NONDEGENERACY AND SATO-TATE DISTRIBUTIONS OF TWO FAMILIES OF JACOBIAN VARIETIES

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Abstract. We consider the curves $y^2 = x^{2^m} - c$ and $y^2 = x^{2^d+1} - cx$ over the rationals where $c \in \mathbb{Q}^\times$. These curves are related via their associated Jacobian varieties in that the Jacobians of the latter appear as factors of the Jacobians of the former. One of the principle aims of this paper is to fully describe their Sato-Tate groups and distributions by determining generators of the component groups. In order to do this, we first prove the nondegeneracy of the two families of Jacobian varieties via their Hodge groups. We then use results relating Sato-Tate groups and twisted Lefschetz groups of nondegenerate abelian varieties to determine the generators of the associated Sato-Tate groups. The results of this paper add new examples to the literature of families of nondegenerate Jacobian varieties and of noncyclic Sato-Tate groups. Furthermore, we compute moment statistics associated to the Sato-Tate groups which can be used to verify the equidistribution statement of the generalized Sato-Tate conjecture by comparing them to moment statistics obtained for the traces in the normalized L-polynomials of the curves.

1. Introduction

In this paper we consider two families of hyperelliptic curves over the rationals

$$C_{2^m} : y^2 = x^{2^m} - c, \qquad C'_{2^d+1} : y^2 = x^{2^d+1} - cx,$$

where $c \in \mathbb{Q}^\times$. These curves are related via their associated Jacobian varieties: the Jacobians of C'_{2^d+1} appear as factors of the Jacobians of C_{2^m}. The authors first studied curves of this form in [12] where we developed an algorithm to compute the identity components of the Sato-Tate groups of their Jacobians. One of the principle aims of this paper is to fully describe their Sato-Tate groups and distributions by determining generators of the component groups. The techniques that were used in [11, 20] apply to curves whose Jacobians are nondegenerate, and so another principle aim of this paper is to prove the nondegeneracy of the two families of Jacobian varieties. Thus, this paper adds new examples to the literature of infinite families of nondegenerate Jacobian varieties and of interesting Sato-Tate groups.

A complex abelian variety A is said to be *nondegenerate* if its (complexified) Hodge ring is generated by divisor classes. While the definition of nondegeneracy is a statement about the Hodge ring, we can see the effects of nondegeneracy in groups constructed from the Hodge structure of the abelian variety: the Mumford-Tate group, the Hodge group, and the Sato-Tate group. The Mumford-Tate group and Hodge group are related to the Hodge ring via an action on certain cohomology groups. The algebraic Sato-Tate conjecture and Mumford-Tate conjecture imply a relationship between the identity component of the Sato-Tate group and the Mumford-Tate and Hodge groups. Thus, it is natural that the nondegeneracy of the Hodge ring has implications for these groups. Furthermore, it follows from [4, Theorem 6.1] and work of Serre in [33, Section 8.3.4] that if A is nondegenerate then the component group $\mathrm{ST}(A)/\mathrm{ST}^0(A)$

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is isomorphic to the Galois group $\mathrm{Gal}(K/F)$, where K is the endomorphism field of A, i.e. the minimal extension over which all the endomorphisms of the abelian variety A are defined. It is for these reasons that it is particularly nice to work with nondegenerate abelian varieties in the context of Sato-Tate distributions, and why it is of interest to find new families of nondegenerate abelian varieties.

The study of Sato-Tate groups began with the original Sato-Tate conjecture for elliptic curves posed around 1960 by Mikio Sato and John Tate (independently). It is a statistical conjecture regarding the distribution of the normalized traces of Frobenius on an elliptic curve. In 2012, the conjecture was generalized to higher dimensional abelian varieties by Serre [33], and determining Sato-Tate groups of abelian varieties is the source of ongoing interest and work. Following the exposition of [18] we discuss the generalized Sato-Tate conjecture. If we let A be a g-dimensional abelian variety over a number field k, we define a compact Lie subgroup of $\mathrm{USp}(2g)$ associated to A and a sequence of conjugacy classes whose characteristic polynomials are the normalized L-polynomials, which are equidistributed in the image of the Haar measure. To be more precise, the L-function associated to the abelian variety A is

$$L(A,s) = \prod_{\mathfrak{p}} L_{\mathfrak{p}}(\mathrm{Norm}(\mathfrak{p})^{-s})^{-1},$$

where we take the product over prime ideals $\mathfrak{p}$ of the ring of integers of k at which A has good reduction. The polynomial $L_{\mathfrak{p}}(T)$ is the characteristic polynomial associated to the reverse Frobenius acting on the ℓ-adic Tate module when ℓ is coprime to $\mathfrak{p}$. Normalizing $L_{\mathfrak{p}}$ so that the roots of the polynomial lie on the circle centered at the origin with norm 1 we can now ask if the resulting polynomials are equidistributed with respect to some measure.

We now specify to the case where A is the Jacobian of a smooth projective curve C. Let X_k be the set of conjugacy classes of $\mathrm{ST}(\mathrm{Jac}(C)_k)$. Let $\{p_i\}_{i\geq 1}$ be an ordering by norm of the set of primes of good reduction for C over k and define a map that sends p_i to x_{p_i} in X_k. We can now state the generalized Sato-Tate conjecture.

Conjecture 1.1. *(Generalized Sato-Tate Conjecture) The sequence $\{x_{p_i}\}_{i\geq 1}$ is equidistributed on X_k with respect to the image on X_k of the Haar measure of* $\mathrm{ST}(\mathrm{Jac}(C)_k)$.

Moreover, this equidistribution statement can be considered as an analogue of the Chebotaryov density theorem (which describes statistically the splitting of primes in a given Galois extension). In this context the Sato-Tate group functions as the Galois group and we then consider the Galois group associated to an arbitrary motive as in [32]. Since the Sato-Tate group of a motive is a compact Lie group, we can retain much (but not all) of the same information by keeping track of the identity component and the group of connected components. Indeed, one way to prove the generalized Sato-Tate conjecture in full, is to prove Langland's principle of functoriality which would give the elements of the motivic Galois group as discussed in [3, pg. 43].

Although the generalized Sato-Tate conjecture is not fully proven, much is known for low dimensional cases. When A is an elliptic curve with complex multiplication (CM), the distribution is precisely one of two depending on whether the CM field is contained in k or not (see [35] for an exposition on this). If the elliptic curve does not admit CM, the Sato-Tate conjecture predicts that exactly one distribution occurs and this has been proven over totally real fields (see [5, 9, 21, 36]). Classification results for dimension 2 and 3 abelian varieties are given in [14] and [18], respectively. The Sato-Tate conjecture was proven for higher dimensional CM abelian varieties in [25], and there has been recent progress on computing Sato-Tate groups of nondegenerate abelian varieties (see, for example, [2, 11, 13, 16, 20, 26, 27]). The main goals of this paper are to prove that the Jacobians of the curves C_{2^m} and C'_{2^d+1} are nondegenerate and to determine their respective Sato-Tate groups. These curves are of particular interest since the Galois groups of their CM fields over $\mathbb{Q}$ are non-cyclic.

Our first main results of the paper are regarding the nondegeneracy of the Jacobian varieties. For example we have the following result (see Theorem 3.4 and Corollaries 3.5 and 3.6).

Theorem 1.2. *Let $m \geq 2$ and $d \geq 1$ be integers, and let $c \in \mathbb{Q}^\times$. Then the Jacobians of the curves $C_{2^m} : y^2 = x^{2^m} - c$ and $C'_{2^d+1} : y^2 = x^{2^d+1} - cx$ are nondegenerate.*

The proof of this result relies on the decomposition of the Jacobian varieties $\mathrm{Jac}(C_{2^m})$ given in Section 3.1 and on analyzing the Hodge groups of the factors $\mathrm{Jac}(C'_{2^d+1})$.

Once we have established nondegeneracy, we are able to give a complete description of the Sato-Tate group for each family of curves. We use similar techniques to those used in [11] and [20], but with the added challenge of considering families of twists. The Jacobians of the curves C'_{2^d+1} appear as factors of the Jacobians of the curves C_{2^m}, and we are able to describe the Sato-Tate group of the former in terms of that of the latter (see Proposition 4.6 and Theorem 4.7).

Theorem 1.3. *For generic $c \in \mathbb{Q}^\times$, the Sato-Tate group of the Jacobian of the curve C_{2^m} : $y^2 = x^{2^m} - c$ satisfies*

$$\mathrm{ST}(\mathrm{Jac}(C_{2^m}) \simeq \left\langle \mathrm{U}(1) \times \left(\mathrm{U}(1)^{2^{m-2}-1}\right)_2, \gamma, \gamma_J, \gamma_c \right\rangle$$

for $m > 2$. The generators of the component group can be written as block diagonal matrices whose diagonal blocks are the generators γ, γ_J, and γ_c in Theorem 4.3 coming from the isogeny factors $\mathrm{Jac}(C'_{2^d+1})$ of $\mathrm{Jac}(C_{2^m})$.

We demonstrate this result with the following example.

