arXiv:2402.04633v1 [math.DG] 7 Feb 2024

A STABILITY RESULT FOR RIEMANNIAN FOLIATIONS

STEPHANE GEUDENS AND FLORIAN ZEISER

Abstract. We show that a Riemannian foliation $\mathcal{F}$ on a compact manifold M is stable, provided that the cohomology group $H^1(\mathcal{F}, N\mathcal{F})$ vanishes. Stability means that any foliation on M close enough to $\mathcal{F}$ is conjugate to $\mathcal{F}$ by means of a diffeomorphism.

Contents

Introduction

On a compact manifold M without boundary, a foliation $\mathcal{F}$ is called *stable* if any other foliation sufficiently close to $\mathcal{F}$ in the C^∞-topology is conjugate to $\mathcal{F}$ under a diffeomorphism of M. Here the topology on the space of k-dimensional foliations is induced by the inclusion

$$\mathrm{Fol}_k(M) \hookrightarrow \Gamma(\mathrm{Gr}_k(M)) : \mathcal{F} \mapsto T\mathcal{F},$$

where $\mathrm{Gr}_k(M) \to M$ is the Grassmannian bundle of k-planes on M (see [11]). When studying stability of a foliation $\mathcal{F}$, the first step is to establish its infinitesimal counterpart. This was done by Heitsch [16], who showed that a smooth path of foliations $\mathcal{F}_t$ with $\mathcal{F}_0 = \mathcal{F}$ gives rise infinitesimally to a one-cocycle in the complex $\left(\Omega^\bullet(\mathcal{F}, N\mathcal{F}), d_\nabla\right)$ of foliated forms with values in the normal bundle $N\mathcal{F}$. Its differential d_∇ is induced by the Bott connection. Moreover, if the path $\mathcal{F}_t$ is generated by applying an isotopy to $\mathcal{F}$, then the corresponding one-cocycle is exact. Hence, we call $\mathcal{F}$ *infinitesimally stable* if $H^1(\mathcal{F}, N\mathcal{F})$ vanishes.

Question. *When does infinitesimal stability imply stability?*

This question is addressed by Hamilton in his unpublished work [15]. Using the Nash-Moser inverse function theorem, he shows that stability is implied by a strong version of infinitesimal stability, namely the existence of tame homotopy operators for the complex $\left(\Omega^\bullet(\mathcal{F}, N\mathcal{F}), d_\nabla\right)$. His main result is that, on a compact manifold M, the latter condition is satisfied by certain Hausdorff foliations, i.e. foliations whose leaf space $M/\mathcal{F}$ is Hausdorff. The leaves of such a foliation are compact, and if M is connected then there exists a generic leaf L_0 such that all leaves in a saturated dense open subset are diffeomorphic to L_0.

Global Reeb-Thurston Stability Theorem (Hamilton [15])**.** *Let M be a compact, connected manifold and $\mathcal{F}$ a Hausdorff foliation on M. If the generic leaf L_0 of $\mathcal{F}$ satisfies $H^1(L_0) = 0$, then $\mathcal{F}$ is stable.*

It is known that vanishing of $H^1(L_0)$ is also necessary for stability of $\mathcal{F}$. This follows from the fact that a leaf without holonomy has a saturated neighborhood U such that $\mathcal{F}|_U$ looks like the foliation by fibers of $L_0 \times D^q \to D^q$, where D^q is a q-disk. Using a non-zero element of $H^1(L_0)$, one can construct an arbitrarily C^k-small perturbation of $\mathcal{F}|_U$ which can be glued with $\mathcal{F}$ outside of U. We refer to [17],[8] for more details.

Using very different techniques, Hamilton's result was generalized by Epstein-Rosenberg to the setting of Hausdorff C^k-foliations with compact leaves, on manifolds that are not necessarily compact [12]. A result similar to the global Reeb-Thurston stability theorem, but for smooth paths of foliations $\mathcal{F}_t$ deforming $\mathcal{F}$ rather than foliations $\mathcal{F}'$ close to $\mathcal{F}$, was obtained by Del Hoyo-Fernandes [8] using the theory of Lie groupoids and Lie algebroids.

This paper stems from an endeavour to extend Hamilton's global Reeb-Thurston stability theorem. Our main result is a positive answer to the question raised above, in the case of *Riemannian* foliations. These are characterized by the existence of a bundle-like Riemannian metric on M [20]. Equivalently, they are locally defined by a Riemannian submersion.

Main Theorem. *Let M be a compact manifold and $\mathcal{F}$ a Riemannian foliation on M such that $H^1(\mathcal{F}, N\mathcal{F}) = 0$. Then $\mathcal{F}$ is stable.*

The foliations appearing in Hamilton's result satisfy the assumptions of our Main Theorem. Note that on a compact manifold, Hausdorff foliations are exactly the Riemannian foliations with all leaves compact, and the stability of such foliations is completely settled by Hamilton's theorem. Hence, in order to find new examples of stable foliations via our Main Theorem, one should look for infinitesimally stable Riemannian foliations with at least one non-compact leaf. We didn't find such foliations, hence we ask:

Question. *Let $\mathcal{F}$ be an infinitesimally stable Riemannian foliation on a compact manifold. Is $\mathcal{F}$ necessarily Hausdorff?*

The proof of our Main Theorem relies on Hamilton's work [15]. In order to give an outline of the proof, we first briefly recall Hamilton's proof of the global Reeb-Thurston stability theorem, which consists of two parts.

Part 1: A choice of Riemannian metric g on M allows us to identify $N\mathcal{F}$ with $T\mathcal{F}^{\perp}$, and it induces an inner product on $\Omega^k(\mathcal{F}, T\mathcal{F}^{\perp})$ given by

$$\langle \alpha \otimes X, \beta \otimes Y \rangle_g := \int_M (\alpha, \beta)_g g(X, Y) dVol_g.$$

Denote by $\|\cdot\|_g$ the associated norm, and let δ_∇ be the formal adjoint of d_∇ with respect to $\langle \cdot, \cdot \rangle_g$. Hamilton first proves the following auxiliary result towards stability.

Stability Theorem (Hamilton [15]). *Let $(M, \mathcal{F})$ be a compact foliated manifold admitting a bundle-like[1] Riemannian metric g. If there exists a constant $C > 0$ such that*

$$\|\eta\|_g \leq C \left(\|d_\nabla \eta\|_g + \|\delta_\nabla \eta\|_g \right) \tag{1}$$

for all $\eta \in \Omega^1(\mathcal{F}, N\mathcal{F})$, then $\mathcal{F}$ is stable.

To obtain this result, Hamiltons shows that the existence of a bundle-like metric together with the estimate (1) implies that the Laplacian $\Delta := d_\nabla \delta_\nabla + \delta_\nabla d_\nabla$ is invertible. He then defines tame homotopy operators for the complex $(\Omega^\bullet(\mathcal{F}, N\mathcal{F}), d_\nabla)$ in degree one, by setting

$$H_1 := \delta_\nabla \circ \Delta^{-1}, \qquad H_2 := \Delta^{-1} \circ \delta_\nabla.$$

[1]Hamilton actually requires the existence of what he calls a holonomy invariant metric, see Def. 4.1. We show in Prop. 4.5 that this is the same thing as a bundle-like metric.

This means that we have a diagram of the form

$$\Gamma(N\mathcal{F}) \underset{H_1}{\overset{d_\nabla}{\rightleftarrows}} \Omega^1(\mathcal{F}, N\mathcal{F}) \underset{H_2}{\overset{d_\nabla}{\rightleftarrows}} \Omega^2(\mathcal{F}, N\mathcal{F})$$

with $d_\nabla \circ H_1 + H_2 \circ d_\nabla = \mathrm{Id}$, and that the C^n-norms of H_1 and H_2 satisfy

$$\|H_i(\eta)\|_n \leq C_n \|\eta\|_{n+r},$$

for some fixed r and constants $C_n > 0$. The existence of these operators allows Hamilton to apply a Nash-Moser type algorithm, leading to the above Stability Theorem.

Part 2: The global Reeb-Thurston stability result follows by applying the above Stability Theorem to Hausdorff foliations $\mathcal{F}$. To do so, Hamilton makes crucial use of the fact that there is a good local model for foliations of this type. Indeed, local Reeb stability implies that every leaf has a saturated neighborhood in which the foliation looks like a quotient of a fiber bundle by a free action of a finite group, the fiber being the generic leaf L_0 of $\mathcal{F}$ [10]. First, this allows him to show that Hausdorff foliations are Riemannian, by constructing bundle-like metrics locally and then gluing them by a partition of unity consisting of basic functions. Second, the existence of such partitions of unity shows that one only needs to prove the estimate (1) in a local model. There, it reduces to proving the same estimate for ordinary de Rham forms on the generic leaf L_0, and this is done by combining the classical Hodge decomposition with the assumption that $H^1(L_0)$ vanishes.

We now outline the proof of our Main Theorem. By Hamilton's Stability Theorem, we only need to show that infinitesimally stable Riemannian foliations satisfy (1). To do so, we take advantage of the fact that a Hodge decomposition for $\Omega(\mathcal{F}, N\mathcal{F})$ has been established in [3] for Riemannian foliations $\mathcal{F}$. This allows us to prove the estimate (1) working globally on the manifold M, hence there is no need anymore for restricting to foliations that have a good local model. Arguing by contradiction, if the estimate (1) didn't hold, then one could find a sequence (α_n) in $\Omega^1(\mathcal{F}, N\mathcal{F})$ with the properties

$$\|d_\nabla \alpha_n\|_g \longrightarrow 0, \quad \|\delta_\nabla \alpha_n\|_g \longrightarrow 0, \quad \|\alpha_n\|_g = 1.$$

We then show that (α_n) has a subsequence which converges to an element in the L^2-closure of the space of harmonic elements. By the Hodge decomposition [3] and the assumption that $H^1(\mathcal{F}, N\mathcal{F})$ vanishes, the limit must be zero. This is impossible since $\|\alpha_n\|_g = 1$.

The main technical difficulty lies in arguing why such a convergent subsequence exists. The result [2, Thm. B] yields a subsequence with the desired properties, but for sequences of ordinary de Rham forms instead of elements in $\Omega^1(\mathcal{F}, N\mathcal{F})$. We solve this issue by using Molino's structure theory for Riemannian foliations [19]. Molino showed that the foliation $\mathcal{F}$ can be lifted to another Riemannian foliation $\widehat{\mathcal{F}}$ on the transverse orthogonal frame bundle $\pi : \widehat{M} \to M$. This way, we get an embedding $\pi^* : \Omega^1(\mathcal{F}, N\mathcal{F}) \hookrightarrow \Omega^1(\widehat{\mathcal{F}}, \pi^* N\mathcal{F})$. Since the pullback bundle $\pi^* N\mathcal{F}$ is trivial as a $T\widehat{\mathcal{F}}$-representation, this allows one to view an element of $\Omega^1(\mathcal{F}, N\mathcal{F})$ as a list of foliated forms on $\widehat{\mathcal{F}}$. Consequently, after lifting the sequence (α_n) to $\widehat{M}$ under the embedding π^*, we are able to use a vector version of [2, Thm. B] in order to pass to a convergent subsequence with the desired properties.

This paper is organized as follows. Section 1 contains the necessary background information about Molino's structure theory for Riemannian foliations. The aim of this section is to set things up correctly for the above described embedding $\pi^* : \Omega^1(\mathcal{F}, N\mathcal{F}) \hookrightarrow \Omega^1(\widehat{\mathcal{F}}, \pi^* N\mathcal{F})$ to have the right properties. In Section 2, we prove our Main Theorem by establishing the estimate (1). In Section 3, we make some comments about the scope of our Main Theorem.

In the Appendix we reconcile two terminologies, by showing that the "holonomy-invariant metrics" defined by Hamilton in [15] agree with the modern notion of bundle-like metric.

Acknowledgements. Part of this work was done while visiting the Max Planck Institute for Mathematics in Bonn, which we would like to thank for its hospitality and financial support. S.G. acknowledges support from the UCL Institute for Mathematical and Statistical Sciences (IMSS). We would like to thank Ioan Mărcuţ for sending us a copy of Hamilton's paper [15] and for useful discussions. We also thank Jesús Álvarez López, Aziz El Kacimi Alaoui and Vladimir Slesar for e-mail exchanges and for their interest in this project.

1. Auxiliary results involving Molino theory

In this section, we recall Molino's structure theory of Riemannian foliations, and we prove some auxiliary results which are needed to show that (1) holds under suitable conditions. Molino showed that a Riemannian foliation $\mathcal{F}$ on M can be lifted to a simpler Riemannian foliation $\widehat{\mathcal{F}}$ on the transverse orthogonal frame bundle $\pi : \widehat{M} \to M$. Using the associated embedding $\pi^* : \Omega^1(\mathcal{F}, N\mathcal{F}) \hookrightarrow \Omega^1(\widehat{\mathcal{F}}, \pi^* N\mathcal{F})$ will facilitate the proof of (1), since $\pi^* N\mathcal{F}$ is trivial as a $T\widehat{\mathcal{F}}$-representation. For this approach to work, we need to make sure that π^* preserves norms. This requires the construction of a suitable bundle-like metric on $\widehat{M}$.

The strategy of lifting to $\widehat{M}$ was used in [3] to derive the existence of a Hodge decomposition for foliated forms with coefficients from the same result with trivial coefficients.

1.1. Structure theory of Riemannian foliations.

We start by recalling some of Molino's results concerning Riemannian foliations. The material in this subsection can be found in [19] and [18]; we will borrow notation and terminology mostly from the latter.

Definition 1.1. Given a foliated manifold $(M, \mathcal{F})$, a Riemannian metric g on M is *bundle-like* if for any open $U \subset M$ and all vector fields $Y, Z \in \mathfrak{X}(U)$ that are projectable and orthogonal to the leaves, the function $g(Y, Z)$ is basic on U. A foliation $\mathcal{F}$ on M is called *Riemannian* if $(M, \mathcal{F})$ admits a bundle-like metric.

This means that the foliation $\mathcal{F}$ is locally given by the fibers of a Riemannian submersion. Examples of Riemannian foliations include simple foliations, suspensions of isometries and foliations given by the orbits of a Lie group action by isometries.

