i=1,2} p^{i(\gamma-u)+u/2} i\eta_i\beta_2^{i-1} g(\beta_1)$$
$$+ \sum_{i\geqslant 0} p^{i(\gamma-u)}(\theta_i\beta_1^i - \eta_i\beta_2^i).$$

Substituting this expansion into (2.12), we see that

$$\mathfrak{C} = p^{\gamma}\left(\frac{t_1^{1/2}}{p}\right)^{\gamma}\left(\frac{t_2^{1/2}}{p}\right)^{\gamma} \sideset{}{^*}\sum_{1\leqslant\beta_1\leqslant p^{u/2}} e\left(\frac{f(\beta_1)}{p^{\gamma}}\right) \sum_{1\leqslant\alpha_1\leqslant p^{u/2}} e\left(\frac{h(\beta_1)\alpha_1}{p^{u/2}}\right), \tag{2.13}$$

where

$$f(\beta_1) = -\sum_{i=1,2} p^{i(\gamma-u)+u/2} i\eta_i\beta_2^{i-1} g(\beta_1) + \sum_{i\geqslant 0} p^{i(\gamma-u)}(\theta_i\beta_1^i - \eta_i\beta_2^i)$$

and

$$h(\beta_1) = \sum_{i=0,1} (i+1)\beta_1^i\left(\theta_{i+1} - \eta_{i+1}\lambda_1\overline{\lambda_2}(\overline{\beta_1}\beta_2)^{i+2}\right)p^{i(\gamma-u)}. \tag{2.14}$$

Executing the linear α_1-sum, it follows

$$\mathfrak{C} \ll p^{\gamma+u/2} \sideset{}{^*}\sum_{\substack{\beta_1(p^{u/2})\\ h(\beta_1)=0(p^{u/2})}} 1.$$

It remains to count the solutions to $h(\beta_1) = 0(p^{u/2})$.

If $\gamma - u > u/2$, then from the expression (2.14), it follows that $h(\beta_1) = 0(p^{u/2})$ implies

$$\theta_1 - \eta_1\lambda_1\overline{\lambda_2}(\overline{\beta_1}\beta_2)^2 = 0\ (p^{u/2}) \Rightarrow \theta_1 - \eta_1\lambda_1\overline{\lambda_2}(\lambda_1\overline{\lambda_2} - \overline{\lambda_2}m\beta_1)^{-2} = 0\ (p^{u/2}).$$

Since $(\theta_1\eta_1, p) = 1$, the last relation forces $\overline{\theta_1}\eta_1\lambda_1\overline{\lambda_2}$ to be a quadratic residue mod $p^{u/2}$ in which case, we get

$$\overline{\lambda_2}m\beta_1 = \lambda_1\overline{\lambda_2} \pm (\overline{\theta_1}\eta_1\lambda_1\overline{\lambda_2})^{1/2}\ (p^{u/2}).$$

This determines β_1 modulo $p^{u/2-\min\{u/2,\nu_p(m)\}}$, and hence, we have at most $O(p^{\nu_p(m)})$ solutions for $\beta_1(p^{u/2})$ and the lemma follows.

So, we can assume $\gamma - u \leqslant u/2$. This forces

$$\theta_1 - \eta_1\lambda_1\overline{\lambda_2}(\overline{\beta_1}\beta_2)^2 = 0\ (p^{\gamma-u}) \Rightarrow \theta_1 - \eta_1\lambda_1\overline{\lambda_2}(\lambda_1\overline{\lambda_2} - \overline{\lambda_2}m\beta_1)^{-2} = 0\ (p^{\gamma-u}). \tag{2.15}$$

Suppose $\nu_p(m) \geqslant \gamma - u (\geqslant u/4)$, then the last congruence becomes

$$\theta_1\lambda_1 = \eta_1\lambda_2\ (p^{\gamma-u}),\ \text{i.e.}\ t_1^{-3/2}s_1\lambda_1 = t_2^{-3/2}s_2\lambda_2\ (p^{\gamma-u}).$$

In this case, we use the trivial bound to get

$$\mathfrak{C} \ll p^{\gamma+u/2} \sum_{\substack{\beta_1(p^{u/2}) \\ h(\beta_1)=0(p^{u/2})}}^{*} 1 \ll p^{\gamma+u}.$$

This proves the second part of the lemma. In the case $\nu_p(m) < \gamma - u$, (2.15) determines β_1 modulo $p^{\gamma-u-\nu_p(m)}$, and say that c is the corresponding solution. Denote $r = \gamma - u - \nu_p(m)$, $h_i(c) = (i+1)(\theta_{i+1} - \eta_{i+1}\lambda_1\overline{\lambda_2}(\lambda_1\overline{\lambda_2} - \overline{\lambda_2}mc)^{-i-2})$, then for $\lambda \in \mathbb{Z}$,

$$h(p^r\lambda + c) = h_0(c) - \eta_0\lambda_1\overline{\lambda_2}^2(\lambda_1\overline{\lambda_2} - \overline{\lambda_2}mc)^{-3}mp^r\lambda + p^r\lambda h_1(c)p^{\gamma-u} + ch_1(c)p^{\gamma-u}\ (p^{u/2}).$$

Dividing the right-hand side by $p^{\gamma-u}$, $h(p^r\lambda + c) = 0\ (p^{u/2})$ boils down to

$$\lambda(h_1(c)p^r - \eta_0\lambda_1\overline{\lambda_2}^2(\lambda_1\overline{\lambda_2} - \overline{\lambda_2}mc)^{-3}(mp^r/p^{\gamma-u})) + h_0(c)/p^{\gamma-u} + ch_1(c) = 0\ (p^{u/2-(\gamma-u)}).$$

Note that the coefficient attached to λ is co-prime to p, and consequently, λ is determined modulo $p^{u/2-(\gamma-u)}$. Combining, it follows that β_1 is determined modulo $p^{u/2-\nu_p(m)}$, and therefore,

$$\mathfrak{C} \ll p^{\gamma+u/2} \sum_{\substack{\beta(p^{u/2}) \\ h(\beta_1)=0(p^{u/2})}}^{*} 1 \ll p^{\gamma+u/2}p^{\nu_p(m)},$$

which is the first part of the lemma.

This completes the proof of the lemma when u is even. When u is odd, in (2.7), we decompose the $a_j's$ as $p^{(u+1)/2}\alpha_j + \beta_j, 1 \leqslant \alpha_j \leqslant p^{(u-1)/2}, 1 \leqslant \beta_j \leqslant p^{(u+1)/2}$ and proceed identically as above. This way we gain a $p^{1/2}$ factor in the α_1-sum but loose a factor of p in the β_1-sum since the linear sum α_1-sum in (2.13) will now only determine $h(\beta_1)$ modulo $p^{(u-1)/2}$. This will result in an extra factor of $p^{1/2}$ the final estimate as indicated in the statement of the lemma. □

Lemma 2.5. *With the notations of (2.6), we have*

$$\mathfrak{C}_{1,1} = \sum_{\substack{a_1,a_2(p) \\ \lambda_1\overline{a_1}-\lambda_2\overline{a_2}=m(p)}}^{*}\sum^{*} S(1,\overline{s_1a_1+t_1},p)\overline{S}(1,\overline{s_2a_2+t_2},p) \ll p^{3/2} + p^2\delta_{\begin{pmatrix} m=0(p) \\ t_1=t_2(p) \\ \lambda_1 s_1=\lambda_2 s_2(p)\end{pmatrix}}.$$

Proof. Consider the linear transformations

$$\delta_1 = \begin{pmatrix} 0 & 1 \\ s_1 & t_1 \end{pmatrix}, \delta_2 = \begin{pmatrix} 0 & 1 \\ s_2 & t_2 \end{pmatrix} \text{ and } \delta_3 = \begin{pmatrix} \lambda_2 & 0 \\ -m & \lambda_1 \end{pmatrix}.$$

Then, we can recast $\mathfrak{C}_{1,1}$ as

$$\mathfrak{C}_{1,1} = \sum_{a_1(p)}^{*} S(1,\delta_1(a_1),p)\overline{S}(1,\delta_2\delta_3(a_1),p).$$

Note that $\det(\delta_i) \neq 0\ (p)$. Hence,

$$\mathfrak{C}_{1,1} = \sum_{a_1(p)}^{*} S(1,a_1,p)\overline{S}(1,\delta_2\delta_3\delta_1^{-1}(a_1),p). \tag{2.16}$$

We are in position to apply the estimates from [5]. Propositions 3.3 and 3.4 amount to the following. Given $\delta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $ad - bc \neq 0\ (p)$, then

$$\sum_{\alpha(p)}^{*} S(1,\alpha;p)\overline{S}(1,\gamma(\alpha);p) \ll p^{3/2} + p^2\delta_{(a-d=b=c=0(p))}. \tag{2.17}$$

In our situation (2.16), we have

$$\delta_2\delta_3\delta_1^{-1} = s_1^{-1}\begin{pmatrix} mt_1 + \lambda_1 s_1 & -m \\ s_1t_2\lambda_1 - t_1(s_2\lambda_2 - t_2m) & s_2\lambda_2 - t_2m \end{pmatrix}.$$

The relation $a - d = b = c = 0\ (p)$ from the (2.17) translates into $m = 0\ (p), \lambda_1 s_1 = \lambda_2 s_2\ (p)$ and $t_1 = t_2\ (p)$. The lemma follows. □

3 | PROOF OF THEOREM 1.2

Here, q is a square-free number and $(q,b) = 1$. We are interested in

$$S = \sum_{n\in\mathcal{N}}\sum_{m\geqslant 1} \alpha_n\lambda(m)\tilde{\mathrm{Kl}}_3(mnb,q)V(m/M). \tag{3.1}$$

Without loss of generality, we can assume that the sum over n above is restricted to $(n,q) = 1$, for if $(n,q) = d$, the hyper-Kloosterman sum degenerates to

$$\tilde{\mathrm{Kl}}_3(mnb,q) = \frac{d}{q} \cdot \tilde{\mathrm{Kl}}_3(m(n/d)b\overline{d}, q/d),$$

with which one arrives at a sum similar to (3.1) with a smaller modulus q/d and a smaller n-sum with length N/d. Hence, it is enough to consider

$$S = \sideset{}{^*}\sum_{n \in \mathcal{N}} \sum_{m \geqslant 1} \alpha_n \lambda(m) \tilde{\mathrm{Kl}}_3(mnb,q) V(m/M),$$

where the '$*$' over the n-sum denotes $(n,q) = 1$. For simplicity, we set

$$K(m) = \tilde{\mathrm{Kl}}_3(mb,q).$$

We begin by separating the coefficients $\lambda(m)$ and $K(mn)$ using the delta symbol. Due to structural reasons, the sizes of the moduli appearing in the delta expansion play no essential role in our approach and only act as a set of auxiliary variables. Hence, we do not require any non-trivial delta symbol expansion and simply use the additive characters with large moduli. This simplifies many of the forthcoming calculations. This is not a new observation and was previously exploited in [1] in the context of the subconvexity problem for $GL(2)$.

Next, we note that a direct application of the delta symbol fails to beat the trivial bound at a certain diagonal contribution. To overcome this, we consider an amplified version of S that introduces more harmonics into the analysis. Let $L \geqslant 1$, which will be chosen later, and $\mathcal{L}$ be the set of primes in $[L, 2L]$ co-prime to q. Note that

$$\sum_{\ell \in \mathcal{L}} |\lambda(\ell)|^2 = \sum_{\substack{\ell \sim L, \ell \text{ prime} \\ (\ell,q)=1}} |\lambda(\ell)|^2 = \sum_{\substack{\ell \sim L \\ \ell \text{ prime}}} |\lambda(\ell)|^2 - \sum_{\substack{p|q \\ p \sim L}} |\lambda(p)|^2 \sim L + O(q^{\epsilon} L^{14/64}), \tag{3.2}$$

using the $GL(2)$ prime number theorem and the Kim–Sarnak bound for individual $GL(2)$ coefficients. Hence, using the Hecke relation

$$\lambda(\ell)\lambda(m) = \lambda(m\ell) + \lambda(m/\ell)\delta_{\ell|m},$$

and the asymptotic (3.2), we see that

$$|S| \ll |\tilde{S}| + O(MN/L), \tag{3.3}$$

where

$$\tilde{S} = \frac{1}{L} \sum_{\ell \in \mathcal{L}} \overline{\lambda(\ell)} \sideset{}{^*}\sum_{n \in \mathcal{N}} \sum_{m \geqslant 1} \alpha_n \lambda(m\ell) K(mn) V(m/M).$$

We have used the Ramanujan bound on average $\sum_{n \leqslant x} |\lambda(n)|^2 \ll x$ and the well-known Deligne's estimate $K(m) \ll 1$ for the last assertion. The rest of this section is devoted to the estimation of $\tilde{S}$. Let $\mathcal{C}$ be the set primes in $[C, 2C]$, with C such that

$$qC > 100ML.$$

Since there is no restriction on the upper bound for C, a suitable large C will ensure that $(c, q\ell) = 1$ for all $c \in \mathscr{C}, \ell \in \mathscr{L}$. Due to the above inequality, we can write $\tilde{S}$ as

$$\tilde{S} = \frac{1}{CL} \sum_{\ell \in \mathscr{L}} \overline{\lambda(\ell)} \sum_{c \in \mathscr{C}} \sideset{}{^*}\sum_{n \in \mathscr{N}} \alpha_n \sum_{\substack{m_1, m_2 \geqslant 1 \\ qc | m_1 - m_2 \ell}} \lambda(m_1) K(m_2 n) V(m_1/(M\ell)) V_1(m_2/M) \times e(q^\epsilon (m_1 - m_2\ell)/ML),$$

where V_1 is another smooth function compactly supported in $\mathbb{R}_{>0}$ such that $V_1(x) = 1, x \in \text{supp}(V)$. The artificial twist by $e(q^\epsilon(m_1 - m_2\ell)/ML)$ allows us keep the length of the dual sums in their generic range. This turns out to be crucial in certain counting arguments of the paper, especially when $M \gg q$. The additional restriction modulo q in $qc|(m_1 - m_2\ell)$ acts as a conductor lowering mechanism. Detecting the congruence condition using additive characters, we obtain

$$\tilde{S} = \frac{1}{qCL} \sideset{}{^*}\sum_{n \in \mathscr{N}} \alpha_n \sum_{\ell \in \mathscr{L}} \overline{\lambda(\ell)} \sum_{c \in \mathscr{C}} \frac{1}{c} \sum_{a(qc)} \sum_{m_1, m_2 \geqslant 1} \lambda(m_1) K(m_2 n) e(a(m_1 - m_2\ell)/qc) \times V(m_1/(M\ell)) V_1(m_2/M) e(q^\epsilon(m_1 - m_2\ell)/ML).$$

