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A simplicial version of the 2–dimensional Fulton–MacPherson operad

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We define an operad in Top, called FM_2^W. The spaces in FM_2^W come with CW decompositions such that the operad compositions are cellular. In fact, each space in FM_2^W is the realization of a simplicial set. We expect, but do not prove here, that FM_2^W is isomorphic to the 2–dimensional Fulton–MacPherson operad FM_2. Our construction is connected to the author's work on the symplectic $(A_\infty, 2)$–category, and suggests a strategy toward equipping the symplectic cochain complex with the structure of a homotopy Batalin–Vilkovisky algebra.

18M75, 55P48; 53D37

1 Introduction

Getzler and Jones [1994] introduced the Fulton–MacPherson operad

$$\mathrm{FM}_2 = (\mathrm{FM}_2(k))_{k \geq 1}, \tag{1}$$

where $\mathrm{FM}_2(k)$ is the compactification à la Fulton and MacPherson [1994] of the configuration space of k distinct labeled points in $\mathbb{R}^2$, modulo translations and dilations. Getzler and Jones proposed in the same paper a collection of cellular decompositions of the spaces in FM_2, such that these decompositions are compatible with the operad maps $\circ_i : \mathrm{FM}_2(k) \times \mathrm{FM}_2(l) \to \mathrm{FM}_2(k+l-1)$. These decompositions formed the basis for a significant amount of work related to the Deligne conjecture, including a proof in [Getzler and Jones 1994] of that conjecture.

Unfortunately, Tamarkin found an error in Getzler and Jones' decomposition. In particular, in the 9–dimensional space $\mathrm{FM}_2(6)$, there are two disjoint open 6–cells C_1 and C_2 with the property that $\overline{C}_1 \cap C_2$ is nonempty, as described in [Voronov 2000, Section 1.2.2]. Salvatore [2022] used meromorphic differentials to construct cellular decompositions of the spaces in FM. His approach is completely different from Getzler and Jones'.

We construct an operad of CW complexes, which we conjecture to be isomorphic in Top to FM_2. Under this expected isomorphism, our decompositions are refinements of Getzler and Jones' attempted decompositions. The context for the current paper is the author's program (as developed in [Bottman 2015; 2019a; 2019b; 2020; Bottman and Carmeli 2021; Bottman and Oblomkov 2019; Bottman and Wehrheim 2018]) to construct Symp, the symplectic $(A_\infty, 2)$–category. Specifically, the author plans to use the decompositions of FM that we construct here to understand the axioms for identity 1–morphisms

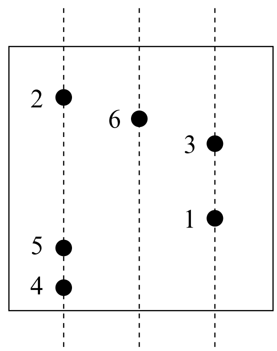

Figure 1

in an $(A_\infty, 2)$–category. In the context of Symp, this suggests a strategy toward endowing symplectic cohomology with a chain-level homotopy Gerstenhaber (and eventually, homotopy BV) algebra structure that is finite in each arity, thus answering Conjecture 2.6.1 of [Abouzaid 2015]. We note that our approach is compatible with the operations in Symp, unlike Salvatore's; in addition, we expect our approach to generalize to the Fulton–MacPherson operad of any dimension.

1.1 Getzler and Jones' attempted decomposition

Getzer and Jones' attempted decomposition is an adaptation to the case of FM_2 of Fox and Neuwirth's decomposition [1962] of the one-point compactification of the configuration space $(\mathbb{R}^2)^k \setminus \Delta$ of k points in $\mathbb{R}^2$, where Δ is the fat diagonal. A Fox–Neuwirth cell corresponds to a choice of which subsets of the points $p_1, \dots, p_k$ should be vertically aligned, the left-to-right order in which these subsets of points should appear, and the top-to-bottom order in which each subset of the points should appear. For instance, Figure 1 is a real-codimension-3 cell in $((\mathbb{R}^2)^6 \setminus \Delta)^*$. Getzler and Jones observed that the Fox–Neuwirth cells are invariant under translations and dilations, and moreover that one can define a similar type of cell for the boundary locus. The elements in the boundary of $\mathrm{FM}_2(k)$ are trees of "screens", and these "boundary cells" are defined by partitioning and ordering the points on each of the screen in the same way as with Fox–Neuwirth cells.

1.2 Tamarkin's counterexample

As described in [Voronov 2000], Tamarkin observed a way in which Getzler and Jones' supposed decomposition fails. Consider $\mathrm{FM}_2(6)$, the open locus of which parametrizes configurations of six distinct points in $\mathbb{R}^2$, up to translations and dilations. Next, we consider the two 6–cells C_1 and C_2 in Figure 2 (we omit the numberings). The j^{th} bubble in C_2 (for $j = 1, 2$) carries a modulus λ_j defined in the following way: by translating and dilating, we can move the left and right lines to $x = 0$ and $x = 1$, respectively; we then denote by λ_j the position of the middle line. The intersection $\overline{C}_1 \cap C_2$ is the codimension-1 locus in C_2 in which $\lambda_1 = \lambda_2$. What Getzler and Jones proposed is therefore not a cellular decomposition, because the intersection of the closures of two distinct n cells should be contained in the $(n-1)$–skeleton.

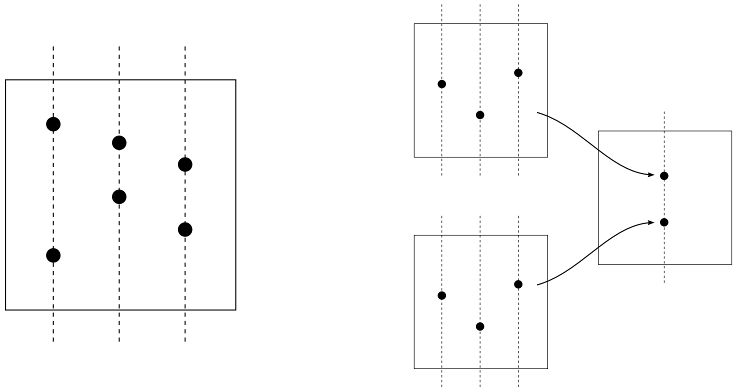

Figure 2

In our construction, C_1, C_2, and $\overline{C}_1 \cap C_2$ will each be a union of cells.

1.3 An overview of our construction

We construct a collection of CW complexes $\mathrm{FM}_2^W(k)$ and maps

$$\circ_i : \mathrm{FM}_2^W(k) \times \mathrm{FM}_2^W(l) \to \mathrm{FM}_2^W(k+l-1) \quad \text{for } 1 \le i \le k. \tag{2}$$

Here is our main result:

Main Theorem *The spaces* $(\mathrm{FM}_2^W(k))_{k\ge 1}$ *together with the composition operations* $\circ_i$ *form a non-*Σ *operad, and the composition maps*

$$\circ_i : \mathrm{FM}_2^W(k) \times \mathrm{FM}_2^W(l) \to \mathrm{FM}_2^W(k+l-1) \tag{3}$$

are cellular.

We will now give a brief overview of the definition of $\mathrm{FM}_2^W(k)$.

(i) First, we define a "W–version" $W_{\boldsymbol{n}}^W$ of the 2–associahedra by the analogy

$$K_r : W(\mathrm{Ass}) \quad :: \quad W_{\boldsymbol{n}} : W_{\boldsymbol{n}}^W. \tag{4}$$

Here K_r is the $(r-2)$–dimensional associahedron, and $W(\mathrm{Ass})$ is the Boardman–Vogt W–construction applied to the associative operad, which is defined in terms of metric stable trees and yields an operad of CW complexes that is isomorphic to the associahedral operad K in Top. $W_{\boldsymbol{n}}$ is an $(|\boldsymbol{n}|+r-3)$–dimensional 2–associahedron, and $W_{\boldsymbol{n}}^W$ is a CW complex that we define in Section 2 in terms of metric stable tree-pairs and which we expect to be homeomorphic to $W_{\boldsymbol{n}}$. We then refine the CW structure on $W_{\boldsymbol{n}}^W$ to a simplicial decomposition.

