arXiv:2403.16300v1 [math.SG] 24 Mar 2024

FORMAL POISSON (CO)HOMOLOGY OF THE LEFSCHETZ SINGULARITY

LAURAN TOUSSAINT, FLORIAN ZEISER

Abstract. We compute the formal Poisson cohomology groups of a real Poisson structure π on $\mathbb{C}^2$ associated to the Lefschetz singularity $(z_1, z_2) \mapsto z_1^2 + z_2^2$. In particular we correct an erroneous computation in the literature. The definition of π depends on a choice of volume form. Using the main result we formally classify all Poisson structure arising from different choices of volume forms.

1. Introduction

A Poisson structure on a smooth manifold M is a bivector field $\pi \in \mathfrak{X}^2(M)$ such that

$$[\pi, \pi] = 0,$$

where $[\cdot, \cdot]$ denotes the Schouten bracket (see e.g. [LPV13]). As first observed by Lichnerowicz [Lic77], any Poisson manifold (M, π) is naturally equipped with a differential on the space of mulivector fields

$$d_\pi := [\pi, \cdot] : \ \mathfrak{X}^\bullet(M) \to \mathfrak{X}^{\bullet+1}(M).$$

The vanishing of the bracket is equivalent to $\mathrm{d}_\pi^2 = 0$, and the associated cohomology $H^\bullet(M, \pi)$ is called *Poisson cohomology*. The groups $H^\bullet(M, \pi)$ contain important information about the Poisson manifold (M, π). For example:

- $H^0(M, \pi)$ consists of functions on M constant along the leaves of the foliation $\mathcal{F}$;
- $H^1(M, \pi), [\cdot, \cdot])$ can be seen as the Lie algebra of infinitesimal outer automorphisms of (M, π);
- $H^2(M, \pi)$ controls infinitesimal deformations modulo those induced by diffeomorphisms.

For a detailed exposition of Poisson geometry and Poisson cohomology we refer the reader to [DZ05; LPV13; CFM21]. Despite their importance, Poisson cohomology is notoriously hard to compute due to the lack of general methods for the computation. The difficulty comes from the fact that $(\mathfrak{X}^\bullet(M), \mathrm{d}_\pi)$ is generally not an elliptic complex. Assuming that π is regular, i.e. all leaves of $\mathcal{F}$ have the same dimension, Xu computed the Poisson cohomology of such Poisson structures in [Xu92]. This, together with the existence of a Mayer-Vietoris sequence due to Vaisman [Vai94], implies that the fundamental problem for the computation lies around neighborhoods of singular leaves.

To this end observe that given a Poisson structure π on M and a singular point $p \in M$ of π, i.e. $\pi_p = 0$, we have a short exact sequence of complexes

$$0 \to (\mathfrak{X}_p^\bullet(M), \mathrm{d}_\pi) \hookrightarrow (\mathfrak{X}^\bullet(M), \mathrm{d}_\pi) \xrightarrow{j_p^\infty} (\mathfrak{X}^\bullet(M)/\mathfrak{X}_p^\bullet(M) \simeq \mathbb{R}[[T_p^*M]] \otimes \wedge^\bullet T_pM, \mathrm{d}_{j_p^\infty \pi}) \to 0$$

resulting in a long exact sequence in cohomology. Here j_p^∞ is the infinite jet map at p, and $\mathfrak{X}_p^\bullet(M)$ denotes multivector fields whose infinite jet at p is zero. The equivalence for the quotient is due to Borel's lemma. A similar short exact sequence can be obtained when the singular locus of π is a higher dimensional submanifold, see [HZ23].

The short exact sequence above was first used by Abreu & Ginzburg [Gin96] and Roytenberg [Roy02]. Its usefulness lies in the fact that if π vanishes in a polynomial way then we can compute the cohomology of $(\mathfrak{X}_p^\bullet(M), \mathrm{d}_\pi)$ as if π was regular. As such we are left with the computation of the complex

$$(\mathbb{R}[[T_p^*M]] \otimes \wedge^\bullet T_pM, \mathrm{d}_{j_p^\infty \pi}).$$

We call its cohomology the formal Poisson cohomology of π at $p \in M$, denoted by $H^\bullet_{F_p}(M, \pi)$. The methods used to compute $H^\bullet_{F_p}(M, \pi)$ are more algebraic in nature (see e.g. [Mon02; Pic06; Pel09]).

The Poisson structure under consideration in this paper are defined as follows. First recall that any fibration $f : (M^m, \mu_M) \to (N^{m-2}, \mu_N)$ between oriented manifolds (M, μ_M) and (N, μ_N) defines a Poisson structure π_{f,μ_M,μ_N} on M due to [GSV14][Thm 2.6] by:

$$\pi_{f,\mu_M,\mu_N}(\mathrm{d}g, \mathrm{d}h)\mu_M := \mathrm{d}g \wedge \mathrm{d}h \wedge f^*\mu_N \qquad \text{for} \quad g, h \in C^\infty(M). \tag{1}$$

Note that π_{f,μ_M,μ_N} depends on the fibration $f : M \to N$ and on the choice of volume forms on M and N. Poisson structures defined by different volume forms admit the same singular foliation $\mathcal{F}$ with:

- 2-dimensional leaves through every $x \in M \setminus \mathrm{Crit}(f)$;
- 0-dimensional leaves for every $x \in \mathrm{Crit}(f)$.

However, the symplectic form along the leaves will vary for different choices of volume forms. The study of such Poisson structures has been of increased interest lately (see e.g. [STV19; BO22]). Our Poisson structure of interest is induced by a fibration with a *Lefschetz singulartiy*:

$$f : \mathbb{R}^4 \to \mathbb{R}^2 \qquad \text{with} \qquad f = (f_1, f_2) = \big(x_1^2 - x_2^2 + x_3^2 - x_4^2,\, 2(x_1x_2 + x_3x_4)\big) \tag{2}$$

Definition 1.1. *Let $\pi = \pi_{f,\mu_M,\mu_N}$ be the Poisson structure on $\mathbb{R}^4$ defined by (1), (2) and the standard volume forms in $\mathbb{R}^4$ and $\mathbb{R}^2$, respectively.*

In this paper we take the first step in the computation of $H^\bullet(\mathbb{R}^4, \pi)$ by computing the formal Poisson cohomology of π at the origin which we denote by $H^\bullet_F(\mathbb{R}^4, \pi)$. We do so by means of its Poisson homology, i.e. on a Poisson manifold (M^m, π) we have another differential on the space of forms

$$\delta_\pi : \Omega^\bullet(M) \to \Omega^{\bullet-1}(M) \qquad \delta_\pi := \iota_\pi \circ \mathrm{d} - \mathrm{d} \circ \iota_\pi$$

introduced by Kozsul [Kos85] and Brylinski [Bry88]. Here d denotes the de-Rham differential and ι the contraction between mulitvector fields and forms. The homology associated to $(\Omega^\bullet(M), \delta_\pi)$ is called Poisson homology and denoted by $H_\bullet(M, \pi)$. Given a volume form $\mu \in \Omega^m(M)$ on M the relation between Poisson (co)homology can be encoded as follows. First note that μ induces the isomorphism

$$\star := \mu^\flat : \mathfrak{X}^\bullet(M) \xrightarrow{\sim} \Omega^\bullet(M), \qquad X \mapsto \iota_X(\mu). \tag{3}$$

We define a vector field $X_\mu \in \mathfrak{X}^1(M)$, the *modular vector field* of (M, π) and μ (see [Wei97]), by

$$\star X_\mu := \mathrm{d} \star (\pi).$$

The vector field X_μ is Poisson, i.e. $\mathrm{d}_\pi X_\mu = 0$ and the class $\mathrm{mod}(M, \pi) = [X_\mu] \in H^1(M, \pi)$ is independent of the chosen volume form μ. Using the isomorphism $\star$ we obtain the relation

$$\delta_\pi = \star \circ (\mathrm{d}_\pi + X_\mu \wedge \cdot) \circ \star^{-1}, \tag{4}$$

between the Poisson differentials d_π and δ_π (see [LPV13][Proposition 4.18]). We call (M, π) *unimodular* if $\mathrm{mod}(M, \pi) = 0 \in H^1(M, \pi)$. It is well-known that (M, π) is unimodular iff there exists a volume form μ with $X_\mu = 0$. In particular, for all unimodular Poisson structures we obtain an isomorphism

$$H_\bullet(M, \pi) \xrightarrow{\sim} H^{m-\bullet}(M, \pi). \tag{5}$$

In particular, for $(\mathbb{R}^4, \pi)$ from Definition 1.1 we have

$$\star(\pi) = \mathrm{d}f_1 \wedge \mathrm{d}f_2,$$

and hence $(\mathbb{R}^4, \pi)$ is unimodular. Note that the isomorphism induced by (3) descends via j_0^∞ to an isomorphism between formal vector fields and forms, which we denote by

$$\mathfrak{X}^\bullet_f := \mathcal{R} \otimes \wedge^\bullet \mathbb{R}^4 \qquad \text{and} \qquad \Omega^\bullet_f := \mathcal{R} \otimes \wedge^\bullet (\mathbb{R}^4)^*.$$

Here $\mathcal{R} := \mathbb{R}[[x_1, \dots, x_4]]$ denotes the ring of formal power series. Therefore, (5) descends to the formal setting and we obtain an isomorphism between formal Poisson (co)homology

$$H^F_\bullet(\mathbb{R}^4, \pi) \xrightarrow{\sim} H_F^{4-\bullet}(\mathbb{R}^4, \pi),$$

which allows us to compute formal Poissson cohomology by means of its homological counterpart. Since from here on out we refer to Poisson (co)homology only in the formal setting, we drop the sub- and superscript F, respectively throughout the rest of the paper.

The main result is the following:

Theorem 1.2. *The formal Poisson homology groups $H_\bullet(\mathbb{R}^4,\pi)$ of $(\mathbb{R}^4,\pi)$ are uniquely described in the various degrees as follows:*

- *in degree 0, $H_0(\mathbb{R}^4,\pi)$ has unique representatives of the form*
$$p+\sum_{i=1}^{4} a_i x_i$$
where $p\in\mathbb{R}[[f_1,f_2]]$ and $a_i\in\mathbb{R}[[x_2^2,x_4]]$;
- *the group $H_1(\mathbb{R}^4,\pi)$ has unique representatives of the form*
$$\sum_{j=1}^{2} p_j\zeta_j+q_j\mathrm{d}f_j+\sum_{i=1}^{4}\mathrm{d}(a_ix_i)+b_ix_i\mathrm{d}f_1$$
where $p_j,q_j\in\mathbb{R}[[f_1,f_2]]$ and $a_i,b_i\in\mathbb{R}[[x_2^2,x_4]]$ and

$$\zeta_1:=\frac{1}{2}(-x_3\mathrm{d}x_1+x_4\mathrm{d}x_2+x_1\mathrm{d}x_3-x_2\mathrm{d}x_4),\qquad \zeta_2:=\frac{1}{2}(-x_4\mathrm{d}x_1-x_3\mathrm{d}x_2+x_2\mathrm{d}x_3+x_1\mathrm{d}x_4); \tag{6}$$

- *in degree 2, $H_2(\mathbb{R}^4,\pi)$ has unique representatives of the form*
$$p\zeta_1\wedge\zeta_2+q\mathrm{d}f_1\wedge\mathrm{d}f_2+\sum_{i=1}^{2}p_i\mathrm{d}(f_1\zeta_i)+q_i\mathrm{d}\zeta_i+\sum_{i=1}^{4}\mathrm{d}(a_ix_i)\wedge\mathrm{d}f_1$$
where $p,p_i,q,q_i\in\mathbb{R}[[f_1,f_2]]$ and $a_i\in\mathbb{R}[[x_2^2,x_4]]$;
- *representatives of $H_3(\mathbb{R}^4,\pi)$ are uniquely described by*
$$\sum_{i=1}^{2}p_i\zeta_2\wedge\mathrm{d}\zeta_i+q_i\mathrm{d}f_1\wedge\mathrm{d}\zeta_i$$
where $p_i,q_i\in\mathbb{R}[[f_1,f_2]]$;
- *in degree 4, elements in $H_4(\mathbb{R}^4,\pi)$ are of the form*
$$\mathbb{R}[[f_1,f_2]]\mu.$$

Remark 1.3. *The groups $H_\bullet(\mathbb{R}^4,\pi)$ from Theorem 1.2, or equivalently $H^\bullet(\mathbb{R}^4,\pi)$, were erroneously computed in [BV20]. We outline the problem of their computation in section 4.1.*

Using the isomorphism (5) we describe the corresponding cohomology groups in Section 2. In particular, we can use the result for the second formal cohomology to classify Poisson structures arising from the fibration in (2) via (1). We obtain the following result.

Corollary 1.4. *Any Poisson structure obtained from a different volume form is formally equivalent to a Poisson structure obtained from*
$$\mu_p=(c+p)\mu$$
for $p\in\langle f_1,f_2\rangle_{\mathbb{R}[[f_1,f_2]]}$ and $c\in\mathbb{R}_+$ and any p and c.

Note that the space of forms $\Omega^\bullet$ comes naturally equipped with another differential, the de-Rham differential d. One has the relation
$$\mathrm{d}\circ\delta_\pi+\delta_\pi\circ\mathrm{d}=0.$$
Therefore we have a bidifferential complex $(\Omega^\bullet_f,\mathrm{d},\delta_\pi)$ and an induced differential complex in Poisson homology $(H_\bullet(\mathbb{R}^4,\pi),\mathrm{d})$. It's cohomology are described as follows.

Corollary 1.5. *For the complex $(H_\bullet(\mathbb{R}^4,\pi),\mathrm{d})$ we find*
$$H^k_{DR}(H_\bullet(\mathbb{R}^4,\pi),\mathrm{d})=\begin{cases}\mathbb{R} & \text{if } k=0\\ 0 & \text{else.}\end{cases}$$

Strategy of the proof of Theorem 1.2. Roughly speaking the proof of Theorem 1.2 is as follows. We start by computing the kernel of δ_π. In turn this allows us to compute the dimension of the Poisson homology groups. Together with the Poisson homology groups these kernels fit into short exact sequences. This allows us to compute the Hilbert-Poincare series associated to the formal Poisson homology groups, and hence obtain their dimensions. The last step consists of finding an explicit set of generators of the right dimension.

In principle all the arguments consist of explicit computations. However, a recurring complication in these computations is that the following property does not always hold. Given $\beta \in \Omega^k_f$, does

$$\mathrm{d}f_1 \wedge \mathrm{d}f_2 \wedge \beta = 0 \quad \text{imply that} \quad \beta = \sum_{i=1}^{2} \mathrm{d}f_i \wedge \alpha_i \quad \text{for some} \quad \alpha_i \in \Omega^{k-1}_f?$$

The failure of this property to hold is measured by the groups $\mathcal{D}^k(\mathrm{d}f_1, \mathrm{d}f_2)$, first defined by [Rha54] and generalized by Saito [Sai76]. It turns out that for $k = 1$ this group vanishes. On the other hand $\mathcal{D}^2(\mathrm{d}f_1, \mathrm{d}f_2)$ is non-zero, which plays an important role in the computations.

There appears to be a close relation between the generators of $\mathcal{D}(\mathrm{d}f_1, \mathrm{d}f_2)$ and those of $H_\bullet(\mathbb{R}^4, \pi)$. For example, $\mathrm{d}\zeta_i$ appear both in Theorem 1.2, and as the generators of $\mathcal{D}^2(\mathrm{d}f_1, \mathrm{d}f_2)$, see Proposition 4.3. Similarly, the appearance of the classes represented by coefficients a_i and b_i in $\mathbb{R}[[x_2^2, x_4]]$ appears to be related to the description of the division groups. We do not explore this connection further but leave it as an option question:

Question 1.6. *Let $f_1, \ldots, f_{n-2} : \mathbb{R}^n \to \mathbb{R}$ be smooth functions. What is the precise relation between the groups $\mathcal{D}^i(\mathrm{d}f_1, \ldots, \mathrm{d}f_{n-2})$ and the formal Poisson (co)homology of the Jacobi-Poisson structure on $\mathbb{R}^n$ induced by the functions $f_1, \ldots, f_{n-2}$?*

Organization of the paper. Recall from (5) that the formal Poisson homology and cohomology groups are isomorphic. In Section 2 we start by giving a cohomology description of the main theorem, and a geometric interpretation of some of the generators. We also partially describe the Gerstenhaber algebra structure of the groups $H^\bullet(\mathbb{R}^4, \pi)$.

In Section 3 we discuss some preliminary notions from Poisson geometry (Section 3.1), and algebra (Section 3.2). Most notably, the definition of Jacobi-Poisson structures and an explicit description of δ_π. Most of our computations involve power series, and are done degree wise. To this end we recall, in Section 3.2.1, the definition of Hilbert-Poincare series, which we later use to compute the rank of the Poisson homology groups in each homogeneous degree. We also recall some standard facts from algebra concerning regular sequences in Section 3.2.2, and standard bases in Section 3.2.3.

In Section 4 we recall the notion of division groups, which measure the failure for the division property to hold. The main result of this section is Proposition 4.3 giving an explicit description of $\mathcal{D}^2(\mathrm{d}f_1, \mathrm{d}f_2)$, the second division group associated to the coefficients of $\mathrm{d}f_1$ and $\mathrm{d}f_2$.

Using these results, Section 5 computes the kernel of the Poisson differential δ_π. The results are collected in Proposition 5.1. The proof of this proposition, which is a long but more or less straightforward computation, is the core of the paper. In Section 6 we use the description of $\ker \delta_\pi$ to compute the Hilbert-Poincare series of the Poisson homology groups, see Proposition 6.1.

The proof of Theorem 1.2, which combines the results from Section 5 and Section 6, is given in Section 7. We check that the generators in Theorem 1.2 are indeed in the kernel. Then, using the Hilbert-Poincare series we see that the set of generators have the right dimension. The remainder of the proof consists of showing that none of the generators are in the image of δ_π.

Lastly, Section 8 and Section 9 respectively contain the proofs of Corollary 1.4 and Corollary 1.5.

Acknowledgements. We would like to thank Ioan Mărcuț for bringing the problem to our attention and useful discussions. L. Toussaint is funded by the Dutch Research Council (NWO) on the project "proper Fredholm homotopy theory" (OCENW.M20.195) of the research programm Open Competition ENW M20-3. F. Zeiser would like to thank the Max Planck Institute for Mathematics in Bonn for its hospitality and financial support during the early stages of this project and the group of Thomas Rot at Vrije Universiteit Amsterdam for its hospitality and support during a research visit.

2. An interpretation of the results

In this section we review theorem 1.2 in view of (5), stating the corresponding result for formal Poisson cohomology and interpreting the result. In the first section we discuss the foliation $\mathcal{F}$ associated with π and its relation with $H^0(\mathbb{R}^4,\pi)$. The second section is devoted to the Lie algebra $H^1(\mathbb{R}^4,\pi)$ and in the third section we look at $H^2(\mathbb{R}^4,\pi)$ and the interpretation of some its classes in terms of deformations of π. Finally, in the last section we describe the higher Poisson cohomology groups and some of the additional algebraic structure present for Poisson cohomology.

