Smooth and Proper Maps *

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To André Joyal, on his $(\infty,0)$-th Birthday

Abstract

This is an expository note explaining how the geometric notions of local connectedness and properness are related to the Σ-type and Π-type constructors of dependent type theory.

Contents

1 Introduction

This paper presents how the geometric notions of local connectedness and properness are related to the Σ-types and Π-types constructors of dependent type theory. The purpose is to underline a common structure, with the hope that the parallel will be beneficial to both fields. The style is mostly expository, the main results are proved in external references.

Dependent type theory is based on the notion of family of types indexed by a type, and the basic operations are the reindexing, the sums and the products of such families, which are assumed to always exist. On the other hand, the geometer's toolbox contains methods to study spaces by means of "bundles", that is using families of spaces indexed by the space of interest (vector bundles, sheaves. . .). There also, bundles can be pulled back along a morphism and sometimes pushed forward in two different ways (additively or multiplicatively). The fact that the pushforwards are not always defined is the source of an interesting feature. It distinguishes the classes of maps along which these pushforwards exists: locally connected and proper morphisms.

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In Section 2, we introduce an abstract notion of *smooth* and *proper maps* associated to any category $\mathbb{C}$. This is done in the setting of fibered/indexed categories over a base category $\mathcal{B}$. The smooth (proper) maps are the maps $u : X \to Y$ in $\mathcal{B}$ along which the base change functor $u^* : \mathbb{C}(Y) \to \mathbb{C}(X)$ admits a left adjoint $u_!$ (a right adjoint u_*) compatible with reindexing/base change (*aka* the Beck–Chevalley conditions). Then Section 3 details many examples in logic, category theory, topology, and geometry, where we show how our abstract definitions connect to existing notions of smoothness and properness.

Acknowledgments The authors are happy to dedicate this paper to André Joyal on the occasion of his 80th birthday. The focus on examples is an hommage to André, who has drilled their importance into the head of the first author.

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Conventions The paper is written in the context of ∞-categories [Lur09, Cis19], but we are simplifying the terminology and simply say "category" for ∞-category and "1-categories" for the truncated notion. We denote by Set (SET) the 1-category of small (large) sets, and by Cat (CAT) the (∞-)category of small (large) (∞-)categories. The categories of functors from C to D is denoted $[C, D]$. The arrow category of a category C is denoted $C^{\to}$.

2 Abstract setting

Let κ be a cardinal and $\mathsf{Set}^{<\kappa}$ be the category of sets of cardinality strictly smaller than κ. The elementary operations on sets are the sums and product of families. For I an arbitrary set, an I-indexed family (I-family for short) in $\mathsf{Set}^{<\kappa}$ is a functor $I \to \mathsf{Set}^{<\kappa}$, the sum and product of I-families are the left and right adjoint to the constant family functor $\mathsf{Set}^{<\kappa} \to [I, \mathsf{Set}^{<\kappa}]$. Given κ, one can ask for what sets I the sum and product of I-families of κ-small sets are κ-small. Let σ be a cardinal such that for any cardinal $\rho < \kappa$ we have $\sigma\rho < \kappa$. Then $\mathsf{Set}^{<\kappa}$ admits sums indexed by objects in $\mathsf{Set}^{<\sigma}$. Let $\Sigma(\kappa)$ be the supremum of all such cardinals σ. We shall call $\Sigma(\kappa)$ the *smooth bound* of κ. The cardinal κ is regular if and only if $\kappa = \Sigma(\kappa)$. Similarly, let π be a cardinal such that for any cardinal $\rho < \kappa$ we have $\rho^\pi < \kappa$. Then $\mathsf{Set}^{<\kappa}$ admits products indexed by objects in $\mathsf{Set}^{<\pi}$. Let $\Pi(\kappa)$ be the supremum of all such cardinals π. We shall call $\Pi(\kappa)$ the *proper bound* of κ. The cardinal κ is inaccessible if and only if $\kappa = \Pi(\kappa) = \Sigma(\kappa)$.

More generally, we can replace the category $\mathsf{Set}^{<\kappa}$ by any category C and extract the classes of sets $\Sigma(C)$ and $\Pi(C)$ indexing the sums and products which exist in C. We shall call $\Sigma(C)$ the *smooth calibration* of C and $\Pi(C)$ the *proper calibration* of C.[1]

We are going to propose an abstract setting for the definition of smooth and proper calibrations and illustrate it with many examples. The category C will be a fibration over some base category B with finite limits, and the calibrations $\Sigma(C)$ and $\Pi(C)$ will be defined as subfibrations of the codomain fibration of B.

2.1 Calibrations and families

We fix a category $\mathcal{B}$ with finite limits. The codomain functor $\mathcal{B}^{\to} \to \mathcal{B}$ is a cartesian fibration and we denote by $\mathbb{B} : \mathcal{B}^{\mathrm{op}} \to \mathsf{CAT}$ the corresponding functor (sending an object X to the slice category $\mathcal{B}_{/X}$). We shall refer to $\mathbb{B}$ as the *universe* of $\mathcal{B}$. Since $\mathcal{B}$ has a terminal object, the terminal functor $1 : \mathcal{B}^{\mathrm{op}} \to \mathsf{CAT}$, is a subfunctor of the universe $\mathbb{B}$. The corresponding fibration is the identity of $\mathcal{B}$. Equivalently, it corresponds to the subcodomain fibration $\mathcal{B} = \mathcal{B}^{\simeq} \subset \mathcal{B}^{\to}$ spanned by the isomorphisms.[2]

[1] The names *smooth* and *proper* are taken from Grothendieck in Pursuing Stacks [Mal05] and the example of left fibrations of categories. The name *calibration* is borrowed from Bénabou [Bén75], even if his notion is slightly different.

[2] We do not distinguish equivalent categories here.