Let M be a compact manifold with a Riemannian foliation $\mathcal{F}$ of codimension q. Fix a bundle-like Riemannian metric g. Upon restricting g to $T\mathcal{F}^{\perp}$, we can consider the *transverse orthogonal frame bundle* $\pi : \widehat{M} \to M$. A point $e \in \widehat{M}_x$ is an orthogonal isomorphism $e : (\mathbb{R}^q, \langle \cdot, \cdot \rangle_{std}) \xrightarrow{\sim} (T_x\mathcal{F}^{\perp}, g|_{T_x\mathcal{F}^{\perp}})$, and $\widehat{M}$ is a principal $O(q)$-bundle for the right action

$$R : \widehat{M} \times O(q) \to \widehat{M}, \quad R_h(e) = e \circ h. \tag{2}$$

One can lift $\mathcal{F}$ to a foliation $\widehat{\mathcal{F}}$ on $\widehat{M}$, which can be described as follows. Two points $e \in \widehat{M}_x$ and $f \in \widehat{M}_y$ lie in the same leaf of $\widehat{\mathcal{F}}$ if x and y lie in the same leaf L of $\mathcal{F}$, and there is a path γ inside L connecting x and y such that the linear holonomy

$$dhol_{\gamma} : T_x\mathcal{F}^{\perp} \to T_y\mathcal{F}^{\perp}$$

takes e to f. Note here that the map $dhol_{\gamma}$ is an isometry, because g is bundle-like. Equipped with the foliation $\widehat{\mathcal{F}}$, the bundle $\widehat{M}$ becomes a *transverse principal $O(q)$-bundle*, i.e.

i) $\widehat{\mathcal{F}}$ is preserved by the $O(q)$-action,

ii) the projection $\pi : \widehat{M} \to M$ maps each leaf $\widehat{L}$ of $\widehat{\mathcal{F}}$ onto a leaf L of $\mathcal{F}$, and the restriction $\pi : \widehat{L} \to L$ is a covering map which is a quotient of the holonomy cover of L.

The foliation $\widehat{\mathcal{F}}$ is transversely parallelizable, i.e. there exist $\widehat{\mathcal{F}}$-projectable vector fields $X_1, \ldots, X_l \in \mathfrak{X}(\widehat{M})$, where $l = \operatorname{codim}(\widehat{\mathcal{F}})$, whose classes $\overline{X_1}, \ldots, \overline{X_l} \in \Gamma(N\widehat{\mathcal{F}})$ form a frame for $N\widehat{\mathcal{F}}$. In fact, $\widehat{\mathcal{F}}$ has a natural transverse parallelism, as we now explain.

Let $\omega \in \Omega^1(\widehat{M}, \mathfrak{o}(q))$ be the transverse Levi-Civita connection on $\widehat{M}$. This is a connection one-form that is $\widehat{\mathcal{F}}$-basic, i.e.

$$T\widehat{\mathcal{F}} \subset \ker \omega \qquad \text{and} \qquad \pounds_X \omega = 0 \quad \text{for all} \quad X \in \Gamma(T\widehat{\mathcal{F}}).$$

Additionally, denote by $\theta \in \Omega^1(\widehat{M}, \mathbb{R}^q)$ the transverse canonical form, which is defined by

$$\theta_e(v) = e^{-1}\left(\mathrm{pr}_{T\mathcal{F}^\perp}(d\pi(v))\right) \qquad \text{for } e \in \widehat{M}_x \text{ and } v \in T_e\widehat{M}.$$

Here $\mathrm{pr}_{T\mathcal{F}^\perp}$ denotes the projection onto $T\mathcal{F}^\perp$. Like the connection form ω, also the canonical form θ is $\widehat{\mathcal{F}}$-basic. It is clear that $\ker \theta_e = (d\pi)^{-1}(T_e\mathcal{F})$, and therefore

$$\omega_e \oplus \theta_e : T_e\widehat{M} \to \mathfrak{o}(q) \oplus \mathbb{R}^q$$

has $T_e\widehat{\mathcal{F}}$ as its kernel. Hence, if $\{\xi_1, \ldots, \xi_{q(q-1)/2}\}$ is a basis of $\mathfrak{o}(q)$ and $\{u_1, \ldots, u_q\}$ is the standard basis of $\mathbb{R}^q$, we obtain sections $\overline{Y_i}, \overline{Z}_j \in \Gamma(N\widehat{\mathcal{F}})$ uniquely determined by

$$\omega(\overline{Y_i}) = 0, \quad \theta(\overline{Y_i}) = u_i \quad \text{and} \quad \omega(\overline{Z}_j) = \xi_j, \quad \theta(\overline{Z}_j) = 0.$$

Moreover, the $\overline{Y_i}, \overline{Z}_j$ are transverse fields, i.e. their representatives are $\widehat{\mathcal{F}}$-projectable. Note that $\overline{Z}_j$ is exactly the class in $\Gamma(N\widehat{\mathcal{F}})$ represented by the fundamental vector field corresponding with $\xi_j \in \mathfrak{o}(q)$. This way, one obtains the *natural transverse parallelism*

$$\{\overline{Y}_1, \ldots, \overline{Y}_q, \overline{Z}_1, \ldots, \overline{Z}_{q(q-1)/2}\}.$$

1.2. **A suitable bundle-like metric on $\widehat{\mathbf{M}}$.** We will now construct a bundle-like metric $\widehat{g}$ on $(\widehat{M}, \widehat{\mathcal{F}})$ that is compatible with the metric g on $(M, \mathcal{F})$, in such a way that the embedding $\pi^* : \Omega^\bullet(\mathcal{F}, T\mathcal{F}^\perp) \hookrightarrow \Omega^\bullet(\widehat{\mathcal{F}}, \pi^*T\mathcal{F}^\perp)$ is isometric for the inner products induced by g and $\widehat{g}$.

Assume the setup from §1.1 and denote by $H := \ker \omega$ the horizontal distribution of the transverse Levi-Civita connection. We will set $\widehat{g}$ on $T\widehat{M} = H \oplus V$ to be an orthogonal sum

$$\widehat{g} := g^{lift} \oplus g_V, \tag{3}$$

where the summands g^{lift} and g_V are defined as follows:

- Since $d\pi|_{H_e} : H_e \to T_{\pi(e)}M$ is an isomorphism for all $e \in \widehat{M}$, we can lift the bundle-like metric g to a fiber metric g^{lift} on H.
- Pick an Ad-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{o}(q)$. The fiber metric g_V is determined by setting at every point $e \in \widehat{M}$,

$$g_V\left(Z_\xi(e), Z_\eta(e)\right) := \langle \xi, \eta \rangle.$$

  Here Z_ξ, Z_η denote the fundamental vector fields corresponding to $\xi, \eta \in \mathfrak{o}(q)$.

The following is well-known; we include a proof for completeness.

Lemma 1.2. *The Riemannian metric $\widehat{g}$ is $O(q)$-invariant and bundle-like for $\widehat{\mathcal{F}}$.*

Proof. Because the $O(q)$-action preserves H and V, we have to check that g^{lift} and g_V are $O(q)$-invariant. Invariance of g^{lift} is clear, and invariance of g_V follows from the computation

$$\begin{aligned} g_V\left(d_eR_h(Z_\xi(e)), d_eR_h(Z_\eta(e))\right) &= g_V\left(Z_{Ad_{h^{-1}}(\xi)}(R_h(e)), Z_{Ad_{h^{-1}}(\eta)}(R_h(e))\right) \\ &= \langle Ad_{h^{-1}}(\xi), Ad_{h^{-1}}(\eta)\rangle \\ &= \langle \xi, \eta\rangle \\ &= g_V\left(Z_\xi(e), Z_\eta(e)\right) \end{aligned}$$

for $e \in \widehat{M}, h \in O(q)$ and $\xi, \eta \in \mathfrak{o}(q)$. Here we made use of the fact that $\langle\cdot,\cdot\rangle$ is Ad-invariant.

To check that $\widehat{g}$ is bundle-like, we pick suitable representatives for the natural transverse parallelism of $(\widehat{M}, \widehat{\mathcal{F}})$, see §1.1. First, let $Z_1, \dots, Z_{q(q-1)/2}$ be the fundamental vector fields corresponding to a basis $\{\xi_1, \dots, \xi_{q(q-1)/2}\}$ of $\mathfrak{o}(q)$. Second, let $Y_1, \dots, Y_q$ be the unique $\widehat{\mathcal{F}}$-projectable vector fields in $\Gamma(T\widehat{\mathcal{F}}^\perp)$ satisfying

$$\omega(Y_i) = 0 \quad \text{ and } \quad \theta(Y_i) = u_i.$$

This way, we get a frame $\{Y_1, \dots, Y_q, Z_1, \dots, Z_{q(q-1)/2}\}$ for $T\widehat{\mathcal{F}}^\perp$ consisting of $\widehat{\mathcal{F}}$-projectable vector fields. Any $\widehat{\mathcal{F}}$-projectable vector field orthogonal to $\widehat{\mathcal{F}}$ is a combination of these vector fields, with $\widehat{\mathcal{F}}$-basic functions as coefficients. Consequently, $\widehat{g}$ is bundle-like as soon as the functions $\widehat{g}(Y_i, Y_j), \widehat{g}(Y_i, Z_j)$ and $\widehat{g}(Z_i, Z_j)$ are $\widehat{\mathcal{F}}$-basic. It is clear that $\widehat{g}(Y_i, Z_j) = 0$ and $\widehat{g}(Z_i, Z_j) = \langle \xi_i, \xi_j\rangle$ are $\widehat{\mathcal{F}}$-basic. To check that $\widehat{g}(Y_i, Y_j)$ is $\widehat{\mathcal{F}}$-basic, pick a point $e \in \widehat{M}$. By definition of the transverse canonical form θ, we know that

$$d\pi(Y_i(e)) = e\big(\theta(Y_i(e))\big) = e(u_i) \in T_{\pi(e)}\mathcal{F}^\perp.$$

Because $e : (\mathbb{R}^q, \langle\cdot,\cdot\rangle_{std}) \xrightarrow{\sim} (T_{\pi(e)}\mathcal{F}^\perp, g|_{T_{\pi(e)}\mathcal{F}^\perp})$ is an orthogonal isomorphism, we get

$$\widehat{g}\big(Y_i(e), Y_j(e)\big) = g\big(d\pi(Y_i(e)), d\pi(Y_j(e))\big) = g\big(e(u_i), e(u_j)\big) = \langle u_i, u_j\rangle_{std} = \delta_{ij}.$$

This shows that also $\widehat{g}(Y_i, Y_j)$ is $\widehat{\mathcal{F}}$-basic, hence it follows that $\widehat{g}$ is bundle-like. □

The isomorphism of Riemannian vector bundles

$$\left(\widehat{M} \times \mathfrak{o}(q), \langle\cdot,\cdot\rangle\right) \xrightarrow{\sim} (V, g_V) : (e, \xi) \mapsto Z_\xi(e)$$

implies that all the fibers of $\pi : \widehat{M} \to M$ have the same volume with respect to the metric induced by $\widehat{g}$. Upon rescaling the inner product $\langle\cdot,\cdot\rangle$ by a constant, we can make sure that the fibers have volume one. This will be important in the sequel.

Corollary 1.3. *We can construct $\widehat{g}$ such that the fibers of $\pi : \widehat{M} \to M$ have volume one.*

We now turn our attention to the embedding

$$\pi^* : \Omega^\bullet(\mathcal{F}, T\mathcal{F}^\perp) \hookrightarrow \Omega^\bullet(\widehat{\mathcal{F}}, \pi^*T\mathcal{F}^\perp). \tag{4}$$

Both sides carry an inner product induced by the metric g and $\widehat{g}$. Explicitly,

$$\begin{aligned} \langle \alpha \otimes X, \beta \otimes Y\rangle_g &:= \int_M (\alpha, \beta)_g g(X, Y) dVol_g, \\ \langle \gamma \otimes \pi^*U, \delta \otimes \pi^*V\rangle_{\widehat{g}^*} &:= \int_{\widehat{M}} (\gamma, \delta)_{\widehat{g}} \pi^*(g(U, V)) dVol_{\widehat{g}}, \end{aligned} \tag{5}$$

for $\alpha \otimes X, \beta \otimes Y \in \Omega^\bullet(\mathcal{F}, T\mathcal{F}^\perp)$ and $\gamma \otimes \pi^*U, \delta \otimes \pi^*V \in \Omega^\bullet(\widehat{\mathcal{F}}, \pi^*T\mathcal{F}^\perp)$. Here we denoted by $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_{\widehat{g}}$ the fiber metrics on $T^*\mathcal{F}$ and $T^*\widehat{\mathcal{F}}$ induced by g and $\widehat{g}$. We now show that the map (4) is compatible with these inner products, provided that $\widehat{g}$ is constructed according to (3) and Cor. 1.3.

Corollary 1.4. *The inclusion*

$$\pi^* : \big(\Omega^\bullet(\mathcal{F}, T\mathcal{F}^\perp), \langle\cdot,\cdot\rangle_g\big) \hookrightarrow \big(\Omega^\bullet(\widehat{\mathcal{F}}, \pi^*T\mathcal{F}^\perp), \langle\cdot,\cdot\rangle_{\widehat{g}^*}\big)$$

is compatible with the inner products.

Proof. Note that in our setting, the projection $\pi : \big(\widehat{M}, \widehat{g}\big) \to (M, g)$ is a surjective Riemannian submersion by construction of $\widehat{g}$. Indeed, we have

$$\ker(d\pi)^\perp = V^\perp = H,$$

and the restriction of $\widehat{g}$ to H is the lift of g. Hence, using a geometric version of Fubini's theorem [21, Chapter II, Thm. 5.6] and the fact that the fibers of $\widehat{M}$ have volume one, we obtain for $\alpha \otimes X$ and $\beta \otimes Y$ in $\Omega^\bullet(\mathcal{F}, T\mathcal{F}^\perp)$ that

$$\begin{aligned}
\big\langle \pi^*\alpha \otimes \pi^*X, \pi^*\beta \otimes \pi^*Y\big\rangle_{\widehat{g}^*} &= \int_{\widehat{M}} (\pi^*\alpha, \pi^*\beta)_{\widehat{g}}\, \pi^*(g(X,Y))\, dVol_{\widehat{g}} \\
&= \int_{\widehat{M}} \pi^*(\alpha,\beta)_g\, \pi^*(g(X,Y))\, dVol_{\widehat{g}} \\
&= \int_{x\in M} (\alpha,\beta)_g\, g(X,Y) \left(\int_{\pi^{-1}(x)} dVol_{\widehat{g}|_{\pi^{-1}(x)}} \right) dVol_g \\
&= \int_M (\alpha,\beta)_g\, g(X,Y)\, dVol_g \\
&= \langle \alpha \otimes X, \beta \otimes Y\rangle_g.
\end{aligned}$$

□

Remark 1.5. An isometry between inner product spaces always extends to an isometry between the completions. In particular, the pullback π^* extends to an isometry of Hilbert spaces between the L^2-completions

$$\pi^* : \big(L^2\Omega^\bullet(\mathcal{F}, T\mathcal{F}^\perp), \langle\cdot,\cdot\rangle_g\big) \hookrightarrow \big(L^2\Omega^\bullet(\widehat{\mathcal{F}}, \pi^*T\mathcal{F}^\perp), \langle\cdot,\cdot\rangle_{\widehat{g}^*}\big).$$

1.3. **Triviality of the pullback bundle.** We now show that the pullback bundle $\pi^*T\mathcal{F}^\perp$ is trivial as a Riemannian vector bundle with flat $T\widehat{\mathcal{F}}$-connection. Hence, using the embedding (4) allows us to drop the $T\mathcal{F}^\perp$-coefficients of elements in $\Omega^1(\mathcal{F}, T\mathcal{F}^\perp)$. The triviality of $\pi^*T\mathcal{F}^\perp$ is claimed (without proof) in [3, §4], where this result is used to obtain a Hodge decomposition for foliated forms with coefficients from the case with trivial coefficients.

