Dispersion and Littlewood's conjecture

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ABSTRACT

Let $\varepsilon > 0$. We construct an explicit, full-measure set of $\alpha \in [0,1]$ such that if $\gamma \in \mathbb{R}$ then, for almost all $\beta \in [0,1]$, if $\delta \in \mathbb{R}$ then there are infinitely many integers $n \geqslant 1$ for which

$$n\|n\alpha - \gamma\| \cdot \|n\beta - \delta\| < \frac{(\log\log n)^{3+\varepsilon}}{\log n}.$$

This is a significant quantitative improvement over a result of the first author and Zafeiropoulos. We show, moreover, that the exceptional set of β has Fourier dimension zero, alongside further applications to badly approximable numbers and to lacunary diophantine approximation. Our method relies on a dispersion estimate and the Three Distance Theorem.

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1. Introduction

Littlewood's conjecture (circa 1930) is a central problem in diophantine approximation. It states that if $\alpha, \beta \in \mathbb{R}$ then

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$$\liminf_{n\to\infty} n\|n\alpha\| \cdot \|n\beta\| = 0,$$

where $\|\cdot\|$ denotes the distance to the nearest integer. The problem lies at the heart of Margulis's measure rigidity conjecture [15] and, as such, is also a holy grail of homogeneous dynamics.

Gallagher [11] determined the multiplicative approximation rate of a generic pair of reals. We write $\mathbb{N} = \mathbb{Z}_{>0}$ throughout.

Theorem 1.1 *(Gallagher, 1962). Let $\psi : \mathbb{N} \to (0,\infty)$ be monotonic, and denote by $W_2^\times(\psi)$ the set of $(\alpha, \beta) \in [0,1]^2$ such that*

$$\|n\alpha\| \cdot \|n\beta\| < \psi(n)$$

holds for infinitely many $n \in \mathbb{N}$. Then the Lebesgue measure of $W_2^\times(\psi)$ is

$$\begin{cases} 0, & \text{if } \displaystyle\sum_{n=1}^{\infty} \psi(n)\log n < \infty \\[2em] 1, & \text{if } \displaystyle\sum_{n=1}^{\infty} \psi(n)\log n = \infty. \end{cases}$$

In particular, for Lebesgue almost all $(\alpha, \beta) \in \mathbb{R}^2$, we have

$$\liminf_{n\to\infty} n(\log n)^2 (\log\log n)\|n\alpha\| \cdot \|n\beta\| = 0.$$

However, for the purpose of proving Littlewood's conjecture, we may assume that α_1, α_2 are badly approximable. Here we recall that $\alpha \in \mathbb{R}$ is *badly approximable* if

$$\inf\{n\|n\alpha\| : n \in \mathbb{N}\} > 0.$$

The set Bad of badly approximable numbers has Lebesgue measure zero so, in some sense, Gallagher's theorem fails to capture the essence of Littlewood's conjecture. With this in mind, Pollington and Velani [18] established that a similar result holds for many badly approximable numbers. We write $\dim_H(\mathcal{A})$ for the Hausdorff dimension of a Borel set $\mathcal{A} \subseteq \mathbb{R}$.

Theorem 1.2 *(Pollington–Velani, 2000). Let $\alpha \in$ Bad. Then there exists $\mathcal{G} \subseteq$ Bad with $\dim_H(\mathcal{G}) = 1$ such that if $\beta \in \mathcal{G}$ then*

$$\liminf_{n\to\infty} n(\log n)\|n\alpha\| \cdot \|n\beta\| \leqslant 1.$$

The assumption that $\alpha \in$ Bad was subsequently relaxed by the first author and Zafeiropoulos [6], and the result made 'fully inhomogeneous'. We require some notation

to state it. Given $\alpha \in [0, 1]$, we consider its partial quotients $a_k(\alpha)$ and its continuants $q_k(\alpha)$. Define

$$\mathcal{K} := \{\alpha \in \mathbb{R} : \Lambda(\alpha) < \infty\} \quad \text{where} \quad \Lambda(\alpha) = \sup \left\{ \frac{\log q_k(\alpha)}{k} : k \geqslant 1 \right\}.$$

The set $\mathcal{K}$ includes almost all real numbers, as $\Lambda(\alpha)$ almost surely equals the Lévy constant:

$$\Lambda(\alpha) = \frac{\pi^2}{12 \log 2} \approx 1.187.$$

Moreover, Bad $\subset \mathcal{K}$, since badly approximable numbers have bounded partial quotients.

Theorem 1.3 (*Chow–Zafeiropoulos, 2021*). *Let* $\alpha \in \mathcal{K}$ *and* $\gamma, \delta \in \mathbb{R}$. *Then there exists* $\mathcal{G} \subseteq$ Bad *with* $\dim_{\mathrm{H}}(\mathcal{G}) = 1$ *such that if* $\beta \in \mathcal{G}$ *then*

$$\liminf_{n \to \infty} n(\log n)\|n\alpha - \gamma\| \cdot \|n\beta - \delta\| \leqslant 1.$$

Haynes, Jensen and Kristensen [13] investigated a similar problem in which $\mathcal{G}$ does not depend on δ. Their main result was sharpened by the second author and Zafeiropoulos [22], and then generalised by the first author and Zafeiropoulos [6].