Definition 2.1.1. A *calibration* is a subuniverse $\mathbb{U} \subset \mathbb{B}$ (i.e. a subfibration). The *constant calibration* is the calibration $1 \to \mathbb{B}$. A calibration is *pointed* if it contains the constant calibration (equivalently, if $\mathcal{U} \subset \mathcal{B}^{\to}$ contains all isomorphisms). A calibration $\mathbb{U} \subset \mathbb{B}$ is *regular* if it is pointed and the corresponding class of maps in $\mathcal{B}$ is closed under composition.

A regular calibration defines a wide subcategory of $\mathcal{B}$ (since it contains all isomorphisms). In fact, regular calibrations are in bijection with wide subcategories of $\mathcal{B}$ whose maps are closed under base change in $\mathcal{B}$.

Definition 2.1.2. Let $\mathcal{U} \subset \mathcal{B}^{\to} \xrightarrow{cod} \mathcal{B}$ be a subfibration of the codomain fibration. From $\mathcal{U}$ and $\mathcal{C}$, the fibration $\mathsf{Fam}_{\mathcal{U}}(\mathcal{C})$ of $\mathcal{U}$-*indexed families of objects in* $\mathcal{C}$ is defined as the functor ϕ in the diagram

$$\begin{array}{ccc} \mathsf{Fam}_{\mathcal{U}}(\mathcal{C}) & \longrightarrow & \mathcal{C} \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{U} & \xrightarrow{dom} & \mathcal{B} \\ \downarrow{\scriptstyle cod} & & \\ \mathcal{B} & & \end{array} \qquad \phi : \mathsf{Fam}_{\mathcal{U}}(\mathcal{C}) \to \mathcal{B}$$

(where the square is fiber product). We denote by $\mathsf{Fam}_{\mathbb{U}}(\mathbb{C})$ the associated functor. Its value at I in $\mathcal{B}$ is the category of pairs $(u : U \to I \in \mathcal{U}, C \in \mathbb{C}(U))$ (with the obvious notion of morphism).

2.2 Sums and products

The family construction associated to the constant calibration is the identity: $\mathbb{C} = \mathsf{Fam}_1(\mathbb{C})$. If $\mathbb{U}$ is a pointed calibration we get a canonical functor $\Delta_{\mathbb{U}} : \mathbb{C} = \mathsf{Fam}_1(\mathbb{C}) \to \mathsf{Fam}_{\mathbb{U}}(\mathbb{C})$.

Definition 2.2.1. Let $\mathbb{U}$ be a pointed calibration. A category $\mathbb{C}$ has $\mathbb{U}$-indexed sums (product) if the canonical functor $\Delta_{\mathbb{U}} : \mathbb{C} \to \mathsf{Fam}_{\mathbb{U}}(\mathbb{C})$ has a left (right) adjoint.

The following result is proved for 1-categories in [Str23]. The statement extends to ∞-categories as well. So do a range of expected results from [Str23, Moe82] about fibrations with internal sums. This has been developed by [BW23, Wei22a] in the type theory of synthetic ∞-categories Riehl–Shulman [RS17, Rie23a] (internally to *any* ∞-topos, see [Shu19, Rie23b, Wei22b]).

Proposition 2.2.2 (Bénabou–Streicher)**.** *If* $\mathbb{U}$ *is a regular calibration, then* $\mathsf{Fam}_{\mathbb{U}}(\mathbb{C})$ *is the free cocompletion of* $\mathbb{C}$ *for sums indexed by objects in* $\mathbb{U}$*. A pointed calibration* $\mathbb{U}$ *is regular if and only if* $\mathbb{U}$ *has* $\mathbb{U}$*-indexed sums.*

Remark 2.2.3. The notion of regular calibration corresponds to the notion of a subuniverse closed under Σ-types in dependent type theory.

2.3 Smooth and proper maps

Definition 2.3.1 (Smooth & proper maps)**.** We fix a functor $\mathbb{C} : \mathcal{B}^{\mathrm{op}} \to \mathsf{CAT}$. A map $u : X \to Y$ in $\mathcal{B}$ is called *left (right) Beck–Chevalley* for $\mathbb{C}$ if for any cartesian square

$$\begin{array}{ccc} \bar{Y} & \xrightarrow{\bar{v}} & Y \\ {\scriptstyle \bar{u}}\downarrow & \lrcorner & \downarrow{\scriptstyle u} \\ \bar{X} & \xrightarrow{v} & X \end{array}$$

the maps u^* and $\bar{u}^*$ have left (right) adjoints

$$\begin{array}{ccc} \mathbb{C}(\bar{Y}) & \xleftarrow{\bar{v}^*} & \mathbb{C}(Y) \\ {\scriptstyle \bar{u}_!}\downarrow\uparrow\downarrow{\scriptstyle \bar{u}_*} & & {\scriptstyle u_!}\downarrow\uparrow\downarrow{\scriptstyle u_*} \\ \mathbb{C}(\bar{X}) & \xleftarrow{v^*} & \mathbb{C}(X) \end{array}$$

and the corresponding mate natural transformation is invertible

$$\bar{u}_! \bar{v}^* \xrightarrow{\sim} v^* u_! \qquad \left(v^* u_* \xrightarrow{\sim} \bar{u}_* \bar{v}^* \right).$$

A map $u : X \to Y$ in $\mathcal{B}$ is called *smooth* (or *stably left Beck–Chevalley*) (*proper*, or *stably right Beck–Chevalley*) if every base change of u is left (right) Beck–Chevalley. The relation of these operations with quantification in logic is recalled in Table 2.

The classes of $\mathbb{C}$-smooth and $\mathbb{C}$-proper maps are closed under base change and define sub-fibrations of the codomain fibration of $\mathcal{B}$. Equivalently, they define regular calibrations $\Sigma(\mathbb{C}) \subset \mathbb{B} \supset \Pi(\mathbb{C})$ of the universe of $\mathbb{B}$. The interest of the notions is in the following result.