We proceed considering the vector bundle $H \cap T\widehat{\mathcal{F}}^\perp$, which is isomorphic to $\pi^*T\mathcal{F}^\perp$ via

$$\varphi : H \cap T\widehat{\mathcal{F}}^\perp \xrightarrow{\sim} \pi^*T\mathcal{F}^\perp : (e, w) \mapsto (e, d\pi(w)). \tag{6}$$

Both sides come with a flat $T\widehat{\mathcal{F}}$-connection and a fiber metric, as we now explain.

First, the Bott connection $\widehat{\nabla}$ on $T\widehat{\mathcal{F}}^\perp \cong N\widehat{\mathcal{F}}$ induces a $T\widehat{\mathcal{F}}$-connection on $H \cap T\widehat{\mathcal{F}}^\perp$. To see why, recall that the transverse Levi-Civita connection ω is $\widehat{\mathcal{F}}$-basic, and therefore $[\Gamma(T\widehat{\mathcal{F}}), \Gamma(H)] \subset \Gamma(H)$. This implies that, for $X \in \Gamma(T\widehat{\mathcal{F}})$ and $Y \in \Gamma(H \cap T\widehat{\mathcal{F}}^\perp)$,

$$\widehat{\nabla}_X Y = \mathrm{pr}_{T\widehat{\mathcal{F}}^\perp}[X, Y] \in \Gamma(H \cap T\widehat{\mathcal{F}}^\perp).$$

Also, the fiber metric g^{lift} on H restricts to $H \cap T\widehat{\mathcal{F}}^{\perp}$. Second, the pullback $\pi^*\nabla$ of the Bott connection ∇ on $T\mathcal{F}^{\perp} \cong N\mathcal{F}$ gives a $T\widehat{\mathcal{F}}$-connection on $\pi^*T\mathcal{F}^{\perp}$. It is determined by requiring that for $v \in T\widehat{\mathcal{F}}$ and $Y \in \Gamma(T\mathcal{F}^{\perp})$,

$$(\pi^*\nabla)_v \pi^* Y := \nabla_{d\pi(v)} Y. \tag{7}$$

Also, the Riemannian metric g restricts to $T\mathcal{F}^{\perp}$, and its pullback π^*g defines a fiber metric on $\pi^*T\mathcal{F}^{\perp}$. It is determined by the requirement that for all $Y_1, Y_2 \in \Gamma(T\mathcal{F}^{\perp})$,

$$(\pi^*g)(\pi^*Y_1, \pi^*Y_2) = \pi^*(g(Y_1, Y_2)).$$

Lemma 1.6. *The vector bundle isomorphism* (6) *is in fact an isomorphism of Riemannian vector bundles with $T\widehat{\mathcal{F}}$-representations*

$$\varphi : \big(H \cap T\widehat{\mathcal{F}}^{\perp}, g^{lift}, \widehat{\nabla}\big) \to \big(\pi^*T\mathcal{F}^{\perp}, \pi^*g, \pi^*\nabla\big).$$

Proof. We first check that φ is compatible with the $T\widehat{\mathcal{F}}$-connections $\widehat{\nabla}$ and $\pi^*\nabla$. Note that $\Gamma(H \cap T\widehat{\mathcal{F}}^{\perp})$ is generated by horizontal lifts Y^{lift} of sections $Y \in \Gamma(T\mathcal{F}^{\perp})$, whereas $\Gamma(\pi^*T\mathcal{F}^{\perp})$ is generated by pullbacks π^*Y of sections $Y \in \Gamma(T\mathcal{F}^{\perp})$. Moreover, the isomorphism φ takes Y^{lift} to π^*Y for any $Y \in \Gamma(T\mathcal{F}^{\perp})$. Hence, by the Leibniz rule for connections, we only need to check that

$$\varphi\left(\widehat{\nabla}_v Y^{lift}\right) = (\pi^*\nabla)_v \pi^*Y \qquad \text{for } v \in T_e\widehat{\mathcal{F}},\ Y \in \Gamma(T\mathcal{F}^{\perp}).$$

We extend v to a local section $X \in \Gamma(T\widehat{\mathcal{F}})$ that is projectable under $\pi : \widehat{M} \to M$. Then

$$\widehat{\nabla}_v Y^{lift} = \mathrm{pr}_{T\widehat{\mathcal{F}}^{\perp}}\big[X, Y^{lift}\big](e),$$

and hence we obtain

$$\begin{aligned}\varphi\left(\widehat{\nabla}_v Y^{lift}\right) &= d\pi\left(\mathrm{pr}_{T\widehat{\mathcal{F}}^{\perp}}\big[X, Y^{lift}\big](e)\right)\\ &= \mathrm{pr}_{T\mathcal{F}^{\perp}}[d\pi(X), Y](\pi(e))\\ &= \nabla_{d\pi(v)}Y\\ &= (\pi^*\nabla)_v\pi^*Y.\end{aligned}$$

Similarly, to see that φ matches the fiber metrics g^{lift} and π^*g, we only need to check that

$$g^{lift}(Y_1^{lift}, Y_2^{lift}) = (\pi^*g)(\pi^*Y_1, \pi^*Y_2) \qquad \text{for } Y_1, Y_2 \in \Gamma(T\mathcal{F}^{\perp}).$$

This equality clearly holds, because both sides are equal to $\pi^*(g(Y_1, Y_2))$. □

Next, we show that $\big(H \cap T\widehat{\mathcal{F}}^{\perp}, g^{lift}, \widehat{\nabla}\big)$ is trivial, as a Riemannian vector bundle with $T\widehat{\mathcal{F}}$-representation. By Lemma 1.6, the same then holds for $\big(\pi^*T\mathcal{F}^{\perp}, \pi^*g, \pi^*\nabla\big)$.

Lemma 1.7. *The vector bundle $H \cap T\widehat{\mathcal{F}}^{\perp}$ has an orthonormal frame of flat sections.*

Proof. We refer to §1.1, where we constructed the natural transverse parallelism for $\widehat{\mathcal{F}}$. In particular, there are unique $\widehat{\mathcal{F}}$-projectable vector fields $Y_1, \dots Y_q \in \Gamma(T\widehat{\mathcal{F}}^{\perp})$ satisfying

$$\omega(Y_i) = 0 \qquad \text{and} \qquad \theta(Y_i) = u_i.$$

These constitute a frame $\{Y_1, \dots Y_q\}$ for $H \cap T\widehat{\mathcal{F}}^{\perp}$ consisting of flat sections for $\widehat{\nabla}$. The fact that $\{Y_1, \dots, Y_q\}$ is orthonormal for g^{lift} was already obtained in the proof of Lemma 1.2, where we showed that

$$g^{lift}(Y_i, Y_j) = \widehat{g}(Y_i, Y_j) = \delta_{ij}.$$

□

Corollary 1.8. *Also* $\left(\pi^*T\mathcal{F}^\perp, \pi^*g, \pi^*\nabla\right)$ *has an orthonormal frame of flat sections.*

Fixing such a frame $\{Y_1, \dots, Y_q\}$ for $\pi^*T\mathcal{F}^\perp$, we get an identification

$$\Omega^\bullet(\widehat{\mathcal{F}}, \pi^*T\mathcal{F}^\perp) \xrightarrow{\sim} \Omega^\bullet(\widehat{\mathcal{F}})^q : \sum_i \alpha_i \otimes Y_i \mapsto (\alpha_1, \dots, \alpha_q). \tag{8}$$

Recall from (5) that we have an L^2 inner product on $\Omega^\bullet(\widehat{\mathcal{F}}, \pi^*T\mathcal{F}^\perp)$. Let $\delta_{\pi^*\nabla}$ denote the formal adjoint of the differential $d_{\pi^*\nabla}$ on $\Omega^\bullet(\widehat{\mathcal{F}}, \pi^*T\mathcal{F}^\perp)$. We also have the similarly defined L^2 inner product $\langle\cdot,\cdot\rangle_{\widehat{g}}$ on $\Omega^\bullet(\widehat{\mathcal{F}})$. Denote by $\delta_{\widehat{\mathcal{F}}}$ the formal adjoint of the foliated de Rham differential $d_{\widehat{\mathcal{F}}}$. The following results are straightforward.

Corollary 1.9. *Under the identification* (8)*, we have that*

i) the differential $d_{\pi^*\nabla}$ *corresponds with* $d_{\widehat{\mathcal{F}}}^{\times q}$.

ii) the induced inner product on $\Omega^\bullet(\widehat{\mathcal{F}})^q$ *is*

$$\big\langle(\alpha_1, \dots, \alpha_q), (\beta_1, \dots, \beta_q)\big\rangle := \sum_i \langle\alpha_i, \beta_i\rangle_{\widehat{g}},$$

iii) the induced codifferential on $\Omega^\bullet(\widehat{\mathcal{F}})^q$ *is given by* $\delta_{\widehat{\mathcal{F}}}^{\times q}$.

1.4. **Properties of the pullback.** The aim of this subsection is to show that the pullback $\pi^* : \Omega^\bullet(\mathcal{F}, T\mathcal{F}^\perp) \hookrightarrow \Omega^\bullet(\widehat{\mathcal{F}}, \pi^*T\mathcal{F}^\perp)$ intertwines differentials and codifferentials.

We remain in the setup of Cor. 1.9, i.e. $\delta_{\pi^*\nabla}$ denotes the formal adjoint of $d_{\pi^*\nabla}$ with respect to the L^2 inner product $\langle\cdot,\cdot\rangle_{\widehat{g}^*}$ on $\Omega^\bullet(\widehat{\mathcal{F}}, \pi^*T\mathcal{F}^\perp)$ and $\delta_{\widehat{\mathcal{F}}}$ is the formal adjoint of $d_{\widehat{\mathcal{F}}}$. We also fix a frame $\{Y_1, \dots, Y_q\}$ for $\pi^*T\mathcal{F}^\perp$ consisting of orthonormal flat sections, which exists by Cor. 1.8. We first prove an auxiliary result.

Lemma 1.10. *For any* $\alpha \otimes Y_i \in \Omega^k(\widehat{\mathcal{F}}, \pi^*T\mathcal{F}^\perp)$*, we have*

$$d_{\pi^*\nabla}(\alpha \otimes Y_i) = \left(d_{\widehat{\mathcal{F}}}\alpha\right) \otimes Y_i \qquad \text{and} \qquad \delta_{\pi^*\nabla}(\alpha \otimes Y_i) = \left(\delta_{\widehat{\mathcal{F}}}\alpha\right) \otimes Y_i. \tag{9}$$

Proof. The first equality is just the Leibniz rule for foliated forms with coefficients in a representation, along with the fact that $d_{\pi^*\nabla}Y_i = 0$.

To prove the second identity, we only need to show that both sides have the same inner product with an element of the form $\beta \otimes Y_j \in \Omega^{k-1}(\widehat{\mathcal{F}}, \pi^*T\mathcal{F}^\perp)$. We compute

$$\begin{aligned}
\big\langle \delta_{\pi^*\nabla}(\alpha \otimes Y_i), \beta \otimes Y_j \big\rangle_{\widehat{g}^*} &= \big\langle \alpha \otimes Y_i, d_{\pi^*\nabla}(\beta \otimes Y_j) \big\rangle_{\widehat{g}^*} \\
&= \big\langle \alpha \otimes Y_i, (d_{\widehat{\mathcal{F}}}\beta) \otimes Y_j \big\rangle_{\widehat{g}^*} \\
&= \big\langle \alpha, d_{\widehat{\mathcal{F}}}\beta \big\rangle_{\widehat{g}} \delta_{ij} \\
&= \big\langle \delta_{\widehat{\mathcal{F}}}\alpha, \beta \big\rangle_{\widehat{g}} \delta_{ij} \\
&= \big\langle (\delta_{\widehat{\mathcal{F}}}\alpha) \otimes Y_i, \beta \otimes Y_j \big\rangle_{\widehat{g}^*},
\end{aligned}$$

where the second equality holds by the first identity in (9), and we used that $\{Y_1, \dots, Y_q\}$ is orthonormal for the fiber metric π^*g on $\pi^*T\mathcal{F}^\perp$. This finishes the proof. □

We now turn our attention to the map $\pi^* : \Omega^\bullet(\mathcal{F}, T\mathcal{F}^\perp) \hookrightarrow \Omega^\bullet(\widehat{\mathcal{F}}, \pi^*T\mathcal{F}^\perp)$. We denote by δ_∇ the formal adjoint of the differential d_∇ on $\Omega^\bullet(\mathcal{F}, T\mathcal{F}^\perp)$ induced by the Bott connection.

Lemma 1.11. *The inclusion* $\pi^* : \Omega^\bullet(\mathcal{F}, T\mathcal{F}^\perp) \hookrightarrow \Omega^\bullet(\widehat{\mathcal{F}}, \pi^*T\mathcal{F}^\perp)$ *satisfies*

$$\pi^* \circ d_\nabla = d_{\pi^*\nabla} \circ \pi^* \qquad \text{and} \qquad \pi^* \circ \delta_\nabla = \delta_{\pi^*\nabla} \circ \pi^*. \tag{10}$$

Proof. To prove the first identity in (10), take $\beta \otimes Z \in \Omega^k(\mathcal{F}, T\mathcal{F}^\perp)$. By the Leibniz rule for d_∇ and $d_{\pi^*\nabla}$, we have the two equations

$$\pi^*(d_\nabla(\beta \otimes Z)) = \pi^*(d_\mathcal{F}\beta) \otimes \pi^* Z + (-1)^k \pi^*\beta \wedge \pi^*(d_\nabla Z),$$
$$d_{\pi^*\nabla}(\pi^*\beta \otimes \pi^* Z) = d_{\widehat{\mathcal{F}}}\pi^*\beta \otimes \pi^* Z + (-1)^k \pi^*\beta \wedge d_{\pi^*\nabla}\pi^* Z.$$

Because π is a foliated map, we know that $\pi^* \circ d_\mathcal{F} = d_{\widehat{\mathcal{F}}} \circ \pi^*$, so it remains to show that

$$\pi^*(d_\nabla Z) = d_{\pi^*\nabla}\pi^* Z.$$

This equality follows immediately from (7), since for $v \in T\widehat{\mathcal{F}}$ we have

$$(d_{\pi^*\nabla}\pi^* Z)(v) = (\pi^*\nabla)_v \pi^* Z = \nabla_{d\pi(v)} Z = (d_\nabla Z)(d\pi(v)) = (\pi^*(d_\nabla Z))(v).$$

To prove the second identity in (10), we note that for $\eta \in \Omega^k(\mathcal{F}, T\mathcal{F}^\perp)$ and $\xi \in \Omega^{k-1}(\mathcal{F}, T\mathcal{F}^\perp)$,

$$\begin{aligned}
\left\langle \delta_{\pi^*\nabla}(\pi^*\eta), \pi^*\xi \right\rangle_{\widehat{g}^*} &= \left\langle \pi^*\eta, d_{\pi^*\nabla}(\pi^*\xi) \right\rangle_{\widehat{g}^*} \\
&= \left\langle \pi^*\eta, \pi^*(d_\nabla \xi) \right\rangle_{\widehat{g}^*} \\
&= \left\langle \eta, d_\nabla \xi \right\rangle_g \\
&= \left\langle \delta_\nabla \eta, \xi \right\rangle_g \\
&= \left\langle \pi^*(\delta_\nabla \eta), \pi^*\xi \right\rangle_{\widehat{g}^*}.
\end{aligned}$$

Here we used the first identity in (10) and Cor. 1.4. This shows that the inner product of $\delta_{\pi^*\nabla}(\pi^*\eta) - \pi^*(\delta_\nabla \eta)$ with any pullback section is zero. Hence, it is enough to show that $\delta_{\pi^*\nabla}(\pi^*\eta) - \pi^*(\delta_\nabla \eta)$ is a pullback section itself, i.e. that for all $\eta \in \Omega^k(\mathcal{F}, T\mathcal{F}^\perp)$:

$$\delta_{\pi^*\nabla}(\pi^*\eta) \in \operatorname{im}\left(\pi^* : \Omega^{k-1}(\mathcal{F}, T\mathcal{F}^\perp) \to \Omega^{k-1}(\widehat{\mathcal{F}}, \pi^* T\mathcal{F}^\perp)\right). \tag{11}$$

As π^* is injective, this can be checked locally on M, i.e. it suffices to show that $\delta_{\pi^*\nabla}(\pi^*(\eta|_U))$ belongs to $\operatorname{im}(\pi^*)$ for any open $U \subset M$. To do so, we use the following.

