KUZNETSOV'S FANO THREEFOLD CONJECTURE VIA HOCHSCHILD-SERRE ALGEBRA

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ABSTRACT. Let Y be a smooth quartic double solid regarded as a degree 4 hypersurface of the weighted projective space $\mathbb{P}(1,1,1,1,2)$. We study the multiplication of Hochschild-Serre algebra of its Kuznetsov component $\mathcal{K}u(Y)$, via matrix factorization. As an application, we give a new disproof of Kuznetsov's Fano threefold conjecture. In appendix, we show kernel of differential of period map for special Gushel-Mukai threefold is of two dimensional by categorical methods, which completes a result in [DIM08, Theorem 7.8].

1. Introduction

Let X be Fano variety whose semi-orthgonal decomposition for bounded derived category is given by

$$D^b(X) = \langle \mathcal{K}u(X), E_1, \ldots, E_n \rangle,$$

where $E_1, \ldots, E_n$ is an exceptional collection of vector bundles over X and $\mathcal{K}u(X)$ is the right orthogonal complement of the collection, called Kuznetsov component. It has been widely believed that the Kuznetsov component encodes the essential birational geometric information of the Fano varieties. Thus extracting geometric information from Kuznetsov components is an important step to understand geometry of Fano varieties. There are numerous way to extract information from Kuznetsov components, which we briefly recall as follows.

1.1. **Stability conditions in Kuznetsov components and moduli space theoretical approach.** One of the most interesting class of Fano varieties are smooth Fano threefolds of Picard rank one of index one and two, whose deformation classes are completely classified in [VI99]. In the paper [BLMS17], the authors construct a stability condition σ in $\mathcal{K}u(X)$ for any such Fano variety X. Denote by $\mathcal{N}(\mathcal{K}u(X))$ the numerical Grothendieck group and fix a numerical class $\mathbf{v} \in \mathcal{N}(\mathcal{K}u(X))$ and consider the Bridgeland moduli space $\mathcal{M}_\sigma(\mathcal{K}u(X), \mathbf{v})$ of (semi)stable object with respect to σ in $\mathcal{K}u(X)$ of numerical class $\mathbf{v}$. The numerical character $\mathbf{v}$ is appropriately chosen such that the corresponding moduli space reconstructs Fano variety of rational curves on X, which is used to reconstruct (birational) isomorphism class of Fano varieties(cf. [BMMS12], [PY22], [GLZ22]).

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1.2. **Topological K-theory of admissible subcategory and Hodge theoretical approach.** Let $\mathcal{A} \subset D^b(X)$ be an admissible subcategory of bounded derived category of a smooth projective variety X. The topological K-theory [Bla16] of dg categories over $\mathbb{C}$ is an additive invariant

$$\mathrm{K}_1^{top} : \mathrm{dg} - cat \to \mathbb{Z} - mod.$$

with Chern character map

$$\mathrm{ch}^{top} : \mathrm{K}_1^{top}(\mathcal{A}_{dg}) \to \mathrm{HP}_1(\mathcal{A}_{dg}).$$

Furthermore $\mathrm{K}_1^{top}(D^{perf}_{dg}(X)) \otimes \mathbb{C} \cong \mathrm{H}^{\mathrm{odd}}(X, \mathbb{C})$, and ch^{top} is the usual Chern character. In particular, the natural splitting from that of X gives a weight one Hodge structure for topological K-group $\mathrm{K}_1^{top}(\mathcal{A})_{tf}$. Namely, the topological Chern character induces,

$$\mathrm{K}_1^{top}(\mathcal{A})_{tf} \otimes \mathbb{C} \cong^{\mathrm{ch}^{top}} \mathrm{HP}_1(\mathcal{A}) \cong \mathrm{HN}_{-1}(\mathcal{A}) \oplus \overline{\mathrm{HN}_{-1}(\mathcal{A})}.$$

Thus, we have a complex torus associated to this weight one Hodge structure. More explicitly,

$$\mathrm{J}(\mathcal{A}) = \frac{\mathrm{HP}_1(\mathcal{A})}{\mathrm{HN}_{-1}(\mathcal{A}) + \mathrm{Im}\,\mathrm{ch}\, \mathrm{K}_1^{top}(\mathcal{A})}.$$