Example 1.4. *We consider the genus $g = 7$ curve $C_{16} : y^2 = x^{16} - c$. In this case $m = 4$ and the isogeny in Equation (2) in Section 3.1 becomes*

$$\mathrm{Jac}(C_{16}) \sim \mathrm{Jac}(C'_3) \times \mathrm{Jac}(C'_5) \times \mathrm{Jac}(C'_9),$$

where C'_n denotes the curve $y^2 = x^n - cx$. The result in Theorem 3.4 proves that the associated Hodge group decomposes as

$$\mathrm{Hg}(\mathrm{Jac}(C_{16})) = \mathrm{Hg}(\mathrm{Jac}(C'_3)) \times \mathrm{Hg}(\mathrm{Jac}(C'_5)) \times \mathrm{Hg}(\mathrm{Jac}(C'_9))$$

and Proposition 4.6 tells us that the identity component of the Sato-Tate group is

$$\begin{aligned} \mathrm{ST}^0(\mathrm{Jac}(C_{16})) &\simeq \underset{\substack{\wr\mid \\ \mathrm{U}(1)}}{\mathrm{ST}^0(\mathrm{Jac}(C'_3))} \times \underset{\substack{\wr\mid \\ \mathrm{U}(1)_2}}{\mathrm{ST}^0(\mathrm{Jac}(C'_5))} \times \underset{\substack{\wr\mid \\ (\mathrm{U}(1)\times\mathrm{U}(1))_2}}{\mathrm{ST}^0(\mathrm{Jac}(C'_9))} \\ &\simeq \mathrm{U}(1) \times (\mathrm{U}(1)^3)_2. \end{aligned}$$

The endomorphism field of $\mathrm{Jac}(C_{16})$ is $\mathbb{Q}(\zeta_{16}, c^{1/g})$; since the Jacobian is nondegenerate, this implies that the component group of the Sato-Tate group is isomorphic to $\mathrm{Gal}(\mathbb{Q}(\zeta_{16}, c^{1/g})/\mathbb{Q})$. For explicit generators of the component group, Theorem 4.7 tells us we can choose $\gamma_J = \mathrm{diag}(J, J, J, J, J, J, J), \gamma_c = \mathrm{diag}(I, \zeta_8^{-1} I, \zeta_8 I, \zeta_{16}^{-3} I, \zeta_{16}^{-1} I, \zeta_{16} I, \zeta_{16}^3 I)$ and $\gamma = \mathrm{diag}(\gamma_3, \gamma_5, \gamma_9)$, where

$$\gamma_3 = I, \qquad \gamma_5 = \begin{pmatrix} 0 & -J \\ J & 0 \end{pmatrix}, \qquad \gamma_9 = \begin{pmatrix} 0 & 0 & I & 0 \\ J & 0 & 0 & 0 \\ 0 & 0 & 0 & -J \\ 0 & I & 0 & 0 \end{pmatrix}$$

are the matrices defined in Theorem 4.3 for the curves C'_3, C'_5, and C'_9 respectively. Hence,

$$\mathrm{ST}(\mathrm{Jac}(C_{16}) \simeq \left\langle \mathrm{U}(1) \times (\mathrm{U}(1)^3)_2,\ \gamma, \gamma_J, \gamma_c \right\rangle$$

where γ is

$$\gamma = \begin{pmatrix} \boxed{I} & & \\ & \begin{array}{|cc|} \hline 0 & -J \\ J & 0 \\ \hline \end{array} & \\ & & \begin{array}{|cccc|} \hline 0 & 0 & I & 0 \\ J & 0 & 0 & 0 \\ 0 & 0 & 0 & -J \\ 0 & I & 0 & 0 \\ \hline \end{array} \end{pmatrix}.$$

Note that $\mathrm{Jac}(C_{16})$ is a CM abelian variety and the CM-type comes from the decomposition of the Jacobian as follows. In the case where $c = 1$, results of [8] tell us that the CM types of the Jacobians come from the embeddings of each of the CM fields into $\mathbb{C}$ and are given by $\mathrm{Jac}(C'_9) : e^{\pi i/8}, e^{3\pi i/8}, e^{5\pi i/8}, e^{7\pi i/8}$, $\mathrm{Jac}(C'_5) : e^{\pi i/4}, e^{3\pi i/4}$, and $\mathrm{Jac}(C'_3) : e^{\pi i/2}$. Since $\mathrm{Jac}(C_{16})$ is nondegenerate, the CM-type of $\mathrm{Jac}(C_{16})$ is the compositum of the CM-type for each of the curves.

A nice feature of this family of curves is that as m grows, the decomposition of $\mathrm{Jac}(C_{2^m})$ simply gains additional factors of the form $\mathrm{Jac}(C'_{2^d+1})$. This leads to recursive definitions of the generators of the Sato-Tate group. To see this, we provide the next example.

Example 1.5. *Letting $c = 1$, we consider the genus 15 curve $C_{32} : y^2 = x^{32} - 1$. In this case $m = 5$ and the isogeny in Equation (2) becomes*

$$\mathrm{Jac}(C_{32}) \sim \mathrm{Jac}(C'_3) \times \mathrm{Jac}(C'_5) \times \mathrm{Jac}(C'_9) \times \mathrm{Jac}(C'_{17}).$$

Proposition 4.6 tells us that the identity component of the Sato-Tate group is

$$\begin{aligned} \mathrm{ST}^0(\mathrm{Jac}(C_{32})) &\sim \underset{\substack{\wr| \\ \mathrm{U}(1)}}{\mathrm{ST}^0(\mathrm{Jac}(C'_3))} \times \underset{\substack{\wr| \\ \mathrm{U}(1)_2}}{\mathrm{ST}^0(\mathrm{Jac}(C'_5))} \times \underset{\substack{\wr| \\ (\mathrm{U}(1)\times\mathrm{U}(1))_2}}{\mathrm{ST}^0(\mathrm{Jac}(C'_9))} \times \underset{\substack{\wr| \\ (\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(1))_2}}{\mathrm{ST}^0(\mathrm{Jac}(C'_{17}))} \\ &\simeq \mathrm{U}(1) \times (\mathrm{U}(1)^6)_2. \end{aligned}$$

For explicit generators of the component group, Theorem 4.7 tells us we can choose $\gamma_J = \mathrm{diag}(J, J, \ldots, J)$ and $\gamma = \mathrm{diag}(\gamma_3, \gamma_5, \gamma_9, \gamma_{17})$ whose entries are the matrices defined in Theorem 4.3 for the curves C'_3, C'_5, C'_9, and C'_{17} respectively. Hence,

$$\mathrm{ST}(\mathrm{Jac}(C_{32}) \simeq \left\langle \mathrm{U}(1) \times (\mathrm{U}(1)^6)_2,\ \gamma, \gamma_J \right\rangle$$

where γ is

$$\gamma = \begin{pmatrix}
I &&&&&&&&&&&&&& \\
& 0 & -J &&&&&&&&&&&& \\
& J & 0 &&&&&&&&&&&& \\
&&& 0 & 0 & I & 0 &&&&&&&& \\
&&& J & 0 & 0 & 0 &&&&&&&& \\
&&& 0 & 0 & 0 & -J &&&&&&&& \\
&&& 0 & I & 0 & 0 &&&&&&&& \\
&&&&&&& 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
&&&&&&& 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \\
&&&&&&& 0 & 0 & 0 & J & 0 & 0 & 0 & 0 \\
&&&&&&& 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
&&&&&&& 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\
&&&&&&& 0 & 0 & 0 & 0 & -J & 0 & 0 & 0 \\
&&&&&&& I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
&&&&&&& 0 & 0 & 0 & 0 & 0 & I & 0 & 0
\end{pmatrix}.$$

Organization of the Paper. In Section 2 we review and establish necessary facts about Sato-Tate groups and nondegeneracy. In Section 3, we prove the nondegeneracy of the Jacobians of C_{2^m} and C'_{2^d+1}. We achieve this for $\mathrm{Jac}(C_{2^m})$ in part by using the decomposition of the Jacobian given in Section 3.1 and by analyzing the Hodge groups of the factors $\mathrm{Jac}(C'_{2^d+1})$. This enables us to use the techniques of [11, 20] to determine generators of the Sato-Tate groups of the Jacobians of the curves C_{2^m} and C'_{2^d+1} in Section 4. We gave explicit examples of some of these generators and the Sato-Tate groups in the Example 1.4 and Example 1.5 above. In Section 5, we compute moment statistics associated to the Sato-Tate groups of the Jacobians of C_{2^m} and C'_{2^d+1}. These moment statistics can be used to verify the equidistribution statement of the generalized Sato-Tate conjecture by comparing them to moment statistics obtained for the traces a_i in the normalized L-polynomial. Note that the numerical moment statistics are an approximation since one can only ever compute them up to some prime. It is of interest to those dealing with equidistribution statements to compare how close these two computations are. We compare the moments in Section 5 for various values of c.

Notation and conventions. We briefly summarize the basic notation and terminology used in this paper. Let C be a smooth projective curve defined over $\mathbb{Q}$. We write $\mathrm{End}(\mathrm{Jac}(C)_k)$ for the ring of endomorphisms of the Jacobian of C that are defined over the field k. Let $K := K_C$ denote the minimal extension $L/\mathbb{Q}$ over which all the endomorphisms of the abelian variety $\mathrm{Jac}(C)$ are defined, i.e. the minimal extension for which $\mathrm{End}(\mathrm{Jac}(C)_L) \simeq \mathrm{End}(\mathrm{Jac}(C)_{\overline{\mathbb{Q}}})$; the field K is called the endomorphism field of $\mathrm{Jac}(C)$.

We denote the Sato-Tate group of the Jacobian of C by $\mathrm{ST}(\mathrm{Jac}(C)) := \mathrm{ST}(\mathrm{Jac}(C)_{\mathbb{Q}})$ with identity component denoted $\mathrm{ST}^0(\mathrm{Jac}(C))$ and component group $\mathrm{ST}(\mathrm{Jac}(C))/\mathrm{ST}^0(\mathrm{Jac}(C))$. We will write ζ_m for a primitive m^{th} root of unity. For any rational number x whose denominator is coprime to an integer r, $\langle x\rangle_r$ denotes the unique representative of x modulo r between 0 and $r-1$.

Define the two matrices

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The symplectic form considered throughout the paper is given by $\mathrm{diag}(J, \ldots, J)$. Lastly, for any positive integer n, we define the following subgroups of the unitary symplectic group $\mathrm{USp}(2n)$.

$$\mathrm{U}(1)_n := \left\langle \mathrm{diag}(\underbrace{u, \overline{u}, \ldots, u, \overline{u}}_{n-\text{times}}) : u \in \mathbb{C}^\times, |u| = 1 \right\rangle$$

and

$$\mathrm{U}(1)^n := \left\langle \mathrm{diag}(u_1, \overline{u_1}, \ldots, u_n, \overline{u_n}) : u_i \in \mathbb{C}^\times, |u_i| = 1 \right\rangle.$$

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2. Background and Preliminaries

2.1. An ℓ-adic Construction of the Sato-Tate Group.

We follow the exposition of [11, 19] and [35, Section 3.2]. See also [33, Chapter 8].