Proposition 2.3.2. *The smooth (proper) calibration of $\mathbb{C}$ is the largest calibration for which $\mathbb{C}$ admits sums (products).*

Proof. Unfolding the condition that $\Delta_{\mathbb{U}} : \mathbb{C} \to \mathsf{Fam}_{\mathbb{U}}(\mathbb{C})$ has a left (right) adjoint, we find that the left adjoint exists if and only if all maps in $\mathcal{U}$ are smooth (proper), see [Str23]. ☐

The characterization of the smooth and proper maps can be quite difficult in practice. In the setting where the category $\mathbb{C}$ has a forgetful functor into the universe $\mathbb{B}$ (typically, when $\mathbb{C}$ classifies objects in $\mathbb{B}$ with an extra structure, notably when $\mathbb{C}$ is a calibration) stricter notions of smooth and proper maps can be defined which are easier to characterize in practice.

Definition 2.3.3 (Strict smoothness/properness)**.** Let $\mathbb{C} : \mathcal{B}^{\mathrm{op}} \to \mathsf{CAT}$ be equipped with a natural transformation $U : \mathbb{C} \subset \mathbb{B}$ to the universe of $\mathbb{B}$. For $f : X \to Y$ be a map in $\mathcal{B}$, we have a commutative square

$$\begin{array}{ccc} \mathbb{C}(D) & \xrightarrow{U_D} & \mathsf{Cat}_{/D} \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathbb{C}(C) & \xrightarrow{U_C} & \mathsf{Cat}_{/C} \,. \end{array}$$

We shall say that a smooth map is *strictly smooth* (or that a proper map is *strictly proper*) if the mate $f_! U_C \to U_D f_!$ (the mate $U_D f_* \to f_* U_C$) is invertible. Essentially, this says that a map is strictly smooth if $f_!$ can be computed by composition with f in $\mathcal{B}$, and that it is strictly proper if f_* can be computed by exponentiation along f in $\mathcal{B}$.

Remark 2.3.4. We shall see that such maps are sometimes easier to characterize in practice. For proper maps, the definition implies that they are exponentiable maps (since the dependent product must exist for the codomain fibration).

Lemma 2.3.5. *If $\mathbb{C} \subset \mathbb{B}$ is a regular calibration, then a $\mathbb{C}$-smooth map is strict if and only if it is in $\mathcal{C} \subset \mathcal{B}^{\to}$.*

Proof. A map $u : X \to Y$ is strict if composition with u sends maps $r : \bar{X} \to X$ in $\mathcal{C}$ to maps $\bar{X} \to Y$ in $\mathcal{C}$. Applied to $r = id_X$, this implies that u is in $\mathcal{C}$ and that every strict smooth maps is in $\mathcal{C}$. The converse is true by regularity of $\mathbb{C}$. ☐

A *modality* on the category $\mathcal{B}$ is a (unique) factorization system $(\mathcal{L}, \mathcal{R})$ such that the factorization (or equivalently the left class $\mathcal{L}$) is stable under base change. In this situation, both classes $\mathcal{L}$ and $\mathcal{R}$ define calibrations $\mathbb{L} \subset \mathbb{B} \supset \mathbb{R}$ and $\mathbb{R}$ is even a reflective calibration (with the reflection given by the factorization).

Lemma 2.3.6. *If $\mathbb{C} = \mathbb{R} \subset \mathbb{B}$ is the subuniverse of the right class of a modality on $\mathcal{B}$, then every map in $\mathcal{B}$ is $\mathbb{C}$-smooth.*

Proof. For a map $u : X \to Y$ in $\mathcal{B}$, and $r : \bar{X} \to X$ in $\mathcal{R}$, the left adjoint $u_!(r)$ is given by the right map of the $(\mathcal{L}, \mathcal{R})$-factorization of the map composite map $\bar{X} \to X \to Y$. It satisfies the Beck–Chevalley condition because the factorization is stable under base change. The last statement is Lemma 2.3.5. ☐

Lemmas 2.3.5 and 2.3.6 together provide examples where smooth and strictly smooth maps do not coincide.

We consider now the case of a *weak* factorization system $(\mathcal{L},\mathcal{R})$ on $\mathcal{B}$. The class $\mathcal{R}$ is still closed under base change and define a regular calibration $\mathbb{R}\subset\mathbb{B}$, but not the class $\mathcal{L}$.

Lemma 2.3.7. *If $\mathbb{C}=\mathbb{R}\subset\mathbb{B}$ if the regular calibration associated to the right class of a weak factorization system $(\mathcal{L},\mathcal{R})$ on $\mathcal{B}$, then a $\mathbb{C}$-proper map u is strict if and only if for any base change $\bar{u}\to u$, the functor $\bar{u}^*$ preserves the class $\mathcal{L}$. In particular, if the factorization system $(\mathcal{L},\mathcal{R})$ is a modality, then every proper map is strict.*

Proof. A proper map u is strict if for any base change $\bar{u}\to u$, the functor $\bar{u}_*$ preserves the maps in $\mathcal{R}$, but this is equivalent to $\bar{u}^*$ preserving the class $\mathcal{L}$. And when $(\mathcal{L},\mathcal{R})$ is a modality, every $\bar{u}^*$ preserves the class $\mathcal{L}$. □

Definition 2.3.8 (Acyclic and localic maps)**.** A map $u:X\to Y$ in $\mathcal{B}$ is called $\mathbb{C}$-*pre-acyclic* if $u^*:\mathbb{C}(Y)\to\mathbb{C}(X)$ is an equivalence. A map $u:X\to Y$ in $\mathcal{B}$ is called $\mathbb{C}$-*acyclic* if every base change of u is $\mathbb{C}$-pre-acyclic. We denote by $\mathrm{A}(\mathbb{C})$ (Alpha) the codomain subfibration of $\mathbb{C}$-acyclic maps. Acyclic maps are always both smooth and proper. A map is called $\mathbb{C}$-*localic* if it is right orthogonal to $\mathbb{C}$-acyclic maps. We denote by $\Lambda(\mathbb{C})$ the codomain subfibration of $\mathbb{C}$-localic maps.