2.1. **The foliation and the Casimir functions.**

For the geometric interpretation let's take a closer look at the foliation $\mathcal{F}$ induced by π. From the introduction we know that $\mathcal{F}$ is closely related to the fibers of the fibration f from (2). In particular, for $(x,y)\neq 0\in\mathbb{R}^2$, the fiber $f^{-1}(x,y)$ is connected and diffeomorphic to a cylinder.

$$f^{-1}(x,y)\simeq S^1\times\mathbb{R}$$

The preimage of the origin under f consists of three leaves. The set $f^{-1}(0)\setminus\{0\}$ has two connected components both of which are diffeomorphic to cylinders. The origin $0\in\mathbb{R}^4$ is the sole critical point of f and is therefore a leaf of dimension 0.

$$f^{-1}(0)=S^1\times\mathbb{R}\cup\{0\}\cup S^1\times\mathbb{R}$$

The leaf space $\mathbb{R}^4/\mathcal{F}$ is, as a topological space, given by $\mathbb{R}^2$ where we have three distinct points instead of the origin. However, continuous functions on $\mathbb{R}^4/\mathcal{F}$ can not distinguish these three points. Viewing f_1 and f_2 as the coordinate functions on the leaf space we obtain for formal functions on the leaf space precisely what we expect, i.e./ power series in these coordinates. That is, in degree 0 we have

$$H^0(\mathbb{R}^4,\pi)=\mathbb{R}[[f_1,f_2]].$$

Remark 2.1. *In fact one can use the discussion above to show that we have an isomorphism*

$$C^\infty(\mathbb{R}^2)\to H^0(\mathbb{R}^4,\pi),\quad g\mapsto g(f_1,f_2).$$

Here $H^0(\mathbb{R}^4,\pi)$ refers to the Poisson cohomology over smooth multivector fields (see [MZ23]).

2.2. **The Lie algebra $H^1(\mathbb{R}^4,\pi)$.**

To describe the Poisson cohomology group in degree one let E_i and T_i, $i=1,2$, be the real and imaginary part of the complex vector fields

$$E=z\partial_z+w\partial_w\qquad\text{and}\qquad T:=z\partial_w-w\partial_z;$$

where $(z,w)=(x_1+ix_2,x_3+ix_4)$.

Corollary 2.2. *In degree one $H^1(\mathbb{R}^4,\pi)$ is the free $H^0(\mathbb{R}^4,\pi)$-module generated by*

$$E_1,E_2,T_1,T_2.$$

This follows immediately from the relations

$$\star E_1=\zeta_i\wedge\mathrm{d}\zeta_i,\quad \star E_2=-\zeta_1\wedge\mathrm{d}\zeta_2=\zeta_2\wedge\mathrm{d}\zeta_1,\tag{7}$$

$$\star T_1=-\frac{1}{4}\mathrm{d}f_i\wedge\mathrm{d}\zeta_i,\quad \star T_2=-\frac{1}{4}\mathrm{d}f_2\wedge\mathrm{d}\zeta_1=\frac{1}{4}\mathrm{d}f_1\wedge\mathrm{d}\zeta_2\tag{8}$$

The Lie bracket on $H^1(\mathbb{R}^4,\pi)$ is induced by the Lie bracket for vector fields. In order to describe the Lie algebra structure, we note that

$$[E_1,E_2]=[T_1,T_2]=[E_i,T_j]=0.$$

Hence the brackets in cohomology are fully described by the relations

$$\mathcal{L}_{E_1}(f_i)=f_i,\quad \mathcal{L}_{E_2}f_1=f_2,\quad \mathcal{L}_{E_2}f_2=-f_1,\quad \mathcal{L}_{T_i}f_j=0.\tag{9}$$

Geometrically, (9) means that the Lie algebra

$$\mathfrak{g}_N:=\{p_iE_i\mid p_i\in\mathbb{R}[[f_1,f_2]]\}$$

describes Poisson vector fields who project non-trivially to the leaf space, i.e. E_1 is mapped to the Euler vector field and E_2 to the rotational vector field. The vector fields T_i preserve the leaves and form a commutative Lie subalgebra of Poisson vector fields tangent to $\mathcal{F}$ which we denote by

$$\mathfrak{g}_T := \{p_i T_i \mid p_i \in \mathbb{R}[[f_1, f_2]]\}$$

As a Lie algebra, the first formal Poisson cohomology is given by

$$H^1(\mathbb{R}^4, \pi) \simeq \mathfrak{g}_T \rtimes \mathfrak{g}_N.$$

2.3. $H^2(\mathbb{R}^4, \pi)$**: Infinitesimal deformations.** Combining Theorem 1.2 with the isomorphism from Equation (5), we obtain the following description of the formal Poisson cohomology in degree 2. Here (and in the rest of this section) we use the following notation:

$$W_i := \star^{-1}(\mathrm{d}\zeta_i).$$

Corollary 2.3. *$H^2(\mathbb{R}^4, \pi)$ has unique representatives of the form*

$$pE_1 \wedge E_2 + q\pi + \sum_{i=1}^{2} p_i E_i \wedge T_1 + q_i W_i + \star^{-1}\left(\sum_{i=1}^{4} \mathrm{d}(a_i x_i) \wedge \mathrm{d}f_1\right)$$

where $p, p_i, q, q_i \in \mathbb{R}[[f_1, f_2]]$ and $a_i \in \mathbb{R}[[x_2^2, x_4]]$.

This follows from the identities

$$\star^{-1}(\zeta_1 \wedge \zeta_2) = -E_1 \wedge E_2 \tag{10}$$

$$\star^{-1}(\mathrm{d}f_1 \wedge \zeta_1) = -4E_i \wedge T_i + f_i W_i \quad \text{and} \quad \star^{-1}(\mathrm{d}f_1 \wedge \zeta_2) = 4E_1 \wedge T_2 + f_1 W_2 = -4E_2 \wedge T_1 + f_2 W_1$$

To study this result in more detail we use the filtration on $\mathfrak{X}^\bullet_f$ induced by f. We set

$$\mathfrak{X}^\bullet_{f,1} := \ker \iota_{\mathrm{d}f_1 \wedge \mathrm{d}f_2} \cap \mathfrak{X}^\bullet_f \quad \text{and} \quad \mathfrak{X}^\bullet_{f,2} := \ker \iota_{\mathrm{d}f_1} \cap \ker \iota_{\mathrm{d}f_2} \cap \mathfrak{X}^\bullet_f \tag{11}$$

Note that, since f_1 and f_2 are Casimir functions, d_π preserves $\mathfrak{X}^\bullet_{f,1}$ and $\mathfrak{X}^\bullet_{f,2}$, respectively. Infinitesimal deformations which preserves $\mathcal{F}$ are governed by cohomology classes represented by elements in $\mathfrak{X}^2_{f,2}$ up to coboundaries. It follows from the proof of the degree 2 part of Theorem 1.2, together with the identity

$$\iota_\alpha(\star^{-1}\beta) = (-1)^{k(4-k)} \star^{-1} (\beta \wedge \alpha) \tag{12}$$

for $\alpha \in \Omega^k, \beta \in \Omega^l$ and $0 \le k \le n-l$, that the only such cohomology classes are of the form $q\pi$ for $q \in \mathbb{R}[[f_1, f_2]]$. All such classes can be realized by a formal Poisson deformation of π of the form

$$\pi_t := (1 + tq)\pi$$

More generally, from (1) we know that any deformation of the volume form μ induces a deformation of Poisson structures preserving the foliation. By Corollary 1.4 we conclude that any Poisson structure obtained in this way is formally equivalent to one as described above.

On the other end of the spectrum of deformation we have those which induce a symplectic structure almost everywhere. There are two different classes of such deformations. First, the deformations represented by elements in $\mathfrak{X}^2_f / \mathfrak{X}^2_{f,1}$. By Corollary 2.3 such deformations are infinitesimally described by the classes $pE_1 \wedge E_2$. Indeed, any bivector

$$\pi_t := \pi + tpE_1 \wedge E_2$$

is a Poisson bivector whose rank is 4 away from the origin. This can be easily seen from the identities

$$\pi = -8\, T_1 \wedge T_2 \quad \text{and} \quad 16E_1 \wedge E_2 \wedge T_1 \wedge T_2 = (f_1^2 + f_2^2)\partial_1 \wedge \partial_2 \wedge \partial_3 \wedge \partial_4,$$

the fact that E_i and T_j commute and since $T_j \in \mathfrak{X}^1_{f,2}$ by (9). The other deformations we want to consider here are of the form

$$\pi_{1,t} = \pi + tW_1 \quad \text{and} \quad \pi_{2,t} = \pi + tW_2.$$

These bivectors are Poisson and we obtain a symplectic structure on $\mathbb{R}^4$ for $t > 0$ since

$$\pi \wedge W_i = 0 \quad \text{and} \quad W_i \wedge W_i = 2\partial_1 \wedge \partial_2 \wedge \partial_3 \wedge \partial_4.$$

2.4. **Higher degrees and the algebraic structure.** Next we consider deformations of the form

$$\pi_{i,t} = \pi + tpW_i. \tag{13}$$

To this end we first need to compute $H^3(\mathbb{R}^4, \pi)$ and $H^4(\mathbb{R}^4, \pi)$.

From Theorem 1.2 we obtain the following:

Corollary 2.4. *The formal Poisson cohomology of π satisfy:*

- *the group $H^3(\mathbb{R}^4, \pi)$ has unique representatives of the form*
$$\sum_{j=1}^{2} p_j E_j \wedge W_1 + q_j T_j \wedge W_1 + \sum_{i=1}^{4} b_i x_i T_1 \wedge W_1 + \star^{-1}\mathrm{d}(a_i x_i)$$
where $p_j, q_j \in \mathbb{R}[[f_1, f_2]]$ and $a_i, b_i \in \mathbb{R}[[x_2^2, x_4]]$;
- *the group $H^4(\mathbb{R}^4, \pi)$ has unique representatives of the form*
$$\left(p + \sum_{i=1}^{4} a_i x_i\right) \partial_1 \wedge \partial_2 \wedge \partial_3 \wedge \partial_4$$
where $p \in \mathbb{R}[[f_1, f_2]]$, $a_i \in \mathbb{R}[[x_2^2, x_4]]$ and where we write $\partial_j = \partial_{x_j}$.

The statement follows from the following identities

$$\star^{-1}\zeta_1 = E_1 \wedge W_1 = -E_2 \wedge W_2 \quad \text{and} \quad \star^{-1}\zeta_2 = E_1 \wedge W_2 = E_2 \wedge W_1$$
$$\star^{-1}\mathrm{d}f_1 = -4T_1 \wedge W_1 = 4T_2 \wedge W_2 \quad \text{and} \quad \star^{-1}\mathrm{d}f_2 = -4T_1 \wedge W_2 = -4T_2 \wedge W_1$$

Going back to the deformations from (13) we obtain, for $i = 1$ that:

$$[\pi_{1,t}, \pi_{1,t}] = 8t^2 p(\partial_x p T_1 \wedge W_1 + \partial_y p T_2 \wedge W_1)$$

In order to make the bivector Poisson, we would need to add a second order element in t, i.e. consider a deformation of the form

$$\pi_{1,t} = \pi + tpW_1 + t^2 W$$

for some $W \in \mathfrak{X}^2_f$ such that

$$[\pi, W] = 4p(\partial_x p T_1 \wedge W_1 + \partial_y p T_2 \wedge W_1).$$

However, the right side represents a non-trivial class in $H^3(\mathbb{R}^4, \pi)$ iff $p \in \mathbb{R}[[f_1, f_2]]$ is not constant and hence it is not possible to find such a W for non-constant p. The same argument shows that $\pi_{2,t}$ does not define a deformation of Poisson structures.

The algebraic structure in cohomology. The cohomology groups $H^\bullet(\mathbb{R}^4, \pi)$ carry a rich algebraic structure, as both the wedge product and the Schouten bracket

$$\wedge : \mathfrak{X}^k \times \mathfrak{X}^l \to \mathfrak{X}^{k+l} \quad \text{and} \quad [\cdot, \cdot] : \mathfrak{X}^k \times \mathfrak{X}^l \to \mathfrak{X}^{k+l-1}$$

descend to cohomology, inducing a Gerstenhaber algebra structure for Poisson cohomology:

$$\left(H^\bullet(\mathbb{R}^4, \pi), \wedge, [\cdot, \cdot]\right).$$

We distinguish two classes of representatives, those with coefficients in $\mathbb{R}[[f_1, f_2]]$ and those with coefficients in $\mathbb{R}[[x_2^2, x_4]]$. For the former, we note that the wedge product and the bracket can be easily deduced from the various relations described above. Describing the precise algebraic structure of representatives with coefficients in $\mathbb{R}[[x_2^2, x_4]]$ is harder and we do not attempt to describe them here. We only want to point out that the corresponding modules, as modules over $H^0(\mathbb{R}^4, \pi)$ are not free.

Explicitly, we have:

Proposition 2.5. *For cohomology classes represented by some $a_i, b_i \in \mathbb{R}$ we have:*

$$f_1 a_i x_i = -2(x_2^2 + x_4^2) a_i x_i \qquad \textit{and} \qquad f_1 b_i x_i = -2(x_2^2 + x_4^2) b_i x_i$$

in terms of the representatives in Theorem 1.2. Moreover, we have the relations

$$a(f_1 x_1 + f_2 x_2) = a(f_2 x_1 - x_2 f_1) = a(f_1 x_3 + f_2 x_4) = a(f_2 x_3 - f_1 x_4) = 0$$

in cohomology for $a \in \mathbb{R}$ and similarly for the classes represented by b_i.

Proof. Degree 4: We note that

$$x_1 f_1 \mu = -2(x_2^2 + x_4^2) x_1 \mu + \mathrm{d}f_1 \wedge \mathrm{d}f_2 \wedge \nu_1$$

where

$$\nu_1 := \frac{1}{4}\big(x_3 \mathrm{d}x_2 \wedge \mathrm{d}x_3 - x_4 \mathrm{d}x_1 \wedge \mathrm{d}x_3 + x_1 \mathrm{d}x_3 \wedge \mathrm{d}x_4\big)$$

which is closed and hence there exists a primitive $\alpha_1 \in \Omega^1(\mathcal{R})$. Similarly we obtain

$$x_1 f_2 \mu = -2(x_2^2 + x_4^2) x_2 \mu + \mathrm{d}f_1 \wedge \mathrm{d}f_2 \wedge \nu_2$$

for the closed two form

$$\nu_2 := \frac{1}{2}\big(- 2x_4 \mathrm{d}x_1 \wedge \mathrm{d}x_4 - x_4 \mathrm{d}x_1 \wedge \mathrm{d}x_3 + x_2 \mathrm{d}x_3 \wedge \mathrm{d}x_4\big).$$

Moreover, we have

$$(x_1 f_1 + x_2 f_2)\mu = \mathrm{d}f_1 \wedge \mathrm{d}f_2 \wedge \nu$$

for the closed two-form

$$\nu := \frac{1}{4}\big(x_4 \mathrm{d}x_2 \wedge \mathrm{d}x_4 - x_3 \mathrm{d}x_1 \wedge \mathrm{d}x_4 + x_1 \mathrm{d}x_3 \wedge \mathrm{d}x_4\big)$$

The other cases follow along the same lines.

Degree 2: We make use of the primitives in degree 4. Consider the 2-cocylce given by

$$\star W = \mathrm{d}(x_1 f_1 + 2(x_2^2 + x_4^2)x_1) \wedge \mathrm{d}f_2.$$

Setting $\star X := -\mathrm{d}f_1 \wedge \nu_1$ we obtain that

$$W = \mathrm{d}_\pi X.$$

Next consider the 2-cocylce given by

$$\star W = \mathrm{d}(x_1 f_1 + x_2 f_2)\mathrm{d}f_2.$$

If we set $\star X := -\mathrm{d}f_1 \wedge \nu$ then we obtain that

$$W = \mathrm{d}_\pi X.$$

The other cases follow similarly using the elements from the proof in degree 4.

Degree 3: Consider the 3-cocylce given by

$$\star T = (x_1 f_1 + 2(x_2^2 + x_4^2)x_1)\mathrm{d}f_1.$$

Let α_1 be a primitiv of ν_1. Then we obtain for $\star W := -\mathrm{d}f_1 \wedge \alpha_1$ that

$$T = \mathrm{d}_\pi W.$$

Next consider the 3-cocylce given by

$$\star T = (x_1 f_1 + x_2 f_2)\mathrm{d}f_1.$$

Let α be a primitiv of a closed 2-form ν and define $\star W := -\mathrm{d}f_1 \wedge \alpha$. Then we obtain that

$$T = \mathrm{d}_\pi W.$$

Similarly we can treat the cocycle

$$\star T = \mathrm{d}(x_1 f_1 + 2(x_2^2 + x_4^2)x_1).$$

Defining $\star W := -\nu_1$ we obtain

$$T = \mathrm{d}_\pi W.$$

The other cases follow similarly using the primitives from the proof in degree 4. □

3. Preliminaries

In this section we recall some facts from Poisson geometry and (homological) algebra. The first section is dedicated to Poisson geometry. In particular, we recall the definition of Jacobi-Poisson structures and describe the Poisson structure we are studying. In the second part we recall several notions from algebra: Hilbert-Poincare series, regular sequences and standard bases.

3.1. **Jacobi Poisson structures.** We recall the notion of Jacobi-Poisson structures as introduce in [Dam89] attributed to Flaschka and Ratiu. See also [GMP93]. In particular, we give an explicit formula for π from Definition 1.1 and we compute the Poisson differential d_π in the various degrees.

Given a vector space $\mathbb{R}^n$ with standard volume form μ and $n-2$ functions $f_1, \dots f_{n-2} \in C^\infty(\mathbb{R}^n)$ one can define a Poisson structure π on $\mathbb{R}^n$ by the following relation for the associated Poisson bracket:

$$\{g, h\}\mu := \mathrm{d}g \wedge \mathrm{d}h \wedge \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_{n-2} \qquad \text{for } g, h \in C^\infty(\mathbb{R}^n).$$

Poisson structures of this form are called *Jacobi-Poisson* structures. Note that the leaves of such Poisson structures have dimension at most 2. For a generalization with leaves of higher dimension see [DP12]. For some properties of Jacobi-Poisson structures and a generalization to manifolds we refer to [Prz01] and [GSV14]. One property is the following:

Lemma 3.1. *Any Jacobi-Poisson structure $\pi \in \mathfrak{X}^2(\mathbb{R}^n)$ is unimodular.*

Proof. We observe that using (3) we have the identity

$$\star(\pi) = \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_{n-2}.$$

Hence using the definition of the modular vector field the result follows. □

As a consequence of the Lemma together with (4) we obtain an isomorphism of complexes:

$$\star : (\mathfrak{X}^\bullet(\mathbb{R}^n), \mathrm{d}_\pi) \xrightarrow{\sim} (\Omega^{n-\bullet}(\mathbb{R}^n), \delta_\pi).$$

Poisson structure associated to a Lefschetz fibration. Consider $f : \mathbb{R}^4 \to \mathbb{R}^2$ as in (2) together with the standard volume forms μ_4 and μ_2 on $\mathbb{R}^4$ and $\mathbb{R}^2$ respectively, given by

$$\mu_4 = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4 \qquad \text{and} \qquad \mu_2 = \mathrm{d}y_1 \wedge \mathrm{d}y_2.$$

We often write μ instead of μ_4. Then π from Definition 1.1 is a Jacobi-Poisson structure and satisfies

$$\star(\pi) = \mathrm{d}f_1 \wedge \mathrm{d}f_2.$$

Explicitly, the Poisson bivector π is given by

$$\begin{aligned}\frac{1}{4}\pi = \quad &(x_1^2 + x_2^2)\partial_3 \wedge \partial_4 - (x_1x_4 - x_2x_3)(\partial_1 \wedge \partial_3 + \partial_2 \wedge \partial_4) \\ &+ (x_3^2 + x_4^2)\partial_1 \wedge \partial_2 + (x_1x_3 + x_2x_4)(\partial_2 \wedge \partial_3 - \partial_1 \wedge \partial_4)\end{aligned}$$

As an immediate consequence we can describe the Poisson differential δ_π on differential forms.