In the case of X being a smooth Fano threefold with $\mathcal{A}$ being the Kuznetsov component $\mathcal{K}u(X)$ as the orthogonal complement of an exceptional collection of vector bundles, by [JLLZ21, Lemma 3.9] $\mathrm{J}(\mathcal{K}u(X)) \cong J(X)$ as polarised abelian varieties. which was proved in [Per22, Section 5]. Similar construction is generalized for any smooth and proper dg category and even to arbitrary dg category in [CMHL$^+$23] and [LXZ24]. On the other hand, in the similar spirit, topological K-theory and noncommutative Hodge theory(cf. [Bla16] and [Per22]) is applied to admissible subcategory of Fano fourfolds in [BP23] and surfaces in [DJR23] to recover Mukai lattice for K3 category and primitive cohomology for surfaces respectively. As application, (birational) categorical Torelli theorem are proved for various varieties.

1.3. **Hochschild-Serre algebra and algebraic approach.** Let $\mathcal{A}$ be a smooth and proper dg category, one can naturally attach a bi-graded algebra

$$\mathcal{A}_S = \bigoplus_{m,n \in \mathbb{Z}} \mathrm{Hom}(\mathrm{Id}, S^m[n])$$

with multiplication map

$$\mathrm{Hom}(\mathrm{Id}, S^{m_1}[n_1]) \times \mathrm{Hom}(\mathrm{Id}, S^{m_2}[n_2]) \xrightarrow{\times} \mathrm{Hom}(\mathrm{Id}, S^{m_1+m_2}[n_1+n_2])$$

given by the composition

$$\mathrm{Id} \xrightarrow{b} \mathrm{Id} \circ S^{m_2}[n_2] \xrightarrow{a \circ \mathrm{Id}} S^{m_1}[n_1] \circ S^{m_2}[n_2] = S^{m_1+m_2}[n_1+n_2] \ ,$$

for $(a, b) \in \mathrm{Hom}(\mathrm{Id}, S^{m_1}[n_1]) \times \mathrm{Hom}(\mathrm{Id}, S^{m_2}[n_2])$. It was studied in [Orl03] and [Căl05], [Cal03] independently when $\mathcal{A}$ is bounded derived category $D^b(X)$ of coherent sheaves on a smooth projective variety X, where they prove basic property of this algebra. Moreover, in [BO01], the author uses a sub-algebra of $D^b(X)_S$, which is isomorphic to anti-canonical ring of a smooth

Fano variety X to reconstruct the variety itself. Recently this algebra is revisited in [BFK23] and [LZ23] under the name *Hochschild-Serre algebra* for admissible subcategory of $D^b(X)$. In particular, the authors of [LZ23] establish a sub-algebra of $\mathcal{K}u(X)_S$ in the case of smooth (weighted) hypersurface of (weighted) projective spaces, which recovers the Jacobian ring of X. Thus a categorical Torelli theorem is proved for those (weighted) hypersurfaces.

1.4. **Kuznetsov's Fano threefold conjecture.** Denote by $\mathcal{MF}_d^i$ the moduli space of smooth Fano threefold of index i and degree d. In [Kuz09, Conjecture 3.7], the author proposed a surprising conjecture relating the non-trivial admissible subcategories of two families of smooth Fano threefolds.

Conjecture 1.1. *There is a correspondence $\mathcal{Z}_d \subset \mathcal{MF}_d^2 \times \mathcal{MF}_{4d+2}^i$, such that for any pair $(Y_d, X_{4d+2}) \in \mathcal{Z}_d$, there is an equivalence of categories*

$$\mathcal{K}u(Y_d) \simeq \mathcal{K}u(X_{4d+2}).$$

The conjecture is proved for $d = 3, 4$ and 5 in [Kuz09]. For remaining cases, there were numerous evidences suggesting that the conjecture might be false. Thus instead of proving this conjecture, people are trying to disprove it. To do this, the natural idea would be looking at the information(moduli spaces, Hodge theory, algebra etc.)extracting from $\mathcal{K}u(Y_d)$ and $\mathcal{K}u(X_{4d+2})$ respectively and then show that they are different. Indeed, in [Zha20], the author adopts the moduli theoretical approach described in Section 1.1 to study particular Bridgeland moduli spaces canonically constructed from $\mathcal{K}u(Y_2)$ and $\mathcal{K}u(X_{10})$ respectively and shows that they are not isomorphic to each other. Independently, in [BP23], the authors apply the Hodge theoretical approach in Section 1.2. They look at the Mukai-Hodge lattice of two K3 categories constructed from equivariant categories of $\mathcal{K}u(Y_2)$ and $\mathcal{K}u(X_{10})$ respectively and show that the Hodge isometry does not exist. In the current paper, we adopt another perspective described in 1.3. Namely, we look at Hochschild-Serre algebra of dg-enhancement of Kuznetsov component of quartic double solid and Gushel-Mukai threefold. It turns out that they are not isomorphic to each other and the proof is very simple.

