IWASAWA INVARIANTS IN RESIDUALLY REDUCIBLE HIDA FAMILIES

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In memory of Joël Bellaïche

ABSTRACT. We study the variation of μ-invariants of modular forms in a cuspidal Hida family in the case that the family intersects an Eisenstein family. We allow for intersections that occur because of "trivial zeros" (that is, because p divides an Euler factor) as in Mazur's Eisenstein ideal paper, and pay special attention to the case of the 5-adic family passing through the elliptic curve $X_0(11)$.

1. Introduction

1.1. **Congruences from L-values.** In [BP19], the first author and Bellaïche studied the μ-invariants of Hida families of cuspidal eigenforms which admit congruences with Eisenstein series. The congruences studied there were ones that arose from p-divisibilities of L-values, in this case, Bernouli numbers. For instance, consider the case of tame level $N = 1$ where $p \mid B_k$, the k-th Bernoulli number. In this situation, there is a cuspidal Hida family which has non-trivial intersections with an Eisenstein Hida family with the intersections occuring at the p-adic weights which are zeroes of the p-adic ζ-function (which is non-trivial as $p \mid B_k$).

Assuming certain Hecke algebras were Gorenstein, it was shown in [BP19] that the μ-invariants in the cuspidal family blew up as one approached these intersection points. Moreover, precise formulas were conjectured about the values of these μ-invariants which in favorable situations (*i.e.* when the cuspidal Hida family was rank 1 over weight space) were given simply as the p-adic valuation of certain special values of the p-adic ζ-function. When $U_p - 1$ generates the Eisenstein ideal, these conjectures were shown to hold for the branch of the Λ-function with trivial tame character, and the λ-invariants in these families were shown to be identically zero.

1.2. **Congruences from Euler factors.** In this paper, we aim to treat the analogous situation where congruences arise because of divisibility of Euler factors. For example, consider the case of N a prime such that $N \equiv 1 \pmod{p}$. By Mazur's famous result [Maz77], as long as $p > 3$, there is always a cuspidal eigenform in $S_2(\Gamma_0(N))$ congruent to the unique ordinary Eisenstein series of weight 2 and level N. The most famous example of this congruence is when $p = 5$ and $N = 11$. In this case, there is a unique such cuspidal eigenform and it is exactly the modular form corresponding to the elliptic curve $X_0(11)$.

The ordinary p-stabilization of this form to level $Np = 55$ lives in a Hida family that is rank 1 over weight space. However, there is a key difference in this situation from the case treated in [BP19] where there was a unique Eisenstein family to consider. In our situation, there are two Eisenstein families; namely, one where

$U_N = 1$ for all weights and one where $U_N = N^{k-1}$. Since $N \equiv 1 \pmod{p}$, these two families are congruent modulo p. Moreover, the localization of the Hida Hecke algebra which corresponds to these two families together with the cuspidal family is not Gorenstein and the methods of [BP19] do not apply.

To avoid this problem, we employ a method already used in [Oht14] and [WWE21] where we remove U_N from our Hecke algebra and replace it with w_N, the Atkin-Lehner involution. While the U_N-eigenvalues of the two Eisenstein families are congruent, the w_N-eigenvalues of these families are 1 and -1, and the congruence is broken. (It is verified in Appendix A that Hida theory still works as expected for these modified Hida Hecke algebras.)

We are interested in the Eisenstein family with w_N-eigenvalue -1 as our cuspidal family has w_N-eigenvalue equal to -1. The localization of the modified Hecke algebra corresponding to this Eisenstein family together with our cuspidal family turns out to be Gorenstein and we are good shape to generalize the methods of [BP19].

We state our results in the specific case of $p = 5$ and $N = 11$. To set up a little bit of notation, let $L_p^+(k,s)$ denote the (plus) two-variable p-adic L-function over $\mathbb{Q}_\infty$, the cyclotomic $\mathbb{Z}_p$-extension, corresponding to the Hida family through $X_0(11)$.

Theorem 1.1. *Let $N = 11$, $p = 5$, and let k be an integer with $k \equiv 2 \pmod{4}$. Let f_k denote the unique ordinary eigenform in $S_k(\Gamma_0(Np))$ whose residual representation is $1 \oplus \omega$, where ω is the mod-p cyclotomic character. Then*

$$L_p^+(k,s) = (N^{k/2} - 1) \cdot U(k,s)$$

where $U(k,s)$ is a unit a power series in k and s. In particular, the Iwasawa invariants of f_k are given by

(1) $\mu(f_k) = \mathrm{val}_p(a_p(f_k) - 1) = 1 + \mathrm{val}_p(k)$ and
(2) $\lambda(f_k) = 0$.

Note that considering weight 2 in the above theorem yields a μ-invariant of 1. This conclusion corresponds to the famous fact that $\mu(X_0(11)) = 1$ for $p = 5$ (see [Maz72, Section 10]). Further, the above theorem implies that as one moves closer and closer to weight 0 in this Hida family, the μ-invariants blow up linearly. Theorem 1.1 is a special case of Theorem 1.2 below.

1.3. **The more general case.** We now discuss the key ingredients that go into Theorem 1.1 to unravel what is special about this case with $(N,p) = (11,5)$ and weight 2, and what can be said in greater generality. To this end, we now let N and p be prime numbers with $N \equiv 1 \pmod{p}$, and let k_0 be an even integer with $0 < k_0 < p-1$. Let $\mathbb{T}_\mathfrak{m}$ denote the completion of the Hida Hecke algebra with tame level N at the maximal ideal given by the residual representation $\overline{\rho} = 1 \oplus \omega^{k_0}$ and with Atkin–Lehner sign $w_N = -1$; let $\mathbb{T}^0_\mathfrak{m}$ denote the quotient of $\mathbb{T}_\mathfrak{m}$ that acts faithfully on cuspforms. Let $\mathcal{I}_\mathfrak{m} \subseteq \mathbb{T}_\mathfrak{m}$ denote the Eisenstein ideal.

The following facts hold in the setting of Theorem 1.1, where $p = 5$, $N = 11$, and $k_0 = 2$:

(1) $\mathbb{T}_\mathfrak{m}$ and $\mathbb{T}^0_\mathfrak{m}$ are Gorenstein
(2) $\mathcal{I}_\mathfrak{m}$ is generated by $U_p - 1$
(3) $p \nmid \frac{B_{k_0}}{k_0}$
(4) $N \equiv 1 \pmod{p}$ but $N \not\equiv 1 \pmod{p^2}$

(5) the rank of $\mathbb{T}^0_{\mathfrak{m}}$ over the Iwasawa algebra Λ equals 1

We will now examine each of these facts in turn and how (if at all) they are used in the proof of Theorem 1.1; then we will state the general version Theorem 1.2 of the theorem. We first note that (2) implies (1) (see Lemma 2.10). In fact, (1) holds whenever $\mathcal{I}_{\mathfrak{m}}$ is principal.

Condition (2) is crucial to obtain as precise a result as in Theorem 1.1. However, we make use of several recent results [Deo23, Wak23, WWE21] about the structure of these Hecke algebras which provide a numerical criterion that is equivalent to (2), as we now explain. Let $\log_N : \mathbb{F}_N^\times \to \mathbb{F}_p$ be a surjective homomorphism (noting that $p \mid (N-1)$, this is a reduction modulo p of a choice of discrete logarithm). Further, let ζ denote a primitive p-th root of unity in $\mathbb{F}_N^\times$. As recalled in Section 5.4 below, the results of [Deo23, Wak23, WWE21] imply that if

$$\sum_{i=1}^{p-1} i^{k_0-2} \log_N(1-\zeta^i) \neq 0 \text{ in } \mathbb{F}_p,$$

then $\mathcal{I}_{\mathfrak{m}}$ is a principal ideal, and moreover, if further

$$p \text{ is not a } p\text{-th power modulo } N,$$

then $U_p - 1$ generates $\mathcal{I}_{\mathfrak{m}}$. Note that when $k_0 \equiv 2 \pmod{p-1}$, the sum appearing in the first condition $\sum_{i=1}^{p-1} i^{k_0-2} \log_N(1-\zeta^i)$ simplifies to $\log_N(p)$, so the first condition is equivalent to the second condition. In particular, in the case $(N, p, k_0) = (11, 5, 2)$, the fact that 5 is not a 5-th power modulo 11 implies (2) (and thus (1)) holds.

Moving on, condition (3) is necessary to ensure that only Euler-factor-type congruences appear. If (3) fails, then there will be a mix of congruences from L-values (as in [BP19]) and from Euler factors and the situation is more complicated. Although it seems very interesting to understand this situation, it is outside the scope of this paper.

Condition (4) is not a serious one. It is used to compute the valuation of $N^{k/2}-1$. In general, we have

$$\mathrm{val}_p(N^{k/2}-1) = \mathrm{val}_p(N-1) + \mathrm{val}_p(k).$$

Condition (5) is important in that without it one cannot simply speak of f_k as there is no longer a unique cuspidal eigenform in our Hida family in each weight. Further, without this condition, the crossing points of the Eisenstein family and the cuspidal family cannot be as finely controlled. Nonetheless, we still have a two-variable p-adic L-function which simply equals $U_p - 1$ up to a unit. Thus we can obtain a formula for the μ-invariants in terms of the U_p-eigenvalues of each form, but we can only give a formula that only depends on k and N for the sum of the μ-invariants of all the forms in weight k in our Hida family.

We note that the same results on the structure of Hecke algebras [Deo23, Wak23, WWE21, WWE20] provide a numerical condition that forces condition (5) to hold. Namely, $\mathbb{T}^0_{\mathfrak{m}}$ has rank 1 over Λ if

$$\prod_{i=1}^{N-1} i^{\left(\sum_{j=1}^{i-1} j^{k_0-1}\right)} \text{ is not a } p\text{-th power modulo } N.$$

Further, if there is a single Galois conjugacy class of modular forms in our Hida family in a given weight, then all of their Iwasawa invariants are the same and we can completely control the situation in this case. On the one hand, we know of no numerical criterion to force there to be a single Galois conjugacy class in a fixed weight. But on the other hand, examples where there are more than one conjugacy class seem rare. When $p = 5$, there are 163 primes N less than 5000 for which $N \equiv 1 \pmod 5$. Of these, 48 have $\mathbb{T}^0_{\mathfrak{m}}$ with rank greater than 1, but only 6 have more than one Galois conjugacy class of forms in weight 2.

We now state our main result in more general terms. Some notation: Let $L_p^+(\mathfrak{m})$ denote the (plus) two-variable p-adic L-function attached to our Hida family and write $L_p^+(\mathfrak{m}, \omega^0)$ for its branch corresponding to $\mathbb{Q}_\infty$, the cyclotomic $\mathbb{Z}_p$-extension—that is, the branch with trivial tame character. For an eigenform g in our Hida family, write ϖ_g for a uniformizer of the normalization of the ring generated by the Hecke-eigenvalue of g.

Theorem 1.2. *Assume that*

(1) $p \nmid \frac{B_{k_0}}{k_0}$,

(2) $\displaystyle\sum_{i=1}^{p-1} i^{k_0-2} \log_N(1-\zeta^i) \neq 0$ *in* $\mathbb{F}_p$, *and*

(3) p *is not a* p*-th power modulo* N.

Then $L_p^+(\mathfrak{m}, \omega^0)$ *has the simple form*

$$L_p^+(\mathfrak{m}, \omega^0) = (U_p - 1) \cdot U$$

where U *is a unit. In particular, for every form* g *in our Hida family,*

$$\mu(g) = \mathrm{ord}_{\varpi_g}(a_p(g) - 1) \text{ and } \lambda(g) = 0.$$

Further, for every integer k *with* $k \equiv k_0 \pmod{p-1}$,

$$\sum_g \mu(g) = \mathrm{val}_p(N-1) + \mathrm{val}_p(k)$$

where the sum is over all Galois conjugacy classes of weight k *forms in our Hida family.*

Theorem 1.2 is stated, and proven, in a more precise form in Theorem 5.20 below.

1.4. **Normalizations.** Like the results in [BP19], the above theorem can be thought of as explaining μ-invariants through p-adic variation. For any individual form, a positive μ-invariant can almost be thought of as an error in normalization. One can simply change the complex period defining the p-adic L-function by a power of p to simply force a μ-invariant to be 0. However, in the family one now sees the μ-invariants as arising from valuations of special values of p-adic analytic functions. In the setting of [BP19], in the rank 1 case, the relevant analytic function was the p-adic ζ-function (which itself has μ-invariant 0). In this paper, the relevant function is simply $\langle N \rangle^{1/2} - 1$ where $\langle N \rangle$ is the element in Λ which specializes to N^k in weight k. Note that $\langle N \rangle^{1/2} - 1$ also has μ-invariant 0.

With that said, one can wonder why these two-variable p-adic L-functions are divisible by these analytic functions that depends only on k and not on the cyclotomic variable s. Indeed, the origin of this project was an attempt to reconcile the results of [BP19] with results of the second author which said that these μ-invariants were identically 0 along the family!

The reconciliation of these results is that there are two natural normalizations of the p-adic L-function and one leads to positive μ-invariants which blow up in the family and the other leads to μ-invariants that are 0 everywhere. We now explain.

In the construction of the two-variable p-adic L-function, one considers a certain explicit class in

$$\mathcal{L}_p^+(\mathfrak{m}) \in X[\![\mathbb{Z}_p^\times]\!] := \varprojlim_n H^1(Y_1(Np^r), \mathbb{Z}_p)_{\mathfrak{m}}^{-,\mathrm{ord}}[\![\mathbb{Z}_p^\times]\!]$$

built out of modular symbols where the upper "ord" denotes the ordinary projector and $\mathfrak{m}$ denotes the completion at the maximal ideal corresponding to our Hida family (the change in sign in the notation is due to Poincaré duality: we think of the negative-signed cohomology class $\mathcal{L}_p^+(\mathfrak{m})$ as a *functional* on modular symbols with positive sign §3). We refer to this class as the two-variable p-adic L-*symbol*.

Typically, when constructing p-adic L-functions, one extracts a function from a symbol by choosing a basis for the space of symbols. In this case, that would involve choosing a basis for the space X as a $\mathbb{T}_{\mathfrak{m}}$-module. However, the space X is not necessarily cyclic as a $\mathbb{T}_{\mathfrak{m}}$-module; in fact, by Ohta's Eichler–Shimura isomorphism for Hida families [Oht99], it is known that X is a dualizing module for $\mathbb{T}_{\mathfrak{m}}$. Hence X is free of rank 1 over $\mathbb{T}_{\mathfrak{m}}$ if and only if $\mathbb{T}_{\mathfrak{m}}$ is a Gorenstein ring. In fact, if $\mathbb{T}_{\mathfrak{m}}$ is Gorenstein, then there is a canonical generator $\{0, \infty\} \in X$ and there is a unique $L_p(\mathfrak{m}) \in \mathbb{T}_{\mathfrak{m}}[\![\mathbb{Z}_p^\times]\!]$ such that

$$\mathcal{L}_p^+(\mathfrak{m}) = L_p^+(\mathfrak{m}) \cdot \{0, \infty\}.$$

The element $L_p^+(\mathfrak{m})$ can be viewed as a two-variable p-adic L-function. Indeed, the cyclotomic variable comes from the fact that $L_p^+(\mathfrak{m})$ lives in a group algebra over $\mathbb{Z}_p^\times$ while the weight variable is parametrized by $\mathrm{Spec}(\mathbb{T}_{\mathfrak{m}})$. If f is a classical form in our Hida family and $\mathfrak{p}_f$ is the corresponding height 1 prime ideal of $\mathbb{T}_{\mathfrak{m}}$, then the image of $L_p^+(\mathfrak{m})$ in $\mathbb{T}_{\mathfrak{m}}/\mathfrak{p}_f[\![\mathbb{Z}_p^\times]\!]$ is the one-variable p-adic L-function of f.

As discussed in length in [BP19], the specializations of $L_p^+(\mathfrak{m})$ to Eisenstein forms in the family always vanish and this is the reason why the μ-invariants blow up in the family. We write $L_p^{+,\mathrm{mod}}(\mathfrak{m})$ for the image of $L_p^+(\mathfrak{m})$ in $\mathbb{T}^0_{\mathfrak{m}}[\![\mathbb{Z}_p^\times]\!]$ and we refer to $L_p^{+,\mathrm{mod}}(\mathfrak{m})$ as the two-variable p-adic L-function with the *modular* normalization. Here "modular" is referring to the fact that this L-function first arose from a Hecke-algebra on the full space of modular forms.

On the other hand there is another natural normalization. A closer examination of the definition of the p-adic L-symbol shows that it actually lives in the cohomology of the compact modular curve:

$$\mathcal{L}_p^+(\mathfrak{m}) \in X^0[\![\mathbb{Z}_p^\times]\!] := \varprojlim_n H^1(X_1(Np^r), \mathbb{Z}_p)_{\mathfrak{m}}^{-,\mathrm{ord}}[\![\mathbb{Z}_p^\times]\!].$$

If we instead assume that $\mathbb{T}^0_{\mathfrak{m}}$, the cuspidal Hecke algebra localized at $\mathfrak{m}$, is Gorenstein, then we again have X^0 is free of rank 1 over $\mathbb{T}^0_{\mathfrak{m}}$; however, in this case there is no natural generator of this space. Nonetheless, pick a generator e of X^0, and write

$$\mathcal{L}_p^+(\mathfrak{m}) = L_p^{+,\mathrm{cusp}}(\mathfrak{m}) \cdot e$$

for a unique $L_p^{+,\mathrm{cusp}}(\mathfrak{m}) \in \mathbb{T}^0_{\mathfrak{m}}[\![\mathbb{Z}_p^\times]\!]$. We call this a two-variable p-adic L-function with the *cuspidal* normalization as it was defined directly from the cuspidal Hecke algebra. It is well-defined up to a unit in $\mathbb{T}^0_{\mathfrak{m}}$.