Proposition 3.2. *The Poisson differential δ_π is given as follows.*

- *In degree 1 we get for $\alpha \in \Omega^1(\mathbb{R}^4)$ that*
$$\delta_\pi(\alpha) = \iota_\pi(\mathrm{d}\alpha)$$
- *For a 2-form $\beta \in \Omega^2(\mathbb{R}^4)$ we obtain the formula*
$$\delta_\pi(\beta) = \iota_{\star^{-1}\mathrm{d}\beta}(\mathrm{d}f_1 \wedge \mathrm{d}f_2) - \mathrm{d}\iota_\pi(\beta)$$
- *In degree 3, the differential of $\gamma \in \Omega^3(\mathbb{R}^4)$ is given by*
$$\delta_\pi\gamma = \iota_\pi(\mathrm{d}\gamma) - \mathrm{d}\iota_{\star^{-1}\gamma}(\mathrm{d}f_1 \wedge \mathrm{d}f_2)$$
- *For the top degree we obtain for any $g \in C^\infty(\mathbb{R}^4)$ that*
$$\delta_\pi(g\mu) = \mathrm{d}g \wedge \mathrm{d}f_1 \wedge \mathrm{d}f_2$$

The proof follows from the definition of δ_π and the identity

$$\iota_V(\alpha) = (-1)^{k(n-l)}\iota_{\star^{-1}\alpha}(\star V),$$

for $V \in \mathfrak{X}^k(\mathbb{R}^n)$ and $\alpha \in \Omega^l(\mathbb{R}^m)$ with $k \le l \le m$.

Remark 3.3. *One can generalize the formulas for δ_π in Proposition 3.2 to describe the differentials associated to any Jacobi-Poisson structure.*

3.2. **(Homological) Algebra.** This section intends to provide the algebraic background needed for the paper. We first recall the definition of the Hilbert-Poincare series together with an example which we will use later on. After a brief recap of regular sequences in the second section, we recall the notion of standard bases in the third part. As an application, we provide a couple of examples which will be used in later computations.

3.2.1. *Hilbert-Poincare series.* A reference for the background material in this sequence is [AM69].

Let $\mathcal{R}$ be a commutative ring and C a class of $\mathcal{R}$-modules. A function $\lambda : C \to \mathbb{Z}$ is called *additive* if for every short exact sequence of $\mathcal{R}$-modules in C:

$$0 \to M' \to M \to M'' \to 0 \qquad \Rightarrow \qquad \lambda(M) = \lambda(M') + \lambda(M'').$$

Example 3.4. *For $\mathcal{R} = \mathbb{R}$ and C the class of all finite dimensional $\mathbb{R}$-vector spaces, $\lambda = \dim$ is a additive function.*

Proposition 3.5. *Assume we have an exact sequence of $\mathcal{R}$-modules in C*

$$0 \to M_0 \to M_1 \to \cdots \to M_n \to 0$$

such that all kernels of the homomorphisms are in C, then for any additive function λ on C:

$$\sum_{i=0}^{n} (-1)^i \lambda(M_i) = 0.$$

Let $\mathcal{R} = \oplus_{n=0}^{\infty} \mathcal{R}_n$ be a Noetherian graded ring. Note that $\mathcal{R}_0$ is Noetherian and let $y_1, \ldots, y_s$ be the generators of $\mathcal{R}$ as an $\mathcal{R}_0$-algebra, of degrees $k_1, \ldots, k_r$, respectively. Let $\mathcal{M} = \oplus_{n=0}^{\infty} \mathcal{M}_n$ be a finitely-generated, graded $\mathcal{R}$-module. Note in particular that this implies that all $\mathcal{M}_i$ are finitely generated $\mathcal{R}_0$-modules. Moreover, let λ be an additive function on the class of all finitely-generated $\mathcal{R}_0$-modules. The *Hilbert-Poincare series* of M with respect to λ is the power series

$$HP(\mathcal{M}, t) := \sum_{i=0}^{\infty} \lambda(\mathcal{M}_i) t^i.$$

In general the Hilbert-Poincare series can be described as follows:

Theorem 3.6 ([AM69, Thm 11.1])**.** *$HP(\mathcal{M}, t)$ is a rational function in t of the form*

$$\frac{p(t)}{\Pi_{j=1}^{r}(1 - t^{k_j})} \qquad \textit{for} \;\; p(t) \in \mathbb{Z}[t].$$

For any graded module $\mathcal{M}$ we denote by $\mathcal{M}(-a)$ the module with the degree shifted by $a \in \mathbb{N}$, i.e.

$$\mathcal{M}(-a)_d = \begin{cases} \mathcal{M}_{d-a} & \text{if } \; 0 \leq d - a \\ 0 & \text{else.} \end{cases}$$

Hence by definition of the Hilbert-Poincare series we obtain for a degree shift the relation

$$HP(\mathcal{M}(-a), t) = t^{-a} HP(\mathcal{M}, t)$$

Another consequence for the Hilbert-Poincare series using Proposition 3.5 is the following.

Corollary 3.7. *Let $\mathcal{R}^1, \ldots, \mathcal{R}^n$ be Noetherian graded rings with $\mathcal{R}_0^1 = \cdots = \mathcal{R}_0^n = \mathcal{R}_0$ which are all finitely generated as an $\mathcal{R}_0$-algebra and let $\mathcal{M}^1, \ldots, \mathcal{M}^n$ be finitely-generated, graded $\mathcal{R}^1, \ldots, \mathcal{R}^n$-modules respectively, fitting into an exact sequence*

$$0 \to \mathcal{M}^1 \to \mathcal{M}^2 \to \cdots \to \mathcal{M}^n \to 0$$

such that all the homomorphisms preserve the grading and all kernels of the homomorphisms in the various degrees are in C. For $\lambda = \dim$ as in Example 3.4 we obtain using Proposition 3.5 that

$$\sum_{i=0}^{n} (-1)^i HP(\mathcal{M}^i, t) = 0.$$

An example which we will also use later on is that of formal forms on $\mathbb{R}^n$:

Example 3.8. *Denote by $\mathcal{R} := \mathbb{R}[[x_1, \ldots, x_n]]$ the ring of formal power series in n-variables and by $\Omega^\bullet(\mathcal{R})$ the DGA of differental forms with coefficient in $\mathcal{R}$, i.e.*

$$\Omega^\bullet(\mathcal{R}) = \wedge^\bullet(\mathbb{R}^n)^* \otimes \mathcal{R}$$

The degree of x_i is 1 and the degree of $\mathrm{d}x_i$ is -1 for all $1 \leq i \leq n$. Then we obtain

$$HP_{\Omega^m(\mathcal{R})}(t) = \binom{n}{m} \frac{t^m}{(1-t)^n}$$

3.2.2. *Regular sequences.* Here we follow [Pel09]. Let $\mathcal{R}$ be a commutative ring and $\mathcal{M}$ be a finitely generated $\mathcal{R}$-module. We recall that we call an element $r \in \mathcal{R}$ a *non-zero divisor* on $\mathcal{M}$ if

$$\forall m \in \mathcal{M}: \ rm = 0 \quad \Rightarrow m = 0,$$

and a $\mathcal{M}$-*regular sequence* is a sequence $r_1, \ldots, r_q \in \mathcal{R}$ such that r_i is a non-zero-divisor in the quotient

$$\mathcal{M}/\langle r_1, \ldots, r_{i-1}\rangle \cdot \mathcal{M},$$

where $\langle r_1, \ldots, r_{i-1}\rangle \subset \mathcal{R}$ denotes the ideal generated by r_j for $1 \leq j \leq i-1$. If $\mathcal{M} = \mathcal{R}$ then we simply call it a regular sequence.

Example 3.9. *The polynomials $x_1x_3 + x_2x_4$ and $x_1x_4 - x_2x_3$ form a regular sequence in $\mathbb{R}[[x_1, x_2, x_3, x_4]]$.*

Lemma 3.10 (see [Wei94])**.** *Let $r_1, \ldots, r_n$ be a regular sequence in $\mathcal{R}$, then the Koszul complex*

$$0 \to \wedge^0(\mathcal{R}^n) \xrightarrow{\wedge\alpha} \ldots \xrightarrow{\wedge\alpha} \wedge^n(\mathcal{R}^n) \to \mathcal{R}/\langle r_1, \ldots, r_n\rangle\mathcal{R} \to 0$$

is exact, where $\alpha = \sum r_i e_i$ and $(e_1, \ldots, e_n)$ is a free basis of $\mathcal{R}^n$.

Corollary 3.11. *The operations $\wedge \mathrm{d}f_1$ and $\wedge \mathrm{d}f_2$ are exact in the sense of the above lemma.*

3.2.3. *Standard bases.* The notion of standard bases was introduced by Hironka in [Hir64] and Buchberger [Buc65]. For more details see e.g [Eis95][chapter 15] or [GP08][chapter 1]. We only give a brief summary of the most relevant facts for us. We will use standard bases to compute ideal membership and intersection of ideals in the ring of power series in an easy way.

Remark 3.12. *All our computations are done for the various degrees of homogeneous polynomials. Hence it would be enough to use Gröbner basis. However, due to our general setting we decide to recap the theory for power series.*

Throughout this subsection let $\mathcal{R} := \mathbb{R}[[x_1, \ldots, x_n]]$. On $\mathcal{R}$ we consider the local ordering obtained from the following monomial ordering (see [GP08][section 1.2]):

$$x^\alpha > x^\beta \quad \Leftrightarrow \quad |\alpha| < |\beta| \ \text{ or } \ |\alpha| = |\beta| \text{ and } \exists 1 \leq i \leq n: \ \alpha_1 = \beta_1, \ldots, \alpha_{i-1} = \beta_{i-1}, \alpha_i > \beta_i.$$

For $f \in \mathcal{R}$ written as

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha$$

and any subset $G \subset \mathcal{R}$ we define

$$LM(f) := \max\{x^\alpha | a_\alpha \neq 0\}, \quad LC(f) = \{a_\alpha | LM(f) = x^\alpha\} \quad \text{ and } \quad L(G) := \langle LM(g) | g \in G \setminus \{0\}\rangle.$$

A standard basis is defined as follows.

Definition 3.13. *Let $\mathcal{I} \subset \mathcal{R}$ be an ideal. A standard basis of $\mathcal{I}$ is a finite set $G \subset \mathcal{I}$ with*

$$G \subset \mathcal{I} \qquad \textit{and} \qquad L(\mathcal{I}) = L(G).$$

Moreover we need the following definitions.

Definition 3.14. *Let $G \subset \mathcal{R}$.*

(1) *$f \subset \mathcal{R}$ is called reduced with respect to G if no monomial of f is contained in $LG(G)$.*
(2) *G is called reduced if $0 \neq G$, for all $f \neq g \in G$ then $LM(f) \nmid LM(g)$ and for all $g \in G$, $LC(g) = 1$ and $g - LM(g)$ is reduced in G.*

Now we have the following proposition.

Proposition 3.15. *Let $f, f_1, \ldots, f_m \in \mathcal{R}$ then there exist $q_1, \ldots, q_m, r \in \mathcal{R}$ such that*

$$f = \sum_{i=1}^{m} q_i f_i + r$$

and r is reduced with respect to $\{f_1, \ldots, f_m\}$ and for all $i = 1, \ldots m$ we have $LM(q_i f_i) \leq LM(f)$.

Moreover, if $\{f_1, \ldots, f_m\}$ is reduced, then r is unique.

Proof. The existence of r follows from [GP08][Theorem 6.4.1]. Uniqueness follows by comparing the monomials of two different r's. $\square$

For $f_1, \ldots f_m \in \mathcal{R}$ as above we call r the normal form of $f \in \mathcal{R}$, i.e.

$$NF(f|\{f_1, \ldots f_m\}) := r.$$

Corollary 3.16. *Let $\mathcal{I} \subset \mathcal{R}$ be an ideal, $G \subset \mathcal{I}$ a reduced standard basis of $\mathcal{I}$. Then for any $f \in \mathcal{R}$:*

$$f \in \mathcal{I} \qquad \Leftrightarrow \qquad NF(f, G) = 0.$$

Proof. For a proof see [GP08][Lemma 1.6.7]. $\square$

The following applications will be useful later. Let $\mathcal{R} := \mathbb{R}[[x_1, x_2, x_3, x_4]]$ and let $\mathcal{J} \subset \mathcal{R}$ be the ideal defined by

$$\mathcal{J} = \mathcal{J}(\mathrm{d}f_1, \mathrm{d}f_2) = \langle G := \{x_1^2 + x_2^2, x_3^2 + x_4^2, x_1x_3 + x_2x_4, x_1x_4 - x_2x_3\}\rangle_{\mathcal{R}}. \tag{14}$$

We have the following two results.

Lemma 3.17. *Consider the module $\mathcal{M}$ over $\mathbb{R}[[x_2^2, x_4]]$ defined by*

$$\mathcal{M} := \langle x_1, x_2, x_3, x_4 \rangle_{\mathbb{R}[[x_2^2, x_4]]}.$$

Then:

$$\mathcal{J} \cap \mathbb{R}[[f_1, f_2]] = \mathcal{J} \cap (\mathbb{R}[[f_1, f_2]] + \mathcal{M}) = (f_1^2 + f_2^2)\mathbb{R}[[f_1, f_2]] \qquad \text{and} \qquad \mathbb{R}[[f_1, f_2]] \cap \mathcal{M} = \{0\}.$$

Proof. Note that G is reduced, hence we can use Corollary 3.16. Moreover, $\mathcal{J}$ is generated by homogeneous polynomials. Hence it is enough to check the statement for homogeneous degree polynomials. For odd homogeneous degree $n = 2m + 1$ we only have contributions from $\mathcal{M}$. We write $h \in \mathcal{M}$ as

$$h = \sum_{j=1}^{4} x_j \sum_{i=0}^{m} a_i^j x_2^{2i} x_4^{2(m-i)}.$$

Then we obtain by exchanging x_1x_4 with x_2x_3 that

$$\begin{aligned} r := NF(h|G) = a_m^1 x_1 x_2^{2m} + x_3 \left(\sum_{i=0}^{m} a_i^3 x_2^{2i} x_4^{2(m-i)} + \sum_{i=0}^{m-1} a_i^1 x_2^{2i+1} x_4^{2(m-i)-1} \right) \\ + \sum_{i=0}^{m} \left(a_i^2 x_2^{2i+1} x_4^{2(m-i)} + a_i^4 x_2^{2i} x_4^{2(m-i)+1} \right). \end{aligned}$$

Hence we have

$$h \in \mathcal{J} \quad \Leftrightarrow \quad r = 0 \quad \Leftrightarrow \quad h = 0.$$

For the even case $n = 2m > 0$ we observe that

$$f_1^2 + f_2^2 = (x_1^2 + x_2^2)^2 + (x_3^2 + x_4^2)^2 + 2(x_1x_3 + x_2x_4)^2 - 2(x_1x_4 - x_2x_3)^2,$$

hence it is enough to consider $g \in \mathbb{R}[[f_1, f_2]]$ of the form

$$g = g_1 f_1^m + g_2 f_1^{m-1} f_2.$$

We have

$$r_g = NF(g|G) = g_1(-2)^m (x_2^2 + x_4^2)^m + g_2(-1)^{m-1} 2^m \left(x_1 x_2^{2m-1} + x_3 \frac{(x_2^2 + x_4^2)^m - x_2^{2m}}{x_4} \right). \tag{15}$$

We write an element $h \in \mathcal{M}$ as

$$h = \sum_{j=1}^{4} x_j \sum_{i=0}^{m-1} a_i^j x_2^{2i} x_4^{2(m-i)-1}.$$

Note that

$$r_h = x_3 \sum_{i=0}^{m-1} \Big(a_i^3 x_2^{2i} x_4^{2(m-i)-1} + a_i^1 x_2^{2i+1} x_4^{2(m-i-1)} \Big) + \sum_{i=0}^{m-1} \Big(a_i^2 x_2^{2i+1} x_4^{2(m-i)-1} + a_i^4 x_2^{2i} x_4^{2(m-i)+1} \Big)$$

and we conclude

$$g + h \in \mathcal{J} \quad \Leftrightarrow \quad r_g + r_h = 0 \quad \Leftrightarrow \quad g = h = 0.$$

□

Lemma 3.18. *Let $g \in \mathbb{R}[[f_1, f_2]]$ and $(p, (q_1, q_2)) \in \mathbb{R}[[x_2]] \oplus \mathbb{R}[[x_2, x_4]]^2$ be such that*

$$g \cdot \mu = \mathrm{d}f_1 \wedge \mathrm{d}(x_1 p + x_3 q_1 + q_2)(\mathrm{d}x_1 \wedge \mathrm{d}x_4 + \mathrm{d}x_2 \wedge \mathrm{d}x_3) \mod \mathcal{J} \cdot \mu,$$

where μ is the standard volume form on $\mathbb{R}^4$. Then we have:

$$x_2 p + x_4 q_1, q_2 \in \mathbb{R}[[x_2^2 + x_4^2]] \qquad \textit{and} \qquad g \in (f_1^2 + f_2^2)\mathbb{R}[[f_1, f_2]],$$

and the same result holds if we replace $\mathrm{d}f_1$ by $\mathrm{d}f_2$.

Proof. By a direct computation we obtain that the right hand side of the statement equals:

$$2\Big(- x_2 q_1 - x_3(x_1 \partial_2 p + x_3 \partial_2 q_1 + \partial_2 q_2) + x_1(x_3 \partial_4 q_1 + \partial_4 q_2) + x_4 p \Big)\mu.$$

Note that again it is enough to check the statement for homogeneous polynomials. We set

$$p = p_0 x_2^{n-1}, \quad q_1 = \sum_{i=0}^{n-1} a_i x_2^i x_4^{n-1-i}, \quad q_2 = \sum_{i=0}^{n} b_i x_2^i x_4^{n-i}.$$

Then we obtain for the right hand side:

$$\begin{aligned} r_R := NF(- x_2 q_1 &- x_3(x_1 \partial_2 p + x_3 \partial_2 q_1 + \partial_2 q_2) + x_1(x_3 \partial_4 q_1 + \partial_4 q_2) + x_4 p | G) = \\ &- \sum_{i=0}^{n-1} a_i x_2^{i+1} x_4^{n-1-i} + (n-1) p_0 x_2^{n-1} x_4 + \sum_{i=0}^{n-1} i a_i x_2^{i-1} x_4^{n-i+1} - \sum_{i=0}^{n} i b_i x_2^{i-1} x_3 x_4^{n-i} \\ &- \sum_{i=0}^{n-1} (n-i-1) a_i x_2^{i+1} x_4^{n-1-i} + \sum_{i=0}^{n-2} (n-i) b_i x_2^{i+1} x_3 x_4^{n-i-2} + b_{n-1} x_1 x_2^{n-1} + p_0 x_2^{n-1} x_4 \\ = n p_0 x_2^{n-1} x_4 &+ a_1 x_4^n - a_{n-1} x_2^n - 2 a_{n-2} x_2^{n-1} x_4 - \sum_{i=0}^{n-3} \left((n-i) a_i - (i+2) a_{i+2} \right) x_2^{i+1} x_4^{n-1-i} \\ &- b_1 x_3 x_4^{n-1} + b_{n-1} x_1 x_2^{n-1} + \sum_{i=0}^{n-2} \left((n-i) b_i - (i+2) b_{i+2} \right) x_2^{i+1} x_3 x_4^{n-i-2}. \end{aligned}$$

If $n = 2m+1$ is odd then g does not contribute and we obtain the condition

$$\begin{aligned} (2m+1-i) a_i = -(i+2) a_{i+2},\ 0 \le i \le 2m-2, \quad a_1 = a_{2m} = 0, \quad 2 a_{2m-1} = (2m+1) p_0 \\ (i+2) b_{i+2} = (2m+1-i) b_i,\ 0 \le i \le 2m-1, \quad b_1 = b_{2m} = 0, \end{aligned}$$

which imply that $p = q_1 = q_2 = 0$.