Theorem 1.4 (*Chow–Zafeiropoulos, 2021*). *Let* $\varepsilon > 0$, $\alpha \in \mathcal{K}$ *and* $\gamma \in \mathbb{R}$. *Then there exists* $\mathcal{G} \subseteq$ Bad *with* $\dim_{\mathrm{H}}(\mathcal{G}) = 1$ *such that if* $\beta \in \mathcal{G}$ *and* $\delta \in \mathbb{R}$ *then*

$$n\|n\alpha - \gamma\| \cdot \|n\beta - \delta\| < \frac{(\log\log\log n)^{\varepsilon+1/2}}{\sqrt{\log n}}$$

has infinitely many solutions $n \in \mathbb{N}$.

We present the following stronger version.

Theorem 1.5. *Let* $\varepsilon > 0$, $\alpha \in \mathcal{K}$ *and* $\gamma \in \mathbb{R}$. *Then there exists* $\mathcal{G} \subseteq$ Bad *with* $\dim_{\mathrm{H}}(\mathcal{G}) = 1$ *such that if* $\beta \in \mathcal{G}$ *and* $\delta \in \mathbb{R}$ *then*

$$n\|n\alpha - \gamma\| \cdot \|n\beta - \delta\| < \frac{(\log\log n)^{3+\varepsilon}}{\log n} \tag{1.1}$$

has infinitely many solutions $n \in \mathbb{N}$.

The quantitative threshold here is close to that of Pollington and Velani's in Theorem 1.2.
For $\delta \in \mathbb{R}$, define

$$\mathrm{Bad}(\delta) = \{\beta \in \mathbb{R} : \inf\{n\|n\beta - \delta\| : n \in \mathbb{N}\} > 0\},$$

and set

$$\mathcal{B} = \{(\beta, \delta) \in \mathbb{R}^2 : \beta \in \text{Bad} \cap \text{Bad}(\delta)\}.$$

The short argument in [6, Section 5], based on Marstrand's slicing theorem [9, Theorem 5.8] and a result of Tseng [23] on twisted approximation, delivers the following consequence.

Corollary 1.6. *Let $\varepsilon > 0$, $\alpha \in \mathcal{K}$ and $\gamma \in \mathbb{R}$. Then there exists $\mathcal{G} \subseteq \mathcal{B}$ with $\dim_H(\mathcal{G}) = 2$ such that if $(\beta, \delta) \in \mathcal{G}$ then (1.1) has infinitely many solutions $n \in \mathbb{N}$.*

The upshot is that here β is δ-inhomogeneously badly approximable, which is "natural and desirable" for these problems, as discussed in the introduction of [19].

The Fourier decay of certain measures is an essential ingredient in the proof of Theorem 1.5. Before presenting further applications, we contextualise this using the notion of Fourier dimension. For Borel $\mathcal{A} \subseteq \mathbb{R}$, write $\mathcal{M}(\mathcal{A})$ for the set of Borel probability measures μ supported on a compact subset of $\mathcal{A}$. For $\mu \in \mathcal{M}(\mathcal{A})$ and $\xi \in \mathbb{R}$, we write

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}} e(-\xi x) d\mu(x)$$

for its Fourier transform at ξ, where $e(y) = e^{2\pi i y}$. If $\mathcal{A} \subseteq [0, 1]$ then, for $\xi \in \mathbb{Z}$, this matches the definition of the ξth Fourier coefficient [16, §3]. The *Fourier dimension* of $\mathcal{A}$ is

$$\dim_F(\mathcal{A}) = \sup\{s \leqslant 1 : \exists \mu \in \mathcal{M}(\mathcal{A}) \quad \widehat{\mu}(\xi) \ll_s (1 + |\xi|)^{-s/2} \quad (\xi \in \mathbb{R})\}.$$

Remark 1.7. *Here $f \ll g$ means that $|f| \leqslant C|g|$ pointwise, for some C, and the subscript s means that C is allowed to depend on s. We will use this, as well as other Vinogradov and Bachmann–Landau notations, liberally throughout.*

Theorem 1.8. *Let $\varepsilon > 0$, $\alpha \in \mathcal{K}$ and $\gamma \in \mathbb{R}$. Denote by $\mathcal{E}$ the set of $\beta \in [0, 1]$ for which there exists $\delta \in \mathbb{R}$ such that (1.1) has only finitely many solutions $n \in \mathbb{N}$. Then $\dim_F(\mathcal{E}) = 0$.*

This follows immediately from the following assertion.

Theorem 1.9. *Let $\varepsilon > 0$, $\alpha \in \mathcal{K}$ and $\gamma \in \mathbb{R}$. Denote by $\mathcal{E}$ the set of $\beta \in [0, 1]$ for which there exists $\delta \in \mathbb{R}$ such that (1.1) has only finitely many solutions $n \in \mathbb{N}$. Let $\mu \in \mathcal{M}([0, 1])$, and assume that*

$$\widehat{\mu}(\xi) \ll (1 + |\xi|)^{-\tau} \qquad (\xi \in \mathbb{Z}) \tag{1.2}$$

for some $\tau > 0$. Then $\mu(\mathcal{E}) = 0$.