When both $\mathbb{T}_{\mathfrak{m}}$ and $\mathbb{T}^0_{\mathfrak{m}}$ are Gorenstein, then both definitions $L_p^{+,\mathrm{mod}}(\mathfrak{m})$ and $L_p^{+,\mathrm{cusp}}(\mathfrak{m})$ make sense and one can ask how they are related. Following ideas of Ohta [Oht05], we show that both $\mathbb{T}_{\mathfrak{m}}$ and $\mathbb{T}^0_{\mathfrak{m}}$ are Gorenstein if and only if the Eisenstein ideal $\mathcal{I}_{\mathfrak{m}}$ is principal. Moreover, we show that $L_p^{+,\mathrm{cusp}}(\mathfrak{m})$ divides $L_p^{+,\mathrm{mod}}(\mathfrak{m})$ and that the ratio is a generator of the Eisenstein ideal. Note that in Theorem 1.2, $U_p - 1$ generates the Eisenstein ideal and this is exactly the analytic factor that causes the μ-invariants to grow in the family. In that theorem, the modular normalization is used. If we instead used the cuspidal normalization, we would lose that factor of $U_p - 1$ and all of the μ-invariants would be 0.

1.5. **Further questions.** In this paper, we only consider that analytic side of Iwasawa theory, in that we only consider p-adic L-functions. There is a parallel algebraic side, involving Selmer groups. This algebraic side is particularly interesting in view of conjectures of Greenberg [Gre01] regarding algebraic μ-invariants of elliptic curves. We are currently working to prove the algebraic analog of Theorem 1.2.

We also restrict our attention to the case where the tame level N is prime and where $p \nmid B_{k_0}$. This restriction on the level is natural in view of our goal to understand the context for the example of $X_0(11)$. However, this assumption has the drawback of limiting the types of examples we can explore: for instance, $X_0(11)$ is the only elliptic curve we know of for which Theorem 1.2 applies. Many of the techniques developed in this paper are general and can be applied to other situations as soon as the relevant results on the structure of Hecke algebras are known. Some interesting situations to consider include:

- N is squarefree. (There are some results on the structure of Hecke algebras proven in [WWE21].)
- N is the square of a prime. (This situation was considered in [LW22].)
- N is prime but $p \mid B_{k_0}$, so there are congruences of both "L-value" and "Euler-factor" type.

In this greater generality, it will usually not be true that the Eisenstein ideal is principal, so it is unlikely that one can hope for results as precise as Theorem 1.2. We make some conjectures (see Conjectures 4.8 and 5.22) regarding what is happening with Iwasawa invariants in some cases where the Eisenstein ideal fails to be principal. We hope that these conjectures can serve as a starting point for investigating these more general situations.

1.6. **Structure of paper.** In Section 2, we begin with an axiomatic approach to describing Hecke algebras, Eisenstein ideals, and p-adic L-functions when there is a unique Eisenstein series parametrized by our Hecke algebra. This axiomatic approach applies equally to the case where the congruences arise by p-divisibility of L-values or of Euler factors and studies the two possible normalizations discussed in §1.4. In Section 3, we review the construction of the two-variable p-adic L-symbol $\mathcal{L}_p^+(\mathfrak{m})$. In Section 4, we consider the setting of [BP19]. We reprove the main results of [BP19] as an illustration of the axiomatics of Section 2 and also make conjectures about what happens when the Eisenstein ideal is not principal. In Section 5, we move to the case of trivial tame character and prime level. This section contains our main results on analytic μ-invariants. Finally, in Appendix A, we verify that Hida theory works equally well for Hecke algebras in which U_N has been replaced by w_N.

Acknowledgements: We dedicate this paper to the memory of Joël Bellaïche. The impact of Joël's mathematics on this paper, from his pioneering work on pseudorepresentations to the paper [BP19], will be clear to the reader. Even the genesis of this paper goes back to a meeting between the authors and Joël, about [BP19] and generalizations, at the Glenn Stevens birthday conference in 2014. At the same conference, Joël first introduced the second author to Carl Wang-Erickson, which led to a fruitful collaboration that provided many of the results that make this paper possible, especially [WWE20, WWE17, WWE21].

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2. Axiomatics

In Section 4, we consider the same situation as [BP19], with a primitive tame character, and in Section 5, we consider the non-primitive setting of Mazur's Eisenstein ideal. The two situations have a lot of commonalities, and to emphasize this, in this section, we consider a purely abstract situation which covers both of these at once.

The main point is that the standard construction of the two-variable p-adic L-function gives not a *function* but an element of a two-variable modular symbols space. In both situations, the modular symbols space is known to be isomorphic to the *dualizing module* of the cuspidal Hecke algebra. If this dualizing module is isomorphic to the Hecke algebra (*i.e.* if the cuspidal Hecke algebra is Gorenstein), then the symbol gives a function. The p-adic L-function can also be thought of as an element of the larger dualizing module of the full Hecke algebra, and if that Hecke algebra is Gorenstein, then the symbol again gives a function. If both Hecke algebras are Gorenstein, this gives two different ways to produce a function from the symbol, and the two functions are different by a normalizing factor.

Much of this discussion revolves around commutative algebra questions of whether one or the other, or both, or these Hecke algebras is Gorenstein. In this section, we discuss these commutative algebra issues in the abstract.

2.1. Set up.

The first key property common to both of the situations we consider is that, though there are congruences between cuspidal families and an Eisenstein family, there are no congruences between different Eisenstein families. We axiomatize this as follows.

Definition 2.1. A *single-Eisenstein Hecke algebra set up* is the data of $(\mathcal{O}, A, R, \mathcal{E}, r_0)$ where

- $\mathcal{O}$ is a complete discrete valuation ring with uniformizer ϖ and residue field k,
- A is a flat, local $\mathcal{O}$-algebra that is a complete commutative Noetherian local complete intersection ring with maximal ideal $\mathfrak{m}_A$,
- R is a commutative local A-algebra,
- $\mathcal{E} : R \to A$ is an A-algebra homomorphism, and

- $r_0 \in R$ is an element the annihilates $\ker(\mathcal{E})$

satisfying the conditions

- R is a free A-module of finite rank d,
- $R/\mathrm{Ann}_R(\ker(\mathcal{E}))$ is a free A-module of rank $d-1$,
- r_0 generates $\mathrm{Ann}_R(\ker(\mathcal{E}))$ as an A-module, and
- $\mathcal{E}(r_0) \in A$ is a not a zero divisor.

Given this set up, let $I = \ker(\mathcal{E})$, let $R^0 = R/\mathrm{Ann}_R(I)$, and let $I^0 \subset R^0$ be the image of I in R^0.

Remark 2.2. In applications, $\mathcal{O}$ will be $\mathbb{Z}_p$ or a finite extension of $\mathbb{Z}_p$, A will be a ring of diamond operators, R will be a localization of a Hecke algebra at an Eisenstein maximal ideal, $\mathcal{E}$ will be the action of R on an A-family of Eisenstein series, and r_0 will be a Hecke operator with the property, for a modular form f, the coefficient $a_1(r_0 f)$ of q in the q-expansion of $r_0 f$ equals the residue of f at a cusp. If $\mathcal{E}$ is the only family of Eisenstein series supported by R, then I is the Eisenstein ideal and R^0 is maximal the quotient of R that acts faithfully on cuspforms.

Let $(\mathcal{O}, A, R, \mathcal{E}, r_0)$ be a single-Eisenstein Hecke algebra set up. Note that the map $\mathcal{E}$ induces an isomorphism $R/I \cong A$, making A into an R-module. Moreover, the map $A \to \mathrm{Ann}_R(I)$ given by $1 \mapsto r_0$ is an isomorphism of R-modules, so that for all $r \in R$,

$$r \cdot r_0 = \mathcal{E}(r) r_0. \tag{2.3}$$

Note also that the action of R on I factors through R^0, so that if $r \in R^0$ and $y \in I$, then ry is a well-defined element of I.

The second key property is that the p-adic L-function is given as an element of a module that is dual to the cuspidal Hecke algebra.

Definition 2.4. Let $(\mathcal{O}, A, R, \mathcal{E}, r_0)$ be a single-Eisenstein Hecke algebra set up. Then an *L-symbol set up* is the data of a triple $(X, \phi, x, \mathcal{L})$ where

- X is a R-module,
- $\phi : X \to A$ is an A-linear functional on X, with kernel $X^0 = \ker(\phi)$,
- $x \in X$ is an element such that $\phi(x) = 1$, and
- $\mathcal{L} \in \ker(\phi) \otimes_{\mathcal{O}} \mathcal{O}[\![\mathbb{Z}_p^\times]\!]$,

satisfying the condition that

- there is an isomorphism $f : X \xrightarrow{\sim} \mathrm{Hom}_A(R, A)$ of R-modules sending X^0 to $\mathrm{Hom}_A(R^0, A)$.

Note that, for all $g \in \mathrm{Hom}_A(R, A)$ and $r \in R$, the element $(r - \mathcal{E}(r))g$ is in $\mathrm{Hom}_A(R^0, A)$ by (2.3). This implies that, for all $y \in X$, the element $(r - \mathcal{E}(r))y$ is in X^0, so that ϕ must be an isomorphism of R-modules:

$$\phi(ry) = \mathcal{E}(r)\phi(y).$$

Recall that a local ring is called *Gorenstein* if it has a dualizing module that is free of rank 1. For our purposes, the important facts to remember are that local complete intersection rings are Gorenstein, and that if $B \to B'$ is a finite flat local ring homomorphism and M is a dualizing module for B, then $\mathrm{Hom}_B(B', M)$ is a dualizing module for B'. In particular, in an L-symbol set up, X is a dualizing module for R and X^0 is a dualizing module for R^0.

2.2. **Algebra around Gorensteinness.** Let $(\mathcal{O}, A, R, \mathcal{E}, r_0)$ be a single-Eisenstein Hecke algebra set up. The composite map

$$R \xrightarrow{\mathcal{E}} A \to A/\mathcal{E}(r_0)A$$

factors through R^0. Let $A^0 = A/\mathcal{E}(r_0)A$ and let $\mathcal{E}^0 : R^0 \to A^0$ denote the induced map. Since the two quotient maps $R \to R^0 \xrightarrow{\mathcal{E}^0} A^0$ and $R \xrightarrow{\mathcal{E}} A \to A^0$ are equal, there is a map to the fiber product:

$$R \to R^0 \times_{A^0} A.$$

Lemma 2.5. *The map $R \to R^0 \times_{A^0} A$ is an isomorphism of A-algebras. In particular, the natural maps $R^0/I^0 \to A^0$ and $I \to I^0$ are isomorphisms of R-modules.*

Proof. Consider the commutative diagram of R-modules with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Ar_0 & \longrightarrow & R & \longrightarrow & R^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \mathcal{E}} & & \downarrow{\scriptstyle \mathcal{E}^0} & & \\
0 & \longrightarrow & A\mathcal{E}(r_0) & \longrightarrow & A & \longrightarrow & A^0 & \longrightarrow & 0.
\end{array}$$

The leftmost vertical arrow is an isomorphism: it is surjective by definition and injective since $\mathcal{E}(r_0)$ is assumed to be a non-zero-divisor. By the five-lemma, the map $\ker(\mathcal{E}) \to \ker(\mathcal{E}^0)$ is an isomorphism. This is enough to imply that the map $R \to R^0 \times_{A^0} A$ is an isomorphism. Indeed, the kernel is easily seen to equal $\ker(Ar_0 \to A\mathcal{E}(r_0))$, which is zero. To see it is surjective, let $(t^0, a) \in R^0 \times_{A^0} A$, and choose a lift $t \in R$ of t. Since $\mathcal{E}(t) \equiv \mathcal{E}^0(t^0) \equiv a \mod A\mathcal{E}(r_0)$, it follows that $\mathcal{E}(t) - a \in A\mathcal{E}(r_0)$, so $\mathcal{E}(t) - a = a'\mathcal{E}(r_0)$ for some $a' \in A$. Then $t - a'r_0$ maps to $(t^0, \mathcal{E}(t - a'r_0)) = (t^0, a)$. □

Now let $(X, \phi, x, \mathcal{L})$ be an L-symbols set up. The following gives criteria for R to be Gorenstein.

Lemma 2.6. *The following are equivalent:*

1. *(1) R is Gorenstein,*
2. *(2) X is a free R-module of rank 1,*
3. *(3) X is cyclic as an R-module,*
4. *(4) X is generated by x as an R-module,*
5. *(5) The map $I \to X^0$ given by $t \mapsto t \cdot x$ is an isomorphism of R-modules.*

Proof. The equivalence of (1) and (2) is the definition of Gorenstein, and clearly (2) implies (3). Assume (3), and let $y \in X$ be a generator of X as an R-module. Let $r \in R$ be such that $ry = x$. Then

$$1 = \phi(x) = \phi(ry) = \mathcal{E}(r)\phi(y).$$

This implies $\mathcal{E}(r) \in A^\times$, so, since R is local, $r \in R^\times$. Since $x = ry$, this implies (4). Since X is a faithful R-module, (4) implies (2). Lastly, the equivalence of (4) and (5) follows by applying the five-lemma to the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & R & \xrightarrow{\mathcal{E}} & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle r \mapsto r\cdot x} & & \| & & \\
0 & \longrightarrow & X^0 & \longrightarrow & X & \xrightarrow{\phi} & A & \longrightarrow & 0.
\end{array} \tag{2.7}$$

□

It follows from the lemma that if R is Gorenstein, then I is a dualizing module for R^0. In particular, if R and R^0 are both Gorenstein, then I is principal. We will see that the converse is true as well. First, we require the following lemma on the structure of R when I is principal.

Lemma 2.8. *Assume that I is a principal ideal with a generator $t \in I$. Let $F(X) \in A[X]$ be the characteristic polynomial of the A-linear endomorphism*

$$R^0 \to R^0,\ r \mapsto tr.$$

Then there are A-algebra isomorphisms

$$A[X]/(F(X)) \xrightarrow{\sim} R^0,\ A[X]/(XF(X)) \xrightarrow{\sim} R$$

given by $X \mapsto t$. Moreover, $F(X)$ is a distinguished polynomial with $F(0)A = \mathcal{E}(r_0)A$.

Proof. Let ϕ denote the A-algebra homomorphism

$$A[X] \to R,\ X \mapsto t.$$

First note that ϕ is surjective. To see this, it suffices, by Nakayama's lemma to prove that $\phi \otimes_A A/\mathfrak{m}_A$ is surjective, so we may assume that A is a field. In that case, R is a d-dimensional algebra over the field A with maximal ideal I, so $I^d = 0$. Fix $r \in R$ and note that $r - \mathcal{E}(r) \in I$, so there is $r_1 \in R$ such that $r = \mathcal{E}(r) + r_1 t$. Let $r_0 = R$ and inductively choose $r_i \in R$ such that

$$r_i = \mathcal{E}(r_i) + r_{i+1}t.$$

Then

$$r = \mathcal{E}(r_0) + \mathcal{E}(r_1)t + \cdots + \mathcal{E}(r_{d-1})t^{d-1},$$

so r is in the image of ϕ.

Composing ϕ with the surjection $R \twoheadrightarrow R^0$ yields a surjective map $\phi^0 : A[X] \to R^0$. Then ϕ^0 factors through $A[X]/(F(X))$ by the Cayley-Hamilton Theorem, and, since $F(X)$ is monic, $A[X]/(F(X))$ is a free A-module of rank equal to the degree of $F(X)$. Since R^0 is a free A-module of rank equal to $\deg(F)$, this implies that ϕ^0 is an isomorphism. Since R^0 is local, it follows that $F(X)$ is distinguished. The remaining parts follow from the isomorphism $R \cong R^0 \times_{A^0} A$ of Lemma 2.5. □

Example 2.9. Suppose that $R^0 = A$. Then Lemma 2.5 implies that $I^0 = \mathcal{E}(r_0)A$ and that I is generated by $r_0 - \mathcal{E}(r_0)$. Then there is an isomorphism

$$A[X]/(X^2 - \mathcal{E}(r_0)X) \xrightarrow{\sim} R$$

given by $X \mapsto r_0 - \mathcal{E}(r_0)$.

Lemma 2.10. *The following are equivalent:*

1. *I^0 is a principal ideal.*
2. *I is a principal ideal.*
3. *I is a free R^0-module of rank 1.*
4. *Both R and R^0 are local complete intersection rings.*
5. *Both R and R^0 are Gorenstein.*
6. *There is $t \in I$ such that $t \cdot x$ is a generator of X^0 as a R^0-module.*

Furthermore, if all of these statements are true, then (6) is true for all generators t of I and every such t is a generator.

Proof. By Lemma 2.5, the map $I \to I^0$ is an isomorphism, so (1) implies (2). Since I is a faithful R^0-module, (2) implies (3), and clearly (3) implies (1).

Now assume (2). By Lemma 2.8, there are isomorphisms

$$R^0/\mathfrak{m}_A R^0 \cong k[X]/(X^{d-1}),\ R/\mathfrak{m}_A R \cong k[X]/(X^d),$$

so $R^0/\mathfrak{m}_A R^0$ and $R/\mathfrak{m}_A R$ are visibly local complete intersection rings. Since A is a local complete intersection and $A \to R^0$ and $A \to R$ are flat, this implies (4) (see [Sta, Tag 09Q7]). Moreover, (4) implies (5) by general algebra (see [Sta, Tag 0DW6]).

Now assume (5). Since R^0 is Gorenstein, there is a generator $x^0 \in X^0$ of X^0 as an R^0-module. By Lemma 2.6 (part (5)) implies that x^0 is of the form $x^0 = tx$ for some $t \in I$, proving (6).