<u>Claim:</u> The pullback $\pi^* : \Omega^\bullet(\mathcal{F}) \to \Omega^\bullet(\widehat{\mathcal{F}})$ intertwines $\delta_\mathcal{F}$ and $\delta_{\widehat{\mathcal{F}}}$.

Working in an open $U \subset M$, fix a tangential orientation for $\mathcal{F}$. Then also $\widehat{\mathcal{F}}$ inherits a tangential orientation on $\pi^{-1}(U)$. We get Hodge star operators $\star_\mathcal{F}$ and $\star_{\widehat{\mathcal{F}}}$. Because

$$d\pi : \left(T_e\widehat{\mathcal{F}}, \widehat{g}\right) \to \left(T_{\pi(e)}\mathcal{F}, g\right)$$

is an orientation preserving isometry, its dual $(d\pi)^*$ intertwines the Hodge stars $\star_\mathcal{F}$ and $\star_{\widehat{\mathcal{F}}}$. The pullback π^* also intertwines $d_{\widehat{\mathcal{F}}}$ and $d_\mathcal{F}$ because π is a foliated map. Hence, using the formula [3, eq. 17] for the leafwise coderivative, we obtain for $\beta \in \Omega^k(\mathcal{F})$:

$$\delta_{\widehat{\mathcal{F}}}(\pi^*\beta) = (-1)^{pk+p+1} \star_{\widehat{\mathcal{F}}} d_{\widehat{\mathcal{F}}} \star_{\widehat{\mathcal{F}}} (\pi^*\beta) = (-1)^{pk+p+1}\pi^*(\star_\mathcal{F} d_\mathcal{F} \star_\mathcal{F} \beta) = \pi^*(\delta_\mathcal{F}\beta),$$

where we denoted $p = \dim \mathcal{F} = \dim \widehat{\mathcal{F}}$. This proves the claim.

We now finish the proof of the lemma. Since $T\mathcal{F}^\perp$ has a local frame of projectable vector fields, we may assume that $\eta = \beta \otimes Z$, where $Z \in \Gamma(T\mathcal{F}^\perp)$ is projectable. We can write

$$\pi^* Z = \sum_i f_i Y_i,$$

where $\{Y_1,\dots,Y_q\}$ is the fixed flat, orthonormal frame of $\pi^*T\mathcal{F}^\perp$. The first identity in (10) along with $d_\nabla Z=0$ implies that $d_{\pi^*\nabla}\pi^*Z=0$. Since also $d_{\pi^*\nabla}Y_i=0$, it follows that

$$\sum_i d_{\widehat{\mathcal{F}}} f_i\otimes Y_i=0,$$

and since $\{Y_1,\dots,Y_q\}$ is a frame for $\pi^*T\mathcal{F}^\perp$, this shows that the f_i are $\widehat{\mathcal{F}}$-basic. By Cor. 1.9, we see that under the identification (8), $\delta_{\pi^*\nabla}(\pi^*\eta)$ corresponds to

$$\delta_{\widehat{\mathcal{F}}}^{\times q}\big(f_1\pi^*\beta,\dots,f_q\pi^*\beta\big)=\big(f_1\delta_{\widehat{\mathcal{F}}}(\pi^*\beta),\dots,f_q\delta_{\widehat{\mathcal{F}}}(\pi^*\beta)\big)=\big(f_1\pi^*(\delta_{\mathcal{F}}\beta),\dots,f_q\pi^*(\delta_{\mathcal{F}}\beta)\big) \tag{12}$$

where the first equality holds because f_i is $\widehat{\mathcal{F}}$-basic, and the second equality holds by the claim above. Transporting (12) back under the identification (8), we obtain that

$$\delta_{\pi^*\nabla}(\pi^*\eta)=\sum_i f_i\pi^*(\delta_{\mathcal{F}}\beta)\otimes Y_i=\pi^*(\delta_{\mathcal{F}}\beta)\otimes\pi^*Z=\pi^*(\delta_{\mathcal{F}}\beta\otimes Z).$$

This shows that the statement (11) holds, hence the proof is finished. □

1.5. **Averaging on the transverse orthogonal frame bundle.** As mentioned before, we aim to prove the estimate (1) by pulling back elements of $\Omega^\bullet(\mathcal{F},T\mathcal{F}^\perp)$ to $\Omega^\bullet(\widehat{\mathcal{F}},\pi^*T\mathcal{F}^\perp)$. The last ingredient that we need is a way to return in the opposite direction. This will be done by averaging elements of $\Omega^\bullet(\widehat{\mathcal{F}},\pi^*T\mathcal{F}^\perp)$ with respect to the $O(q)$-action.

We find it more convenient to describe this operation on the complex $\Omega^\bullet(\widehat{\mathcal{F}},H\cap T\widehat{\mathcal{F}}^\perp)$ instead. This is equivalent by Lemma 1.6, since the map φ induces an isometry of complexes

$$\varphi:\Big(\Omega^\bullet(\widehat{\mathcal{F}},H\cap T\widehat{\mathcal{F}}^\perp),\langle\cdot,\cdot\rangle_{\widehat{g}_H},d_{\widehat{\nabla}}\Big)\to\Big(\Omega^\bullet(\widehat{\mathcal{F}},\pi^*T\mathcal{F}^\perp),\langle\cdot,\cdot\rangle_{\widehat{g}^*},d_{\pi^*\nabla}\Big), \tag{13}$$

where $\langle\cdot,\cdot\rangle_{\widehat{g}^*}$ was defined in (5) and $\langle\cdot,\cdot\rangle_{\widehat{g}_H}$ is defined by

$$\langle\alpha\otimes X,\beta\otimes Y\rangle_{\widehat{g}_H}=\int_{\widehat{M}}(\alpha,\beta)_{\widehat{g}}\,g^{lift}(X,Y)dVol_{\widehat{g}}.$$

We now argue that there is a well-defined averaging operation on $\Omega^\bullet(\widehat{\mathcal{F}},H\cap T\widehat{\mathcal{F}}^\perp)$. Note that the given $O(q)$-action $R:\widehat{M}\times O(q)\to\widehat{M}$ satisfies the following:

- it preserves the foliation $\widehat{\mathcal{F}}$ (see §1.1),
- it preserves H, being the kernel of the connection 1-form ω,
- it preserves the Riemannian metric $\widehat{g}$ (see §1.2).

It follows that the $O(q)$-action preserves $H\cap T\widehat{\mathcal{F}}^\perp$, hence it induces an action

$$O(q)\times\Omega^\bullet(\widehat{\mathcal{F}},H\cap T\widehat{\mathcal{F}}^\perp)\to\Omega^\bullet(\widehat{\mathcal{F}},H\cap T\widehat{\mathcal{F}}^\perp):(h,\alpha\otimes X)\mapsto R_h^*(\alpha)\otimes R_h^*(X).$$

Consequently, we get a well-defined averaging map

$$Av:\Omega^\bullet(\widehat{\mathcal{F}},H\cap T\widehat{\mathcal{F}}^\perp)\to\Omega^\bullet_{inv}(\widehat{\mathcal{F}},H\cap T\widehat{\mathcal{F}}^\perp):\xi\mapsto\int_{O(q)}(R_h^*\xi)dh, \tag{14}$$

where dh is the normalized bi-invariant Haar measure on $O(q)$ and $\Omega^\bullet_{inv}(\widehat{\mathcal{F}},H\cap T\widehat{\mathcal{F}}^\perp)$ is the subspace of invariant forms, i.e.

$$\Omega^k_{inv}(\widehat{\mathcal{F}},H\cap T\widehat{\mathcal{F}}^\perp)=\big\{\xi\in\Omega^k(\widehat{\mathcal{F}},H\cap T\widehat{\mathcal{F}}^\perp)|\ \forall h\in O(q):\ R_h^*\xi=\xi\big\}.$$

Invariant forms descend to elements of $\Omega^\bullet(\mathcal{F},T\mathcal{F}^\perp)$, i.e. $\Omega^k_{inv}(\widehat{\mathcal{F}},H\cap T\widehat{\mathcal{F}}^\perp)=\pi^*\Omega^k(\mathcal{F},T\mathcal{F}^\perp)$.

Lemma 1.12. *The $O(q)$-action on $\Omega^\bullet(\widehat{\mathcal{F}},H\cap T\widehat{\mathcal{F}}^\perp)$ is orthogonal with respect to $\langle\cdot,\cdot\rangle_{\widehat{g}_H}$.*

Proof. We first remark that $O(q)$-invariance of $\widehat{g}$ implies that also $dVol_{\widehat{g}}$ is $O(q)$-invariant (see [21, II.5.(III)]). Hence, for $h \in O(q)$ and $\alpha \otimes X, \beta \otimes Y \in \Omega^k(\widehat{\mathcal{F}}, H \cap T\widehat{\mathcal{F}}^\perp)$, we have

$$\begin{aligned}
\langle R_h^*(\alpha \otimes X), R_h^*(\beta \otimes Y)\rangle_{\widehat{g}_H} &= \int_{\widehat{M}} (R_h^*\alpha, R_h^*\beta)_{\widehat{g}}\; g^{lift}(R_h^* X, R_h^* Y) dVol_{\widehat{g}} \\
&= \int_{\widehat{M}} R_h^*(\alpha, \beta)_{\widehat{g}}\; R_h^*\left(g^{lift}(X, Y)\right) dVol_{\widehat{g}} \\
&= \int_{\widehat{M}} (\alpha, \beta)_{\widehat{g}}\; g^{lift}(X, Y) dVol_{\widehat{g}} \\
&= \langle \alpha \otimes X, \beta \otimes Y\rangle_{\widehat{g}_H},
\end{aligned}$$

using in turn the $O(q)$-invariance of $\widehat{g}, g^{lift}$ and $dVol_{\widehat{g}}$. This proves the statement. □

Denote by $d_{\widehat{\nabla}}$ the differential induced on $\Omega^\bullet(\widehat{\mathcal{F}}, H \cap T\widehat{\mathcal{F}}^\perp)$ by the Bott connection $\widehat{\nabla}$, and its formal adjoint with respect to $\langle \cdot, \cdot \rangle_{\widehat{g}_H}$ by $\delta_{\widehat{\nabla}}$. Lemma 1.12 has the following consequence.

Lemma 1.13. *The operators $d_{\widehat{\nabla}}$ and $\delta_{\widehat{\nabla}}$ commute with the $O(q)$-action, i.e. for all $h \in O(q)$:*

$$d_{\widehat{\nabla}} \circ R_h^* = R_h^* \circ d_{\widehat{\nabla}}, \qquad \textit{and} \qquad \delta_{\widehat{\nabla}} \circ R_h^* = R_h^* \circ \delta_{\widehat{\nabla}}.$$

Proof. To obtain the first identity, we argue as in the proof of Lemma 1.11. First note that R_h^* commutes with the foliated differential $d_{\widehat{\mathcal{F}}}$, because R_h preserves $\widehat{\mathcal{F}}$. Hence, the Leibniz rule for $d_{\widehat{\nabla}}$ reduces the proof to showing that for $X \in \Gamma(H \cap T\widehat{\mathcal{F}}^\perp)$, we have

$$d_{\widehat{\nabla}} R_h^* X = R_h^* d_{\widehat{\nabla}} X. \tag{15}$$

To do so, we will pair both sides with an arbitrary element $V \in \Gamma(T\widehat{\mathcal{F}})$. We get

$$\begin{aligned}
\left(R_h^* d_{\widehat{\nabla}} X\right)(V) &= R_h^*\left(d_{\widehat{\nabla}} X((R_h)_* V)\right) \\
&= R_h^*\left(\widehat{\nabla}_{(R_h)_* V} X\right) \\
&= R_h^*\left(\mathrm{pr}_{T\widehat{\mathcal{F}}^\perp}\left[(R_h)_* V, X\right]\right) \\
&= \mathrm{pr}_{T\widehat{\mathcal{F}}^\perp}\left[V, R_h^* X\right] \\
&= \widehat{\nabla}_V R_h^* X \\
&= \left(d_{\widehat{\nabla}} R_h^* X\right)(V)
\end{aligned}$$

using in the fourth equality that R_h^* preserves each summand in the direct sum $T\widehat{\mathcal{F}} \oplus T\widehat{\mathcal{F}}^\perp$. This shows that the equality (15) holds, which finishes the proof of the first identity.