(ii) Toward our construction of $\mathrm{FM}_2^W(k)$, we decompose $\mathrm{FM}_2(k)$ into Getzler–Jones cells, then identify each open Getzler–Jones cell with a product of open 2–associahedra. We then replace each such product by the corresponding product of interiors of the spaces $W_{\boldsymbol{n}}^W$ described in the previous step. This product comes with a decomposition into products of simplices, and we refine this to a simplicial structure. Finally, we attach these decomposed Getzler–Jones cells together to produce $\mathrm{FM}_2^W(k)$. This part of the construction appears in Section 3.

The essential property of $\mathrm{FM}_2^W(k)$ that we must verify is that our CW decomposition is valid. It is clear that our putative open cells disjointly decompose our space, and that they are homeomorphic to open balls. The only nontrivial check we need to make is that the n–cells are attached to the $(n-1)$–skeleton. This is where Getzler and Jones' attempted decomposition fails: the 6–cell C_1 that we described in Section 1.2 is not attached to the 5–skeleton. Our decomposition satisfies this property by construction: we attach a given n–cell by taking a closed n–simplex, then attaching it to the existing skeleton via quotient maps from the boundary $(n-1)$–simplices to the $(n-1)$–skeleton. In fact, the boundary of an n–cell is a union of cells of dimension at most $n-1$.

1.4 The relationship between our construction and Symp

The genesis of the construction of FM_2^W was a connection between the symplectic $(A_\infty, 2)$–category Symp and E_2 suggested by Jacob Lurie in 2016. (The construction of Symp is a long-term project of the author, building on work of Ma'u, Wehrheim, and Woodward; see [Bottman 2015; 2019a; 2019b; 2020; Bottman and Carmeli 2021; Bottman and Wehrheim 2018; Ma'u et al. 2018].) We can express this connection concretely, via a collection of maps

$$f_\sigma^W : W_{\boldsymbol{n}}^W \to \mathrm{FM}_2^W(|\boldsymbol{n}|), \tag{5}$$

where σ is a 2–*permutation*, as defined in Section 3.2. The idea of this map is very simple. The map f_σ forgets the data of the lines, then labels the points according to the 2–permutation σ. Then f_σ extends continuously to the boundary of $W_{\boldsymbol{n}}$; it is an embedding on the interior of its domain, but contracts some boundary cells.

Example 1.1 In Figure 3, we depict W_{111} and its image under an appropriate map f_σ. More precisely, we depict their nets — to "assemble" both CW complexes, one would cut them out, then glue together like-numbered edges. As is evident, most of the 2–cells of W_{111} are contracted by f_σ.

While it would take us too far afield to explain the relationship between FM_2 and Symp (and their W–counterparts) in detail, let us indicate the basic idea. Symp, being an $(A_\infty, 2)$–category, assigns to a chain in a 2–associahedron $W_{\boldsymbol{n}}$ an operation on 2–morphisms. (For instance, the objects of Symp are symplectic manifolds, and given two objects M_0 and M_1, the 1–morphism category is $\mathrm{Fuk}(M_0^- \times M_1)$; 2–associahedra W_n, where n is a single positive integer, act on this Fukaya category by the usual A_∞–operations.) The current definition of an $(A_\infty, 2)$–category, appearing in [Bottman and Carmeli 2021], does not equip identity 1–morphisms with all the possible structure. Indeed, when defining operations on 2–morphisms in the situation where some of the 1–morphisms are identities, those 1–morphisms should be allowed to be "moved past" the other 1–morphisms. To make this precise, one exactly needs to understand the maps f_σ, and to equip their targets with a CW structure so that f_σ is cellular. One way to

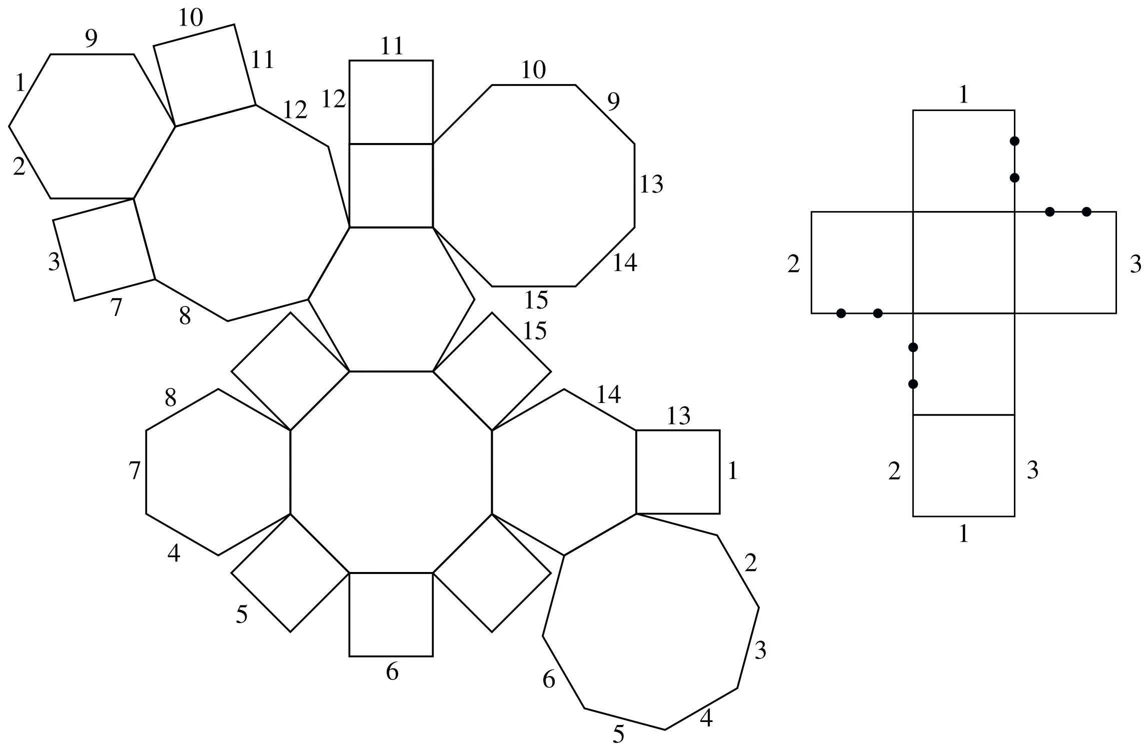

Figure 3

proceed toward this goal is to first decompose FM_2^W so that f_σ^W is cellular, and next construct coherent homeomorphisms $W_{\boldsymbol{n}} \cong W_{\boldsymbol{n}}^W$ and $\mathrm{FM}_2(k) \cong \mathrm{FM}_2^W(k)$.

The following result therefore shows the way toward a connection between the symplectic $(A_\infty, 2)$–category and FM_2^W. It is an immediate consequence of our construction of $W_{\boldsymbol{n}}^W$ and $\mathrm{FM}_2^W(k)$, and it forms the content of Remark 3.14.

Proposition *Fix* $r \geq 1$, $\boldsymbol{n} \in \mathbb{Z}_{\geq 0}^r \setminus \{\mathbf{0}\}$, *and a* 2*–permutation* σ *of type* $\boldsymbol{n}$*. Then the associated map*

$$f_\sigma^W : W_{\boldsymbol{n}}^W \to \mathrm{FM}_2^W(|\boldsymbol{n}|) \tag{6}$$

is cellular.

1.5 Future directions

The author plans to develop several aspects of the current paper. In particular:

- With several collaborators, the author plans to extend this work to produce cellular decompositions of FM_k^W for all $k \geq 1$, and to show that FM_k^W is isomorphic to FM_k in Top.
- This paper can be construed as a way of incorporating identity 1–morphisms into the symplectic $(A_\infty, 2)$–category. The author plans to formalize this in future work on the algebra of $(A_\infty, 2)$–categories.

- We plan to upgrade this work to give a cellular model for the framed analogue of the Fulton–MacPherson operad. This suggests a way of endowing symplectic cohomology with a chain-level BV algebra structure, which is the subject of Conjecture 2.6.1 of [Abouzaid 2015].

Acknowledgments

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Jacob Lurie drew an analogy that suggested to the author that there must be a link between $(A_\infty, 2)$–categories and E_2–algebras. Alexander Voronov explained to the author the colorful history surrounding this problem. A conversation with Naruki Masuda, Hugh Thomas, and Bruno Vallette led the author to think about replacing FM_2 with a "W–construction version" thereof. The author thanks Dean Barber, Michael Batanin, Sheel Ganatra, Ezra Getzler, Mikhail Kapranov, Ben Knudsen, Paolo Salvatore, and Dev Sinha for their interest and encouragement.