Now assume (6). Then the leftmost vertical map in (2.7) is surjective. By the snake lemma, the center vertical map is also surjective, and, since X is a faithful R-module, this implies that the center vertical map (and hence all the vertical maps by the 5 lemma) in (2.7) are isomorphisms. By (6), the composite map

$$R^0 \xrightarrow{r \mapsto rt} I \xrightarrow{a \mapsto ax} X^0$$

is surjective. Since the second map is an isomorphism, this implies that t generates I, proving (2). □

2.3. **Algebra around L-functions.** Let $(\mathcal{O}, A, R, \mathcal{E}, r_0)$ be a single-Eisenstein Hecke algebra set up and let $(X, \phi, x, \mathcal{L})$ be an L-symbol set up for it. In this section, we define different elements of $R[\![\mathbb{Z}_p^\times]\!]$, under different Gorenstein hypotheses, which we think of as normalizations of the L-function (as in Section 1.4) associated to the L-symbol set up. When both R and R^0 are Gorenstein, then there are two distinct normalizations and we compare them.

Lemma 2.11. *Suppose that R is Gorenstein. Then there is a unique $L \in R[\![\mathbb{Z}_p^\times]\!]$ such that $\mathcal{L} = L \cdot x$. Moreover, L is in the subset $I[\![\mathbb{Z}_p^\times]\!] \subset R[\![\mathbb{Z}_p^\times]\!]$.*

Proof. By Lemma 2.6, the map

$$R[\![\mathbb{Z}_p^\times]\!] \xrightarrow{r \mapsto rx} X \otimes_{\mathcal{O}} \mathcal{O}[\![\mathbb{Z}_p^\times]\!]$$

is an isomorphism and it induces an isomorphism $I[\![\mathbb{Z}_p^\times]\!] \to X^0 \otimes_{\mathcal{O}} \mathcal{O}[\![\mathbb{Z}_p^\times]\!]$. Then L is the preimage of $\mathcal{L}$ under the second isomorphism. □

Definition 2.12. If R is Gorenstein, the element $L \in R[\![\mathbb{Z}_p^\times]\!]$ of Lemma 2.11 is called the *modular normalization of the p-adic L-function.*

Lemma 2.13. *Suppose that R^0 is Gorenstein, so that X^0 is a free R^0-module of rank 1. Then, for each generator e of X^0, there is a unique $L_e^0 \in R^0[\![\mathbb{Z}_p^\times]\!]$ such that $\mathcal{L} = L_e^0 \cdot e$. Moreover, the class of L_e^0 in the quotient multiplicitive monoid $R^0[\![\mathbb{Z}_p^\times]\!]/(R^0)^\times$ is independent of the choice of generator e.*

Proof. The first part is clear since $X^0 \otimes_{\mathcal{O}} \mathcal{O}[\![\mathbb{Z}_p^\times]\!]$ is a free $R^0[\![\mathbb{Z}_p^\times]\!]$-module with generator e. If f is another generator for X^0, then $f = Ue$ for a unique $U \in (R^0)^\times$, and

$$L_e^0 \cdot e = \mathcal{L} = L_f^0 \cdot f = L_f^0 \cdot Ue$$

so $L_e^0 = L_f^0 \cdot U$ and the second part follows. □

Definition 2.14. If R^0 is Gorenstein, let L^0 denote the class of L^0_e in $R^0[\![\mathbb{Z}_p^\times]\!]/(R^0)^\times$ for some choice of e (Lemma 2.13 says that L^0 is independent of the choice). The class L^0 is called the *cuspidal normalization of the p-adic L-function.*

If both R and R^0 are Gorenstein, then there are two different normalizations L and L^0. By Lemma 2.10, both R and R^0 are Gorenstein if and only if I is principal; the following lemma essentially says that the two normalizations L and L^0 differ by a generator of I.

Lemma 2.15. *Suppose that I is principal.*

(1) There is a unique element $L \in I[\![\mathbb{Z}_p^\times]\!]$ such that $\mathcal{L} = L \cdot x$.
(2) For each generator $t \in I$, the element $tx \in X^0$ is a generator for X^0 as a R^0-module, and there is a unique element $L^0_{tx} \in R^0[\![\mathbb{Z}_p^\times]\!]$ such that $\mathcal{L} = L^0_{tx} \cdot tx$.
(3) For each generator $t \in I$, there is an equality $L = L^0_{tx} \cdot t$ of elements of $I[\![\mathbb{Z}_p^\times]\!]$.

Proof.

(1) By Lemma 2.10, R is Gorenstein, so (1) follows from Lemma 2.11.
(2) By Lemma 2.10, $tx \in X^0$ is a generator for X^0 as a T^0-module, and (2) follows from Lemma 2.13.
(3) By Lemmas 2.5 and 2.10, the map $R \to I$ given by $1 \mapsto t$ factors through an isomorphism $R^0 \xrightarrow{\sim} I$. Since $L \in I[\![\mathbb{Z}_p^\times]\!]$, there is a unique element $L_t \in T^0[\![\mathbb{Z}_p^\times]\!]$ such that $L = L_t \cdot t$. Then, by (1), $\mathcal{L} = L_t \cdot tx$. On the other hand, by (2), L^0_{tx} is the unique element of $T^0[\![\mathbb{Z}_p^\times]\!]$ satisfying $\mathcal{L} = L^0_{tx} \cdot tx$. Hence $L_t = L^0_{tx}$, and (3) follows.

□

Note that $R[\![\mathbb{Z}_p^\times]\!]$ (and similarly $R^0[\![\mathbb{Z}_p^\times]\!]$) is a semi-local ring with components labeled by the characters of $(\mathbb{Z}/p\mathbb{Z})^\times$; choosing a generator of $1+p\mathbb{Z}_p$, each component is isomorphic to a power series ring $R[\![u]\!]$. For $j \in \{0,\dots,p-2\}$ and $f \in R[\![\mathbb{Z}_p^\times]\!]$, let $f(\omega^j) \in R[\![u]\!]$ denote the image of f in the ω^j-component. If $c : R[\![u]\!] \to R$ and is a R-algebra homomorphism, we often write $f(\omega^j, c) \in R$ instead of $c(f(\omega^j))$, and think of this "as evaluation at c".

Lemma 2.16. *Suppose that I is principal and that there is a $j \in \{0,\dots,p-2\}$ and an R-algebra homomorphism $c : R[\![u]\!] \twoheadrightarrow R$ such that $L(\omega^j, c) \in I$ is a generator. Let $t = L(\omega^j, c)$ and let $L^0_{tx} \in R^0[\![\mathbb{Z}_p^\times]\!]$ be as in Lemma 2.15(2). Then:*

(1) $L(\omega^j) = L(\omega^j, c) \cdot L^0_{tx}(\omega^j)$, and
(2) $L^0_{tx}(\omega^j) \in (R^0[\![u]\!])^\times$.

In particular, $L^0(\omega^j) \in (R^0[\![u]\!])^\times/(R^0)^\times$ and $L(\omega^j) \equiv L(\omega^j, c) \pmod{(R^0[\![u]\!])^\times}$.

Proof. By Lemma 2.15 (3), there is an equality $L(\omega^j) = L^0_{tx}(\omega^j)t.$, which proves (1). Applying the homomorphism c yields

$$t = L(\omega^j, c) = c(L^0_{tx}(\omega^j))t.$$

Since I is a free R^0-module, this implies that $c(L^0_{tx}(\omega^j)) = 1$ and hence that $L^0_{tx}(\omega^j) \in (R^0[\![u]\!])^\times$ as $R^0[\![u]\!]$ is local. This proves (2). □

2.4. **Content and μ-invariant.** These results about the relationships between the modular and cuspidal normalizations have implications about their content and μ-invariants. We first recall the definitions of these concepts.

Definition 2.17. Let B be a commutative ring B. The *content* of a power series $f = \sum a_i(f)u^i \in B[\![u]\!]$ is the ideal $\mathrm{content}_B(f) \subset B$ generated by all the coefficients $a_i(f)$; the series f *has unit content* if $\mathrm{content}_B(f) = B$. Note that, for every $b \in B$, there is an equality

$$\mathrm{content}_B(bf) = b \cdot \mathrm{content}_B(f), \tag{2.18}$$

and, in particular, that the content of f depends only on the image of f in $B[\![u]\!]/B^\times$.

If B is local and f has unit content, then $a_i(f) \in R^\times$ for some i, and the λ-*invariant* $\lambda(f) \in \mathbb{Z}$ is defined as the minimal i such that $a_i(f) \in R^\times$. Note that f is a unit if and only if it has unit content and $\lambda(f) = 0$.

Finally, if B is a DVR with uniformizer ϖ, then the μ-*invariant* $\mu(f) \in \mathbb{Z}$ is defined to the unique integer n such that $\mathrm{content}_R(f) = \varpi^n R$. In this case, f has unit content if and only if $\mu(f) = 0$.

Note that the values of a power series function are in the content ideal. In other words, if $c : B[\![u]\!] \to B$ is B-algebra homomorphism, then $c(f) \in \mathrm{content}_B(f)$. Indeed, $c(f) = \sum a_i(f)c(u)^i$ and each $a_i(f)$ is in $\mathrm{content}_B(f)$.

Now let $(\mathcal{O}, A, R, \mathcal{E}, r_0)$ be a single-Eisenstein Hecke algebra set up and let $(X, \phi, x, \mathcal{L})$ be an L-symbol set up for it.

Lemma 2.19. *Assume that I is principal, fix $j \in \{0, \dots, p-1\}$, and let $c\colon R[\![u]\!] \to R$ be an R-algebra homomorphism.*

(1) $L(\omega^j, c) \in \mathrm{content}_R(L(\omega^j))$.
(2) There is an equality $\mathrm{content}_R(L(\omega^j)) = \mathrm{content}_R(L^0(\omega^j))I$ of ideals in R.
(3) If $L(\omega^j, c)$ is a generator of I, then $\mathrm{content}_R(L(\omega^j)) = I$ and $L^0(\omega^j)$ is a unit.

Proof. (1) is the general fact that the values of a power series function are in the content ideal. (2) and (3) follow from Lemmas 2.15 and 2.16, respectively. □

To discuss μ-invariants, we must work over a DVR. We now fix an $\mathcal{O}$-algebra homomorphism

$$w : A \to \mathcal{O},$$

which, in our applications, will correspond to fixing a weight. Let $R_w = R \otimes_{A,w} \mathcal{O}$ and I_w be the image of I in R_w, and similarly for R_w^0 and I_w^0.

Assume that I is principal and fix $j \in \{0, \dots, p-2\}$. Let $t \in I$ be a generator and let $F(X) \in A[X]$ be the characteristic polynomial of t, as in Lemma 2.8. Let $F_w(X) \in \mathcal{O}[X]$ denote the image of $F(X)$ under w. Let $F_w(X) = \prod_{i=1}^r F_{w,i}(X)$ be the factorization of $F_w(X)$ in $\mathcal{O}[X]$ into irreducible polynomials, and let $\mathcal{O}_{w,i}$ be the normalization of $\mathcal{O}[X]/(F_{w,i}(X))$, and let $\varpi_{w,i}$ be a uniformizer in $\mathcal{O}_{w,i}$ and let $X_{w,i}$ be the image of X in $\mathcal{O}_{w,i}$. Then, by Lemma 2.8, the normalization of R_w^0 is isomorphic to $\prod_{i=1}^r \mathcal{O}_{w,i}$. For $i = 1, \dots, r$, let $L_{w,i}(\omega^j) \in \mathcal{O}_{w,i}[\![u]\!]$ be the image of $L(\omega^j)$ under the map

$$R[\![u]\!] \to R_w[\![u]\!] \to \mathcal{O}_{w,i}[\![u]\!].$$

Remark 2.20. In our applications, the irreducible factors $F_{w,i}(X)$ correspond to the cuspidal eigenforms of weight w that are congruent to the given Eisenstein series.

The rings $\mathcal{O}_{w,i}$ are the valuation ring in their corresponding Hecke fields, and the elements $L_{w,i}(\omega^j)$ are their p-adic L-functions.

Lemma 2.21. *With the notation as in the previous paragraph, suppose there is a surjective R-algebra homomorphism $c : R[\![u]\!] \to R$ such that $L(\omega^j, c) = t$.*

(1) There is an equality

$$\mathrm{val}_\varpi(F_w(0)) = \sum_{i=1}^{r} \mu(L_{w,i}(\omega^j)).$$

(2) Suppose that there is an integer $M > 0$ such that $\mathrm{val}_\varpi(F_w(0)) > Mr$. Then there exists an i such that $\mu(L_{w,i}(\omega^j)) > M$.

Proof. By Lemma 2.16, there is an equality

$$L(\omega^j) = L(\omega^j, c)L^0_{tx}(\omega^j) = tL^0_{tx}(\omega^j)$$

and $L^0_{tx}(\omega^j) \in R^0[\![u]\!]^\times$. Since, for each $i = 1, \dots, r$, the map

$$R \to R_w \to \mathcal{O}_{w,i}$$

sends t to $X_{w,i}$, it follows that $L_{w,i}(\omega^j) = X_{w,i}U_i$ for a unit $U_i \in \mathcal{O}_{w,i}[\![u]\!]^\times$. In particular, the μ-invariant of $L_{w,i}(\omega^j)$ is the valuation of $X_{w,i}$:

$$\mu(L_{w,i}(\omega^j)) = \mathrm{val}_{\varpi_{w,i}}(X_{w,i}).$$

Since $\mathcal{O}_{w,i}$ is the normalization of $\mathcal{O}[X]/(F_{w,i}(X))$, the valuation of $X_{w,i}$ is given by

$$\mathrm{val}_{\varpi_{w,i}}(X_{w,i}) = \mathrm{val}_\varpi(F_{w,i}(0)).$$

Combining the last two equalities with the fact that $F_w(0) = \prod_i F_{w,i}(0)$ gives

$$\mathrm{val}_\varpi(F_w(0)) = \sum_{i=1}^{r} \mathrm{val}_\varpi(F_{w,i}(0)) = \sum_{i=1}^{r} \mu(L_{w,i}(\omega^j)),$$

which proves (1). Part (2) is clear from (1). □

Example 2.22. Suppose that $R^0 = A$. Then, by Example 2.9, $F(X) = X + \mathcal{E}(r_0)$, so $F_w(X) = X - w(\mathcal{E}(r_0))$ for every choice of w, and $\mu(L_w(\omega^j)) = \mathrm{val}_\varpi(w(\mathcal{E}(r_0)))$.

3. Two-variable p-adic L-symbols

The singular cohomology groups

$$H^1(Y_1(Np^r), \mathbb{Z}_p) \text{ and } H^1(X_1(Np^r), \mathbb{Z}_p)$$

can be respectively identified with the homology groups

$$H_1(Y_1(Np^r), \{\text{cusps}\}, \mathbb{Z}_p) \text{ and } H_1(X_1(Np^r), \mathbb{Z}_p)$$

as in [Sha11, Proposition 3.5]. We will make this identification implicitly and write $\{\alpha, \beta\}_r$ for the cohomology class corresponding to the geodesic connecting α to β for α and β in $\mathbb{P}^1(\mathbb{Q})$. These groups can also be identified with étale (co)homology groups, but one has to be careful about Galois actions, since Poincaré duality has a one-Tate-twist in it—this is all discussed in [Sha11, Section 3.5].

Write $H^1(Y_1(Np^r), \mathbb{Z}_p)^{\pm,\mathrm{ord}}$ for the ordinary subspace of this cohomology group with sign $\pm$ and set $\{\alpha, \beta\}_r^{\pm,\mathrm{ord}}$ to be the projection of $\{\alpha, \beta\}_r$ to this subspace.

Set $H^1_\Lambda(Y_1(N)) := \varprojlim_r H^1(Y_1(Np^r), \mathbb{Z}_p)^{\mathrm{ord}}$ and analogously define $H^1_\Lambda(X_1(N))$. We can then write down the two-variable p-adic L-symbol explictly as follows:

$$\mathcal{L}_p^{\pm} := \varprojlim_r \left(\sum_{a \in (\mathbb{Z}/p^r\mathbb{Z})^\times} U_p^{-r}\{\infty, a/p^r\}_r^{\mp,\mathrm{ord}} \otimes [a] \right)_r \in H^1_\Lambda(Y_1(N))^{\mp} \otimes \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!].$$

(The sign change here is intentional: we want to consider $\mathcal{L}_p$ as a *functional* on cohomology classes (*i.e.* a homology class), and the sign change appears because of the one-twist mentioned above in Poincaré duality. With this convention, $\mathcal{L}_p^{\pm}$ is a functional on cohomology classes of the same sign). By [Oht99, Proposition 4.3.4], we have that $\{\infty, a/p^r\}_r^{\mathrm{ord}}$ is in $H^1(X_1(Np^r), \mathbb{Z}_p)^{\mathrm{ord}}$ and thus $\mathcal{L}_p^{\pm}$ actually lives in $H^1_\Lambda(X_1(N))^{\mp} \otimes \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]$.

Lastly, let $\mathbb{T}$ denote the Hida Hecke algebra (to be defined more carefully in the following section). Then $H^1_\Lambda(Y_1(N))^{\pm}$ is a $\mathbb{T}$-module and for any maximal ideal $\mathfrak{m} \subseteq \mathbb{T}$, we have that $H^1_\Lambda(Y_1(N))^{\pm}_{\mathfrak{m}}$ is a direct summand of $H^1_\Lambda(Y_1(N))^{\pm}$. We write $\mathcal{L}_p^{\pm}(\mathfrak{m})$ for the projection of $\mathcal{L}_p^{\pm}$ to $H^1_\Lambda(Y_1(N))^{\pm}_{\mathfrak{m}}[\![\mathbb{Z}_p^\times]\!]$.