1.5. **Main Results.** Let Y be a smooth quartic double solid and X be a smooth Gushel-Mukai threefold. Denote by $\mathcal{K}u(Y)_S, \mathcal{K}u(X)_S$ the Hochschild-Serre algebra of dg-enhancement of $\mathcal{K}u(Y)$ and $\mathcal{K}u(X)$ respectively. Consider the multiplication

$$\text{Hom}(\text{Id}, S^2_{\mathcal{K}u(Y)}[-2]) \times \text{Hom}(\text{Id}, S_{\mathcal{K}u(Y)}[-1]) \to \text{Hom}(\text{Id}, S^3_{\mathcal{K}u(Y)}[-3]) \cong \text{HH}_1(\mathcal{K}u(Y)), \tag{1}$$

and associated map

$$\gamma_Y : \text{HH}^2(\mathcal{K}u(Y)) \to \text{Hom}(\text{HH}_{-1}(\mathcal{K}u(Y)), \text{HH}_1(\mathcal{K}u(Y))).$$

Then we show

Theorem 1.2. *The kernel of the map γ_Y is one dimensional.*

On the other hand, from [JLLZ22, Theorem 4.6], we know the map

$$\gamma_X : \mathrm{HH}^2(\mathcal{K}u(X)) \to \mathrm{Hom}(\mathrm{HH}_{-1}(\mathcal{K}u(X)), \mathrm{HH}_1(\mathcal{K}u(X)))$$

is injective for all ordinary Gushel-Mukai threefold X. Assuming $\mathcal{K}u(Y) \simeq \mathcal{K}u(X)$ and the equivalence is induced by a Fourier-Mukai functor. Then by [JLLZ22, Theorem 4.8], $\mathrm{Ker}(\gamma_Y) \cong \mathrm{Ker}(\gamma_X) = 0$, which is a contradiction. As an immediate corollary, we have

Corollary 1.3. *For any Gushel Mukai threefold X and quartic double solid Y, the categories $\mathcal{K}u(X)$ and $\mathcal{K}u(Y)$ are never equivalent. In particular, the Conjecture 1.1 for $d = 2$ fails.*

Remark 1.4. By [KP18, Lemma 3.8] and [KP19, Theorem 1.6], for any special Gushel-Mukai threefold X', there exists an ordinary one X such that $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$, thus the Conjecture 1.1 for $d = 2$ is reduced to the case of ordinary Gushel-Mukai threefold.

1.6. **Organization of the paper.** In Section 2, we recall the terminology of category of graded matrix factorization $\mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \mathcal{O}(d), \omega)$ with $\mathbb{C}^*$-action on $\mathbb{A}^{n+1}$ of weight $(a_0, \ldots, a_n)$. Then we describe the multiplications of Hochschild-Serre algebra for the matrix factorization. In Section 3, we describe the multiplication of Hochschild-Serre algebra for Kuznetsov component of a smooth quartic double solid and prove Theorem 1.2, as a corollary, we disproof Kuznetsov's Fano threefold conjecture. In Appendix 4, we extend a classical result [DIM08, Theorem 7.8] to special Gushel-Mukai threefold.

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2. DG CATEGORY OF GRADED MATRIX FACTORIZATIONS

In this section, we recall the terminology of *dg*-category of matrix factorization. We follow the context in [BFK14]. We refer the reader to [Kel06] for the basic of *dg* categories. Denote by Hqe(dg-cat) the localized dg-cat with respect to the quasi-equivalences of *dg* categories. Let (X, G, L, ω) be a quadruple where X is a quasi-projective variety with G action, where G is a reductive algebraic group, L is a G-equivariant line bundle and ω is a G-invariant section of L. Our main example is $(\mathbb{A}^{n+1}, \mathbb{C}^*, \mathcal{O}(d), \omega)$. The $\mathbb{C}^*$ action on $\mathbb{A}^{n+1}$ is given by $\lambda\cdot(x_0, x_1, \cdots, x_n) = (\lambda^{a_0}\cdot x_0, \lambda^{a_1}\cdot x_1, \cdots, \lambda^{a_n}\cdot x_n)$, $a_0, a_1, \cdots, a_n$ are integers such that $\gcd(a_0, a_1, \cdots, a_n) = 1$. $\mathcal{O}(d)$ is the trivial line bundle twisted with the character $\mathbb{X}_d : \mathbb{C}^* \to \mathbb{C}^*, \quad \lambda \mapsto \lambda^d$. ω is a $\mathbb{C}^*$-invariant section of $\mathcal{O}(d)$. Namely ω is a degree d polynomial, the weight of variable x_i is a_i.