For $n = 2m$ is even we note that

$$f_1^2 + f_2^2 = (x_1^2 + x_2^2)^2 + (x_3^2 + x_4^2)^2 + 2(x_1 x_3 + x_2 x_4)^2 - 2(x_1 x_4 - x_2 x_3)^2.$$

Hence it is enough to consider $g = g_1 f_1^m + g_2 f_1^{m-1} f_2$ as in the previous lemma. From (15) we obtain the conditions

$$2(m-i) a_{2i} = 2(i+1) a_{2(i+1)}, \quad (2(m-i)-1) a_{2i+1} - (2i+3) a_{2i+3} = (-2)^{m-1} \binom{m}{i+1} g_1,$$

for $0 \le i \le m-2$ and

$$a_1 = -(-2)^{m-1} g_1, \quad a_{2m-1} = (-2)^{m-1} g_1, \quad 2m p_0 = 2 a_{2m-2},$$

as well as

$$\begin{aligned} 2(i+1)b_{2(i+1)} &= 2(m-i)b_{2i}, \qquad 0 \le i \le m-1, \\ (2i+3)b_{2i+3} &= (2(m-i)-1)b_{2i+1} + (-2)^{m-1}\binom{m}{i+1}g_2, \quad 0 \le i \le m-2, \\ b_1 = (-2)^{m-1}g_2 \qquad & b_{2m-1} = -(-2)^{m-1}g_2, \end{aligned}$$

which implies the statement. The result for $\mathrm{d}f_2$ follows along the same lines. □

4. Division groups

Consider the space $\Omega^\bullet(M)$ of differential forms on a smooth manifold M. It is possible to divide differential forms in the following sence: let $\alpha \in \Omega^1(M)$ be nowhere vanishing then for any $\beta \in \Omega^\bullet(M)$,

$$\beta \wedge \alpha = 0 \quad \Longrightarrow \quad \beta = \mu \wedge \alpha,$$

for some $\mu \in \Omega^{\bullet-1}(M)$. This property is extremely useful when manipulating forms but does not hold in general for the exterior algebra of a module. To measure the failure, de Rham [Rha54] introduced the following groups.

Consider a free $\mathcal{R}$-module $\mathcal{M}$ with basis $(e_1, \dots, e_n)$. Fix elements $\alpha_1, \dots, \alpha_k \in \mathcal{M}$. The quotient

$$\mathcal{D}^p(\alpha_1, \dots, \alpha_k) := \{\beta \in \Lambda^p\mathcal{M} \mid \beta \wedge \alpha_1 \wedge \dots \wedge \alpha_k = 0\} / \sum_{i=1}^k \left(\alpha_i \wedge \Lambda^{p-1}\mathcal{M}\right),$$

is called the **p-th division group** associated to $(\alpha_1, \dots, \alpha_k)$.

We make use of a result by Saito [Sai76] about these groups. To state the results we first define the following. Use the basis elements to write

$$\alpha_1 \wedge \dots \wedge \alpha_k = \sum_{1 \le i_1 < \dots < i_k < \le n} a_{i_1 \cdots i_k} e_{i_1} \wedge \dots \wedge e_{i_k},$$

with $a_{i_1 \cdots i_k} \in \mathcal{R}$, and define $\mathcal{J} = \mathcal{J}(\alpha_1, \dots, \alpha_k) \subset \mathcal{R}$, the ideal generated by the coefficients $a_{i_1, \cdots, i_k}$.

Definition 4.1. *The* depth $\operatorname{depth}_{\mathcal{R}}(I, \mathcal{M})$ *of an ideal $I \subset \mathcal{R}$ on a finitely generated module $\mathcal{M}$ is the supremum of the lengths of all $\mathcal{M}$-regular sequences of elements of I.*

Proposition 4.2 ([Sai76])**.** *The division groups satisfy:*

$$\mathcal{D}^p(\alpha_1, \dots, \alpha_k) = 0,$$

for all $0 \le p < \operatorname{depth}(\mathcal{J}, \mathcal{R})$.

Now, let us return to the setting of Lefschetz singularities. We introduce the notation:

$$\beta_1 := \mathrm{d}\zeta_1 = \mathrm{d}x_1 \wedge \mathrm{d}x_3 - \mathrm{d}x_2 \wedge \mathrm{d}x_4 \quad \text{and} \quad \beta_2 := \mathrm{d}\zeta_2 = \mathrm{d}x_1 \wedge \mathrm{d}x_4 + \mathrm{d}x_2 \wedge \mathrm{d}x_3,$$

where ζ_i were defined in (6). We have the following result.

Proposition 4.3. *The depth of $\mathcal{J} = \mathcal{J}(\mathrm{d}f_1, \mathrm{d}f_2)$ is 2, and there is an isomorphism:*

$$\begin{aligned} I: \quad \mathbb{R} \oplus \mathbb{R}[[x_2]] \oplus \mathbb{R}[[x_2, x_4]]^2 &\xrightarrow{\sim} \quad \mathcal{D}^2(\mathrm{d}f_1, \mathrm{d}f_2) \\ (c, p, (q_1, q_2)) \quad &\mapsto [c\beta_1 + (px_1 + q_1x_3 + q_2)\beta_2] \end{aligned}$$

4.1. **Consequence of Proposition 4.3.** In this subsection we explain the problem of the claimed computation for the formal Poisson cohomology for Lefschetz singularities in [BV20].

Let us first recall the definition of an isolated complete intersection singularity. Set $\mathcal{R} = \mathbb{R}[[x_1, \dots, x_n]]$ and consider polynomials $p_1, \dots, p_k \in \mathbb{R}[x_1, \dots, x_n]$ with zero constant term. Define $\mathcal{I} = \mathcal{I}(p_1, \dots, p_k)$ to be the ideal generated by

$$\langle p_1, \dots, p_k, \det(\partial_{j_i} p_i)_{1 \le i \le k} \rangle.$$

Definition 4.4. *$(p_1, \dots, p_k)$ has an isolated complete intersection singularity (ICIS) if $(p_1, \dots, p_k)$ is a regular sequence in $\mathcal{R}$ and $\mathcal{R}/\mathcal{I}$ is a finite dimensional $\mathbb{R}$-vector space.*

Let us define the ideal $\mathcal{J} = \mathcal{J}(p_1, \ldots, p_k)$ by its generators

$$\mathcal{J} := \langle \det(\partial_{j_i} p_i)_{1 \le i \le k} \rangle_{\mathcal{R}}.$$

The following is due to [Loo84][Proof of Proposition (4.4)]:

Proposition 4.5. *If $(p_1, \ldots, p_k)$ is an ICIS, then the depth of the ideal $\mathcal{J}$ in $\mathcal{R}$ is $k+1$.*

For the computations in [BV20] it is assumed that (f_1, f_2) is an ICIS (p. 16, last paragraph). This assumption is crucial for the computations as done in [Pel09]. However, combining 4.3 and Proposition 4.5 implies that (f_1, f_2) is not an ICIS. In fact, as we shall see, the non-vanishing of $\mathcal{D}^2$ seems to be main factor for some of the cohomology groups being non-free modules over H^0.

4.2. **Proposition 4.3: The proof and a Corollary.** We start with the proof of Proposition 4.3. To see that $\operatorname{depth}(\mathcal{J}, \mathcal{R}) = 2$ observe that

$$\beta_i \wedge \mathrm{d}f_1 \wedge \mathrm{d}f_2 = 0, \quad i = 1, 2.$$

Furthermore, the coefficients of β_i are homogenenous of degree 0, whereas the coefficients of $\mathrm{d}f_i$ are homogeneous of degree 1. This implies that β_1 and β_2 define a non-zero classes in $\mathcal{D}^2(\mathrm{d}f_1, \mathrm{d}f_2)$. Hence, by Proposition 4.2 we have $\operatorname{depth}(\mathcal{J}, \mathcal{R}) \le 2$. Hence Example 3.9 implies that $\operatorname{depth}(\mathcal{J}, \mathcal{R}) = 2$.

We note first that the map is well-defined. Let $\omega \in \Omega^2(\mathcal{R})$ be of the form

$$\omega = \sum_{i<j} \omega_{ij} \mathrm{d}x_i \wedge \mathrm{d}x_j \qquad \text{with } \omega_{ij} \in \mathcal{R}.$$

The wedge product of ω with $\mathrm{d}f_1 \wedge \mathrm{d}f_2$ is described by the function

$$\begin{aligned} \star^{-1}(\frac{1}{4}\omega \wedge \mathrm{d}f_1 \wedge \mathrm{d}f_2) = \quad & (x_1^2 + x_2^2)\omega_{34} - (x_1x_4 - x_2x_3)(\omega_{13} + \omega_{24}) \\ & + (x_3^2 + x_4^2)\omega_{12} + (x_1x_3 + x_2x_4)(\omega_{23} - \omega_{14}). \end{aligned} \tag{16}$$

We note that for $\gamma \in \Omega^1(\mathcal{R})$ of the form

$$\gamma = \sum_i \gamma_i \mathrm{d}x_i$$

we obtain that

$$\begin{aligned} \frac{1}{2}\mathrm{d}f_1 \wedge \gamma = \quad & (x_1\gamma_2 + x_2\gamma_1)\mathrm{d}x_1 \wedge \mathrm{d}x_2 + (x_3\gamma_4 + x_4\gamma_3)\mathrm{d}x_3 \wedge \mathrm{d}x_4 \\ & + (x_1\gamma_3 - x_3\gamma_1)\mathrm{d}x_1 \wedge \mathrm{d}x_3 + (x_4\gamma_2 - x_2\gamma_4)\mathrm{d}x_2 \wedge \mathrm{d}x_4 \\ & + (x_1\gamma_4 + x_4\gamma_1)\mathrm{d}x_1 \wedge \mathrm{d}x_4 - (x_2\gamma_3 + x_3\gamma_2)\mathrm{d}x_2 \wedge \mathrm{d}x_3. \end{aligned} \tag{17}$$

Similarly we have for $\delta \in \Omega^1(\mathcal{R})$ that

$$\begin{aligned} \frac{1}{2}\mathrm{d}f_2 \wedge \delta = \quad & (x_2\delta_2 - x_1\delta_1)\mathrm{d}x_1 \wedge \mathrm{d}x_2 + (x_4\delta_4 - x_3\delta_3)\mathrm{d}x_3 \wedge \mathrm{d}x_4 \\ & + (x_2\delta_3 - x_4\delta_1)\mathrm{d}x_1 \wedge \mathrm{d}x_3 + (x_1\delta_4 - x_3\delta_2)\mathrm{d}x_2 \wedge \mathrm{d}x_4 \\ & + (x_2\delta_4 - x_3\delta_1)\mathrm{d}x_1 \wedge \mathrm{d}x_4 + (x_1\delta_3 - x_4\delta_2)\mathrm{d}x_2 \wedge \mathrm{d}x_3. \end{aligned} \tag{18}$$

We want to show that for any $\omega \in Z^2$ we can find γ and δ such that

$$\bar{\omega} = \omega + \mathrm{d}f_1 \wedge \gamma + \mathrm{d}f_2 \wedge \delta \tag{19}$$

is of the form as described in Proposition 4.3.

Let us assume that $\omega \in Z^2$ or equivalently that

$$\star^{-1}(\omega \wedge \mathrm{d}f_1 \wedge \mathrm{d}f_2) = 0.$$

Idea: We show in steps how we can simplify (parts) of the coefficients of ω as follows:

(1) We make choices for (parts) of the coefficients of γ_i in (17) and δ_j in (18) to show that these simplifications can be achieved.
(2) We continue in the next step with a choice for γ_i and δ_j based upon the simplification for ω which we verified in the previous steps.

Notation: To keep notation simple:

- We write $I \in \mathbb{N}_0^4$ with a subscript to indicate which entries vary over non-zero entries, e.g. I_{24} stands for $(0, i_2, 0, i_4)$ with $i_2, i_4 \in \mathbb{N}_0$. We simply write I for I_{1234};
- For $r \in \mathcal{R}$ the expression $r(I)$ refers to the coefficient of the term $x_1^{i_1} x_2^{i_2} x_3^{i_3} x_4^{i_4}$ for r;
- If $J = (j_1, j_2, j_3, j_4)$ happens to be such that at least one of the j_i is negative we set $r(J) := 0$.

We note first that the coefficients ω_{12} and ω_{34} satisfy:

$$\omega_{12}(I_{34}) = 0 \qquad \text{and} \qquad \omega_{34}(I_{12}) = 0.$$

This follows immediately from (16) since the only non-zero elements in the subrings $\mathbb{R}[[x_1, x_2]]$ and $\mathbb{R}[[x_3, x_4]]$ can come from the corresponding parts of ω_{34} and ω_{12}, respectively.

Claim 1: We may assume that the coefficients ω_{12} and ω_{34} satisfy:

$$\omega_{12} = \omega_{34} = 0. \tag{20}$$

We can choose

$$\begin{aligned}
\delta_1(I) &:= \frac{1}{2}\omega_{12}(I + e_1) + \gamma_2(I) + (\gamma_1 + \delta_2)(I + e_1 - e_2), \\
\delta_2(I_{234}) &:= -\frac{1}{2}\omega_{12}(I_{234} + e_2) - \gamma_1(I_{234}), \qquad \delta_4(I_{124}) := -\frac{1}{2}\omega_{34}(I_{124} + e_4) - \gamma_3(I_{124}) \\
\delta_3(I) &:= \frac{1}{2}\omega_{34}(I + e_3) + \gamma_4(I) + (\gamma_3 + \delta_4)(I + e_3 - e_4).
\end{aligned} \tag{21}$$

By comparing the coefficients in (17), (18) and (19) this implies the claim.

Claim 2: We may assume that

$$\omega_{13} + \omega_{24} = 0 = \omega_{23} - \omega_{14} \quad \text{i.e.} \quad r = 0.$$

By Example 3.9 the sequence $\{x_1x_3 + x_2x_4, x_1x_4 - x_2x_3\} \subset \mathcal{R}$ is regular. Since $\mathcal{R}$ is a unique factorization domain, the vanishing of (16) together with (20) implies the existence of $r \in \mathcal{R}$ such that

$$\omega_{13} + \omega_{24} = (x_1x_3 + x_2x_4)r \quad \text{and} \quad \omega_{23} - \omega_{14} = (x_1x_4 - x_2x_3)r.$$

We choose

$$\gamma_1(I + e_1) := (\gamma_3 + \delta_4)(I + e_3) - \delta_2(I + e_1) - \frac{1}{2}r.$$

Summing up the $\mathrm{d}x_1 \wedge \mathrm{d}x_3$ and the $\mathrm{d}x_2 \wedge \mathrm{d}x_4$-components of (17) and (18) yields

$$(x_1x_3 + x_2x_4)\big((\gamma_3 + \delta_4)(I + e_3) - (\gamma_1 + \delta_2)(I + e_1)\big).$$

Similarly, we obtain for the $\mathrm{d}x_1 \wedge \mathrm{d}x_4$-components minus the $\mathrm{d}x_2 \wedge \mathrm{d}x_3$-components that

$$-(x_1x_4 - x_2x_3)\big((\gamma_3 + \delta_4)(I + e_3) - (\gamma_1 + \delta_2)(I + e_1)\big)$$

implying the claim.

Claim 3: We may assume that

$$\omega_{13} \in \mathbb{R}.$$

Note that the sum of the $\mathrm{d}x_1 \wedge \mathrm{d}x_3$-components from (17) and (18) is given by

$$x_1\gamma_3 - x_3\gamma_1 + x_2\delta_3 - x_4\delta_1 = x_1\gamma_3(I_{124}) - x_3\gamma_1(I_{234}) + x_2\gamma_4 - x_4\gamma_2 + x_1x_3(\delta_2(I + e_1) - \delta_4(I + e_3)).$$

Hence by taking

$$
\begin{aligned}
\gamma_2(I_4) &:= \frac{1}{2}\omega_{13}(I_4+e_4), \qquad \gamma_4(I_{24}) := -\frac{1}{2}\omega_{13}(I_{24}+e_2)+\gamma_2(I_{24}+e_2-e_4),\\
\gamma_3(I_{124}) &:= -\frac{1}{2}\omega_{13}(I_{124}+e_1)-\gamma_4(I_{124}+e_1-e_2)+\gamma_2(I_{124}+e_1-e_4),\\
\gamma_1(I_{234}) &:= \frac{1}{2}\omega_{13}(I_{234}+e_3)+\gamma_4(I_{234}+e_3-e_2)-\gamma_2(I_{234}+e_3-e_4),\\
\delta_4(I+e_3) &:= \frac{1}{2}\omega_{13}(I+e_1+e_3)+\delta_2(I+e_1)+\gamma_4(I+(1,-1,1,0))-\gamma_2(I+(1,0,1,-1))
\end{aligned}
\tag{22}
$$

we obtain the claim.

To simplify ω_{14} we add the $\mathrm{d}x_1\wedge \mathrm{d}x_4$-components of (17) and (18), which yields

$$
x_1\gamma_4+x_4\gamma_1+x_2\delta_4-x_3\delta_1 = \begin{array}{l}(x_1^2+x_2^2)\gamma_4(I_{124}+e_1)+(x_1x_3+x_2x_4)\gamma_4(I+e_3)\\ -(x_3^2+x_4^2)\gamma_2(I+e_3)+(x_1x_4-x_2x_3)(\gamma_3(I+e_3)+\gamma_2(I_{24}+e_2)).\end{array}
\tag{23}
$$

Note that all the different coefficients are independent of each other. Moreover, we have

$$
\mathcal{J}=\mathcal{R}\cdot(x_3^2+x_4^2)+\mathcal{R}\cdot(x_1x_3+x_2x_4)+\mathcal{R}\cdot(x_1x_4-x_2x_3)+\mathbb{R}[[x_1,x_2,x_4]]\cdot(x_1^2+x_2^2).
\tag{24}
$$

Hence the result follows from:

Claim 4: The following map is an isomorphism of $\mathbb{R}$-vector spaces:

$$
\begin{array}{rccc}
m: & \mathbb{R}[[x_2]]\oplus\mathbb{R}[[x_2,x_4]]^2 & \xrightarrow{\sim} & \mathcal{R}/\mathcal{J}\\
& (p,(q_1,q_2)) & \mapsto & [px_1+q_1x_3+q_2].
\end{array}
$$

We first show that the map m is injective. In order to do this we use that the linear map:

$$
\begin{array}{rcl}
l:\mathbb{R}[[x_1,x_2]]\oplus\mathbb{R}[[x_3,x_4]]\oplus\mathcal{R}^2 & \to & \mathcal{J}\\
(p_1,p_2,(r_1,r_2)) & \mapsto & (x_1^2+x_2^2)p_1+(x_3^2+x_4^2)p_2+(x_1x_3+x_2x_4)r_1+(x_1x_4-x_2x_3)r_2
\end{array}
$$

is surjective with kernel given by

$$
(0,0,((x_1x_4-x_2x_3)\cdot r,-(x_1x_3+x_2x_4)\cdot r)) \qquad \text{for } r\in\mathcal{R}.
$$

This can be seen from Example 3.9 and the relations

$$
\begin{aligned}
x_3(x_1^2+x_2^2)-x_1(x_1x_3+x_2x_4)+x_2(x_1x_4-x_2x_3)&=0,\\
x_4(x_1^2+x_2^2)-x_2(x_1x_3+x_2x_4)-x_1(x_1x_4-x_2x_3)&=0,\\
x_1(x_3^2+x_4^2)-x_3(x_1x_3+x_2x_4)-x_4(x_1x_4-x_2x_3)&=0,\\
x_2(x_3^2+x_4^2)-x_4(x_1x_3+x_2x_4)+x_3(x_1x_4-x_2x_3)&=0.
\end{aligned}
$$

Let $(p,(q_1,q_2))\in\mathbb{R}[[x_2]]\oplus\mathbb{R}[[x_2,x_4]]^2$ such that $m(p,(q_1,q_2))=0$, i.e.

$$
\exists(p_1,p_2,(r_1,r_2))\in\mathbb{R}[[x_1,x_2]]\oplus\mathbb{R}[[x_3,x_4]]\oplus\mathcal{R}^2: \quad l(p_1,p_2,(r_1,r_2))=px_1+q_1x_3+q_2.
$$

Comparing the terms in $\mathbb{R}[[x_1,x_2]]$ and $\mathbb{R}[[x_3,x_4]]$ we immediately obtain

$$
p=0 \qquad q_1(I_4)=0, \qquad q_2(I_2)=0 \qquad q_2(I_4)=0 \quad \text{and} \quad p_1=p_2=0.
$$

Hence we can write the image of m uniquely as

$$
x_2x_3\tilde{q}_1+x_2x_4\tilde{q}_2 \qquad \text{for some } \tilde{q}_1,\tilde{q}_2\in\mathbb{R}[[x_2,x_4]].
$$

But such an element is in the image of l iff it is zero, hence m is indeed injective.