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2 A "W–version" of the 2–associahedra

In this section, we construct a "W–version" of the 2–associahedra. (The 2–associahedra were originally defined in [Bottman 2019a].) This is an essential ingredient in our definition of $\mathrm{FM}_2^W(k)$, which will appear in Section 3.

2.1 A warm-up: K^W, ie $W(\mathrm{Ass})$, ie a W–version of the associahedra

In this subsection, we recall a certain operad, which we will denote by $K^W = (K_r^W)_{r\geq 1}$. This is simply the Boardman–Vogt W–construction applied to the associative operad Ass. We construct only K^W rather than recalling the general definition of the W–construction, because this one-off construction will be a useful warm-up to our construction of W^W later in this section. As noted in [Barber 2013], K^W is isomorphic in Top to the associahedral operad K.

The following proposition summarizes what we will prove about K^W:

Proposition 2.1 *The spaces $(K_r^W)_{r\geq 1}$ form a non-Σ operad of CW complexes, and the composition maps*

$$\circ_i \colon K_r^W \times K_s^W \to K_{r+s-1}^W \tag{7}$$

defined in Definition 2.11 are cellular.

We will prove Proposition 2.1 at the end of the current subsection.

We begin with a definition of rooted ribbon trees. Stable rooted ribbon trees with r leaves index the strata of the associahedron K_r, and they will be an integral part of the definition of K_r^W.

Definition 2.2 [Bottman 2019a, Definition 2.2] A *rooted ribbon tree* (RRT) is a tree T with a choice of a root $\alpha_{\mathrm{root}} \in T$ and a cyclic ordering of the edges incident to each vertex; we orient such a tree toward the root. We say that a vertex α of an RRT T is *interior* if the set $\mathrm{in}(\alpha)$ of its incoming neighbors is nonempty, and we denote the set of interior vertices of T by T_{int}. An RRT T is *stable* if every interior vertex has at least two incoming edges. We define K_r^{tree} to be the set of all isomorphism classes of stable rooted ribbon trees with r leaves.

We denote the i^{th} leaf of an RRT T by λ_i^T. For any $\alpha, \beta \in T$, $T_{\alpha\beta}$ denotes those vertices γ such that the path $[\alpha, \gamma]$ from α to γ passes through β. We define $T_\alpha := T_{\alpha_{\mathrm{root}}\alpha}$.

Remark 2.3 Ribbon trees (resp. rooted ribbon trees) are often referred to as planar trees (resp. planted trees).

Next, we define a version of RRTs with internal edge lengths:

Definition 2.4 A *metric RRT* $(T, (\ell_e))$ is the data of

- an RRT T, and
- for every edge e of T not incident to a leaf (but possibly incident to the root), a length $\ell_e \in [0, 1]$.

We call this a *metric RRT of type* T.

Now we will define a "dimension" function d on stable RRTs:

Definition 2.5 [Bottman 2019a, Definition 2.4] For T a stable RRT in K_r^{tree}, we define its *dimension* $d(T) \in [0, r-2]$ like so:

$$d(T) := r - \#T_{\mathrm{int}} - 1. \tag{8}$$

Definition 2.6 Given a stable tree T, the *cell associated to* T is denoted by C_T and is defined to consist of all metric RRTs of type T.

Note that we can canonically identify C_T with the closed cube of dimension equal to the number of internal edges of T. That is:

$$C_T \cong [0, 1]^{\#T_{\mathrm{int}}-1} = [0, 1]^{r-2-d(T)}. \tag{9}$$

As we will see, K_r^W is $(r-2)$–dimensional; it follows that $d(T)$ is the codimension of C_T in K_r^W. (The unfortunate clash of terminology between "dimension" and "codimension" is due to the fact that, in K_r, the cell indexed by T has dimension $d(T)$.)

We now define K_r^W by taking the union of the cells C_T for T any stable RRT with r leaves, then collapsing edges of length 0.

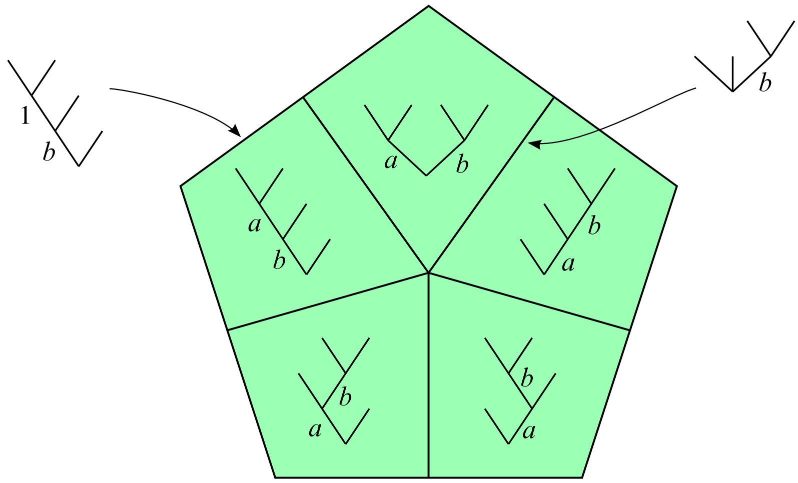

Figure 4

Definition 2.7 Given $r \geq 1$, we define K_r^W to be the following quotient:

$$K_r^W := \Big(\bigsqcup_{T \in K_r^{\mathrm{tree}}} C_T \Big) \Big/ \sim. \tag{10}$$

Here $\sim$ identifies $(T, (\ell_e))$ and $(T', (\ell'_e))$ if, after collapsing all edges e of T with $\ell_e = 0$ and all edges e of T' with $\ell'_e = 0$, both metric RRTs reduce to the same metric RRT $(T'', (\ell''_e))$.

Example 2.8 In Figure 4, we depict the CW complex K_4^W. Note that this is a refinement of K_4, which (as a CW complex) is a pentagon. We have labeled the open top cells by the metric stable RRTs that they parametrize, where each a and b is allowed to vary in $[0, 1]$. The closed top cells are glued together along the cells where some of the edge lengths are 0 — for instance, we have indicated how the top and top-right cubes are joined along the internal edge of the pentagon where the edge length b in both cells becomes 0. The boundary of K_r^W is the union of cells where at least one edge length is 1.

Finally, we define a simplicial refinement of the CW structure on K_r^W. To approach this, we note that if P is the poset $\{0, 1\}^k$, where $\sigma_1 < \sigma_2$ if σ_2 can be gotten by changing some of the 0s of σ_1 to 1s, then the nerve of P is a simplicial decomposition of the cube $[0, 1]^k$. More concretely, the top simplices are the sets of the form

$$\{(x_1, \dots, x_k) \in [0, 1]^k \mid 0 < x_{\sigma(1)} < \cdots < x_{\sigma(k)} < 1\}, \tag{11}$$

where σ is a permutation on k letters. The remaining simplices are the result of replacing some of these inequalities by equalities.

Definition 2.9 We refine the CW structure on K_r^W by decomposing each cell C_T in K_r^W like so: we make the identification $C_T \cong [0, 1]^{r-2-d(T)}$, then perform the simplicial decomposition described in the previous paragraph. This refinement equips K_r^W with a simplicial decomposition.

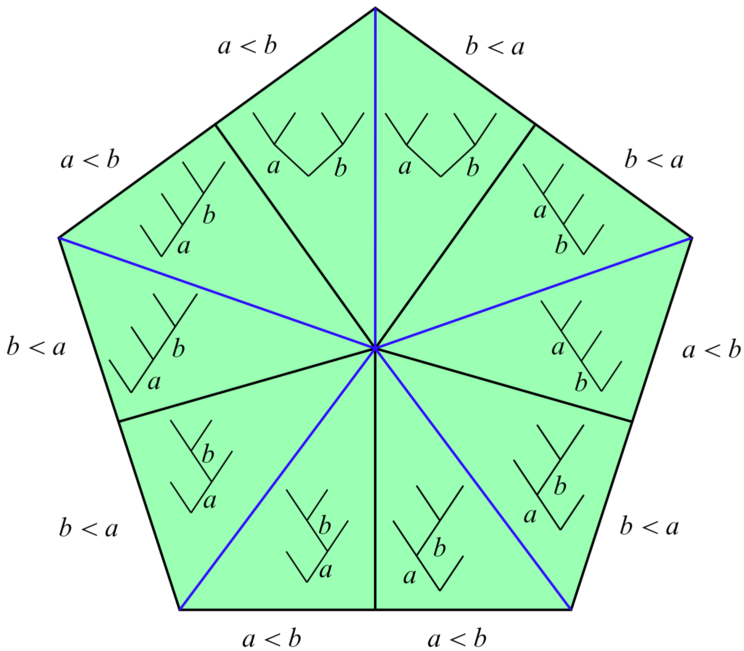

Figure 5

Example 2.10 In Figure 5, we depict the simplicial complex K_4^W. This is the refinement of our initial cubical CW decomposition of K_r^W gotten by subdividing each of the five squares into two triangles. We indicate the new edges by coloring them blue.