We have *dg* category $\mathrm{Fact}(X, G, L, \omega)$, whose objects are a quadruple $(\mathcal{E}_{-1}, \mathcal{E}_0, \Phi_{-1}, \Phi_0)$, where $\mathcal{E}_{-1}$ and $\mathcal{E}_0$ are G-equivariant quasi-coherent sheaves, $\Phi_{-1} : \mathcal{E}_0 \to \mathcal{E}_{-1} \otimes L$ and $\Phi_0 : \mathcal{E}_{-1} \to \mathcal{E}_0$

are morphism of G-equivariant sheaves such that

$$\Phi_{-1} \circ \Phi_0 = \omega.$$
$$(\Phi_0 \otimes L) \circ \Phi_{-1} = \omega.$$

The space of morphisms in $\mathrm{Fact}(X, G, L, \omega)$ are the internal Hom of G-equivariant sheaves while extending the pairs of morphisms to certain $\mathbb{Z}$-graded complexes. We point out the reference [BFK14] for interested reader. There is a category $\mathrm{Acycli}(\mathrm{Fact}(X, G, L, \omega))$ which imitates acyclic complexes in category of complexes of sheaves. The absolute derived category $D^{abs}[\mathrm{Fact}(X, G, L, \omega)]$ is the homotopy category of dg quotient $\frac{\mathrm{Fact}(X,G,L,\omega)}{\mathrm{Acyclic}(\mathrm{Fact}(X,G,L,\omega))} \in$ Hqe(dg-cat). Let $\mathrm{Inj}(X, G, L, \omega) \subset \mathrm{Fact}(X, G, L, \omega)$ be the dg sub-category whose components are G-equivariant injective quasi-coherent sheaves. We write $[\mathcal{A}]$ as the homotopic categoy of any dg category $\mathcal{A}$.

Lemma 2.1. *The composition* $\mathrm{Inj}(X, G, L, \omega) \to \mathrm{Fact}(X, G, L, \omega) \to \frac{\mathrm{Fact}(X,G,L,\omega)}{\mathrm{Acyclic}(\mathrm{Fact}(X,G,L,\omega))}$ *induces an equivalence of homotopic categories*

$$[\mathrm{Inj}(X, G, L, \omega)] \cong [\frac{\mathrm{Fact}(X, G, L, \omega)}{\mathrm{Acyclic}(\mathrm{Fact}(X, G, L, \omega))}] := D^{abs}[\mathrm{Fact}(X, G, L, \omega)]$$

Let $\mathrm{Inj}_{\mathrm{coh}}(X, G, L, \omega) \subset \mathrm{Inj}(X, G, L, \omega)$ be a dg sub-category whose objects are quasi-isomorphic to objects with coherent components in category $\mathrm{Fact}(X, G, L, \omega)$.

Define shifting functor

$$[1] : (\mathcal{E}_{-1}, \mathcal{E}_0, \Phi_{-1}, \Phi_0) \mapsto (\mathcal{E}_0, \mathcal{E}_{-1} \otimes L, -\Phi_0, -\Phi_{-1} \otimes L).$$

With cone construction, the homotopic category $[\mathrm{Inj}_{\mathrm{coh}}(X, G, L, \omega)]$ is a triangulated category which is equivalent to category of graded matrix factorization in [Orl09] for $(\mathbb{A}^{n+1}, \mathbb{C}^*, \mathcal{O}(d), \omega)$.