Applying Theorem 1.9 with μ as Lebesgue measure furnishes the following result.

Corollary 1.10. *Let $\varepsilon > 0$, $\alpha \in \mathcal{K}$ and $\gamma \in \mathbb{R}$. Then, for almost all $\beta \in \mathbb{R}$, if $\delta \in \mathbb{R}$ then* (1.1) *has infinitely many solutions $n \in \mathbb{N}$.*

This sharpens [6, Theorem 1.5], which has roughly $(\log n)^{\varepsilon - 1/2}$ on the right hand side.

In the course of our work, we also establish some results in lacunary approximation.

Theorem 1.11. *Let $n_1, n_2, \ldots$ be a lacunary sequence of positive integers, let $\varepsilon > 0$, and let μ be as in Theorem 1.9. Then, for μ-almost all $\beta \in [0, 1]$, if $\delta \in \mathbb{R}$ then*

$$\|n_t \beta - \delta\| < \frac{(\log t)^{3+\varepsilon}}{t} \tag{1.3}$$

has infinitely many solutions $t \in \mathbb{N}$.

Applying Theorem 1.11 with μ as Lebesgue measure delivers the following result.

Corollary 1.12. *Let $n_1, n_2, \ldots$ be a lacunary sequence of positive integers, and let $\varepsilon > 0$. Then, for almost all $\beta \in \mathbb{R}$, if $\delta \in \mathbb{R}$ then* (1.3) *has infinitely many solutions $t \in \mathbb{N}$.*

We will also be able to infer the following badly approximable analogue.

Theorem 1.13. *Let $n_1, n_2, \ldots$ be a lacunary sequence of positive integers, and let $\varepsilon > 0$. Then there exists $\mathcal{G} \subseteq \mathrm{Bad}$ with $\dim_\mathrm{H}(\mathcal{G}) = 1$ such that if $\beta \in \mathcal{G}$ and $\delta \in \mathbb{R}$ then* (1.3) *has infinitely many solutions $t \in \mathbb{N}$.*

Once more, we apply the short argument in [6, Section 5], giving rise to the following result about inhomogeneously badly approximable numbers.

Corollary 1.14. *Let $n_1, n_2, \ldots$ be a lacunary sequence of positive integers, and let $\varepsilon > 0$. Then there exists $\mathcal{G} \subseteq \mathcal{B}$ with $\dim_\mathrm{H}(\mathcal{G}) = 2$ such that if $(\beta, \delta) \in \mathcal{G}$ then* (1.3) *has infinitely many solutions $t \in \mathbb{N}$.*

We stress that our results are all uniform in the shift δ, which limits the set of available techniques. More is known when β is allowed to depend on δ. We state below a special case of [5, Theorem 1.8].

Theorem 1.15 *(Chow–Technau, 2021+). Let $\alpha \in \mathbb{R}$ be irrational and non-Liouville, and let $\gamma, \delta \in \mathbb{R}$. Let $\psi : \mathbb{N} \to (0, \infty)$ be monotonic with*

$$\sum_{n=1}^{\infty} \psi(n) \log n = \infty.$$

Then, for almost all $\beta \in \mathbb{R}$, there exist infinitely many $n \in \mathbb{N}$ such that

$$\|n\alpha - \gamma\| \cdot \|n\beta - \delta\| < \psi(n).$$

For example, taking $\psi(n) = 1/(n \log n \cdot \log \log n)$ gives

$$\liminf_{n \to \infty} n(\log n)\|n\alpha - \gamma\| \cdot \|n\beta - \delta\| = 0$$

in this context. Theorem 1.15 was proved using the Bohr set machinery developed in [3,4], and settled a conjecture of Beresnevich, Haynes and Velani [2].

The lacunary theory is also well understood when the shift is fixed. We state a special case of [19, Theorem 1].

Theorem 1.16 (*Pollington–Velani–Zafeiropoulos–Zorin, 2022*). *Let $\delta \in \mathbb{R}$, let $n_1, n_2, \ldots$ be a lacunary sequence of positive integers, and let μ be as in Theorem 1.9. Let $\psi : \mathbb{N} \to [0,1]$, and denote by $\mathcal{W}(\psi)$ the set of $\beta \in [0,1]$ such that*

$$\|n_t\beta - \delta\| \leqslant \psi(t)$$

has infinitely many solutions $t \in \mathbb{N}$. Then

$$\mu(\mathcal{W}(\psi)) = \begin{cases} 0, & \text{if } \displaystyle\sum_{t=1}^{\infty} \psi(t) < \infty \\[2em] 1, & \text{if } \displaystyle\sum_{t=1}^{\infty} \psi(t) = \infty. \end{cases}$$

Theorem 1.11 is quantitatively close to the threshold in Theorem 1.16 — as a function of t — and it comes with the 'shift-uniformity' attribute: the set of allowed β does not depend on δ in Theorem 1.11 (unlike Theorem 1.16, which does not have this attribute).