Now that we have constructed the spaces K_r^W, we can prove Proposition 2.1, which states that (K_r^W) is a non-Σ operad and that the operad maps are cellular.

Definition 2.11 Fix r, s, and $i \in [1,r]$. We wish to define the composition map

$$\circ_i \colon K_r^W \times K_s^W \to K_{r+s-1}^W. \tag{12}$$

We do so cell by cell. That is, fix cells $C_T \subset K_r^W$ and $C_{T'} \subset K_s^W$. Define T'' to be the result of grafting T' to the i^{th} leaf of T. Then we define $\circ_i$ on $C_T \times C_{T'}$ like so: given collections of edge lengths on T and T', combine them to produce a collection of edge lengths on T'', where we assign to the single newly formed interior edge the length 1.

Proof of Proposition 2.1 Fix r, s, and $i \in [1,r]$, and consider the composition map

$$\circ_i \colon K_r^W \times K_s^W \to K_{r+s-1}^W. \tag{13}$$

To show that $\circ_i$ is cellular, let's consider the restriction of $\circ_i$ to a product $C_T \times C_{T'}$ of closed cubes, for $T \in K_r$ and $T' \in K_s$. Denote by T'' the tree obtained by grafting the root of T' to the i^{th} leaf of T. Then $\circ_i$ includes $C_T \times C_{T'}$ into $C_{T''}$ as the face gotten by requiring the outgoing edge of the root of T' to have length 1. The CW structure of this face of $C_{T''}$ is finer than that of $C_T \times C_{T''}$, so $\circ_i$ is indeed cellular. □

2.2 Metric tree-pairs and the definition of $W_{\boldsymbol{n}}^W$

Just as we defined K_r^W to be the parameter space of metric stable RRTs, we will define $W_{\boldsymbol{n}}^W$ to parametrize metric stable tree-pairs. The definition of metric stable tree-pairs is somewhat involved, so we devote the current subsection to this definition.

Before defining metric stable tree-pairs, we recall the definition of stable tree-pairs:

Definition 2.12 [Bottman 2019a, Definition 3.1] A *stable tree-pair of type* $\boldsymbol{n}$ is a datum $2T = T_b \xrightarrow{f} T_s$, with T_b, T_s, and f described below:

- The *bubble tree* T_b is an RRT whose edges are either solid or dashed, which must satisfy these properties:
  - The vertices of T_b are partitioned as $V(T_b) = V_{\text{comp}} \sqcup V_{\text{seam}} \sqcup V_{\text{mark}}$, where
    - every $\alpha \in V_{\text{comp}}$ has at least 1 solid incoming edge, no dashed incoming edges, and either a dashed or no outgoing edge;
    - every $\alpha \in V_{\text{seam}}$ has zero or more dashed incoming edges, no solid incoming edges, and a solid outgoing edge; and
    - every $\alpha \in V_{\text{mark}}$ has no incoming edges and either a dashed or no outgoing edge.

    We partition $V_{\text{comp}} =: V_{\text{comp}}^1 \sqcup V_{\text{comp}}^{\geq 2}$ according to the number of incoming edges of a given vertex.
  - **Stability** If α is a vertex in V_{comp}^1 and β is its incoming neighbor, then $\#\text{in}(\beta) \geq 2$; if α is a vertex in $V_{\text{comp}}^{\geq 2}$ and $\beta_1, \dots, \beta_l$ are its incoming neighbors, then there exists j with $\#\text{in}(\beta_j) \geq 1$.
- The *seam tree* T_s is an element of K_r^{tree}.
- The *coherence map* is a map $f: T_b \to T_s$ of sets having these properties:
  - f sends root to root, and if $\beta \in \text{in}(\alpha)$ in T_b, then either $f(\beta) \in \text{in}(f(\alpha))$ or $f(\alpha) = f(\beta)$.
  - f contracts all dashed edges, and every solid edge whose terminal vertex is in V_{comp}^1.
  - For any $\alpha \in V_{\text{comp}}^{\geq 2}$, f maps the incoming edges of α bijectively onto the incoming edges of $f(\alpha)$, compatibly with $<_\alpha$ and $<_{f(\alpha)}$.
  - f sends every element of V_{mark} to a leaf of T_s, and if $\lambda_i^{T_s}$ is the i^{th} leaf of T_s, then $f^{-1}\{\lambda_i^{T_s}\}$ contains n_i elements of V_{mark}, which we denote by $\mu_{i1}^{T_b}, \dots, \mu_{in_i}^{T_b}$.

We denote by $W_{\boldsymbol{n}}^{\text{tree}}$ the set of isomorphism classes of stable tree-pairs of type $\boldsymbol{n}$. Here an isomorphism from $T_b \xrightarrow{f} T_s$ to $T_b' \xrightarrow{f'} T_s'$ is a pair of maps $\varphi_b: T_b \to T_b'$ and $\varphi_s: T_s \to T_s'$ that fit into a commutative square in the obvious way and that respect all the structure of the bubble trees and seam trees.

Next, we define metric stable tree-pairs. This notion is more subtle than that of metric stable RRTs, because we must impose conditions on the edge-lengths. (This should be compared to Bottman and Oblomkov's similar constraints [2019, Section 3], imposed in order to define local charts on a complexified version of $W_{\boldsymbol{n}}$.)

Definition 2.13 A *metric stable tree-pair* $(2T, (L_e), (\ell_e))$ is the following data:

- $2T$ is a stable tree-pair.
- We have, for every interior dashed edge e of T_b, a length $L_e \in [0,1]$, and, for every interior edge e of T_s, a length $\ell_e \in [0,1]$, subject to the following coherence conditions (where for convenience we set $L_\alpha := L_e$ for $\alpha \in V_{\mathrm{comp}}(T_b) \setminus \{\alpha_{\mathrm{root}}\}$ and e the outgoing edge of α, and similarly for the edge-lengths in T_s):
  - For every $\alpha_1, \alpha_2 \in V^{\geq 2}_{\mathrm{comp}}(T_b)$ and $\beta \in V^1_{\mathrm{comp}}(T_b)$ with $f(\alpha_1) = f(\alpha_2) = f(\beta)$, we require

$$\max_{\gamma \in [\alpha_1, \beta)} L_\gamma = \max_{\gamma \in [\alpha_2, \beta)} L_\gamma. \tag{14}$$

  - For every $\rho \in V_{\mathrm{int}}(T_s) \setminus \{\rho_{\mathrm{root}}\}$ and $\alpha \in V^{\geq 2}_{\mathrm{comp}}(T_b)$ with $f(\alpha) = \rho$, we require

$$\ell_\rho = \max_{\gamma \in [\alpha, \beta_\alpha)} L_\gamma, \tag{15}$$

  where we define β_α to be the first element of $V^{\geq 2}_{\mathrm{comp}}(T_b)$ that the path from α to α_{root} passes through.

Finally, we recall the *dimension* of a stable tree-pair. Similarly to the dimension of a stable RRT, this will be the codimension in $W^W_{\boldsymbol{n}}$ of the cell corresponding to the stable tree-pair in question.

Definition 2.14 [Bottman 2019a, Definition 3.3] For $2T$ a stable tree-pair, we define the *dimension* $d(2T) \in [0, |\boldsymbol{n}| + r - 3]$ like so:

$$d(2T) := |\boldsymbol{n}| + r - \#V^1_{\mathrm{comp}}(T_b) - \#(T_s)_{\mathrm{int}} - 2. \tag{16}$$

We are now prepared to define $W^W_{\boldsymbol{n}}$, the "W–version" of the 2–associahedron. We will define $W^W_{\boldsymbol{n}}$ by attaching together the cells C_{2T}, which consist of metric stable tree-pairs.

Definition 2.15 Given a stable tree-pair $2T$, the *cell associated to* $2T$ is the collection of all metric stable tree-pairs of type $2T$. We denote this cell by C_{2T}.