The name 'acyclic' is motivated by the following result. The name 'localic' is motivated by an application to topos theory (see third example of Section 3.3). In a ∞-topos $\mathcal{E}$, an acyclic class is a class of maps containing all isomorphisms, closed under composition and base change, and under colimits in the arrow category of $\mathcal{E}$ [ABFJ22, Definition 3.2.8]. If $(\mathcal{L},\mathcal{R})$ is a (unique) factorization system on $\mathcal{E}$, the class $\mathcal{L}$ is acyclic if and only if $(\mathcal{L},\mathcal{R})$ is a modality.

Lemma 2.3.9. *If $\mathcal{B}$ is an ∞-topos and if $\mathbb{C}:\mathcal{B}^{\mathrm{op}}\to\mathsf{CAT}$ sends colimits to limits, then the class of $\mathbb{C}$-acyclic maps is an acyclic class.*

Proof. It is easy to see from the definition that $\mathrm{A}(\mathbb{C})$ contains all isomorphisms, is closed under composition and base change. It is also closed under small colimits in $\mathcal{B}^{\to}$ since the functor $\mathbb{C}$ send colimits to limits and $\mathrm{A}(\mathbb{C})$ is the inverse image of the class $\mathsf{CAT}^{\simeq}\subset\mathsf{CAT}^{\to}$ which is closed under limits. □

When $\mathcal{B}$ is a topos and the $\mathbb{C}$-acyclic and $\mathbb{C}$-localic maps form a factorization system, they define a modality on $\mathcal{B}$.

3 Examples

3.1 Set theory and type theory

The examples are summarized in Table 1.

Any category C represents a functor $\mathcal{B}^{\mathrm{op}}=\mathsf{Set}^{\mathrm{op}}\to\mathsf{CAT}$, sending I to C^I. In this setting, all conditions on maps can be computed fiberwise. A set I is smooth (proper) if the coproduct $\coprod_I:C^I\to C$ (product functor $\prod_I:C^I\to C$) exists. A map is smooth (proper) if its fibers are smooth (proper) sets. If C is the terminal category, every set is smooth and proper. If C is the initial category, the smooth (proper) maps are the surjections. A set I is acyclic if $C\to C^I$ is an equivalence. If C is the terminal category, every set is acyclic (and every map is acyclic). If C is the empty category, the acyclic sets are the non-empty sets (the acyclic maps are surjections and the localic maps are the injections). If C is otherwise, the only acyclic sets are the singletons (the acyclic maps are the bijections and every maps is localic).

The second example is the one detail in the introduction of Section 2. It is of the previous kind if and only if κ is a regular cardinal.

If we consider now the example of the fibration of subsets (or of injections), every map $u:I\to J$ is smooth and proper. The functors $u_!$ and u_* are the two direct images of subsets, classically related to

Table 1: Examples from set theory and type theory

$\mathcal{B}$	$\mathbb{C}$	$\Sigma(\mathbb{C})$	$\Pi(\mathbb{C})$	$\mathrm{A}(\mathbb{C})$	$\Lambda(\mathbb{C})$
Cat. of sets	a category C	sets I for which $\coprod_I$ exists in C	sets I for which $\prod_I$ exists in C	(see text)	
Cat. of sets	maps with κ-small fibers ($\kappa \geq 2$)	maps with Σ-small fibers for $\Sigma = \sup\{\sigma \mid \rho < \kappa \Rightarrow \rho.\sigma < \kappa\}$	maps with Π-small fibers for $\Pi = \sup\{\pi \mid \rho < \kappa \Rightarrow \rho^\pi < \kappa\}$	bijections	all maps
Cat. of sets	subsets ($\kappa = 2$)	all maps	all maps	bijections	all maps
$\mathcal{B}$	hyperdoctrine $\mathcal{B}^{\mathrm{op}} \to \mathsf{Poset}$	all maps	all maps	?	?
$\mathcal{B}$	π-clan structure $\mathbb{D} \subset \mathbb{B}$	strict smooth = all maps in $\mathcal{D}$	strict proper $\supset$ all maps in $\mathcal{D}$	?	?
A category $\mathcal{E}$ with a subobject classifier Ω	dominance $\mathcal{O} \subset \Omega$	smooth = overt maps strict smooth = maps in $\mathcal{O}$	proper = proper maps str. proper = ?	(see text)	
A 1-topos $\mathcal{E}$	Grothendieck topology $\Omega_j \subset \Omega$	smooth = all maps strict smooth = closed monos	proper = "quasi-compact" maps	maps inverted by the localization	relative sheaves
$\mathcal{S}$, the ∞-category of ∞-groupoids	subcategory $\mathcal{S}^{<\kappa}$ of κ-small ∞-groupoids	strict : maps with κ-small fibers (if κ regular)	strict : maps with κ-small fibers (if κ inacc.)	?	?
An ∞-topos $\mathcal{E}$	subuniverse $\mathbb{T}_n$ of n-truncated objects	the whole universe (strict smooth maps are the n-truncated ones)	the whole universe (all maps are strictly proper)	$(n+1)$-conn. maps	$(n+1)$-trunc. maps
An ∞-topos $\mathcal{E}$	subuniverse $\mathbb{R} \subset \mathbb{E}$ of modal types (for a modality $(\mathcal{L}, \mathcal{R})$ on $\mathcal{E}$)	the whole universe (strict maps are those in $\mathbb{R}$)	?	décalage class of $\mathcal{L}$ [ABFJ24]	right class of the décalage modality of [ABFJ23]
An ∞-topos $\mathcal{E}$	subuniverse $\mathbb{F} \subset \mathbb{E}$ of sheaves assoc. to a lex loc. $\mathcal{F} = \mathcal{E}[\mathcal{W}^{-1}]$	the whole universe (strict maps are those in $\mathbb{F}$)	?	maps in $\mathcal{W}$	maps in $\mathbb{F}$

existential and universal quantifiers ($u_!A = \{j \mid \exists a \in A, a \in u^{-1}(j)\}$, $u_*A = \{j \mid \forall a \in A, a \in u^{-1}(j)\}$). More generally, the setting of sets and subsets could be replaced by a hyperdoctrine in the sense of Lawvere [Law69, Law70, See83]. We recall the correspondence between logical quantifiers and the adjoints to base change in Table 2.