The analogous inhomogeneous multiplicative approximation problem with uniformity in both γ and δ has also been considered. Shapira [21] was the first to obtain such a result. Subsequently, Gorodnik and Vishe [12] demonstrated the following quantitative refinement.

Theorem 1.17 (*Gorodnik–Vishe, 2016*). *There exists $\kappa > 0$ such that, for almost all pairs $(\alpha, \beta) \in \mathbb{R}^2$,*

$$\liminf_{|n| \to \infty} |n|(\log \log \log \log \log |n|)^{\kappa}\|n\alpha - \gamma\| \cdot \|n\beta - \delta\| = 0$$

for any $\gamma, \delta \in \mathbb{R}$.

Our results are quantitatively much stronger than this. However, uniformity in both γ and δ presents a further challenge that our method seems unable to overcome.

2. Lacunary Diophantine inequalities

Various authors counted solutions to certain lacunary diophantine equations. We state, for imminent use, a variant of a lemma of Rudnick and Zaharescu, see [20, Lemma 2.2], in which the dependence on the number of variables is made explicit.

Lemma 2.1. *Let* $A_1 \geqslant A_2 \geqslant \cdots \geqslant A_s$ *be positive integers. Then, for any* $b \in \mathbb{Z}$ *and any* $Y \in \mathbb{N}$, *there are at most* $8^s Y^{s-1}$ *integer vectors* $\mathbf{y} \in [-Y, Y]^s$ *such that*

$$|A_1 y_1 + \cdots + A_s y_s + b| \leqslant A_1.$$

Proof. This is a straightforward adaptation of the proof of [20, Lemma 2.1]. The only change is to use an explicit version of the Lipschitz principle. The points lie in the region

$$\Omega = \{\mathbf{y} \in [-Y, Y]^s : |y_1 + \lambda_2 y_2 + \cdots + \lambda_s y_s + c| \leqslant 1\},$$

for some $\lambda_2, \ldots, \lambda_s \in (0, 1]$ and some $c \in \mathbb{R}$. By [1, Theorem 1.1], the number of integer points in Ω is at most

$$2^s Y^{s-1} + \sum_{j=0}^{s-1} \binom{s}{j} (2Y)^j \leqslant 8^s Y^{s-1},$$

as claimed. $\square$

By adapting the proof of [20, Lemma 2.3], we establish the following estimate for the number of solutions to a certain diophantine inequality. The result is sharp up to logarithmic factors, owing to possible diagonal solutions. Unlike [20, Lemma 2.3], it is uniform in the number of variables. It is also more general, in that it deals with inequalities and the ranges are allowed to differ. Our argument is slightly more refined, so we present it in full.

Lemma 2.2. *Let* $a(1), a(2), \ldots$ *be a lacunary sequence of positive integers, and let* $r \in (1, 2]$ *be such that*

$$a(n + 1) > r a(n) \qquad (n \in \mathbb{N}).$$

Let $s \in \mathbb{N}$, *let* $Z \geqslant Y \geqslant 2$, *and let* $K \in [0, a(Z)/8]$. *Assume that* $4s \leqslant Z$. *Then there exists a constant* $R = R(r) > 1$ *so that are at most*

$$s! R^s \max\{(YZ \log Z)^{s/2}, Y^{s-1} Z (\log Z)^{s-1}\}$$

integer solutions to

$$|y_1 a(z_1) + \cdots + y_s a(z_s)| \leqslant K$$

such that

$$0 < |y_j| \leqslant Y, \quad Z < z_j \leqslant 2Z \qquad (1 \leqslant j \leqslant s).$$

Proof. It suffices to count solutions with

$$z_1 \geqslant \cdots \geqslant z_s.$$

To each solution we associate a partition $(B_1, \ldots, B_\ell)$ of the indices as follows. The set B_1 comprises those j such that $z_j \geqslant z_1 - \frac{2 \log Z}{\log r}$. With $j_2 = \max B_1 + 1$, the set B_2 comprises those $j \geqslant j_2$ such that $z_j \geqslant z_{j_2} - \frac{2 \log Z}{\log r}$, and so on. We fix such a partition, giving

$$z_{j_k} < z_{j_{k-1}} - \frac{2 \log Z}{\log r} \leqslant z_{j_{k-1}} \qquad (2 \leqslant k \leqslant \ell)$$

and hence

$$a(z_{j_k}) < \frac{a(z_{j_{k-1}})}{Z^2} \qquad (2 \leqslant k \leqslant \ell).$$

Next, we count $(\mathbf{y}, \mathbf{z})$ corresponding to the partition $(B_1, \ldots, B_\ell)$.

We start by choosing z_j for $j \in B_k$ whenever $|B_k| \geqslant 2$. If there are t such values of k, then there are at most

$$Z^t \left(\frac{3 \log Z}{\log r} \right)^{s-\ell}$$

choices here. Next, we choose y_j whenever $B_k = \{z_j\}$. There are at most $(2Y)^{\ell-t}$ choices here.

By the triangle inequality, we have $|B_1| \geqslant 2$ and

$$|y_1 a(z_1) + \cdots + y_{j_2-1} a(z_{j_2-1})| < a(z_1).$$

This has at most $8^{j_2-1} Y^{j_2-2}$ solutions $(y_1, \ldots, y_{j_2-1})$, by Lemma 2.1.