Note that we can identify C_{2T} with the subset of the cube $[0,1]^k$ defined by the equalities (14) and (15), where k is the number of interior dashed edges of T_b plus the number of interior edges of T_s.

Definition 2.16 Fix $r \geq 1$ and $\boldsymbol{n} \in \mathbb{Z}^r_{\geq 0} \setminus \{\boldsymbol{0}\}$. We define $W^W_{\boldsymbol{n}}$ similarly to how we defined $K^W_{\boldsymbol{r}}$ in Definition 2.7:

$$W^W_{\boldsymbol{n}} := \Bigl(\bigsqcup_{2T \in W^{\mathrm{tree}}_{\boldsymbol{n}}} C_{2T} \Bigr) \Big/ {\sim}.$$

The quotient here is somewhat subtler than the quotient that appeared in Definition 2.7, specifically when it comes to T_b. In T_s, we simply contract any edges of length 0. We indicate in Figure 6 how to perform the necessary contractions in T_b when some edge-lengths are 0. The reader should think of the left contraction as undoing a type-1 move (as in [Bottman 2019a, Section 3.1]), whereas the right contraction undoes either a type-2 or a type-3 move. Note that we are using the coherences enforced in

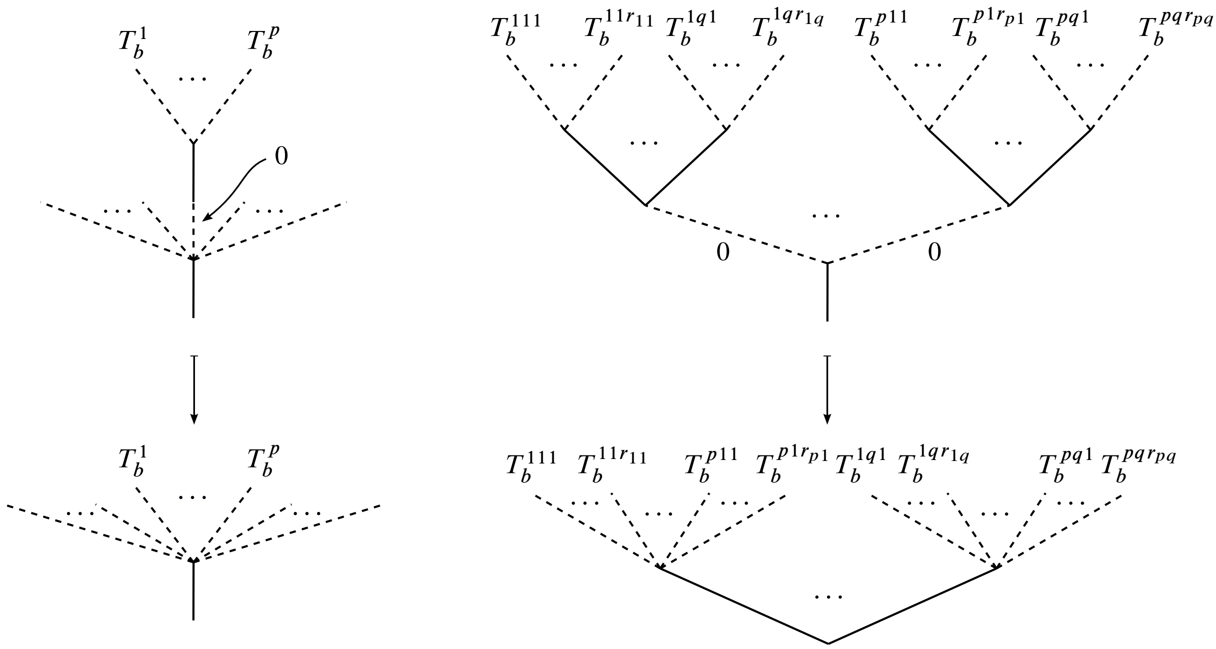

Figure 6

Definition 2.13 — for instance, these mean that we do not have to consider a situation as in the right-hand side of the above figure, but where only some of the edge-lengths in this portion of T_b are 0.

Example 2.17 In Figure 7, we depict the CW complex W_{21}^W. Each of the parameters a and b lie in $[0, 1]$; they do not have the same meaning across different cells. The eight interior edges (resp. sixteen boundary edges) correspond to the loci in the top cells where a parameter goes to 0 (resp. to 1).

Finally, we refine the CW structure on $W_{\boldsymbol{n}}^W$ to a simplicial decomposition.

Lemma 2.18 *Fix a stable tree-pair $2T$. For every simplex S in the standard simplicial decomposition of $[0, 1]^k \supset C_{2T}$, S is either contained in C_{2T} or disjoint from it. The collection of such simplices that are contained in C_{2T} form a simplicial decomposition of C_{2T}.*

Proof Fix a simplex S. S is defined by a collection of equalities and inequalities of the form

$$0 * x_{\sigma(1)} * \cdots * x_{\sigma(k)} * 1, \tag{17}$$

where each "$*$" is either a "$<$" or an "$=$" and where σ is a permutation on k letters. After imposing these (in)equalities, the left- and right-hand sides of the equalities (14) and (15) become single variables. This collection of equalities will either be always satisfied or never satisfied, depending on the constraints in (17). Depending on which of these is the case, S is either contained in C_{2T} or disjoint from it.

It follows immediately that the collection of simplices that are contained in C_{2T} form a simplicial decomposition of C_{2T}. □

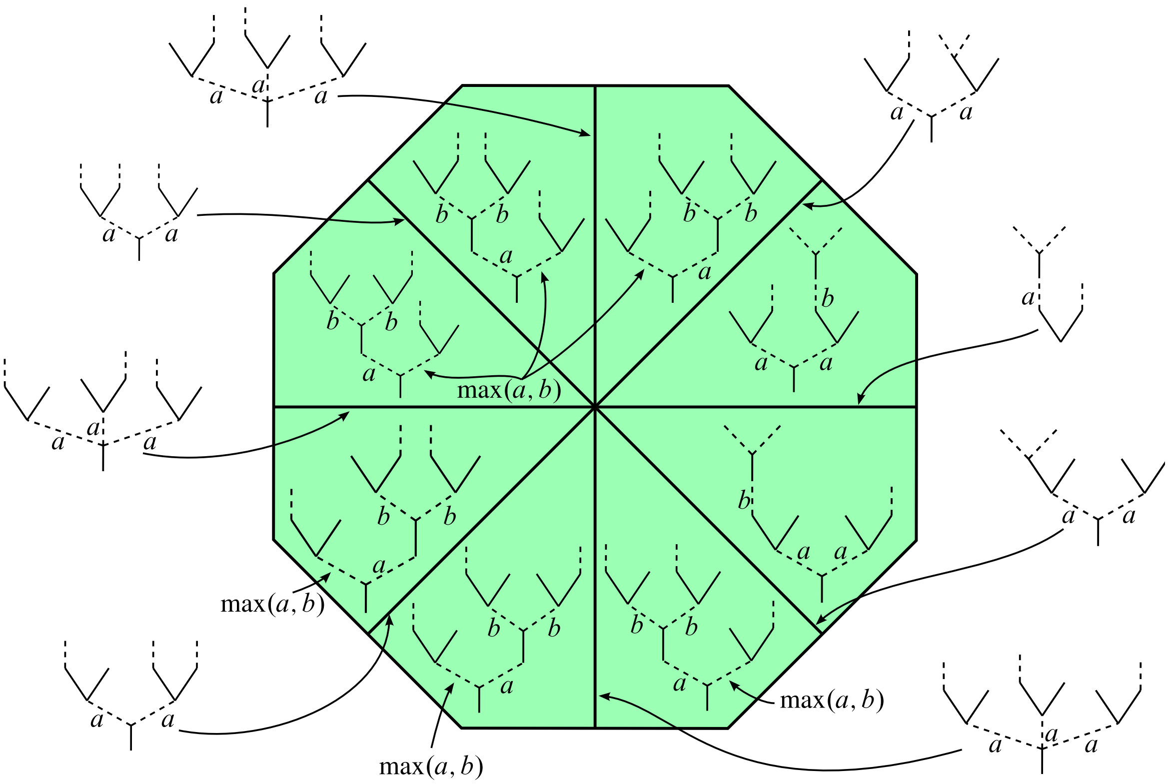

Figure 7

Example 2.19 In Figure 8, we illustrate the closed cell in W_{40}^W associated to the underlying tree-pair of the (top-dimensional) metric tree-pair shown on the right. The restriction on the lengths $a, b, c, d \in [0, 1]$ is that they must satisfy $\max(a, b) = \max(c, d)$; as a result, this cell has the CW type of a square pyramid.