Since $(c, q) = 1$, we can split the above sum as

$$\tilde{S} = \sum_{d|q} S(d) + \mathcal{S}, \tag{3.4}$$

where

$$S(d) = \frac{1}{qCL} \sideset{}{^*}\sum_{n \in \mathscr{N}} \alpha_n \sum_{\ell \in \mathscr{L}} \overline{\lambda(\ell)} \sum_{c \in \mathscr{C}} \frac{1}{c} \sideset{}{^*}\sum_{a(dc)} \sum_{m_1, m_2 \geqslant 1} \lambda(m_1) K(m_2 n) e(a(m_1 - m_2\ell)/dc) \, V(m_1/(M\ell)) V_1(m_2/M) e(q^\epsilon(m_1 - m_2\ell)/ML), \tag{3.5}$$

and

$$\mathcal{S} = \frac{1}{qCL} \sideset{}{^*}\sum_{n \in \mathscr{N}} \alpha_n \sum_{\ell \in \mathscr{L}} \overline{\lambda(\ell)} \sum_{c \in \mathscr{C}} \frac{1}{c} \sum_{a(q)} \sum_{m_1, m_2 \geqslant 1} \lambda(m_1) K(m_2 n) e(a(m_1 - m_2\ell)/q) \, V(m_1/(M\ell)) V_1(m_2/M) e(q^\epsilon(m_1 - m_2\ell)/ML).$$

A trivial estimation of $\mathcal{S}$ yields

$$\mathcal{S} \ll M^2 N / C.$$

Hence, we can ignore the contribution $\mathcal{S}$ because C is allowed to be arbitrary large. The rest of the section is devoted to the estimation of $S(d), d|q$.

3.1 Dualisation

In the cuspidal case, the Voronoi summation transforms the m_1-sum in $S(d)$ into

$$\sum_{m_1 \geqslant 1} \lambda(m_1) e\left(\frac{am_1}{dc}\right) V(m_1/(M\ell)) e(q^\epsilon m_1/ML) = \frac{ML}{dc} \sum_{\tilde{m}_1 \geqslant 1} \lambda(\tilde{m}_1) e\left(\frac{\pm \bar{a}\tilde{m}_1}{dc}\right) I_1^{\pm}(\tilde{m}_1, c), \tag{3.6}$$

where $I_1^{\pm}(\tilde{m}_1, c) = (ML)^{-1} H^{\pm}(\tilde{m}_1/d^2c^2)$, $H^{\pm}$ as in Lemma 2.3. Note that in each case, $I_1^{\pm}(\tilde{m}_1, c)$ will be roughly of the form

$$I_1^{\pm}(\tilde{m}_1, c) \approx \text{(constant factors)} \int_{\mathbb{R}} V(y) e(q^\epsilon y) K\left(\frac{\sqrt{ML\tilde{m}_1 y}}{dc}\right) dy,$$

where $K(\cdots)$ is one of the Bessel functions appearing in Lemma 2.3. Since the order of these Bessel functions is fixed for us, $K(x)$ will oscillate like $e(x)$ (see [21], p. 206). Hence, by repeated integration by parts, we can conclude that $I_1^{\pm}(\tilde{m}_1, c)$ is negligibly small unless

$$\tilde{m}_1 \asymp q^\epsilon d^2 c^2/ML, \tag{3.7}$$

in which case the jth derivative is trivially bounded by

$$|\tilde{m}_1|^j \frac{\partial^j I_1^{\pm}(\tilde{m}_1, c)}{\partial \tilde{m}_1{}^j} \ll_{j,\epsilon} q^{j\epsilon}. \tag{3.8}$$

The Poisson summation transforms the m_2-sum in (3.5) into

$$\begin{aligned} &\sum_{m_2 \sim M} K(m_2 n) e\left(\frac{-am_2\ell}{dc}\right) V_1(m_2/M) e(-q^\epsilon m_2 \ell/ML) \\ &= \frac{M}{qc} \sum_{\alpha(qc)} K(\alpha n) e\left(\frac{-a\alpha\ell}{dc}\right) \sum_{\tilde{m}_2 \in \mathbb{Z}} e\left(\frac{-\tilde{m}_2\alpha}{qc}\right) I_2(\tilde{m}_2, c), \end{aligned}$$

where

$$I_2(\tilde{m}_2, c) = \int_{\mathbb{R}} V_1(x) e\left(\frac{q^\epsilon \ell}{L} - \frac{M\tilde{m}_2 x}{qc}\right) dx.$$

Again, from repeated integration by parts, it follows that $I_2(\tilde{m}_2, c)$ is negligible unless

$$|\tilde{m}_2| \asymp q^{1+\epsilon} c/M$$

and the jth derivative is bounded by

$$|\tilde{m}_2|^j \frac{\partial^j I_2(\tilde{m}_2, c)}{\partial \tilde{m}_2{}^j} \ll_{j,\epsilon} q^{j\epsilon}. \tag{3.9}$$

Combining the above two transformations, we see that $S(d)$ can be replaced by

$$S(d) = \frac{M^2}{q^2dC} \sum_{n\in\mathcal{N}}^{*} \alpha_n \sum_{\ell\in\mathcal{L}} \overline{\lambda(\ell)} \sum_{c\in\mathcal{C}} \frac{1}{c^3} \sum_{\tilde{m}_1 \asymp d^2c^2/ML} \sum_{|\tilde{m}_2|\asymp qc/M} \lambda(\tilde{m}_1)\mathfrak{C}(\cdots)J(\tilde{m}_1,\tilde{m}_2,c), \tag{3.10}$$

where

$$J(\tilde{m}_1,\tilde{m}_2,c) = I_1^{\pm}(\tilde{m}_1,c)I_2(\tilde{m}_2,c),$$

and

$$\mathfrak{C}(\cdots) = \sum_{a(dc)}^{*} \sum_{\alpha(qc)} K(\alpha n) e\left(-\frac{a\alpha\ell}{dc} - \frac{\tilde{m}_2\alpha}{qc} \pm \frac{\overline{a}\tilde{m}_1}{dc}\right). \tag{3.11}$$

Note that from (3.8) and (3.9), we have

$$|\tilde{m}_1|^{j_1}|\tilde{m}_2|^{j_2} \frac{\partial^{j_1}\partial^{j_2}J(\tilde{m}_1,\tilde{m}_2,c)}{\partial\tilde{m}_1{}^{j_1}\partial\tilde{m}_2{}^{j_2}} \ll_{j_1,j_2} q^{(j_1+j_2)\epsilon}. \tag{3.12}$$

Remark 2. In the case of Eisenstein coefficients, there is an additional '0th' term in the right-hand side of (3.6) which we now briefly show has a small contribution towards B. From Lemma 2.3, the main term will roughly be of the form

$$\frac{ML}{dc} I(c),$$

where $I(c)$ is an integral transform with $I(c) \ll 1$. Hence, if $S_0(d)$ denotes the contribution of the main term towards $S(d)$, then

$$S(d) \ll \frac{M^2}{q^2dC} \sum_{n\in\mathcal{N}}^{*} \alpha_n \sum_{\ell\in\mathcal{L}} \overline{\lambda(\ell)} \sum_{c\in\mathcal{C}} \frac{1}{c^3} \sum_{\tilde{m}_2 \ll qc/M} \tilde{\mathfrak{C}}(\cdots)I(c)I_2(\tilde{m}_2,c), \tag{3.13}$$

where $\tilde{\mathfrak{C}}(\cdots)$ is the simpler character sum

$$\tilde{\mathfrak{C}}(\cdots) = \sum_{a(dc)}^{*} \sum_{\alpha(qc)} K(\alpha n) e\left(-\frac{a\alpha\ell}{dc} - \frac{\tilde{m}_2\alpha}{qc}\right).$$

It can be easily shown that $\tilde{\mathfrak{C}}(\cdots) \ll qc$. Trivially estimating (3.13), we therefore obtain

$$S_0(d) \ll ML/(dC).$$

From the freedom of choosing C, it follows that $S_0(d)$ will have a negligible contribution.

Let us come back to the generic case (3.10). Dividing the $\tilde{m}_1$-sum into dyadic blocks $\tilde{m}_1 \sim M_1 \asymp d^2C^2/ML$ with the localising factors $W(\tilde{m}_1/M_1)$, we get

$$S(d) \ll \sup_{M_1 \asymp d^2C^2/ML} S(d, M_1), \tag{3.14}$$

where

$$S(d, M_1) = \frac{M^2}{q^2 dC} \sum_{n\in\mathcal{N}}^{*} \alpha_n \sum_{\ell\in\mathcal{L}} \overline{\lambda(\ell)} \sum_{c\in\mathcal{C}} \frac{1}{c^3} \sum_{\tilde{m}_1\in\mathbb{Z}} W(\tilde{m}_1/M_1) \sum_{|\tilde{m}_2|\asymp qC/M} \lambda(\tilde{m}_1)\mathfrak{C}(\cdots)J(\tilde{m}_1, \tilde{m}_2, c). \tag{3.15}$$

3.2 | Simplifying the character sum

Splitting the α (qc) sum (3.11) using the Chinese remainder theorem and executing the modulo c part, we obtain the congruence relation

$$a = -\tilde{m}_2\overline{\ell(q/d)}\ (c),$$

and we are left with

$$\mathfrak{C}(\cdots) = c \cdot e\left(\frac{\pm q\overline{d}^2\ell\overline{\tilde{m}_2}\tilde{m}_1}{c}\right) \sum_{a(d)}^{*} \sum_{\alpha(q)} K(c\alpha n) e\left(-\frac{a\alpha\ell}{d} - \frac{\tilde{m}_2\alpha}{q} \pm \frac{\overline{ca}\tilde{m}_1}{d}\right).$$

Substituting the definition

$$K(c\alpha n) = \frac{1}{q}\sum_{\beta(q)}^{*} e\left(\frac{\beta}{q}\right) \sum_{l(q)}^{*} e\left(\frac{lc\alpha nb + \overline{l\beta}}{q}\right)$$

and executing the α (q) sum, we obtain

$$l = \overline{cnb}(\tilde{m}_2 + a(q/d)\ell)\ (q).$$

Substituting we get

$$\mathfrak{C}(\cdots) = c \cdot e\left(\frac{\pm q\overline{d}^2\ell\overline{\tilde{m}_2}\tilde{m}_1}{c}\right) \sum_{a(d)}^{*} S(1, cnb\overline{(\tilde{m}_2 + a(q/d)\ell)}; q) e\left(\frac{\overline{ca}\tilde{m}_1}{d}\right). \tag{3.16}$$

Observe that $\mathfrak{C}(\cdots)$ is additive w.r.t. $\tilde{m}_1$ (c).

3.3 | Cauchy–Schwarz and Poisson

Applying Cauchy–Schwarz inequality to (3.15) keeping the $\tilde{m}_1$ sum outside and everything else inside the absolute value square, we see that

$$S(d, M_1) \ll \frac{M^2}{q^2 dC^3} \cdot dC/(ML)^{1/2} \cdot \Omega^{1/2}, \tag{3.17}$$

where

$$\Omega = \sum_{\tilde{m}_1 \in \mathbb{Z}} W(\tilde{m}_1/M_1)$$

$$\times \left| \sum_{n \in \mathcal{N}}^{*} \alpha_n \sum_{\ell \in \mathcal{L}} \overline{\lambda(\ell)} \sum_{c \in \mathcal{C}} \sum_{|\tilde{m}_2| \asymp dC/M} e\left(\frac{\pm q \overline{d}^2 \ell \overline{\tilde{m}_2} \tilde{m}_1}{c} \right) \mathfrak{C}_1(n, \tilde{m}_1, \tilde{m}_2, \ell, c) J(\tilde{m}_1, \tilde{m}_2, c) \right|^2,$$

where $\mathfrak{C}_1(\cdots)$ is $\mathfrak{C}(\cdots)$ in (3.16) without the first factor c. Opening the absolute value square, we obtain

$$\Omega = \sum_{\ell_1, \ell_2 \sim L} \lambda(\ell_1) \overline{\lambda(\ell_2)} \sum_{n_1, n_2 \in \mathcal{N}}^{*} \alpha_{n_1} \overline{\alpha_{n_2}} \sum_{c_1, c_2 \in \mathcal{C}} \sum_{|\tilde{m}_2|, |\tilde{m}_3| \asymp dC/M}$$

$$\sum_{\tilde{m}_1 \in \mathbb{Z}} e\left(\frac{q \overline{d}^2 \ell_1 \overline{\tilde{m}_2} \tilde{m}_1}{c_1} - \frac{q \overline{d}^2 \ell_2 \overline{\tilde{m}_3} \tilde{m}_1}{c_2} \right) \mathfrak{C}_1(n_1, \tilde{m}_1, \tilde{m}_2, \ell_1, c_1) \overline{\mathfrak{C}_1(n_2, \tilde{m}_1, \tilde{m}_3, \ell_2, c_2)}$$

$$\times J(\tilde{m}_1, \tilde{m}_2, c_1) \overline{J(\tilde{m}_1, \tilde{m}_3, c_2)} W(\tilde{m}_1/M_1). \tag{3.18}$$