4. The primitive case

In this section, we consider the case Eisenstein families with a *primitive* tame character. In this case, congruences modulo p between Eisenstein series and cuspforms arise because p divides an L-value, as in [Rib76]. In the next section, we will consider the case of *trivial* tame character and congruences that occur because p divides an Euler-factor, as in [Maz77]. This primitive case is the same setting that was considered in [BP19] and we obtain similar results. The main novelty is that we highlight the role played by the two possible normalizations, modular and cuspidal, which allows us to obtain results and conjectures when the Eisenstein ideal is not assumed to be principal.

4.1. **Setup.** Let $p \geq 5$ and let N be an integer with $p \nmid N\varphi(N)$. Let $\mathfrak{H}$ denote the p-adic Hida Hecke algebra of tame level $\Gamma_1(N)$. It is an algebra over $\mathbb{Z}_p[\![(\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{Z}_p^\times]\!]$ generated by T_q for primes $q \nmid Np$ and U_ℓ for $\ell \mid Np$. The Eisenstein ideal $\mathcal{I} \subset \mathfrak{H}$ is the ideal generated by $T_q - (1 + \langle q\rangle q^{-1})$ for $q \nmid Np$ and by $U_\ell - 1$ for $\ell \mid Np$.

Let $\mathfrak{m} \subset \mathfrak{H}$ be a (Good Eisen) maximal ideal, in the sense of [BP19, §3.1], and let $\mathbb{T}_\mathfrak{m}$ be the completion of $\mathfrak{H}$ at $\mathfrak{m}$. This determines a character $\theta_\mathfrak{m} : (\mathbb{Z}/Np\mathbb{Z})^\times \to \overline{\mathbb{Q}}_p^\times$, as explained in *loc. cit.*, and the (Good Eisen) condition implies that the tame part $\theta_\mathfrak{m}|_{(\mathbb{Z}/N\mathbb{Z})^\times}$ of $\theta_\mathfrak{m}$ is primitive. Let $\mathcal{O}_\mathfrak{m}$ be the valuation ring in the p-adic field generated by the values of $\theta_\mathfrak{m}$ and let $\varpi \in \mathcal{O}_\mathfrak{m}$ be a uniformizer. Let $\Lambda_\mathfrak{m} = \mathcal{O}_\mathfrak{m}[\![T]\!]$; it is a regular local flat $\mathcal{O}_\mathfrak{m}$-algebra, and $\mathbb{T}_\mathfrak{m}$ is a finite flat local $\Lambda_\mathfrak{m}$. By [BP19, Lemma 3.1], there is a $\Lambda_\mathfrak{m}$-algebra homomorphism $\mathrm{Eis}_\mathfrak{m} : \mathbb{T}_\mathfrak{m} \to \Lambda_\mathfrak{m}$ with kernel $\mathcal{I}_\mathfrak{m}$ that satisfies $\mathrm{Ann}_{\mathbb{T}_\mathfrak{m}}(\mathcal{I}_\mathfrak{m}) \cong \Lambda_\mathfrak{m}$ and is generated by the Hecke operator T_0 determined by $a_1(T_0 f) = a_0(f)$.

It is straightforward to see that $(\mathcal{O}, A, R, \mathcal{E}, r_0) = (\mathcal{O}_\mathfrak{m}, \Lambda_\mathfrak{m}, \mathbb{T}_\mathfrak{m}, \mathrm{Eis}_\mathfrak{m}, T_0)$ is a single-Eisenstein Hecke algebra set up in the sense of Definition 2.1. We note that $\mathcal{E}(r_0) = \mathrm{Eis}_\mathfrak{m}(T_0)$ is the constant term of the Eisenstein family, which in this case is the Kubota-Leopoldt series $L_p(\psi_\mathfrak{m}^{-1}, \kappa) \in \Lambda_\mathfrak{m}$ of [BP19, §3.8].

We seek now to form our L-symbol set up as in Definition 2.4. Let $C_1(Np^r)$ denote the cusps of $X_1(Np^r)$; following [FK12, Section 1.3.2], cusps lying over the cusp 0 of $X_0(Np^r)$ will be called *0-cusps*. Let $\mathcal{C}_r = \ker(\mathbb{Z}_p[C_1(Np^r)(\mathbb{C})] \xrightarrow{\Sigma} \mathbb{Z}_p)$ where Σ is the augmentation map. There is an exact sequence

$$0 \to H^1(X_1(Np^r,\mathbb{Z}_p) \to H^1(Y_1(Np^r,\mathbb{Z}_p) \xrightarrow{\partial} \mathcal{C}_r \to 0, \tag{4.1}$$

where ∂ is the boundary map at the cusps, which satisfies $\partial(\{\alpha,\beta\}_r) = \alpha - \beta$. By [Oht03, Proposition (3.1.2)], the localized inverse limit $\mathcal{C}_\mathfrak{m} = (\varprojlim_r(\mathcal{C}_r))_\mathfrak{m}$ is free of rank one as a Λ-module, generated by the projection of the class of $0-\infty$ to the $\mathfrak{m}$-part. Taking inverse limit and localization of the sequences (4.1) then yields an exact sequence

$$0 \to H^1_\Lambda(X_1(N))_\mathfrak{m} \to H^1_\Lambda(Y_1(N))_\mathfrak{m} \xrightarrow{\phi} \Lambda_\mathfrak{m} \to 0 \tag{4.2}$$

where $\phi(\{0,\infty\}) = 1$. This is the exact sequence of [Oht03, Theorem (1.5.5) (III)] (see also [FK12, Section 6.2.5] for the description of ϕ as the boundary at 0-cusps).

Since $\{0,\infty\} \in H^1_\Lambda(Y_1(N))^-_\mathfrak{m}$, taking minus-parts of (4.2) yields an exact sequence

$$0 \to H^1_\Lambda(X_1(N))^-_\mathfrak{m} \to H^1_\Lambda(Y_1(N))^-_\mathfrak{m} \xrightarrow{\phi} \Lambda_\mathfrak{m} \to 0. \tag{4.3}$$

Proposition 4.4. *Taking $(X,\phi,x,\mathcal{L}) = (H^1_\Lambda(Y_1(N))^-_\mathfrak{m}, \phi, \{0,\infty\}, \mathcal{L}_p^+(\mathfrak{m}))$ gives an L-symbol set up.*

Proof. The exact sequence (4.3) implies that $\ker(\phi) = H^1_\Lambda(X_1(N))^-_\mathfrak{m}$. By [FK12, §1.7.13 and Proposition 6.3.5], $H^1_\Lambda(Y_1(N))^-_\mathfrak{m}$ is a dualizing module for $\mathbb{T}_\mathfrak{m}$ and $H^1_\Lambda(X_1(N))^-_\mathfrak{m}$ is a dualizing module for $\mathbb{T}^0_\mathfrak{m}$. Moreover, these isomorphism are compatible in the sense that the isomorphism $H^1_\Lambda(Y_1(N))^-_\mathfrak{m} \cong \mathrm{Hom}_{\Lambda_\mathfrak{m}}(\mathbb{T}_\mathfrak{m},\Lambda_\mathfrak{m})$ sends $H^1_\Lambda(X_1(N))^-_\mathfrak{m}$ isomorphically to $\mathrm{Hom}_{\Lambda_\mathfrak{m}}(\mathbb{T}^0_\mathfrak{m},\Lambda_\mathfrak{m})$ (see [Oht03, Diagram (3.5.3)]). Since $\phi(\{0,\infty\}) = 1$ and $\mathcal{L}_p^+(\mathfrak{m}) \in H^1_\Lambda(X_1(N))^-_\mathfrak{m} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]$, this completes the verification. □

4.2. **Results.** Applying the results of §2 yields the following theorem.

Theorem 4.5.

(1) *If $\mathbb{T}_\mathfrak{m}$ is Gorenstein, then there is a unique $L_p^+(\mathfrak{m}) \in \mathbb{T}_\mathfrak{m}[\![\mathbb{Z}_p^\times]\!]$ such that $\mathcal{L}_p^+(\mathfrak{m}) = L_p^+(\mathfrak{m}) \cdot \{0,\infty\}$. Moreover,* $\mathrm{content}(L_p^+(\mathfrak{m})) \subseteq \mathcal{I}_\mathfrak{m}$.

(2) *Suppose that $\mathbb{T}^0_\mathfrak{m}$ is Gorenstein, so that $H^1_\Lambda(X_1(N))^+_\mathfrak{m}$ is free of rank 1 over $\mathbb{T}^0_\mathfrak{m}$. Then for every generator e of $H^1_\Lambda(X_1(N))^+_\mathfrak{m}$, there is a unique $L_p^+(\mathfrak{m})^0_e \in \mathbb{T}^0_\mathfrak{m}[\![\mathbb{Z}_p^\times]\!]$ such that $\mathcal{L}_p^+(\mathfrak{m}) = L_p^+(\mathfrak{m})^0_e \cdot e$.*

(3) *If $\mathcal{I}_\mathfrak{m}$ is a principal ideal with generator t, then $\mathbb{T}_\mathfrak{m}$ and $\mathbb{T}^0_\mathfrak{m}$ are Gorenstein and $H^1_\Lambda(X_1(N))^+_\mathfrak{m}$ is generated by $t\{0,\infty\}$. Moreover, there is an equality*

$$L_p^+(\mathfrak{m}) = L_p^+(\mathfrak{m})^0_{t\{0,\infty\}} t$$

in $\mathcal{I}_\mathfrak{m}[\![\mathbb{Z}_p^\times]\!]$.

Proof. Part (1) is Lemma 2.11. Part (2) is Lemma 2.13. Part (3) is Lemma 2.15. □

Remark 4.6. Part (3) is a mild refinement of [BP19, Theorem 3.14].

For an integer $k \geq 2$, let $k : \Lambda_\mathfrak{m} \to \mathcal{O}_\mathfrak{m}$ be the weight-k specialization, so that $\mathbb{T}_\mathfrak{m} \otimes_{\Lambda_\mathfrak{m},k} \mathcal{O}_\mathfrak{m} = \mathbb{T}_{\mathfrak{m},k}$, the Hecke algebra acting on weight-k forms. The normalization of $\mathbb{T}_{\mathfrak{m},k}$ is $\mathbb{T}_{\mathfrak{m},k} \to \prod_{i=1}^{r_k} \mathcal{O}_{f_{k,i}}$, where $\{f_{k,i} \mid i = 1,\dots r_k\}$ is a complete list of Galois-orbits of modular eigenforms with coefficients in $\overline{\mathbb{Q}}_p$ that are residually Eisenstein, and $\mathcal{O}_{f_{k,i}}$ are the valuation rings in their respective p-adic Hecke eigenfields. Let $\varpi_{f_{k,i}} \in \mathcal{O}_{f_{k,i}}$ be a uniformizer. Recall that the image of $L_p^+(\mathfrak{m})$ in $\mathcal{O}_{f_{k,i}}[\![\mathbb{Z}_p^\times]\!]$ is $L_p^+(f_{k,i})$, the one-variable p-adic L-function of $f_{k,i}$.

We write $L_p^+(\mathfrak{m},\omega^j)$ for the projection of $L_p^+(\mathfrak{m})$ to the ω^j-component of $\mathbb{T}_\mathfrak{m}[\![\mathbb{Z}_p^\times]\!]$ and likewise write $L_p^+(\mathfrak{m},\omega^j)_e^0$ for the projection of $L_p^+(\mathfrak{m})_e^0$ to the ω^j-component of $\mathbb{T}^0_\mathfrak{m}[\![\mathbb{Z}_p^\times]\!]$. The following analyzes the ω^0-components of these p-adic L-functions and is a mild refinement of the results of [BP19, §3.7].

Theorem 4.7. *Suppose that U_p-1 generates $\mathcal{I}_\mathfrak{m}$ and let $e=(U_p-1)\{0,\infty\}$. Then*

1. *$L_p^+(\mathfrak{m},\omega^0)=L_p^+(\mathfrak{m},\omega^0)_e^0\cdot(U_p-1)$ and $L_p^+(\mathfrak{m},\omega^0)_e^0\in(\mathbb{T}^0_\mathfrak{m}[\![u]\!])^\times$.*
2. *For every weight k and index i, the μ- and λ-invariants of $f_{k,i}$ are*
$$\mu(L_p^+(f_{k,i},\omega^0))=\mathrm{val}_{\varpi_{f_{k,i}}}(a_p(f_{k,i})-1) \text{ and } \lambda(L_p^+(f_{k,i},\omega^0))=0.$$
3. *For every weight k, the sum of the μ-invariants is given by*
$$\sum_{i=1}^{r_k}\mu(L_p^+(f_{k,i},\omega^0))=\mathrm{val}_\varpi(L_p(\psi_\mathfrak{m}^{-1},k)).$$
4. *For every integer M, there is a weight k and index i such that*
$$\mu(L_p^+(f_{k,i},\omega^0))>M.$$

Proof. Let $\mathbf{1}:\Lambda_\mathfrak{m}\to\mathcal{O}_\mathfrak{m}$ be evaluation at the trivial character. The argument of [BP19, Theorem 3.15] shows that, if $\mathbb{T}_\mathfrak{m}$ is Gorenstein, then
$$L_p^+(\mathfrak{m},\mathbf{1})=L_p^+(\mathfrak{m},\omega^0,\mathbf{1})=(U_p-1)v$$
for $v\in(\mathbb{T}_\mathfrak{m})^\times$. Thus part (1) follows from Lemma 2.16 and immediately implies (2), whereas part (3) follows from Lemma 2.21(1). Finally let $M>0$ be an integer, and choose an integer k close enough to a zero of $L_p(\psi_\mathfrak{m}^{-1},\kappa)$ that
$$\mathrm{val}_\varpi(L_p(\psi_\mathfrak{m}^{-1},k)))>M\mathrm{rank}_\Lambda(\mathbb{T}^0_\mathfrak{m}).$$
Since $\mathrm{rank}_\Lambda(\mathbb{T}^0_\mathfrak{m})\geq r_k$, part (4) follows from Lemma 2.21(2). □

4.3. **Conjectures.** We make the following conjecture.

Conjecture 4.8.

1. *Suppose that $\mathbb{T}_\mathfrak{m}$ is Gorenstein. For each j, there is an equality*
$$\mathrm{content}_{\Lambda_\mathfrak{m}}(L_p^+(\mathfrak{m},\omega^j))=\mathcal{I}_\mathfrak{m}.$$
2. *Suppose that $\mathbb{T}^0_\mathfrak{m}$ is Gorenstein and let e be a generator of $H^1_\Lambda(X_1(N))^-_\mathfrak{m}$ as a $\mathbb{T}^0_\mathfrak{m}$-module. Then, for each j, $L_p^+(\mathfrak{m},\omega^j)_e^0$ has unit content.*

We wish to compare this conjecture with [BP19, Conjecture 3.16]. For a height-one prime $\mathfrak{p}\subset\mathbb{T}^0_\mathfrak{m}$, let $\mathcal{O}_\mathfrak{p}$ be the normalization of $\mathbb{T}^0_\mathfrak{m}/\mathfrak{p}$ (which is a DVR) and let $\varpi_\mathfrak{p}$ denote a uniformizer in $\mathcal{O}_\mathfrak{p}$. Let $L_p^+(\mathfrak{p})_e^0, L_p^+(\mathfrak{p})\in\mathcal{O}_\mathfrak{p}[\![\mathbb{Z}_p^\times]\!]$ denote the images of $L_p^+(\mathfrak{m})_e^0$ and $L_p^+(\mathfrak{m})$ (supposing they exist). If $\mathcal{I}_\mathfrak{m}$ is generated by $\mathfrak{t}$, let $\mathfrak{t}_\mathfrak{p}$ denote the image of $\mathfrak{t}$ in $\mathcal{O}_\mathfrak{p}$.

Conjecture 4.9 (Bellaïche-Pollack)**.** *Suppose that $\mathcal{I}_\mathfrak{m}$ is generated by $\mathfrak{t}$. Then, for every height-one prime $\mathfrak{p}\subset\mathbb{T}^0_\mathfrak{m}$, there is an equality $\mu(L_p^+(\mathfrak{p},\omega^j))=\mathrm{val}_{\varpi_\mathfrak{p}}(\mathfrak{t}_\mathfrak{p})$ for each j.*

Lemma 4.10. *Suppose that $H^1_\Lambda(X_1(N))^-_\mathfrak{m}$ is generated by an element e as a $\mathbb{T}^0_\mathfrak{m}$-module. The following are equivalent:*

(1) $L_p^+(\mathfrak{m},\omega^j)_e^0$ has unit content.
(2) for some height-one prime $\mathfrak{p}\subset\mathbb{T}^0_{\mathfrak{m}}$, the μ-invariant $\mu(L_p^+(\mathfrak{p},\omega^j)_e^0)$ vanishes.
(3) for all height-one primes $\mathfrak{p}\subset\mathbb{T}^0_{\mathfrak{m}}$, the μ-invariant $\mu(L_p^+(\mathfrak{p},\omega^j)_e^0)$ vanishes.

Proof. Clear since $\mathbb{T}^0_{\mathfrak{m}}$ is a local ring. □

Lemma 4.11. *Suppose that $\mathcal{I}_{\mathfrak{m}}$ is generated by $\mathfrak{t}$ and let $e=\mathfrak{t}\{0,\infty\}$. Then for every height-one prime $\mathfrak{p}\subset\mathbb{T}^0_{\mathfrak{m}}$, there is an equality*

$$\mu(L_p^+(\mathfrak{p},\omega^j))=\mu(L_p^+(\mathfrak{p},\omega^j)_e^0)+\mathrm{val}_{\varpi_{\mathfrak{p}}}(\mathfrak{t}_{\mathfrak{p}}).$$

In particular, $\mu(L_p^+(\mathfrak{p},\omega^j))\geq \mathrm{val}_{\varpi_{\mathfrak{p}}}(\mathfrak{t}_{\mathfrak{p}})$, with equality if and only if $\mu(L_p^+(\mathfrak{p},\omega^j)_e^0)$ is zero.