We indicate the simplicial refinement of this cell: the square pyramid is subdivided into eight 3–simplices, which are defined by imposing inequalities and equalities as shown in this figure.

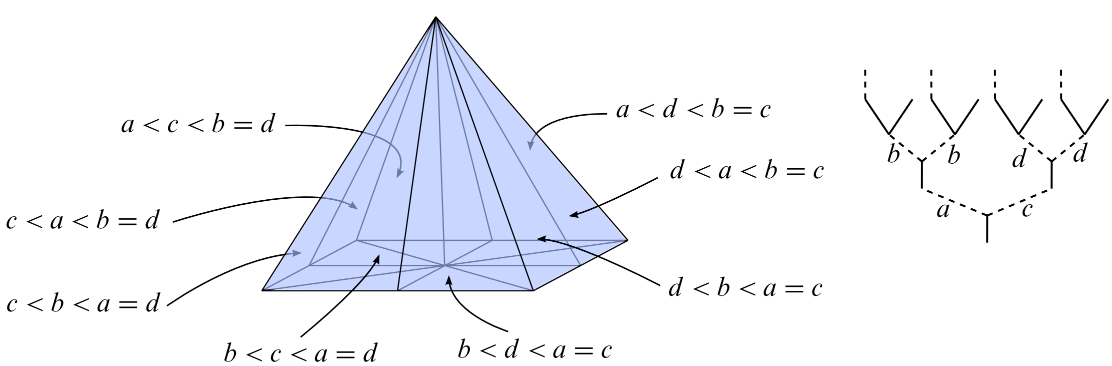

Figure 8

3 The construction of FM_2^W

In this final section, we will construct a collection of CW complexes $(\mathrm{FM}_2^W(k))_{k\geq 1}$ and a collection of operations

$$\circ_i : \mathrm{FM}_2^W(k) \times \mathrm{FM}_2^W(l) \to \mathrm{FM}_2^W(k+l-1) \tag{18}$$

such that these data form an operad.

We will now give an overview of our construction of $\mathrm{FM}_2^W(k)$. This is an expansion of step (ii) in the overview we gave in Section 1.3, and we label the parts accordingly:

(iia) Each open Getzler–Jones cell in $\mathrm{FM}_2(k)$ can be identified with a product of open 2–associahedra, ie a product of the form $\mathring{W}_{\boldsymbol{m}^1} \times \cdots \times \mathring{W}_{\boldsymbol{m}^a}$ (where "$\mathring{X}$" is our notation for the interior of a space X). For each such open cell, we replace these 2–associahedra by their W–construction equivalents thusly: $\mathring{W}^W_{\boldsymbol{m}^1} \times \cdots \times \mathring{W}^W_{\boldsymbol{m}^a}$. This product comes with the product CW structure, and we refine this in a way that endows $\mathring{W}^W_{\boldsymbol{m}^1} \times \cdots \times \mathring{W}^W_{\boldsymbol{m}^a}$ with the structure of a simplicial complex.

(iib) While an open Getzler–Jones cell can be identified with a product $\mathring{W}_{\boldsymbol{m}^1} \times \cdots \times \mathring{W}_{\boldsymbol{m}^a}$ of 2–associahedra, their compactifications (in $\mathrm{FM}_2(k)$ and $W_{\boldsymbol{m}^1} \times \cdots \times W_{\boldsymbol{m}^a}$, respectively) are different: the compactification of the former is smaller than the compactification of the latter. This is reflected in how we glue our products $\mathring{W}^W_{\boldsymbol{m}^1} \times \cdots \times \mathring{W}^W_{\boldsymbol{m}^a}$ together. Specifically, we perform this gluing by applying a quotient map to each simplex in the boundary of $W^W_{\boldsymbol{m}^1} \times \cdots \times W^W_{\boldsymbol{m}^a}$. This quotient map is closely related to the maps $f_\sigma : W_{\boldsymbol{n}} \to \mathrm{FM}_2(k)$ that we described in Section 1.4: they reflect the fact that the compactification used to define $W_{\boldsymbol{n}}$ allows lines with no marked points, whereas the compactification of a Getzler–Jones cell does not allow this.

The following is the main result of this section, which we stated in the introduction and record again here:

Main Theorem *The spaces* $(\mathrm{FM}_2^W(k))_{k\geq 1}$ *together with the composition operations* $\circ_i$ *defined in Definition 3.11 form a non-*Σ *operad, and the composition maps*

$$\circ_i : \mathrm{FM}_2^W(k) \times \mathrm{FM}_2^W(l) \to \mathrm{FM}_2^W(k+l-1) \tag{19}$$

are cellular.

Proof Combine Lemmata 3.12 and 3.13 below. □

3.1 Quotient maps on 2–associahedra

Before we can define the quotient involved in (24), we will define for every cell F in $\partial W^W_{\boldsymbol{n}}$ a map q_F from F to a certain product of 2–associahedra, where this target will vary for difference choices of F. We begin with two preliminary definitions:

Definition 3.1 Fix $r \geq 1$ and $\boldsymbol{n} \in \mathbb{Z}^r_{\geq 0} \setminus \{\boldsymbol{0}\}$, and fix $i \in [1, r]$ such that $n_i = 0$. Define $\tilde{\boldsymbol{n}} := (n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_r)$. We then define a map of posets $\pi_i^{\mathrm{tree}} : W_{\boldsymbol{n}}^{\mathrm{tree}} \to W_{\tilde{\boldsymbol{n}}}^{\mathrm{tree}}$ by applying the following procedure to $2T = T_b \xrightarrow{f} T_s \in W_{\boldsymbol{n}}^{\mathrm{tree}}$:

(i) Denote by e_0 the edge in T_s incident to the i^{th} leaf $\lambda_i^{T_s}$. If e is a solid edge in T_b that is mapped identically under f to e_0, then we delete e. Next, we delete e_0. We modify f in the obvious way.

(ii) After performing these deletions, our tree-pair may no longer be stable. We rectify this in T_b (resp. T_s) by performing the contractions indicated on the left (resp. right):

Specifically, we perform these contractions as many times as necessary for the tree-pair to be stable.

Denoting the end result of this procedure by $\widetilde{2T}$, we define $\pi_i^{\mathrm{tree}}(2T) := \widetilde{2T}$.

Next, we define another map of posets. Fix $r \geq 1$ and $\boldsymbol{n} \in \mathbb{Z}^r_{\geq 0} \setminus \{\boldsymbol{0}\}$. Denote by $\tilde{\boldsymbol{n}}$ the result of deleting all the zeroes from $\boldsymbol{n}$, and set $\tilde{r}$ to be the length of $\tilde{\boldsymbol{n}}$. We define $\pi^{\mathrm{tree}}\colon W_{\boldsymbol{n}}^{\mathrm{tree}} \to W_{\tilde{\boldsymbol{n}}}^{\mathrm{tree}}$ by applying the map π_i^{tree} once for each i with $n_i = 0$.

It is not hard to check that the choices implicit in this definition do not matter, and that the resulting maps are indeed maps of posets.

Definition 3.2 Fix $r \geq 1$ and $\boldsymbol{n} \in \mathbb{Z}^r_{\geq 0} \setminus \{\boldsymbol{0}\}$. We define a map $\pi^W\colon W_{\boldsymbol{n}}^W \to W_{\tilde{\boldsymbol{n}}}^W$ in the same fashion as π^{tree}, with the provision that when we contract adjacent edges of lengths ℓ_1 and ℓ_2 (whether in T_b or T_s) we equip the resulting edge with length $\max(\ell_1, \ell_2)$.

Next, we recall a W–version analogue of two properties of the 2–associahedra:

***W*–version analogue of the forgetful property of** [Bottman 2019a, Theorem 4.1] Fix $r \geq 1$ and $\boldsymbol{n} \in \mathbb{Z}^r_{\geq 0} \setminus \{\boldsymbol{0}\}$. There is a surjection $W_{\boldsymbol{n}}^W \to K_r^W$ which sends a metric stable tree-pair $(T_b \xrightarrow{f} T_s, (L_e), (\ell_e))$ to the metric stable RRT $(T_s, (\ell_e))$.