Denote by

$$\{1\} = - \otimes \mathcal{O}_{\mathbb{A}^{n+1}}(1) : \mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{\mathrm{n}+1}, \mathbb{C}^*, \mathcal{O}(\mathrm{d}), \omega) \to \mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{\mathrm{n}+1}, \mathbb{C}^*, \mathcal{O}(\mathrm{d}), \omega)$$

the twisting functor which maps

$$\mathcal{E}_{-1} \xrightarrow{\Phi_0} \mathcal{E}_0 \xrightarrow{\Phi_{-1}} \mathcal{E}_{-1}(d)$$

to

$$\mathcal{E}_{-1}(1) \xrightarrow{\Phi_0(1)} \mathcal{E}_0(1) \xrightarrow{\Phi_{-1}(1)} \mathcal{E}_{-1}(d+1)$$

Clearly, we have equality of functors $\{d\} := \{1\}^d = [2]$.

Let $X \subset \mathbb{P}(a_1, a_2, \cdots, a_n)$ be a smooth hypersurface of degree $d \le n$ defined by ω. Let

$$\mathcal{K}u(X) := \left\langle \mathcal{O}_X, \mathcal{O}_X(1), \cdots, \mathcal{O}_X(\sum_{j=0}^{n} a_j - 1 - d) \right\rangle^{\perp}.$$

Roughly speaking, $Ku(X)$ is identified with the essential subcategory of B-branes of X in Physics. If X a is Calabi-Yau variety, $Ku(X) = D^b(X)$. On LG side, the category $[\mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \mathcal{O}(d), \omega)]$ is identified with the category of B-branes of Landau-Ginzburg model.

Physically B-branes of X and LG model are naturally equivalent, which was proved by Orlov [Orl09] mathematically. Namely, we have equivalence

$$\mathcal{K}u(X) \cong [\mathrm{Inj}_{\mathrm{coh}}(X, G, L, \omega)].$$

Consider the natural enhancement $\mathrm{Inj}_{\mathrm{coh}}(X)$, and let $\mathcal{K}u_{dg}(X)$ be a dg subcategory that enhance $\mathcal{K}u(X)$. Orlov' σ /LG correspondence can be lifted to be equivalence of dg categories.

Theorem 2.2. *[BFK14, Theorem 6.13] There is an equivalence in* Hqe(dg-cat)*,*

$$\Phi : \mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega) \cong \mathcal{K}u_{dg}(X).$$

According to [BFK14], the natural functors can be reinterpreted as kernels of Fourier-Mukai transforms, and the natural transformations between these functors are morphism of kernels. We write $\Delta(m)$ as the kernel of functor $- \otimes \mathcal{O}_{\mathbb{A}^{n+1}}(m)$.

Lemma 2.3. *[FK18, Theorem 1.2] The Serre functor of* $[\mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega)]$ *is* $- \otimes \mathcal{O}_{\mathbb{A}^{n+1}}(\sum_{j=0}^{n} -a_j)[n+1]$.

Proof. Since $\mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \mathcal{O}(d), \omega) \cong \mathcal{K}u_{dg}(X)$, the category $\mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \mathcal{O}(d), \omega)$ is smooth and proper, or by [FK18, lemma 2.11, 2.14]. Then there is a smooth proper dg algebra A such that $[\mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \mathcal{O}(d), \omega)] \cong D^{perf}(A)$, the arguments in [FK18, Theorem 2.18] show the Serre functor is

$$- \otimes \omega_{\mathbb{A}^{n+1}}[n + 1 - \dim \mathbb{C}^* + 1] = - \otimes \mathcal{O}_{\mathbb{A}^{n+1}}(\sum_{j=0}^{n} -a_j)[n + 1].$$

□

Next we recall a key theorem in [BFK14, Theorem 1.2]. For $g \in \mathbb{C}^*$, we write W_g as the conormal sheaf of fixed locus $(\mathbb{A}^{n+1})^g$ and k_g the character of $\det(W_g)$. We write $H^\bullet(d\omega_g)$ as the Koszul cohomology of the Jacobian ideal of $\omega_g := \omega|_{(\mathbb{A}^{n+1})^g}$. Let $\gamma = e^{\frac{2\pi i}{d}}$, and $\mu_d = \langle 1, \gamma, \gamma^2, \cdots, \gamma^{d-1} \rangle$.

Proposition 2.4. *[BFK14, Theorem 5.9] Assume ω has only isolated singularty at 0, then*

$$\mathrm{Hom}(\Delta, \Delta(m)[t]) \cong \Big(\bigoplus_{g \in \mu_d,\ t - \mathrm{rk}\, W_g \textbf{ is even}} \mathrm{Jac}(\omega_g)(m - k_g + d(\frac{t - \mathrm{rk}\, W_g}{2})) \Big)^{\mathbb{C}^*}$$

We refer the reader to [BFK14] for details of computation. We describe the multiplication under the isomorphism of Theorem 2.4. To make this self contain, we introduce some notions used in the proof.