Let A be a nonsingular projective g-dimensional variety over $\mathbb{C}$. For any prime ℓ, we define the Tate module $T_\ell := \varprojlim_n A[\ell^n]$; this is a free $\mathbb{Z}_\ell$-module of rank $2g$. We define the rational Tate module $V_\ell := T_\ell \otimes_{\mathbb{Z}} \mathbb{Q}$, which is a $\mathbb{Q}_\ell$-vector space of dimension $2g$. The Galois action on the Tate module is given by an ℓ-adic representation

$$\rho_{A,\ell} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{Aut}(V_\ell) \cong \mathrm{GL}_{2g,\mathbb{Q}_\ell}. \tag{1}$$

The ℓ-adic monodromy group of A, denoted $G_{A,\ell}$, is the Zariski closure of the image of this map in $\mathrm{GL}_{2g,\mathbb{Q}_\ell}$, and we define $G^1_{A,\ell} := G_{A,\ell} \cap \mathrm{Sp}_{2g,\mathbb{Q}_\ell}$.

Conjecture 2.1 (Algebraic Sato-Tate Conjecture)**.** [4, Conjecture 2.1] *There is an algebraic subgroup* $\mathrm{AST}(A)$ *of* GSp_{2g} *over* $\mathbb{Q}$*, called the algebraic Sato-Tate group of* A*, such that the connected component of the identity* $\mathrm{AST}^0(A)$ *is reductive and, for each prime* ℓ*,* $G^1_{A,\ell} = \mathrm{AST}(A) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$.

When this conjecture holds, we define the Sato-Tate group of A, denoted $\mathrm{ST}(A)$, to be a maximal compact Lie subgroup of $G^1_{A,\ell} \otimes_{\mathbb{Q}_\ell} \mathbb{C} = \mathrm{AST}(A) \otimes_{\mathbb{Q}} \mathbb{C}$ contained in $\mathrm{USp}(2g)$. It is conjectured that $\mathrm{ST}(A)$ is, up to conjugacy in $\mathrm{USp}(2g)$, independent of the choice of the prime ℓ and of the embedding of $\mathbb{Q}_\ell$ in $\mathbb{C}$. While the Sato-Tate group is a compact Lie group, it may not be connected [17]. We denote the connected component of the identity (also called the identity component) of $\mathrm{ST}(A)$ by $\mathrm{ST}^0(A)$.

There are many cases where the algebraic Sato-Tate conjecture is known to be true. Banaszak and Kedlaya prove that the conjecture holds for all abelian varieties of dimension at most 3 [4, Theorem 6.11], for many examples of simple abelian varieties [4, Theorem 6.9], and, using results of Serre [31], for all abelian varieties of CM type [4, Theorem 6.6]. There are examples of infinite families of higher dimensional Jacobian varieties for which the algebraic Sato-Tate conjecture is known to be true in [11, 17, 20]. Furthermore, Cantoral-Farfán and Commelin [7] proved that the algebraic Sato-Tate conjecture holds whenever the Mumford-Tate conjecture holds for an abelian variety.

2.2. **Hodge and Mumford Tate Groups.** In this section, we follow the exposition in [19] and Chapters 1 and 17 of [6].

Let A be a nonsingular projective variety over $\mathbb{C}$. We denote the first homology group of A by $V(A) := H_1(A, \mathbb{Q})$ and its dual (the first cohomology group) by $V^*(A) := H^1(A, \mathbb{Q})$. The complex vector space $V(A)_{\mathbb{C}}$ has a weight -1 Hodge structure, i.e., a decomposition $V(A)_{\mathbb{C}} = V(A)^{-1,0} \oplus V(A)^{0,-1}$ where $\overline{V(A)^{-1,0}} = V(A)^{0,-1}$. This corresponds to the following weight 1 Hodge structure of its dual

$$V^*_{\mathbb{C}} = H(A)^{1,0} \oplus H(A)^{0,1},$$

where $V(A)^{-1,0} = H^{1,0}(A)^*$ and $V(A)^{0,-1} = H^{0,1}(A)^*$. The notation in the decomposition of $H^1(A, \mathbb{Q})$ is defined by $H^{a,b}(A) = H^a(\Omega^b_A)$, where Ω^b_A is the sheaf of holomorphic b-forms on A. We can also define $H^{a,b}(A)$ by

$$H^{a,b}(A) \simeq \bigwedge^a H^{1,0}(A) \otimes \bigwedge^b H^{0,1}(A).$$

Hodge structures of weight n, $H^n(A, \mathbb{C}) \simeq \bigwedge^n H^1(A, \mathbb{C})$, satisfy $H^n(A, \mathbb{C}) \simeq \bigoplus_{a+b=n} H^{a,b}(A)$.

The Hodge structure determines a representation $\mu_{\infty,A} : \mathbb{G}_m(\mathbb{C}) \to \mathrm{GL}(V_{\mathbb{C}})$ acting as multiplication by z on $V(A)^{-1,0}$ and trivially on $V(A)^{0,-1}$. With this setup, we define the Mumford-Tate group of A, denoted $\mathrm{MT}(A)$, to be the smallest $\mathbb{Q}$-algebraic subgroup of GL_V such that $\mu_{\infty,A}(\mathbb{G}_m(\mathbb{C})) \subseteq \mathrm{MT}(A)_{\mathbb{C}}$. We define the Hodge group of A to be the connected component of the identity of $\mathrm{MT}(A) \cap \mathrm{SL}_V$.

The Hodge group can also be formed by restricting the representation $\mu_{\infty,A}$ to the circle group $\mathbb{S}^1 := \{z \in \mathbb{C} \mid |z| = 1\}$. With this setup, the Hodge group is the smallest $\mathbb{Q}$-algebraic subgroup of GL_V such that $\mu_{\infty,A}(\mathbb{S}^1) \subseteq \mathrm{Hg}(A)_{\mathbb{C}}$. The image of this restriction of $\mu_{\infty,A}$ lies in $\mathrm{SL}(V_{\mathbb{C}})$, and so the Hodge group is a $\mathbb{Q}$-algebraic subgroup of SL_V. In fact, one can show that the image of the representation $\mu_{\infty,A}$ is contained in the symplectic group $\mathrm{GSp}(V_{\mathbb{C}})$. Hence, $\mathrm{MT}(A)$ and $\mathrm{Hg}(A)$ are $\mathbb{Q}$-algebraic subgroups of GSp_V and Sp_V, respectively.

There are some useful identities for Hodge groups of products of abelian varieties. For $n \geq 1$, we can identify $\mathrm{Hg}(A^n)$ with $\mathrm{Hg}(A)$ and the action is performed diagonally on $V(A^n) = (V(A))^n$. More generally, for $n_1, n_2, \ldots, n_t \geq 1$, $\mathrm{Hg}(A_1^{n_1} \times A_2^{n_2} \times \cdots \times A_t^{n_t})$ is isomorphic to $\mathrm{Hg}(A_1 \times A_2 \times \cdots \times A_t)$. Even more generally, if A and B are abelian varieties then $\mathrm{Hg}(A \times B) \subseteq \mathrm{Hg}(A) \times \mathrm{Hg}(B)$ (see [19] for examples where this containment is strict).

2.3. **Nondegenerate Abelian Varieties.** We denote the (complexified) Hodge ring of A by

$$\mathscr{B}^*(A) := \sum_{d=0}^{\dim(A)} \mathscr{B}^d(A),$$

where $\mathscr{B}^d(A) = (H^{2d}(A, \mathbb{Q}) \cap H^{d,d}(A)) \otimes \mathbb{C}$ is the $\mathbb{C}$-span of Hodge cycles of codimension d on A. The subring of $\mathscr{B}^*(A)$ generated by the divisor classes, i.e. generated by $\mathscr{B}^1(A)$, is

$$\mathscr{D}^*(A) := \sum_{d=0}^{\dim(A)} \mathscr{D}^d(A),$$

where $\mathscr{D}^d(A)$ is the $\mathbb{C}$-span of classes of intersection of d divisors. Note that we always have the containment $\mathscr{D}^*(A) \subseteq \mathscr{B}^*(A)$, and we say that an abelian variety A is *nondegenerate* if the containment is an equality, i.e. its Hodge ring is generated by divisor classes.

Results of Hazama [24] show that A is stably nondegenerate if and only if the rank of the Hodge group is maximal, which implies a similar statement for the Mumford-Tate group (A is stably nondegenerate if $\mathscr{D}^*(A^n) = \mathscr{B}^*(A^n)$ for all $n > 0$). Hazama also proves an interesting result regarding products of abelian varieties: if A and B are both stably nondegenerate, then the only scenario in which $A \times B$ could be degenerate is if at least one of the simple factors is of type IV in the Albert's classification.

The relationship between divisor classes and Hodge cycles is relevant to the Hodge Conjecture. Let $\mathscr{C}^d(A)$ be the subspace of $\mathscr{B}^d(A)$ generated by the classes of algebraic cycles on A of codimension d. Then

$$\mathscr{D}^d(A) \subseteq \mathscr{C}^d(A) \subseteq \mathscr{B}^d(A)$$

and the Hodge Conjecture for A asserts that every Hodge cycle is algebraic: $\mathscr{C}^d(A) = \mathscr{B}^d(A)$ for all d (see [1, 34]). One way to prove the Hodge Conjecture in codimension d is to prove the equality $\mathscr{D}^d(A) = \mathscr{B}^d(A)$. However this equality does not always hold, even when the Hodge conjecture is known to be true.

An example of this phenomenon is given by Shioda [34] (and worked out in greater detail in [11, 19, 28]). The Jacobian of the curve $y^2 = x^9 - 1$ satisfies the Hodge Conjecture and is a 4-dimensional abelian variety that is isogenous to the product of two nondegenerate abelian varieties: a CM elliptic curve E and a 3-dimensional absolutely simple CM abelian variety A. However, Shioda shows that the Hodge ring in dimension 2 contains exceptional cycles (those not generated by divisor classes). Furthermore, Lombardo shows that the Mumford-Tate group of the Jacobian is degenerate and the degeneracy of the Sato-Tate group is examined in [11, 19]. See [19] for more examples of this.

Nondegeneracy also plays a role in the component group of the Sato-Tate group. In general, for an abelian variety A/F we have a canonical surjection

$$\mathrm{ST}(A)/\,\mathrm{ST}^0(A) \to \mathrm{Gal}(K/F),$$

where K is the endomorphism field of A. If A is nondegenerate then this surjection is an isomorphism (see, for example, [4, Section 6] and Theorem 2.16a of [14]). On the other hand, the surjection is not necessarily an isomorphism if A is degenerate. In general, we have an isomorphism $\mathrm{ST}(A)/\,\mathrm{ST}^0(A) \simeq \mathrm{Gal}(L/F)$, where L is the minimal Galois extension of F for which $\mathrm{ST}(A_L)$ is connected. The field L is the fixed field of the kernel of a map induced by the ℓ-adic representation $\rho_{A,\ell}$ in Equation (1) (see [35, Theorem 3.12]). See [19, Section 5.2] for an example of a Jacobian variety for which the field L is larger than its endomorphism field.