***W*–version analogue of the recursive property of** [Bottman 2019a, Theorem 4.1] Fix a stable tree-pair $2T = T_b \xrightarrow{f} T_s \in W_{\boldsymbol{n}}^{\mathrm{tree}}$. There is an inclusion of CW complexes

$$\Gamma_{2T}\colon \prod_{\substack{\alpha\in V^1_{\mathrm{comp}}(T_b)\\ \mathrm{in}(\alpha)=(\beta)}} W^W_{\#\mathrm{in}(\beta)} \times \prod_{\rho\in V_{\mathrm{int}}(T_s)} \prod^{K^W_{\#\mathrm{in}(\rho)}}_{\substack{\alpha\in V^{\geq 2}_{\mathrm{comp}}(T_b)\cap f^{-1}\{\rho\}\\ \mathrm{in}(\alpha)=(\beta_1,\dots,\beta_{\#\mathrm{in}(\rho)})}} W^W_{\#\mathrm{in}(\beta_1),\dots,\#\mathrm{in}(\beta_{\#\mathrm{in}(\alpha)})} \hookrightarrow W^W_{\boldsymbol{n}}, \tag{20}$$

where the superscript on one of the product symbols indicates that it is a fiber product with respect to the maps in the description of the forgetful property above.

The map Γ_{2T} defined in [Bottman 2019a], which is defined for the posets $W_{\boldsymbol{n}}^{\mathrm{tree}}$, is defined by attaching stable tree-pairs together in a way specified by the stable tree-pair $2T$. This map is similar, but we are attaching together *metric* stable tree-pairs. We assign the length 1 to the edges along which we attach the trees. (The image of Γ_{2T} is a union of cells in $\partial W_{\boldsymbol{n}}^W$.)

We can now define the quotient maps q_F on $W_{\boldsymbol{n}}^W$:

Definition 3.3 Fix $r \geq 1$, $\boldsymbol{n} \in \mathbb{Z}_{\geq 0}^r \setminus \{\boldsymbol{0}\}$, a stable type-$\boldsymbol{n}$ tree-pair $\widetilde{2T}$, and a face F of the associated cell $C_{\widetilde{2T}}$ in $W_{\boldsymbol{n}}^W$ with the property that F lies in $\partial W_{\boldsymbol{n}}^W$. (Equivalently, the metric tree-pairs in F have at least one length that is identically equal to 1.) The *quotient map associated to* F is a map q_F from F to a product of 2–associahedra. Given a metric stable tree-pair $(2T, (L_e), (\ell_e))$, we define its image under π in the following fashion:

(i) Break up T_b and T_s along the edges that are identically 1 in F. Equivalently, choose $2T$ of minimal dimension with the property that F lies in the image of Γ_{2T}, then identify F as a top cell in a product of fiber products of the following form:

$$\prod_{\substack{\alpha \in V_{\mathrm{comp}}^1(T_b) \\ \mathrm{in}(\alpha) = (\beta)}} W_{\#\mathrm{in}(\beta)}^W \times \prod_{\rho \in V_{\mathrm{int}}(T_s)} \prod_{\substack{\alpha \in V_{\mathrm{comp}}^{\geq 2}(T_b) \cap f^{-1}\{\rho\} \\ \mathrm{in}(\alpha) = (\beta_1, \ldots, \beta_{\#\mathrm{in}(\rho)})}}^{K_{\#\mathrm{in}(\rho)}^W} W_{\#\mathrm{in}(\beta_1), \ldots, \#\mathrm{in}(\beta_{\#\mathrm{in}(\alpha)})}^W. \tag{21}$$

As a result, we obtain a list of metric stable tree-pairs, which we can regard as lying inside a product $W_{\boldsymbol{m}^1}^W \times \cdots \times W_{\boldsymbol{m}^a}^W$.

(ii) We then apply the map π^W to each of the factors in the product just recorded, hence producing an element of $W_{\widetilde{\boldsymbol{m}}^1}^W \times \cdots \times W_{\widetilde{\boldsymbol{m}}^a}^W$. (As in Definitions 3.1 and 3.2, $\widetilde{\boldsymbol{m}}^i$ denotes the result of removing the 0s from $\boldsymbol{m}^i$.)

Note that for two cells F_1 and F_2 in the boundary of $W_{\boldsymbol{n}}^W$, the targets of q_{F_1} and q_{F_2} are typically different.

Example 3.4 In Figure 9, we illustrate several things about W_{21}^W. Initially, W_{21}^W is an octagon, decomposed into eight squares; this is indicated by the black lines. The simplicial refinement divides each square into two 2–simplices. We have indicated the metric tree-pairs that correspond to each of the eight squares, as well as those corresponding to the sixteen 1–simplices that comprise ∂W_{21}^W. (Some dashed edges are not labeled; these should be interpreted as having length $\max(a, b)$.)

Finally, we have indicated the behavior of the quotient maps on W_{21}^W. These maps are the identity on every edge except for those indicated in red. Each pair of red edges is contracted to a point. One reflection of this is that in Example 1.1, the octagons in W_{111} are taken to the (cellular) hexagons in the Getzler–Jones cell indicted on the right.

3.2 The construction of $\mathrm{FM}_2^W(k)$

In this subsection, we tackle the construction of $\mathrm{FM}_2^W(k)$. First, we will describe our version of the Getzler–Jones cells. Next, we will explain how to glue these spaces together.

To define the Getzler–Jones cells, we must introduce 2–*permutations*, which will allow us to enforce the alignment and ordering of special points on screens as in Figure 1.

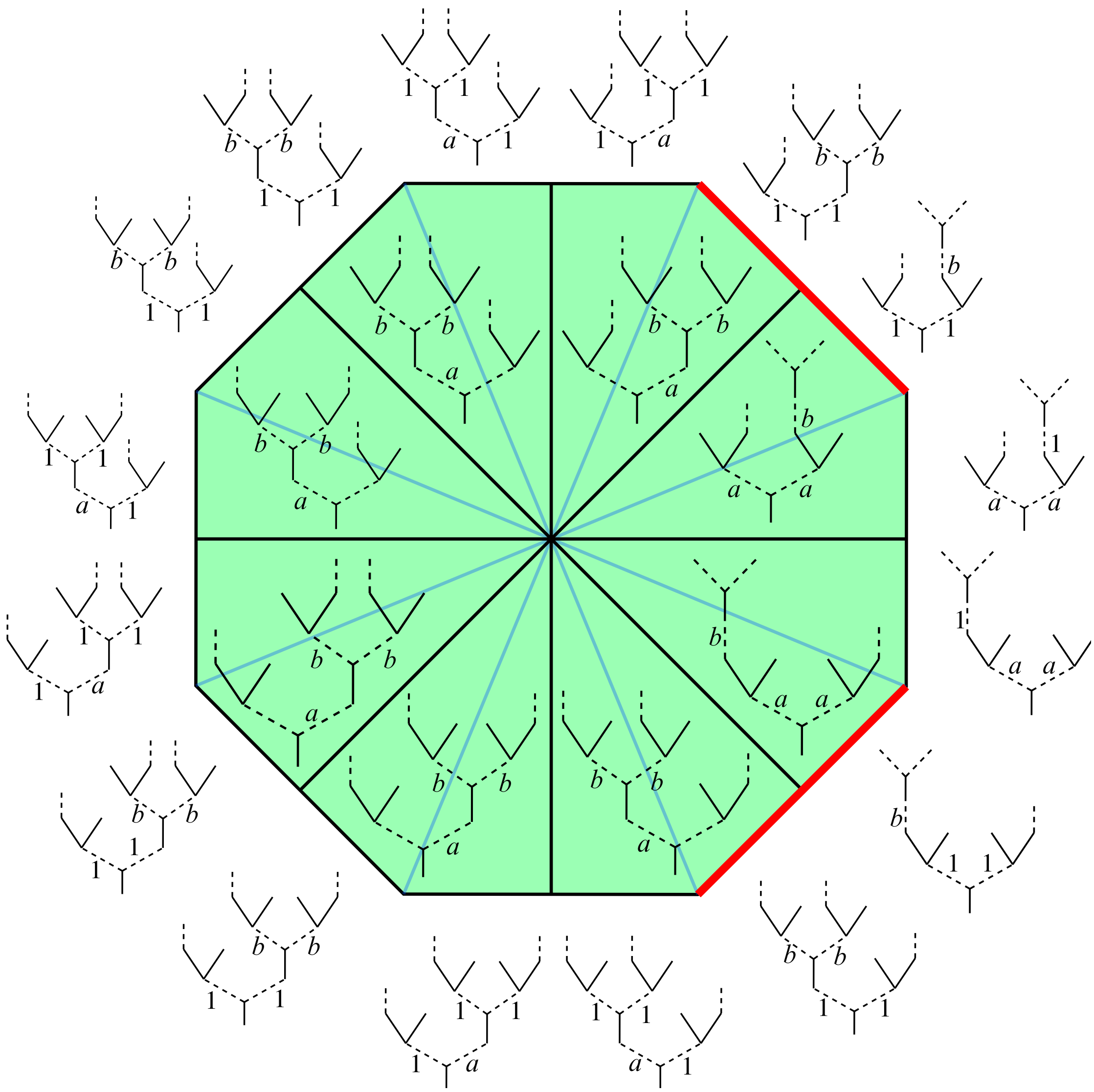

Figure 9

Definition 3.5 Fix a finite set A. A 2*–permutation* σ *on* A is the data

- an ordered decomposition

$$A = A_1 \sqcup \cdots \sqcup A_r, \tag{22}$$

  where A_r is allowed to be empty, and

- for each i, a linear order on A_i.