A final application of the Poisson summation formula transforms the $\tilde{m}_1$-sum above into

$$\frac{M_1}{dc_1c_2} \sum_{k (pc_1c_2)} e\left(\frac{q \overline{d}^2 \ell_1 \overline{\tilde{m}_2} k}{c_1} - \frac{q \overline{d}^2 \ell_2 \overline{\tilde{m}_3} k}{c_2} \right) \mathfrak{C}_1(n_1, k, \tilde{m}_2, \ell_1, c_1) \overline{\mathfrak{C}_1(n_2, k, \tilde{m}_3, \ell_2, c_2)}$$

$$\times \sum_{\tilde{m}_4 \in \mathbb{Z}} e\left(\frac{-\tilde{m}_4 k}{dc_1c_2} \right) \mathcal{I}(\tilde{m}_2, \tilde{m}_3, \tilde{m}_4, c_1, c_2) \tag{3.19}$$

$$= \frac{M_1}{d} \sum_{\tilde{m}_4 \in \mathbb{Z}} \mathfrak{C}_2(\cdots) \cdot \mathcal{I}(\tilde{m}_2, \tilde{m}_3, \tilde{m}_4, c_1, c_2) \cdot \delta_{c_2 \ell_1 \overline{\tilde{m}_2} - c_1 \ell_2 \overline{\tilde{m}_3} = \overline{(q/d)} \tilde{m}_4 (c_1c_2)},$$

where

$$\mathcal{I}(\tilde{m}_2, \tilde{m}_3, \tilde{m}_4, c_1, c_2) = \int_{\mathbb{R}} W(x) J(M_1 x, \tilde{m}_2, c_1) \overline{J(M_1 x, \tilde{m}_3, c_2)} e(-M_1 \tilde{m}_4 x / dc_1c_2) dx \tag{3.20}$$

and

$$\mathfrak{C}_2(\cdots) = \sum_{k(d)} \mathfrak{C}_1(n_1, k, \tilde{m}_2, \ell_1, c_1) \overline{\mathfrak{C}_1(n_2, k, \tilde{m}_3, \ell_2, c_2)} e\left(\frac{-\overline{c_1c_2} \tilde{m}_4 k}{d} \right).$$

Equation (3.12) and repeated integration by parts in (3.20) allow us to truncate

$$|\tilde{m}_4| \ll dC^2/M_1.$$

Substituting the transformation (3.19) into (3.18), we obtain

$$\Omega = \frac{M_1}{d} \sum_{\ell_1,\ell_2 \sim L} \lambda(\ell_1)\overline{\lambda(\ell_2)} \sum_{n_1,n_2 \in \mathcal{N}}^{*} \alpha_{n_1}\overline{\alpha_{n_2}} \sum_{c_1,c_2 \in \mathcal{C}} \sum_{\tilde{m}_2,\tilde{m}_3 \asymp dC/M} \sum_{\substack{\tilde{m}_4 \ll dC^2/M_1 \\ c_2\ell_1\overline{\tilde{m}_2} - c_1\ell_2\overline{\tilde{m}_3} = \overline{(q/d)}\tilde{m}_4 (c_1c_2)}} \times \mathfrak{C}_2(\cdots) \cdot \mathcal{I}(\tilde{m}_2, \tilde{m}_3, \tilde{m}_4, c_1, c_2). \tag{3.21}$$

It remains to estimate the character sum $\mathfrak{C}_2(\cdots)$. Substituting the definition (3.16) and executing the k (d)-sum in $\mathfrak{C}_2(\cdots)$, we obtain

$$\mathfrak{C}_2(\cdots) = d \sum_{\substack{a_1,a_2(d) \\ c_2\overline{a_1} - c_1\overline{a_2} = \tilde{m}_4(d)}}^{*}\sum^{*} S(1, c_1n_1\overline{(\tilde{m}_2 + a_1(q/d)\ell_1)}; q)\overline{S}(1, c_2n_2\overline{(\tilde{m}_3 + a_2(q/d)\ell_2)}; q). \tag{3.22}$$

Lemma 3.1. *Let $\mathfrak{C}_2(\cdots)$ as in (3.22). Then*

$$\mathfrak{C}_2(\cdots) \ll qd^{3/2} \sum_{k|d} k^{1/2} \delta_{\begin{pmatrix} \tilde{m}_4 = 0(k) \\ n_1c_1\tilde{m}_3 = n_2c_2\tilde{m}_2(k) \\ n_1c_1^2\ell_2 = n_2c_2^2\ell_1(k)\end{pmatrix}}. \tag{3.23}$$

Proof. Since q is square-free, we can split the Kloosterman sums modulo q/d and d and get

$$\mathfrak{C}_2(\cdots) = d \cdot S(1, \overline{d}^2 c_1n_1b\overline{\tilde{m}_2}; q/d)\overline{S}(1, \overline{d}^2 c_2n_2b\overline{\tilde{m}_3}; q/d) \cdot \mathfrak{C}_3, \tag{3.24}$$

where

$$\mathfrak{C}_3 = \sum_{\substack{a_1,a_2(d) \\ c_2\overline{a_1} - c_1\overline{a_2} = \tilde{m}_4(d)}}^{*}\sum^{*} S(1, \overline{(q/d)}^2 c_1n_1b\overline{(\tilde{m}_2 + a_1(q/d)\ell_1)}; d)\overline{S}(1, \overline{(q/d)}^2 c_2n_2b\overline{(\tilde{m}_3 + a_2(q/d)\ell_2)}; d).$$

Suppose $d = d_1d_2 \cdots d_l$, where each d_i is prime. Then $\mathfrak{C}_3$ further factorises as

$$\mathfrak{C}_3 = \prod_{i=1}^{l} K_i, \tag{3.25}$$

where

$$K_i = \sum_{\substack{a_1,a_2(d_i) \\ c_2\overline{a_1} - c_1\overline{a_2} = \tilde{m}_4(d_i)}}^{*}\sum^{*} S(1, \overline{(q/d_i)}^2 c_1n_1b\overline{(\tilde{m}_2 + a_1(q/d)\ell_1)}; d_i) \times \overline{S}(1, \overline{(q/d_i)}^2 c_2n_2b\overline{(\tilde{m}_3 + a_2(q/d)\ell_2)}; d_i).$$

We apply the estimates from Lemma 2.5 with the parameters

$$(s_1, t_1) = ((q/d_i)^2(q/d)\overline{c_1n_1b}\ell_1, (q/d_i)^2\overline{c_1n_1b}\tilde{m}_2), \quad (s_2, t_2)$$
$$= ((q/d_i)^2(q/d)\overline{c_2n_2b}\ell_2, (q/d_i)^2\overline{c_2n_2b}\tilde{m}_3),$$
$$(\lambda_1, \lambda_2) = (c_2, c_1),$$

and $m = \tilde{m}_4$. The congruences $m = 0\,(p), t_1 = t_2\,(p)$ and $\lambda_1 s_1 = \lambda_2 s_2\,(p)$ then translate into $\tilde{m}_4 = 0\,(d_i), n_1 c_1 \tilde{m}_3 = n_2 c_2 \tilde{m}_2\,(d_i)$ and $n_1 c_1^2 \ell_1 = n_2 c_2^2 \ell_2\,(d_i)$, respectively. Hence, Lemma 2.5 gives

$$K_i \ll d_i^{3/2} \left(1 + d_i^{1/2} \delta_{\begin{pmatrix} \tilde{m}_4 = 0(d_i) \\ n_1 c_1 \tilde{m}_3 = n_2 c_2 \tilde{m}_2(d_i) \\ n_1 c_1^2 \ell_2 = n_2 c_2^2 \ell_1(d_i) \end{pmatrix}} \right).$$

Plugging in these estimates in (3.25), we obtain

$$\mathfrak{C}_3 \ll d^{3/2} \prod_{i=1}^{k} \left(1 + d_i^{1/2} \delta_{\begin{pmatrix} \tilde{m}_4 = 0(d_i) \\ n_1 c_1 \tilde{m}_3 = n_2 c_2 \tilde{m}_2(d_i) \\ n_1 c_1^2 \ell_2 = n_2 c_2^2 \ell_1(d_i) \end{pmatrix}} \right).$$

Since d_i's are pairwise co-prime, the congruences can be clubbed together to yield

$$\mathfrak{C}_3 \ll d^{3/2} \sum_{k|d} k^{1/2} \delta_{\begin{pmatrix} \tilde{m}_4 = 0(k) \\ n_1 c_1 \tilde{m}_3 = n_2 c_2 \tilde{m}_2(k) \\ n_1 c_1^2 \ell_2 = n_2 c_2^2 \ell_1(k) \end{pmatrix}}.$$

The lemma follows after plugging the last estimate into (3.24) and using the Weil's bound for the remaining two Kloosterman sums. □

We proceed to estimate the contribution of the zero and non-zero frequencies in Ω.

3.3.1 | The zero frequency

Assuming $(c_i, \ell_i) = 1$, when $\tilde{m}_4 = 0$, the congruence

$$c_2 \ell_1 \overline{\tilde{m}_2} - c_1 \ell_2 \overline{\tilde{m}_3} = \overline{(q/d)} \tilde{m}_4 (c_1 c_2)$$

in (3.21) implies

$$c_1 = c_2 = c \quad \text{and} \quad \ell_2 \tilde{m}_2 = \ell_1 \tilde{m}_3(c). \tag{3.26}$$

Therefore, from Lemma 3.1, we get

$$\mathfrak{C}_2(\cdots) \ll q d^{3/2} \sum_{k|d} k^{1/2} \delta_{\begin{pmatrix} n_1 \tilde{m}_3 = n_2 \tilde{m}_2(k) \\ n_1 \ell_2 = n_2 \ell_1(k) \end{pmatrix}} \tag{3.27}$$

in the case of zero frequency. Let Ω_0 denote the contribution of the zero frequency towards Ω (3.21). Then, from the above estimate for the character sum, we get

$$\Omega_0 \ll \frac{M_1}{d} \cdot q d^{3/2} \cdot \sum_{\ell_1, \ell_2 \sim L} |\lambda(\ell_1) \overline{\lambda(\ell_2)}| \sum_{n_1, n_2 \in \mathcal{N}} |\alpha_{n_1} \overline{\alpha_{n_2}}| \sum_{c \in \mathscr{C}} \sum_{k|d} k^{1/2} \sum_{\substack{\tilde{m}_2, \tilde{m}_3 \asymp dC/M \\ \ell_2 \tilde{m}_2 = \ell_1 \tilde{m}_3(c)}} \delta_{\begin{pmatrix} n_1 \tilde{m}_3 = n_2 \tilde{m}_2(k) \\ n_1 \ell_2 = n_2 \ell_1(k) \end{pmatrix}}.$$

Given n_1, n_2, ℓ_1, ℓ_2 and $\tilde{m}_2$, $\tilde{m}_3$ is determined modulo kc from the two congruence conditions. Hence, the number the $(\tilde{m}_2, \tilde{m}_3)$ pairs satisfying the congruence is at most $dC/M(1 + d/(kM))$.

Therefore,

$$\Omega_0 \ll \frac{M_1}{d} \cdot qd^{3/2} \cdot \sum_{\ell_1,\ell_2\sim L} |\lambda(\ell_1)\overline{\lambda(\ell_2)}| \sum_{c\in\mathscr{C}} \sum_{k|d} \sum_{\substack{n_1,n_2\in\mathscr{N} \\ n_1\ell_1=n_2\ell_2(k)}} k^{1/2}(dC/M)(1+d/(kM)). \tag{3.28}$$

Before proceeding further, we use the inequality $|\lambda(\ell_1)\overline{\lambda(\ell_2)}| \ll |\lambda(\ell_1)|^2 + |\lambda(\ell_2)|^2$, and due to symmetry, we consider the contribution of first term only. Now given (n_1, ℓ_1), there are at most $(1 + NL/k)$ many (n_2, ℓ_2) pairs satisfying the congruence in (3.28). Hence,

$$\begin{aligned}\Omega_0 &\ll \frac{M_1}{d} \cdot qd^{3/2} \cdot \sum_{\ell_1\sim L} |\lambda(\ell_1)|^2 \sum_{c\in\mathscr{C}} \sum_{k|d} \sum_{n_1\in\mathscr{N}} k^{1/2}(dC/M)(1+d/(kM))(1+NL/k) \\ &\ll \frac{M_1}{d} \cdot qd^{3/2} \cdot LCN(dC/M) \sum_{k|d} k^{1/2}(1+d/(kM))(1+NL/k) \\ &\ll \frac{M_1}{d} \cdot qd^{3/2} \cdot LCN(dC/M)\Big(d^{1/2} + (d/M) + NL + (d/M)NL\Big).\end{aligned} \tag{3.29}$$

Note that the last three terms inside the parenthesis of the last line are dominated by $NL(1 + d/M)$. Substituting the upper bound $M_1 \ll d^2C^2/ML$, we then obtain

$$\Omega_0 \ll qd^4NC^4/M^2 + qd^{7/2}N^2LC^4(1+d/M)/M^2. \tag{3.30}$$

3.3.2 | Non-zero frequencies

Let $\Omega_{\neq 0}$ denote the contribution of the non-zero frequencies $\tilde{m}_4 \neq 0$ towards Ω (3.21). We use the estimate

$$\mathfrak{C}_2(\cdots) \ll qd^{3/2} \sum_{k|d} k^{1/2}\delta_{(\tilde{m}_4=0(k))}$$

from Lemma 3.1 in this case. With this bound in (3.21), we get

$$\Omega_{\neq 0} \ll \frac{M_1}{d} \cdot qd^{3/2} \cdot \sum_{\ell_1,\ell_2\sim L} |\lambda(\ell_1)\overline{\lambda(\ell_2)}| \sum_{n_1,n_2\in\mathscr{N}} \sum_{c_1,c_2\in\mathscr{C}} \sum_{k|d} \sum_{\substack{\tilde{m}_2,\tilde{m}_3\asymp dC/M \\ c_2\ell_1\overline{\tilde{m}_2}-c_1\ell_2\overline{\tilde{m}_3}=\overline{(q/d)}\tilde{m}_4(c_1c_2)}} \sum_{\tilde{m}_4\ll dC^2/M_1} k^{1/2}\delta_{k|\tilde{m}_4}.$$

We write $\tilde{m}_4 = k\lambda, \lambda \ll dC^2/(M_1k), \lambda \neq 0$ and rewrite the above as

$$\begin{aligned}\Omega_{\neq 0} \ll \frac{M_1}{d} \cdot qd^{3/2} \cdot &\sum_{\ell_1,\ell_2\sim L} |\lambda(\ell_1)\overline{\lambda(\ell_2)}| \sum_{n_1,n_2\in\mathscr{N}} \sum_{c_1,c_2\in\mathscr{C}} \sum_{k|d} k^{1/2} \sum_{\lambda\ll dC^2/(M_1k)} \\ \times &\sum_{\substack{\tilde{m}_2\asymp dC/M \\ c_2\ell_1\overline{\tilde{m}_2}-c_1\ell_2\overline{\tilde{m}_3}=\overline{(q/d)}k\lambda(c_1c_2)}} \sum_{\tilde{m}_3\asymp dC/M} 1.\end{aligned} \tag{3.31}$$

The number of pairs $(\tilde{m}_2, \tilde{m}_3)$ satisfying the congruence modulo c_1c_2 in (3.31) is at most $(c_2\ell_1, k\lambda)(c_1\ell_2, k\lambda)(1 + d/M)^2$. Recall that $(c_i, q) = (\ell_j, q) = 1$ and consequently $(c_il_j, k) = 1$.