Furthermore, for nondegenerate abelian varieties, we can determine explicit generators of the component group of the Sato-Tate group through the twisted Lefschetz group (see [4, Theorem 6.6]). The twisted Lefschetz group, denoted $\mathrm{TL}(A)$, is a closed algebraic subgroup of Sp_{2g} defined by

$$\mathrm{TL}(A) := \bigcup_{\tau \in \mathrm{Gal}(\overline{F}/F)} \mathrm{L}(A)(\tau),$$

where $\mathrm{L}(A)(\tau) := \{\gamma \in \mathrm{Sp}_{2g} \mid \gamma\alpha\gamma^{-1} = \tau(\alpha) \text{ for all } \alpha \in \mathrm{End}(A_{\overline{F}})_{\mathbb{Q}}\}$.

Proposition 2.2. [4] *The algebraic Sato-Tate Conjecture holds for nondegenerate abelian varieties A with* $\mathrm{AST}(A) = \mathrm{TL}(A)$.

We rely on this result in Section 4 when we compute generators of the Sato-Tate groups of the two families of Jacobian varieties we focus on in this paper.

3. Nondegeneracy

In this section we prove that the Jacobian varieties we are studying are nondegenerate. We do this by first determining the decomposition of the Jacobians in Section 3.1 and then by analyzing the Hodge groups of the factors in the decomposition in Section 3.2. The results in this section allow us to use the twisted Lefschetz group to determine generators of the Sato-Tate group for each Jacobian in Section 4.

3.1. **Decompositions of the Jacobians.** Here we specify known results for decompositions of Jacobians to the families we are studying in this paper. We show that the Jacobians of curves of the form $C_{2^m} : y^2 = x^{2^m} - c$ decompose into products of Jacobians of curves of the form $C'_{2^d+1} : y^2 = x^{2^d+1} - cx$, which gives the motivation for studying these two specific families of curves together.

We first recall the following result, which we will use to describe the decomposition of the Jacobians of the curves C_{2^m}.

Theorem 3.1. [12, Theorem 4.3] *Let $v_2 \colon \mathbb{Q}^* \to \mathbb{Z}$ denote the 2-adic valuation map. Let $C \colon y^2 = x^{2g+2}+c$ be a hyperelliptic curve of genus g and write $k := v_2(g+1)$. Then we have the following isogeny over $\mathbb{Q}$*

$$\mathrm{Jac}(C) \sim \mathrm{Jac}(y^2 = x^{(g+1)/2^k} + c)^2 \times \prod_{i=0}^{k-1} \mathrm{Jac}(y^2 = x^{(g+1)/2^i+1} + cx).$$

The genus of the curve $y^2 = x^{2^m} - c$ is $g = 2^{m-1} - 1$, and so applying Theorem 3.1 to this curve and adjusting the indices yields

$$\mathrm{Jac}(y^2 = x^{2^m} - c) \sim \prod_{d=1}^{m-1} \mathrm{Jac}(y^2 = x^{2^d+1} - cx). \tag{2}$$

The Jacobians of dimension greater than 1 (i.e., when $d > 1$) that appear in this decomposition can be factored further. When $c = 1$, results of [8] prove that the Jacobian of the curve C'_{2^d+1} is isogenous to the square of a simple abelian variety and that this simple abelian variety is the Jacobian of a hyperelliptic curve with CM by $\mathbb{Q}(\zeta + \zeta^{2^d-1})$, where ζ is a primitive 2^{d+1}th root of unity. Combining the above with the identity $\zeta^{2^d-1} = -\overline{\zeta}$ yields the following result.

Proposition 3.2. *The Jacobian of $C_{2^m} : y^2 = x^{2^m} - 1$ factors over $\overline{\mathbb{Q}}$ as the product*

$$\mathrm{Jac}(y^2 = x^{2^m} - 1) \sim (y^2 = x^3 - x) \times \prod_{d=2}^{m-1} X_d^2,$$

where each X_d is a simple Jacobian variety of dimension 2^{d-2} with CM by $\mathbb{Q}(\zeta_{2^{d+1}} - \overline{\zeta}_{2^{d+1}})$ and $\zeta_{2^{d+1}}$ is a primitive 2^{d+1}th root of unity.

The Jacobian $\mathrm{Jac}(C_{2^m})$ decomposes similarly for generic $c \in \mathbb{Q}^\times$ since twists are isomorphic over $\overline{\mathbb{Q}}$.

Remark 3.3. *Just as in* [17, Remark 5.4]*, we note that the Sato-Tate group of* $\mathrm{Jac}(C_{2^m})$ *is not isomorphic to the direct sum of the Sato-Tate groups of its factors. Indeed, it cannot be the case since as we show in Section 3.2 these are all nondegenerate abelian varieties and so, by the discussion in Section 2.3, the component groups of the Sato-Tate groups are isomorphic to the Galois groups of the CM fields over $\mathbb{Q}$. However, the Galois group associated to* $\mathrm{Jac}(C_{2^m})$ *is not isomorphic to the direct product of the Galois groups of the factors. This further demonstrates the need to give an explicit description for* $\mathrm{ST}(C_{2^m})$ *and* $\mathrm{ST}(C'_{2^d+1})$ *in terms of generators.*

3.2. **Nondegeneracy Results.** Our main result of this section is the following theorem.

Theorem 3.4. *The Jacobian of* $y^2 = x^{2^m} - 1$ *is nondegenerate.*

Proof. We will prove that the Hodge group of $\mathrm{Jac}(y^2 = x^{2^m} - 1)$ is maximal, which is equivalent to proving that its Hodge ring is generated by divisor classes (see [24, Theorem 1.2]). The decomposition in Proposition 3.2 implies that $\mathrm{Hg}(\mathrm{Jac}(y^2 = x^{2^m} - 1)) = \mathrm{Hg}(X_1 \times X_2^2 \cdots \times X_{m-1}^2)$, where X_1 denotes the elliptic curve $y^2 = x^3 - x$. Recall from Section 2.2 that, for an abelian variety A, we have $\mathrm{Hg}(A^n) = \mathrm{Hg}(A)$ for $n \geq 1$. Hence, to prove the nondegeneracy of $\mathrm{Jac}(y^2 = x^{2^m} - 1)$ it suffices to prove that

$$\mathrm{Hg}(X_1 \times X_2^2 \cdots \times X_{m-1}^2) = \mathrm{Hg}(X_1) \times \mathrm{Hg}(X_2) \times \cdots \times \mathrm{Hg}(X_{m-1}).$$

For $d \geq 2$, let $\zeta_{2^{d+1}}$ be a primitive 2^{d+1}th root of unity. Let $F_d := \mathbb{Q}(\zeta_{2^{d+1}} - \overline{\zeta}_{2^{d+1}})$ denote the CM field of the simple factor X_d and let $F_1 := \mathbb{Q}(\zeta_4) = \mathbb{Q}(i)$ denote the CM field of the elliptic curve X_1. As shown in [8, Proposition 4.7], the field F_d, for $d \geq 1$, contains no proper CM subfield. Hence, for distinct simple factors X_{d_1} and X_{d_2} with $d_1 < d_2$, there is no embedding $F_{d_1} \hookrightarrow F_{d_2}$.

Each X_d is a CM abelian variety, and so its Hodge group is a commutative algebraic torus. We can, therefore, write $\mathrm{Hg}(X_d) = U_{F_d}$, where the notation U_{F_d} denotes the torus associated to the field F_d.

Now let $X = X_{d_1} \times \cdots \times X_{d_r}$, with each $d_k > 1$, be a subproduct of $X_1 \times \cdots \times X_{m-1}$. For any $d_1 < d_2, d_3, \ldots, d_r$, the abelian variety X_{d_1} is not a factor of X and its dimension is less than or equal to the dimension of each factor of X. We will prove that

$$\mathrm{Hg}(X_{d_1} \times X) = \mathrm{Hg}(X_{d_1}) \times \mathrm{Hg}(X).$$

Since the simple abelian varieties $X_{d_2}, \ldots, X_{d_r}$ are nonisogenous, the CM field of their product X will be the compositum of their CM fields. Hence, the center of the endomorphism algebra of X is the product $F_{d_2} \times \cdots \times F_{d_r}$ and the center of $\mathrm{Hg}(X)$ is contained in $U_{F_{d_2}} \times \cdots \times U_{F_{d_r}}$.

As noted in Lemma 3.6 of [29], if $\mathrm{Hg}(X_{d_1} \times X)$ is strictly contained in $\mathrm{Hg}(X_{d_1}) \times \mathrm{Hg}(X)$ then the center of $\mathrm{Hg}(X)$ contains an algebraic torus that is $\mathbb{Q}$-isogenous to $\mathrm{Hg}(X_{d_1})$. This would imply that there is a homomorphism

$$U_{F_{d_1}} \to U_{F_{d_2}} \times \cdots \times U_{F_{d_r}}$$

with finite kernel. However this is not possible since F_{d_1} does not embed into any of the fields F_{d_i}. Thus, we must have

$$\mathrm{Hg}(X_{d_1} \times X) = \mathrm{Hg}(X_{d_1}) \times \mathrm{Hg}(X).$$

Given that the above holds for any subproduct X, we can conclude that

$$\mathrm{Hg}(X_1 \times \cdots \times X_{m-1}) = \mathrm{Hg}(X_1) \times \cdots \times \mathrm{Hg}(X_{m-1}),$$

which proves that the Jacobian of $y^2 = x^{2^m} - 1$ is nondegenerate. □

Corollary 3.5. *Let* $c \in \mathbb{Q}^\times$. *The Jacobian of* $C_{2^m} : y^2 = x^{2^m} - c$ *is nondegenerate.*

Proof. Nondegeneracy is invariant under twisting. □

Corollary 3.6. *Let* $c \in \mathbb{Q}^\times$ *and* $d \geq 1$. *The Jacobian of* $C_{2^d+1} : y^2 = x^{2^d+1} - cx$ *is nondegenerate.*

Proof. Factors of a nondegenerate abelian variety must also be nondegenerate. Alternatively, we can prove this by noting that these Jacobian varieties are squares of nondegenerate abelian varieties. □

3.2.1. *Nondegeneracy via Mumford-Tate Groups.* We can also study nondegeneracy via the Mumford-Tate groups of the Jacobian varieties. Consider a product $A = A_1 \times A_2 \times \cdots \times A_n$ of nonisogenous abelian varieties. We wish to know when one of the canonical projections $\mathrm{MT}(A) \longrightarrow \mathrm{MT}(A_i)$ is an isomorphism, an isogeny, or neither. If any of the projections is an isomorphism or an isogeny then the Mumford-Tate group of A is degenerate, and it is nondegenerate otherwise (see Section 3.1 of [19] for more details).