Proof. Clear from Lemma 2.15. □

Proposition 4.12. *Suppose that $\mathcal{I}_{\mathfrak{m}}$ is principal and let $j_0\in\mathbb{Z}/(p-1)\mathbb{Z}$. Then the following are equivalent:*
(1) Conjecture 4.8 (1) for $j=j_0$.
(2) Conjecture 4.8 (2) for $j=j_0$.
(3) Conjecture 4.9 for $j=j_0$.
Moreover, if U_p-1 generates $\mathcal{I}_{\mathfrak{m}}$, then these conjectures are all true for $j=0$.

Proof. Let $\mathfrak{t}$ be a generator of $\mathcal{I}_{\mathfrak{m}}$ and let $e=\mathfrak{t}\{0,\infty\}$. Theorem 4.5(3) implies $L_p^+(\mathfrak{m},\omega^{j_0})=\mathfrak{t}\cdot L_p^+(\mathfrak{m},\omega^{j_0})_e^0$, so by the multiplicative property (2.18) of content, it follows that

$$\mathrm{content}_{\Lambda_{\mathfrak{m}}}(L_p^+(\mathfrak{m},\omega^{j_0}))=\mathfrak{t}\cdot\mathrm{content}_{\Lambda_{\mathfrak{m}}}(L_p^+(\mathfrak{m},\omega^{j_0})_e^0)=\mathcal{I}_{\mathfrak{m}}\cdot\mathrm{content}_{\Lambda_{\mathfrak{m}}}(L_p^+(\mathfrak{m},\omega^{j_0})_e^0).$$

This makes the equivalence of (1) and (2) clear.

Now assume (2), so that $L_p^+(\mathfrak{m},\omega^{j_0})_e^0$ has unit content, and let $\mathfrak{p}\subset\mathbb{T}^0_{\mathfrak{m}}$ be a height-one prime. Then $\mu(L_p^+(\mathfrak{p},\omega^{j_0})_e^0)=0$ by Lemma 4.10. It then follows from Lemma 4.11 that $\mu(L_p^+(\mathfrak{m},\omega^{j_0}))=\mathrm{val}_{\varpi_{\mathfrak{p}}}(\mathfrak{t}_{\mathfrak{p}})$, proving (3).

Now assume (3), so that $\mu(L_p^+(\mathfrak{m},\omega^{j_0}))=\mathrm{val}_{\varpi_{\mathfrak{p}}}(\mathfrak{t}_{\mathfrak{p}})$ for all $\mathfrak{p}$. Then $\mu(L_p^+(\mathfrak{p},\omega^{j_0})_e^0)$ vanishes by Lemma 4.11, and so $L_p^+(\mathfrak{m},\omega^{j_0})_e^0$ has unit content by Lemma 4.10, proving (2).

For the last claim, if U_p-1 generates $\mathcal{I}_{\mathfrak{m}}$, then Theorem 4.7 implies that $L_p^+(\mathfrak{m},\omega^0)_e^0$ is a unit and Conjecture 4.8 (2) is clear for $j=0$. □

5. Mazur case

In this section, we consider the case of tame level $\Gamma_0(N)$ for a prime N. This is, in some sense, the opposite of the previous section: whereas in Section 4 we considered forms with *primitive* tame character, in this section we consider *trivial* tame character. The main difference is that the constant term of the relevant Eisenstein series are multiplied by an Euler factor at N and congruences can occur because p divides that Euler factor. We focus on these kinds of congruences in the most interesting case: when $N\equiv 1 \pmod p$. This includes the case considered by Mazur in his original article [Maz77] on the Eisenstein ideal, and so we refer to this set up as the "Mazur case". We make use of recent advances [WWE20, Wak23, WWE21, Deo23, Lec18, Lec21] about the Gorenstein property for the relevant Hecke algebras. This allows us to replace the Gorenstein assumptions in the results of Section 4 with some numerical criteria.

Another difference between this setup and the primitive case considered in Section 4 is in the way we specify a unique Eisenstein family. In tame level $\Gamma_0(N)$, the space of ordinary Eisenstein families has rank 2. A generic basis of the space is given by U_N-eigenforms: one where U_N acts by 1 and one where U_N acts by N^{k-1}. Since $N \equiv 1 \pmod{p}$, these two families are congruent and cannot be separated by localizing a maximal ideal of the Hecke algebra. To resolve this issue, following an idea of Ohta [Oht14] as in [WWE21], we replace the U_N operator in the Hecke algebra by the Atkin-Lehner involution w_N. The two w_N-eigenvector ordinary Eisenstein families are not congruent, so there is a unique Eisenstein family after localizing at a maximal ideal in this new Hecke algebra.

The following subsection summarizes the results of Appendix A where it is verified that Hida theory works as expected for these modified Hecke algebras. After that, we verify that there is a unique ordinary Eisenstein family where w_N acts by -1 and we compute the constant term of its q-expansion. We explain how to use Ohta's results on Λ-adic Eichler-Shimura for tame level $\Gamma_1(N)$ to prove the same results for tame level $\Gamma_0(N)$. We then establish various numerical criteria that guarantee that our Hecke algebras are Gorenstein and further ones that guarantee that $U_p - 1$ generates the Eisenstein ideal. With these results in hand, we apply the axiomatic setup of §2 to deduce our main analytic results.

5.1. **Hida theory with Atkin-Lehner operators.** Fix an even integer k_0 with $0 < k_0 < p-1$. For integers $r \geq 0, k \geq 2$ with $k \equiv k_0 \bmod p-1$, let $\mathfrak{H}_{k,Np^r}$ denote subalgebra of $\mathrm{End}_{\mathbb{Z}_p}(M_k(\Gamma_0(N)\cap\Gamma_1(p^r), \mathbb{Z}_p))$ generated by operators T_q for primes $q \nmid Np$, together with w_N and U_p. Let $\mathfrak{h}_{k,Np^r}$ denote the image of $\mathfrak{H}_{k,Np^r}$ in $\mathrm{End}_{\mathbb{Z}_p}(S_k(\Gamma_0(N)\cap\Gamma_1(p^r), \mathbb{Z}_p))$. Let $\mathfrak{H}_k^{\mathrm{ord}}$ denote the inverse limit over r of the ordinary part $\mathfrak{H}_{k,Np^r}^{\mathrm{ord}}$ of $\mathfrak{H}_{k,Np^r}$, and similarly for $\mathfrak{h}_k^{\mathrm{ord}}$. The main result proven in Appendix A is that these algebras satisfy the main theorems of Hida theory, just as for the Hecke algebras with U_N-operators:

- (independence of weight) $\mathfrak{H}_k^{\mathrm{ord}}$ and $\mathfrak{h}_k^{\mathrm{ord}}$ are independent of k (and depend only on k_0), and so can be denoted simply by $\mathfrak{H}^{\mathrm{ord}}$ and $\mathfrak{h}^{\mathrm{ord}}$,
- (freeness over Λ) the algebras $\mathfrak{H}^{\mathrm{ord}}$ and $\mathfrak{h}^{\mathrm{ord}}$ are free Λ-modules of finite rank,
- (duality) the pairing $(f, T) \mapsto a_1(Tf)$ is a perfect duality between $\mathfrak{H}^{\mathrm{ord}}$ and Λ-adic modular forms (and similarly for $\mathfrak{h}^{\mathrm{ord}}$ and cuspforms),
- (control) the natural maps $\mathfrak{H}^{\mathrm{ord}} \to \mathfrak{H}_{k,Np^r}^{\mathrm{ord}}$ induce isomorphisms $\mathfrak{H}^{\mathrm{ord}}/\omega_{r,k} \to \mathfrak{H}_{k,Np^r}^{\mathrm{ord}}$ for a particular $\omega_{r,k} \in \Lambda$ (and similarly for $\mathfrak{h}^{\mathrm{ord}}$).

5.2. **Eisenstein series.** Let $\mathfrak{m} \subseteq \mathfrak{H}^{\mathrm{ord}}$ denote the maximal ideal corresponding to the residual representation $1 \oplus \omega^{k_0-1}$ and which contains $w_N + 1$. Write $\mathbb{T}_{\mathfrak{m}}$ for the completion of $\mathfrak{H}^{\mathrm{ord}}$ at $\mathfrak{m}$ and $\mathbb{T}_{\mathfrak{m}}^0$ for the completion of $\mathfrak{h}^{\mathrm{ord}}$ at $\mathfrak{m}$. Then both $\mathbb{T}_{\mathfrak{m}}$ and $\mathbb{T}_{\mathfrak{m}}^0$ are modules over $\Lambda = \mathbb{Z}_p[\![T]\!]$ the Iwasawa algebra.

Let E^{ord} denote the family of Eisenstein series whose specialization to a weight $k \equiv k_0 \pmod{p-1}$ is E_k^{ord}, the unique ordinary Eisenstein series of weight k and level p whose constant term is $-(1-p^{k-1})\frac{B_k}{2k}$. We wish to promote this family to an eigenfamily of level Np where w_N acts by -1. To this end, note that it is easy to compute the action of w_N on E_k^{ord} thought of as a form of level Np as this form is old at N. Indeed, for any form f of level Np, we have $w_N f = N^{1-k/2} \cdot f\big|_k \left(\begin{smallmatrix} Na & b \\ Np & Nd \end{smallmatrix}\right)$

where this matrix has determinant N and thus

$$\begin{aligned} w_N E_k^{\mathrm{ord}}(z) &= N^{1-k/2} \cdot \left(E_k^{\mathrm{ord}}\big|_k \left(\begin{smallmatrix} Na & b \\ Np & Nd \end{smallmatrix} \right) \right)(z) \\ &= N^{1-k/2} \cdot \left(E_k^{\mathrm{ord}}\big|_k \left(\begin{smallmatrix} a & b \\ p & Nd \end{smallmatrix} \right) \left(\begin{smallmatrix} N & 0 \\ 0 & 1 \end{smallmatrix} \right) \right)(z) \\ &= N^{1-k/2} \cdot \left(E_k^{\mathrm{ord}}\big|_k \left(\begin{smallmatrix} N & 0 \\ 0 & 1 \end{smallmatrix} \right) \right)(z) \\ &= N^{k/2} \cdot E_k^{\mathrm{ord}}(Nz). \end{aligned}$$

With this formula in hand, an easy computation shows that $\mathcal{E}_k^{\pm}(q) := E_k^{\mathrm{ord}}(q) \pm N^{k/2} E_k^{\mathrm{ord}}(q^N)$ is a w_N-eigenform with eigenvalue ± 1. In particular, there is a unique Eisenstein series in $M_k(\Gamma_0(Np))$ where w_N acts with sign ± 1. Further note that the constant term of $\mathcal{E}_k^{\pm}(q)$ is given by $-(1 \pm N^{k/2})(1-p^{k-1})\frac{B_k}{2k}$.

Let $\mathcal{E}^-$ denote the family of Eisenstein series that in weight $k \equiv k_0 \pmod{p-1}$ specializes to $\mathcal{E}_k^-$. Let $\mathrm{Eis} : \mathbb{T}_{\mathfrak{m}} \to \Lambda$ be the homomorphism corresponding to $\mathcal{E}^-$ and let $\mathcal{I}_{\mathfrak{m}}$ denote the kernel of this map. As $\mathcal{E}^-$ is the unique Eisenstein family parametrized by $\mathbb{T}_{\mathfrak{m}}$, the ideal $\mathrm{Ann}_{\mathbb{T}_{\mathfrak{m}}}(\mathcal{I}_{\mathfrak{m}})$ is free of rank one as a Λ-module, is generated by the Hecke operator T_0 determined by $a_1(T_0 f) = a_0(f)$, and $\mathbb{T}^0_{\mathfrak{m}} = \mathbb{T}_{\mathfrak{m}}/T_0\mathbb{T}_{\mathfrak{m}}$.

Taken together, all this implies that $(\mathcal{O}, A, R, \mathcal{E}, r_0) = (\mathbb{Z}_p, \Lambda, \mathbb{T}_{\mathfrak{m}}, \mathrm{Eis}, T_0)$ is a single-Eisenstein Hecke algebra set up. Further, the element $\xi = \mathrm{Eis}(T_0)$ corresponds to the constant term of $\mathcal{E}^-$ which is $\zeta_{p,k_0} \cdot (1 - \langle N \rangle^{1/2})$ where $\zeta_{p,k_0} \in \Lambda$ is the ω^{k_0}-branch of the p-adic ζ-function and $\langle N \rangle \in \Lambda$ is the element which specializes to N^k in weight k.

5.3. **Λ-adic Eichler-Shimura for tame level $\Gamma_0(N)$.** Consider the Λ-adic étale cohomology groups $H^1_\Lambda(Y_0(N))$ and $H^1_\Lambda(X_0(N))$ with tame level $\Gamma_0(N)$, defined as

$$H^1_\Lambda(Y_0(N)) = \varprojlim H^1(Y(\Gamma_0(N) \cap \Gamma_1(p^r)), \mathbb{Z}_p)^{\mathrm{ord},(k_0)},$$

and similarly for $X_0(N)$, where the superscript (k_0) means the ω^{k_0}-eigenspace for the diamond-operator action of $(\mathbb{Z}/p\mathbb{Z})^\times$. In this section, we use Ohta's results on the structure of $H^1_\Lambda(Y_1(N))$ and $H^1_\Lambda(X_1(N))$ to deduce analogous results for $H^1_\Lambda(Y_0(N))$ and $H^1_\Lambda(X_0(N))$. The main input is the following result about the structure of $H^1_\Lambda(Y_1(N))$ as a $\mathbb{Z}_p[\Delta]$-module, where $\Delta = \Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^\times$ is the group of diamond operators of level N.

Lemma 5.1. *The Λ-adic cohomology group $H^1_\Lambda(Y_1(N))$ is a projective $\mathbb{Z}_p[\Delta]$-module, and the natural maps*

$$H^1_\Lambda(Y_1(N))_\Delta \to H^1_\Lambda(Y_0(N)) \to H^1_\Lambda(Y_1(N))^\Delta$$

are isomorphisms.

Proof. It is enough to prove the result at level $\Gamma_1(Np^r)$ for fixed r; taking inverse limits gives the result for Λ-adic cohomology. Let $Y_1 = Y_1(Np^r)$ and $Y_0 = Y(\Gamma_0(N) \cap \Gamma_1(p^r))$.

First note that $H^0(Y_1, \mathbb{Z}_p)^{\mathrm{ord}}$ is zero because U_p acts by p on $H^0(Y_1, \mathbb{Z}_p)$. Then the following are evident:

(1) $Y_1 \to Y_0$ is an étale covering with Galois group Δ,
(2) the cohomology $H^i(Y_1, \mathbb{Z}_p)^{\mathrm{ord}}$ is only supported in degree $i = 1$, and
(3) $H^1(Y_1, \mathbb{Z}_p)^{\mathrm{ord}}$ is p-torsion-free.

These three facts are enough to imply that $H^1(Y_1, \mathbb{Z}_p)^{\mathrm{ord}}$ is a projective $\mathbb{Z}_p[\Delta]$-module, as follows. By the comparison isomorphism between étale and Betti cohomology, it is enough to show that the singular cohomology is projective. Let $C^\bullet(Y_1, \mathbb{Z}_p)$ be the complex of singular cochains on Y_1; by (1), it is a bounded complex of flat $\mathbb{Z}_p[\Delta]$-modules. Hecke operators act on $C^\bullet(Y_1, \mathbb{Z}_p)$, and, since the ordinary projector is an idempotent, the complex $C^\bullet(Y_1, \mathbb{Z}_p)^{\mathrm{ord}}$ is still a bounded complex of flat $\mathbb{Z}_p[\Delta]$-modules and its cohomology groups are $H^i(Y_1, \mathbb{Z}_p)^{\mathrm{ord}}$. Hence, there is a perfect complex $C^\bullet$ of $\mathbb{Z}_p[\Delta]$-modules and a quasi-isomorphism

$$C^\bullet \simeq C^\bullet(Y_1, \mathbb{Z}_p)^{\mathrm{ord}}$$

(see [Mum08, Lemma II.5.1, pg. 47]). By (2), this implies that $H^1(Y_1, \mathbb{Z}_p)^{\mathrm{ord}}$ has finite projective dimension over $\mathbb{Z}_p[\Delta]$, which, together with (3), implies that it is projective (see [Bro94, Theorems 8.10 and 8.12, pg. 152], for instance, or use the Auslander-Buchsbaum formula).

The composition of the two maps

$$H^1(Y_1, \mathbb{Z}_p)^{\mathrm{ord}}_\Delta \to H^1(Y_0, \mathbb{Z}_p)^{\mathrm{ord}} \to (H^1(Y_1, \mathbb{Z}_p)^{\mathrm{ord}})^\Delta$$

is equal to the map induced by multiplication by the norm element of $\mathbb{Z}_p[\Delta]$ on $H^1(Y_1, \mathbb{Z}_p)^{\mathrm{ord}}$, which is an isomorphism since $H^1(Y_1, \mathbb{Z}_p)^{\mathrm{ord}}$ is $\mathbb{Z}_p[\Delta]$-projective. The map $H^1(Y_0, \mathbb{Z}_p)^{\mathrm{ord}} \to (H^1(Y_1, \mathbb{Z}_p)^{\mathrm{ord}})^\Delta$ is clearly injective, so this completes the proof. □

This lemma allows us to to prove the following analog of Ohta's Λ-adic Eichler-Shimura isomorphisms.