Table 2: Quantifiers and direct images

	Indexing object	Families	change of index	left image	right image
Predicate logic	variables	predicates	substitution	$\exists$	$\forall$
Dependent type theory	contexts	dependent types	substitution	Σ	Π
Category theory: fibration $C \to B$	object in base	object in fiber	base change u^*	$u_!$	u_*

Another example related to type theory is that of a category $\mathcal{B}$ with a π-clan structure in the sense of Joyal [Joy17]. The clan structure distinguishes a class of maps in $\mathcal{D} \subset \mathcal{B}$ closed under base change and containing all isomorphisms, or equivalently a regular calibration $\mathbb{D} \subset \mathbb{B}$. This implies that strict smooth maps are exaclty the maps in $\mathcal{D}$. And the definition if the π-clan structure says exactly that every map in $\mathcal{D}$ is strictly proper [Joy17, Definition 2.4.1].

If $\mathcal{B}$ is a cartesian closed 1-category with a subobject classifier (e.g. 1-topos), we consider the example of a *dominance*, which is a regular calibration of monomorphisms classified by a subobject $\mathcal{O} \subset \Omega$ of the subobject classifier [Esc04b, Hyl91]. Intuitively, the object $\mathcal{O}$ classifies some subobjects meant to be "open" in the sense of topology, and the exponential $\mathcal{O}^A$ is the "space" of open subspaces. The smooth (proper) maps define a notion of open (compact) maps in the sense of topology. The strictly smooth maps are those classified by $\mathcal{O}$ (Lemma 2.3.5). The strictly proper maps are those maps for which the direct image can be computed within the posets of all subobjects. Acyclic and localic maps are considered in the literature, but only with the base change property along cartesian projections (and not arbitrary maps). They are called $\mathcal{O}$-*equable* maps and $\mathcal{O}$-*replete* maps in *op. cit.*.

When $\mathcal{B}$ is a 1-topos, any Grothendieck topology defines a dominance $\Omega_j \subset \Omega$ where Ω_j is the classifier of closed monos. In this case, all maps are smooth, the closed monos are the strictly smooth maps. The acyclic maps are the maps inverted in the localization by the topology, and the localic maps are the relative sheaves. A map $A \to 1$ is proper if and only if A it is quasi-compact (every covering family has a covering finite subfamily). General proper maps can be described by a relative version of the same condition.

The ∞-category $\mathcal{S}$ of ∞-groupoids is a higher categorical model for dependent type theory with univalence: precisely, the small sub-universes correspond to univalent maps in $\mathcal{S}$. It seems difficult to describe the smooth, proper, acyclic and localic maps for an arbitrary subuniverse, but it is easier with strict smooth and proper maps. We shall only consider the subuniverse of κ-small spaces. When κ is regular, the strict smooth maps are the maps with κ-small fibers by Lemma 2.3.5. When κ is inaccessible, the strict proper maps are the maps with κ-small fibers by Lemma 2.3.7. In particular, if κ is the inaccessible cardinal bounding the size of small objects, all maps are smooth and proper. In this case, the acyclic maps are reduced to equivalences and all maps are localic.

In the example of an ∞-topos $\mathcal{E}$, we denote the universe by $\mathbb{E}$.[3] By definition of an ∞-topos, the functor

[3]The same example could be presented in the setting of 1-topoi, but the formalism of ∞-topoi is just easier.

$\mathbb{E} : \mathcal{E}^{\mathrm{op}} \to \mathsf{CAT}$ sends colimits of $\mathcal{E}$ to limits in CAT, and the interesting subuniverses are those satisfying a similar condition (corresponding to local classes of maps [Lur09, Proposition 6.1.3.7]).

For $n \geq -2$, we denote $\mathcal{C}_n$ and $\mathcal{T}_n$ the classes of n-connected and n-truncated maps. For example, the class $\mathcal{C}_{-1}$ and $\mathcal{T}_{-1}$ are the classes of surjections and monomorphisms. Every maps can be factored uniquely into an n-connected maps followed by an n-truncated maps, and this factorization is stable under base change. The class $\mathcal{T}_n$ contained all isomorphisms, is closed under composition, and is local. It defines a regular calibration $\mathbb{T}_n \subset \mathbb{E}$ which is also a reflective subuniverse (where the reflection is given by the factorization). Every map is smooth for $\mathbb{T}$, and the strict smooth maps are those in $\mathbb{T}$. Every map is strictly proper (hence proper) for $\mathbb{T}$ because dependent products preserve the truncation level of objects. The $\mathbb{T}_n$-acyclic maps and $\mathbb{T}_n$-localic maps can be characterized as the $(n+1)$-connected maps and $(n+1)$-truncated maps (see below).