We demonstrate this for the example $C_{16} : y^2 = x^{16} - 1$. As noted in Equation 2, the Jacobian is isogenous to the product

$$\mathrm{Jac}(C'_9) \times \mathrm{Jac}(C'_5) \times \mathrm{Jac}(C'_3).$$

In order to study canonical projections of the Mumford-Tate group, we build the matrix M representing the map $N_1^*\phi_1^* + N_2^*\phi_2^* + N_3^*\phi_3^*$ described in Section 3.1 of [19]. The maps N_i^* are norm maps and the ϕ_i^* are maps on character groups associated to the CM fields of the factors. The matrix M is obtained by concatenating matrices M_1, M_2, M_3 corresponding to the factors $\mathrm{Jac}(C'_9), \mathrm{Jac}(C'_5)$, and $\mathrm{Jac}(C'_3)$, respectively. This yields the matrix

$$M = \left(\begin{array}{cccccccc|cccc|cc}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1
\end{array}\right), \tag{3}$$

where the vertical lines are included to visually separate the M_i matrices. In order to determine if a projection $\mathrm{MT}(\mathrm{Jac}(C_{16})) \to \mathrm{MT}(A)$, where A is a factor of $\mathrm{Jac}(C_{16})$, is an isomorphism or isogeny (or neither), we look at the space spanned of the kernel of M and the subspace of $\mathbb{Z}^8 \times \mathbb{Z}^4 \times \mathbb{Z}^2$ corresponding to the factor A. In each case, we have verified in Sage [30] that the index of this space in $\mathbb{Z}^8 \times \mathbb{Z}^4 \times \mathbb{Z}^2$ is infinite, and so we conclude that $\mathrm{Jac}(C_{16})$ has nondegenerate Mumford-Tate group.

We used this strategy to verify that the Mumford-Tate group is nondegenerate for several examples of the form $y^2 = x^{2^m} - 1$. Proving that this holds in general for curves in this family would provide an alternate proof of the result in Theorem 3.4.

4. Sato-Tate Groups

The main results of this section are Theorems 4.3 and 4.7 where we compute generators of the Sato-Tate groups of the Jacobians of the curves C'_{2^d+1} and C_{2^m}. For both families of curves, we obtain the component group of the Sato-Tate group by computing the twisted Lefschetz groups (recall the results of Proposition 2.2).

We begin by again considering the decomposition of the Jacobian of $y^2 = x^{2^m} - c$, now with the goal of describing its complex uniformization.

4.1. **Explicit Maps to Lower Dimensional Jacobians.** Theorem 3.1 and Equation (2) of Section 3 give us a decomposition of the Jacobian of the curve $y^2 = x^{2^m} - c$ for $c \in \mathbb{Q}^\times$. In this subsection we give the maps to the lower dimensional Jacobian varieties since we will need these for our work in Section 4.3.1.

Let C denote the curve $y^2 = x^{2g+2} - c$. We can write the following nonconstant morphisms defined over $\mathbb{Q}$ to the lower genus curves $C' : y^2 = x^{g+2} - cx$ and $\tilde{C} : y^2 = x^{g+1} - c$:

$$\phi : C \to C', \quad (x, y) \mapsto (x^2, xy),$$
$$\tilde{\phi} : C \to \tilde{C}, \quad (x, y) \mapsto (x^2, y).$$

The curves $y^2 = x^{2g+2} - c$ in this paper are quite special: we are specifically working with curves $y^2 = x^{2^m} - c$, which have odd genus $g = 2^{m-1} - 1$. This implies that the lower genus curves C' are of the form $y^2 = x^{2^{m-1}+1} - cx$ and $\tilde{C}$ are of the form $y^2 = x^{2^{m-1}} - c$. We can use the same process as above to write maps from $y^2 = x^{2^{m-1}} - 1$ to lower genus curves. In general, for $2 \leq d \leq m-1$, there are maps from $C_{2^m} : y^2 = x^{2^m} - c$ to $C'_{2^d+1} : y^2 = x^{2^d+1} - cx$ and $C_{2^d} : y^2 = x^{2^d} - c$ given by

$$\phi_d : C_{2^m} \to C'_{2^d+1}, \quad (x, y) \mapsto (x^{2^{m-d}}, x^{2^{m-d-1}} y), \tag{4}$$
$$\tilde{\phi}_d : C_{2^m} \to \tilde{C}_{2^d}, \quad (x, y) \mapsto (x^{2^{m-d}}, y). \tag{5}$$

There is an additional map ϕ_d when $d = 1$. This maps C_{2^m} to the elliptic curve $y^2 = x^3 - cx$, which is isogenous to the curve $y^2 = x^4 - c$ obtained from the map $\tilde{\phi}_2$. For the factorization in Equation (2), we chose the model $y^2 = x^3 - cx$ for the elliptic curve factor and so the maps we will work with are all of the form appearing in Equation (4).

For $1 \leq d \leq m-1$, the genus of the curve $C'_{2^d+1} : y^2 = x^{2^d+1} - cx$ is $g_d = 2^{d-1}$. A basis for the space of regular differential 1-forms of $\mathrm{Jac}(C'_{2^d+1})$ is given by $\{x^j dx/y\}_{j=0}^{g_d-1}$. Pulling back these 1-forms via the map $\phi_d : C_{2^m} \to C'_{2^d+1}$ yields $\phi_d^*(x^j dx/y) = x^{d_j} dx/y$, where $d_j = 2^{m-d-1}(2j+1)-1$. Letting $\omega_{d,j} = x^{d_j} dx/y$, we see that the map ϕ_d corresponds to the inclusion

$$\mathbb{C}\left\langle \{\omega_{d,j}\}_{j=0}^{g_d-1} \right\rangle \hookrightarrow \mathbb{C}\left\langle \{x^i dx/y\}_{i=0}^{g-1} \right\rangle, \tag{6}$$

where $\mathbb{C}\left\langle \{x^i dx/y\}_{i=0}^{g-1} \right\rangle = H^0((C_{2^m})_{\mathbb{C}}, \Omega^1_{\mathbb{C}})$ can be identified with the complex uniformization of $\mathrm{Jac}(C_{2^m})$.

It is straightforward to verify that distinct values of d lead to distinct subspaces $\mathbb{C}\left\langle \{x^{d_j} dx/y\} \right\rangle$ and that the subspaces span $H^0((C_{2^m})_{\mathbb{C}}, \Omega^1_{\mathbb{C}})$. Note that when $d = m-1$, the exponents d_j are all even and the inclusion in Equation (6) yields all of the differentials in $H^0((C_{2^m})_{\mathbb{C}}, \Omega^1_{\mathbb{C}})$ with even exponent for x.

4.2. **Sato-Tate groups for curves of the form $y^2 = x^{2^d+1} - cx$.** Let C'_{2^d+1} denote the curve $y^2 = x^{2^d+1} - cx$, where $d \geq 2$, and so the genus of C'_{2^d+1} is a power of 2: $g_d = 2^{d-1}$.

4.2.1. *Preliminaries.* Throughout this subsection, let $\zeta_{2^{d+1}}$ be a primitive 2^{d+1}th root of unity. The group of automorphisms of C'_{2^d+1} is generated by the automorphisms

$$\alpha_d(x, y) = (\zeta_{2^{d+1}}^2 x, \zeta_{2^{d+1}} y)$$
$$\beta_d(x, y) = \left(-\frac{c^{1/g_d}}{x}, \frac{c^b y}{x^{g_d+1}} \right),$$

where $b = (g_d + 1)/(2g_d) = (2^{d-1} + 1)/2^d$ and β^2 is the hyperelliptic involution. Thus, the endomorphism field of $\mathrm{Jac}(C'_{2^d+1})$ is $K = \mathbb{Q}(\zeta_{2^{d+1}}, c^{1/2g_d})$ and we have $\mathrm{End}(\mathrm{Jac}(C'_{2^d+1})_K) = \mathrm{End}(\mathrm{Jac}(C'_{2^d+1})_{\overline{\mathbb{Q}}})$. Thus, in this subsection, we say that $c \in \mathbb{Q}^\times$ is generic if $[K : \mathbb{Q}]$ is maximal, i.e. $[K : \mathbb{Q}] = 2^d \cdot 2g_d = 2^{2d}$.

In Corollary 3.6 we proved that the Jacobian variety $\mathrm{Jac}(C'_{2^d+1})$ is nondegenerate. This allows us to conclude that $\mathrm{ST}(\mathrm{Jac}(C'_{2^d+1}))/\,\mathrm{ST}^0(\mathrm{Jac}(C'_{2^d+1})) \simeq \mathrm{Gal}(K/\mathbb{Q})$. The following lemma gives a generating set for this Galois group.