Let Z be a quasi-projective variety with G action, G is an algebraic group. Let H be a subgroup of G. We have an action of H on $G \times Z$ defined by

$$\tau : H \times G \times Z \to G \times Z, \quad (h, g, z) \mapsto (g \cdot h^{-1}, h \cdot z).$$

The fppf quotient of $G \times X$ of H is a seperated algebraic space, whcih is denoted as $G \times^H Z$, see [BFK14, Lemma 2.12]. Consider morphisms

$$l : Z \to G \times^H Z, \quad x \mapsto (e, x).$$

$$\alpha : G \times^H Z \to Z, \quad (g, x) \mapsto gx.$$

First, the pull back functor l^* define an equivalence of equivariant quasi-coherent sheaves. Namely

$$l^* : \mathrm{Qcoh}_G\, G \times^H Z \to \mathrm{Qcoh}_H\, Z.$$

is an equivalence [Tho87, Lemma 1.3]. We write $\alpha_* : \mathrm{Qcoh}_G\, G \times^H Z \to \mathrm{Qcoh}_H\, Z$ as the push forward functor of α, and $\alpha^* : \mathrm{Qcoh}_H\, Z \to \mathrm{Qcoh}_G\, G \times^H Z$ as the pull pack functor of α.

Definition 2.5.

$$\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}} := \alpha_* \circ (l^*)^{-1} : \mathrm{Qcoh}_H\, Z \to \mathrm{Qcoh}_G\, Z.$$

$$\mathrm{Res}_H^G := l^* \circ \alpha^* : \mathrm{Qcoh}_G\, Z \to \mathrm{Qcoh}_H\, Z.$$

We still write Ind_H^G and Res_H^G as derived functors of derived categories of equivariant sheaves. Ind_H^G is right adjoint functor of Res_H^G. In our case, $Z = \mathbb{A}^{n+1} \times \mathbb{A}^{n+1}$, $G = \mathbb{C}^* \times_{\mathbb{C}^*} \mathbb{C}^* = \{(g_1, g_2) \in \mathbb{C}^* \times \mathbb{C}^* | g_1^d = g_2^d\}$, and $H = \mathbb{C}^* \subset G$ via diagonal embedding. The G action on Z is given by

$$(g_1, g_2) \cdot (x_1, \cdots, x_{n+1}, y_1, \cdots, y_{n+1}) = (g_1^{a_1} x_1, \cdots, g_1^{a_{n+1}} x_{n+1}, g_2^{a_1} y_1, \cdots, g_2^{a_{n+1}} y_{n+1})$$

By definition,

$$\mathrm{Hom}(\Delta, \Delta(m)[t]) \cong \mathrm{Hom}(\mathrm{Ind}_{\mathbb{C}^*}^{\mathbb{C}^* \times_{\mathbb{C}^*} \mathbb{C}^*} \Delta_* \mathcal{O}_{\mathbb{A}^{n+1}}, \mathrm{Ind}_{\mathbb{C}^*}^{\mathbb{C}^* \times_{\mathbb{C}^*} \mathbb{C}^*} \Delta_* \mathcal{O}_{\mathbb{A}^{n+1}}(m)[t]).$$

The multiplication

$$\Phi : \mathrm{Hom}(\Delta, \Delta(m_1)[t_1]) \times \mathrm{Hom}(\Delta, \Delta(m_2)[t_2]) \to \mathrm{Hom}(\Delta, \Delta(m_1 + m_2)[t_1 + t_2])$$

maps (a, b) to ab is the composition
(2)

$$\mathrm{Ind}_{\mathbb{C}^*}^{\mathbb{C}^* \times_{\mathbb{C}^*} \mathbb{C}^*} \Delta_* \mathcal{O}_{\mathbb{A}^{n+1}} \xrightarrow{b} \mathrm{Ind}_{\mathbb{C}^*}^{\mathbb{C}^* \times_{\mathbb{C}^*} \mathbb{C}^*} \Delta_* \mathcal{O}_{\mathbb{A}^{n+1}}(m_2)[t_2] \xrightarrow{a} \mathrm{Ind}_{\mathbb{C}^*}^{\mathbb{C}^* \times_{\mathbb{C}^*} \mathbb{C}^*} \Delta_* \mathcal{O}_{\mathbb{A}^{n+1}}(m_1 + m_2)[t_1 + t_2] .$$