More generally, we define a modality on $\mathcal{E}$ as a (unique) factorization system $(\mathcal{L}, \mathcal{R})$ on $\mathcal{E}$ such that both classes are stable under base change. Then both classes $\mathcal{L}$ and $\mathcal{R}$ are local by [ABFJ20, Proposition 3.6.5] and define "good" subuniverses $\mathbb{L}, \mathbb{R} \subset \mathbb{E} : \mathcal{E}^{\mathrm{op}} \to \mathsf{CAT}$. We shall consider the case $\mathbb{C} = \mathbb{R}$. Since the class $\mathcal{R}$ is closed under composition and base change, Lemma 2.3.5 shows that the strict smooth maps are all maps in $\mathcal{R}$. But the subuniverse actually admits sums indexed by all maps in $\mathcal{E}$. Given a map $u : X \to Y$ in $\mathcal{E}$ and a map $r : \bar{X} \to X$ in $\mathcal{R}$, the $(\mathcal{L}, \mathcal{R})$-factorization of the composite map $\bar{X} \to X \to Y$ gives maps $\bar{X} \to \bar{Y}$ in $\mathcal{L}$ and $r' : \bar{Y} \to Y$ in $\mathcal{R}$. The image of r by $u_!$ is the map $\bar{Y} \to Y$. The Beck–Chevalley condition holds because the factorization is stable under base change. A more conceptual way to see this is to say that $\mathbb{R}$ is in fact a reflective subuniverse (where the reflection is given by the factorization) and therefore complete under all sums existing in $\mathbb{E}$.

In this example, it seems difficult to describe the proper maps without further assumption on $\mathbb{R}$. However, the acyclic maps are exactly the maps in the *décalage* of the class $\mathcal{L}$ [ABFJ24, 2.2.7] (they are also the 'fiberwise $\mathbb{R}$-equivalences' of [ABFJ22, Def.3.3.1]). In particular, they form an acyclic class in $\mathcal{E}$ [ABFJ22, Theorem 3.3.9]. In this context, acyclic and localic maps define a modality on $\mathcal{E}$ which is detailled in [ABFJ23, Theorem 2.3.32].

Finally, any left-exact localization of ∞-topoi $q^* : \mathcal{E} \to \mathcal{E}[\mathcal{W}^{-1}]$ provide a modality $(\mathcal{W}, \mathcal{F})$ on $\mathcal{E}$, where $\mathcal{W}$ is the class if maps inverted by q^* and $\mathcal{F}$ is the class of "relative sheaves" [RSS19, ABFJ22]. Such a modality is left-exact in the sense that the factorization of a map preserves finite limits (in the arrow category). In fact, there is a bijection between left-exact localization and left-exact modalities. Let $\mathbb{W} \subset \mathbb{E} \supset \mathbb{F}$ be the corresponding subuniverses. If $\mathbb{E}' : \mathcal{E}[\mathcal{W}^{-1}]^{\mathrm{op}} \to \mathsf{CAT}$ is the universe of the ∞-topos $\mathcal{E}[\mathcal{W}^{-1}]$, one can show that $\mathbb{F} = \mathbb{E}' \circ q^* : \mathcal{E}^{\mathrm{op}} \to \mathsf{CAT}$. Since left-exact modalities are fixed by décalage [ABFJ24, Lemma 2.4.6 (1)], the acyclic (localic) maps coincides with the class $\mathcal{W}$ ($\mathcal{F}$).

3.2 Category theory

We consider now example from category theory. The study of fibrations of categories was the motivation of Grothendieck to introduce his notion of smooth and proper functors. The examples are summarized in Table 3.

The first example is that of the codomain fibration of a category with finite limits. There, every map is smooth and the proper maps are exactly the exponentiable maps. Essentially by definition, all the projections $X \times Y \to X$ are proper if and only if the category $\mathcal{B}$ is cartesian closed, and all the maps are proper if and only if the category $\mathcal{B}$ is locally cartesian closed. The acyclic maps are the isomorphisms and every map is localic.

The second example is the particular case of the first one where $\mathcal{B} = \mathsf{Cat}$. It will be the base of all the other examples. We denote by $\mathbb{C}\mathsf{at}$ the universe of Cat (which can be thought of either as the codomain fibration, or as the slice functor $C \mapsto \mathsf{Cat}_{/C}$). The category Cat is cartesian closed but not locally cartesian closed. The exponentiable functors are the Conduché fibrations (see [AF20, Lemma 1.11] for a higher categorical account). The acyclic maps are the equivalences of categories and every functor is localic.

The following examples will study various kinds of "fibrations" condition on functors. This will lead to consider functors $\mathbb{C} : \mathsf{Cat}^{\mathrm{op}} \to \mathsf{CAT}$ with a natural forgetful morphism $\mathbb{C} \to \mathbb{C}\mathsf{at}$ into the universe of Cat (or equivalently a morphism between the corresponding fibrations over Cat).

Table 3: Examples from category theory

$\mathcal{B}$	$\mathbb{C}$	$\Sigma(\mathbb{C})$	$\Pi(\mathbb{C})$
A lex category	the codomain fibration	all maps	exponentiable maps
Cat. of categories	all functors	all functors	Conduché fib.
Cat. of categories	left fibrations	smooth = ? strictly smooth = left fib.	proper = ? strictly proper = Lurie-proper functor (= Grothendieck-smooth)
Cat. of categories	right fibrations	smooth = ? strictly smooth = right fib.	proper = ? strictly proper = Lurie-smooth functors (= Grothendieck-proper)
Cat. of categories	cocartesian fibrations	smooth = ? strictly smooth = cocart. fib.	proper = ? strictly proper = Cond. + u^* pres. ff left adj.
Cat. of categories	cartesian fibrations	smooth = ? strictly smooth = cart. fib.	proper = ? strictly proper = Cond. + u^* pres. ff right adj.