Lemma 4.1. *Let $d \geq 1$, the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_{2^{d+1}}, c^{1/2g_d})/\mathbb{Q})$ equals $\langle \sigma_{-1}, \sigma_5, \tau\rangle$, where τ is defined by $\tau(c^{1/2g_d}) = \zeta_{2^{d+1}}^2 c^{1/2g_d}$ and σ_a denotes the Galois automorphism $\sigma_a(\zeta_{2^{d+1}}) = \zeta_{2^{d+1}}^a$.*

Proof. The proof of this result relies on simple abstract algebra facts that can be found in [10]. First, it is clear that τ generates the Galois automorphism group of $c^{1/2g_d}$. The Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_{2^{d+1}})/\mathbb{Q})$ is isomorphic to $(\mathbb{Z}/2^{d+1}\mathbb{Z})^\times$ via $a \mapsto \sigma_a$. The group $(\mathbb{Z}/2^{d+1}\mathbb{Z})^\times$ is generated by two elements, one of order 2 and one of order 2^{d-1}. We can choose 5 as the element of order 2^{d-1}. Since no power of 5 is congruent to $-1 \pmod{2^{d+1}}$ for $d \geq 1$, we can choose -1 as our generator of order 2. Thus, we can choose σ_{-1}, σ_5 as generators of $\mathrm{Gal}(\mathbb{Q}(\zeta_{2^{d+1}})/\mathbb{Q})$. □

The final preliminary detail we need before computing generators of the Sato-Tate group is a description of the endomorphisms of $\mathrm{Jac}(C'_{2^d+1})$. We compute pullbacks of the regular 1-forms $\omega_j = x^j dx/y$, where $j = 0, 1, \ldots, g_d - 1$, with respect to the curve automorphisms α_d and β_d and find that

$$\alpha_d^*(\omega_j) = \zeta_{2^{d+1}}^{2j+1}\omega_j \tag{7}$$
$$\beta_d^*(\omega_j) = (-1)^j c^{\delta_j}\omega_{g-1-j}, \tag{8}$$

where $\delta_j = (2j - 2^{d-1} + 1)/2^d$. By taking the symplectic basis of $H_1(\mathrm{Jac}(C'_{2^d+1})_\mathbb{C}, \mathbb{C})$ corresponding the above basis of regular 1-forms (with respect to the skew-symmetric basis $\mathrm{diag}(J, J, \ldots, J)$), the endomorphisms corresponding to these automorphisms are

$$\alpha_d = \mathrm{diag}(Z_d, Z_d^3, Z_d^5 \ldots, Z_d^{2g_d-1}) \tag{9}$$
$$\beta_d = \mathrm{antidiag}(D_0, -D_1, D_2, \ldots, -D_{g_d-1}), \tag{10}$$

where $Z_d = \mathrm{diag}(\zeta_{2^{d+1}}, \overline{\zeta}_{2^{d+1}})$ and $D_j = \mathrm{diag}(c^{\delta_j}, c^{\delta_j})$.

4.2.2. *Sato-Tate group results.* We begin with a result for the identity component of the Sato-Tate group, which confirms Conjecture 6.13 of [12] for this family of curves.

Proposition 4.2. *Let $d \geq 2$ and $C'_{2^d+1} : y^2 = x^{2^d+1} - cx$. Then*

$$\mathrm{ST}^0(\mathrm{Jac}(C'_{2^d+1})) \simeq \left(\mathrm{U}(1)^{2^{d-2}}\right)_2.$$

When $d = 1$, $\mathrm{Jac}(C'_{2^d+1})$ is an elliptic curve with CM, and so $\mathrm{ST}^0(\mathrm{Jac}(C'_{2^d+1})) \simeq \mathrm{U}(1)$.

Proof. The result for $d = 1$ is well-known. When $d \geq 2$, each of the Jacobians $\mathrm{Jac}(C'_{2^d+1})$ are isogenous over $\overline{\mathbb{Q}}$ to squares of simple abelian varieties with CM and are of dimension 2^{d-1} (see Section 3.1). Since $\mathrm{Jac}(C'_{2^d+1})$ is nondegenerate (Corollary 3.6), the result follows. □

Theorem 4.3. *Let g_d be the genus of the curve $C'_{2^d+1} : y^2 = x^{2^d+1} - cx$, where $d \geq 1$, and let ζ be a primitive 2^{d+1}th root of unity. Up to conjugation in $\mathrm{USp}(2g_d)$, the Sato-Tate group of the Jacobian of the curve for generic $c \in \mathbb{Q}^\times$ satisfies*

$$\mathrm{ST}(\mathrm{Jac}(C'_{2^d+1})) \simeq \mathrm{ST}^0(\mathrm{Jac}(C'_{2^d+1})) \rtimes \mathrm{Gal}(\mathbb{Q}(\zeta, c^{1/g_d})/\mathbb{Q}).$$

The generators of the component group can be taken to be $\gamma_c = \mathrm{diag}(\zeta^{1-g_d}I, \zeta^{3-g_d}I, \ldots, \zeta^{(2g_d-1)-g_d}I)$, $\gamma_J = \mathrm{diag}(J, J, \ldots, J)$, *and* γ, *where the* 2×2 *block entries of* γ *are given by*

$$\gamma_{i,j} = \begin{cases} I & \text{if } 2j-1 = \langle 5(2i-1)\rangle_{2^{d+1}}, \\ J & \text{if } j \leq \frac{g_d}{2} \text{ and } 2j-1 = 2^{d+1} - \langle 5(2i-1)\rangle_{2^{d+1}}, \\ -J & \text{if } j > \frac{g_d}{2} \text{ and } 2j-1 = 2^{d+1} - \langle 5(2i-1)\rangle_{2^{d+1}}, \\ 0 & \text{otherwise,} \end{cases}$$

for $1 \leq i, j \leq g_d$.

Proof. For $d = 1$, it is well-known that $\mathrm{ST}(\mathrm{Jac}(C'_{2^d+1})) = N(\mathrm{U}(1))$. From the above construction we obtain the matrices $\gamma_c = I$, $\gamma_J = J$, and $\gamma = I$, and we see that $N(\mathrm{U}(1)) \simeq \langle \mathrm{U}(1), J\rangle$.

For $d \geq 2$ we use the techniques implemented in [11, 20] to compute the twisted Lefschetz group of the Jacobian. Applying Proposition 2.2 then yields the desired result.

Recall from Section 4.2.1 that the endomorphism field of $\mathrm{Jac}(C'_{2^d+1})$ is $K = \mathbb{Q}(\zeta_{2^{d+1}}, c^{1/2g_d})$. Lemma 4.1 gives the three generators of the Galois group, and we now consider the action of each generator on the endomorphisms α_d and β_d.

First note that the action on β_d is trivial for both σ_{-1} and σ_5. The block entries of α_d are odd powers of the 2×2 matrix $Z_d = \mathrm{diag}(\zeta, \overline{\zeta})$. The Galois element σ_{-1} acts on blocks of this form as follows:

$${}^{\sigma_{-1}}Z_d = \mathrm{diag}(\overline{\zeta}_{2^{d+1}}, \zeta_{2^{d+1}}) = \overline{Z}_d.$$

Thus, σ_{-1} acts on α_d by conjugating each entry: ${}^{\sigma_{-1}}\alpha_d = \overline{\alpha}_d$. Since $JZ_dJ^{-1} = Z_d$ and $J^{-1} = -J$, we find that $\gamma_J\alpha_d\gamma_J^{-1} = {}^{\sigma_{-1}}\alpha_d$ where $\gamma_J = \mathrm{diag}(J, J, \ldots, J)$. Furthermore, one can check that $\gamma_J\beta_d\gamma_J^{-1} = \beta_d = {}^{\sigma_{-1}}\beta_d$. Since α_d and β_d generate the endomorphism ring of $\mathrm{Jac}(C'_{2^d+1})$, this confirms that γ_J is in the twisted Lefschetz group.

We now consider the action of the Galois element σ_5, which can be described by

$${}^{\sigma_5}Z_d = \mathrm{diag}(\zeta_{2^{d+1}}^5, \overline{\zeta_{2^{d+1}}}^5).$$

The block entry appearing in the i^{th} diagonal block is Z_d^{2i-1}. Note that both $2i-1$ and $\langle 5(2i-1)\rangle_{2^{d+1}}$ are odd, positive integers. Thus, the action of σ_5 can be described by

$${}^{\sigma_5}Z_d^{2i-1} = \begin{cases} Z_d^{\langle 5(2i-1)\rangle_{2^{d+1}}} & \text{if } \langle 5(2i-1)\rangle_{2^{d+1}} \leq 2g_d, \\ \overline{Z}_d^{2^{d+1}-\langle 5(2i-1)\rangle_{2^{d+1}}} & \text{if } \langle 5(2i-1)\rangle_{2^{d+1}} > 2g_d. \end{cases}$$

Based on this, and using the techniques developed in [11] and [20], we find that the γ defined in the statement of the theorem satisfies $\gamma\alpha_d\gamma^{-1} = {}^{\sigma_5}\alpha_d$. Furthermore, one can show that $\gamma\beta_d\gamma^{-1} = \beta_d = {}^{\sigma_5}\beta_d$.

Finally, we have $\gamma_c\alpha_d\gamma_c^{-1} = \alpha_d$ since these are all diagonal matrices, and so $\gamma_c\alpha_d\gamma_c^{-1} = {}^{\tau}\alpha_d$. Recall the notation $D_j = \mathrm{diag}(c^{\delta_j}, c^{\delta_j})$ from Section 4.2.1. The j^{th} block of $\gamma_c\beta_d\gamma_c^{-1}$ is given by $(\zeta^{2j-g_d+1}I)D_j(\zeta^{2j-g+1}I) = \zeta^{2(2j-g+1)}D_j$, which equals ${}^{\tau}D_j$. Hence, $\gamma_c\beta_d\gamma_c^{-1} = {}^{\tau}\beta_d$. □

Remark 4.4. *Taking non-generic values of* c *only affects the form of* γ_c. *To treat these cases, we simply replace* ζ *with* ζ^a *when* $c \in \mathbb{Q}(\zeta_{2^{d+1}})^{\times a} \setminus \mathbb{Q}(\zeta_{2^{d+1}})^{\times 2a}$. *Note that* a *divides* $4g_d$ *and so it must be a power of 2.*

4.3. **Sato-Tate groups for curves of the form** $y^2 = x^{2^m} - c$**.** Let C_{2^m} denote the curve $y^2 = x^{2^m} - c$ with $m > 2$. The genus of C_{2^m} is always odd: $g = 2^{m-1} - 1$. The main result of this section is Theorem 4.7 which gives an explicit description of the Sato-Tate group of $\mathrm{Jac}(C_{2^m})$. This is a particularly nice result since we express the generators of $\mathrm{ST}(\mathrm{Jac}(C_{2^m}))$ in terms of the Sato-Tate groups of the factors of $\mathrm{Jac}(C_{2^m})$.