The second identity is now obtained from the computation

$$\begin{aligned}
\left\langle (\delta_{\widehat{\nabla}} \circ R_h^*)(\alpha \otimes X), \beta \otimes Y\right\rangle_{\widehat{g}_H} &= \left\langle R_h^*(\alpha \otimes X), d_{\widehat{\nabla}}(\beta \otimes Y)\right\rangle_{\widehat{g}_H} \\
&= \left\langle \alpha \otimes X, (R_{h^{-1}}^* \circ d_{\widehat{\nabla}})(\beta \otimes Y)\right\rangle_{\widehat{g}_H} \\
&= \left\langle \alpha \otimes X, (d_{\widehat{\nabla}} \circ R_{h^{-1}}^*)(\beta \otimes Y)\right\rangle_{\widehat{g}_H} \\
&= \left\langle \delta_{\widehat{\nabla}}(\alpha \otimes X), R_{h^{-1}}^*(\beta \otimes Y)\right\rangle_{\widehat{g}_H} \\
&= \left\langle (R_h^* \circ \delta_{\widehat{\nabla}})(\alpha \otimes X), \beta \otimes Y\right\rangle_{\widehat{g}_H}
\end{aligned}$$

for $\alpha \otimes X \in \Omega^k(\widehat{\mathcal{F}}, H \cap T\widehat{\mathcal{F}}^\perp)$ and $\beta \otimes Y \in \Omega^{k-1}(\widehat{\mathcal{F}}, H \cap T\widehat{\mathcal{F}}^\perp)$. Here the second and last equality rely on Lemma 1.12, and the third equality holds by the first part of the proof. □

Note that $d_{\widehat{\nabla}}$ and $\delta_{\widehat{\nabla}}$ commute with the Haar integral since they are differential operators and derivatives commute with the Haar integral by the dominated convergence theorem. Hence, Lemma 1.13 immediately implies the following.

Corollary 1.14. *The averaging map* (14) *commutes with* $d_{\widehat{\nabla}}$ *and* $\delta_{\widehat{\nabla}}$*, i.e. we have*

$$d_{\widehat{\nabla}} \circ Av = Av \circ d_{\widehat{\nabla}}, \qquad \text{and} \qquad \delta_{\widehat{\nabla}} \circ Av = Av \circ \delta_{\widehat{\nabla}}.$$

At last, we show that the averaging map extends to the L^2-completion.

Lemma 1.15. *Averaging is a bounded linear operator on* $\big(\Omega^k(\widehat{\mathcal{F}}, H \cap T\widehat{\mathcal{F}}^\perp), \|\cdot\|_{\widehat{g}_H}\big)$*, since*

$$\|Av(\alpha \otimes X)\|_{\widehat{g}_H} \leq \|\alpha \otimes X\|_{\widehat{g}_H}.$$

Consequently, it extends to a bounded linear operator on the L^2*-completion*

$$Av : \big(L^2\Omega^k(\widehat{\mathcal{F}}, H \cap T\widehat{\mathcal{F}}^\perp), \|\cdot\|_{\widehat{g}_H}\big) \to \big(L^2\Omega^k(\widehat{\mathcal{F}}, H \cap T\widehat{\mathcal{F}}^\perp), \|\cdot\|_{\widehat{g}_H}\big).$$

Proof. First note that for all $\xi, \eta \in \Omega^k(\widehat{\mathcal{F}}, H \cap T\widehat{\mathcal{F}}^\perp)$ and $h \in O(q)$, we have

$$\int_{O(q)} \langle R_h^*\xi, \eta\rangle_{\widehat{g}_H} dh = \left\langle \int_{O(q)} (R_h^*\xi) dh, \eta \right\rangle_{\widehat{g}_H}$$

due to Fubini's theorem. Using this, we get

$$\begin{aligned}
\big\langle Av(\alpha \otimes X), Av(\alpha \otimes X)\big\rangle_{\widehat{g}_H} &= \int_{O(q)} \int_{O(q)} \big\langle R_h^*(\alpha \otimes X), R_k^*(\alpha \otimes X)\big\rangle_{\widehat{g}_H} dh dk \\
&\leq \int_{O(q)} \int_{O(q)} \big\|R_h^*(\alpha \otimes X)\big\|_{\widehat{g}_H} \big\|R_k^*(\alpha \otimes X)\big\|_{\widehat{g}_H} dh dk \\
&= \|\alpha \otimes X\|_{\widehat{g}_H}^2,
\end{aligned}$$

where we used the Cauchy-Schwarz inequality and Lemma 1.12. This proves the lemma. ☐

At last, we use the isomorphism (13) to transport the averaging map to $\Omega^\bullet(\widehat{\mathcal{F}}, \pi^*T\mathcal{F}^\perp)$. We summarize here its main properties. The first two are consequences of Cor. 1.14 and Lemma 1.15, whereas the last one holds because it is true for smooth sections.

Corollary 1.16. *There is a well-defined averaging map* Av *on* $\Omega^\bullet(\widehat{\mathcal{F}}, \pi^*T\mathcal{F}^\perp)$ *satisfying:*

(1) $d_{\pi^*\nabla} \circ Av = Av \circ d_{\pi^*\nabla}$ *and* $\delta_{\pi^*\nabla} \circ Av = Av \circ \delta_{\pi^*\nabla}$*,*
(2) Av *is bounded, hence it extends continuously to* $L^2\Omega^\bullet(\widehat{\mathcal{F}}, \pi^*T\mathcal{F}^\perp)$*,*
(3) *for all* $\eta \in L^2\Omega^\bullet(\mathcal{F}, T\mathcal{F}^\perp)$*, we have*

$$Av(\pi^*\eta) = \pi^*\eta.$$

2. Proof of the Main Theorem

This section is devoted to the proof of the Main Theorem stated in the introduction. Note that by Hamilton's Stability Theorem, it is enough to prove the following proposition.

Proposition 2.1. *Let* M *be a compact manifold with a Riemannian foliation* $\mathcal{F}$ *and bundle-like metric* g*. If* $H^1(\mathcal{F}, T\mathcal{F}^\perp) = 0$*, then there exists a constant* $C > 0$ *such that*

$$\|\alpha\|_g \leq C\left(\|d_\nabla \alpha\|_g + \|\delta_\nabla \alpha\|_g\right), \qquad \forall \alpha \in \Omega^1(\mathcal{F}, T\mathcal{F}^\perp). \tag{16}$$

We briefly outline the proof. Assuming by contradiction that the estimate (16) does not hold, we get a sequence (α_n) in $\Omega^1(\mathcal{F}, T\mathcal{F}^\perp)$ with the properties

$$\|\alpha_n\|_g = 1, \quad \|d_\nabla \alpha_n\|_g \longrightarrow 0, \quad \|\delta_\nabla \alpha_n\|_g \longrightarrow 0.$$

We then construct a subsequence converging to an element in the L^2-closure of the space of harmonic elements. We reach a contradiction, combining the assumption that $H^1(\mathcal{F}, T\mathcal{F}^\perp)$ vanishes with the Hodge decomposition for $\Omega^\bullet(\mathcal{F}, T\mathcal{F}^\perp)$. We first recall the latter.

Theorem 2.2 ([3, Cor. C])**.** *Let $(M, \mathcal{F}, g)$ be a closed manifold with a Riemannian foliation and a bundle-like metric. Define the Laplace operator $\Delta_\nabla := d_\nabla\delta_\nabla + \delta_\nabla d_\nabla$ on $\Omega^\bullet(\mathcal{F}, T\mathcal{F}^\perp)$. We then have the leafwise Hodge decomposition*

$$\Omega^\bullet(\mathcal{F}, T\mathcal{F}^\perp) = \ker\Delta_\nabla \oplus \overline{\operatorname{im}\Delta_\nabla} = \big(\ker d_\nabla \cap \ker\delta_\nabla\big) \oplus \overline{\operatorname{im} d_\nabla} \oplus \overline{\operatorname{im}\delta_\nabla},$$

hence $\overline{H}(\mathcal{F}, T\mathcal{F}^\perp)$ can be canonically identified with $\ker\Delta_\nabla$.

The statement involves the leafwise reduced cohomology $\overline{H}(\mathcal{F}, T\mathcal{F}^\perp)$, which is defined as $\overline{H}(\mathcal{F}, T\mathcal{F}^\perp) = \ker d_\nabla / \overline{\operatorname{im} d_\nabla}$, where the closure is taken in the C^∞-topology.

Remark 2.3. The result [3, Cor. C] is actually more general, as it holds for foliated forms with values in an arbitrary Riemannian vector bundle V with flat Riemannian $T\mathcal{F}$-connection. The authors give a detailed proof for the case with trivial coefficients, and in [3, §4] they sketch how lifting to the transverse orthogonal frame bundle reduces the proof of the general case to the trivial case. Our Section 1 fills in the details of [3, §4], for the case $V = T\mathcal{F}^\perp$.

In order to pass to a convergent subsequence of (α_n), we lift the sequence to the transverse orthogonal frame bundle via the pullback $\pi^* : \Omega^\bullet(\mathcal{F}, T\mathcal{F}^\perp) \hookrightarrow \Omega^\bullet(\widehat{\mathcal{F}}, \pi^* T\mathcal{F}^\perp)$. Doing so, it becomes a sequence of vectors in $\Omega^1(\widehat{\mathcal{F}})$, and this allows us to obtain the desired subsequence by applying a variation of the technical convergence result [2, Thm. B].

2.1. **A convergent subsequence.** We first recall [2, Thm. B] and adapt it to our needs.

Let $(M, \mathcal{F})$ be a compact manifold with a Riemannian foliation, and fix a bundle-like metric g on M. Denote by $\langle\cdot,\cdot\rangle_g$ the L^2-inner product on $\Omega^\bullet(M)$, and by $\big(L^2\Omega(M), \langle\cdot,\cdot\rangle_g\big)$ the Hilbert space of square-integrable differential forms. The splitting $TM = T\mathcal{F} \oplus T\mathcal{F}^\perp$ induces a decomposition

$$\Omega^k(M) = \bigoplus_{u+v=k} \Gamma\left(\wedge^u (T\mathcal{F}^\perp)^* \otimes \wedge^v T^*\mathcal{F}\right), \tag{17}$$

which in turn yields a bi-grading on $\Omega(M)$. The de Rham derivative d and its formal adjoint δ split into bi-homogeneous components

$$d = d_{0,1} + d_{1,0} + d_{2,-1}, \qquad \delta = \delta_{0,-1} + \delta_{-1,0} + \delta_{-2,1},$$

where the double index indicates the bi-degree of the component in question. Here $\delta_{-i,-j}$ is the formal adjoint of $d_{i,j}$. Denote by Δ_0 the Laplacian in leafwise direction, and by $\mathcal{H}$ its space of harmonic elements, i.e.

$$\Delta_0 = d_{0,1}\delta_{0,-1} + \delta_{0,-1}d_{0,1}, \qquad \mathcal{H} = \ker\Delta_0.$$

We introduce operators d_h and δ_h, defined by rescaling d and δ in the transverse direction:

$$d_h := d_{0,1} + h d_{1,0} + h^2 d_{2,-1}, \quad \delta_h := \delta_{0,-1} + h\delta_{-1,0} + h^2\delta_{-2,1}, \tag{18}$$

where $h > 0$ is a parameter. The corresponding Laplace and Dirac operators are given by

$$\Delta_h := d_h\delta_h + \delta_h d_h, \qquad \text{and} \qquad D_h := d_h + \delta_h. \tag{19}$$

One can show that D_h is formally self-adjoint and $D_h^2 = \Delta_h$. We can now state a particular case of [2, Thm. B]. The latter is more general than the statement we recall here, as it gives a more detailed conclusion when the rate of convergence in (20) below is specified.

Theorem 2.4 ([2])**.** *Let $(M, \mathcal{F}, g)$ be a closed manifold equipped with a Riemannian foliation and a bundle-like metric. Assume $(\omega_n) \subset \Omega^k(M)$ is a sequence satisfying $\|\omega_n\|_g = 1$ and*

$$\langle \Delta_{h_n}\omega_n, \omega_n\rangle_g \longrightarrow 0, \qquad \text{for some} \;\; h_n \downarrow 0. \tag{20}$$

Then a subsequence of (ω_n) strongly converges to an element $\omega \in \overline{\mathcal{H}} \subset L^2\Omega^k(M)$.

We need a vector version of the above result. We equip $(\Omega^k(M))^q$ with the direct sum inner product induced by $\langle\cdot,\cdot\rangle_g$, which we still denote by $\langle\cdot,\cdot\rangle_g$. Its completion is $(L^2\Omega^k(M))^q$.

Corollary 2.5. *Let $(M, \mathcal{F}, g)$ be a closed manifold equipped with a Riemannian foliation and a bundle-like metric. Assume $(\omega_n) \subset (\Omega^k(M))^q$ is a sequence satisfying $\|\omega_n\|_g = 1$ and*

$$\left\langle \Delta_{h_n}^{\times q}\omega_n, \omega_n\right\rangle_g \longrightarrow 0, \qquad \text{for some} \;\; h_n \downarrow 0.$$

Then a subsequence of (ω_n) strongly converges to an element $\omega \in \overline{\mathcal{H}}^q \subset (L^2\Omega^k(M))^q$.

Proof. The proof is by induction on q. If $q = 1$, then the statement is exactly Theorem 2.4. We show that if the statement holds for $q-1$, then it holds for q. Let $\omega_n = (\omega_n^1, \dots, \omega_n^q)$ be a sequence as in the statement. Since $\|\omega_n\|_g = 1$ for all n, there exists $i \in \{1, \dots, q\}$ such that, after passing to a subsequence,

$$\|\omega_n^i\|_g \geq \frac{1}{q}.$$

Without loss of generality, we assume that $i = q$. Note that the condition

$$\left\langle \Delta_{h_n}^{\times q}\omega_n, \omega_n\right\rangle_g \longrightarrow 0$$

implies that for any $j \in \{1, \dots, q\}$, we have

$$\left\langle \Delta_{h_n}\omega_n^j, \omega_n^j\right\rangle_g \longrightarrow 0.$$

Theorem 2.4 implies that upon passing to a subsequence, $\omega_n^q/\|\omega_n^q\|_g$ strongly converges to an element $\omega^q \in \overline{\mathcal{H}} \subset L^2\Omega^k(M)$. Since the sequence $\|\omega_n^q\|_g$ is bounded above by 1 and below by $1/q$, it has a subsequence which converges to some $c \in [\frac{1}{q}, 1]$. Hence, after passing to a subsequence, we obtain that ω_n^q strongly converges to $c\,\omega^q \in \overline{\mathcal{H}} \subset L^2\Omega^k(M)$.

Now we look at $\tilde{\omega}_n = (\omega_n^1, \dots, \omega_n^{q-1}) \in \big(\Omega^k(M)\big)^{q-1}$. We distinguish between two cases.

(1) If there exists some $0 < d$ such that $d \leq \|\tilde{\omega}_n\|_g$ for all n, then $\tilde{\omega}_n/\|\tilde{\omega}_n\|_g$ satisfies the statement for $q-1$, and after passing to a subsequence, we get that $\tilde{\omega}_n$ converges strongly to some $\tilde{\omega} \in \overline{\mathcal{H}}^{q-1} \subset \big(L^2\Omega^k(M)\big)^{q-1}$.
(2) Else, after passing to a subsequence, $\tilde{\omega}_n$ converges to $0 \in \overline{\mathcal{H}}^{q-1} \subset \big(L^2\Omega^k(M)\big)^{q-1}$.