Hence,

$$\Omega_{\neq 0} \ll \frac{M_1}{d} \cdot qd^{3/2}(1+d/M)^2 \cdot \sum_{\ell_1,\ell_2 \sim L} |\lambda(\ell_1)\overline{\lambda(\ell_2)}| \sum_{n_1,n_2 \in \mathcal{N}} \sum_{k|d} k^{1/2} \sum_{\lambda \ll dC^2/(M_1 k)} (\ell_1,\lambda)(\ell_2,\lambda) \sum_{c_1,c_2 \in \mathcal{C}} (c_2,\lambda)(c_1,\lambda).$$

We next execute the (c_1, c_2)-sum with the bound C^2, the λ-sum with bound the $(\ell_1,\ell_2)(dC^2/M_1 k)$ and then the (n_1, n_2)-sum with the bound N^2. We arrive at

$$\begin{aligned}\Omega_{\neq 0} &\ll \frac{M_1}{d} \cdot qd^{3/2}(1+d/M)^2 N^2 C^2 (dC^2/M_1) \cdot \sum_{\ell_1,\ell_2 \sim L} |\lambda(\ell_1)\overline{\lambda(\ell_2)}|(\ell_1,\ell_2) \\ &\ll \frac{M_1}{d} \cdot qd^{3/2}(1+d/M)^2 N^2 C^2 (dC^2/M_1)\left(L \sum_{\ell \sim L} |\lambda(\ell)|^2 + \sum_{\ell_1,\ell_2 \sim L} |\overline{\lambda(\ell_1)}\lambda(\ell_2)| \right) \\ &\ll qd^{3/2} C^4 N^2 L^2 (1+d/M)^2.\end{aligned} \tag{3.32}$$

From (3.30) and (3.32), we get

$$\begin{aligned}\Omega = \Omega_0 + \Omega_{\neq 0} &\ll qd^4 NC^4/M^2 + qd^{3/2}C^4N^2L^2(1+d/M)^2 + qd^{7/2}N^2LC^4(1+d/M)/M^2 \\ &= qd^4NC^4/M^2 + qd^{7/2}C^4N^2L^2(1+M/d)^2/M^2 + qd^{9/2}N^2LC^4(1+M/d)/M^3.\end{aligned} \tag{3.33}$$

3.4 | Optimal choice for L

Substituting the last estimate into (3.17), we arrive at

$$S(d, M_1) \ll \frac{d^2 M^{1/2} N^{1/2}}{q^{3/2} L^{1/2}} + \frac{d^{7/4} M^{1/2} N L^{1/2}}{q^{3/2}}(1+M/d) + \frac{d^{9/4} N}{q^{3/2}}(1+M/d)^{1/2}.$$

Therefore, from (3.14) and (3.4), it follows

$$\tilde{S} \ll \frac{q^{1/2} M^{1/2} N^{1/2}}{L^{1/2}} + q^{1/4} M^{1/2} N L^{1/2}(1+M/q) + q^{3/4} N (1+M/q)^{1/2}.$$

Equating the first two terms, we obtain

$$L = q^{1/4} N^{-1/2} (1+M/q)^{-1}.$$

Note that this choice makes sense because the right-hand side is $\gg 1$ due to the assumption $N \leqslant q^{1/2}(1+M/q)^{-2}$ in Theorem 1.2. With the above choice, we obtain

$$\tilde{S} \ll M^{1/2} N^{3/4} q^{3/8} (1+M/q)^{1/2} + N q^{3/4} (1+M/q)^{1/2}.$$

Substituting the above in (3.3), we finally obtain

$$S \ll M^{1/2} N^{3/4} q^{3/8} (1+M/q)^{1/2} + M N^{3/2} q^{-1/4} (1+M/q) + N q^{3/4} (1+M/q)^{1/2}.$$

4 | PROOF OF THEOREM 1.3

Here, $q = p^\gamma, \gamma \geqslant 2$ and $p > 2$. We proceed slightly differently in this case. Instead of using the entire modulus q for the conductor lowering mechanism, we only use a part p^r, where $r < q$ is chosen optimally later. This serves two purposes: it simplifies certain counting arguments arising from the character sum estimates, and more importantly, it introduces more terms in the 'diagonal' while having a lesser impact in the off-diagonals as compared to the case of amplification.

Note that we can assume $(n, p) = 1$ since otherwise the trace function vanishes. As earlier, let $\mathscr{C}$ be the set primes in $[C, 2C]$, with C such that

$$p^r C > 100M. \tag{4.1}$$

We choose a large C such that $(c, q) = 1$ for all $c \in \mathscr{C}$. Due to the above relation, we can recast S as

$$S = \frac{1}{C}\sum_{c\in\mathscr{C}}\sideset{}{^*}\sum_{n\in\mathscr{N}}\alpha_n \sum_{\substack{m_1\sim M\\ m_2\sim M\\ p^r c|(m_1-m_2)}}\sum \lambda(m_1)K(m_2 n)V(m_1/M)V_1(m_2/M)e(q^\epsilon(m_1-m_2)/M).$$

Detecting the congruence condition using additive characters, we obtain

$$S = \frac{1}{p^r C}\sideset{}{^*}\sum_{n\in\mathscr{N}}\alpha_n \sum_{c\in\mathscr{C}}\frac{1}{c}\sum_{a(p^r c)}\sum_{\substack{m_1\sim M\\ m_2\sim M}}\sum \lambda(m_1)K(m_2 n)e(a(m_1-m_2)/p^r c)$$
$$V(m_1/M)V_1(m_2/M)e(q^\epsilon(m_1-m_2)/M).$$

Breaking the a $(p^r c)$ sum into Ramanujan sums, we obtain the decomposition

$$S = \sum_{0\leqslant k\leqslant r} S(k) + \mathcal{S},$$

where

$$S(k) = \frac{1}{p^r C}\sideset{}{^*}\sum_{n\in\mathscr{N}}\alpha_n \sum_{c\in\mathscr{C}}\frac{1}{c}\sideset{}{^*}\sum_{a(p^{r-k} c)}\sum_{\substack{m_1\sim M\\ m_2\sim M}}\sum \lambda(m_1)K(m_2 n)e(a(m_1-m_2)/p^{r-k} c)$$
$$V(m_1/M)V_1(m_2/M)e(q^\epsilon(m_1-m_2)/M), \tag{4.2}$$

and

$$\mathcal{S} = \frac{1}{p^r C}\sideset{}{^*}\sum_{n\in\mathscr{N}}\alpha_n \sum_{c\in\mathscr{C}}\frac{1}{c}\sum_{a(p^{r-k})}\sum_{\substack{m_1\sim M\\ m_2\sim M}}\sum \lambda(m_1)K(m_2 n)e(a(m_1-m_2)/p^{r-k})$$
$$V(m_1/M)V_1(m_2/M)e(q^\epsilon(m_1-m_2)/M).$$

Note that a trivial estimation yields

$$\mathcal{S} \ll M^2 N/C,$$

and therefore can ignored since C is allowed to be arbitrary large. The rest of paper is devoted to the estimation of $S(k), 0 \leqslant k \leqslant r$.

4.1 | Dualisation

Arguing similarly as in Remark 2, we can assume that we are in the cuspidal case. The Voronoi summation transforms the m_1-sum in (4.2) into

$$\sum_{m_1 \geqslant 1} \lambda(m_1) e\left(\frac{am_1}{p^{r-k}c}\right) V(m_1/M) e(q^\epsilon m_1/M) = \frac{M}{p^{r-k}c} \sum_{\tilde{m}_1 \geqslant 1} \lambda(\tilde{m}_1) e\left(\frac{\pm \bar{a}\tilde{m}_1}{p^{r-k}c}\right) I_1^{\pm}(\tilde{m}_1, c),$$

where $I_1^{\pm}(\tilde{m}_1, c) = M^{-1} H^{\pm}(\tilde{m}_1/p^{2(r-k)}c^2)$, $H^{\pm}$ as in Lemma 2.3. Due to the same reasons as in (3.7), one can truncate $\tilde{m}_1$-sum (up to a negligible error) to $\tilde{m}_1 \asymp p^{2(r-k)+2\epsilon} C^2/M$.

With the application of the Poisson summation formula, the m_2-sum in (4.2) becomes

$$\sum_{m_2 \geqslant 1} K(m_2 n) e\left(\frac{-am_2}{p^{r-k}c}\right) V_1(m_2/M) = \frac{M}{p^\gamma c} \sum_{\alpha(p^\gamma c)} K(\alpha n) e\left(\frac{-a\alpha}{p^{r-k}c}\right) \sum_{\tilde{m}_2 \ll p^\gamma c/M} e\left(\frac{-\tilde{m}_2 \alpha}{p^\gamma c}\right) I_2(\tilde{m}_2, c),$$

where

$$I_2(\tilde{m}_2, c) = \int_{\mathbb{R}} V_1(x) e(q^\epsilon x - M\tilde{m}_2 x/p^\gamma c) dx.$$

One can again restrict the $\tilde{m}_2$-sum to $\tilde{m}_2 \asymp p^{\gamma+\epsilon} C/M$.

Combining the above two transformations, we see that $S(k)$ can be replaced by

$$S(k) = \frac{M^2}{p^{\gamma+2r-k} C} \sum_{n \in \mathcal{N}}^{*} \alpha_n \sum_{c \in \mathcal{C}} \frac{1}{c^3} \sum_{\tilde{m}_1 \asymp p^{2(r-k)} C^2/M} \sum_{\tilde{m}_2 \asymp p^\gamma C/M} \lambda(\tilde{m}_1) \mathfrak{C}(\cdots) J(\tilde{m}_1, \tilde{m}_2, c), \tag{4.3}$$

where

$$J(\tilde{m}_1, \tilde{m}_2, c) = I_1^{\pm}(\tilde{m}_1, c) I_2(\tilde{m}_2, c),$$

and

$$\mathfrak{C}(\cdots) = \sum_{a(p^{r-k}c)}^{*} \sum_{\alpha(p^\gamma c)} K(\alpha n) e\left(-\frac{a\alpha}{p^{r-k}c} - \frac{\tilde{m}_2 \alpha}{p^\gamma c} \pm \frac{\bar{a}\tilde{m}_1}{p^{r-k}c}\right). \tag{4.4}$$

As in (3.12), $J(\tilde{m}_1, \tilde{m}_2, c)$ satisfies

$$|\tilde{m}_1|^{j_1} |\tilde{m}_2|^{j_2} \frac{\partial^{j_1} \partial^{j_2} J(\tilde{m}_1, \tilde{m}_2, c)}{\partial \tilde{m}_1{}^{j_1} \partial \tilde{m}_2{}^{j_2}} \ll_{j_1, j_2, \epsilon} p^{(j_1+j_2)\epsilon\gamma}. \tag{4.5}$$

Dividing the $\tilde{m}_1$-sum in (4.3) into dyadic blocks $\tilde{m}_1 \sim M_1 \asymp p^{2(r-k)} C^2/M$ and inserting localising factor $W(\tilde{m}_1/M_1)$, we get

$$S(k) \ll \sup_{M_1 \asymp p^{2(r-k)} C^2/M} S(k, M_1), \tag{4.6}$$

where

$$S(k,M_1)=\frac{M^2}{p^{\gamma+2r-k}C}\sum_{n\in\mathcal{N}}^{*}\alpha_n\sum_{c\in\mathscr{C}}\frac{1}{c^3}\sum_{\tilde{m}_1\geqslant 1}W(\tilde{m}_1/M_1)\sum_{\tilde{m}_2\asymp p^\gamma C/M}\lambda(\tilde{m}_1)\mathfrak{C}(\cdots)J(\tilde{m}_1,\tilde{m}_2,c). \quad (4.7)$$

4.2 | Simplifying the character sum

Splitting the $\alpha\ (pc)$ sum in (4.4) using the Chinese remainder theorem and executing the modulo c part, we obtain the congruence relation

$$a=-\overline{p}^{\gamma-r+k}\tilde{m}_2\ (c'),$$

and we are left with

$$\mathfrak{C}(\cdots)=ce\left(\frac{\pm\overline{p}^{2(r-k)}p^\gamma\overline{\tilde{m}_2}\tilde{m}_1}{c}\right)\sum_{a(p^{r-k})}^{*}\sum_{\alpha(p^\gamma)}K(c\alpha n)e\left(-\frac{a\alpha}{p^{r-k}}-\frac{\tilde{m}_2\alpha}{p^\gamma}\pm\frac{\overline{ca}\tilde{m}_1}{p^{r-k}}\right).$$