4.3.1. *Preliminaries.* Throughout this subsection, let ζ_{2^m} be a primitive 2^mth root of unity. The group of automorphisms of C_{2^m} is generated by the automorphisms

$$\alpha(x,y) = (\zeta_{2^m} x, y)$$
$$\beta(x,y) = \left(\frac{c^{1/(g+1)}}{x}, \frac{ic^{1/2}y}{x^{g+1}}\right).$$

Note that, once again, β^2 is the hyperelliptic involution. Thus, the endomorphism field of $\mathrm{Jac}(C_{2^m})$ is $K = \mathbb{Q}(\zeta_{2^m}, c^{1/(g+1)})$ and we have $\mathrm{End}(\mathrm{Jac}(C_{2^m})_K) \simeq \mathrm{End}(\mathrm{Jac}(C_{2^m})_{\overline{\mathbb{Q}}})$. Thus, in this subsection, we say that $c \in \mathbb{Q}^\times$ is generic if $[K:\mathbb{Q}]$ is maximal, i.e. $[K:\mathbb{Q}] = 2^{m-1}(g+1) = 2^{2m-2}$.

In Theorem 3.4 we proved that the Jacobian variety $\mathrm{Jac}(C_{2^m})$ is nondegenerate. This allows us to conclude that $\mathrm{ST}(\mathrm{Jac}(C_{2^m}))/\mathrm{ST}^0(\mathrm{Jac}(C_{2^m})) \simeq \mathrm{Gal}(K/\mathbb{Q})$. Applying Lemma 4.1 yields $\mathrm{Gal}(K/\mathbb{Q}) = \langle \sigma_{-1}, \sigma_5, \tau_m\rangle$, where τ_m is defined by $\tau_m(c^{1/(g+1)}) = \zeta_{2^m}^2 c^{1/(g+1)} = \zeta_{2^{m-1}} c^{1/(g+1)}$.

The final preliminary detail we need before computing generators of the Sato-Tate group is a description of the endomorphisms of $\mathrm{Jac}(C_{2^m})$. Recall the notation $\omega_{d,j}$ from Section 4.1: for $1 \le d \le m-1$ and $0 \le j \le 2^{d-1}-1$

$$\omega_{d,j} = x^{d_j}\frac{dx}{y}, \tag{11}$$

where $d_j = 2^{m-d-1}(2j+1)-1$. The complex uniformization of $\mathrm{Jac}(C_{2^m})$ can be identified with $H^0((C_{2^m})_{\mathbb{C}}, \Omega^1_{\mathbb{C}}) = \mathbb{C}\langle\{\omega_{d,j}\}\rangle$. We compute pullbacks of the regular 1-forms $\omega_{d,j}$ with respect to the curve automorphisms α and β and obtain the following.

Lemma 4.5. *For for $1 \le d \le m-1$ and $0 \le j \le 2^{d-1}-1$,*

$$\alpha^*(\omega_{d,j}) = \zeta_{2^m}^{d_j+1}\omega_{d,j} \tag{12}$$
$$\beta^*(\omega_{d,j}) = c^{(2j+1-2^{d-1})/2^d} i\omega_{d,j'}, \tag{13}$$

where $j' = 2^{d-1}-1-j$.

Proof. We easily show the validity of Equation (12):

$$\alpha^*(\omega_{d,j}) = \alpha^*(x^{dj}dx/y) = \frac{\zeta_{2^m}^{d_j+1}x^{d_j}dx}{y} = \zeta_{2^m}^{d_j+1}\omega_{d,j}.$$

To prove that Equation (13) holds, first note that

$$\beta^*(\omega_{d,j}) = \beta^*(x^{d_j}dx/y) = \frac{(c^{1/(g+1)}/x)^{d_j}d(c^{1/(g+1)}/x)}{c^{1/2}iy/x^{g+1}} = c^{(d_j+1)/(g+1)-1/2}ix^{g-1-d_j}dx/y.$$

The exponent for c in the above expression can be rewritten as follows

$$\frac{d_j+1}{g+1} - \frac{1}{2} = \frac{2^{m-d-1}(2j+1)-2^{m-2}}{2^{m-1}} = \frac{2j+1-2^{d-1}}{2^d}.$$

Furthermore,

$$g-1-d_j = 2^{m-1}-2-(2^{m-d-1}(2j+1)-1) = 2^{m-d-1}(2(2^{d-1}-j-1)+1)-1,$$

which equals $d_{j'}$ when $j' = 2^{d-1}-j-1$. Hence, $\beta^*(\omega_{d,j}) = c^{(2j+1-2^{d-1})/2^d}ix^{d_{j'}}dx/y$, which proves the desired result. ∎

We take the symplectic basis of $H_1(\mathrm{Jac}(C_{2^m})_{\mathbb{C}}, \mathbb{C})$ corresponding the basis $\{\omega_{d,j}\}_{d,j}$ of regular 1-forms (with respect to the skew-symmetric basis $\mathrm{diag}(J, J, \dots, J)$) to give explicit descriptions of the endomorphisms of $\mathrm{Jac}(C_{2^m})$. The power of ζ_{2^m} appearing in the pullback of α satisfies $d_j + 1 = 2^{m-(d+1)}(2j+1)$. This yields $\zeta_{2^m}^{d_j+1} = ((\zeta_{2^m})^{2^{m-(d+1)}})^{2j+1}$, which equals $(\zeta_{2^{d+1}})^{2j+1}$. These are the powers of $\zeta_{2^{d+1}}$ that appear in the endomorphism α_d in Equation (9). Thus, the endomorphism corresponding to α is a block diagonal matrix whose block entries are given by the diagonal matrices α_d for $1 \le d \le m-1$, written in increasing order on d:

$$\alpha = \mathrm{diag}(\alpha_1, \alpha_2, \dots, \alpha_{m-1}). \tag{14}$$

We can define the endomorphism β in a similar manner. Note that, for $1 \le d \le m-1$, the exponents for c coincide with those appearing in β_d from Equation (10). We find that the endomorphism β is a block diagonal matrix whose block entries are nearly the antidiagonal matrices β_d. Letting Z be the $(2g \times 2g)$-matrix $Z = \mathrm{diag}(i, -i, i, -i, \dots, i, -i)$, the endomorphism β can be written as

$$\beta = Z \cdot \mathrm{diag}(\beta_1, \beta_2, \dots, \beta_{m-1}). \tag{15}$$

4.3.2. *Sato-Tate group results.*

Proposition 4.6. *For $m > 2$,*

$$\mathrm{ST}^0(\mathrm{Jac}(C_{2^m})) \simeq \mathrm{U}(1) \times \left(\mathrm{U}(1)^{2^{m-2}-1}\right)_2.$$

Proof. Using the decomposition of the Jacobians in Equation (2) we have

$$\mathrm{Jac}(y^2 = x^{2^m} - c) \sim (y^2 = x^3 - cx) \times \prod_{d=2}^{m-1} \mathrm{Jac}(y^2 = x^{2^d+1} - cx).$$

It is well known that $\mathrm{ST}^0(y^2 = x^3 - cx) \simeq \mathrm{U}(1)$. For the case where $d > 1$, by Proposition 4.2 we have that the identity component of the Jacobians of each of the factors is

$$\mathrm{ST}^0(\mathrm{Jac}(C'_{2^d+1})) \simeq \left(\mathrm{U}(1)^{2^{d-1}}\right)_2.$$

Since the Jacobian of C_m is nondegenerate (Theorem 3.4), the conclusion follows. □

Theorem 4.7. *For generic $c \in \mathbb{Q}^\times$ and $m > 2$, the Sato-Tate group of the Jacobian of the curve $C_{2^m} : y^2 = x^{2^m} - c$ satisfies*

$$\mathrm{ST}(\mathrm{Jac}(C_{2^m})) \simeq \mathrm{ST}^0(\mathrm{Jac}(C_{2^m})) \rtimes \mathrm{Gal}(\mathbb{Q}(\zeta_{2^m}, c^{1/(g+1)})/\mathbb{Q}).$$

The generators of the component group can be written as block diagonal matrices whose diagonal blocks are the generators γ, γ_J, and γ_c in Theorem 4.3 coming from the isogeny factors $\mathrm{Jac}(C'_{2^d+1})$ of $\mathrm{Jac}(C_{2^m})$.

Proof. The proof follows from Theorem 4.3, the action of the Galois generators $\sigma_{-1}, \sigma_5, \tau_m$, and from the definitions of α and β in Equations (14) and (15). □

Remark 4.8. *Taking non-generic values of c only affects the form of γ_c. See Remark 4.4 for information on how the matrices γ_c change for different values of c.*

Examples 1.4 and 1.5 in the Introduction make explicit the generators that we obtain from this construction for the curves $y^2 = x^{16} - c$ and $y^2 = x^{32} - 1$.

5. Moment Statistics

In this section, we describe the distributions of the coefficients of the characteristic polynomial of random conjugacy classes in the Sato-Tate groups in Theorems 4.3 and 4.7. These moment statistics can be used to verify the equidistribution statement of the generalized Sato-Tate conjecture by comparing them to moment statistics obtained for the coefficients of the normalized L-polynomial. The numerical moment statistics are an approximation since one can only ever compute them up to some bound.

5.1. **Computation Techniques.** The techniques described in this section are adapted from [11, 15, 20]. We define the nth moment (centered at 0) of a probability density function to be the expected value of the nth power of the values, i.e. $M_n[X] = E[X^n]$. The moments of a function or distribution can be used to understand the shape of the graph or data. For example, the first moment M_1 is the average value (or expected value of the function), the second moment M_2 gives the variance, and the third moment M_3 describes the skewness of the distribution.

We first define the Haar measure on the groups that we obtain for the identity components of the Sato-Tate groups. From Propositions 4.2 and 4.6 we see that the possible groups are products of the groups $\mathrm{U}(1)$ and $\mathrm{U}(1)_2$. For each of these groups, we are interested in the pushforward of the Haar measure onto the set of conjugacy classes $\mathrm{conj}(\mathrm{U}(1))$ or $\mathrm{conj}(\mathrm{U}(1)_2)$.