In both cases $\tilde{\omega}_n$ converges to an element $\tilde{\omega} \in \overline{\mathcal{H}}^{q-1} \subset \big(L^2\Omega^k(M)\big)^{q-1}$, hence ω_n converges to $(\tilde{\omega}, c\,\omega^q) \in \overline{\mathcal{H}}^q \subset \big(L^2\Omega^k(M)\big)^q$. This finishes the proof. □

2.2. **The proof of Proposition 2.1.** We argue by contradiction. If there exists no $C > 0$ such that (16) holds, then for each $n \in \mathbb{N}$ we can find $\alpha_n \in \Omega^1(\mathcal{F}, T\mathcal{F}^\perp)$ such that

$$\|\alpha_n\|_g > n\left(\|d_\nabla \alpha_n\|_g + \|\delta_\nabla \alpha_n\|_g\right).$$

Dividing α_n by $\|\alpha_n\|_g$, we can assume that $\|\alpha_n\|_g = 1$. Consequently, (α_n) satisfies

$$\|d_\nabla \alpha_n\|_g \longrightarrow 0, \quad \|\delta_\nabla \alpha_n\|_g \longrightarrow 0, \quad \|\alpha_n\|_g = 1. \tag{21}$$

Step 1: Pass to a convergent subsequence of (α_n).

First, we lift α_n to the transverse orthogonal frame bundle $\widehat{M}$ via the isometric embedding

$$\pi^* : \left(\Omega^1(\mathcal{F}, T\mathcal{F}^\perp), \langle\cdot,\cdot\rangle_g\right) \hookrightarrow \left(\Omega^1(\widehat{\mathcal{F}}, \pi^* T\mathcal{F}^\perp), \langle\cdot,\cdot\rangle_{\widehat{g}^*}\right)$$

from Cor. 1.4. Setting $\beta_n := \pi^*(\alpha_n)$, we obtain a sequence which by Lemma 1.11 satisfies

$$\|d_{\pi^*\nabla} \beta_n\|_{\widehat{g}^*} \longrightarrow 0, \quad \|\delta_{\pi^*\nabla} \beta_n\|_{\widehat{g}^*} \longrightarrow 0, \quad \|\beta_n\|_{\widehat{g}^*} = 1. \tag{22}$$

Cor. 1.9 gives an identification between $\Omega^1(\widehat{\mathcal{F}}, \pi^* T\mathcal{F}^\perp)$ and $\Omega^1(\widehat{\mathcal{F}})^q$ under which $d_{\pi^*\nabla}$ and $\delta_{\pi^*\nabla}$ correspond with $d_{\widehat{\mathcal{F}}}^{\times q}$ and $\delta_{\widehat{\mathcal{F}}}^{\times q}$, and so that $\Omega^1(\widehat{\mathcal{F}})^q$ carries the direct sum inner product induced by $\langle\cdot,\cdot\rangle_{\widehat{g}}$. Hence, we can regard the β_n as vectors $(\beta_n^1, \dots, \beta_n^q) \in \Omega^1(\widehat{\mathcal{F}})^q$ satisfying

$$\|d_{\widehat{\mathcal{F}}}^{\times q} \beta_n\|_{\widehat{g}} \longrightarrow 0, \quad \|\delta_{\widehat{\mathcal{F}}}^{\times q} \beta_n\|_{\widehat{g}} \longrightarrow 0, \quad \|\beta_n\|_{\widehat{g}} = 1. \tag{23}$$

Extending the β_n^j by zero on $T\widehat{\mathcal{F}}^\perp$, we can view them as one-forms on $\widehat{M}$.

We now want to pass to a convergent subsequence of (β_n), using Cor. 2.5. Let (h_n) be a sequence of positive real numbers, left unspecified for now. We compute

$$\left\langle \Delta_{h_n}^{\times q} \beta_n, \beta_n \right\rangle_{\widehat{g}} = \sum_{i=1}^q \langle \Delta_{h_n} \beta_n^i, \beta_n^i \rangle_{\widehat{g}} = \sum_{i=1}^q \left\| D_{h_n} \beta_n^i \right\|_{\widehat{g}}^2. \tag{24}$$

Recalling the formulas (18),(19) and using the triangle inequality, (24) is bounded by

$$\sum_{i=1}^q \left(\left\|(d_{0,1} + \delta_{0,-1})\beta_n^i\right\|_{\widehat{g}} + h_n \left\|(d_{1,0} + \delta_{-1,0})\beta_n^i\right\|_{\widehat{g}} + h_n^2 \left\|(d_{2,-1} + \delta_{-2,1})\beta_n^i\right\|_{\widehat{g}} \right)^2. \tag{25}$$

Since $d_{2,-1}$ and $\delta_{-2,1}$ are differential operators of order zero [1, Lemma 1.1], we have that

$$\left\langle (d_{2,-1} + \delta_{-2,1})\beta_n^i, (d_{2,-1} + \delta_{-2,1})\beta_n^i \right\rangle_{\widehat{g}} \le C \langle \beta_n^i, \beta_n^i \rangle_{\widehat{g}} \le C,$$

for some $C > 0$. In the last inequality, we used that $\|\beta_n^i\|_{\widehat{g}} \le \|\beta_n\|_{\widehat{g}} = 1$. We also recall that

$$\left\|(d_{0,1} + \delta_{0,-1})\beta_n^i\right\|_{\widehat{g}} \longrightarrow 0$$

by (23). Hence, to make sure that (25) goes to zero, it suffices to pick (h_n) such that

$$h_n \longrightarrow 0 \qquad \text{and} \qquad h_n \left\|(d_{1,0} + \delta_{-1,0})\beta_n^i\right\|_{\widehat{g}} \longrightarrow 0 \quad \text{for } i = 1, \dots, q.$$

Setting for instance

$$h_n := \frac{1}{n(1 + \max_{i \in \{1,\dots,q\}} \left\|(d_{1,0} + \delta_{-1,0})\beta_n^i\right\|_{\widehat{g}})},$$

these requirements are satisfied, and therefore we have

$$\left\langle \Delta_{h_n}^{\times q} \beta_n, \beta_n \right\rangle_{\widehat{g}} \longrightarrow 0 \qquad \text{and} \qquad h_n \longrightarrow 0.$$

Hence, we can apply Cor. 2.5. Denoting $\mathcal{H} = \ker \Delta_0$ and passing to a subsequence of (β_n), we obtain an element $\beta \in \overline{\mathcal{H}}^q \subset \big(L^2\Omega^1(\widehat{M})\big)^q$ such that

$$\beta_n \longrightarrow \beta \;\text{ in }\; \Big(\big(L^2\Omega^1(\widehat{M})\big)^q, \|\cdot\|_{\widehat{g}}\Big).$$

Next, we consider the leafwise Laplacian $\Delta_{\widehat{\mathcal{F}}}^{\times q} = d_{\widehat{\mathcal{F}}}^{\times q}\delta_{\widehat{\mathcal{F}}}^{\times q} + \delta_{\widehat{\mathcal{F}}}^{\times q} d_{\widehat{\mathcal{F}}}^{\times q}$ on the space $\Omega^1(\widehat{\mathcal{F}})^q$, the completion of which is its closure $\big(L^2\Omega^1(\widehat{\mathcal{F}})\big)^q$ in $\big(L^2\Omega^1(\widehat{M})\big)^q$. Since $\beta_n \to \beta$, it is clear that $\beta \in \big(L^2\Omega^1(\widehat{\mathcal{F}})\big)^q$. We claim that actually

$$\beta \in \overline{\ker \Delta_{\widehat{\mathcal{F}}}^{\times q}} \subset \big(L^2\Omega^1(\widehat{\mathcal{F}})\big)^q. \tag{26}$$

To see this, recall that both summands in $T^*\widehat{M} = T^*\widehat{\mathcal{F}} \oplus (T\widehat{\mathcal{F}}^\perp)^*$ are orthogonal with respect to $\langle\cdot,\cdot\rangle_{\widehat{g}}$. Hence, the restriction map $r : \Big(\Omega^1(\widehat{M})^q, \|\cdot\|_{\widehat{g}}\Big) \to \Big(\Omega^1(\widehat{\mathcal{F}})^q, \|\cdot\|_{\widehat{g}}\Big)$ is bounded, and therefore it extends to a bounded linear operator between the completions

$$r : \Big(\big(L^2\Omega^1(\widehat{M})\big)^q, \|\cdot\|_{\widehat{g}}\Big) \to \Big(\big(L^2\Omega^1(\widehat{\mathcal{F}})\big)^q, \|\cdot\|_{\widehat{g}}\Big)$$

Now, because $\beta \in \overline{\mathcal{H}}^q$ there exists a sequence $\gamma_n \in \mathcal{H}^q$ such that $\gamma_n \to \beta$, and hence

$$r(\gamma_n) \longrightarrow \beta \;\text{ in }\; \Big(\big(L^2\Omega^1(\widehat{\mathcal{F}})\big)^q, \|\cdot\|_{\widehat{g}}\Big). \tag{27}$$

The Laplacian Δ_0 preserves bi-degrees and it agrees with $\Delta_{\widehat{\mathcal{F}}}$ on $\Omega^{0,\bullet}(\widehat{M}) = \Omega^\bullet(\widehat{\mathcal{F}})$, hence

$$r \circ \Delta_0^{\times q} = \Delta_{\widehat{\mathcal{F}}}^{\times q} \circ r : \Omega^1(\widehat{M})^q \to \Omega^1(\widehat{\mathcal{F}})^q.$$

Therefore, the fact that $\gamma_n \in \ker \Delta_0^{\times q}$ implies that $r(\gamma_n) \in \ker \Delta_{\widehat{\mathcal{F}}}^{\times q}$, hence $\beta \in \overline{\ker \Delta_{\widehat{\mathcal{F}}}^{\times q}}$.

Having now established (26), we transport the convergence statement $\beta_n \to \beta$ back under the isometric isomorphism $\Omega^1(\widehat{\mathcal{F}})^q \cong \Omega^1(\widehat{\mathcal{F}}, \pi^*T\mathcal{F}^\perp)$ from Cor. 1.9, or rather its continuous extension to the L^2-completions. Setting $\Delta_{\pi^*\nabla} := d_{\pi^*\nabla}\delta_{\pi^*\nabla} + \delta_{\pi^*\nabla}d_{\pi^*\nabla}$, we obtain that

$$\beta_n \longrightarrow \beta \;\in \overline{\ker \Delta_{\pi^*\nabla}} \subset L^2\Omega^1(\widehat{\mathcal{F}}, \pi^*T\mathcal{F}^\perp). \tag{28}$$

At last, recall that $\beta_n = \pi^*(\alpha_n)$, and that by Remark 1.5,

$$\pi^* : \big(L^2\Omega^1(\mathcal{F}, T\mathcal{F}^\perp), \langle\cdot,\cdot\rangle_g\big) \hookrightarrow \big(L^2\Omega^1(\widehat{\mathcal{F}}, \pi^*T\mathcal{F}^\perp), \langle\cdot,\cdot\rangle_{\widehat{g}^*}\big)$$

is an isometric embedding of Hilbert spaces. In particular, its image is closed. Hence, there exists some uniquely determined $\alpha \in L^2\Omega^1(\mathcal{F}, T\mathcal{F}^\perp)$ with $\beta = \pi^*(\alpha)$, and (28) implies

$$\alpha_n \longrightarrow \alpha \;\text{ in }\; L^2\Omega^1(\mathcal{F}, T\mathcal{F}^\perp). \tag{29}$$

Step 2: Derive a contradiction using the Hodge decomposition.

We first claim that $\alpha \in \overline{\ker \Delta_\nabla} \subset L^2\Omega^1(\mathcal{F}, T\mathcal{F}^\perp)$. Note that since $\beta = \pi^*(\alpha) \in \overline{\ker \Delta_{\pi^*\nabla}}$ there exists a sequence of harmonic elements $\gamma_n \in \ker \Delta_{\pi^*\nabla}$ such that

$$\gamma_n \longrightarrow \pi^*(\alpha) \;\text{ in }\; L^2\Omega^1(\widehat{\mathcal{F}}, \pi^*T\mathcal{F}^\perp).$$

Since Av is continuous and $Av(\pi^*\alpha) = \pi^*(\alpha)$ (see Cor. 1.16 (2) & (3)), this implies that

$$Av(\gamma_n) \longrightarrow \pi^*(\alpha) \;\text{ in }\; L^2\Omega^1(\widehat{\mathcal{F}}, \pi^*T\mathcal{F}^\perp). \tag{30}$$

Because $Av(\gamma_n)$ is $O(q)$-invariant, there are unique $\eta_n \in \Omega^1(\mathcal{F}, T\mathcal{F}^\perp)$ with

$$Av(\gamma_n) = \pi^*(\eta_n) \tag{31}$$

Therefore, the convergence (30) together with the fact that π^* is an isometric implies that

$$\eta_n \longrightarrow \alpha \text{ in } L^2\Omega^1(\mathcal{F}, T\mathcal{F}^\perp). \quad (32)$$

Next, since Av commutes with $\Delta_{\pi^*\nabla}$ by Cor. 1.16 (1) and $\gamma_n \in \ker \Delta_{\pi^*\nabla}$, we also have $Av(\gamma_n) \in \ker \Delta_{\pi^*\nabla}$. Moreover, according to Lemma 1.11, we know that

$$\pi^* \circ \Delta_\nabla = \Delta_{\pi^*\nabla} \circ \pi^*.$$

Hence, applying $\Delta_{\pi^*\nabla}$ to both sides in (31) and using that π^* is injective, we get that

$$\Delta_\nabla \eta_n = 0.$$

Hence, (32) implies that $\alpha \in \overline{\ker \Delta_\nabla} \subset L^2\Omega^1(\mathcal{F}, T\mathcal{F}^\perp)$, as claimed.

Finally, our assumption that $H^1(\mathcal{F}, T\mathcal{F}^\perp) = 0$ implies that also $\overline{H}^1(\mathcal{F}, T\mathcal{F}^\perp) = 0$. Hence, Thm. 2.2 ensures that $\ker \Delta_\nabla$ is trivial, and therefore $\alpha = 0$. In conclusion, we constructed a sequence (α_n) in $\Omega^1(\mathcal{F}, T\mathcal{F}^\perp)$ which by (29) and (21) satisfies

$$\alpha_n \longrightarrow 0 \qquad \text{and} \qquad \|\alpha_n\|_g = 1.$$

This is impossible, hence our initial assumption is wrong. This shows that there does exist a constant $C > 0$ so that the estimate (16) is satisfied, which finishes the proof. □

Remark 2.6. In the above proof, we actually only needed that $\overline{H}^1(\mathcal{F}, N\mathcal{F}) = 0$, and this has a remarkable consequence. It shows that for a Riemannian foliation $\mathcal{F}$ on a compact manifold M, the vanishing of $\overline{H}^1(\mathcal{F}, N\mathcal{F})$ is equivalent with the vanishing of $H^1(\mathcal{F}, N\mathcal{F})$. Indeed, one implication being obvious, assume that $\overline{H}^1(\mathcal{F}, N\mathcal{F})$ vanishes. Then the estimate (16) is satisfied by the proof we just gave, and Hamilton's Stability Theorem (stated in the introduction) implies that $H^1(\mathcal{F}, N\mathcal{F})$ vanishes.