Proposition 5.2. *There are split-exact sequences of* $\mathbb{T}_\mathfrak{m}$*-modules*

$$0 \to \mathbb{T}^0_\mathfrak{m} \to H^1_\Lambda(Y_0(N))_\mathfrak{m} \to \mathrm{Hom}_\Lambda(\mathbb{T}_\mathfrak{m}, \Lambda) \to 0$$
$$0 \to \mathbb{T}^0_\mathfrak{m} \to H^1_\Lambda(X_0(N))_\mathfrak{m} \to \mathrm{Hom}_\Lambda(\mathbb{T}^0_\mathfrak{m}, \Lambda) \to 0.$$

Proof. Let $\mathfrak{H}_1^{\mathrm{ord}}$ be the p-adic Hida Hecke algebra of tame level $\Gamma_1(N)$ and let $\mathfrak{h}_1^{\mathrm{ord}}$ be its cuspidal quotient. Let $m_{\Lambda,1}$ and $S_{\Lambda,1}$ be the spaces of Λ-adic modular forms and cuspforms, respectively, of tame level $\Gamma_1(N)$, and let m_Λ and S_Λ be the corresponding spaces of tame level $\Gamma_0(N)$.

Ohta's Λ-adic Eichler-Shimura isomorphisms [Oht99, Oht00] imply that there are exact sequences of $\mathfrak{H}_1^{\mathrm{ord}}$-modules

$$0 \to \mathfrak{h}_1^{\mathrm{ord}} \to H^1_\Lambda(Y_1(N)) \to m_{\Lambda,1} \to 0 \tag{5.3}$$
$$0 \to \mathfrak{h}_1^{\mathrm{ord}} \to H^1_\Lambda(X_1(N)) \to S_{\Lambda,1} \to 0. \tag{5.4}$$

These sequences are defined using the action of $G_{\mathbb{Q}_p}$ and since $k_0 \neq 1$ (because k_0 is even) the actions on the sub and quotient are distinguished, so the sequences split as $\mathfrak{H}_1^{\mathrm{ord}}$-modules (see [Oht03, Section 3.4] and [FK12, Section 6.3.12]); *a fortiori*, they split as $\mathbb{Z}_p[\Delta]$-modules (since the diamond operators are in $\mathfrak{H}_1^{\mathrm{ord}}$). Lemma 5.1 and the splitting of (5.3) imply that $\mathfrak{h}_1^{\mathrm{ord}}$ and $m_{\Lambda,1}$ are $\mathbb{Z}_p[\Delta]$-projective. Since the dual of a projective $\mathbb{Z}_p[\Delta]$-module is projective, this implies that $\mathfrak{H}_1^{\mathrm{ord}}$ and $S_{\Lambda,1}$ are projective $\mathbb{Z}_p[\Delta]$-modules. Then the sequence (5.4) implies that $H^1_\Lambda(X_1(N))$ is also $\mathbb{Z}_p[\Delta]$-projective. Just as in the proof of Lemma 5.1, this implies that the natural map

$$H^1_\Lambda(X_0(N)) \to H^1_\Lambda(X_1(N))^\Delta$$

is an isomorphism. Hence taking Δ-invariants of the split-exact sequences (5.3) and (5.4) yields exact sequences

$$0 \to (\mathfrak{h}_1^{\mathrm{ord}})^\Delta \to H^1_\Lambda(Y_0(N)) \to (m_{\Lambda,1})^\Delta \to 0 \tag{5.5}$$

$$0 \to (\mathfrak{h}_1^{\mathrm{ord}})^\Delta \to H^1_\Lambda(X_0(N)) \to (S_{\Lambda,1})^\Delta \to 0. \tag{5.6}$$

Now note that the natural maps $m_\Lambda \to (m_{\Lambda,1})^\Delta$ and $S_\Lambda \to (S_{\Lambda,1})^\Delta$ are isomorphisms because a modular form of level $\Gamma_1(N)$ that is invariant under the diamond operators is also a form of level $\Gamma_0(N)$. This implies that the dual map

$$\mathrm{Hom}_\Lambda(S_{\Lambda,1},\Lambda)_\Delta \to \mathrm{Hom}_\Lambda(S_\Lambda,\Lambda), \tag{5.7}$$

is an isomorphism. But, by duality (see Theorem A.3 and Theorem A.9), there are isomorphisms $\mathrm{Hom}_\Lambda(S_{\Lambda,1},\Lambda) \cong \mathfrak{h}_1^{\mathrm{ord}}$ and $\mathrm{Hom}_\Lambda(S_\Lambda,\Lambda) \cong \mathfrak{h}^{\mathrm{ord}}$. Moreover, since $\mathfrak{h}_1^{\mathrm{ord}}$ is $\mathbb{Z}_p[\Delta]$-free, the norm map induces an isomorphism

$$(\mathfrak{h}_1^{\mathrm{ord}})_\Delta \cong (\mathfrak{h}_1^{\mathrm{ord}})^\Delta.$$

Combining this isomorphism with the duality isomorphisms and (5.7) gives a string of isomorphisms

$$(\mathfrak{h}_1^{\mathrm{ord}})^\Delta \cong (\mathfrak{h}_1^{\mathrm{ord}})_\Delta \cong \mathrm{Hom}_\Lambda(S_{\Lambda,1},\Lambda)_\Delta \cong \mathrm{Hom}_\Lambda(S_\Lambda,\Lambda) \cong \mathfrak{h}^{\mathrm{ord}}.$$

Hence (5.5) and (5.6) are isomorphic to

$$0 \to \mathfrak{h}^{\mathrm{ord}} \to H^1_\Lambda(Y_0(N)) \to m_\Lambda \to 0 \tag{5.8}$$

$$0 \to \mathfrak{h}^{\mathrm{ord}} \to H^1_\Lambda(X_0(N)) \to S_\Lambda \to 0. \tag{5.9}$$

Localizing at $\mathfrak{m}$ then completes the proof. □

Just as in Section 4.1, there is an exact sequence

$$0 \to H^1_\Lambda(X_0(N)) \to H^1_\Lambda(Y_0(N)) \xrightarrow{\phi} \Lambda \to 0.$$

where $\phi(\{0,\infty\}) = 1$. Since $\{0,\infty\} \in H^1_\Lambda(Y_0(N))^-$, this implies that there is an exact sequence

$$0 \to H^1_\Lambda(X_0(N))^- \to H^1_\Lambda(Y_0(N))^- \xrightarrow{\phi} \Lambda \to 0.$$

Proposition 5.10. *Taking* $(X,\phi,x,\mathcal{L}) = (H^1_\Lambda(Y_0(N))^-,\phi,\{0,\infty\},\mathcal{L}_p^+(\mathfrak{m}))$ *gives an L-symbol set up.*

Proof. Just as in the proof of Proposition 4.4, given Proposition 5.2. □

5.4. **Gorenstein results and generators of the Eisenstein ideal.** In this section, we give some numerical criteria for when the Eisenstein ideal $\mathcal{I}_\mathfrak{m}$ is principal (and thus $\mathbb{T}_\mathfrak{m}$ and $\mathbb{T}^0_\mathfrak{m}$ are Gorenstein), when $\mathcal{I}_\mathfrak{m}$ is generated by $U_p - 1$, and when $\mathbb{T}^0_\mathfrak{m}$ has rank 1 over Λ.

Let $\mathbb{T}^{\mathrm{tame}}_{\mathfrak{m},k}$ and $\mathbb{T}_{\mathfrak{m},k}$ denote the respective Hecke algebras of $M_k(\Gamma_0(N))^{\mathrm{ord}}$ and $M_k(\Gamma_0(Np))^{\mathrm{ord}}$ and let $\mathbb{T}^{0,\mathrm{tame}}_{\mathfrak{m},k}$ and $\mathbb{T}^0_{\mathfrak{m},k}$ denote the respective Hecke algebras of $S_k(\Gamma_0(N))^{\mathrm{ord}}$ and $S_k(\Gamma_0(Np))^{\mathrm{ord}}$. Here all of these Hecke algebras are defined to contain w_N rather than U_N.

Theorem 5.11. *If* $k > 2$*, there are isomorphisms* $\mathbb{T}_{\mathfrak{m},k} \cong \mathbb{T}^{\mathrm{tame}}_{\mathfrak{m},k}$ *and* $\mathbb{T}^0_{\mathfrak{m},k} \cong \mathbb{T}^{0,\mathrm{tame}}_{\mathfrak{m},k}$*. For* $k = 2$*, the same conclusions hold when* p *is not a* p*-th power modulo* N*.*

Proof. This result is easy is $k \neq 2$. Indeed, an eigenform f of weight k which is p-new has $a_p(f) = \pm p^{\frac{k-2}{2}}$ and thus is ordinary if and only if $k = 2$. In particular, when $k \neq 2$, none of these ordinary Hecke algebras grow from level N to level Np.

The case of $k = 2$ is much deeper and follows from the results of the second author and Wang-Erickson, namely [WWE21, Theorem 1.4.5 and Proposition A.3.1] and rely on the hypothesis that p is not a p-th power modulo N. □

Returning to the setting of §5.2, we have $\mathbb{T}_\mathfrak{m}$ and $\mathbb{T}^0_\mathfrak{m}$ are the localized Hida Hecke algebras corresponding to the residual representation $1 \oplus \omega^{k_0-1}$ and we now give criteria for when these rings are Gorenstein. To this end, let g denote a generator of $\mathbb{F}_N^\times$ and define $\log_N : \mathbb{F}_N^\times \to \mathbb{F}_p$ by $\log_N(g^a) = a \pmod{p}$. Note that this map is well-defined as a is defined modulo $N-1$ and $p \mid N-1$. Further, fix ζ_p a p-th root of unity in $\mathbb{F}_N^\times$.

Theorem 5.12. *Assume that both of the following two conditions hold:*

(1) $p \nmid B_{k_0}$, *and*

(2) $\displaystyle\sum_{i=1}^{p-1} i^{k_0-2} \log_N(1-\zeta_p^i) \neq 0$ *in* $\mathbb{F}_p$.

Then $\mathcal{I}_\mathfrak{m}$ *is a principal ideal. In particular,* $\mathbb{T}_\mathfrak{m}$ *and* $\mathbb{T}^0_\mathfrak{m}$ *are Gorenstein and* $\mathbb{T}_{\mathfrak{m},k}$ *and* $\mathbb{T}^0_{\mathfrak{m},k}$ *are Gorenstein for all* $k \equiv k_0 \pmod{p-1}$.

Proof. The results of [Deo23] imply that $\mathcal{I}_\mathfrak{m}$ is principal (see, in particular, [Deo23, Remark 5.7]). Once we know $\mathcal{I}_\mathfrak{m}$ is principal, then Lemma 2.10 implies that $\mathbb{T}_\mathfrak{m}$ and $\mathbb{T}^0_\mathfrak{m}$ are Gorenstein and Theorem A.9 implies that $\mathbb{T}_{\mathfrak{m},k}$ and $\mathbb{T}^0_{\mathfrak{m},k}$ are Gorenstein for all $k \equiv k_0 \pmod{p-1}$. □

Remark 5.13. We note that if $k_0 \equiv 2 \pmod{p-1}$, then

$$\sum_{i=1}^{p-1} i^{k_0-2} \log_N(1-\zeta_p^i) = \sum_{i=1}^{p-1} \log_N(1-\zeta_p^i) = \log_N\left(\prod_{i=1}^{p=1} 1-\zeta_p^i\right) = \log_N(p).$$

In particular, when $k_0 \equiv 2 \pmod{p-1}$, the hypotheses of Theorem 5.12 reduce simply to asking that p is not a p-th power modulo N

We now give criteria for when $U_p - 1$ generates the Eisenstein ideal.

Theorem 5.14. *Assume the hypotheses of Theorem 5.12. Then* p *is not a* p*-th power modulo* N *if and only if* $U_p - 1$ *generates the Eisenstein ideal.*

Proof. First note that $U_p - 1$ generates the Eisenstein ideal $\mathcal{I}_\mathfrak{m}$ of $\mathbb{T}_\mathfrak{m}$ if and only if $U_p - 1$ generates the Eisenstein ideal $\mathcal{I}_{\mathfrak{m},k}$ of $\mathbb{T}_{\mathfrak{m},k}$ for one (equivalently any) $k \equiv k_0 \pmod{p-1}$. Thus it suffices to work in $\mathbb{T}_{\mathfrak{m},k}$ to establish the above theorem.

To this end, choose $k \equiv k_0 \pmod{p-1}$ with $k > 2$. By Theorem 5.12, $\mathcal{I}_{\mathfrak{m},k}$ is principal, and hence there is a surjective homomorphism $\mathbb{Z}_p[x] \to \mathbb{T}_{\mathfrak{m},k}$ sending x to a generator of $\mathcal{I}_{\mathfrak{m},k}$. This map induces an isomorphism $\mathbb{T}_{\mathfrak{m},k}/p\mathbb{T}_{\mathfrak{m},k} \cong \mathbb{F}_p[x]/(x^r)$ for some $r \geq 1$. On the other hand, [Wak23, Proposition 4.4.2] gives a surjective homomorphism $\phi : \mathbb{T}_{\mathfrak{m},k} \to \mathbb{F}_p[\epsilon]/(\epsilon^2)$ such that

$$\begin{aligned} T_\ell &\mapsto 1 + \ell^{k_0-1} + \epsilon(\ell^{k_0-1} - 1)\log_N(\ell), \\ U_p &\mapsto 1 - \epsilon \log_N(p). \end{aligned}$$

Hence, an element $t \in \mathcal{I}_{\mathfrak{m},k}$ is a generator if and only if $\phi(t) \neq 0$. It follows that $U_p - 1$ generates $\mathcal{I}_{\mathfrak{m},k}$ if only if p is not a p-th power modulo N. □

Remark 5.15. We note that when $k_0 \equiv 2 \pmod{p-1}$, all of the hypotheses of Theorem 5.14 again reduce to simply assuming that p is not a p-th power modulo N.

Lastly, we give a criteria for when $\mathbb{T}^0_{\mathfrak{m}}$ has rank 1 over Λ.

Theorem 5.16. *Assume the hypotheses of Theorem 5.14 and that*

$$\prod_{i=1}^{N-1} i^{\left(\sum_{j=1}^{i-1} j^{k_0-1}\right)} \tag{5.17}$$

is not a p-th power modulo N. Then $\mathbb{T}^0_{\mathfrak{m}}$ has rank 1 over Λ.

Proof. By Theorem A.9, to compute the rank of $\mathbb{T}^0_{\mathfrak{m}}$ over Λ, it suffices to compute the rank of $\mathbb{T}^0_{\mathfrak{m},k}$ over $\mathbb{Z}_p$ for any $k \equiv k_0 \pmod{p-1}$. The theorem then follows from [Deo23, Corollary B]. □

Remark 5.18. In [Lec18, pg. 36], it is verified that when $k_0 \equiv 2 \pmod{p-1}$ the quantity in (5.17) is a p-th power modulo N if and only if Merel's number $\prod_{i=1}^{\frac{N-1}{2}} i^i$ is a p-th power modulo N.

5.5. **Results.** By Proposition 5.10, $(H^1_\Lambda(Y_0(N))^-, \phi, \{0,\infty\}, \mathcal{L}^+_p(\mathfrak{m}))$ is an L-symbol set up for the single-Eisenstein Hecke algebra set up $(\mathbb{Z}_p, \Lambda, \mathbb{T}_{\mathfrak{m}}, \mathcal{E}^-, T_0)$. Applying the results of Section 2 to this set up, we obtain a theorem which is essentially identical to Theorem 4.5:

Theorem 5.19.

1. *(1) If $\mathbb{T}_{\mathfrak{m}}$ is Gorenstein, then there is a unique $L^+_p(\mathfrak{m}) \in \mathbb{T}_{\mathfrak{m}}[\![\mathbb{Z}_p^\times]\!]$ such that $\mathcal{L}^+_p(\mathfrak{m}) = L^+_p(\mathfrak{m}) \cdot \{0,\infty\}$. Moreover,* $\mathrm{content}(L^+_p(\mathfrak{m})) \subseteq \mathcal{I}_{\mathfrak{m}}$.
2. *(2) Suppose that $\mathbb{T}^0_{\mathfrak{m}}$ is Gorenstein, so that $H^1_\Lambda(X_0(N))^+_{\mathfrak{m}}$ is free of rank 1 over $\mathbb{T}^0_{\mathfrak{m}}$. Then, for every generator e of $H^1_\Lambda(X_0(N))^+_{\mathfrak{m}}$, there is a unique $L^+_p(\mathfrak{m})^0_e \in \mathbb{T}^0_{\mathfrak{m}}[\![\mathbb{Z}_p^\times]\!]$ such that $\mathcal{L}^+_p(\mathfrak{m}) = L^+_p(\mathfrak{m})^0_e \cdot e$.*
3. *(3) If $\mathcal{I}_{\mathfrak{m}}$ is a principal ideal with generator t, then $\mathbb{T}_{\mathfrak{m}}$ and $\mathbb{T}^0_{\mathfrak{m}}$ are Gorenstein and $H^1_\Lambda(X_0(N))^+_{\mathfrak{m}}$ is generated by $t\{0,\infty\}$. Moreover, there is an equality*
$$L^+_p(\mathfrak{m}) = L^+_p(\mathfrak{m})^0_{t\{0,\infty\}} t$$
in $\mathcal{I}_{\mathfrak{m}}[\![\mathbb{Z}_p^\times]\!]$.

Note that Theorem 5.12 gives criteria for when the hypothesis of part (3) holds. We now move on to the case where $U_p - 1$ generates the Eisenstein ideal. We use notation analogous to that used in Section 4.2. In particular, for a fixed weight k, the forms $f_{k,1}, \dots, f_{k,r_k}$ are a complete list of Galois-conjugacy-classes of eigenforms for $\mathbb{T}_{\mathfrak{m},k}$ and $\varpi_{f_{k,i}}$ denotes a uniformizer in the p-adic Hecke field of $f_{k,i}$.