We start with the unitary group $\mathrm{U}(1)$ and consider the trace map tr on a random element $U \in \mathrm{U}(1)$ defined by $z := \mathrm{tr}(U) = u + \overline{u} = 2\cos(\theta)$, where $u = e^{i\theta}$. From here we see that $dz = 2\sin(\theta)d\theta$ and

$$\mu_{\mathrm{U}(1)} = \frac{1}{\pi}\frac{dz}{\sqrt{4-z^2}} = \frac{1}{\pi}d\theta$$

gives a uniform measure of $\mathrm{U}(1)$ on $\theta \in [-\pi, \pi]$ (see [35, Section 2]). We can deduce the following pushforward measures

$$\mu_{\mathrm{U}(1)^n} = \prod_{i=1}^{n}\frac{1}{\pi}\frac{dz_i}{\sqrt{4-z_i^2}} = \prod_{i=1}^{n}\frac{1}{\pi}d\theta_i \quad \text{and} \quad \mu_{(\mathrm{U}(1)_2)^n} = \prod_{i=1}^{n}\frac{1}{\pi}\frac{dz_i}{\sqrt{4-z_i^2}} = \prod_{i=1}^{n}\frac{1}{\pi}d\theta_i.$$

Note that though the measure $\mu_{(\mathrm{U}(1)_2)^n}$ is expressed the same as the measure $\mu_{\mathrm{U}(1)^n}$, we will get a different distribution since in the former case each eigenangle θ_i occurs with multiplicity 2. These can be generalized even further to give measures for the groups appearing as identity components in Propositions 4.2 and 4.6.

We now define the moment sequence $M[\mu]$, where μ is a positive measure on some interval $I = [-d, d]$. The n^{th} moment $M_n[\mu]$ is the expected value of ϕ_n with respect to μ, where ϕ_n is the function $z \mapsto z^n$. It is therefore given by $M_n[\mu] = \int_I z^n \mu(z)$. Computing this for the measures $\mu_{\mathrm{U}(1)}$ and $\mu_{\mathrm{U}(1)_2}$ yields $M_n[\mu_{\mathrm{U}(1)}] = \binom{n}{n/2}$ and $M_n[\mu_{\mathrm{U}(1)_2}] = 2^n\binom{n}{n/2}$, where $\binom{n}{n/2} = 0$ if n is odd. We can take binomial convolutions of these moment identities to compute moments for the groups appearing as identity components in Propositions 4.2 and 4.6.

More generally, let B be a random element of a compact subgroup of $\mathrm{USp}(2g)$ and let $a_i := a_i(B)$ denote the i^{th} coefficient of the characteristic polynomial of B. The n^{th} moment $M_n[a_i]$ is the expected value of a_i^n and we can compute this by integrating against the Haar measure. We obtain moment statistics for the full Sato-Tate group of an abelian variety by taking the average of the moments for $U \cdot B$ for each element B in the component group, where U is a random element of the identity component. We work out an explicit example in Section 5.2 to demonstrate how these techniques work in practice.

The moment statistics coming from the Sato-Tate group can then be compared to numerical moment statistics coming from the coefficients of the normalized L-polynomial of the abelian variety. The numerical moments in Sections 5.2 and 5.3 were computed for primes up to 2^{22} using an algorithm described in [22] and [23]. The bound was chosen to balance two competing interests: accuracy and computation time. While it would be ideal to be able to use a larger bound for the prime p, we found that this bound was sufficient for comparison to the moments coming from the Sato-Tate group. On the other hand, this limitation provides extra motivation to compute explicit generators for the Sato-Tate group in order to give an accurate description of the limiting distributions of the coefficients of normalized L-polynomials.

5.2. **Genus 4 Example.** We begin with the genus 4 family $y^2 = x^9 - cx$. For generic c, we use Theorem 4.3 to determine that component group of Sato-Tate group is generated by the block matrices

$$\begin{pmatrix} 0 & 0 & I & 0 \\ J & 0 & 0 & 0 \\ 0 & 0 & 0 & -J \\ 0 & I & 0 & 0 \end{pmatrix},$$

$\mathrm{diag}(J, J, J, J)$, and $\mathrm{diag}(\zeta_{16}^{-3} I, \zeta_{16}^{-1} I, \zeta_{16} I, \zeta_{16}^{3} I)$. The identity component is the connected group $(\mathrm{U}(1) \times \mathrm{U}(1))_2$.

Let $U = \mathrm{diag}(u_0, \overline{u}_0, u_1, \overline{u}_1, u_0, \overline{u}_0, u_1, \overline{u}_1)$ where $u_j = e^{i\theta_j}$, be an element in the identity component. The elements of the Sato-Tate group that contribute to the a_1-moment statistics are those of the form $g_k = U \cdot \gamma_c^k$, where $k \in [0, 7]$. We compute the characteristic polynomial of each g_k and find that the n^{th} moment $M_n[a_i(g_k)]$ is given by

$$\frac{2^{n-2}}{\pi^2} \int_0^{2\pi} \int_0^{2\pi} \left(\cos\left(\theta_0 - \frac{6\pi k}{16}\right) + \cos\left(\theta_0 + \frac{2\pi k}{16}\right) + \cos\left(\theta_1 - \frac{2\pi k}{16}\right) + \cos\left(\theta_1 + \frac{6\pi k}{16}\right) \right)^n d\theta_0 \, d\theta_1,$$

We then compute the moment statistics $M_n[\mu_1]$ of the full Sato-Tate group by averaging over the size of the group. A similar process can be used to compute moments $M_n[\mu_i]$ for the higher traces a_i.

For nongeneric c, we adjust this by taking a 2nd, 4th, or 8th power of the last generator if c is a square, 4th power, or 8th power in $\mathbb{Q}(\zeta_{16})^\times$, respectively (see Remark 4.4). In these cases, we restrict the above computation to the appropriate values of k. Tables 1 - 4 below give (rounded) numerical a_1-moment statistics for several values of c and the corresponding exact μ_1-moment statistics coming from the Sato-Tate group. The odd moments $M_n[\mu_1] = 0$ for all odd n, so we omit those values from the table. The errors in the even moments are consistent with those for the odd moments and would be improved by choosing a larger bound for p.

a_1-moments				
c	M_2	M_4	M_6	M_8
3	0.995	20.626	612.971	23153.8
5	0.997	20.806	626.22	24061.5
6	0.991	20.593	620.942	23994.3
7	0.997	20.802	626.461	24069.3
μ_1-moments				
	1	21	640	24955

TABLE 1. Table of a_1- and μ_1-moments ($p < 2^{22}$) for some generic $c \in \mathbb{Q}(\zeta_{16})$.

a_1-moments				
c	M_2	M_4	M_6	M_8
2	0.989	26.486	972.585	42552.8
8	0.989	26.486	972.585	42552.8
32	0.989	26.486	972.585	42552.8
μ_1-moments				
	1	27	1000	44100

TABLE 2. Table of a_1- and μ_1-moments ($p < 2^{22}$) for some values $c \in \mathbb{Q}(\zeta_{16})^{\times 2} \setminus \mathbb{Q}(\zeta_{16})^{\times 4}$.

a_1-moments				
c	M_2	M_4	M_6	M_8
4	0.987	35.287	1557.64	75827.6
64	0.987	35.287	1557.64	75827.6
μ_1-moments				
	1	36	1600	78400

TABLE 3. Table of a_1- and μ_1-moments ($p < 2^{22}$) for some values $c \in \mathbb{Q}(\zeta_{16})^{\times 4} \setminus \mathbb{Q}(\zeta_{16})^{\times 8}$.

a_1-moments				
c	M_2	M_4	M_6	M_8
1	1.989	71.299	3154.55	153942
16	1.989	71.299	3154.55	153942
μ_1-moments				
	2	72	3200	156800

TABLE 4. Table of a_1- and μ_1-moments ($p < 2^{22}$) for some values $c \in \mathbb{Q}(\zeta_{16})^{\times 8}$.

5.3. **Higher Genus Examples.** In this section we present moment statistics for some higher genus curves. Computing the numerical moments becomes more difficult as the genus grows and limits the bound we can use on the primes. In order to compute numerical statistics using primes up to 2^{22} for the curves $y^2 = x^{16} - 1$ and $y^2 = x^{32} - 1$, we relied on the factorizations of their Jacobians and L-polynomials. For example, for the curve $y^2 = x^{16} - 1$, we use Equation (2) to write

$$\mathrm{Jac}(y^2 = x^{16} - 1) \sim \mathrm{Jac}(y^2 = x^3 - x) \times \mathrm{Jac}(y^2 = x^5 - x) \times \mathrm{Jac}(y^2 = x^9 - x).$$

This tells us that the L-polynomial of $\mathrm{Jac}(y^2 = x^{16} - 1)$ (for good primes p) factors into the product of L-polynomials of the factor curves. The a_1-trace of the normalized L-polynomial of $\mathrm{Jac}(y^2 = x^{16} - 1)$ is therefore the sum of the a_1-traces of the factors, which we were able to compute. The numerical and Sato-Tate moments are given in Table 5.

	M_2	M_4	M_6	M_8
a_1	4.9749	238.008	20277.7	2203730
μ_1	5	243	21170	2358755

TABLE 5. Table of a_1- and μ_1-moments ($p < 2^{22}$) for $y^2 = x^{16} - 1$.

We were able to directly compute moment statistics for $y^2 = x^{17} - x$ for primes up to 2^{22}. The numerical and Sato-Tate moments for the a_1-trace are given in Table 6.

	M_2	M_4	M_6	M_8
a_1	1.98265	164.813	19727.9	2861530
μ_1	2	168	20480	3041920

TABLE 6. Table of a_1- and μ_1-moments ($p < 2^{22}$) for $y^2 = x^{17} - x$.

For the curve $y^2 = x^{32} - 1$, we once again made use of the factorization of the Jacobian and its L-polynomial. From Equation (2) we have

$$\mathrm{Jac}(y^2 = x^{32} - 1) \sim \mathrm{Jac}(y^2 = x^3 - x) \times \mathrm{Jac}(y^2 = x^5 - x) \times \mathrm{Jac}(y^2 = x^9 - x) \times \mathrm{Jac}(y^2 = x^{17} - x)$$

and so we add the a_1-traces of the factors to obtain the a_1-trace for $y^2 = x^{32} - 1$. The numerical and Sato-Tate moments are given in Table 7.

	M_2	M_4	M_6	M_8
a_1	6.94177	698.749	150219	46647200
μ_1	7	723	159190	49909475

TABLE 7. Table of a_1- and μ_1-moments ($p < 2^{22}$) for $y^2 = x^{32} - 1$.

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