We conclude the proof by comparing dimensions of the domain and codomain of m for a fixed homogeneous degree in $\mathcal{R}/\mathcal{J}$. We denote by $\mathcal{R}_d$ and $\mathcal{J}_d$ the corresponding subvector spaces generated by polynomials of homogeneous degree $d\in\mathbb{N}_0$. Using the fact that for $n\in N$ and $d\in\mathbb{N}_0$ we have

$$
\dim\mathbb{R}[[x_1,\dots,x_n]]_d=\binom{n-1+d}{n-1},
$$

and the description of $\mathcal{J}$ via l we obtain for the dimension of $\mathcal{J}_d$ for $2 \leq d$ that

$$\text{for } 2 \leq d < 4: \quad \dim \mathcal{J}_d = 2\binom{d-1}{1} + 2\binom{d+1}{3},$$
$$\text{and else}: \quad \dim \mathcal{J}_d = 2\binom{d-1}{1} + 2\binom{d+1}{3} - \binom{d-1}{3}.$$

Finally, a direct computation implies that for any $d \in \mathbb{N}$ we have

$$\dim \mathcal{R}_d/\mathcal{J}_d = \dim \mathcal{R}_d - \dim \mathcal{J}_d = 2(d+1),$$

which proves the claim by comparing it with $\dim(m^{-1}(\{\mathcal{R}_d/\mathcal{J}_d\}))$. Hence we proved Proposition 4.3.

Corollary 4.6. *The isomorphism from Proposition 4.3 satisfies the relations:*

$$I^{-1}([f_1 \cdot I(0,p,(q_1,q_2))]) = -2(0, x_2^2 p, (x_2x_4p + (x_2^2+x_4^2)q_1, (x_2^2+x_4^2)q_2))$$
$$I^{-1}([f_2 \cdot I(0,p,(q_1,q_2))]) = 2(0, q_2(I_2)x^{I_2+e_2}, (x_4q_2 + x_2^2x_4^{-1}(q_2 - q_2(I_2)x^{I_2}), -(x_2p + x_4q_1)(x_2^2+x_4^2)))$$

for any $(p,(q_1,q_2)) \in \mathbb{R}[[x_2]] \oplus \mathbb{R}[[x_2,x_4]]^2$. Additionally, we have

$$[f_1\beta_1 - f_2\beta_2] = 0 = [f_2\beta_1 + f_1\beta_2].$$

In particular, for any $(p,(q_1,q_2)) \in \mathbb{R}[[x_2]] \oplus \mathbb{R}[[x_2,x_4]]^2$ which satisfy

$$px_2 + x_4q_1 = a(x_2^2+x_4^2)^m \qquad \text{and} \qquad q_2 = b(x_2^2+x_4^2)^m$$

for some $a, b \in \mathbb{R}$, we obtain that

$$[(x_1p + x_3q_1)\beta_2] = [c_1 f_1^{m-1} f_2 \beta_2] \quad \text{and} \quad [q_2\beta_2] = [c_2 f_1^m \beta_2],$$

for $c_1 := (-2)^{-m}a$ and $c_2 := 2^{-m}b$.

Proof. The first two equations follow by a direct computation using (23) and its induced partition for $\mathcal{J}$ as stated in (24). The last two equalities are obtained from a direct computation and choices for γ and δ as described in (22) and (21). $\square$

5. Kernel of the Poisson differential

The aim of this section is to compute the kernel of the Poisson differential $\delta_\pi : \Omega_f^\bullet \to \Omega_f^{\bullet-1}$. We prove the following:

Proposition 5.1. *For the kernel of the differential δ_π from Proposition 3.2 we obtain the following results for formal differential forms. The statements are for $g, g_i \in \mathcal{R}$ and $p, p_i, q_i \in \mathbb{R}[[f_1, f_2]]$.*

- *For $\alpha \in \Omega_f^1$ we obtain that*

$$\alpha \in \ker \delta_\pi \quad \Leftrightarrow \quad \alpha = \mathrm{d}g_0 + \sum_{i=1}^2 p_i\zeta_i + g_i \mathrm{d}f_i. \tag{25}$$

- *In degree 2 we get for $\omega \in \Omega_f^2$ that*

$$\omega \in \ker \delta_\pi \quad \Leftrightarrow \quad \omega = p\zeta_1 \wedge \zeta_2 + g\mathrm{d}f_1 \wedge \mathrm{d}f_2 + \sum_{i=1}^2 \mathrm{d}f_1 \wedge p_i\zeta_i + \mathrm{d}(q_i\zeta_i) + \mathrm{d}g_i \wedge \mathrm{d}f_i. \tag{26}$$

- *For a formal 3-form $\gamma \in \Omega_f^3$ we obtain*

$$\gamma \in \ker \delta_\pi \quad \Leftrightarrow \quad \gamma = \mathrm{d}f_1 \wedge \mathrm{d}f_2 \wedge \mathrm{d}g + \sum_{i=1}^2 q_i\epsilon_i + \mathrm{d}f_1 \wedge \mathrm{d}\big(p_i\zeta_i\big) \tag{27}$$

  where $\epsilon_i = \star E_i$.
- *In the top degree we have for $g \in \mathcal{R}$ that*

$$g\mu \in \ker \delta_\pi \quad \Leftrightarrow \quad g \in \mathbb{R}[[f_1, f_2]]. \tag{28}$$

For top degree forms this is a consequence of [Pel09][Proposition 3.1], since $\mathcal{D}^1 = 0$. In the other degrees, Pelap uses the vanishing of $\mathcal{D}^2$, which doesn't hold in our case. Therefore, we need an adaptation which takes the non-triviality of $\mathcal{D}^2$ into account. The sections below provide proofs for the various degree and are ordered according to their logical dependence:

$$\text{degree } 1 \;\Rightarrow\; \text{degree } 3 \Rightarrow\; \text{degree } 2.$$

Therefore, the proof for degree 1 is at the core of the proof of Proposition 5.1, which we obtain by an induction argument. Another important ingredient for the proof in any degree is a good understanding of the group $\mathcal{D}^2$, as described in Proposition 4.3 and Corollary 4.6. We have two more observations:

- First we note that the standard scalar multiplication $m_t : \mathbb{R}^4 \to \mathbb{R}^4$ on $\mathbb{R}^4$ satisfies
$$\delta_\pi \circ m_t^* = m_t^* \circ \delta_\pi.$$
Hence δ_π maps homogeneous degree coefficients to homogeneous degree coefficients and it is enough to prove the statements for coefficients of a fixed homogeneous degree.
- It is easy to verify the implications $\Leftarrow$ in Proposition 5.1 using Proposition 3.2. Therefore, we really only need to show that $\Rightarrow$ holds. To do so it is enough to show that
$$\ker \delta_\pi \mod V = \{0\}, \tag{29}$$
where V denotes the vector space generated by elements on the right hand side of Proposition 5.1. Whenever we look at this quotient we refer to it by $\operatorname{mod} V$.

That being said, we are ready to start the proof of Proposition 5.1.

5.1. **Proof of Degree 1.** For $\alpha \in \Omega^1_f$ recall from Proposition 3.2 that

$$\delta_\pi(\alpha) = \mathrm{d}\alpha \wedge \mathrm{d}f_1 \wedge \mathrm{d}f_2.$$

Hence it suffices to prove that

$$\mathrm{d}\alpha \wedge \mathrm{d}f_1 \wedge \mathrm{d}f_2 = 0 \quad \Leftrightarrow \quad \alpha = \mathrm{d}g_0 + \sum_{i=1}^{2} p_i(f_1, f_2)\zeta_i + g_i \mathrm{d}f_i.$$

If the coefficients of α are of homogeneous degree $n+1$, i.e. $m_t^*\alpha = t^{n+2}\alpha$, the following lemma will allow us to derive the statement by induction over even and odd degrees, respectively.

Lemma 5.2. *For $n = 0$ there exist $c_1, c_2 \in \mathbb{R}$ such that*

$$\mathrm{d}(\alpha - (c_1\zeta_1 + c_2\zeta_2)) = 0.$$

For $1 \leq n = 2m$ there exist $c_1, c_2 \in \mathbb{R}$ such that

$$\mathrm{d}(\alpha - (c_1 f_1^m + c_2 f_1^{m-1} f_2)\zeta_2) = \alpha_1 \wedge \mathrm{d}f_1 + \alpha_2 \wedge \mathrm{d}f_2.$$

Moreover, for $1 \leq n = 2m+1$ we have

$$\mathrm{d}\alpha = \alpha_1 \wedge \mathrm{d}f_1 + \alpha_2 \wedge \mathrm{d}f_2.$$

In all cases we may assume

$$m_t^*\alpha_i = t^n \alpha_i \qquad \textit{for} \quad i = 1, 2.$$

Proof. From Proposition 4.3 we obtain that

$$\mathrm{d}\alpha \wedge \mathrm{d}f_1 \wedge \mathrm{d}f_2 = 0 \quad \Rightarrow \quad \mathrm{d}\alpha = I(c, p, (q_1, q_2)) + \sum_{i=1}^{2} \alpha_i \wedge \mathrm{d}f_i$$

for some $c \in \mathbb{R}, p \in \mathbb{R}[[x_2]], q \in \mathbb{R}[[x_2, x_4]]^2$ and $\alpha_i \in \Omega^1_f$. Note that since α has coefficient of homogeneous degree $n+1$ we may assume that the coefficients of $I(c, p, q)$ and α_i are homogeneous of degree n and $n-1$, respectively. Wedging the equation for $\mathrm{d}\alpha$ with $\mathrm{d}f_1$ and applying d yields

$$\mathrm{d}f_1 \wedge \mathrm{d}I(c, p, q) = \mathrm{d}f_1 \wedge \mathrm{d}f_2 \wedge \mathrm{d}\alpha_2. \tag{30}$$

If $n = 0$, then the statement follows immediately from the fact that $\mathrm{d}\zeta_i = \beta_i$. Hence we assume $n > 0$. Then (30) implies the hypothesis of Lemma 3.18 for $g \cdot \mu = 0$, which in turn implies that

$x_2p + x_4q_1, q_2 \in R[[x_2^2 + x_4^2]]$. Hence Corollary 4.6 implies $[(x_1p + x_3q_1)\beta_2] = [c_1f_1^{m-1}f_2\beta_2]$ and $[q_2\beta_2] = [c_2f_1^m\beta_2]$ so that

$$I(0,p,q) = [(c_1f_1^{m-1}f_2 + c_2f_1^m)\beta_2] \in \mathcal{D}^2.$$

Finally, since $\mathrm{d}\zeta_i = \beta_i$ we get

$$I(0,p,q) = [\mathrm{d}((c_1f_1^{m-1}f_2 + c_2f_1^m)\zeta_2)] \in \mathcal{D}^2,$$

and therefore by changing α_i we obtain the statement. □

We prove the statement first for the case of $n = 2m-1$ being odd. In the base case, i.e. $m = 0$ it is straightforward to see that $\tau = \mathrm{d}g$ for g of homogeneous degree 1. Hence we assume

$$m_t^*\alpha = t^{2m+3}\alpha \qquad \text{and} \qquad \mathrm{d}f_1 \wedge \mathrm{d}f_2 \wedge \mathrm{d}\alpha = 0.$$

From Lemma 5.2 we get that

$$\mathrm{d}\alpha = \sum_{i=1}^{2} \alpha_i \wedge \mathrm{d}f_i \quad \text{with} \quad m_t^*\alpha_i = t^{2m+1}\alpha_i \quad \text{and} \quad \mathrm{d}f_1 \wedge \mathrm{d}f_2 \wedge \mathrm{d}\alpha_i = 0 \quad \text{for} \quad i = 1,2.$$

We distinguish two cases. For $m = 1$ the induction hypothesis implies that

$$\mathrm{d}\alpha = \mathrm{d}g_1 \wedge \mathrm{d}f_1 + \mathrm{d}g_2 \wedge \mathrm{d}f_2,$$

and hence we can conclude that

$$\alpha = \mathrm{d}g_0 + g_1\mathrm{d}f_1 + g_2\mathrm{d}f_2,$$

For $1 < m$ the induction hypothesis yields

$$\mathrm{d}\alpha = \mathrm{d}g_1 \wedge \mathrm{d}f_1 + \mathrm{d}g_2 \wedge \mathrm{d}f_2 + g\mathrm{d}f_1 \wedge \mathrm{d}f_2,$$

for some $g \in \mathcal{R}$. Taking d of this expression we see that $g\mu \in \ker \delta_\pi$. Hence $g \in \mathbb{R}[[f_1, f_2]]$ and we can absorb g into g_1 and g_2. Therefore we obtain

$$\alpha = \mathrm{d}g_0 + g_1\mathrm{d}f_1 + g_2\mathrm{d}f_2$$

for some $g_0, g_1, g_2 \in \mathcal{R}$, which proves the Proposition for n odd.

For the even case let $n = 2m$. The base case ($m = 0$) can be checked by hand, where we obtain that

$$\alpha = \mathrm{d}g_0 + \sum_{i=1}^{2} p_i\zeta_i + g_i\mathrm{d}f_i$$

with $g_0 \in \mathcal{R}$ homogeneous of degree 2 and $p_i, g_i \in \mathbb{R}$ for $i = 1,2$. For α satisfying

$$m_t^*\alpha = t^{2m+2}\alpha \qquad \text{and} \qquad \mathrm{d}f_1 \wedge \mathrm{d}f_2 \wedge \mathrm{d}\alpha = 0$$

we get from Lemma 5.2 that

$$\mathrm{d}\alpha = \mathrm{d}\left((c_1f_1^m + c_2f_1^{m-1}f_2)\zeta_2\right) + \sum_{j=1}^{2}(\mathrm{d}g_j + \sum_{i=1}^{2} p_i^j\zeta_i) \wedge \mathrm{d}f_j + g\mathrm{d}f_1 \wedge \mathrm{d}f_2.$$

Here the $p_i^j \in \mathbb{R}[[f_1, f_2]]$ are homogeneous of degree $2m-2$, $g_j \in \mathcal{R}$ are homogeneous of degree $2m$ for $i,j \in \{1,2\}$ and $g \in \mathcal{R}$ is homogeneous of degree $2m-2$. Hence we have (see (29))

$$\mathrm{d}\alpha = \sum_{i,j=1}^{2} p_i^j\zeta_i \wedge \mathrm{d}f_j + g\mathrm{d}f_1 \wedge \mathrm{d}f_2 \qquad \mod \mathrm{d}V. \tag{31}$$

Lemma 5.3. *We may assume that*

$$\mathrm{d}\alpha = \sum_{i=1}^{2} p_i\zeta_i \wedge \mathrm{d}f_1 + g\mathrm{d}f_1 \wedge \mathrm{d}f_2 \qquad \mod \mathrm{d}V.$$

Proof. Note that we have the relations

$$\mathrm{d}f_1 \wedge \zeta_1 - \mathrm{d}f_2 \wedge \zeta_2 = \mathrm{d}(f_1\zeta_1 - f_2\zeta_2) \quad \text{and} \quad \mathrm{d}f_2 \wedge \zeta_1 + \mathrm{d}f_1 \wedge \zeta_2 = \mathrm{d}(f_2\zeta_1 + f_1\zeta_2).$$

Hence for any $p, q \in \mathbb{R}[[f_1, f_2]]$ we have:

$$\begin{aligned}(p + f_2\partial_y p)\mathrm{d}f_2 \wedge \zeta_2 - f_1\partial_y p \mathrm{d}f_2 \wedge \zeta_1 &= (p + f_1\partial_x p)\mathrm{d}f_1 \wedge \zeta_1 - f_2\partial_x p \mathrm{d}f_1 \wedge \zeta_2 && \mod \mathrm{d}V,\\ (q + f_2\partial_y q)\mathrm{d}f_2 \wedge \zeta_1 + f_1\partial_y q \mathrm{d}f_2 \wedge \zeta_2 &= -(q + f_1\partial_x q)\mathrm{d}f_1 \wedge \zeta_2 - f_2\partial_x q \mathrm{d}f_1 \wedge \zeta_1 && \mod \mathrm{d}V.\end{aligned}$$

Adding those two equations, we want to determine the image of

$$\begin{pmatrix} p \\ q \end{pmatrix} \mapsto \begin{pmatrix} q - f_1\partial_y p + f_2\partial_y q \\ p + f_2\partial_y p + f_1\partial_y q \end{pmatrix}.$$

By looking at the maximal powers of f_2 in the image and inductively decreasing, we obtain that the map is injective, and hence surjective, thus proving the claim. ☐

Applying d to equation (31) yields

$$0 = \mathrm{d}(\sum_{i=1}^{2} p_i\zeta_i) \wedge \mathrm{d}f_1 + \mathrm{d}g \wedge \mathrm{d}f_1 \wedge \mathrm{d}f_2.$$

To show that this implies (25) we first prove an auxiliary Lemma.

Lemma 5.4. *For any $g \in \mathcal{R}$ we have*

$$\mathcal{L}_{T_1} g, \mathcal{L}_{T_2} g \in \mathbb{R}[[f_1, f_2]] \quad \Rightarrow \quad \mathcal{L}_{T_1} g = \mathcal{L}_{T_2} g = 0.$$

Proof. Note that we can focus on even homogeneous degrees only. Hence we can view g as an element in ring generated by

$$\begin{gathered} f_1,\ f_2,\ x_1^2 + x_2^2 + x_3^2 + x_4^2,\ x_1^2 - x_2^2 - x_3^2 + x_4^2,\ x_1^2 + x_2^2 - x_3^2 - x_4^2,\\ 2(x_1x_2 - x_3x_4),\ 2(x_1x_3 + x_2x_4),\ 2(x_1x_3 - x_2x_4),\ 2(x_1x_4 + x_2x_3),\ 2(x_1x_4 - x_2x_3).\end{gathered}$$

By a direct computation we can check that $\mathcal{L}_{T_i}$ maps generators to generators. More generally, for a homogeneous monomial of degree n in the above variables we note that either the total degree n_f in the variables of f_1 and f_2 is preserved or the monomial is maped to zero. Therefore it is enough to look at polynomomials in f_1 and f_2, but on such $\mathcal{L}_{T_i}$ acts trivial. ☐

We use the Lemma to show the following.

Corollary 5.5. *Fix $j = 1, 2$. For $p_1, p_2 \in \mathbb{R}[[f_1, f_2]]$ and $g \in \mathcal{R}$ the condition*

$$\mathrm{d}g \wedge \mathrm{d}f_1 \wedge \mathrm{d}f_2 + \sum_{i=1}^{2} \mathrm{d}(p_i\zeta_i) \wedge \mathrm{d}f_j = 0$$

implies $p_1, p_2 = 0$ and $g \in \mathbb{R}[[f_1, f_2]]$.

Proof. Let $j = 1$. Note that we have the relations

$$4\iota_{T_i}\zeta_1 = f_i, \quad 4\iota_{T_1}\zeta_2 = f_2 \quad 4\iota_{T_2}\zeta_2 = -f_1 \qquad \mathcal{L}_{T_i}\zeta_j = 0. \tag{32}$$

Hence contracting with T_1 and T_2, respectively together with (9) yields the equations

$$0 = f_1\partial_y p_1 + f_2\partial_y p_2 + p_2 + 4\mathrm{d}g(T_1) \quad \text{and} \quad 0 = f_2\partial_y p_1 + p_1 - f_1\partial_y p_2 + 4\mathrm{d}g(T_2)$$

and therefore $\mathrm{d}g(T_1), \mathrm{d}g(T_2) \in \mathbb{R}[[f_1, f_2]]$. Hence Lemma 5.4 implies $\mathcal{L}_{T_i}(g) = 0$ and the above equations imply $p_1 = p_2 = 0$. The statement for g then follows from (28). The proof for $j = 2$ follows along the same lines. ☐

Remark 5.6. *Note that the conclusion of Corollary 5.5 also holds under the assumption*

$$\mathrm{d}g \wedge \mathrm{d}f_1 \wedge \mathrm{d}f_2 + \sum_{i=1}^{2} p_i \mathrm{d}\zeta_i \wedge \mathrm{d}f_j = 0$$

The proof follows along the same lines.