Theorem 5.20. *Let $e = (U_p - 1)\{0,\infty\}$. Assume that all of the following three conditions hold:*

- *(a) $p \nmid B_{k_0}$,*
- *(b) $\displaystyle\sum_{i=1}^{p-1} i^{k_0-2} \log_N(1-\zeta_p^i) \neq 0$ in $\mathbb{F}_p$, and*
- *(c) p is not a p-th power modulo N.*

Then the following four statements are all true:

1. *(1) $L^+_p(\mathfrak{m}, \omega^0) = L^+_p(\mathfrak{m}, \omega^0)^0_e \cdot (U_p - 1)$ and $L^+_p(\mathfrak{m}, \omega^0)^0_e \in (\mathbb{T}^0_{\mathfrak{m}}[\![u]\!])^\times$.*

(2) *For every weight k and index i, the μ- and λ-invariants of $f_{k,i}$ are*
$$\mu(L_p^+(f_{k,i},\omega^0)) = \mathrm{val}_{\varpi_{f_{k,i}}}(a_p(f_{k,i})-1) \text{ and } \lambda(L_p^+(f_{k,i},\omega^0)) = 0.$$

(3) *For every weight k, the sum of the μ-invariants is given by*
$$\sum_{i=1}^{r_k} \mu(L_p^+(f_{k,i},\omega^0)) = \mathrm{val}_p(N-1) + \mathrm{val}_p(k).$$

(4) *For every integer M and every weight k such that $\mathrm{val}_p(k) > M\mathrm{rank}_\Lambda(\mathbb{T}^0_\mathfrak{m})$, there exists an index i such that*
$$\mu(L_p^+(f_{k,i},\omega^0)) > M.$$

If, in addition,
$$\prod_{i=1}^{N-1} i^{\left(\sum_{j=1}^{i-1} j^{k_0-1}\right)} \text{ is not a } p\text{-th power},$$
then $\mathrm{rank}_\Lambda \mathbb{T}^0_\mathfrak{m} = 1$ *and*
$$\mu(L_p^+(f_k,\omega^0)) = \mathrm{val}_p(N-1) + \mathrm{val}_p(k),$$
where f_k is the unique form of weight k in the Hida family corresponding to the isomorphism $\mathbb{T}^0_\mathfrak{m} \cong \Lambda$.

Proof. By Theorem 5.14, the assumptions (a)-(c) imply that $U_p - 1$ generates $\mathcal{I}_\mathfrak{m}$. The proof of parts (1)-(4) works verbatim as in the proof of Theorem 4.7, except that the role of the Kubota–Leopold series $L_p(\psi_\mathfrak{m}^{-1},\kappa)$ is played by $\mathcal{E}^-(T_0)$, which equals $\zeta_{p,k_0}(\kappa)(1-N^{\kappa/2})$. In particular, for every weight k,
$$\begin{aligned}\mathrm{val}_\varpi(\mathcal{E}^-(T_0)|_{\kappa=k}) &= \mathrm{val}_\varpi(\zeta_{p,k_0}(k)) + \mathrm{val}_\varpi(1-N^{k/2}) \\ &= \mathrm{val}_\varpi(N-1) + \mathrm{val}_\varpi(k),\end{aligned}$$
because $\mathrm{val}_\varpi(\zeta_{p,k_0}(k)) = 0$ by (a). The last claim follows from Theorem 5.16 and (3). □

Remark 5.21. We note that the relative advantage of Theorem 5.20 over Theorem 4.7 is that all of the hypotheses are simple numerical criteria that can be verified to be true in any given case.

5.6. **Conjectures and evidence.** The below is a re-statement of Conjecture 4.8 but now in the setting of this section.

Conjecture 5.22.

(1) *Suppose that $\mathbb{T}_\mathfrak{m}$ is Gorenstein. For each j, there is an equality*
$$\mathrm{content}(L_p^+(\mathfrak{m},\omega^j)) = \mathcal{I}_\mathfrak{m}.$$

(2) *Suppose that $\mathbb{T}^0_\mathfrak{m}$ is Gorenstein with $H^1_\Lambda(X_1(N))^+_\mathfrak{m}$ generated by an element e as a $\mathbb{T}^0_\mathfrak{m}$-module. Then $L_p^+(\mathfrak{m},\omega^j)^0_e$ has unit content.*

If $\mathcal{I}_\mathfrak{m}$ is principal then both $\mathbb{T}_\mathfrak{m}$ and $\mathbb{T}^0_\mathfrak{m}$ are Gorenstein and the hypotheses of both parts of the above conjecture are satisfied. When $U_p - 1$ generates $\mathcal{I}_\mathfrak{m}$, then the $j=0$ case of this conjecture follows from Theorem 5.20.

We ran numerical tests of this conjecture analogous to what was done in [BP19, Section 3.9]. Namely, for all primes $N < 80$, we ran through all primes $p \geq 5$ such that $N \equiv 1 \pmod p$. For each such pair (N,p), we considered $2 \leq k_0 \leq p-3$

and when the corresponding $\mathbb{T}^0_{\mathfrak{m}}$ had rank 1, we verified the above conjecture for all values of j. The assumption of the rank 1 of $\mathbb{T}^0_{\mathfrak{m}}$ is needed because the method of computation involved iterating U_p on overconvergent modular symbols which converges when there is a unique form in the Hida family in each weight k. This assumption on the rank of $\mathbb{T}^0_{\mathfrak{m}}$ also implies that both $\mathbb{T}^0_{\mathfrak{m}}$ and $\mathbb{T}_{\mathfrak{m}}$ are Gorenstein and so $\mathcal{I}_{\mathfrak{m}}$ is principal in this case. In all, we verified the conjecture (for all allowable values of j) for 35 eigenforms.

Appendix A. Hida theory with tame Atkin-Lehner involutions

We consider a variant of the Hida Hecke algebra with operators w_ℓ at primes ℓ dividing the tame level rather than U_ℓ. Let $N = p\ell_1 \dots \ell_r$ be squarefree, and assume $p > 3$.

A.1. **Review of Hida theory.** In this section, we review Hida theory. All of these results are well-known, see, for example, [Hid86a], [Hid86b], [Hid93, Chapter 7], [Wil88], [FK12, Section 1.5].

A.1.1. *Classical modular forms.* For $r \geq 0$ let Γ_r denote the congruence subgroup $\Gamma_0(N) \cap \Gamma_1(p^r)$. For a ring R, let $M_k(\Gamma_r, R)$ and $S_k(\Gamma_r, R)$ denote the spaces of modular forms and cusp forms, respectively, of weight k and level Γ_r, with coefficients in R. There is an injective q-expansion map

$$\mathrm{expand}_q : M_k(\Gamma_r, R) \to R[\![q]\!]$$

which we denote by $f \mapsto \sum_n a_n(f) q^n$. For $f \in S_k(\Gamma_r, R)$ we have $a_0(f) = 0$. We define

$$m_k(\Gamma_r, R) = \{ f \in M_k(\Gamma_r, Q(R)) \ : \ a_n(f) \in R \text{ for all } n > 0 \}$$

where $Q(R)$ is the localization $S^{-1}R$ where S is the set of non-zero-divisors of R.

There are Hecke operators T_n for $(n, N) = 1$ and U_ℓ for $\ell \mid N$ as well as the Atkin-Lehner involutions w_ℓ for $\ell \mid N$ and diamond operators $\langle n \rangle$ for $(n, N) = 1$. These all act on $m_k(\Gamma_r, R)$ and preserve the submodules $M_k(\Gamma_r, R)$ and $S_k(\Gamma_r, R)$. We let

$$\mathfrak{H}'_{r,k} \subset \mathrm{End}_{\mathbb{Z}_p}(m_k(\Gamma_r, \mathbb{Z}_p)), \ \mathfrak{h}'_{r,k} \subset \mathrm{End}_{\mathbb{Z}_p}(S_k(\Gamma_r, \mathbb{Z}_p))$$

be the $\mathbb{Z}_p$-subalgebras generated by the diamond operators and T_n for $(n, N) = 1$ as well as U_ℓ for $\ell | N$. These are commutative algebras, and the subject of Hida theory.

We let

$$\mathfrak{H}_{r,k} \subset \mathrm{End}_{\mathbb{Z}_p}(m_k(\Gamma_r, \mathbb{Z}_p)), \ \mathfrak{h}_{r,k} \subset \mathrm{End}_{\mathbb{Z}_p}(S_k(\Gamma_r, \mathbb{Z}_p))$$

be the $\mathbb{Z}_p$-subalgebras generated by the diamond operators and T_n for $(n, N) = 1$ as well as U_p and w_ℓ for $\ell | \frac{N}{p}$. These are commutative algebras, and are the main focus in this paper.

Lemma A.1. *Let M be $m_k(\Gamma_r, \mathbb{Z}_p)$ or $S_k(\Gamma_r, \mathbb{Z}_p)$, and let H be $\mathfrak{H}'_{r,k}$ or $\mathfrak{h}'_{r,k}$, respectively. Then M and H are free $\mathbb{Z}_p$-modules of finite rank and the pairing*

$$M \times H \to \mathbb{Z}_p$$

given by $(f, T) \mapsto a_1(Tf)$ is perfect.

A.1.2. *Cohomology.* Let $Y(\Gamma_r)$ be the modular curve with level Γ_r and let $X(\Gamma_r)$ be its compactification. For $k \geq 2$, let $\mathcal{F}_k$ denote the twisted constant p-adic étale sheaf on $Y(\Gamma_r)$ associated to the representation $\mathrm{Sym}^{k-2}\mathrm{Std}$ of GL_2, and denote by the same letter its pushforward to $X(\Gamma_r)$. Consider the cohomology groups $H^1(r,k), H^1_c(r,k), H^1_P(r,k)$ defined as follows

$$H^1(r,k) := H^1(Y(\Gamma_r), \mathcal{F}_k),\ H^1_c(r,k) := H^1_c(Y(\Gamma_r), \mathcal{F}_k),\ H^1_P(r,k) := H^1(X(\Gamma_r), \mathcal{F}_k).$$

By Eichler-Shimura theory, the algebras $\mathfrak{H}'_{r,k}$ and $\mathfrak{H}_{r,k}$ act on $H^1_\dagger(r,k)$ for any $\dagger \in \{\emptyset, c, P\}$, faithfully if $\dagger \in \{\emptyset, c\}$, and factoring exactly through $\mathfrak{h}'_{r,k}$ and $\mathfrak{h}_{r,k}$, respectively, if $\dagger = P$.

Lemma A.2. *For any $k \geq 2$, any $r' \geq r \geq 0$, and any $\dagger \in \{\emptyset, c, P\}$, the trace maps*

$$H^1_\dagger(r',k) \to H^1_\dagger(r,k)$$

commute with the actions of T_ℓ and $\langle \ell \rangle$ for any $\ell \nmid N$, U_ℓ for any $\ell \mid N$, and w_ℓ for any $\ell \mid\mid N$.

Proof. For T_ℓ and U_ℓ, this is proven in [Oht93, Lemma 7.4.1], and essentially the same proof works for w_ℓ. Indeed, just as in that proof, it is enough to that w_ℓ commutes with the natural map

$$H^1_\dagger(r,k) \to H^1_\dagger(r',k)$$

for $\dagger \in \{\emptyset, c\}$. This commutivity is clear because on both spaces w_ℓ is given by the same double coset operator. Explicitly, at any level M with $\ell || M$, w_ℓ is the double coset operator associated to any matrix $W_{\ell,M}$ of the form

$$W_{\ell,M} = \begin{pmatrix} \ell x & y \\ Mz & \ell w \end{pmatrix}$$

such that $\det(W_{\ell,M}) = \ell$ (the operator is independent of the choice of $W_{\ell,M}$). We see that any choice of $W_{\ell,Np^{r'}}$ is also a valid choice for W_{ℓ,Np^r}. □

A.1.3. *Hida theory.* For a $\mathfrak{H}_{r,k}$-module or $\mathfrak{H}'_{r,k}$-module M, let M^{ord} denote the largest direct summand on which U_p acts invertibly. For k fixed, let

$$\mathfrak{H}'^{\mathrm{ord}}_k = \varprojlim_{r \geq 0} \mathfrak{H}'^{\mathrm{ord}}_{r,k},\ \ \mathfrak{h}'^{\mathrm{ord}}_k = \varprojlim_{r \geq 0} \mathfrak{h}'^{\mathrm{ord}}_{r,k}$$

where the transition maps send T_q and U_ℓ to the operator with the same name (this is well-defined by Lemma A.2). These are algebras over the Iwasawa algebra $\Lambda = \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]$, via the diamond operator action. For $a = 1, \ldots, p-1$, and any Λ-module, M, let $M^{(a)}$ denote the direct summand where the torsion subgroup of $\mathbb{Z}_p^\times$ acts by the a-th power of the Teichemuller character. Identify $\Lambda^{(a)}$ with $\mathbb{Z}_p[\![1+p\mathbb{Z}_p]\!]$, and, for any $k \equiv a \pmod p$, let $\omega_{r,k} = [1+p]^{p^r} - (1+p)^{(k-2)p^r} \in \Lambda^{(a)}$.

Let

$$m^{\mathrm{ord}}_{k,\Lambda} = \varprojlim_{r \geq 0} m_k(\Gamma_r, \mathbb{Z}_p)^{\mathrm{ord}},\ \ S^{\mathrm{ord}}_{k,\Lambda} = \varprojlim_{r \geq 0} S_k(\Gamma_r, \mathbb{Z}_p)^{\mathrm{ord}}$$

where the transition maps are the trace maps. These are faithful modules for $\mathfrak{H}'^{\mathrm{ord}}_k$ and $\mathfrak{h}'^{\mathrm{ord}}_k$, respectively. The maps

$$m_k(\Gamma_r, \mathbb{Z}_p)^{\mathrm{ord}} \to \mathbb{Q}_p[(\mathbb{Z}/p^r\mathbb{Z})^\times][\![q]\!]$$

given by

$$f \mapsto \sum_{a \in (\mathbb{Z}/p^r\mathbb{Z})^\times} (\mathrm{expand}_q(\langle a\rangle^{-1} U_p^r w_{Np^r}(f))[a]$$

induce injective Λ-module homomorphisms

$$m_{k,\Lambda}^{\mathrm{ord}} \to Q(\Lambda) + q\Lambda[\![q]\!], \; S_{k,\Lambda}^{\mathrm{ord}} \to q\Lambda[\![q]\!],$$

and we identify these modules with there images in $Q(\Lambda)[\![q]\!]$. The following is known as Hida's control theorem.

Theorem A.3. *The Λ-modules $m_{k,\Lambda}^{\mathrm{ord}}, S_{k,\Lambda}^{\mathrm{ord}}, \mathfrak{H}_k'^{\mathrm{ord}}$ and $\mathfrak{h}_k'^{\mathrm{ord}}$ are independent of the integer $k \geq 2$. We may and do drop the subscript k from the notation, and denote these objects simply by $m_\Lambda^{\mathrm{ord}}, S_\Lambda^{\mathrm{ord}}, \mathfrak{H}'^{\mathrm{ord}}$ and $\mathfrak{h}'^{\mathrm{ord}}$.*

Let M be m_Λ^{ord} or S_Λ^{ord} and let H be $\mathfrak{H}'^{\mathrm{ord}}$ or $\mathfrak{h}'^{\mathrm{ord}}$, respectively. Let $M_{r,k}$ and $H_{r,k}$ be the fixed weight and level versions. We have:

(1) *M and H are free Λ-modules of finite rank.*
(2) *The pairing*
$$M \times H \to \Lambda,$$
given by $(f, T) \mapsto a_1(Tf)$, is perfect.
(3) *For any $k \geq 2$ and $r \geq 0$, the natural maps*
$$M/\omega_{r,k}M \to M_{r,k}, \; H/\omega_{r,k}H \to H_{r,k}$$
are isomorphisms.

A.2. **Hida theory with Atkin-Lehner operators.** In this section, we prove the analog of the control theorem for the algebras $\mathfrak{H}_{r,k}^{\mathrm{ord}}$ and $\mathfrak{h}_{r,k}^{\mathrm{ord}}$.

For an element $\epsilon = (\epsilon_1, \ldots, \epsilon_r) \in \{\pm 1\}^r$ and an $\mathfrak{H}_{r,k}$-module M, let M^ϵ denote the summand of M on which w_{ℓ_i} acts by ϵ_i for $i = 1, \ldots, r$.

A.2.1. *Atkin-Lehner theory, and reducing from level N to level N/p.* We will frequently make use of the following result, which is a variant of Atkin-Lehner theory (c.f. [AL70, Theorem 1]) that allows for more-general coefficient rings. It was first proven by Mazur [Maz77, Lemma II.5.9, pg. 83] in the case $M > 5$ is prime, and in the general case by Ohta [Oht14, Corollary (2.1.4)].

Lemma A.4. *Let M be an integer and let $f \in M_k(\Gamma_0(M), R)$ with R a $\mathbb{Z}[1/2M]$-algebra, and suppose that f is an eigenform for all w_ℓ with $\ell | M$ and that $a_n(f) = 0$ for all $(n, M) = 1$. Then f is constant.*

Note that the lemma requires that the level be invertible in the coefficient ring. We want to apply the lemma to rings R where $p \notin R^\times$. The following result, which essentially says that ordinary forms are old-at-p when the weight is at least 3, is useful to remove p from the level. It was proven by Gouvêa [Gou92, Lemma 3] for cusp forms, and the same proof works for modular forms (c.f. [Oht05, Proposition 1.3.2]).