Substituting the definition

$$K(c\alpha n)=\frac{1}{p^\gamma}\sum_{\beta(p^\gamma)}^{*}e\left(\frac{\beta}{p^\gamma}\right)\sum_{l(p^\gamma)}^{*}e\left(\frac{lc\alpha nb+\overline{l}\beta}{p^\gamma}\right)$$

and executing the $\alpha\ (p^\gamma)$ sum we obtain

$$l=\overline{cnb}(\tilde{m}_2+p^{\gamma-r+k}a)\ (p^\gamma).$$

Substituting we get

$$\mathfrak{C}(\cdots)=c\cdot e\left(\frac{\pm\overline{p}^{2(r-k)}p^\gamma\overline{\tilde{m}_2}\tilde{m}_1}{c}\right)\sum_{a(p^{r-k})}^{*}S(1,cnb\overline{(\tilde{m}_2+p^{\gamma-r+k}a)};p^\gamma)e\left(\frac{\overline{ca}\tilde{m}_1}{p^{r-k}}\right). \quad (4.8)$$

4.3 | Cauchy–Schwarz and Poisson

Applying Cauchy–Schwarz inequality to (4.7) keeping the $\tilde{m}_1$ sum outside and everything else inside the absolute value square, we arrive at

$$S(k,M_1)\ll\frac{M^2}{p^{\gamma+2r-k}C^3}\cdot p^{r-k}C/M^{1/2}\cdot\Omega^{1/2}, \quad (4.9)$$

where

$$\Omega=\sum_{\tilde{m}_1\in\mathbb{Z}}W(\tilde{m}_1/M_1)\left|\sum_{n\in\mathcal{N}}^{*}\alpha_n\sum_{c\in\mathscr{C}}\sum_{\tilde{m}_2\asymp p^\gamma C/M}e\left(\frac{\pm\overline{p}^{2(r-k)}p^\gamma\overline{\tilde{m}_2}\tilde{m}_1}{c}\right)\mathfrak{C}_1(n,c,\tilde{m}_1,\tilde{m}_2)J(\tilde{m}_1,\tilde{m}_2,c)\right|^2,$$

where $\mathfrak{C}_1(\cdots)$ is $\mathfrak{C}(\cdots)$ in (4.8) without the first factor c. Opening the absolute value square, we get

$$\Omega = \sum^{*}_{n_1,n_2\in\mathcal{N}} \alpha_{n_1}\overline{\alpha_{n_2}} \sum_{c_1,c_2\in\mathcal{C}} \sum_{\tilde{m}_2,\tilde{m}_3\asymp p^\gamma C/M}$$

$$\sum_{\tilde{m}_1\in\mathbb{Z}} e\left(\frac{\overline{p}^{2(r-k)}p^\gamma\overline{\tilde{m}_2}\tilde{m}_1}{c_1} - \frac{\overline{p}^{2(r-k)}p^\gamma\overline{\tilde{m}_3}\tilde{m}_1}{c_2}\right)\mathfrak{C}_1(n_1,c_1,\tilde{m}_1,\tilde{m}_2)\overline{\mathfrak{C}_1(n_2,c_2,\tilde{m}_1,\tilde{m}_3)}$$

$$\times J(\tilde{m}_1,\tilde{m}_2,c_1)\overline{J(\tilde{m}_1,\tilde{m}_3,c_2)}W(\tilde{m}_1/M_1). \tag{4.10}$$

A final application of the Poisson summation formula transforms the $\tilde{m}_1$ sum into

$$\frac{M_1}{p^{r-k}c_1c_2}\sum_{\beta(p^{r-k}c_1c_2)} e\left(\frac{\overline{p}^{2(r-k)}p^\gamma\overline{\tilde{m}_2}\beta}{c_1} - \frac{\overline{p}^{2(r-k)}p^\gamma\overline{\tilde{m}_3}\beta}{c_2}\right)\mathfrak{C}_1(n_1,c_1,\beta,\tilde{m}_2)\overline{\mathfrak{C}_1(n_2,c_2,\beta,\tilde{m}_3)}$$

$$\times\sum_{\tilde{m}_4\in\mathbb{Z}} e\left(\frac{-\tilde{m}_4\beta}{p^{r-k}c_1c_2}\right)\mathcal{J}(\tilde{m}_2,\tilde{m}_3,\tilde{m}_4,c_1,c_2)$$

$$= \frac{M_1}{p^{r-k}}\sum_{\tilde{m}_4\in\mathbb{Z}} \mathfrak{C}_2(\cdots)\cdot\mathcal{J}(\tilde{m}_2,\tilde{m}_3,\tilde{m}_4,c_1,c_2)\cdot\delta_{c_2\overline{\tilde{m}_2}-c_1\overline{\tilde{m}_3}=\overline{p}^{2(\gamma-r+k)}\tilde{m}_4(c_1'c_2')}, \tag{4.11}$$

where

$$\mathcal{J}(\tilde{m}_2,\tilde{m}_3,\tilde{m}_4,c_1,c_2) = \int_{\mathbb{R}} W(x)J(M_1x,\tilde{m}_2,c_1)\overline{J(M_1x,\tilde{m}_3,c_2)}e(-M_1\tilde{m}_4x/(p^{r-k}c_1c_2))dx \tag{4.12}$$

and

$$\mathfrak{C}_2(\cdots) = \sum_{\beta(p^{r-k})}\mathfrak{C}_1(n_1,c_1,\beta,\tilde{m}_2)\overline{\mathfrak{C}_1(n_2,c_2,\beta,\tilde{m}_3)}e\left(\frac{-\overline{c_1c_2}\tilde{m}_4\beta}{p^{r-k}}\right).$$

Due to (4.5) and repeated integration by parts, (4.12) is negligibly small unless

$$\tilde{m}_4 \ll p^{r-k}C^2/M_1 \ll M/p^{r-k}.$$

Substituting (4.11) in place of the $\tilde{m}_1$-sum in (4.10), we obtain

$$\Omega = \frac{M_1}{p^{r-k}}\sum^{*}_{n_1,n_2\in\mathcal{N}}\alpha_{n_1}\overline{\alpha_{n_2}}\sum_{c_1,c_2\in\mathcal{C}}\sum_{\substack{\tilde{m}_2,\tilde{m}_3\asymp p^\gamma C/M\\ c_2\overline{\tilde{m}_2}-c_1\overline{\tilde{m}_3}=\overline{p}^{(\gamma-r+k)}\tilde{m}_4(c_1c_2)}}\sum_{\tilde{m}_4\ll M/p^{r-k}}\mathfrak{C}_2(\cdots)\cdot\mathcal{J}(\tilde{m}_2,\tilde{m}_3,\tilde{m}_4,c_1,c_2). \tag{4.13}$$

It remains to estimate $\mathfrak{C}_2$. Substituting the definition (4.8) and executing the $\beta(p^\gamma)$ sum, we obtain

$$\mathfrak{C}_2(\cdots) = p^{r-k}\sum^{*}\sum^{*}_{\substack{a_1,a_2(p^{r-k})\\ c_2\overline{a_1}-c_1\overline{a_2}=m_4(p^{r-k})}} S(1,c_1n_1b\overline{(\tilde{m}_2+p^{\gamma-r+k}a_1)};p^\gamma)\overline{S}(1,c_2n_2b\overline{(\tilde{m}_3+p^{\gamma-r+k}a_2)};p^\gamma). \tag{4.14}$$

We proceed for estimating the contribution of the zero and the non-zero frequencies towards (4.13).

4.4 | The zero frequency $\tilde{m}_4 = 0$

Note that from the congruence condition in (4.13), $\tilde{m}_4 = 0$ implies $c_1 = c_2 = c$ and $\tilde{m}_3 = \tilde{m}_2$ (c). We write $\tilde{m}_3 = \tilde{m}_2 + c\lambda$, $\lambda \ll p^\gamma/M$.

Case 1: $n_1 \neq n_2$ or $\lambda \neq 0$

In this case, the trivial estimation of (4.13) turns out to be worse than the non-diagonal contributions in the sub-Weyl range $M \ll q^{2/3}$. Fortunately, we can overcome this by exploiting the extra cancellations in the long $\tilde{m}_2(\asymp p^\gamma C/M)$-sum. Let A_0 denote the contribution of the case under consideration towards (4.13). Then

$$A_0 = \frac{M_1}{p^{r-k}} \sideset{}{^*}\sum_{n_1,n_2 \in \mathcal{N}} \alpha_{n_1}\overline{\alpha_{n_2}} \sum_{c\in\mathscr{C}} \sum_{\lambda \ll p^\gamma/M} \sum_{\tilde{m}_2 \asymp p^\gamma C/M} \mathfrak{C}_2(\cdots)\cdot \mathcal{J}(\tilde{m}_2, \tilde{m}_2 + c\lambda, 0, c, c), \tag{4.15}$$

where from (4.14),

$$\mathfrak{C}_2(\cdots) = p^{r-k} \sideset{}{^*}\sum_{a(p^{r-k})} S(1, cn_1\overline{(\tilde{m}_2 + p^{\gamma-r+k}a)}; p^\gamma)\overline{S}(1, cn_2\overline{(c\lambda + \tilde{m}_2 + p^{\gamma-r+k}a)}; p^\gamma).$$

We apply Poisson summation on the $\tilde{m}_2$-sum and observe that only zero frequency survives since the conductor is p^γ, whereas the length of the $\tilde{m}_2$-sum is $p^\gamma C/M \ggg p^\gamma$ when C is suitable large. Hence, the $\tilde{m}_2$-sum in (4.15) becomes

$$\sum_{\tilde{m}_2 \asymp p^\gamma C/M} \mathfrak{C}_2(\cdots)\cdot \mathcal{J}(\tilde{m}_2, \tilde{m}_2 + c\lambda, 0, c, c)$$
$$= p^{r-k}\cdot \frac{C}{M} \sideset{}{^*}\sum_{a(p^{r-k})} \sideset{}{^*}\sum_{\alpha(p^\gamma)} S(1, cn_1 b\overline{(\alpha + p^{\gamma-r+k}a)}; p^\gamma)\overline{S}(1, cn_2 b\overline{(c\lambda + \alpha + p^{\gamma-r+k}a)}; p^\gamma)\cdot \mathcal{I}(\cdots), \tag{4.16}$$

where

$$\mathcal{I}(\cdots) = \int_{x\sim 1} \mathcal{J}((p^\gamma C/M)x, (p^\gamma C/M)x + c\lambda, 0, c, c)dx.$$

After the change of variables $cn_1 b\overline{(\alpha + p^{\gamma-r+k}a)} \mapsto \alpha$, the right-hand side of (4.16) then becomes

$$\frac{p^{2(r-k)}C}{M} \sideset{}{^*}\sum_{\alpha(p^\gamma)} S(1, \alpha; p^\gamma)\overline{S}(1, \overline{n_1}n_2\alpha\overline{(\alpha\overline{n_1}\lambda + 1)}; p^\gamma)\cdot \mathcal{I}(\cdots) \tag{4.17}$$

and therefore,

$$\sum_{\tilde{m}_2 \asymp p^\gamma C/M} \mathfrak{C}_2(\cdots) \cdot \mathcal{I}(\tilde{m}_2, \tilde{m}_2 + c\lambda, 0, c, c) \ll \frac{p^{2(r-k)}C}{M} \left| \sideset{}{^*}\sum_{\alpha(p^\gamma)} S(1, \alpha; p^\gamma)\overline{S}(1, \overline{n_1} n_2 \alpha \overline{(\alpha \overline{n_1} \lambda + 1)}; p^\gamma) \right|. \tag{4.18}$$

An estimate evaluation of the character sum above can be obtained by following the proof of Lemma 2.4. However, this sum has been already studied in [5] and we quote them directly for simplicity.

Lemma 4.1 (R. Dabrowski and B. Fisher). *For $a \in \mathbb{Z}_p^\times, b \in \mathbb{Z}_p$ and $\gamma \geqslant 1$,*

$$\sideset{}{^*}\sum_{x(p^\gamma)} S(1, x; p^\gamma)\overline{S}(1, ax\overline{(bx+1)}; p^\gamma) \ll p^{3\gamma/2} p^{(\min\{\gamma, \nu_p(a-1), \nu_p(b)\})/2}. \tag{4.19}$$

This is the summary of their Theorem 3.2, Proposition 3.3 and Proposition 3.4, in case of the particular character sum in (4.19). Plugging this estimate in (4.18), we obtain

$$\sum_{\tilde{m}_2 \asymp p^\gamma C/M} \mathfrak{C}_2(\cdots) \cdot \mathcal{I}(\tilde{m}_2, \tilde{m}_2 + c\lambda, 0, c, c) \ll \frac{p^{3\gamma/2+2(r-k)}C}{M} \cdot p^{(\min\{\gamma, \nu_p(n_1-n_2), \nu_p(\lambda)\})/2}.$$

Consequently, (4.15) can be bounded by

$$\begin{aligned} A_0 &\ll \frac{M_1}{p^{r-k}} \cdot \frac{p^{3\gamma/2+2(r-k)}C}{M} \sum_{c \in \mathcal{C}} \sum_{n_1, n_2 \in \mathcal{N}} \sum_{\substack{\lambda \ll p^\gamma/M \\ (n_1-n_2, \lambda) \neq (0,0)}} p^{(\min\{\gamma, \nu_p(n_1-n_2), \nu_p(\lambda)\})/2} \\ &\ll \frac{M_1}{p^{r-k}} \cdot \frac{p^{3\gamma/2+2(r-k)}C}{M} \cdot C \cdot (N^2 p^\gamma/M) \\ &\ll \frac{p^{5\gamma/2+3(r-k)}C^4N^2}{M^3}. \end{aligned} \tag{4.20}$$

Case 2: $n_1 = n_2$ and $\tilde{m}_2 = \tilde{m}_3$

In this case, we use the trivial estimate

$$\mathfrak{C}_2(\cdots) = p^{r-k} \sideset{}{^*}\sum_{a(p^{r-k})} S(1, cn_1\overline{(\tilde{m}_2 + p^{\gamma-r+k}a)}; p^\gamma)\overline{S}(1, cn_2\overline{(c\lambda + \tilde{m}_2 + p^{\gamma-r+k}a)}; p^\gamma) \ll p^{\gamma+2(r-k)}.$$

So, if B_0 denotes the contribution of this case towards (4.13), then

$$B_0 \ll \frac{M_1}{p^{r-k}} \sum_{n_1 \in \mathcal{N}} \sum_{c \in \mathcal{C}} \sum_{\tilde{m}_2 \ll p^\gamma C/M} p^{\gamma+2(r-k)} \ll \frac{M_1}{p^{r-k}} \cdot NC \cdot \frac{p^\gamma C}{M} \cdot p^{\gamma+2(r-k)} \ll \frac{p^{2\gamma+3(r-k)}C^4N}{M^2}. \tag{4.21}$$

Combining (4.20) and (4.21), we obtain

$$\Omega_0 \ll \frac{p^{2\gamma+3(r-k)}C^4N}{M^2} + \frac{p^{5\gamma/2+3(r-k)}C^4N^2}{M^3}. \tag{4.22}$$

4.5 | Non-zero frequencies $m_4 \neq 0$

We divide the $\tilde{m}_4$-sum in (4.13) into cases according to the two parts given by Lemma 2.4 and denote their contribution towards (4.13) by A_1 for the first part, and A_2 for the second part. Note that $u = r - k$ satisfies the hypothesis

$$u \leqslant 4\gamma/5 \tag{4.23}$$

in our final choice of r.