5.2. **Proof of Degree 3.** The argument consists of two steps which are proven below. Suppose that $\gamma \in \Omega^3_f$ satisfies $\delta_\pi \gamma = 0$ then we show the following:

Step 1: Replacing γ by $\gamma - q_1\epsilon_1 - q_2\epsilon_2$ for some $q_1, q_2 \in \mathbb{R}[[f_1, f_2]]$ we may assume that

$$\star^{-1}(\mathrm{d}\gamma) = \iota_{\star^{-1}\gamma}\mathrm{d}f_1 = \iota_{\star^{-1}\gamma}\mathrm{d}f_2 = 0. \tag{33}$$

Step 2: If γ satisfies (33) then there exists $\alpha \in \ker \delta_\pi \subset \Omega^1_f$ such that

$$\gamma = \mathrm{d}f_1 \wedge \mathrm{d}\alpha.$$

Using (25) the last equation can be rewritten as

$$\gamma = \mathrm{d}f_1 \wedge \mathrm{d}\big(f_2\mathrm{d}g + \sum_{i=1}^{2} p_i\zeta_i\big),$$

for $g \in \mathcal{R}$ and $p_1, p_2 \in \mathbb{R}[[f_1, f_2]]$, which finishes the proof.

Proof of Step 1: Let $\gamma \in \Omega^3_f$ be such that $m_t^*\gamma = t^{m+3}\gamma$ and $\delta_\pi\gamma = 0$, i.e.

$$0 = \delta_\pi\gamma = \iota_\pi(\mathrm{d}\gamma) - \mathrm{d}\iota_{\star^{-1}\gamma}(\mathrm{d}f_1 \wedge \mathrm{d}f_2).$$

Wedging the above equation with $\mathrm{d}f_1$ and $\mathrm{d}f_2$ respectively, using Proposition 3.2 and (28) implies

$$\iota_{\star^{-1}\gamma}(\mathrm{d}f_i) = p_i \in \mathbb{R}[[f_1, f_2]],$$

satisfying the equation

$$\star^{-1}(\mathrm{d}\gamma)\mathrm{d}f_1 \wedge \mathrm{d}f_2 = \big((\partial_x p_1)(f_1, f_2) + (\partial_y p_2)(f_1, f_2)\big)\mathrm{d}f_1 \wedge \mathrm{d}f_2 \tag{34}$$

and hence in particular that $\star^{-1}(\mathrm{d}\gamma) \in \mathbb{R}[[f_1, f_2]]$. Using the relations (9)

$$\delta(E_1) = 2, \quad \delta(E_2) = 0,$$

it is easy to see that $\epsilon_i = \star E_i \in \ker \delta_\pi$. Given $q_1, q_2 \in \mathbb{R}[[f_1, f_2]]$, the identities above also imply:

$$\iota_{q_1E_1+q_2E_2}\mathrm{d}f_1 = (xq_1 + yq_2)(f_1, f_2),$$
$$\iota_{q_1E_1+q_2E_2}\mathrm{d}f_2 = (yq_1 - xq_2)(f_1, f_2).$$

Hence, replacing γ by $\gamma - q_1\epsilon_1 - q_2\epsilon_2$, for suitably chosen q_1 and q_2, we may assume that

$$p_1(f_1, f_2) = p_1(f_1) \quad \text{ and } \quad p_2(f_1, f_2) = p_2(f_1).$$

We want to show that $p_1 = p_2 = 0$. Recall that be homogeneity of γ and since f_1 and f_2 are homogeneous of degree 2, we have $m = 2n - 1$ where $n \geq 1$ is the homogeneous degree of p_i as a polynomial in one variable. As such we can write

$$p_1 = c_1x^n \quad \text{and} \quad p_2 = c_2x^n,$$

for some $c_1, c_2 \in \mathbb{R}$. We also write $\gamma = \sum_{i=1}^4 \gamma_i \star(\partial_{x_i})$. We denote by a_i and b_i respectively the coefficient of the $x_1^{2n-1-i}x_2^i$-terms for γ_1 and γ_2, $0 \leq i \leq 2n-1$. With this notation, explicitly computing the $x_1^{2n-i}x_2^i$-terms of Equation (34) yields the following conditions:

$$2a_0 = c_1 \quad \text{ and } \quad 2b_0 = c_2,$$
$$(-1)^j\binom{n}{n-j}c_1 = 2(a_{2j} + b_{2j-1}) \quad \text{ and } \quad (-1)^j\binom{n}{n-j}c_2 = 2(a_{2j-1} + b_{2j}) \qquad \text{for } \ 1 \leq j \leq n,$$
$$a_{2j+1} = -b_{2j} \quad \text{ and } \quad a_{2j} = -b_{2j+1} \quad \text{ for } \ 0 \leq j \leq n-1,$$
$$2b_{2n-1} = (-1)^{n-1}c_1 \quad \text{ and } \quad 2a_{2n-1} = (-1)^n c_2.$$

Using the equations in lines 1-3 we obtain

$$a_{2j+1} = -b_{2j} = c_2\frac{(-1)^j}{2}\sum_{0\leq l\leq j}\binom{n}{l} \quad \text{ and } \quad a_{2j} = -b_{2j+1} = c_2\frac{(-1)^j}{2}\sum_{0\leq l\leq j}\binom{n}{l}$$

for $1 \leq j \leq n-1$, and hence in particular that

$$a_{2n-1} = c_2\frac{(-1)^{n-1}}{2}(2^n - 1) \quad \text{ and } \quad b_{2n-1} = c_1\frac{(-1)^n}{2}(2^n - 1)$$

which yields a contradiction for all $1 \leq n$ unless $c_1 = c_2 = 0$. This completes the proof of Step 1.

Proof of Step 2: Assume γ satisfies (33). Observe that for any $\alpha \in \Omega^1_f$ we have

$$\star \circ \sharp\left(\iota_{\star^{-1}\gamma}(\alpha)\right) = \alpha \wedge \gamma \tag{35}$$

where $\sharp : \Omega^0_f \to \mathfrak{X}^0_f$ is the canonical identification. Hence Corollary 3.11 implies that

$$\gamma = \mathrm{d}f_1 \wedge \beta$$

for some $\beta \in \mathcal{D}^2(\mathrm{d}f_1, \mathrm{d}f_2)$ with $m_t^*\beta = t^{m+1}\beta$. Thus, it remains to show that $\beta = \mathrm{d}\alpha$ with $\alpha \in \ker \delta_\pi$. From Proposition 4.3 we know that

$$\beta = I(c,p,q) + \sum_{i=1}^{2} \mathrm{d}f_i \wedge \alpha_i$$

for some $c \in \mathbb{R}, p \in \mathbb{R}[[x_2]], q \in \mathbb{R}[[x_2,x_4]]^2$ and $\alpha_i \in \Omega^1(\mathcal{R})$. Moreover, since we are interested in γ we may assume that $\alpha_1 = 0$, and rename α_2 to α. Then Equation (33) implies:

$$\mathrm{d}f_1 \wedge \mathrm{d}I(c,p,q) = \mathrm{d}f_1 \wedge \mathrm{d}f_2 \wedge \mathrm{d}\alpha. \tag{36}$$

Hence it suffices to show that

$$\beta = \sum_{i=1}^{2} \mathrm{d}f_i \wedge \alpha_i \mod \mathrm{d}V.$$

That is, we may assume $I(c,p,q) = 0$. If β has constant coefficients, i.e. $m = 1$, then this follows immediately from the fact that $\mathrm{d}\zeta_i = \beta_i$. Hence let's assume β is homogeneous of degree $m > 1$. To see the statement, we recall that the coefficients of $\mathrm{d}f_1 \wedge \mathrm{d}f_2$ are (by definition) in the ideal $\mathcal{J}$ defined in (14), so that (36) is zero mod $\mathcal{J}$. This means the left hand side of (36) satisfies the hypothesis of Lemma 3.18 taking $g \cdot \mu = 0$, which in turn implies that $x_2p + x_4q_1, q_2 \in R[[x_2^2 + x_4^2]]$. Hence $m = 2n + 1$ for $1 \leq n$, otherwise we are done. It then follows from Corollary 4.6 that $[(x_1p + x_3q_1)\beta_2] = [c_1 f_1^{n-1} f_2 \beta_2]$ and $[q_2\beta_2] = [c_2 f_1^n \beta_2]$ so that

$$I(0,p,q) = [(c_1 f_1^{n-1} f_2 + c_2 f_1^n)\beta_2].$$

Finally, since $\mathrm{d}\zeta_i = \beta_i$, and $\mathrm{d}(c_1 f_1^{n-1} f_2 + c_2 f_1^n) = 0 \mod \mathcal{J}$ we obtain

$$I(0,p,q) = [\mathrm{d}((c_1 f_1^{n-1} f_2 + c_2 f_1^n)\zeta_2)],$$

as desired. Hence the proof is complete.

5.3. **Proof of Degree 2.** Let $\beta \in \Omega^2_f$ such that

$$\delta_\pi(\beta) = \iota_{\star^{-1}\mathrm{d}\beta}(\mathrm{d}f_1 \wedge \mathrm{d}f_2) - \mathrm{d}\iota_\pi(\beta) = 0.$$

Applying d to the equation implies that $\mathrm{d}\beta \in \ker \delta_\pi$. By a direct computation we can check that

$$q\zeta_1 \wedge \zeta_2 \in \ker \delta_\pi$$

for any $q \in \mathbb{R}[[f_1, f_2]]$. Moreover, the relations

$$\mathrm{d}f_1 \wedge \zeta_1 \wedge \zeta_2 = f_1\epsilon_2 - f_2\epsilon_1, \quad \mathrm{d}f_2 \wedge \zeta_1 \wedge \zeta_2 = f_2\epsilon_2 + f_1\epsilon_1 \quad \text{and} \quad \mathrm{d}(\zeta_1 \wedge \zeta_2) = 2\epsilon_2 \tag{37}$$

imply that we have

$$\mathrm{d}(q\zeta_1 \wedge \zeta_2) = (2q + f_1\partial_x q + f_2\partial_y q)\epsilon_2 + (f_1\partial_y q - f_2\partial_x q)\epsilon_1. \tag{38}$$

Therefore, by replacing β by $\beta - q\zeta_1 \wedge \zeta_2$ and using (27) we can assume that

$$\mathrm{d}\beta = \mathrm{d}f_1 \wedge \mathrm{d}f_2 \wedge \mathrm{d}g + q\epsilon_1 + \mathrm{d}f_1 \wedge \mathrm{d}\big(p_i\zeta_i\big)$$

for some $q, p_i \in \mathbb{R}[[f_1, f_2]]$ and $g \in \mathcal{R}$. Taking the differential d of this equation yields

$$0 = \mathrm{d}q \wedge \epsilon_1 + 2q\mu = (f_1\partial_x q + f_2\partial_y q + 2q)\mu$$

and hence $q = 0$ and we can conclude that

$$\beta = g\mathrm{d}f_1 \wedge \mathrm{d}f_2 + \mathrm{d}f_1 \wedge p_i\zeta_i + \mathrm{d}\alpha$$

for $p_i \in \mathbb{R}[[f_1, f_2]]$, $g \in \mathcal{R}$ and $\alpha \in \Omega^1_f$. In other words, the map

$$\mathrm{d} : \{\beta = g\mathrm{d}f_1 \wedge \mathrm{d}f_2 + \mathrm{d}f_1 \wedge p_i\zeta_i + \mathrm{d}\alpha\} \cap \ker \delta_\pi \to \{\gamma \in \ker \delta_\pi |\ \gamma \text{ satisfies } (34)\}$$

is surjective. Therefore, it is enough to study its kernel, i.e. we may assume $\beta = \mathrm{d}\alpha$ such that

$$\delta_\pi(\beta) = \iota_{\star^{-1}\mathrm{d}\beta}(\mathrm{d}f_1 \wedge \mathrm{d}f_2) - \mathrm{d}\iota_\pi(\beta) = -\mathrm{d}\iota_\pi(\mathrm{d}\alpha) = 0.$$

By Proposition 3.2 this implies that $\alpha \in \ker \delta_\pi$ and hence (25) implies the statement.

6. Poincare series

In this section we use the Hilbert-Poincare series to determine the dimension of the Poisson homology groups. We achieve this by constructing short exact sequences for the kernels of δ_π as described in Proposition 5.1. We obtain the following result:

Proposition 6.1. *For the Poisson homology spaces we have the following Hilbert-Poincare series:*

$$\begin{aligned} HP_{H_0}(t) &= \frac{4t^2+4t+1}{(1-t^2)^2}, \\ HP_{H_1}(t) &= 4t\frac{t^3+t^2+2t+1}{(1-t^2)^2}, \\ HP_{H_2}(t) &= 2t^2\frac{4t^2+2t+1}{(1-t^2)^2}. \end{aligned}$$

In degree 1 we have the following statement.

Lemma 6.2. *The following sequence is exact:*

$$0 \to \mathbb{R}[[f_1, f_2]] \xrightarrow{\theta_1} \mathcal{R}(-2)^2 \oplus \mathcal{R} \oplus \mathbb{R}[[f_1, f_2]](-2)^2 \xrightarrow{\beta} \ker \delta_{\pi,1} \to 0,$$

where the maps are given by

$$\theta_1 : u \mapsto ((-\partial_{f_1} u, -\partial_{f_2} u), u, 0) \qquad \textit{and} \qquad \sigma_1 : ((g_1, g_2), g_0, (p_1, p_2)) \mapsto \mathrm{d}g_0 + \sum_i p_i(f_1, f_2)\zeta_i + g_i \mathrm{d}f_i.$$

Proof. Injectivity of θ_1 follows from the definition, while surjectivity of σ_1 follows from (25). Let

$$\alpha := \sigma_1(g_1, g_2, g_0, p_1, p_2) = 0.$$

Wedging with $\mathrm{d}f_1$, applying d to both sides and using Corollary 5.5 implies that $p_1 = p_2 = 0$ and $g_2 \in \mathbb{R}[[f_1, f_2]]$. Wedging with $\mathrm{d}f_1 \wedge \mathrm{d}f_2$ implies that $g_0 \in \mathbb{R}[[f_1, f_2]]$ by Proposition 3.2 and (28). Hence the statement follows by contraction with E_1 and E_2 by (9). $\square$

In degree 2 we obtain the following short exact sequences for the kernel.

Lemma 6.3. *The following sequence is exact:*

$$0 \to \mathbb{R}[[f_1, f_2]](-2)^2 \xrightarrow{\theta_2} \mathcal{R}(-2)^2 \oplus \mathcal{R}(-4) \oplus \mathbb{R}[[f_1, f_2]](-4)^3 \oplus \mathbb{R}[[f_1, f_2]](-2)^2 \xrightarrow{\sigma_2} \ker \delta_{\pi,2} \to 0,$$

where the maps are given by

$$\theta_2 : (u_1, u_2) \mapsto ((u_1, u_2), \partial_{f_2} u_1 - \partial_{f_1} u_2, 0, 0) \qquad \textit{and}$$

$$\sigma_2 : ((g_1, g_2), g, (p, p_1, p_2), (q_1, q_2)) \mapsto p\zeta_1 \wedge \zeta_2 + g\mathrm{d}f_1 \wedge \mathrm{d}f_2 + \sum_{i=1}^{2} \mathrm{d}f_1 \wedge p_i\zeta_i + \mathrm{d}(q_i\zeta_i) + \mathrm{d}g_i \wedge \mathrm{d}f_i.$$

Proof. Note that θ_2 is injective, σ_2 is surjective by the description of $\ker \delta_{\pi,2}$ in (26) and we have

$$\operatorname{Im} \theta_2 \subset \ker \sigma_2.$$

Let us determine the kernel of σ_2. To do so, we set

$$\beta := \sigma_2((g_1, g_2), g, (p, p_1, p_2), (q_1, q_2)).$$

We first note that due to (38), (9) and (12) we have

$$\mathrm{d}\beta \wedge (f_2 \mathrm{d}f_1 - f_1 \mathrm{d}f_2) = 2(f_1^2 + f_2^2)(2p + f_1\partial_x p + f_2 \partial_y p)\mu,$$

implying that $p = 0$ if $\beta = 0$. Wedging with $\mathrm{d}f_1$ and applying Corollary 5.5 implies that $q_1 = q_2 = 0$ and $g_2 \in \mathbb{R}[[f_1, f_2]]$. Applying d and using again Corollary 5.5 implies the statement. $\square$

In degree 3 we obtain the following short exact sequence for the kernel of the differential.

Lemma 6.4. *The following sequence is exact:*

$$0 \to \mathbb{R}[[f_1, f_2]](-4) \xrightarrow{\theta_3} \mathcal{R}(-4) \oplus \mathbb{R}[[f_1, f_2]](-4)^4 \xrightarrow{\sigma_3} \ker \delta_{\pi,3} \to 0,$$

where the maps are given by:

$$\theta_3 : u \mapsto (u, 0), \qquad \textit{and} \qquad \sigma_3 : (g, (q_1, q_2, p_1, p_2)) \mapsto \mathrm{d}f_1 \wedge \mathrm{d}f_2 \wedge \mathrm{d}g + \sum_{i=1}^{2} q_i \epsilon_i + \mathrm{d}f_1 \wedge \mathrm{d}(p_i \zeta_i).$$

Proof. The map θ_3 is injective by definition and surjectivity of σ_3 follows from (27). To show that

$$\operatorname{Im} \theta_3 = \ker \sigma_3$$

we define

$$\gamma := \sigma_3(g, (q_1, q_2, p_1, p_2)).$$

Note that by (9) and (12) we have the relations

$$(f_1 \mathrm{d}f_1 + f_2 \mathrm{d}f_2) \wedge \gamma = 2(f_1^2 + f_2^2) q_1 \mu, \qquad \text{and} \qquad (f_2 \mathrm{d}f_1 - f_1 \mathrm{d}f_2) \wedge \gamma = 2(f_1^2 + f_2^2) q_2 \mu.$$

Hence $\gamma = 0$ implies that $q_1 = q_2 = 0$. Therefore Corollary 5.5 implies the statement. $\square$

Now we are ready to proof Proposition 6.1.