If $|B_2| \geqslant 2$ then we have

$$|y_{j_2} a(z_{j_2}) + \cdots + y_{j_3-1} a(z_{j_3-1}) + b| < a(z_2),$$

for some $b \in \mathbb{Z}$ that has been determined. This has at most $8^{j_3-j_2} Y^{j_3-j_2-1}$ solutions $(y_{j_2}, \ldots, y_{j_3-1})$, by Lemma 2.1. Otherwise, $|B_2| = 1$ which implies

$$|y_2 a(z_2) + b| < \frac{a(z_2)}{2}.$$

If $y_2 > 0$ then

$$\frac{b}{2y_2} < a(z_2) < \frac{2b}{y_2},$$

which has at most $5/\log r$ solutions z_2. Similarly, if $y_2 < 0$ then there are $5/\log r$ solutions z_2.

Repeating the argument, our total count for this partition is

$$(240/\log r)^s Y^{s-t} Z^t (\log Z)^{s-\ell}.$$

Because $\ell \geqslant t$, this quantity is at most

$$(240/\log r)^s Z^t (Y \log Z)^{s-t}.$$

As $1 \leqslant t \leqslant s/2$, we obtain

$$(240/\log r)^s \max\{(YZ \log Z)^{s/2}, Y^{s-1} Z (\log Z)^{s-1}\}.$$

Finally, there are at most 2^s partitions, and at most $s!$ permutations of the variables. $\quad\square$

3. Dispersion

We require a quantitative description of how dense a finite set of points is. Whereas previous authors have used discrepancy, we use dispersion. Given a set $\mathcal{A} \subset [0,1)$ of $T \geqslant 2$ elements

$$0 \leqslant a_1 < a_2 < \cdots < a_T < 1,$$

its *dispersion* (the largest gap in the nearest neighbour spacing statistic) is

$$\mathrm{disp}(\mathcal{A}) = \max_{t \leqslant T}(a_{t+1} - a_t) \quad \text{where} \quad a_{T+1} = a_1 + 1,$$

cf. [8, Definition 1.15]. If the elements in $\mathcal{A}$ are chosen independently and uniformly in $[0,1)$ at random, then $\mathrm{disp}(\mathcal{A}) \asymp \frac{\log T}{T}$ with high probability — see David and Nagaraja's monograph on order statistics [7, Section 6.4]. The following result asserts that this is approximately true when $\mathcal{A}$ is a segment of a generically-dilated lacunary sequence.

Theorem 3.1. *Let $n_1, n_2, \ldots$ be a lacunary sequence of positive integers, and let $r \in (1,2]$ be such that*

$$n_{t+1} > r n_t \qquad (t \in \mathbb{N}).$$

Let $\varepsilon > 0$, and let μ, τ be as in Theorem 1.9. Then

$$\mathrm{disp}(\{n_t\beta \bmod 1 : T < t \leqslant 2T\}) \ll_{\beta,r,\varepsilon,\tau} \frac{(\log T)^{3+\varepsilon}}{T}, \qquad (3.1)$$

for μ-almost all $\beta \in [0, 1]$.

As $\varepsilon > 0$ is arbitrary, it suffices to establish (3.1) with the right hand side replaced by D/T, where

$$D = (\log T)^{3+2\varepsilon}.$$

The plan is to show that, generically, intervals of length roughly D/T have the expected number of points. To this end, we use a concentration argument and the union bound.

Let $\omega : \mathbb{R} \to \mathbb{R}_{\geqslant 0}$ be a bump function supported on $[-1, 1]$ such that $\int_{-\infty}^{\infty} \omega(x)\, \mathrm{d}x = 1$. This necessarily has the Fourier decay property

$$\widehat{\omega}(\xi) \ll_N (1 + |\xi|)^{-N}$$

for each $N > 0$, which follows from integration by parts — see for instance [17, Equation (3.1.12)]. Let $c \in (0, 1)$, and define

$$C_T(\beta) = C_{T,c}(\beta) = \sum_{u \in \mathbb{Z}} \sum_{T < t \leqslant 2T} \omega\left(\frac{n_t\beta - c + u}{D/T}\right), \qquad C(\beta) = C_T(\beta) - D.$$

To motivate the use of $C_T(\beta)$ and outline our proof strategy, observe that it is a weighted count of integers $t \in (T, 2T]$ such that

$$n_t\beta \in [c - D/T, c + D/T] \bmod 1.$$

In the final stages of the proof, we will cover $[0, 1)$ by $O(T/D)$ many intervals of length $2D/T$. If $C_T(\beta)$ is positive for each of the centres c, then the largest gap cannot contain an entire such interval, and so we will conclude that the dispersion is bounded by $4D/T$. We will achieve this for almost all β by showing that $C_T(\beta)$ is close to its mean D with high probability.