Lemma A.5. *Let M be any integer that is prime to p and let $k > 2$. Then the map*

$$M_k(\Gamma_1(M), \mathbb{Z}_p)^{\mathrm{ord}} \to M_k(\Gamma_1(M) \cap \Gamma_0(p), \mathbb{Z}_p)^{\mathrm{ord}}$$

is an isomorphism.

A.2.2. *Duality.* The key to the proof of the control theorem will be the following duality result, which is the analog of Lemma A.1.

Proposition A.6. *For any $r \geq 0$ and $k \geq 2$, consider the pairings*

$$(-,-)_{r,k} : m_k(\Gamma_r, \mathbb{Z}_p)^{\mathrm{ord}} \times \mathfrak{H}_{r,k}^{\mathrm{ord}} \to \mathbb{Z}_p, \;\; (-,-)_{r,k}^0 : S_k(\Gamma_r, \mathbb{Z}_p)^{\mathrm{ord}} \times \mathfrak{h}_{r,k}^{\mathrm{ord}} \to \mathbb{Z}_p$$

given by $(f,T) \mapsto a_1(Tf)$.

1. *For any r, k, the resulting maps*
$$\mathfrak{H}_{r,k}^{\mathrm{ord}} \to \mathrm{Hom}_{\mathbb{Z}_p}(m_k(\Gamma_r, \mathbb{Z}_p)^{\mathrm{ord}}, \mathbb{Z}_p), \;\; \mathfrak{h}_{r,k}^{\mathrm{ord}} \to \mathrm{Hom}_{\mathbb{Z}_p}(S_k(\Gamma_r, \mathbb{Z}_p)^{\mathrm{ord}}, \mathbb{Z}_p)$$
are injective.
2. *For $r = 0$ and any $k > 2$, the pairings are perfect.*

For the proof of the proposition, we need a lemma comparing $m_k(\Gamma_0, \mathbb{Z}_p)^{\mathrm{ord},\epsilon}$ to $M_k(\Gamma_0, \mathbb{Z}_p)^{\mathrm{ord},\epsilon}$.

Lemma A.7. *For any $\epsilon \in \{\pm 1\}^r$, there is an exact sequence*

$$0 \to M_k(\Gamma_0, \mathbb{Z}_p)^{\mathrm{ord},\epsilon} \to m_k(\Gamma_0, \mathbb{Z}_p)^{\mathrm{ord},\epsilon} \xrightarrow{a_0} \left(\frac{1}{p^{m_{k,\epsilon}}}\mathbb{Z}_p\right)/\mathbb{Z}_p \to 0$$

for some integer $m_{k,\epsilon} \geq 0$. Moreover, if we define m' as the maximal integer m such that there exists $h \in M_k(\Gamma_0, \mathbb{Z}_p)^{\mathrm{ord},\epsilon}$ with $h \equiv 1 \pmod{p^m}$ and $a_0(h) = 1$, then $m_{k,\epsilon} = m'$.

Proof. We have an exact sequence

$$0 \to M_k(\Gamma_0, \mathbb{Z}_p)^{\mathrm{ord},\epsilon} \to m_k(\Gamma_0, \mathbb{Z}_p)^{\mathrm{ord},\epsilon} \xrightarrow{a_0} \mathbb{Q}_p/\mathbb{Z}_p.$$

Since $m_k(\Gamma_0, \mathbb{Z}_p)^{\mathrm{ord},\epsilon}$ is finitely generated, the image of the a_0 map is a finite cyclic p-group. We define $m_{k,\epsilon} \geq 0$ so that the order of this group is $p^{m_{k,\epsilon}}$, and we get the desired exact sequence.

If $h' \in m_k(\Gamma_0, \mathbb{Z}_p)^{\mathrm{ord},\epsilon}$ satisfies $a_0(h') = \frac{1}{p^{m_{k,\epsilon}}}$, then $h := p^{m_{k,\epsilon}} h' \in M_k(\Gamma_0, \mathbb{Z}_p)^{\mathrm{ord},\epsilon}$ satisfies $h \equiv 1 \pmod{p^{m_{k,\epsilon}}}$ and $a_0(h) = 1$, so $m_{k,\epsilon} \leq m'$. Conversely, if $h \equiv 1 \pmod{p^{m'}}$ and $a_0(h) = 1$, then $h' := p^{-m'}h \in m_k(\Gamma_0, \mathbb{Z}_p)^{\mathrm{ord},\epsilon}$ and $a_0(h') = p^{-m'}$ so $m_{k,\epsilon} \geq m'$. □

Remark A.8. It is known that $m_{k,\epsilon} = 0$ if $(p-1) \nmid k$, but we shall not use this.

Proof of Proposition A.6. We give the proof only for the modular case, the cuspidal case being similar but easier. We can decompose $m_k(\Gamma_0, \mathbb{Z}_p)^{\mathrm{ord}}$ and $\mathfrak{H}_{0,k}^{\mathrm{ord}}$ into eigenspaces for the Aktin-Lehner operators. It is enough to show that, for any choice of ϵ, the induced pairing

$$(-,-)_{r,k} : m_k(\Gamma_r, \mathbb{Z}_p)^{\mathrm{ord},\epsilon} \times \mathfrak{H}_{r,k}^{\mathrm{ord},\epsilon} \to \mathbb{Z}_p,$$

has the desired properties.

We first show that the pairing is perfect with $\mathbb{Q}_p$-coefficients, which will imply (1). Suppose that $f \in M_k(\Gamma_r, \mathbb{Q}_p)^{\mathrm{ord},\epsilon}$ satisfies $(f,T)_{r,k} = 0$ for all $T \in \mathfrak{H}_{r,k}^{\mathrm{ord},\epsilon}[1/p]$. This implies that $a_n(f) = 0$ for all $(n, N/p) = 1$. By Lemma A.4, this implies that f is constant, and hence 0. Conversely, suppose that $T \in \mathfrak{H}_{r,k}^{\mathrm{ord},\epsilon}[1/p]$ satisfies $(f,T)_{r,k} = 0$ for all $f \in M_k(\Gamma_r, \mathbb{Q}_p)^{\mathrm{ord},\epsilon}$. Then for any $g \in M_k(\Gamma_r, \mathbb{Q}_p)^{\mathrm{ord},\epsilon}$, we have $a_n(Tg) = (T_n g, T)_{r,k} = 0$ for all $(n, N/p) = 1$. The same lemma then implies that $Tg = 0$, which implies that $T = 0$ since $\mathfrak{H}_{r,k}^{\mathrm{ord},\epsilon}[1/p]$ acts faithfully on $M_k(\Gamma_r, \mathbb{Q}_p)^{\mathrm{ord},\epsilon}$.

Now we consider (2). By the perfectness of the $\mathbb{Q}_p$-pairing, we have that

$$m_k(\Gamma_0,\mathbb{Z}_p)^{\mathrm{ord},\epsilon} \to \mathrm{Hom}_{\mathbb{Z}_p}(\mathfrak{H}_{0,k}^{\mathrm{ord},\epsilon},\mathbb{Z}_p)$$

is injective. We have to show it is surjective. Let $\phi : \mathfrak{H}_{0,k}^{\mathrm{ord},\epsilon} \to \mathbb{Z}_p$. We see that there is $f' \in M_k(\Gamma_0,\mathbb{Q}_p)^{\mathrm{ord},\epsilon}$ such that $\phi(T) = (T, f')_k$ for all $T \in \mathfrak{H}_{0,k}^{\mathrm{ord},\epsilon}$. This implies that $a_n(f') = \phi(T_n f') \in \mathbb{Z}_p$ for all $(n, N/p) = 1$, where $T_n := U_{p^r} T_{n/p^r}$ if $p^r || n$. If $f' \in m_k(\Gamma_0,\mathbb{Z}_p)^{\mathrm{ord},\epsilon}$, we are done, so assume for a contradiction that it is not. Then there is a minimal $r \geq 1$ such that $f = p^r f' \in m_k(\Gamma_0,\mathbb{Z}_p)^{\mathrm{ord},\epsilon}$. By the minimality of r, we must have $a_n(f) \in \mathbb{Z}_p^\times$ for some $n > 0$ (clearly this n must have $(n, N/p) > 1$).

Assume that $f \in M_k(\Gamma_0,\mathbb{Z}_p)^{\mathrm{ord},\epsilon}$. By Lemma A.5, we may consider $f \in M_k(\Gamma_0(N/p),\mathbb{Z}_p)^{\mathrm{ord},\epsilon}$. Consider the image $\bar{f}$ in $M_k(\Gamma_0(N/p),\mathbb{F}_p)^{\mathrm{ord},\epsilon}$. Since $a_n(f') \in \mathbb{Z}_p$ for all $(n,N) = 1$ and since $r \geq 1$, we have $a_n(\bar{f}) = 0$ for all $(n,N) = 1$. As $N/p \in \mathbb{F}_p^\times$, Lemma A.4 applies and we get that $\bar{f}$ is constant. This implies that $a_n(f) \equiv 0 \pmod p$ for all $n > 0$, a contradiction.

Finally, assume that $f \notin M_k(\Gamma_0,\mathbb{Z}_p)^{\mathrm{ord},\epsilon}$, so $s := -\mathrm{val}_p(a_0(f)) > 0$. Then $g := a_0(f)^{-1} f \in M_k(\Gamma_0,\mathbb{Z}_p)^{\mathrm{ord},\epsilon}$ satisfies $a_0(g) = 1$. By Lemma A.5, we may consider $g \in M_k(\Gamma_0(N/p),\mathbb{Z}_p)^{\mathrm{ord},\epsilon}$. Since $a_n(f') \in \mathbb{Z}_p$ for all $(n,N) = 1$, we have $a_n(g) = 0 \pmod{p^{r+s}}$ for all $(n,N) = 1$. As $N/p \in (\mathbb{Z}/p^{r+s}\mathbb{Z})^\times$, Lemma A.4 implies that $g \pmod{p^{r+s}}$ is the constant $a_0(g) = 1$. By Lemma A.7, and since $r > 0$, this implies that $s < m_{k,\epsilon}$.

Now fix $h \in M_k(\Gamma_0,\mathbb{Z}_p)^{\mathrm{ord},\epsilon}$ such that $h \equiv 1 \pmod{p^{m_{k,\epsilon}}}$ and such that $a_0(h) = 1$. Since $\mathrm{val}_p(a_0(f)) = -s$ and $s < m_{k,\epsilon}$, we see that $a_n(a_0(f)h) \in p\mathbb{Z}_p$ for all $n > 0$. Then letting $f'' = f - a_0(f)h$, we see that $f'' \in M_k(\Gamma_0(N/p),\mathbb{Z}_p)^{\mathrm{ord},\epsilon}$. Moreover, we see that $a_n(f'') = p^r a_n(f') - a_n(a_0(f)h) \equiv 0 \pmod p$, for all $(n, N/p) = 1$. By Lemma A.4 this implies that $f'' \pmod p$ is constant. In particular, for all $n > 0$ we have

$$0 \equiv a_n(f'') \equiv a_n(f) \pmod p,$$

a contradiction. □

A.2.3. *Control theorem.* For k fixed, let

$$\mathfrak{H}_k^{\mathrm{ord}} = \varprojlim_{r\geq 0} \mathfrak{H}_{r,k}^{\mathrm{ord}},\ \mathfrak{h}_k^{\mathrm{ord}} = \varprojlim_{r\geq 0} \mathfrak{h}_{r,k}^{\mathrm{ord}}$$

where the transition maps send T_q and w_ℓ to the operator with the same name (this is well-defined by Lemma A.2). These are algebras over the Iwasawa algebra $\Lambda = \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]$, via the diamond operator action. We can now prove the main theorem, which is a control theorem for the algebras $\mathfrak{H}_k^{\mathrm{ord}}$ and $\mathfrak{h}_k^{\mathrm{ord}}$.

Theorem A.9. *The algebras $\mathfrak{H}_k^{\mathrm{ord}}$ and $\mathfrak{h}_k^{\mathrm{ord}}$ for various k are canonically identified with each other. We can and do drop the subscript k from the notation and simply refer to these algebras as $\mathfrak{H}^{\mathrm{ord}}$ and $\mathfrak{h}^{\mathrm{ord}}$. Moreover:*

(1) There are canonical isomorphisms of Λ-modules $\mathfrak{H}^{\mathrm{ord}} \to \mathfrak{H}'^{\mathrm{ord}}$ and $\mathfrak{h}^{\mathrm{ord}} \to \mathfrak{h}'^{\mathrm{ord}}$. In particular, $\mathfrak{H}^{\mathrm{ord}}$ and $\mathfrak{h}^{\mathrm{ord}}$ are free Λ-modules of finite rank.

(2) The pairings

$$m_\Lambda^{\mathrm{ord}} \times \mathfrak{H}^{\mathrm{ord}} \to \Lambda,\ S_\Lambda^{\mathrm{ord}} \times \mathfrak{h}^{\mathrm{ord}} \to \Lambda,$$

given by $(f,T) \mapsto a_1(Tf)$, are perfect.

(3) *The natural maps*

$$\mathfrak{H}^{\mathrm{ord}}/\omega_{r,k}\mathfrak{H}^{\mathrm{ord}} \to \mathfrak{H}^{\mathrm{ord}}_{r,k}, \ \mathfrak{h}^{\mathrm{ord}}/\omega_{r,k}\mathfrak{h}^{\mathrm{ord}} \to \mathfrak{h}^{\mathrm{ord}}_{r,k}$$

are isomorphisms for all $r \geq 0$ *and* $k \geq 2$.

(4) *The pairings* $(-,-)_{r,k}$ *and* $(-,-)^0_{r,k}$ *of Proposition A.6 are perfect for all* $r \geq 0$ *and* $k \geq 2$.

Proof. We give the proofs only for modular case, the cuspidal cases being similar. First, we see that $\mathfrak{H}^{\mathrm{ord}}_k$ is the subalgebra of $\mathrm{End}_\Lambda(m^{\mathrm{ord}}_{k,\Lambda})$ generated by the operators T_q, U_p and w_ℓ. But by Theorem A.3, $m^{\mathrm{ord}}_{k,\Lambda}$ is independent of k. This shows that these algebras are independent of k.

The map $\mathfrak{H}^{\mathrm{ord}} \to \mathfrak{H}'^{\mathrm{ord}}$ in (1) is given as the composite

$$\mathfrak{H}^{\mathrm{ord}} \to \mathrm{Hom}_\Lambda(m^{\mathrm{ord}}_\Lambda, \Lambda) \xrightarrow{\sim} \mathfrak{H}'^{\mathrm{ord}}$$

where the first map is induced by the pairing in (2), and the second map is the isomorphism of Theorem A.3 (2). Let $X = \mathrm{coker}(\mathfrak{H}^{\mathrm{ord}} \to \mathfrak{H}'^{\mathrm{ord}})$, which is a finitely generated Λ-module.

Similarly, for any fixed k and r, we have a map $\mathfrak{H}^{\mathrm{ord}}_{r,k} \to \mathfrak{H}'^{\mathrm{ord}}_{r,k}$ defined as the composite

$$\mathfrak{H}^{\mathrm{ord}}_{r,k} \to \mathrm{Hom}_{\mathbb{Z}_p}(m_k(\Gamma_r, \mathbb{Z}_p)^{\mathrm{ord}}, \mathbb{Z}_p) \xrightarrow{\sim} \mathfrak{H}'^{\mathrm{ord}}_{r,k},$$

where the first map is given by $(-,-)_{r,k}$ and the second map is the isomorphism given by Lemma A.1. This map is an isomorphism for every r and k if and only if (4) holds.

We have a commutative diagram to compare these maps:

$$\begin{array}{ccc} \mathfrak{H}^{\mathrm{ord}}/\omega_{r,k}\mathfrak{H}^{\mathrm{ord}} & \longrightarrow & \mathfrak{H}'^{\mathrm{ord}}/\omega_{r,k}\mathfrak{H}'^{\mathrm{ord}} \\ \downarrow & & \downarrow \wr \\ \mathfrak{H}^{\mathrm{ord}}_{r,k} & \hookrightarrow & \mathfrak{H}'^{\mathrm{ord}}_{r,k}. \end{array} \tag{A.10}$$

The leftmost vertical arrow is surjective because the operators T_q, U_p and w_ℓ map to the operators of the same name, the lower horizontal arrow is injective by Proposition A.6(1), and the rightmost vertical arrow is an isomorphism by Theorem A.3 (3). For $r = 0$ and $k > 2$, Proposition A.6(2) implies that the lower horizontal arrow is an isomorphism. This implies that the map

$$\mathfrak{H}^{\mathrm{ord}}/\omega_{0,k}\mathfrak{H}^{\mathrm{ord}} \to \mathfrak{H}'^{\mathrm{ord}}/\omega_{0,k}\mathfrak{H}'^{\mathrm{ord}}$$

is surjective for all $k > 2$. In other words, we have $X/\omega_{0,k}X = 0$ for all $k > 2$. The elements $\omega_{0,3}, \omega_{0,4}, \dots, \omega_{0,p+1}$ are in the $p-1$ different maximal ideals of Λ, so $X = 0$ by Nakayama's lemma. This implies that $\mathfrak{H}^{\mathrm{ord}} \to \mathfrak{H}'^{\mathrm{ord}}$ is surjective.

Returning to the diagram (A.10) for arbitrary $r \geq 0$ and $k \geq 2$, we see that the lower horizontal arrow is also surjective, proving (4). Since the map $\mathfrak{H}^{\mathrm{ord}} \to \mathfrak{H}'^{\mathrm{ord}}$ is the inverse limit (for k fixed and r increasing) of these maps, it is also an isomorphism proving (1). By the definition of the map $\mathfrak{H}^{\mathrm{ord}} \to \mathfrak{H}'^{\mathrm{ord}}$, this also proves (2). This shows that all the arrows except the leftmost vertical in (A.10) are isomorphisms, so it is too, proving (3). □

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