Proof of Proposition 6.1. We use the exact sequence of degree preserving maps

$$0 \to \ker \delta_{\pi,i+1} \hookrightarrow \Omega_f^{i+1} \xrightarrow{\delta_{\pi,i+1}} \ker \delta_{\pi,i} \to H_i \to 0. \tag{39}$$

Let us first compute the Hilbert-Poincare series of the kernels of $\delta_{\pi,i}$. Therefore we note that

$$HP_{\mathcal{R}(-n)} = \frac{t^n}{(1-t)^4} \qquad \text{and} \qquad HP_{\mathbb{R}[[f_1,f_2]](-n)} = \frac{t^n}{(1-t^2)^2}$$

for all $n \in \mathbb{N}_0$. For any two graded vector spaces U and V we have that

$$HP_{U \oplus V} = HP_U + HP_V,$$

and for any exact sequence of graded vector spaces

$$0 \to V_0 \to \cdots \to V_k \to 0$$

with degree preserving maps in between, that

$$HP_{V_k} = \sum_{i=0}^{k-1} (-1)^{k-1-i} HP_{V_i}.$$

From Lemma 6.4 we obtain

$$HP_{\ker \delta_{\pi,3}}(t) = \frac{t^4}{(1-t^2)^2} + \frac{t^4}{(1-t)^4}.$$

Similarly, Lemma 6.3 and Lemma 6.2 imply

$$HP_{\ker \delta_{\pi,2}}(t) = \frac{3t^4}{(1-t^2)^2} + \frac{t^4 + 2t^2}{(1-t)^4} \qquad \text{and} \qquad HP_{\ker \delta_{\pi,1}}(t) = \frac{2t^2 - 1}{(1-t^2)^2} + \frac{2t^2 + 1}{(1-t)^4}.$$

Additionally, note that by Example 3.8 we have

$$HP_{\ker \delta_{\pi,0}}(t) = HP_{\Omega_f^0}(t) = \frac{1}{(1-t)^4}, \qquad HP_{\Omega_f^1}(t) = \frac{4t}{(1-t)^4},$$
$$HP_{\Omega_f^2}(t) = \frac{6t^2}{(1-t)^4}, \qquad HP_{\Omega_f^3}(t) = \frac{4t^3}{(1-t)^4}.$$

Hence (39) for $i=2$ implies that

$$\begin{aligned} HP_{H_2}(t) &= HP_{\ker\delta_{\pi,2}}(t) + HP_{\ker\delta_{\pi,3}}(t) - HP_{\Omega^3_f}(t) \\ &= \frac{3t^4}{(1-t^2)^2} + \frac{t^4+2t^2}{(1-t)^4} + \frac{t^4}{(1-t^2)^2} + \frac{t^4}{(1-t)^4} - \frac{4t^3}{(1-t)^4} \\ &= \frac{4t^4}{(1-t^2)^2} + 2t^2\frac{(1-t)^2}{(1-t)^4} \;=\; 2t^2\frac{4t^2+4t+1}{(1-t^2)^2}. \end{aligned}$$

For $i=1$ we get

$$\begin{aligned} HP_{H_1}(t) &= HP_{\ker\delta_{\pi,1}}(t) + HP_{\ker\delta_{\pi,2}}(t) - HP_{\Omega^2_f}(t) \\ &= \frac{2t^2-1}{(1-t^2)^2} + \frac{2t^2+1}{(1-t)^4} + \frac{3t^4}{(1-t^2)^2} + \frac{t^4+2t^2}{(1-t)^4} - \frac{6t^2}{(1-t)^4} \\ &= \frac{3t^4+2t^2-1}{(1-t^2)^2} + \frac{(1-t)^2(1+t)^2}{(1-t)^4} \;=\; 4t\frac{t^3+t^2+2t+1}{(1-t^2)^2}. \end{aligned}$$

Finally, for $i=0$ we compute

$$\begin{aligned} HP_{H_0}(t) &= HP_{\ker\delta_{\pi,0}}(t) + HP_{\ker\delta_{\pi,1}}(t) - HP_{\Omega^1_f}(t) \\ &= \frac{1}{(1-t)^4} + \frac{2t^2-1}{(1-t^2)^2} + \frac{2t^2+1}{(1-t)^4} - \frac{4t}{(1-t)^4} \\ &= \frac{2t^2-1}{(1-t^2)^2} + 2\frac{(1-t)^2}{(1-t)^4} \;=\; \frac{4t^2+4t+1}{(1-t^2)^2}. \end{aligned}$$

□

7. Proof of the main theorem

In this section we prove Theorem 1.2. In each degree the strategy of the proof is the same: by Proposition 6.1 the given set of generators has the right dimension. Hence it is enough to show that none of the elements generated by the given module are in the image of the Poisson differential.

7.1. **Degree 0.** For the proof in degree 0 we need the following auxiliary statement.

Lemma 7.1. *Let $g \in \mathbb{R}[[f_1,f_2]]$, $(c,p,(q_1,q_2)) \in \mathbb{R}\oplus\mathbb{R}[[x_2]]\oplus\mathbb{R}[[x_2,x_4]]^2$ and $\alpha\in\Omega^1_f$ such that:*

$$g\cdot\mu = \mathrm{d}f_1\wedge\mathrm{d}I(c,p,(q_1,q_2)) + \mathrm{d}f_1\wedge\mathrm{d}f_2\wedge\mathrm{d}\alpha.$$

Then $g=0$ and the same statement holds if we exchange the roles of $\mathrm{d}f_1$ and $\mathrm{d}f_2$.

Proof. We prove the statement for g being homogeneous of degree $2n$ by induction on n.

$n=0,1:$ If $g\in\mathbb{R}$ then the statement holds by a degree count of the coefficients. If g is homogeneous of degree 1 then the statement follows from Lemma 3.18.

Induction step: Assume that $g\in\mathbb{R}[[f_1,f_2]]$ is homogeneous of degree $2n$. Then Lemma 3.18 implies

$$x_2p + x_4q_1, q_2 \in \mathbb{R}\cdot(x_2^2+x_4^2)^{n-1}, \qquad \text{and} \qquad g\in(f_1^2+f_2^2)\mathbb{R}[[f_1,f_2]].$$

Hence, by Corollary 4.6 we have:

$$I(c,p,(q_1,q_2)) = (c_1f_1^{n-1}+c_2f_1^{n-2}f_2)\beta_2 + \sum_{i=1}^{2}\mathrm{d}f_i\wedge\alpha_i,$$

for some $c_i\in\mathbb{R}$ and $\alpha_i\in\Omega^1_f$, with $i=1,2$. Therefore, writing g as $(f_1^2+f_2^2)\widetilde{g}$ and replacing α by $\alpha-\alpha_2$, again denoted by α, the original equation becomes

$$(f_1^2+f_2^2)\widetilde{g}\cdot\mu = \mathrm{d}f_1\wedge\mathrm{d}f_2\wedge\mathrm{d}\alpha.$$

Using (9), (12) and (37) we note that

$$2g\mathrm{d}f_1\wedge\mathrm{d}f_2\wedge\zeta_1\wedge\zeta_2 = (f_1^2+f_2^2)g\cdot\mu.$$

Hence combining the two equations above, we obtain by Proposition 4.3 that

$$2\widetilde{g}\zeta_1 \wedge \zeta_2 - \mathrm{d}\alpha = I(0, \widetilde{p}, (\widetilde{q}_1, \widetilde{q}_2)) + \sum_{i=1}^{2} \alpha_i \wedge \mathrm{d}f_i$$

for some $(\widetilde{p}, (\widetilde{q}_1, \widetilde{q}_2)) \in \mathbb{R}[[x_2]] \times \mathbb{R}[[x_2, x_4]]^2$ and $\alpha_i \in \Omega^1_f$. Note that the homogeneous degrees of $\widetilde{p}, \widetilde{q}_1, \widetilde{q}_2$ and α_i are two smaller than those of p, q_1, q_2 and α, respectively. Now taking the de-Rham differential and wedging with $\mathrm{d}f_1$ implies by (9), (12) and (37) that

$$2((f_1^2 + f_2)^2 \partial_y \widetilde{g} + f_2 \widetilde{g}) \cdot \mu = \mathrm{d}f_1 \wedge \mathrm{d}I(0, \widetilde{p}, (\widetilde{q}_1, \widetilde{q}_2)) + \mathrm{d}f_1 \wedge \mathrm{d}f_2 \wedge \mathrm{d}\alpha_2.$$

Therefore, the induction hypothesis implies that

$$(f_1^2 + f_2)^2 \partial_y \widetilde{g} + f_2 \widetilde{g} = 0,$$

and hence $\widetilde{g} = 0$ concluding the induction. The proof for $\mathrm{d}f_2$ follows along the same lines. □

Now we are ready to prove Theorem 1.2 for degree 0. We distinguish between even and odd homogeneous degrees.

<u>The odd degree:</u> The zeroth Poisson homology group is realized by the free $\mathbb{R}[[x_2^2, x_4^2]]$-module generated by the linear monomials. This follows from the description of a boundary by Proposition 3.2 and Lemma 3.17.

<u>The even degree:</u> By a similar argument as in the odd degree we obtain that the coefficients in even degree are contained in the ideal $\mathcal{I}$ if and only if they are of the form

$$(f_1^2 + f_2^2)p$$

where $p \in \mathbb{R}[[f_1, f_2]]$. Hence we only need to check if there exists an $\alpha \in \Omega^1(\mathcal{R})$ such that

$$(f_1^2 + f_2^2)p\mu = \mathrm{d}f_1 \wedge \mathrm{d}f_2 \wedge \mathrm{d}\alpha$$

We show that such an α can only exist if $p = 0$. This follows from Lemma 7.1.

7.2. **Degree 1.** We first prove another auxiliary lemma.

Lemma 7.2. *Let $g \in \mathbb{R}[[f_1, f_2]]$ and $a_i \in \mathbb{R}[[x_2^2, x_4]]$ and $\gamma \in \Omega^3_f$ such that*

$$0 = \left(g - \star^{-1}(\mathrm{d}\gamma)\right) \mathrm{d}f_1 \wedge \mathrm{d}f_2 + \mathrm{d}\left(\left(\sum_{i=1}^{4} a_i x_i - \iota_{\star^{-1}\gamma}(\mathrm{d}f_2)\right) \mathrm{d}f_1 + \iota_{\star^{-1}\gamma}(\mathrm{d}f_1)\mathrm{d}f_2\right).$$

Then $g = a_1 = a_2 = a_3 = a_4 = 0$ and $\gamma \in \ker \delta_\pi$.

Proof. Note that by wedging with $\mathrm{d}f_1$, Proposition 3.2 and (28) imply

$$\iota_{\star^{-1}\gamma}(\mathrm{d}f_1) \in \mathbb{R}[[f_1, f_2]].$$

Using (35), (9) and Corollary 3.11 we obtain that

$$\gamma = r_1\epsilon_1 + r_2\epsilon_2 + \mathrm{d}f_1 \wedge \beta$$

for unique $r_1 \in \mathbb{R}[[f_1, f_2]]$, $r_2 \in \mathbb{R}[[f_2]]$ and some $\beta \in \Omega^2_f$. Wedging with $\mathrm{d}f_2$ and using (9), (35), Proposition 3.2 and (28) implies

$$\left(\sum_{i=1}^{4} a_i x_i\right)\mu + \beta \wedge \mathrm{d}f_1 \wedge \mathrm{d}f_2 \in \mathbb{R}[[f_1, f_2]]\mu.$$

Hence Lemma 3.17 yields

$$a_1 = a_2 = a_3 = a_4 = 0 \qquad \text{and} \qquad \beta \wedge \mathrm{d}f_1 \wedge \mathrm{d}f_2 \in (f_1^2 + f_2^2)\mathbb{R}[[f_1, f_2]]\mu.$$

This implies that

$$\beta = r\zeta_1 \wedge \zeta_2 + \widetilde{\beta} + \mathrm{d}f_1 \wedge \alpha_1 + \mathrm{d}f_2 \wedge \alpha_2,$$

for some $r \in \mathbb{R}[[f_1, f_2]]$, $\alpha_1, \alpha_2 \in \Omega^1_f$ and $\widetilde{\beta}$ representing a non-trivial class in $\mathcal{D}^2(\mathrm{d}f_1, \mathrm{d}f_2)$. Note that

$$r_1\epsilon_1 + r_2\epsilon_2 + r\mathrm{d}f_1 \wedge \zeta_1 \wedge \zeta_2 \in \ker \delta_\pi.$$

Hence it is enough to show

$$\gamma := \mathrm{d}f_1 \wedge (\widetilde{\beta} + \mathrm{d}f_2 \wedge \alpha_2) \in \ker \delta_\pi.$$

In this case, the original equation is equivalent to

$$g \cdot \mu = \mathrm{d}f_1 \wedge \mathrm{d}\widetilde{\beta} + \mathrm{d}f_1 \wedge \mathrm{d}f_2 \wedge \mathrm{d}\alpha_2.$$

Using Lemma 7.1 we obtain $g = 0$, which concludes the proof. □

To prove the statement in degree one, it is, by a dimension count and Proposition 6.1, enough to show that for any non-trivial choice of coefficients $a_j, b_j \in \mathbb{R}[[x_2^2, x_4]]$ and $q_i, p_i \in \mathbb{R}[[f_1, f_2]]$, the 1-cycle

$$\alpha := \mathrm{d}(\sum_{j=1}^{4} a_j x_j) + \sum_{j=1}^{4} b_j x_j \mathrm{d}f_1 + \sum_{i=1}^{2} q_i \mathrm{d}f_i + p_i \zeta_i$$

is not a boundary. That is, for all $\beta \in \Omega_f^2$ we have

$$\widetilde{\alpha} := \alpha - \delta_\pi(\beta) = 0 \quad \Rightarrow \quad a_j = b_j = q_i = p_i = 0.$$

If we take the differential of $\widetilde{\alpha} = 0$ we obtain:

$$0 = \mathrm{d}\widetilde{\alpha} = \mathrm{d}\left(\sum_{i=1}^{2} q_i \mathrm{d}f_i + p_i \zeta_i\right) + \mathrm{d}\left(\sum_{j=1}^{4} b_j x_j + \iota_{\star^{-1}\mathrm{d}\beta}(\mathrm{d}f_2)\right) \wedge \mathrm{d}f_1 - \mathrm{d}\iota_{\star^{-1}\mathrm{d}\beta}(\mathrm{d}f_1) \wedge \mathrm{d}f_2.$$

Wedging with $\mathrm{d}f_1$, Corollary 5.5 implies that $p_i = 0$. Applying Lemma 7.2 for $\gamma = -\mathrm{d}\beta$ implies that $b_j = 0$ for $j = 1, 2, 3, 4$, $\partial_x q_2 = \partial_y q_1$ and $\mathrm{d}\beta \in \ker \delta_\pi$. The arguments from Section 5.3 give

$$\beta = r\zeta_1 \wedge \zeta_2 + g\mathrm{d}f_1 \wedge \mathrm{d}f_2 + \mathrm{d}f_1 \wedge p_i \zeta_i + \mathrm{d}\overline{\alpha},$$

for $r, p_i \in \mathbb{R}[[f_1, f_2]]$, $g \in \mathcal{R}$ and $\overline{\alpha} \in \Omega_f^1$. Note all terms except $\mathrm{d}\overline{\alpha}$ are in $\ker \delta_\pi$. Therefore, we have

$$0 = \widetilde{\alpha} = q_1 \mathrm{d}f_1 + q_2 \mathrm{d}f_2 + \mathrm{d}\left(\iota_{\star^{-1}\mathrm{d}\overline{\alpha}}(\mathrm{d}f_1 \wedge \mathrm{d}f_2) + \sum_{j=1}^{4} a_j x_j\right).$$

Wedging the expression with $\mathrm{d}f_1 \wedge \mathrm{d}f_2$ yields

$$\mathrm{d}\left(\iota_{\star^{-1}\mathrm{d}\overline{\alpha}}(\mathrm{d}f_1 \wedge \mathrm{d}f_2) + \sum_{j=1}^{4} a_j x_j\right) \wedge \mathrm{d}f_1 \wedge \mathrm{d}f_2 = 0.$$

By Proposition 3.2, (28) and (35) this is equivalent to

$$\mathrm{d}\overline{\alpha} \wedge \mathrm{d}f_1 \wedge \mathrm{d}f_2 + \left(\sum_{j=1}^{4} a_j x_j\right) \mu \in \mathbb{R}[[f_1, f_2]]\mu.$$

Then, Lemma 3.17 implies $a_j = 0$ for $j = 1, 2, 3, 4$ and by Lemma 7.1 we have

$$\star \circ \sharp \left(\iota_{\star^{-1}\mathrm{d}\overline{\alpha}}(\mathrm{d}f_1 \wedge \mathrm{d}f_2)\right) = \mathrm{d}\overline{\alpha} \wedge \mathrm{d}f_1 \wedge \mathrm{d}f_2 = 0.$$

Hence $\widetilde{\alpha} = 0$ implies $q_1 = q_2 = 0$ which completes the proof.

7.3. **Degree 2.** We proceed similar as in the previous cases. Consider

$$\beta = p\zeta_1 \wedge \zeta_2 + q\mathrm{d}f_1 \wedge \mathrm{d}f_2 + \sum_{i=1}^{2} p_i \mathrm{d}(f_1 \zeta_i) + q_i \mathrm{d}\zeta_i + \mathrm{d}\left(\sum_{j=1}^{4} a_j x_j\right) \wedge \mathrm{d}f_1,$$

where $p, q, p_i, q_i \in \mathbb{R}[[f_1, f_2]]$ and $a_j \in \mathbb{R}[[x_2^2, x_4]]$. We want to show that

$$0 = \widetilde{\beta} := \beta - \delta_\pi(\gamma) \quad \Rightarrow \quad p = q = p_i = q_i = a_j = 0$$

Replicating the argument for Lemma 6.3, replacing Corollary 5.5 with Remark 5.6 we obtain

$$p = p_1 = p_2 = q_1 = q_2 = 0 \quad \text{and} \quad \iota_{\star^{-1}\gamma}(\mathrm{d}f_1), \star^{-1}\mathrm{d}\gamma \in \mathbb{R}[[f_1, f_2]].$$

As such we are left with

$$0=\widetilde{\beta}=q\mathrm{d}f_1\wedge\mathrm{d}f_2-\iota_\pi(\mathrm{d}\gamma)+\mathrm{d}\left(\iota_{\star^{-1}\gamma}(\mathrm{d}f_2)+\sum_{j=1}^{4}a_jx_j\right)\wedge\mathrm{d}f_1-\mathrm{d}\iota_{\star^{-1}\gamma}(\mathrm{d}f_1)\wedge\mathrm{d}f_2.$$

Hence Lemma 7.2 implies $q=a_j=0$. Counting the elements and comparing them to Proposition 6.1 implies that we have a representative for every cohomology class.

7.4. **Degree 3.** By Proposition 3.2, elements in the image of the differential have the form

$$\mathrm{d}f_1\wedge\mathrm{d}f_2\wedge\mathrm{d}g,$$

for $g\in\mathcal{R}$. Hence we can argue as in the proof of Lemma 6.4.

8. Proof of Corollary 1.4

A Poisson structure coming from a deformation of the volume form is equals $g\pi$ for some $g\in\mathcal{R}$ with positive constant term. Using (3) this corresponds to the 2-form $g\mathrm{d}f_1\wedge\mathrm{d}f_2$.

From the definition of the differential we see that

$$g\mathrm{d}f_1\wedge\mathrm{d}f_2\ \in\ker\delta_\pi$$

By Proposition 3.2 and Theorem 1.2 we can write

$$g\mathrm{d}f_1\wedge\mathrm{d}f_2=\beta+\iota_\pi(\mathrm{d}\gamma)-\mathrm{d}\iota_{\star^{-1}\gamma}(\mathrm{d}f_1\wedge\mathrm{d}f_2)$$

for some $\beta=\beta(a_i,p,p_j,q,q_j)$ representing a class in the second Poisson homology according to Theorem 1.2 and $\gamma\in\Omega^3_f$. Following the first part of the argument from subsection 7.3 we can conclude that $p=p_i=q_i=0$. Using the first part of the proof of Lemma 7.2 we can conclude that $a_i=0$ and

$$\gamma=r_1\epsilon_1+r_2\epsilon_2+\mathrm{d}f_1\wedge(r\zeta_1\wedge\zeta_2+\widetilde{\beta}+\mathrm{d}f_2\wedge\alpha)$$

for unique $r_1\in\mathbb{R}[[f_1,f_2]]$, $r_2\in\mathbb{R}[[f_2]]$, some $r\in\mathbb{R}[[f_1,f_2]]$, $\alpha\in\Omega^1_f$ and where $\widetilde{\beta}$ represents a non-trivial class in $\mathcal{D}^2(\mathrm{d}f_1,\mathrm{d}f_2)$. Note that this implies in particular, that

$$g\mathrm{d}f_1\wedge\mathrm{d}f_2=q\mathrm{d}f_1\wedge\mathrm{d}f_2+\delta_\pi(\gamma)$$

for some $q\in\mathbb{R}[[f_1,f_2]]$ and some $\gamma\in\Omega^3_f$ such that

$$\iota_{\star^{-1}\gamma}(\mathrm{d}f_i)=0.$$

In other words, we can find a vector field $X\in\mathfrak{X}^1_{f,2}$ (see (11)) and $q\in\mathbb{R}[[f_1,f_2]]$ satisfying

$$g\pi=q\pi+\mathrm{d}_\pi(X).$$

This allows us to prove the following which implies Corollary 1.4.