Poisson summation yields

$$C_T(\beta) = \frac{D}{T} \sum_{\ell \in \mathbb{Z}} \sum_{T < t \leqslant 2T} \widehat{\omega}(D\ell/T) e(\ell(n_t\beta - c)).$$

Thus, as $\widehat{\omega}(0) = 1$, we have

$$C(\beta) = \frac{D}{T} \sum_{\ell \neq 0} \sum_{T < t \leqslant 2T} \widehat{\omega}(D\ell/T) e(\ell(n_t\beta - c)).$$

Define

$$L = \frac{T}{(\log T)^{3+\varepsilon}}$$

and

$$C_0(\beta) = \frac{D}{T} \sum_{0<|\ell|\leqslant L} \widehat{\omega}\left(D\ell/T\right) \sum_{T<t\leqslant 2T} e(\ell(n_t\beta - c)). \tag{3.2}$$

Observe from the rapid decay of $\widehat{\omega}$ that

$$C_0(\beta) = C(\beta) + O((\log T)^{-100}).$$

Remark 3.2. Here and in what follows, the implied constants are allowed to depend on ω without specific indication. However, the implied constants will always be independent of the shift parameter c.

We may assume that $T \geqslant 100$. Let $s \in \mathbb{N}$ with $8s \leqslant T$. We begin with a Fourier series expansion

$$|C_0(\beta)|^{2s} = \sum_{k\in\mathbb{Z}} a_k e(k\beta), \tag{3.3}$$

noting that the series on the right is a finite sum. Our goal is to bound

$$\|C_0\|^{2s}_{L^{2s}(\mu)} = \int_0^1 |C_0(\beta)|^{2s} \mathrm{d}\mu(\beta) = \sum_{k\in\mathbb{Z}} a_k \int_0^1 e(k\beta)\mathrm{d}\mu(\beta) = \sum_{k\in\mathbb{Z}} a_k \widehat{\mu}(-k).$$

This will enable us to show that $C_0(\beta)$ is generically small, and we will see from there that $C_T(\beta)$ concentrates around D.

We will evaluate the Fourier coefficients a_k in terms of the Fourier coefficients of $C_0(\beta)$. We compute from (3.2) that

$$C_0(\beta) = \frac{D}{T} \sum_{0<|\ell|\leqslant L} \widehat{\omega}\left(D\ell/T\right) \sum_{T<t\leqslant 2T} e(\ell(n_t\beta - c))$$

$$= \frac{D}{T} \sum_{0<|\ell|\leqslant L} \sum_{T<t\leqslant 2T} \widehat{\omega}\left(D\ell/T\right) e(-\ell c) e(\ell n_t\beta) = \sum_{m\in\mathbb{Z}} b_m e(m\beta),$$

where

$$b_m = \frac{D}{T} \sum_{(\ell,t)} \widehat{\omega}\left(D\ell/T\right) e(-\ell c),$$

and the summation runs over all integer pairs (ℓ, t) such that

$$0 < |\ell| \leqslant L, \qquad T < t \leqslant 2T, \qquad \ell n_t = m.$$

Now

$$
\begin{aligned}
|C_0(\beta)|^{2s} &= \prod_{j \leqslant s} \sum_{m_j, m_{s+j} \in \mathbb{Z}} b_{m_j} \overline{b_{m_{s+j}}} e((m_j - m_{s+j})\beta) \\
&= \left(\frac{D}{T}\right)^{2s} \prod_{j \leqslant s} \sum_{\substack{0 < |\ell_j|, |\ell_{s+j}| \leqslant L \\ T < t_j, t_{s+j} \leqslant 2T}} \widehat{\omega}(D\ell_j/T) \overline{\widehat{\omega}(D\ell_{s+j}/T)} \\
&\qquad\qquad\qquad\qquad \cdot e((\ell_{s+j} - \ell_s)c) e((\ell_j n_{t_j} - \ell_{s+j} n_{t_{s+j}})\beta) \\
&= \sum_{k \in \mathbb{Z}} a_k e(k\beta),
\end{aligned}
$$

where we see that

$$
a_k = \left(\frac{D}{T}\right)^{2s} \sum_{\substack{0 < |\ell_1|, \ldots, |\ell_{2s}| \leqslant L \\ T < t_1, \ldots, t_{2s} \leqslant 2T \\ \sum_{j \leqslant s}(\ell_j n_{t_j} - \ell_{s+j} n_{t_{s+j}}) = k}} \widehat{\omega}(D\ell_j/T) \overline{\widehat{\omega}(D\ell_{s+j}/T)} e((\ell_{s+j} - \ell_s)c).
$$

Note that $|\widehat{\omega}(\xi)| \leqslant \int_{-\infty}^{\infty} \omega(x)\, dx = 1$, for any $\xi \in \mathbb{R}$. Therefore

$$|a_k| \leqslant (D/T)^{2s} N(k) \qquad (k \in \mathbb{Z}),$$

where $N(k)$ counts $(\ell_1, \ldots, \ell_{2s}, t_1, \ldots, t_{2s}) \in \mathbb{Z}^{4s}$ such that

$$0 < |\ell_j| \leqslant L, \quad T < t_j \leqslant 2T \qquad (1 \leqslant j \leqslant 2s) \tag{3.4}$$

and

$$\sum_{j \leqslant s}(\ell_j n_{t_j} - \ell_{s+j} n_{t_{s+j}}) = k.$$

Let $R = R(r) > 1$ be as in Lemma 2.2.