We define the *type* of σ to be the vector $\boldsymbol{n} := (|A_1|, \ldots, |A_r|)$. If σ is a 2–permutation whose type $\boldsymbol{n}$ has no zero entries, then we say that σ *has no empty part*.

Remark 3.6 A type-$(\overbrace{1,\dots,1}^{r})$ 2–permutation is exactly the data of a permutation on r letters. The same is true of a type-(n) 2–permutation.

Next, we define a *Getzler–Jones datum*, the set of which indexes the Getzler–Jones cells in $\mathrm{FM}_2^W(k)$.

Definition 3.7 Fix $k \geq 2$. A *Getzler–Jones datum* consists of

- a stable rooted tree T with k leaves, together with a numbering of its leaves from 1 through k, and
- for every interior vertex $v \in T_{\mathrm{int}}$, a 2–permutation σ on its incoming vertices $V_{\mathrm{in}}(T)$ such that σ has no empty part.

We denote the type of the 2–permutation associated to v by $\boldsymbol{n}(v)$. We will abuse notation and denote the entire Getzler–Jones datum by T.

Finally, we can define the *Getzler–Jones cells of type k*:

Definition 3.8 Fix $k \geq 2$ and a Getzler–Jones datum T. Then we define

(23)
$$\mathrm{GJ}_T := \prod_{v \in T_{\mathrm{int}}} \mathring{W}^W_{\boldsymbol{n}(v)} \quad \text{and} \quad \widetilde{\mathrm{GJ}}_T := \prod_{v \in T_{\mathrm{int}}} W^W_{\boldsymbol{n}(v)}.$$

We call GJ_T the *Getzler–Jones cell* GJ_T *associated to* T, and refer to GJ_T as a *type-k Getzler–Jones cell.* In Lemma 2.18 we equipped $W^W_{\boldsymbol{n}}$ with the structure of a simplicial complex, which induces a CW structure on GJ_T and $\widetilde{\mathrm{GJ}}_T$. We refine these to equip GJ_T and $\widetilde{\mathrm{GJ}}_T$ with simplicial decompositions, in the fashion of Lemma 2.18.

Remark 3.9 The reason why we do not refer to $\widetilde{\mathrm{GJ}}_T$ as a "closed Getzler–Jones cell" is because it is *not* the closure in $\mathrm{FM}_2^W(k)$ of GJ_T. In fact, it is larger than this closure. Our reason for making this second definition is that $\widetilde{\mathrm{GJ}}_T$ will be an integral part of our definition of $\mathrm{FM}_2^W(k)$.

We will define $\mathrm{FM}_2^W(k)$ as a quotient of the following form, where T varies over type-k Getzler–Jones data:

(24)
$$\mathrm{FM}_2^W(k) := \Big(\coprod_T \widetilde{\mathrm{GJ}}_T\Big)\Big/\sim.$$

The remaining ingredient is the collection of maps that we will use to attach these spaces. As a consequence of the definition of these maps, $\mathrm{FM}_2^W(k)$ will decompose as a set into the union of all type-k Getzler–Jones cells.

Finally, we come to the definition of $\mathrm{FM}_2^W(k)$:

Definition 3.10 Fix $k \geq 2$. We construct $\mathrm{FM}_2^W(k)$ like so:

(i) Begin with the following disjoint union, where T varies over type-k Getzler–Jones data:

(25)
$$\coprod_T \widetilde{\mathrm{GJ}}_T.$$

(ii) Fix a type-k Getzler–Jones datum T, and fix a cell F in the boundary of $\widetilde{\mathrm{GJ}}_T = \prod_{v\in T_{\mathrm{int}}} W^W_{\boldsymbol{n}(v)}$. F lies inside a product of cells in the 2–associahedra that comprise $\widetilde{\mathrm{GJ}}_T$ — that is, we may write $F \subset \prod_{v\in T_{\mathrm{int}}} F_v \subset \prod_{v\in T_{\mathrm{int}}} W^W_{\boldsymbol{n}(v)}$, where F_v is a cell in $W^W_{\boldsymbol{n}(v)}$. For every v, we have a map q_v from $W^W_{\boldsymbol{n}(v)}$ to a product of 2–associahedra; by combining these, we obtain a map from F to a product of 2–associahedra. In fact, we can regard the target of this map as a Getzler–Jones cell.

(iii) We take the quotient of the disjoint union in (25) by attaching the constituent spaces together via the maps we defined in the last step.

We define $\mathrm{FM}^W_2(1)$ to be a point.

It is a consequence of the simplicial structure of the $\widetilde{\mathrm{GJ}}_T$ that each $\mathrm{FM}^W_2(k)$ has the structure of a CW complex. As noted above, a result of our definition is that $\mathrm{FM}^W_2(k)$ decomposes as a union of Getzler–Jones cells, over all Getzler–Jones data of type k.

3.3 The operad structure on FM^W_2

Definition 3.11 Fix k, l, and $i \in [1,k]$. We wish to define the map

$$\circ_i : \mathrm{FM}^W_2(k) \times \mathrm{FM}^W_2(l) \to \mathrm{FM}^W_2(k+l-1). \tag{26}$$

To do so, fix Getzler–Jones data T and T' of types k and l, respectively, and fix cells $F \subset \mathrm{GJ}_T$ and $F' \subset \mathrm{GJ}_{T'}$. We will define $\circ_i$ on

$$\mathrm{GJ}_T \times \mathrm{GJ}_{T'} = \prod_{v\in T_{\mathrm{int}}\sqcup T'_{\mathrm{int}}} W^W_{\boldsymbol{n}(v)}. \tag{27}$$

Define T'' to be the result of grafting T' to the i^{th} leaf of T, and completing it to a Getzler–Jones datum in the obvious way. We define $\circ_i$ on $\mathrm{GJ}_T \times \mathrm{GJ}_{T'}$ to be the identification of $\mathrm{GJ}_T \times \mathrm{GJ}_{T'}$ with $\mathrm{GJ}_{T''}$.

Lemma 3.12 *Taken together, the spaces $(\mathrm{FM}^W_2(k))_{k\geq 1}$ together with the composition operations $\circ_i$ form a non-Σ operad.*

Proof This is immediate from the definition. □

Lemma 3.13 *The composition maps*

$$\circ_i : \mathrm{FM}^W_2(k) \times \mathrm{FM}^W_2(l) \to \mathrm{FM}^W_2(k+l-1) \tag{28}$$

are cellular.

Proof This is similar to the proof of Proposition 2.1. □

Remark 3.14 Fix $r \geq 1$, $\boldsymbol{n} \in \mathbb{Z}^r_{\geq 0}\setminus\{\boldsymbol{0}\}$, and a 2–permutation σ of type $\boldsymbol{n}$. Then the associated forgetful map

$$f^W_\sigma : W^W_{\boldsymbol{n}} \to \mathrm{FM}^W_2(|\boldsymbol{n}|) \tag{29}$$

is cellular. This map is defined in the obvious way: we first identify $W_{\boldsymbol{n}}^{W}$ with the corresponding $\widetilde{\mathrm{GJ}}_T$, where T is a Getzler–Jones datum whose associated tree T is a corolla with $|\boldsymbol{n}|$ leaves. Then, we include $\widetilde{\mathrm{GJ}}_T$ into the disjoint union $\bigsqcup_T \widetilde{\mathrm{GJ}}_T$, and finally take the quotient to land in $\mathrm{FM}_2^W(|\boldsymbol{n}|)$.

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