Claim 8.1. *Given $g\in\mathcal{R}$ with positive constant term, there exists a formal diffeomorphism ϕ satisfying*

$$\phi^*(g\pi)=p\pi$$

for some $p\in\mathbb{R}[[f_1,f_2]]$ with positive constant term.

To see this, first observe that the flow ϕ_t of X satisfies:

$$\phi_1^*(g\pi)-g\pi=\int_0^1\frac{\mathrm{d}}{\mathrm{d}t}\phi_t^*(g\pi)\mathrm{d}t=[X,g\pi]=-g\mathrm{d}_\pi X+(\mathcal{L}_Xg)\pi. \tag{40}$$

Let us write g as $g=\sum_{i=0}^\infty g_i$, where g_i is homogeneous of degree i and $g_0>0$. By induction on i we show that if $g_j\in\mathbb{R}[[f_1,f_2]]$ for all $j<i$ then we can change $g\pi$ such that also $g_i\in\mathbb{R}[[f_1,f_2]]$.

We start with $i=1$. From cohomology statement we can find $\widetilde{X}_1$ homogeneous of degree 2 such that

$$g_1\pi=\mathrm{d}_\pi\widetilde{X}_1 \tag{41}$$

Taking $X_1 := \frac{1}{g}\widetilde{X}_1$ we obtain from (40) that:

$$\phi_1^*(g\pi) - g\pi = -g\mathrm{d}_\pi X_1 + \mathcal{O}(|x|^4) = -\mathrm{d}_\pi \widetilde{X}_1 + \mathcal{O}(|x|^4)$$

Hence we may assume $g_1 = 0$. In even degrees (41) will be of the form

$$g_{2j}\pi = q_{2j}\pi + \mathrm{d}_\pi \widetilde{X}_{2j}$$

for some $q_{2j} \in \mathbb{R}[[f_1, f_2]]$ contributing to q. Repeating the argument for increasing $i \in \mathbb{N}$ we map $g\pi$ to $q\pi$.

9. Proof of Corollary 1.5

In degree zero, the result follows from Theorem 1.2 as d maps representatives to representatives.

In degree four, we note that for the image of d we have by (8) that

$$\begin{aligned}\mathrm{d}\left(\sum_{i=1}^{2} p_i\zeta_2 \wedge \mathrm{d}\zeta_i + q_i \mathrm{d}f_1 \wedge \mathrm{d}\zeta_i\right) &= 2p_2\mu + (\partial_x p_1 + \partial_y p_2)\mathrm{d}f_1 \wedge \zeta_2 \wedge \mathrm{d}\zeta_1 + (\partial_x p_2 - \partial_y p_1)\mathrm{d}f_1 \wedge \zeta_2 \wedge \mathrm{d}\zeta_2 \\ &= (2p_2 + f_2(\partial_x p_1 + \partial_y p_2) + f_1(\partial_x p_2 - \partial_y p_1))\mu\end{aligned}$$

which implies that d is surjective in homology.

In degree three, note that from the discussion in degree four, the kernel of d is parameterized by

$$q_1, q_2, p_1 \in \mathbb{R}[[f_1, f_2]]$$

since the differential operator $2 + x\partial_x + y\partial_y$ is invertible on $\mathbb{R}[[x, y]]$. To determine the image of d let $\beta = \beta(a_i, p, p_j, q, q_j)$ be a representative of a class in the second Poisson homology. We can write β as

$$\beta := p\zeta_1 \wedge \zeta_2 + q\mathrm{d}f_1 \wedge \mathrm{d}f_2 + \sum_{i=1}^{2} p_i \mathrm{d}f_1 \wedge \zeta_i + q_i \mathrm{d}\zeta_i + \sum_{i=1}^{4} \mathrm{d}(a_i x_i) \wedge \mathrm{d}f_1$$

for $p, p_i, q, q_i \in \mathbb{R}[[f_1, f_2]]$ and $a_i \in \mathbb{R}[[x_2^2, x_4]]$. Applying d and using (7) and (8) yields

$$\begin{aligned}\mathrm{d}\beta = {}&(\partial_x p \mathrm{d}f_1 + \partial_y p \mathrm{d}f_2) \wedge \zeta_1 \wedge \zeta_2 + 2p\zeta_2 \wedge \mathrm{d}\zeta_1 + \sum_{i=1}^{2} \partial_y p_i \mathrm{d}f_2 \wedge \mathrm{d}f_1 \wedge \zeta_i \\ &+ (\partial_x q_1 + \partial_y q_2 - p_1)\mathrm{d}f_1 \wedge \mathrm{d}\zeta_1 + (\partial_x q_2 - \partial_y q_1 - p_2)\mathrm{d}f_1 \wedge \mathrm{d}\zeta_2\end{aligned}$$

Let $\widetilde{q}_1, \widetilde{q}_2, \widetilde{p}_1 \in \mathbb{R}[[f_1, f_2]]$ and

$$\widetilde{p}_2 := -(2 + x\partial_x + y\partial_y)^{-1}(y\partial_x - x\partial_y)\widetilde{p}_1.$$

Let $\gamma = \gamma(\widetilde{p}_i, \widetilde{q}_i)$ be the corresponding representative of a class in $\ker \mathrm{d}$. We want to determine for which p, p_i, q, q_i, a_i as above and $g \in \mathcal{R}$ we have

$$0 = \gamma + \mathrm{d}\beta + \mathrm{d}g \wedge \mathrm{d}f_1 \wedge \mathrm{d}f_2.$$

Wedging with $\mathrm{d}f_1$ and $\mathrm{d}f_2$ respectively, yields

$$(f_1^2 + f_2^2)\partial_y p + 2f_2 p + f_1\widetilde{p}_2 + f_2\widetilde{p}_1 = 0 = (f_1^2 + f_2^2)\partial_x p + 2f_2 p - f_2\widetilde{p}_2 + f_1\widetilde{p}_1$$

and hence we obtain

$$-\widetilde{p}_1 = 2p + f_1\partial_x p + f_2\partial_y p.$$

Therefore, we may assume $\widetilde{p_1} = \widetilde{p_2} = p = 0$. Contracting the remaining equation with T_1 and T_2 respectively, yields using (32) that

$$\begin{aligned}0 &= \partial_x q_2 - \partial_y q_1 - p_2 - f_1\partial_y p_1 - f_2\partial_y p_2 + \widetilde{q}_1 + 4\mathrm{d}g(T_1) \\ 0 &= \partial_x q_1 + \partial_y q_2 - p_1 - f_2\partial_y p_1 + f_1\partial_y p_2 + \widetilde{q}_2 + 4\mathrm{d}g(T_2)\end{aligned}$$

By Lemma 5.4 this implies $\mathrm{d}g(T_1) = \mathrm{d}g(T_2) = 0$. Hence $\widetilde{q}_1, \widetilde{q}_2$ can be realized by choices of p_2 and p_1, respectively.

In degree two, let's denote a representative of the second Poisson homology by $\beta = \beta(\widetilde{a}_i, \widetilde{q}, \widetilde{q}_j, \widetilde{p}, \widetilde{p}_j)$ as above. Then by the above discussion, $\beta \in \ker \mathrm{d}$ iff $\widetilde{p} = 0$ and $\widetilde{p}_i$ such that

$$\widetilde{p}_2 = (1 + y\partial_y)^{-1}(\partial_x\widetilde{q}_2 - \partial_y\widetilde{q}_1 - f_1\partial_y\widetilde{p}_1) \quad \text{and} \quad \widetilde{p}_1 = (1 + y\partial_y)^{-1}(\partial_x\widetilde{q}_1 + \partial_y\widetilde{q}_2 + f_1\partial_y\widetilde{p}_2).$$

We denote by $\alpha = \alpha(a_i, b_i, q_j, p_j)$ a representative of the first Poisson homology and $\gamma \in \Omega^3_f$. We want to find α and γ such that

$$0 = \beta + \mathrm{d}\alpha + \iota_\pi(\mathrm{d}\gamma) - \mathrm{d}\iota_{\star^{-1}\gamma}(\mathrm{d}f_1 \wedge \mathrm{d}f_2)$$

where $\mathrm{d}\alpha$ is given by

$$\mathrm{d}\alpha = \sum_{j=1}^{2} \mathrm{d}(p_j\zeta_j) + (\partial_x q_2 - \partial_y q_1))\,\mathrm{d}f_1 \wedge \mathrm{d}f_2 + \sum_{i=1}^{4} \mathrm{d}\,(b_i x_i) \wedge \mathrm{d}f_1.$$

Wedging the equation above with $\mathrm{d}f_1$ and contracting with T_1 and T_2 yields by (8) and (32) that

$$0 = f_1\partial_y p_1 + f_2\partial_y p_2 + p_2 + \widetilde{q}_2 + d(\iota_{\star^{-1}\gamma}(\mathrm{d}f_1))(T_1)$$
$$0 = f_2\partial_y p_1 - f_1\partial_y p_2 + p_1 + \widetilde{q}_1 + d(\iota_{\star^{-1}\gamma}(\mathrm{d}f_1))(T_2)$$

Using again Lemma 5.4 implies $d(\iota_{\star^{-1}\gamma}(\mathrm{d}f_1))(T_1) = d(\iota_{\star^{-1}\gamma}(\mathrm{d}f_1))(T_2) = 0$. Hence $\widetilde{q}_1$ and $\widetilde{q}_2$ can be realized by choices of p_1 and p_2, respectively. That is, we may assume $\widetilde{q}_i = \widetilde{p}_i = p_i = 0$. From Lemma 7.2 implies that $\widetilde{a}_i = b_i$ and $\partial_x q_2 = \partial_y q_1$.

Finally, to determine the kernel of d in degree one, we note that from the discussion in degree two we immediately get $p_1 = p_2 = b_i = 0$ and $\partial_x q_2 = \partial_y q_1$. Using Theorem 1.2 we therefore obtain the statement.

References

[AM69] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969, pp. ix+128.

[BO22] N. Bárcenas and J. Torres Orozco. *Poisson cohomology of singular fibrations in dimension 4.* 2022. arXiv: 2208.00084 [math.DG].

[Bry88] J.-L. Brylinski. "A differential complex for Poisson manifolds". In: *J. Differential Geom.* 28.1 (1988), pp. 93–114. ISSN: 0022-040X. URL: http://projecteuclid.org/euclid.jdg/1214442161.

[Buc65] B. Buchberger. *Ein Algorithmus zum Auffinden der Basiselemente des Restklassenringes nach einem nulldimensionalen Polynomideal.* Dissertation an dem Math. Inst. der Universität von Innsbruck. 1965.

[BV20] P. Batakidis and R. Vera. "Poisson cohomology of broken Lefschetz fibrations". In: *Differential Geom. Appl.* 72 (2020), pp. 101661, 28. ISSN: 0926-2245. DOI: 10.1016/j.difgeo.2020.101661. URL: https://doi.org/10.1016/j.difgeo.2020.101661.

[CFM21] M. Crainic, R. L. Fernandes, and I. Mărcuţ. *Lectures on Poisson geometry.* Vol. 217. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, [2021] ©2021, pp. xix+479. ISBN: 978-1-4704-6430-1. DOI: 10.1090/gsm/217. URL: https://doi.org/10.1090/gsm/217.

[Dam89] P. A. Damianou. *Nonlinear Poisson brackets.* Thesis (Ph.D.)–The University of Arizona. ProQuest LLC, Ann Arbor, MI, 1989, p. 109. URL: http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqdiss&rft_dat=xri:pqdiss:8919027.

[DP12] P. A. Damianou and F. Petalidou. "Poisson brackets with prescribed Casimirs". In: *Canad. J. Math.* 64.5 (2012), pp. 991–1018. ISSN: 0008-414X. DOI: 10.4153/CJM-2011-082-2. URL: https://doi.org/10.4153/CJM-2011-082-2.

[DZ05] J.-P. Dufour and N. T. Zung. *Poisson structures and their normal forms.* Vol. 242. Progress in Mathematics. Birkhäuser Verlag, Basel, 2005, pp. xvi+321. ISBN: 978-3-7643-7334-4.

[Eis95] D. Eisenbud. *Commutative algebra.* Vol. 150. Graduate Texts in Mathematics. With a view toward algebraic geometry. Springer-Verlag, New York, 1995, pp. xvi+785. ISBN: 0-387-94268-8; 0-387-94269-6. DOI: 10.1007/978-1-4612-5350-1. URL: https://doi.org/10.1007/978-1-4612-5350-1.

[Gin96] V. L. Ginzburg. "Momentum mappings and Poisson cohomology". In: *Internat. J. Math.* 7.3 (1996), pp. 329–358. ISSN: 0129-167X. DOI: 10.1142/S0129167X96000207. URL: https://doi.org/10.1142/S0129167X96000207.

[GMP93] J. Grabowski, G. Marmo, and A. M. Perelomov. “Poisson structures: towards a classification”. In: *Modern Phys. Lett. A* 8.18 (1993), pp. 1719–1733. ISSN: 0217-7323. DOI: 10.1142/S0217732393001458. URL: https://doi.org/10.1142/S0217732393001458.

[GP08] G.-M. Greuel and G. Pfister. *A* **Singular** *introduction to commutative algebra*. extended. With contributions by Olaf Bachmann, Christoph Lossen and Hans Schönemann, With 1 CD-ROM (Windows, Macintosh and UNIX). Springer, Berlin, 2008, pp. xx+689. ISBN: 978-3-540-73541-0.

[GSV14] L. García-Naranjo, P. Suárez-Serrato, and R. Vera. “Poisson Structures on Smooth 4–Manifolds”. In: *Letters in Mathematical Physics* 105 (June 2014). DOI: 10.1007/s11005-015-0792-8.

[Hir64] H. Hironaka. “Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II”. In: *Ann. of Math. (2)* **79** *(1964), 109–203; ibid. (2)* 79 (1964), pp. 205–326. ISSN: 0003-486X. DOI: 10.2307/1970547. URL: https://doi.org/10.2307/1970547.

[HZ23] D. Hoekstra and F. Zeiser. “Poisson cohomology of 3D Lie algebras”. In: *J. Geom. Phys.* 191 (2023), Paper No. 104862, 38. ISSN: 0393-0440,1879-1662. DOI: 10.1016/j.geomphys.2023.104862. URL: https://doi.org/10.1016/j.geomphys.2023.104862.

[Kos85] J.-L. Koszul. “Crochet de Schouten-Nijenhuis et cohomologie”. In: Numéro Hors Série. The mathematical heritage of Élie Cartan (Lyon, 1984). 1985, pp. 257–271.

[Lic77] A. Lichnerowicz. “Les variétés de Poisson et leurs algèbres de Lie associées”. In: *J. Differential Geometry* 12.2 (1977), pp. 253–300. ISSN: 0022-040X. URL: http://projecteuclid.org/euclid.jdg/1214433987.

[Loo84] E. J. N. Looijenga. *Isolated singular points on complete intersections*. Vol. 77. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1984, pp. xi+200. ISBN: 0-521-28674-3. DOI: 10.1017/CBO9780511662720. URL: https://doi.org/10.1017/CBO9780511662720.

[LPV13] C. Laurent-Gengoux, A. Pichereau, and P. Vanhaecke. *Poisson structures*. Vol. 347. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, Heidelberg, 2013, pp. xxiv+461. ISBN: 978-3-642-31089-8. DOI: 10.1007/978-3-642-31090-4. URL: https://doi.org/10.1007/978-3-642-31090-4.

[Mon02] P. Monnier. “Poisson cohomology in dimension two”. In: *Israel J. Math.* 129 (2002), pp. 189–207. ISSN: 0021-2172. DOI: 10.1007/BF02773163. URL: https://doi.org/10.1007/BF02773163.

[MZ23] I. Mărcuţ and F. Zeiser. “The Poisson cohomology of $\mathfrak{sl}_2^*(\mathbb{R})$”. In: *J. Symplectic Geom.* 21.3 (2023), pp. 603–652. ISSN: 1527-5256.

[Pel09] S. R. T. Pelap. “Poisson (co)homology of polynomial Poisson algebras in dimension four: Sklyanin’s case”. In: *J. Algebra* 322.4 (2009), pp. 1151–1169. ISSN: 0021-8693. DOI: 10.1016/j.jalgebra.2009.05.024. URL: https://doi.org/10.1016/j.jalgebra.2009.05.024.

[Pic06] A. Pichereau. “Poisson (co)homology and isolated singularities”. In: *J. Algebra* 299.2 (2006), pp. 747–777. ISSN: 0021-8693. DOI: 10.1016/j.jalgebra.2005.10.029. URL: https://doi.org/10.1016/j.jalgebra.2005.10.029.

[Prz01] R. Przybysz. “On one class of exact Poisson structures”. In: *J. Math. Phys.* 42.4 (2001), pp. 1913–1920. ISSN: 0022-2488. DOI: 10.1063/1.1346588. URL: https://doi.org/10.1063/1.1346588.

[Rha54] G. de Rham. “Sur la division de formes et de courants par une forme linéaire”. In: *Comment. Math. Helv.* 28 (1954), pp. 346–352. ISSN: 0010-2571,1420-8946. DOI: 10.1007/BF02566941. URL: https://doi.org/10.1007/BF02566941.

[Roy02] D. Roytenberg. “Poisson cohomology of SU(2)-covariant “necklace” Poisson structures on S^2”. In: *J. Nonlinear Math. Phys.* 9.3 (2002), pp. 347–356. ISSN: 1402-9251. DOI: 10.2991/jnmp.2002.9.3.7. URL: https://doi.org/10.2991/jnmp.2002.9.3.7.

[Sai76] K. Saito. “On a generalization of de-Rham lemma”. In: *Ann. Inst. Fourier (Grenoble)* 26.2 (1976), pp. vii, 165–170. ISSN: 0373-0956. URL: http://www.numdam.org/item?id=AIF_1976__26_2_165_0.

[STV19] P. Suárez-Serrato, J. Torres Orozco, and R. Vera. “Poisson and near-symplectic structures on generalized wrinkled fibrations in dimension 6”. In: *Ann. Global Anal. Geom.* 55.4 (2019), pp. 777–804.

[Vai94] I. Vaisman. *Lectures on the geometry of Poisson manifolds*. Vol. 118. Progress in Mathematics. Birkhäuser Verlag, Basel, 1994, pp. viii+205. ISBN: 3-7643-5016-4. DOI: 10.1007/978-3-0348-8495-2. URL: https://doi.org/10.1007/978-3-0348-8495-2.

[Wei94] C. A. Weibel. *An introduction to homological algebra*. Vol. 38. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1994, pp. xiv+450. ISBN: 0-521-43500-5; 0-521-55987-1. DOI: 10.1017/CBO9781139644136. URL: https://doi.org/10.1017/CBO9781139644136.

[Wei97] A. Weinstein. "The modular automorphism group of a Poisson manifold". In: *J. Geom. Phys.* 23.3-4 (1997), pp. 379–394. ISSN: 0393-0440,1879-1662. DOI: 10.1016/S0393-0440(97)80011-3. URL: https://doi.org/10.1016/S0393-0440(97)80011-3.

[Xu92] P. Xu. "Poisson cohomology of regular Poisson manifolds". In: *Ann. Inst. Fourier (Grenoble)* 42.4 (1992), pp. 967–988. ISSN: 0373-0956. URL: http://www.numdam.org/item?id=AIF_1992__42_4_967_0.

Lauran Toussaint,
Vrije Universiteit Amsterdam, 1081 HV Amsterdam, The Netherlands
E-mail address: l.e.toussaint@vu.nl

Florian Zeiser,
University of Illinois at Urbana-Champaign, 61801 Urbana, United States
E-mail address: fzeiser@illinois.edu