Case 1: $(r-k)/2 < \gamma - (r-k)$ or $\nu_p(\tilde{m}_4) < \gamma - (r-k)$

In this case, the first part of Lemma 2.4 gives

$$\mathfrak{C}_2(\cdots) \ll p^{\gamma+3(r-k)/2+1/2} \cdot p^{\nu_p(\tilde{m}_4)}.$$

Substituting this in (4.13), it follows

$$A_1 \ll \frac{M_1}{p^{r-k}} \cdot p^{\gamma+3(r-k)/2+1/2} \sideset{}{^*}\sum_{n_1,n_2\in\mathcal{N}} \sum_{c_1,c_2\in\mathscr{C}} \sum_{\substack{\tilde{m}_2,\tilde{m}_3 \asymp p^\gamma C/M \\ c_2\overline{\tilde{m}_2} - c_1\overline{\tilde{m}_3} = \overline{p}^{(\gamma-r+k)}\tilde{m}_4 (c_1c_2)}} \sum_{\tilde{m}_4 \ll M/p^{r-k}} p^{\nu_p(\tilde{m}_4)}. \tag{4.24}$$

Next, consider the $\tilde{m}_2, \tilde{m}_3$ sum in (4.24). Given $\tilde{m}_4 (\neq 0)$, there are $(c_1, \tilde{m}_4)(c_2, \tilde{m}_4)(1 + p^\gamma/M)^2$ many $(\tilde{m}_2, \tilde{m}_3)$ pairs satisfying the congruence mod c_1c_2. Hence,

$$\begin{aligned} A_1 &\ll \frac{M_1}{p^{r-k}} \cdot p^{\gamma+3(r-k)/2+1/2} \sideset{}{^*}\sum_{n_1,n_2\in\mathcal{N}} \sum_{\tilde{m}_4 \ll M/p^u} p^{\nu_p(\tilde{m}_4)} \sum_{c_1,c_2\in\mathscr{C}} (c_1,\tilde{m}_4)(c_2,\tilde{m}_4)(1+p^\gamma/M)^2 \\ &\ll \frac{M_1}{p^{r-k}} \cdot p^{\gamma+3(r-k)/2+1/2} \cdot N^2C^2\left(1+\frac{p^\gamma}{M}\right)^2 \cdot \frac{M}{p^{r-k}}. \end{aligned} \tag{4.25}$$

Case 2: $(r-k)/2 \geqslant \gamma - (r-k)$ and $\nu_p(\tilde{m}_4) \geqslant \gamma - (r-k)$

In this case, the second part of Lemma 2.4 applies. The condition $t_1^{-3/2}s_1\lambda_1 = t_2^{-3/2}s_2\lambda_2\ (p^{\gamma-u})$ translates to

$$c_1n_2 = c_2n_1(\overline{\tilde{m}_2}\tilde{m}_3)^3\ (p^{\gamma-(r-k)}), \tag{4.26}$$

and we have the estimate

$$\mathfrak{C}_2(\cdots) \ll p^{\gamma+3(r-k)/2+1/2} \cdot p^{(r-k)/2}.$$

We write $\tilde{m}_4 = p^{\gamma-(r-k)}\lambda, \lambda \ll M/p^\gamma$. The congruence condition modulo c_1c_2 in (4.13) then implies

$$c_2 = \tilde{m}_2\lambda\ (c_1) \ \text{ and } \ c_1 = \tilde{m}_3\lambda\ (c_2), \tag{4.27}$$

or in other words,

$$c_2 = c_1\delta_1 + \tilde{m}_2\lambda, \text{ and } c_1 = c_2\delta_2 + \tilde{m}_3\lambda, \tag{4.28}$$

for some $\delta_1, \delta_2 \ll 1 + O\left(\frac{(|\tilde{m}_2|+|\tilde{m}_3|)\lambda}{C}\right)$. Observe that

$$\frac{(|\tilde{m}_2| + |\tilde{m}_3|)\lambda}{C} \ll \frac{p^\gamma C}{MC} \cdot \frac{M}{p^\gamma} \ll 1.$$

Hence, δ_1, δ_2 in (4.28) are bounded, and so,

$$A_2 \ll \frac{M_1}{p^{r-k}} \cdot p^{\gamma+3(r-k)/2+1/2} \cdot p^{(r-k)/2} \sum_{\delta_1,\delta_2 \ll 1} \sum_{\lambda \ll M/p^\gamma} \sum_{\tilde{m}_2,\tilde{m}_3 \asymp p^\gamma C/M} \sum_{c_1,c_2 \in \mathscr{C}} \sideset{}{^\#}\sum_{n_1,n_2 \in \mathcal{N}} 1, \tag{4.29}$$

where '#' denotes the restrictions (4.26) and (4.28). When $\delta_1\delta_2 \neq 1$, note that (4.28) uniquely determines the pair (c_1, c_2). Fixing (c_1, c_2), the sum over n_2 with the restriction (4.26) is then bounded by $(1 + N/p^{\gamma-(r-k)})$, and we see that (4.29) is

$$\begin{aligned} A_2 &\ll \frac{M_1}{p^{r-k}} \cdot p^{\gamma+3(r-k)/2+1/2} \cdot p^{(r-k)/2} \sum_{\delta_1,\delta_2 \ll 1} \sum_{\lambda \ll M/p^\gamma} \sum_{\tilde{m}_2,\tilde{m}_3 \asymp p^\gamma C/M} \sum_{n_1 \in \mathcal{N}} \left(1 + \frac{N}{p^{\gamma-(r-k)}}\right) \\ &\ll \frac{M_1}{p^{r-k}} \cdot p^{\gamma+3(r-k)/2+1/2} \cdot N\left(1 + \frac{N}{p^{\gamma-(r-k)}}\right)\left(\frac{p^\gamma C}{M}\right)^2 \frac{M}{p^\gamma} p^{(r-k)/2}. \end{aligned} \tag{4.30}$$

A comparison shows that the last estimate is the second line of (4.25) times the factor

$$\frac{1}{N}\left(1 + \frac{M}{p^\gamma}\right)^{-2}\left(1 + \frac{N}{p^{\gamma-(r-k)}}\right)p^{3(r-k)/2-\gamma} \ll p^{5(r-k)/2-2\gamma} + \left(1 + \frac{M}{p^\gamma}\right)^{-2}\frac{p^{3(r-k)/2}}{Np^\gamma} \ll 1,$$

since our choice of r will satisfy (see (4.36))

$$p^r \ll \min\left\{(Np^\gamma)^{2/3}\left(1 + \frac{M}{p^\gamma}\right)^{4/3}, p^{4\gamma/5}\right\}. \tag{4.31}$$

Hence, $A_2 \ll A_1$ when $\delta_1\delta_2 \neq 1$. When $\delta_1\delta_2 = 1$, (4.28) will imply $\tilde{m}_2 = \pm\tilde{m}_3$. Since $\tilde{m}_2 > 0$, it follows $\tilde{m}_2 = \tilde{m}_3$, $\delta_1 = \delta_2 = -1$. Consequently, (4.28) and (4.26) becomes

$$c_2 = -c_1 + \tilde{m}_2\lambda \text{ and } c_1(n_1 + n_2) = n_1\tilde{m}_2\lambda \ (p^{\gamma-(r-k)}).$$

Since $(n_1\tilde{m}_2, p) = 1$, the number of c_1 satisfying the last congruence is $\ll p^{\nu_p(\lambda)}C/p^{\gamma-(r-k)}$. Hence, (4.29) in this case becomes

$$\begin{aligned} A_2 &\ll \frac{M_1}{p^{r-k}} \cdot p^{\gamma+3(r-k)/2+1/2} \cdot p^{(r-k)/2} \sum_{\lambda \ll M/p^\gamma} \sum_{\tilde{m}_2 \asymp p^\gamma C/M} \sum_{n_1,n_2 \in \mathcal{N}} p^{\nu_p(\lambda)}C/p^{\gamma-(r-k)} \\ &\ll \frac{M_1}{p^{r-k}} \cdot p^{\gamma+3(r-k)/2+1/2} N^2 \left(\frac{p^\gamma C}{M}\right) \cdot \frac{M}{p^\gamma} \cdot Cp^{3(r-k)/2-\gamma}. \end{aligned}$$

The last estimate is the second line of (4.25) times the factor

$$\frac{p^\gamma}{M}\left(1+\frac{p^\gamma}{M}\right)^{-2} p^{3(r-k)/2-\gamma} \ll p^{3(r-k)/2-\gamma} \ll 1,$$

where we have again invoked (4.31).

We conclude that the non-zero frequencies are dominated by A_1 in (4.25), that is,

$$\begin{aligned}\Omega_{\neq 0} &\ll \frac{M_1}{p^{r-k}} \cdot p^{\gamma+3(r-k)/2+1/2} \cdot N^2C^2\left(1+\frac{p^\gamma}{M}\right)^2 \cdot \frac{M}{p^{r-k}} \\ &\ll p^{\gamma+3(r-k)/2+1/2}N^2C^4\left(1+\frac{p^\gamma}{M}\right)^2.\end{aligned} \tag{4.32}$$

From (4.22) and (4.32), we finally have

$$\Omega \ll \frac{p^{2\gamma+3(r-k)}C^4N}{M^2} + \frac{p^{5\gamma/2+3(r-k)}C^4N^2}{M^3} + p^{\gamma+3(r-k)/2+1/2}N^2C^4\left(1+\frac{p^\gamma}{M}\right)^2.$$

Substituting the last bound into (4.9), we arrive at

$$\begin{aligned}&S(k,M_1)\\ &\ll \frac{M^2}{p^{\gamma+2r-k}C^3} \cdot \frac{p^{r-k}C}{M^{1/2}} \cdot \left(\frac{p^{2\gamma+3(r-k)}C^4N}{M^2} + \frac{p^{5\gamma/2+3(r-k)}C^4N^2}{M^3}\right.\\ &\quad\left. + p^{\gamma+3(r-k)/2+1/2}N^2C^4\left(1+\frac{p^\gamma}{M}\right)^2\right)^{1/2}\\ &\ll p^{r/2-3k/2}M^{1/2}N^{1/2} + p^{\gamma/4+r/2-3k/4}N + p^{-\gamma/2-r/4-3k/4+1/4}M^{3/2}N(1+p^\gamma/M).\end{aligned} \tag{4.33}$$

4.6 | Optimal choice for r

It follows from (4.33) and (4.6) that

$$S \ll p^{r/2}M^{1/2}N^{1/2} + p^{\gamma/2-r/4+1/4}M^{1/2}N(1+M/p^\gamma) + p^{\gamma/4+r/2}N. \tag{4.34}$$

Equating the first two terms, we obtain

$$p^r = p^{2\gamma/3+1/3}N^{2/3}(1+M/p^\gamma)^{4/3}, \tag{4.35}$$

that is,

$$r \approx \lfloor 2/3(\gamma+1+\log_p N(1+M/p^\gamma)^2)\rfloor.$$

But recall from (4.23) that r is assumed to be at most $4\gamma/5$. We choose

$$r = \lfloor \min\{2/3(\gamma+\log_p N(1+M/p^\gamma)^2), 4\gamma/5\}\rfloor. \tag{4.36}$$

So, the third term in (4.34) can be bounded by $p^{13\gamma/20}N$. Note that when $N \leqslant p^{\gamma/5}(1+M/p^\gamma)^{-2}$,

$$2/3(\gamma + \log_p N(1+M/p^\gamma)^2) \leqslant 4\gamma/5,$$

so that (4.35) holds (up to a factor of $p^{5/3}$) and we get

$$S \ll p^{7/12}p^{\gamma/3}M^{1/2}N^{5/6}(1+M/p^\gamma)^{2/3} + p^{13\gamma/20}N,$$

in this case. When $N > p^{\gamma/5}(1+M/p^\gamma)^{-2}$, we have $r = \lfloor 4\gamma/5 \rfloor$ so that the second term in (4.34) dominates the first and we get

$$S \ll p^{1/4}p^{\gamma/2-\lfloor 4\gamma/5\rfloor/4+\epsilon}M^{1/2}N(1+M/p^\gamma) + p^{13\gamma/20}N.$$

Combining, we have the final estimate

$$S \ll p^{7/12}q^{1/3}M^{1/2}N^{5/6}(1+M/q)^{2/3} + \delta_{(N>q^{1/5}(1+M/q)^{-2})}p^{1/4}q^{3/10}M^{1/2}N(1+M/q) + q^{13/20}N, \tag{4.37}$$

where $q = p^\gamma$.