Combining Prop. 2.1 with Hamilton's Stability Theorem yields our Main Theorem.

Main Theorem. *Let M be a compact manifold and $\mathcal{F}$ a Riemannian foliation on M such that $H^1(\mathcal{F}, N\mathcal{F}) = 0$. Then $\mathcal{F}$ is stable.*

We didn't manage to find examples for this result other than the Hausdorff foliations appearing in the Global Reeb-Thurston stability theorem by Hamilton. In fact, there are results indicating that such examples may not be easy to find:

Remark 2.7. Ghys [13] shows that any Riemannian foliation on a compact, simply connected manifold admits an arbitrarily C^0-close foliation by compact leaves. This result has been generalized by Caramello Jr. and Töben [5] to Killing foliations on compact manifolds. It is not clear to the authors whether these approximations can be done in the C^∞-topology. This would imply that stable Killing foliations are necessarily Hausdorff.

In the next section, we show that the Main Theorem cannot yield new examples of stable foliations when $\mathcal{F}$ is one-dimensional and orientable, or when $\mathcal{F}$ has codimension 1 and is co-orientable.

3. Remarks about the scope of the Main Theorem

In this section, we collect some final remarks concerning the scope of our Main Theorem. We show that our result does not cover all known instances of stable foliations, by recalling a class of stable non-Riemannian foliations. We also look at the special case in which $\mathcal{F}$ is oriented of dimension 1, and the case in which $\mathcal{F}$ is co-oriented of codimension 1. We show that in the former case the assumptions of our Main Theorem are never satisfied. In the latter case the assumptions can be satisfied, but they force $\mathcal{F}$ to be Hausdorff.

3.1. **Stable non-Riemannian foliations.** We display a class of non-Riemannian foliations that were shown to be stable in [9], extending a result from [14]. The foliations in question are constructed by suspending certain foliations on tori with an Anosov diffeomorphism.

Example 3.1. [9] We pick a matrix $A \in SL(n, \mathbb{Z})$, where $n \geq 2$, which is diagonalizable over $\mathbb{R}$ with positive eigenvalues. Denote the eigenvalues by

$$\mu_1, \ldots, \mu_p, \lambda_1, \ldots, \lambda_q,$$

where $p + q = n$. We view A as a diffeomorphism of the torus $\mathbb{T}^n$. Pick independent linear vector fields $X_1, \ldots, X_p, Y_1, \ldots, Y_q \in \mathfrak{X}(\mathbb{T}^n)$ such that[2]

$$\begin{cases} A_* X_j = \mu_j X_j, \\ A_* Y_k = \lambda_k Y_k. \end{cases}$$

The foliation $\text{Span}\{X_1, \ldots, X_p\}$ on $\mathbb{T}^n$ is invariant under A, hence the product foliation $\text{Span}\{X_1, \ldots, X_p, \partial_t\}$ on $\mathbb{T}^n \times \mathbb{R}$ descends to a foliation $\mathcal{F}$ on the mapping torus

$$\mathbb{T}_A := \frac{\mathbb{T}^n \times \mathbb{R}}{(\theta, t) \sim (A(\theta), t + 1)}. \tag{33}$$

Assume moreover that the matrix A satisfies the following two conditions:

(1) The eigenvalues λ_k and the quotients λ_k/μ_j are different from 1.
(2) There is a basis of $\mathbb{R}^n$ given by eigenvectors $v_1, \ldots, v_p, w_1, \ldots, w_q$ of A (corresponding respectively to the eigenvalues $\mu_1, \ldots, \mu_p, \lambda_1, \ldots, \lambda_q$) with the property that for any $i = 1, \ldots, p$, the coordinates $v_i^1, \ldots, v_i^n$ of v_i are linearly independent over $\mathbb{Q}$.

Under these assumptions, also the eigenvalues $\mu_1, \ldots, \mu_p$ are different from 1, and therefore A is an Anosov diffeomorphism of $\mathbb{T}^n$. If the eigenvalues of A are all different, then condition (2) is satisfied exactly when the characteristic polynomial of A is irreducible over $\mathbb{Q}$.

If (1) and (2) are satisfied, then the proof of [9, Thm. 2.2] gives tame homotopy operators for the complex $\big(\Omega^\bullet(\mathcal{F}, N\mathcal{F}), d_\nabla\big)$. It follows that $\mathcal{F}$ is stable (using Nash-Moser).

Claim: The foliation $\mathcal{F}$ is not Riemannian.

Let $\{\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q\}$ be the dual frame of $\{X_1, \ldots, X_p, Y_1, \ldots, Y_q\}$. Then

$$\big\{dt, \mu_1^{-t}\alpha_1, \ldots, \mu_p^{-t}\alpha_p, \lambda_1^{-t}\beta_1, \ldots, \lambda_q^{-t}\beta_q\big\} \tag{34}$$

is a frame for $T^*\mathbb{T}_A$; these are indeed one-forms on $\mathbb{T}^n \times \mathbb{R}$ invariant under the identification in (33). If g would be a bundle-like metric on $(\mathbb{T}_A, \mathcal{F})$, then the bilinear form g^T given by

$$g^T(Z_1, Z_2) := g\big(\text{pr}_{T\mathcal{F}^\perp} Z_1, \text{pr}_{T\mathcal{F}^\perp} Z_2\big), \qquad \forall Z_1, Z_2 \in \mathfrak{X}(\mathbb{T}_A)$$

would be a transverse metric, i.e.

$$\ker g^T = T\mathcal{F}, \qquad \text{and} \qquad \pounds_Z g^T = 0 \qquad \text{for all } \ Z \in \Gamma(T\mathcal{F}). \tag{35}$$

Expressing g^T in the frame (34) gives an equation of the form

$$g^T = \sum_{i,j} f_{ij}(\theta, t)\lambda_i^{-t}\lambda_j^{-t}\beta_i \otimes \beta_j,$$

where f_{ij} satisfies

$$f_{ij}\big(A(\theta), t + 1\big) = f(\theta, t). \tag{36}$$

[2]Put differently, X_j (resp. Y_k) is a vector field whose coefficients are constant, given by the components of an eigenvector of A for the eigenvalue μ_j (resp. λ_k).

We now assert that, if f_{ij} is not identically zero, then $\lambda_i\lambda_j = 1$. Since $\pounds_{\partial_t} g^T$ vanishes by assumption (35), we obtain the identity

$$f_{ij}\big(A(\theta),t\big)\lambda_i^{-t}\lambda_j^{-t} = f_{ij}\big(A(\theta),t+1\big)\lambda_i^{-t-1}\lambda_j^{-t-1} = f_{ij}(\theta,t)\lambda_i^{-t-1}\lambda_j^{-t-1},$$

where we also used (36). This equality implies that

$$\lambda_i\lambda_j f_{ij}\big(A(\theta),t\big) = f_{ij}(\theta,t). \tag{37}$$

Now take a rational point $\theta := (\theta_1,\dots,\theta_n) \in \mathbb{Q}^n/\mathbb{Z}^n$. We claim that θ is periodic for A. Indeed, $\theta = (p_1/q,\dots,p_n/q)$ for some $0 \le p_j < q$, and applying iterates of A yields

$$A^m(\theta) \in \big\{(q_1/q,\dots,q_n/q):\ 0 \le q_j < q\big\} \qquad \text{for any } m \in \mathbb{N},$$

which is a finite set. So (37) implies that for $(\theta,t) \in \mathbb{Q}^n/\mathbb{Z}^n \times \mathbb{R}$, there exists $k \in \mathbb{N}$ with

$$(\lambda_i\lambda_j)^k f_{ij}(\theta,t) = f_{ij}(\theta,t). \tag{38}$$

If f_{ij} is not identically zero, then there exists $(\theta,t) \in \mathbb{Q}^n/\mathbb{Z}^n \times \mathbb{R}$ with $f_{ij}(\theta,t) \neq 0$. Hence, (38) along with the fact that all eigenvalues are positive, yields that $\lambda_i\lambda_j = 1$.

Now, evaluating g^T on the vector fields $\lambda_1^t Y_1,\dots,\lambda_q^t Y_q \in \mathfrak{X}(\mathbb{T}_A)$ gives

$$f_{ii} = g^T\big(\lambda_i^t Y_i, \lambda_i^t Y_i\big) = g\big(\mathrm{pr}_{T\mathcal{F}^\perp}(\lambda_i^t Y_i), \mathrm{pr}_{T\mathcal{F}^\perp}(\lambda_i^t Y_i)\big) > 0.$$

Hence, the assertion just proved implies that $\lambda_i^2 = 1$, and since the eigenvalues are positive this gives $\lambda_i = 1$. This contradicts the assumption (1) above, hence $\mathcal{F}$ is not Riemannian.

A simple example of the type described above is the following. Let $A \in SL(2,\mathbb{Z})$ be any hyperbolic matrix, i.e. without eigenvalues on the unit circle. This implies that the eigenvalues are real, and by the rational root theorem they must be irrational. We assume that they are positive, so they are given by

$$\lambda > 1 > \lambda^{-1} > 0.$$

Since $\lambda, \lambda^{-1} \in \mathbb{R} \setminus \mathbb{Q}$, the characteristic polynomial of A is irreducible over $\mathbb{Q}$, and the assumptions (1) and (2) of [9] are satisfied. Hence, picking a linear vector field X on $\mathbb{T}^2$ defined by an eigenvector of A, the associated suspension foliation $\mathcal{F}$ on $\mathbb{T}_A$ is stable. This stability result was first obtained in [14], by purely geometric arguments. A common choice of matrix A is

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \in SL(2,\mathbb{Z}).$$

3.2. **Dimension one.** We consider the particular case of one-dimensional tangentially oriented Riemannian foliations, i.e. Riemannian flows. Examples are suspensions of isometries. We show that such foliations never satisfy the assumptions of our Main Theorem.

We will make use of the fact that there is a normal form for Riemannian flows on compact manifolds due to Carrière [6, II.C Proposition 3]. We first recall this result. Let M^{n+1} be a compact manifold with a Riemannian flow $\mathcal{F}$. Fix a leaf L and denote by $\overline{L}$ its closure. There exists a saturated neighborhood V of $\overline{L}$ such that:

1. V is diffeomorphic to $S^1 \times \mathbb{T}^k \times D^{n-k}$, through a diffeomorphism sending $\overline{L}$ to $S^1 \times \mathbb{T}^k \times \{0\}$. Here D^{n-k} is the unit ball around the origin in $\mathbb{R}^{n-k}$.
2. The flow $\mathcal{F}$ restricted to V is conjugated to the flow obtained by suspension of a diffeomorphism γ of $\mathbb{T}^k \times D^{n-k}$ of the form
$$\gamma(x,y) = \big(R(x), A(y)\big)$$
where R is an irrational translation of $\mathbb{T}^k$ and A is a rotation of $\mathbb{R}^{n-k}$.

Proposition 3.2. *Let M^{n+1} be a compact manifold and $\mathcal{F}$ a Riemannian flow on M. Then the reduced cohomology group $\overline{H}^1(\mathcal{F}, N\mathcal{F})$ is nonzero.*

Proof. First consider the case in which all leaves of $\mathcal{F}$ are circles. If $\overline{H}^1(\mathcal{F}, N\mathcal{F})$ would vanish, then $\mathcal{F}$ would be stable by our Main Theorem and Remark 2.6. However, it is known that Hausdorff foliations whose generic leaf has nonzero first de Rham cohomology group are not stable [17],[8]. Hence, $\overline{H}^1(\mathcal{F}, N\mathcal{F})$ must be nonzero.

In the rest of the proof, we assume that $\mathcal{F}$ has a non-closed leaf. Let $\mathcal{F}$ be generated by the vector field $X \in \mathfrak{X}(M)$. Since $\mathcal{F}$ is oriented, we can use the duality result [3, Cor. C] which states that $\overline{H}^1(\mathcal{F}, N\mathcal{F}) \cong \overline{H}^0(\mathcal{F}, N^*\mathcal{F})$. Now note that

$$\overline{H}^0(\mathcal{F}, N^*\mathcal{F}) = H^0(\mathcal{F}, N^*\mathcal{F}) = \{\alpha \in \Gamma(N^*\mathcal{F}) : \pounds_X \alpha = 0\} = \Omega^1_{bas}(\mathcal{F}),$$

hence it suffices to construct a non-zero basic one-form on $(M, \mathcal{F})$. To do so, we use the Carrière normal form recalled above. Fix a non-closed leaf L and consider the local model around $\overline{L}$, that is

$$V \cong S^1 \times \mathbb{T}^k \times D^{n-k}$$

with $k > 0$. It follows that the space of basic one-forms $\Omega^1_{bas}(V)$ can be identified with the space $\Omega^1_{inv}(\mathbb{T}^k \times D^{n-k})$ consisting of γ-invariant one-forms on $\mathbb{T}^k \times D^{n-k}$. There exists a non-zero γ-invariant one-form on $\mathbb{T}^k \times D^{n-k}$. For instance, let φ denote one of the angle coordinates on $\mathbb{T}^k$ and consider $d\varphi \in \Omega^1(\mathbb{T}^k \times D^{n-k})$. This form is γ-invariant because $d\varphi$ is invariant under translations of $\mathbb{T}^k$. Hence, there exists a nonzero form $\alpha \in \Omega^1_{bas}(\mathcal{F}|_V)$.

It remains to extend α as a basic one-form on $(M, \mathcal{F})$. In case $\overline{L} = M$, then $V = M$ hence there is nothing to prove. If $\overline{L} \neq M$ then we have an open cover $\{V, M \setminus \overline{L}\}$ consisting of saturated open subsets. We can take a partition of unity $\{\rho, \sigma\}$ subordinate to this cover, such that ρ and σ are basic functions [4, Lemma 2.2]. Then $\rho\alpha \in \Omega^1_{bas}(\mathcal{F})$ is a non-zero basic one-form defined on all of M. This finishes the proof. □

3.3. **Codimension one.** In this subsection, we consider compact connected manifolds M equipped with a foliation $\mathcal{F}$ defined by a closed one-form $\theta \in \Omega^1(M)$. It is well-known that either $\mathcal{F}$ is given by the fibers of a fibration over S^1 or all leaves of $\mathcal{F}$ are dense [7, § 9.3], depending on the group of periods

$$P(\theta) = \left\{ \int_\sigma \theta : \ [\sigma] \in H_1(M) \right\}.$$

Since $P(\theta)$ is a subgroup of $(\mathbb{R}, +)$, it is either discrete or dense. In the former case, $\mathcal{F}$ is given by a fibration over S^1, and in the latter case all leaves of $\mathcal{F}$ are dense.