Let us start with the the functor $\mathsf{LFib} : \mathsf{Cat}^{\mathrm{op}} \to \mathsf{CAT}$ sending a category C to its category $\mathsf{LFib}(C) = [C, \mathcal{S}]$ of left fibrations. This functor is essentially representable by the category $\mathcal{S}$ (up to a size issue). The Grothendieck construction provides a natural transformation $\mathsf{LFib}(C) \to \mathsf{Cat}_{/C}$ and we can talk about strict notions. Recall that the left fibrations are the right class of a (unique) factorization system on Cat, where the left class is that of initial functors. The characterization of smooth and proper maps is open. The previous factorization system is not stable under base change and we cannot apply Lemma 2.3.6. But the strict smooth maps are exactly the left fibrations themselves. The class of proper functors contains right fibrations And we can use Lemma 2.3.7 to characterize the strict proper maps as the exponentiable functors $u : C \to D$ such that for any base change $\bar{u} \to u$, the pullback along $\bar{u}$ preserves the class of initial functors. This is exactly the notion of proper functor of [Lur09, dual of Definition 4.1.2.9] and the notion of smooth functor of Grothendieck in Pursuing Stacks (and also in [Joy08, 21.1] and [Cis19, dual of Definition 4.4.1]).[4] In particular, right fibrations are strict proper functors for LFib. In the dual situation of right fibrations, the notion of smooth and proper are reversed.

Finally, we can look at the example of cocartesian (and cartesian) fibrations. The functor of interest is $C \mapsto \mathsf{CcFib}(C) = [C, \mathsf{Cat}]$. Up to size issues, it is represented by Cat itself. The class of cocartesian fibrations is almost the right class of a weak factorization system on Cat (in 1-categories, it is the right class of an algebraic factorization system [BG14]). The corresponding left class is the class of fully faithful left adjoint functor. Any functor $f : C \to D$ admits a factorization $C \to f \downarrow D \to D$ into a fully faithful left adjoint functor followed by a cocartesian fibration (where $f \downarrow D$ if the fiber product of $C \to D \xleftarrow{dom} D^{\to}$). Again, the smooth and proper maps are difficult to find. The strict smooth maps are the cocartesian fibrations themselves. The previous factorization is enough to apply Lemma 2.3.7 to characterize the strictly proper

[4] The reversal of names is due to a preference for covariant or contravariant functors in the definitions. Grothendieck's convention uses presheaves, we have preferred to use covariant functors.

maps as the exponentiable functors $u : C \to D$ such that u^* preserves fully faithful left adjoint functors. This class includes that of cartesian fibrations. In the dual situation of right fibrations, the notion of smooth and proper are reversed.

In all the "fibration" examples, the acyclic maps are the equivalences and every functor is localic. A functor $u : C \to D$ induces an equivalence $u^* : \mathcal{S}^D \to \mathcal{S}^C$ if and only if it is a Morita equivalence. The acyclic maps are Morita equivalences which are stable under base change. This implies that they must be (essentially) surjective functors. Since a Morita equivalence is always a fully faithful, this implies that this is an actual equivalence.

3.3 Topology and geometry

We now consider examples where the base category $\mathcal{B}$ is a category of topological objects. We are going to see a tight connection between direct images functors, logical quantifiers, and classical topological conditions.

The examples are summarized in Table 4.

Table 4: Examples from topology and geometry

$\mathcal{B}$	$\mathbb{C}$	$\Sigma(\mathbb{C})$	$\Pi(\mathbb{C})$	$A(\mathbb{C})$	$\Lambda(\mathbb{C})$
Cat. of locales	all maps	all maps	exponentiable maps	homeo-morphisms	all maps
Cat. of locales	open immersions (rep. by Sierpiński space)	open maps (strict: open immersions)	proper maps (strict: locally compact and proper)	homeo-morphisms	all maps
Cat. of 1-topoi	open immersions (rep. by Sierpiński topos)	open maps [Joh02]	proper maps [MV00] (strict: exponentiable + proper)	hyper-connected morphisms	localic morphisms
Cat. of 1-topoi	étale maps (rep. by object classifier)	locally connected maps [Joh02]	tidy maps [MV00]	equivalences	all morphisms
Cat. of ∞-topoi	étale maps (rep. by object classifier)	locally contractible maps [MW23a]	proper maps [MW23b]	equivalences	all morphisms
Cat. of ∞-topoi	n-tr. étale maps (rep. by n-truncated object classifier)	locally n-connected maps (by methods similar to [MW23a])	n-proper maps (by methods similar to [MW23b])	hyper-$(n+1)$-con-nected morphisms	$(n+1)$-localic morphisms
Cat. of schemes	torsion sheaves	smooth map ⊃ smooth morphisms [FK88, Thm 7.3]	proper map ⊃ proper morphisms [FK88, Thm 6.1]	?	?

In the first example, $\mathcal{B}$ = Locale is the category of locales[5] with the codomain fibration. Every map is smooth, and the proper maps are the exponentiable ones. The exponentiable locales are the locally compact ones (i.e. the locales such that $\mathrm{Op}(X)$ is a continuous lattice) [Joh82, Theorem 4.11]. Acyclic maps are the isomorphisms, and every map is localic.

The next example is that of the functor $\mathsf{Locale}^{\mathrm{op}} \to \mathsf{CAT}$ sending a locale X to its frame $\mathrm{Op}(X)$ of open domains. This functor is represented by the Sierpiński space. Let us say that a map is an *open immersion* if it is isomorphism to the inclusion of an open. The fibration corresponding to $\mathrm{Op}(-)$ is the subfibration of the codomain fibration of Locale spanned by open immersions. This shows that the notion of strict smooth and proper maps makes sense.

The smooth morphisms are almost by definition the open morphisms of topology (this can be shown directly, or deduced from the similar result for topoi [Joh02, Lemma C.3.1.10], see also [Esc04a]). By Lemma 2.3.5 the strict smooth maps are the open immersions that are also open, but that is the case of every open immersion. The proper morphisms are the proper morphisms of topology (i.e. the universally closed morphisms, this can be seen directly by considering left adjoints on the posets $\mathrm{Op}(X)^{\mathrm{op}}$, or deduced from topos theory [Joh02, Lemma C.3.2.81], see also [Esc04a, Esc20]). The characterization of strict proper maps is open. An open immersion is proper if and only if it is isomorphic to the inclusion of a clopen. The computation of acyclic and localic maps is straighforward.