Lemma 3.3. *Let $T \geqslant 100$, and let $s \in \mathbb{N}$ with $8s \leqslant T$. Let n_T be as in Theorem 3.1, and let $K \in [0, n_T/8]$. For $k \in \mathbb{Z}$, let a_k be as in* (3.3). *Then*

$$\sum_{|k| \leqslant K} |a_k| \leqslant \frac{(2s)! R^{2s} D^{2s}}{(\log T)^{2s + s\varepsilon}}.$$

Proof. Observe that

$$\sum_{|k| \leqslant K} |a_k| \leqslant (D/T)^{2s} \mathfrak{N},$$

where $\mathfrak{N}$ counts $(\ell_1, \ldots, \ell_{2s}, t_1, \ldots, t_{2s}) \in \mathbb{Z}^{4s}$ in the ranges (3.4) such that

$$\left| \sum_{j \leqslant s} (\ell_j n_{t_j} - \ell_{s+j} n_{t_{s+j}}) \right| \leqslant K.$$

By Lemma 2.2, we thus have

$$\sum_{|k| \leqslant K} |a_k| \leqslant (D/T)^{2s} (2s)! R^{2s} \frac{T^{2s}}{(\log T)^{2s+s\varepsilon}} = \frac{(2s)! R^{2s} D^{2s}}{(\log T)^{2s+s\varepsilon}}. \qquad \square$$

Observe that

$$\sum_{k \in \mathbb{Z}} |a_k| \leqslant (D/T)^{2s} \sum_{k \in \mathbb{Z}} N(k) \leqslant (D/T)^{2s} (2LT)^{2s} = (2DL)^{2s}. \qquad (3.5)$$

Proof of Theorem 3.1. We use Lemma 3.3 for the small indices, and the estimates (1.2), (3.5) for the large ones. This gives

$$\|C_0\|_{L^{2s}(\mu)}^{2s} = \sum_{k \in \mathbb{Z}} a_k \widehat{\mu}(-k) \ll \sum_{|k| \leqslant K} |a_k| + K^{-\tau} \sum_{|k| > K} |a_k|$$

$$\leqslant \frac{(2s)! R^{2s} D^{2s}}{(\log T)^{2s+s\varepsilon}} + \frac{(2DL)^{2s}}{K^\tau}$$

for any $K \in [0, n_T/8]$. If T is sufficiently large in terms of r, τ, then we can choose

$$s = \left\lfloor \frac{\log T}{\sqrt{\log \log T}} \right\rfloor, \qquad K = T^{2s/\tau},$$

whence

$$\|C_0\|_{L^{2s}(\mu)}^{2s} \ll \frac{(2s)! R^{2s} D^{2s}}{(\log T)^{2s+s\varepsilon}}.$$

Consequently

$$\mu(\{\beta \in [0,1] : |C_T(\beta) - D| > D/2\}) \ll \frac{(2s)!(2R)^{2s}}{(\log T)^{2s+s\varepsilon}}.$$

Let us now choose a set $\mathcal{C} = \mathcal{C}(T) \subset [0,1)$ of cardinality $O(T/D)$ such that the balls of radius D/T centred in $\mathcal{C}$ cover $[0,1)$. As

$$\frac{(2s)!(2R)^{2s}}{(\log T)^{2s+s\varepsilon}} = \exp\left(2s \log s + O(s) - (2+\varepsilon)s \log \log T\right)$$

$$= \exp(2 \log T \sqrt{\log \log T} + o(\log T) - (2+\varepsilon) \log T \sqrt{\log \log T})$$

$$= \exp(o(\log T) - \varepsilon \log T \sqrt{\log \log T}) = T^{o(1) - \varepsilon \sqrt{\log \log T}}.$$

Recalling that $C_T(\beta) = C_{T,c}(\beta)$, we see that

$$\sum_{T=1}^{\infty} \sum_{c \in \mathcal{C}} \mu(\{\beta \in [0,1] : |C_{T,c}(\beta) - D| > D/2\})$$

$$\ll \sum_{T=1}^{\infty} T \frac{(2s)!(2R)^{2s}}{(\log T)^{2s+s\varepsilon}} = \sum_{T=1}^{\infty} T^{1+o(1) - \varepsilon \sqrt{\log \log T}} \ll \sum_{T=1}^{\infty} T^{-2} < \infty.$$

Thus, by the first Borel–Cantelli lemma, for μ-almost all $\beta \in [0,1]$, if $T \in \mathbb{N}$ is sufficiently large in terms of $\beta, r, \varepsilon, \tau$ then

$$C_{T,c}(\beta) > 0 \qquad (c \in \mathcal{C}).$$

In particular, for each $c \in \mathcal{C}$ there exists $t = t(c) \in (T, 2T]$ such that

$$\|n_t \beta - c\| \ll D/T.$$

Therefore

$$\mathrm{disp}(\{n_t \beta \bmod 1 : T < t \leqslant 2T\}) \ll D/T,$$

as claimed. $\square$

4. Completing the proofs

In this section, we combine Theorem 3.1 with existing tools to establish Theorems 1.5, 1.9, 1.11 and 1.13. Let $f : \mathbb{N} \to [0,1]$ be non-increasing with the doubling property $f(t) \ll f(2t)$, and assume that Theorem 3.1 holds with (3.1) replaced by

$$\mathrm{disp}(\{n_t \beta \bmod 1 : T < t \leqslant 2T\}) = o(f(T)) \qquad (T \to \infty). \qquad (4.1)$$

By Theorem 3.1, applied with $\varepsilon/2$ in place of ε, we can take

$$f(t) = \frac{(\log t)^{3+\varepsilon}}{t}.$$

We now show that Theorem 1.11 holds, more generally, with $o(f(t))$ in place of $(\log t)^{3+\varepsilon}/t$.