5 | AN ALTERNATIVE ESTIMATE

We will use the above estimates for N going up to certain threshold. For N larger, we get better estimates simply by applying Cauchy–Schwarz inequality followed by Poisson summation in the m-sum. Recall that

$$S = \sum_{n\in\mathcal{N}}\sum_{m\geqslant 1}\alpha_n\lambda(m)K(mn)V(m/M), \tag{5.1}$$

where

$$K(m) = \tilde{\mathrm{Kl}}_3(mb,q) = \frac{1}{q}\sum_{x(q)}^{*} e\left(\frac{mbx}{q}\right)S(1,\overline{x};q).$$

Lemma 5.1. *For $q = p^\gamma, \gamma \geqslant 1$, we have*

$$S \ll MN^{1/2} + M^{1/2}Nq^{1/4}(1+M/q)^{1/2}.$$

To see this, we apply Cauchy–Schwarz inequality to (5.1) keeping the m-sum outside to get

$$S \ll \frac{M^{1/2}}{q}\left(\sum_{m\in\mathbb{Z}}V(m/M)\left|\sum_{n\in\mathcal{N}}\alpha_n\sum_{x(q)}^{*}e\left(\frac{xmnb}{q}\right)S(1,\overline{x};q)\right|^2\right)^{1/2}. \tag{5.2}$$

Opening the absolute value square and dualising the m-sum using the Poisson summation formula, we arrive at

$$\sum_{m\in\mathbb{Z}} V(m/M)\left|\sum_{n\in\mathcal{N}}\alpha_n \sum_{x(q)}^{*} e\left(\frac{xmnb}{q}\right)S(1,\overline{x};q)\right|^2$$
$$=\frac{M}{q}\sum_{n_1,n_2\in\mathcal{N}}\alpha_{n_1}\overline{\alpha}_{n_2}\sum_{x_1,x_2(q)}^{*} S(1,\overline{x_1};q)\overline{S}(1,\overline{x_2};q)\sum_{r(q)} e\left(\frac{rb(n_1x_1-n_2x_2)}{q}\right)\sum_{\tilde{m}\in\mathbb{Z}} e\left(\frac{-\tilde{m}r}{q}\right)I(\tilde{m})$$
$$=M\sum_{n_1,n_2\in\mathcal{N}}\alpha_{n_1}\overline{\alpha}_{n_2}\sum_{\tilde{m}\in\mathbb{Z}} C(n_1,n_2,\tilde{m})\cdot I(\tilde{m}), \tag{5.3}$$

where

$$I(\tilde{m}) = \int_{\mathbb{R}} V(x)e(-M\tilde{m}x/q)dx$$

and

$$C(n_1,n_2,\tilde{m}) = \sum_{x(q)}^{*} S(1,\overline{x};q)\overline{S}(1,\overline{(n_1\overline{n_2}x+\overline{n_2}b\tilde{m})};q). \tag{5.4}$$

It is clear that $I(\tilde{m})$ is negligibly small unless $\tilde{m} \ll q/M$.

It remains to estimate the character sum $C(\cdots)$. In the case of prime power moduli, an explicit evaluation of the character sum C as a function of $(\tilde{m}, n_1, n_2)$ can be obtained by following the proof of Lemma 2.4 or otherwise. If $\alpha_n = 1$, as required for our application, this evaluation can be used to non-trivially bound one of the n_1, n_2, or $\tilde{m}$-sum in (5.3) using an exponent pair estimate. However, since we are not interested in this improvement for the purposes of this paper, we use the ready-made estimates available in [5].

Lemma 5.2. *For any $q \geqslant 1$ and $C(n_1, n_2, \tilde{m})$ as in (5.4), we have*

$$C(n_1,n_2,\tilde{m}) \ll q^{3/2}\sum_{k|q} k^{1/2}\delta_{\left(\substack{n_1=n_2(k)\\ \tilde{m}=0(k)}\right)}.$$

Proof. Let us factorise q into product of prime powers $q = \prod_{1\leqslant i\leqslant} q_i$, where $q_i = p_i^{\gamma_i}$ and p_i's are prime. Then, by repeated use of the well-known multiplicative property of the Kloosterman sums ([16], eq. (1.59)), we get

$$S(1,\overline{x};q)\overline{S}(1,\overline{(n_1\overline{n_2}x+\overline{n_2}b\tilde{m})};q) = \prod_{1\leqslant i\leqslant l} S(1,\overline{(q/q_i)}^2\overline{x},q_i)\overline{S}(1,\overline{(q/q_i)}^2\overline{(n_1\overline{n_2}x+\overline{n_2}\tilde{m})},q_i).$$

Splitting the residue classes $x(q)$ in (5.4) using the Chinese Remainder Theorem, it then follows

$$C(n_1,n_2,\tilde{m}) = \prod_{1\leqslant i\leqslant l} K_i, \tag{5.5}$$

where

$$K_i = \sum_{x(q_i)}^{*} S(1, \overline{(q/q_i)}^2 \overline{x}, q_i) \overline{S}(1, \overline{(q/q_i)}^2 \overline{(n_1 \overline{n_2} x + \overline{n_2} \tilde{m})}, q_i)$$
$$= \sum_{x(q_i)}^{*} S(1, x, q_i) \overline{S}(1, \overline{n_1} n_2 x \overline{((q/q_i)^2 \overline{n_1} \tilde{m} x + 1)}, q_i).$$

We can now apply estimates for K_i from Lemma 4.19 giving us

$$K_i \ll q_i^{3/2} p_i^{(\min\{\gamma_i, \nu_{p_i}(n_1 - n_2), \nu_{p_i}(b)\})/2} \ll q_i^{3/2} \sum_{k | q_i} k^{1/2} \delta_{\binom{n_1 = n_2(k)}{\tilde{m} = 0(k)}}.$$

The lemma follows after substituting these estimates for K_i into (5.5) and gluing the congruences. □

Plugging in the estimate from Lemma 5.2 into (5.3), we obtain

$$\sum_{m \in \mathbb{Z}} V(m/M) \left| \sum_{n \in \mathcal{N}} \alpha_n \sum_{x(q)}^{*} e\left(\frac{xmnb}{q}\right) S(1, \overline{x}; q) \right|^2 \ll M q^{3/2} \sum_{k | q} k^{1/2} \sum_{n_1, n_2 \in \mathcal{N}} \sum_{\tilde{m} \ll q/M} \delta_{\binom{n_1 = n_2(k)}{\tilde{m} = 0(k)}}$$
$$\ll M q^{3/2} \sum_{k | q} k^{1/2} N (1 + N/k)(1 + q/Mk)$$
$$\ll M N q^{3/2} \sum_{k | q} (q^{1/2} + N(1 + q/M))$$
$$\ll M N q^2 + N^2 q^{5/2} (1 + M/q).$$

Final substitution into (5.2) yields

$$S \ll M N^{1/2} + M^{1/2} N q^{1/4} (1 + M/q)^{1/2}. \tag{5.6}$$

This completes the proof Lemma 5.1.

6 | THE APPLICATION: PROOF OF THEOREM 1.1

Let $q \geqslant 1, a \in \mathbb{Z}$ such that $(a, q) = 1$. We are interested in the asymptotic of

$$S = \sum_{\substack{n \leqslant X \\ n = a(q)}} d_3(n).$$

Detecting $n = a\ (q)$ using additive characters, we obtain

$$S = \frac{1}{q} \sum_{\alpha(q)} \sum_{n \leqslant X} d_3(n) e\left(\frac{\alpha(n - a)}{q}\right).$$

Splitting into Ramanujan sums, we get

$$S = \sum_{d|q} S(d), \tag{6.1}$$

where

$$S(d) = \frac{1}{q} \sideset{}{^*}\sum_{\alpha(d)} \sum_{n \leqslant X} d_3(n) e\left(\frac{\alpha(n-a)}{d}\right).$$

Fix $A > 0$. Choose a smooth function $w(x)$ such that $w(x) = 1$ for $x \in [X^{1-\epsilon/2}, X + X^{1-\epsilon}]$ and $\mathrm{supp}(w) \subseteq [X^{1-\epsilon}, X + X^{1-\epsilon/2}]$ and satisfying

$$x^j w^{(j)}(x) \ll_{\epsilon,j} X^{j\epsilon},$$

for $j \geqslant 0$. Smoothing the n-sum in S_k using the weight function w, we obtain

$$S(d) = \frac{1}{q} \sideset{}{^*}\sum_{\alpha(d)} \sum_{n \geqslant 1} d_3(n) w(n) e\left(\frac{\alpha(n-a)}{d}\right) + O(X^{1-\epsilon}/q). \tag{6.2}$$

The Voronoi summation formula (2.1) for d_3 transforms the n-sum above into

$$\begin{aligned}
&\sum_{n \geqslant 1} d_3(n) w(n) e\left(\frac{\alpha n}{d}\right) \\
&= \frac{1}{d} \int_0^\infty P(\log y, d) w(y) dy \\
&+ \frac{d}{2\pi^{3/2}} \sum_{\pm} \sum_{r|d} \sum_{m \geqslant 1} \frac{1}{rm} \sum_{r_1|r} \sum_{r_2|\frac{r}{r_1}} \sigma_{0,0}(r/(r_1 r_2), m) S(\pm m, \overline{\alpha}; d/r) \Phi_{\pm}(mr^2/d^3).
\end{aligned} \tag{6.3}$$

Substituting into (6.2), we obtain

$$S(d) = M(d) + E(d) + O(X^{1-\epsilon}/q), \tag{6.4}$$

where

$$M(d) = \frac{1}{qd} \left(\int_0^\infty P(\log y, d) w(y) dy \right) \sideset{}{^*}\sum_{\alpha(d)} e(-\alpha a/d) = \frac{\mu(d)}{qd} \int_0^\infty P(\log y, d) w(y) dy, \tag{6.5}$$

and

$$E(d) = \frac{d^2}{2\pi^{3/2} q} \sum_{\pm} \sum_{r|d} \sum_{m \geqslant 1} \frac{1}{rm} \sum_{r_1|r} \sum_{r_2|\frac{r}{r_1}} \sigma_{0,0}(r/(r_1 r_2), m) K_{r,d}(m) \Phi_{\pm}(mr^2/d^3), \tag{6.6}$$

where

$$K_{r,d}(m) = \frac{1}{d} \sideset{}{^*}\sum_{\alpha(d)} e(-a\alpha/d) S(\pm m, \overline{\alpha}; d/r).$$

Write $d = d_0 d_1$, where d_0 is the square-free and d_1 is the square-full part. Then note that $K_{r,d}(m)$ vanishes unless $r|d_0$ in which case we have

$$K_{r,d}(m) = \frac{\mu(r)}{d} \sideset{}{^*}\sum_{\alpha(d/r)} e(-\bar{r}a\alpha/(d/r))S(\pm m, \bar{\alpha}; d/r).$$

Recall from (2.1) that

$$\sigma_{0,0}(r/(r_1r_2), m) = \sum_{t|(r/(r_1r_2),m)} \mu(t)d_3(m/t). \tag{6.7}$$

We fix the divisor $t|(r/(r_1r_2))$ in (6.7) and push the m-sum in (6.6) inside to see that

$$E(d) \ll \frac{d^2}{q} \sum_{r|d_0} \frac{1}{r^2} \sum_{t|r} \frac{1}{t}|C(d,r,t)|, \tag{6.8}$$

where

$$C(d,r,t) = \sum_{m\geqslant 1} \frac{d_3(m)}{m} \tilde{\mathrm{Kl}}_3(mb, d/r)\Phi_{\pm}(mtr^2/d^3),$$

with $b = \pm\bar{r}ta$. Now from Lemma 2.2, it follows that the m-sum above is negligibly small unless $m \ll d^3/(tr^2X)$. Also, if we define

$$\psi(m) = (\min\{mtr^2X/d^3, 1\})^{-1}\Phi_{\pm}(mtr^2/d^3),$$

then from the same lemma, we have

$$y^j\psi^{(j)}(y) \ll_j 1.$$

Hence, we can write

$$C(d,r,t) = \sum_{m\ll d^3/(tr^2X)} \frac{\min\{mtr^2X/d^3, 1\}}{m} \cdot d_3(m)\tilde{\mathrm{Kl}}_3(mb, d/r)\psi(m).$$

Dividing the m-sum above into dyadic blocks $m \sim Y, Y \ll d^3/(tr^2X)$, we see that

$$C(d,r,t) \ll \frac{\min\{Ytr^2X/d^3, 1\}}{Y} \sup_{Y\ll d^3/(tr^2X)} |C(d,r,t,Y)| \ll \frac{tr^2X}{d^3} \sup_{Y\ll d^3/(tr^2X)} |C(d,r,t,Y)|,$$

where

$$C(d,r,t,Y) = \sum_{m\sim Y} d_3(m)\tilde{\mathrm{Kl}}_3(mb, d/r).$$

Substituting the last inequality into (6.8), we conclude

$$E(d) \ll \frac{X}{qd} \sup_{\substack{r|d_0, t|r \\ Y\ll d^3/(tr^2X)}} |C(d,r,t,Y)|. \tag{6.9}$$

We proceed for the estimation of $C(d, r, t, Y)$. We do this by converting it into a bilinear sum as in Theorem 1.2 with $N \ll Y^{1/3}$ using the symmetry in the factorisation of $d_3(m)$. Expanding $d_3(m)$ into product of three variables and introduction dyadic partition in each of the variables, we get

$$C(d,r,t,Y) \ll \sup_{\substack{N_1,N_2,N_3>0\\ N_1N_2N_3\sim Y}} \left| \sum_{n_1,n_2,n_3} \mathrm{Kl}_3(n_1n_2n_3b, d/r)V(n_1/N_1)V(n_2/N_2)V(n_3/N_3) \right|. \tag{6.10}$$