In the third example, $\mathcal{B}$ is the category of 1-topoi and geometric morphisms [Joh02, AJ21]. If we look at the codomain fibration, the only non-trivial class of maps is that of exponentiable maps [JJ82].

Another important fibration is the one of open sub-1-topoi (or open immersions), sending a 1-topos X to the poset of subterminal objects in its category of sheaves $\mathrm{Sh}(\mathsf{X})$ (its dual 1-logos in the sense of [AJ21]). This fibration is represented by the Sierpiński topos (dual to the 1-logos $\mathsf{Set}^{\to}$). The identification of smooth maps as open morphisms of 1-topoi is done in [Joh02, Theorem C.3.1.28], and that of proper maps as proper morphisms of 1-topoi is done in [MV00, Corollary I.5.9] and in [Joh02, Theorem C.3.1.28]. Both references use a "weak" version of Beck–Chevalley conditions (where the mate transformation is only monic) for arbitrary sheaves. But restricted to subterminal objects this condition recovers the one from Definition 2.3.1 and this can be shown to be equivalent to the weak condition (see [Joh02, Proposition A.4.1.17] and [MV00, Proposition 3.2]).

Open immersions can be composed and define a regular calibration on Topos. Then, Lemma 2.3.5 shows that the strict smooth maps are exactly the open immersion of 1-topoi. The characterization of strict proper maps is open. An open immersion of 1-topoi is proper if it is also the inclusion of a closed sub-1-topoi (corresponding to a decidable subterminal object). Interestingly, in this example the notion of acyclic and localic maps recover the hyperconnected and localic morphisms of topoi (this is in fact the example motivating the name 'localic'). The proof is straightforward for hyperconnected maps, and the fact that the right orthogonal to hyperconnected morphisms are the localic morphisms is [Joh02, Lemma A.4.6.4].

The next example is the case of the fibration of étale maps over the category of 1-topoi, sending a 1-topos X to its category of sheaves of sets (its 1-logos) $\mathrm{Sh}(\mathsf{X})$. This fibration is representable by the "object classifier" or the "topos line" (that we shall denote A, its logos of sheaves is $[\mathsf{finset}, \mathsf{Set}]$) and the existence of sums and product for the étale fibration can be interpreted as sums and product and product structure on this object. In this case, the smooth maps are the locally connected morphisms of 1-topoi [Joh02, Corollary C.3.3.16], and the proper maps are the tidy morphisms of 1-topoi, see [MV00, Corollary III.4.9] or [Joh02, Corollary C.3.4.11]. The strict smooth maps are all the étale morphisms. The characterization of strict proper maps is open. In this setting, the acyclic maps are the equivalences and every map is localic.

Interestingly, not every étale map is proper: the intersection of the two classes is the class of finite maps. This means that the "topos line" does not have arbitrary products indexed by itself, but only finite products. This might be surprising since category of sheaves are locally cartesian closed, but dependent products are not preserved by geometric morphisms, only finite products are.

The generalization of the previous results to ∞-topoi is quite rich! For the case of the codomain fibration, exponentiable ∞-topoi have been characterized in [Lur17, Theorem 21.1.6.12] and [AL19].

[5] Using locales instead of topological spaces simplify the computations. Similar considerations are true for sober spaces.

The case of the fibration of étale morphisms has been studied recently by Martini and Wolf. If $\mathcal{B} = \mathsf{Topos}_\infty$ is the category of ∞-topoi and $\mathbb{C}(\mathsf{X}) = \mathrm{Sh}_\infty(\mathsf{X})$ is the category of higher sheaves (the ∞-logos dual to X), then the smooth maps are the locally contractible morphisms of ∞-topoi [MW23a], and the proper maps are the proper morphisms of ∞-topoi [Lur09, MW23b]. The acyclic maps are equivalences and every map is localic.

It is interesting to restrict the fibration of higher sheaves (aka higher etale morphisms) to *n-truncated* sheaves only (the case $n = -1$ recovers the fibration of open immersion). In this case, it is easy to adapt the results of Martini and Wolf to characterize the smooth maps as some *locally n-connected* morphisms of ∞-topoi[6] and the proper maps are the *n-proper* morphisms of ∞-topoi[7] The strict smooth maps are the n-truncated étale morphisms of topoi. The corresponding acyclic and localic maps define notions of hyper-$(n+1)$-connected morphisms and $(n+1)$-localic morphisms, and every geometric morphism should factors into a hyper-$(n+1)$-connected morphism followed by a $(n+1)$-localic morphism. Moreover, an ∞-topos X should be n-localic in the sense of [Lur09, Definition 6.4.5.8] if and only if the morphism $\mathsf{X} \to 1$ is n-localic in the previous sense.

Étale maps define a regular calibration of the universe of Topos_∞ and notion of strict smooth and proper maps can be defined. By Lemma 2.3.5, the strict smooth maps are exactly the etale maps. The strict proper maps are exponentiable maps u whose direct image u_* preserves etale maps. their characterization is open.

The last example of Table 4 is in algebraic geometry, where the base category is the category Sch of (noetherian) schemes, and the functor $\mathsf{Sch}^{\mathrm{op}} \to \mathsf{CAT}$ is the one sending a scheme X to its category of torsion sheaves [FK88]. The characterization of smooth and proper maps in general is open, but [FK88, Theorems 6.1 & 7.3] show that smooth morphisms of schemes are smooth maps, and that proper morphisms of schemes are proper maps. This setting is the one that inspired Grothendieck to name his notions of smooth and proper functors [Mal05]. The characterization of acyclic and localic maps is an open question.

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