Proof of Theorem 1.11. For μ-almost all $\beta \in [0,1]$, we have (4.1). Thus, if $T \in \mathbb{N}$ then there exists $t \in (T, 2T]$ such that

$$\|n_t \beta - \delta\| = o(f(T)).$$

In particular, there exists an increasing sequence of positive integers t along which

$$\|n_t \beta - \delta\| = o(f(t)). \quad \square$$

We require the following variant of a lemma of the first author and Zafeiropoulos, which draws its power from a quantitative version of the Three Distance Theorem.

Lemma 4.1 *(Chow–Zafeiropoulos, 2021). Let $\alpha \in \mathcal{K}$ and $\gamma \in \mathbb{R}$. Then there exists a sequence $n_1, n_2, \ldots$ of positive integers such that*

$$8^t \leqslant n_t \leqslant 4e^{6\Lambda(\alpha)t}, \quad n_t \|n_t \alpha - \gamma\| \leqslant 8, \quad n_{t+1} > 2n_t \quad (t \in \mathbb{N}).$$

Proof. See [6, Lemma 2.1], and its proof (to see that $n_{t+1} > 2n_t$). $\quad \square$

We now show that Theorem 1.9 holds, more generally, with $f(\log n)$ in place of $(\log \log n)^{3+\varepsilon}/\log n$ in (1.1).

Proof of Theorem 1.9. Let $\alpha \in \mathcal{K}$ and $\gamma \in \mathbb{R}$, and let $n_1, n_2, \ldots$ be as in Lemma 4.1. By Theorem 1.11, for μ-almost all $\beta \in \mathbb{R}$, if $\delta \in \mathbb{R}$ then

$$\|n_t \beta - \delta\| = o(f(t))$$

along some infinite sequence of $t \in \mathbb{N}$. As $\log n_t \ll_\alpha t$, and f is non-increasing with the doubling property, we have $f(t) \ll_\alpha f(\log n_t)$, whence

$$n_t \cdot \|n_t \alpha - \gamma\| \cdot \|n_t \beta - \delta\| = o(f(\log n_t))$$

along this sequence. $\quad \square$

We now show that Theorem 1.5 holds, more generally, with $f(\log n)$ in place of $(\log \log n)^{3+\varepsilon}/\log n$ in (1.1).

Proof of Theorem 1.5. Let $s < 1$. Kaufman [14] constructed

$$\mu \in \mathcal{M}(\mathrm{Bad} \cap [0,1])$$

with the following two key properties:

(i) (Frostman dimension) For any interval $I \subseteq [0, 1]$, we have

$$\mu(I) \ll_s \lambda(I)^s,$$

where λ denotes Lebesgue measure.

(ii) (Polynomial Fourier decay) We have

$$\widehat{\mu}(\xi) \ll (1 + |\xi|)^{-7/10^4} \qquad (\xi \in \mathbb{R}).$$

We showed that Theorem 1.9 holds, more generally, with $f(\log n)$ in place of $(\log \log n)^{3+\varepsilon} / \log n$ in (1.1). Applying this to the measure μ gives $\mu(\mathcal{E}) = 0$, where $\mathcal{E}$ is the set of $\beta \in [0, 1]$ for which there exists $\delta \in \mathbb{R}$ such that

$$n \|n\alpha - \gamma\| \cdot \|n\beta - \delta\| < f(\log n)$$

has only finitely many solutions $n \in \mathbb{N}$. Choosing $\mathcal{G} = \text{Bad} \cap [0, 1] \setminus \mathcal{E}$, we have $\mu(\mathcal{G}) = 1 > 0$. Now the mass distribution principle [10, Chapter 4] reveals that $\dim_{\text{H}}(\mathcal{G}) \geqslant s$. As s can be taken arbitrarily close to 1, we must have $\dim_{\text{H}}(\mathcal{G}) = 1$. □

Theorem 1.13 follows in the same way, applying Theorem 1.11 instead of Theorem 1.9.

Closing remarks. In light of the aforementioned random heuristic, it is conceivable that one might obtain $(\log T)^{1+\varepsilon} / T$ on the right hand side of (3.1). This would be interesting in its own right, and would also strengthen our other results. Getting the optimal order of the dispersion will likely involve lengthy arguments, as this extremal order statistic is non-trivial to compute even with perfectly independent random variables at hand — a luxury we are not granted in the diophantine setting!

Declaration of competing interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

Data availability

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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