By symmetry, we can assume $N_1 \leqslant N_2 \leqslant N_3$. Note that this forces $N_1 \ll Y^{1/3}$. Gluing $n_2n_3 = m$, we obtain

$$\sum_{n_1,n_2,n_3} \mathrm{Kl}_3(n_1n_2n_3b, d/r)V(n_1/N_1)V(n_2/N_2)V(n_3/N_3) = \sum_{n_1\sim N_1} \sum_{m\sim Y/N_1} a(m)\mathrm{Kl}_3(mn_1b, d/r), \tag{6.11}$$

where

$$a(m) = \sum_{b|m} V(b/N_2)V(m/bN_3).$$

Using the Mellin inversion

$$V(x) = \int_{(\sigma)} \tilde{V}(s)x^{-s}\, ds, \tag{6.12}$$

we can further write

$$a(m) = \int\int \tilde{V}(s_1)\tilde{V}(s_2)N_2^{s_1}N_3^{s_2}m^{-s_2}\sigma_{s_2-s_1}(m)\, ds_1\, ds_2.$$

Note that since V is a nice weight function, we can restrict the contour in (6.12) to $|s| \ll X^{\epsilon}$ up to a negligible error. Feeding all these information into the right hand of (6.11), we obtain

$$\sum_{n_1\sim N_1} \sum_{m\sim Y/N_1} a(m)\mathrm{Kl}_3(mn_1b, d/r) \ll \sup_{|s_i|\ll X^{\epsilon}} \left| \sum_{n_1\sim N_1} \sum_{m\sim Y/N_1} \sigma_{s_1}(m)m^{s_2}\mathrm{Kl}_3(mn_1b, d/r) \right|.$$

Substituting the last relation into (6.10), we finally obtain

$$C(d,r,t,Y) \ll \sup_{\substack{N\ll Y^{1/3}\\ |s_i|\ll X^{\epsilon}}} |S(Y/N, N)|, \tag{6.13}$$

where

$$S(M,N) = \sum_{n\sim N} \sum_{m\sim M} \sigma_{s_1}(m)m^{s_2}\mathrm{Kl}_3(mnb, d/r).$$

We are now position to apply our estimates for bilinear sums obtained in previous sections.

6.1 | Square-free moduli

Here, d/r is square-free. We want to apply the estimates from Theorem 1.2 and Lemma 5.1 to $S(M,N)$ with the parameters $q = d/r, M = Y/N$. For this, we need to first verify the hypothesis $N \leqslant q^{1/2}(1+M/q)^{-2}$ of Theorem 1.2. Note that $N \gg q^{1/2}$ translates to $N \gg (d/r)^{1/2}$ which implies

$$Y^{1/3} \gg (d/r)^{1/2} \Rightarrow (d^3/(tr^2X))^{1/3} \gg (d/r)^{1/2} \Rightarrow d \gg X^{2/3},$$

which is not the case since $d \leqslant q \leqslant X^{2/3}$ in our final choice of $q(\leqslant X^{1/2+1/30-\varepsilon})$. Similarly, $NM^2 \gg q^{5/2}$ will imply $d \gg X^{7/4}$ which is also not the case. Hence, the condition $N \leqslant q^{1/2}(1+M/q)^{-2}$ is satisfied so that from Theorem 1.2 and Lemma 5.1, we obtain

$$S(Y/N,N) \ll \begin{cases} Y^{1/2}N^{1/4}\mathfrak{q}^{3/8} + Y/(N^{1/4}\mathfrak{q}^{1/8}) + YN^{1/4}/\mathfrak{q}^{1/4} + Y^2/(N^{1/2}\mathfrak{q}^{5/4}) + Nq^{3/4} + Y^{1/2}\mathfrak{q}^{1/4}/N^{1/2}, \\ Y/N^{1/2} + Y^{1/2}N^{1/2}\mathfrak{q}^{1/4} + Y\mathfrak{q}^{-1/4}, \end{cases} \tag{6.14}$$

where $\mathfrak{q} = d/r$. Our job now is to optimally choose bounds between the two lines in (6.14) depending on the size of N. Note that since $N \ll Y^{1/3}$, the second term in the second bound of (6.14) is $\ll Y^{2/3}\mathfrak{q}^{1/4}$. Similarly, the last term in the first bound is clearly $\ll Y^{2/3}\mathfrak{q}^{1/4}$. Also, note that $Y/N^{1/2} \gg Y/\mathfrak{q}^{-1/4}$ since $N \ll \mathfrak{q}^{1/2}$ as pointed out earlier. So, we can write

$$S(Y/N,N) \ll \sum_{i=1}^{5} A_i + Y^{2/3}\mathfrak{q}^{1/4},$$

where

$$A_1 = \min\{Y^{1/2}N^{1/4}\mathfrak{q}^{3/8}, Y/N^{1/2}\}, A_2 = \min\{Y/(N^{1/4}\mathfrak{q}^{1/8}), Y/N^{1/2}\},$$

$$A_3 = \min\{YN^{1/4}/\mathfrak{q}^{1/4}, Y/N^{1/2}\}, A_4 = \min\{Y^2/(N^{1/2}\mathfrak{q}^{5/4}), Y/N^{1/2}\},$$

and

$$A_5 = \min\{N\mathfrak{q}^{3/4}, Y/N^{1/2}\}.$$

A_1 attains its largest value when the two terms inside the parenthesis are equal, that is, when $N = Y^{2/3}/\mathfrak{q}^{1/2}$, which gives

$$A_1 \leqslant Y^{2/3}\mathfrak{q}^{1/4}. \tag{6.15}$$

Similarly arguing, we obtain

$$A_2 \leqslant Y/\mathfrak{q}^{1/4},\ A_3 \leqslant Y/\mathfrak{q}^{1/6}, A_4 \leqslant Y^2/\mathfrak{q}^{5/4}, A_5 \leqslant Y^{2/3}\mathfrak{q}^{1/4}.$$

Hence,

$$S(Y/N,N) \ll Y^{2/3}\mathfrak{q}^{1/4} + Y/\mathfrak{q}^{1/6} + Y^2/\mathfrak{q}^{5/4}. \tag{6.16}$$

Substituting this in (6.13) and then in (6.9), we obtain

$$
\begin{aligned}
E(d) &\ll \frac{X}{qd} \sup_{\substack{r|d_0,t|r \\ Y \ll d^3/(tr^2X)}} (Y^{2/3}(d/r)^{1/4} + Y(d/r)^{-1/6} + Y^2(d/r)^{-5/4}) \\
&\ll \frac{X}{qd}(X^{-2/3}d^{9/4} + X^{-1}d^{17/6} + X^{-2}d^{19/4}) \\
&\ll X^{1/3}q^{1/4} + q^{5/6} + X^{-1}q^{11/4},
\end{aligned} \tag{6.17}
$$

where he used the upper bound $d \ll q$ in the last line. The last line of (6.17) is $O(X^{1-\epsilon}/q)$ for $q \leqslant X^{1/2+1/30-\epsilon}$ and therefore $E(d) \ll X^{1-\epsilon}/q$ for $q \leqslant X^{1/2+1/30-\epsilon}$. Hence, from (6.4) and (6.1), it follows

$$
S = \sum_{d|q} M(d) + O(X^{1-\epsilon}/q), \tag{6.18}
$$

for square free $q \leqslant X^{1/2+1/30-\epsilon}$.

6.2 | Prime power moduli

Here, $q = p^\gamma, \gamma \geqslant 2$ and so $d/r = p^k$. Without loss of generality, we can assume $k \geqslant 2$ since for $k = 1$, we can use the estimate (6.16) for $S(Y/N, N)$ to arrive at the same bound (6.17). Furthermore, when $k \geqslant 2$ note that $r = 1$ since r has to divide the square-free part of d which is 1 in this case. For $d = p^k, k \geqslant 2$, using the estimate from (4.37) and Lemma 5.1, we obtain

$$
S(Y/N, N) \ll \begin{cases} p^{7/12}Y^{1/2}N^{1/3}d^{1/3} + p^{7/12}Y^{7/6}/(N^{1/3}d^{1/3}) \\ \qquad + p^{1/4}Y^{1/2}N^{1/2}d^{3/10} + p^{1/4}Y^{3/2}/(N^{1/2}d^{7/10}) + N\mathfrak{q}^{13/20}, \\ Y/N^{1/2} + Y^{1/2}N^{1/2}d^{1/4} + Y\mathfrak{q}^{-1/4}, \end{cases} \tag{6.19}
$$

where $d = p^k, k \geqslant 2$. As earlier, we use $N \ll Y^{1/3}$ to bound the second term of second bound in (6.19) by $Y^{2/3}d^{1/4}$ and ignore the third term due to the inequality $N \ll d^{1/2}$. Hence, this time we get

$$
S(Y/N, N) \ll Y^{2/3}d^{1/4} + A_1 + A_2 + A_3 + A_4 + A_5, \tag{6.20}
$$

where

$$
A_1 = \min\{p^{7/12}Y^{1/2}N^{1/3}d^{1/3}, Y/N^{1/2}\},\ A_2 = \min\{p^{7/12}Y^{7/6}/(N^{1/3}d^{1/3}), Y/N^{1/2}\},
$$

$$
A_3 = \min\{p^{1/4}Y^{1/2}N^{1/2}d^{3/10}, Y/N^{1/2}\},\ A_4 = \min\{p^{1/4}Y^{3/2}/(N^{1/2}d^{7/10}), Y/N^{1/2}\},
$$

and

$$
A_5 = \min\{N\mathfrak{q}^{13/20}, Y/N^{1/2}\}.
$$

Arguing as in (6.15), we obtain the following estimates for A_i:

$$
A_1 \leqslant Y^{7/10}d^{1/5}p^{7/20},\ A_2 \leqslant Y^{3/2}d^{-1}p^{7/4},\ A_3 \leqslant Y^{3/4}d^{3/20}p^{1/8},\ A_4 \leqslant Y^{3/2}d^{-7/10}p^{1/4}
$$

and

$$A_5 \leqslant Y^{2/3}d^{13/60}.$$

Using these estimates for A_i in (6.20), we obtain

$$S(Y/N, N) \ll Y^{2/3}d^{1/4} + Y^{7/10}d^{1/5}p^{7/20} + Y^{3/2}d^{-1}p^{7/4} + Y^{3/4}d^{3/20}p^{1/8} + Y^{3/2}d^{-7/10}p^{1/4} + Y^{2/3}d^{13/60}.$$

Substituting the last estimate into (6.13) and then in (6.9), we obtain

$$\begin{aligned} E(d) &\ll \frac{X}{qd} \sup_{Y \ll d^3/X} \Big(Y^{2/3}d^{1/4} + Y^{7/10}d^{1/5}p^{7/20} + Y^{3/2}d^{-1}p^{7/4} + Y^{3/4}d^{3/20}p^{1/8} \\ &\qquad + Y^{3/2}d^{-7/10}p^{1/4} + Y^{2/3}d^{13/60} \Big) \\ &\ll \frac{X}{qd} \Big(X^{-2/3}d^{9/4} + X^{-7/10}d^{23/10}p^{7/20} + X^{-3/2}d^{7/2}p^{7/4} + X^{-3/4}d^{12/5}p^{1/8} \\ &\qquad + X^{-3/2}d^{19/5}p^{1/4} + X^{-2/3}d^{133/60} \Big) \\ &= X^{1/3}q^{1/4} + X^{3/10}q^{3/10}p^{3/20} + X^{-1/2}q^{3/2}p^{7/4} + X^{1/4}q^{2/5}p^{1/8} \\ &\qquad + X^{-1/2}q^{9/5}p^{1/4} + X^{1/3}q^{13/60}. \end{aligned} \tag{6.21}$$

The last line in (6.21) is $O(X^{1-\epsilon}/q)$ when $q \ll X^{1/2+1/30-\epsilon}$ and $\gamma \geqslant 28$. The exponent $1/2 + 1/30$ and the power $\gamma \geqslant 28$ is determined by the '$X^{1/3}q^{1/4}$' and the '$X^{3/10}q^{3/10}p^{3/20}$' terms, respectively.

Remark 3. The main contributing term '$X^{1/3}q^{1/4}$' originates from the '$Y^{1/2}N^{1/2}d^{1/4}$' term in the second line of (6.19). Thus, it is evident that any improvement in this term, which corresponds to the off-diagonal contribution in (5.3), would result in an improvement in the exponent of distribution.

Hence, from (6.4) and (6.1), it follows

$$S = \sum_{d|q} M(d) + O(X^{1-\epsilon}/q), \tag{6.22}$$

for $q = p^\gamma \leqslant X^{1/2+1/30-\epsilon}$ and $\gamma \geqslant 28$.

Finally, from (6.18) and (6.22), it follows

$$S = \sum_{\substack{n \leqslant X \\ n = a(q)}} d_3(n) = \sum_{d|q} M(d) + O(X^{1-\epsilon}/q), \tag{6.23}$$

for $q \leqslant X^{1/2+1/30-\epsilon}$, where q is either square free or $q = p^\gamma, \gamma \geqslant 28$. Note that the $M(d)$'s, which are given by (6.5), are independent of the residue class a (q). Hence, summing the expression

(6.23) over all the co-prime residue classes $a\ (q)$, we obtain

$$\sum_{\substack{n\leqslant X\\(n,q)=1}} d_3(n) = \phi(q)\left(\sum_{d|q} M(d)\right) + O(\phi(q)X^{1-\epsilon}/q),$$

from which it follows

$$\sum_{d|q} M(d) = \frac{1}{\phi(q)} \sum_{\substack{n\leqslant X\\(n,q)=1}} d_3(n) + O(X^{1-\epsilon}/q).$$

Theorem 1.1 follows after substituting the last expression for $\sum_{d|q} M(d)$ into (6.23).

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