Lemma 3.3. *Let M be a compact, connected manifold and $\mathcal{F}$ a foliation on M defined by a closed one-form $\theta \in \Omega^1(M)$. If $H^1(\mathcal{F}) = 0$, then $\mathcal{F}$ is given by a fibration over S^1.*

Proof. Assume by contradiction that the leaves of $\mathcal{F}$ are dense. Consider the short exact sequence of complexes

$$0 \longrightarrow (\Omega^\bullet_{\mathcal{F}}(M), d) \longrightarrow \big(\Omega^\bullet(M), d\big) \xrightarrow{r} \big(\Omega^\bullet(\mathcal{F}), d_{\mathcal{F}}\big) \longrightarrow 0,$$

where $\Omega^\bullet_{\mathcal{F}}(M)$ consists of the forms which vanish when pulled back to the leaves of $\mathcal{F}$. We get a long exact sequence in cohomology

$$\cdots \longrightarrow H^1_{\mathcal{F}}(M) \longrightarrow H^1(M) \longrightarrow H^1(\mathcal{F}) \longrightarrow \cdots$$

Since $H^1(\mathcal{F})$ vanishes by assumption, we get that the map $H^1_{\mathcal{F}}(M) \to H^1(M)$ is surjective.

We claim that $H^1_{\mathcal{F}}(M) = \mathbb{R}[\theta]$, which then implies that also $H^1(M) = \mathbb{R}[\theta]$. To prove the claim, note that any element $\alpha \in \Omega^1_{\mathcal{F}}(M)$ is of the form $\alpha = f\theta$ for some $f \in C^\infty(M)$. If α is moreover closed, then $df \wedge \theta = 0$, hence f is leafwise constant. Since the leaves of $\mathcal{F}$ are dense by assumption, f must be constant. This confirms that $H^1(M) = H^1_{\mathcal{F}}(M) = \mathbb{R}[\theta]$.

A theorem by Tischler [22] states that M is a fiber bundle over S^1. Let us denote the projection by $\pi : M \to S^1$, and the angle form on S^1 by $d\varphi \in \Omega^1(S^1)$. Since $H^1(M) = \mathbb{R}[\theta]$, we know that $[\theta] = c[\pi^* d\varphi]$ for some non-zero constant $c \in \mathbb{R}$. But this implies that the period group satisfies $P(\theta) = cP(\pi^* d\varphi)$. Since $P(\pi^* d\varphi)$ is discrete, it follows that also $P(\theta)$ is discrete. This is impossible since the leaves of $\mathcal{F}$ are assumed to be dense. □

As a consequence, we obtain that codimension one co-orientable Riemannian foliations satisfying the assumptions of our Main Theorem are necessarily Hausdorff. Hence in this situation, all instances of stable foliations obtained via our Main Theorem are already covered by Hamilton's global Reeb-Thurston stability result.

Corollary 3.4. *Let M be a compact, connected manifold and $\mathcal{F}$ a co-orientable Riemannian foliation of codimension one. If $H^1(\mathcal{F}, N\mathcal{F})$ vanishes, then $\mathcal{F}$ is Hausdorff.*

Proof. A codimension one co-orientable Riemannian foliation $\mathcal{F}$ has a closed defining one-form $\theta \in \Omega^1(M)$ by [23, Thm. 7.3 (ii)]. A vector field $Z \in \mathfrak{X}(M)$ satisfying $\theta(Z) = 1$ trivializes $N\mathcal{F}$ as a $T\mathcal{F}$-representation. Consequently, $H^1(\mathcal{F}, N\mathcal{F}) \cong H^1(\mathcal{F})$. Applying Lemma 3.3 yields the conclusion. □

4. Appendix

In [15], Hamilton considers foliated manifolds $(M, \mathcal{F})$ admitting what he calls a "holonomy invariant metric". We show that such a metric coincides with the classical notion of "bundle-like metric" [20]. Hence, in modern language, Hamilton requires $\mathcal{F}$ to be Riemannian [19].

Definition 4.1 (Hamilton [15])**.** Given a foliated manifold $(M, \mathcal{F})$, a Riemannian metric g on M is *holonomy invariant* if for any open $U \subset M$ and all basic functions $f, h \in C^\infty_{bas}(U)$, the function $(df, dh)_g$ is again basic.

In Def. 4.1 above, we consider g as a metric on the cotangent bundle T^*M via

$$(\alpha, \beta)_g := g(\alpha^\sharp, \beta^\sharp)$$

for all $\alpha, \beta \in \Omega^1(M)$. Here the vector field $\gamma^\sharp$ corresponding with a one-form γ is defined by requiring that $g(\gamma^\sharp, X) = \gamma(X)$ for all $X \in \mathfrak{X}(M)$.

Remark 4.2. In a foliated chart $(x_1, \ldots, x_k, y_1, \ldots, y_l)$ such that $T\mathcal{F} = \mathrm{Span}\{\partial_{x_1}, \ldots, \partial_{x_k}\}$, holonomy invariance just means that $(dy_i, dy_j)_g$ only depends on the y-coordinates.

Definition 4.3 ([20],[19])**.** Given a foliated manifold $(M, \mathcal{F})$, a Riemannian metric g on M is *bundle-like* if for any open $U \subset M$ and all vector fields $Y, Z \in \mathfrak{X}(U)$ that are projectable and orthogonal to the leaves, the function $g(Y, Z)$ is basic on U. A foliation $\mathcal{F}$ on M is called *Riemannian* if $(M, \mathcal{F})$ admits a bundle-like metric.

Remark 4.4. i) It is instructive to check what the bundle-like condition means in local coordinates. Pick a foliated chart $(x_1, \ldots, x_k, y_1, \ldots, y_l)$ such that $T\mathcal{F} = \mathrm{Span}\{\partial_{x_1}, \ldots, \partial_{x_k}\}$. Decomposing ∂_{y_a} in the direct sum

$$TM = T\mathcal{F} \oplus T\mathcal{F}^\perp,$$

there exist unique functions f_a^i such that

$$V_a := \partial_{y_a} - \sum_{i=1}^{k} f_a^i \partial_{x_i} \in \Gamma(T\mathcal{F}^\perp). \tag{39}$$

Note that the vector fields V_a are projectable. Consequently, the Riemannian metric g is bundle-like exactly when $g(V_a, V_b)$ is basic for all $a, b \in \{1, \dots, l\}$.

ii) Put differently, a Riemannian metric g is bundle-like exactly when its restriction to $T\mathcal{F}^\perp \cong N\mathcal{F}$ is parallel for the Bott connection, i.e.

$$X(g(Y,Z)) = g\big(\mathrm{pr}_{T\mathcal{F}^\perp}[X,Y], Z\big) + g\big(Y, \mathrm{pr}_{T\mathcal{F}^\perp}[X,Z]\big) \tag{40}$$

for all $X \in \Gamma(T\mathcal{F})$ and $Y, Z \in \Gamma(T\mathcal{F}^\perp)$. Indeed, the condition (40) holds as soon as it holds for all Y, Z in a local frame for $T\mathcal{F}^\perp$. Such a frame can be chosen to consist of projectable vector fields (see part i) above), in which case the right hand side of (40) vanishes. It follows that (40) holds exactly when g is bundle-like.

Both Def. 4.1 and Def. 4.3 require that g locally induces a Riemannian metric on the leaf space $M/\mathcal{F}$. We include a detailed proof showing that the definitions are indeed equivalent.

Proposition 4.5. *Let $(M, \mathcal{F})$ be a foliated manifold and g a Riemannian metric on M. Then holonomy invariance of g is equivalent with g being bundle-like.*

Proof. First assume that g is holonomy invariant. If $(x_1, \dots, x_k, y_1, \dots, y_l)$ is a foliated chart such that $T\mathcal{F} = \mathrm{Span}\{\partial_{x_1}, \dots, \partial_{x_k}\}$, then clearly $\{dy_1^\sharp, \dots, dy_l^\sharp\}$ is a local frame for $T\mathcal{F}^\perp$. Moreover, $dy_i^\sharp$ is projectable because g is holonomy invariant. To see this, we have to check that for a locally defined basic function h, the function $dy_i^\sharp(h)$ is again basic. We have

$$dy_i^\sharp(h) = dh(dy_i^\sharp) = g\big(dh^\sharp, dy_i^\sharp\big) = (dh, dy_i)_g,$$

and the latter is indeed basic since h, y_i are basic and g is holonomy invariant. To check that g is bundle-like, we take locally defined projectable vector fields $Y, Z \in \Gamma(T\mathcal{F}^\perp)$, i.e.

$$Y = \sum_{i=1}^{l} f_i dy_i^\sharp \qquad \text{and} \qquad Z = \sum_{j=1}^{l} h_j dy_j^\sharp.$$

Since Y, Z and $dy_1^\sharp, \dots, dy_l^\sharp$ are projectable, the functions f_i, h_j are basic. Hence

$$g(Y,Z) = \sum_{i,j=1}^{l} f_i h_j g\big(dy_i^\sharp, dy_j^\sharp\big) = \sum_{i,j=1}^{l} f_i h_j (dy_i, dy_j)_g$$

is basic, because g is holonomy invariant. This proves that g is bundle-like.

Conversely, assume that g is bundle-like. By Remark 4.4 ii), we then know that g satisfies

$$X(g(Y,Z)) = g\big(\mathrm{pr}_{T\mathcal{F}^\perp}[X,Y], Z\big) + g\big(Y, \mathrm{pr}_{T\mathcal{F}^\perp}[X,Z]\big) \tag{41}$$

for $X \in \Gamma(T\mathcal{F})$ and $Y, Z \in \Gamma(T\mathcal{F}^\perp)$. To show that g is holonomy invariant, it suffices to take a foliated chart $(x_1, \dots, x_k, y_1, \dots, y_l)$ and check that $dy_i^\sharp$ is projectable. Indeed, then

$$(dy_i, dy_j)_g = g\big(dy_i^\sharp, dy_j^\sharp\big)$$

would be basic since g is bundle-like. To see that $dy_i^\sharp$ is projectable, we take a locally defined $X \in \Gamma(T\mathcal{F})$ and check that $[X, dy_i^\sharp] \in \Gamma(T\mathcal{F})$. The latter follows if we show that

$$g\big([X, dy_i^\sharp], V_j\big) = 0, \tag{42}$$

where $V_1, \dots, V_l$ are the vector fields defined in (39). Here we use that the V_j form a local frame for $T\mathcal{F}^\perp$. Using that the V_j are also projectable, (42) follows from the computation

$$g\big([X, dy_i^\sharp], V_j\big) = g\big(\mathrm{pr}_{T\mathcal{F}^\perp}[X, dy_i^\sharp], V_j\big) = X(g(dy_i^\sharp, V_j)) - g\big(dy_i^\sharp, \mathrm{pr}_{T\mathcal{F}^\perp}[X, V_j]\big) = 0.$$

Here the first equality uses that $V_j \in \Gamma(T\mathcal{F}^\perp)$, the second one uses (41) and the third one follows from $g(dy_i^\sharp, V_j) = \delta_{ij}$ and $[X, V_j] \in \Gamma(T\mathcal{F})$. Hence, g is holonomy invariant. □

References

[1] J. Álvarez López, *The basic component of the mean curvature of Riemannian foliations*, Ann. Glob. Anal. Geom. **10**(2), p. 179-194, 1992.

[2] J. Álvarez López and Y. Kordyukov, *Adiabatic limits and spectral sequences for Riemannian foliations*, Geom. Funct. Anal. **10**(5), p. 977-1027, 2000.

[3] J. Álvarez López and Y. Kordyukov, *Long Time Behavior of Leafwise Heat Flow for Riemannian Foliations*, Compos. Math. **125**(2), p. 129-153, 2001.

[4] V. Belfi, E. Park and K. Richardson, *A Hopf index theorem for foliations*, Diff. Geom. Appl. **18**(3), p. 319-341, 2003.

[5] F.C. Caramello Jr. and D. Töben, *Positively curved Killing foliations via deformations*, Trans. Amer. Math. Soc. **372**(11), p. 8131-8158, 2019.

[6] Y. Carrière, *Flots riemanniens*, Astérisque **116**, p. 31-52, 1984.

[7] L. Conlon, *Differentiable Manifolds: A First Course*, Birkhäuser Advanced Texts, Birkhäuser Boston, 1993.

[8] M. Del Hoyo and R.L. Fernandes, *On deformations of compact foliations*, Proc. Amer. Math. Soc. **147**(10), p. 4555-4561, 2019.

[9] A. El Kacimi Alaoui and M. Nicolau, *A class of C^∞-stable foliations*, Ergod. Theory Dyn. Syst. **13**(4), p. 697-704, 1993.

[10] D.B.A. Epstein, *Foliations with all leaves compact*, Ann. Inst. Fourier **26**(1), p. 265-282, 1976.

[11] D.B.A. Epstein, *A topology for the space of foliations*, Geometry and Topology, p. 132-150, Lecture Notes in Mathematics, Volume 597, Springer Berlin, 1977.

[12] D.B.A. Epstein and H. Rosenberg, *Stability of compact foliations*, Geometry and Topology, p. 151-160, Lecture Notes in Mathematics, Volume 597, Springer Berlin, 1977.

[13] E. Ghys, *Feuilletages riemanniens sur les variétés simplement connexes*, Ann. Inst. Fourier **34**(4), p. 203-223, 1984.

[14] E. Ghys and V. Sergiescu, *Stabilité et conjugaison différentiable pour certains feuilletages*, Topology **19**(2), p. 179-197, 1980.

[15] R. Hamilton, *Deformation theory of foliations*, unpublished, 1978.

[16] J.L. Heitsch, *A cohomology for foliated manifolds*, Comment. Math. Helv. **50**, p. 197-218, 1975.

[17] R. Langevin and H. Rosenberg, *On stability of compact leaves and fibrations*, Topology **16**(1), p. 107-111, 1977.

[18] I. Moerdijk and J. Mrčun, *Introduction to Foliations and Lie Groupoids*, Cambridge Studies in Advanced Mathematics, Volume 91, Cambridge University Press, 2003.

[19] P. Molino, *Riemannian foliations*, Progress in Mathematics, Volume 73, Birkhäuser Boston, 1988.

[20] B.L. Reinhart, *Foliated manifolds with bundle-like metrics*, Ann. of Math. **69**(1), p. 119-132, 1959.

[21] T. Sakai, *Riemannian Geometry*, Translations of Mathematical Monographs, Volume 149, American Mathematical Society, Providence RI, 1996.

[22] D. Tischler, *On fibering certain foliated manifolds over S^1*, Topology **9**(2), p. 153-154, 1970.

[23] P. Tondeur, *Foliations on Riemannian manifolds*, Universitext, Springer-Verlag New York, 1988.

University College London, Department of Mathematics, 25 Gordon Street, London WC1H 0AY, United Kingdom

Email address: `s.geudens@ucl.ac.uk`

University of Illinois at Urbana-Champaign, Department of Mathematics, 1409 W. Green Street, Urbana, IL 61801, USA

Email address: `fzeiser@illinois.edu`