Without loss of generality, we assume $m_1 = m_2 = t_1 = t_2 = 0$. The sequence (1) is equivalent to

$$\mathbb{L}\Delta^* \mathrm{Res}_{\mathbb{C}^*}^{\mathbb{C}^* \times_{\mathbb{C}^*} \mathbb{C}^*} \mathrm{Ind}_{\mathbb{C}^*}^{\mathbb{C}^* \times_{\mathbb{C}^*} \mathbb{C}^*} \Delta_* \mathcal{O}_{\mathbb{A}^{n+1}} \xrightarrow{\quad b \quad} \mathbb{L}\Delta^* \mathrm{Res}_{\mathbb{C}^*}^{\mathbb{C}^* \times_{\mathbb{C}^*} \mathbb{C}^*} \mathrm{Ind}_{\mathbb{C}^*}^{\mathbb{C}^* \times_{\mathbb{C}^*} \mathbb{C}^*} \Delta_* \mathcal{O}_{\mathbb{A}^{n+1}} \xrightarrow{\quad a \quad} \mathcal{O}_{\mathbb{A}^{n+1}}$$

Here a, b are regarded as $\mathbb{C}^*$ equivariant morphism via diagonal embedding $\mathbb{C}^* \hookrightarrow \mathbb{C}^* \times_{\mathbb{C}^*} \mathbb{C}^*$. The morphism b here is $\mathbb{L}\Delta^* \mathrm{Res}_{\mathbb{C}^*}^{\mathbb{C}^* \times_{\mathbb{C}^*} \mathbb{C}^*} \Delta^*(b) : \mathbb{L}\Delta^* \mathrm{Res}_{\mathbb{C}^*}^{\mathbb{C}^* \times_{\mathbb{C}^*} \mathbb{C}^*} \mathrm{Ind}_{\mathbb{C}^*}^{\mathbb{C}^* \times_{\mathbb{C}^*} \mathbb{C}^*} \mathcal{O}_{\mathbb{A}^{n+1}} \to \mathbb{L}\Delta^* \mathrm{Res}_{\mathbb{C}^*}^{\mathbb{C}^* \times_{\mathbb{C}^*} \mathbb{C}^*} \mathrm{Ind}_{\mathbb{C}^*}^{\mathbb{C}^* \times_{\mathbb{C}^*} \mathbb{C}^*} \Delta_* \mathcal{O}_{\mathbb{A}^{n+1}}$.

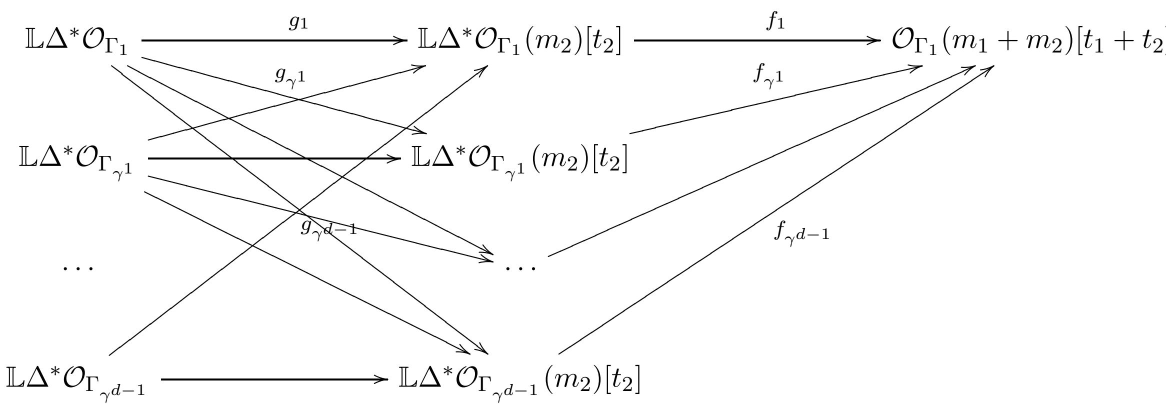
Next, $\mathrm{Ind}_{\mathbb{C}^*}^{\mathbb{C}^* \times_{\mathbb{C}^*} \mathbb{C}^*} \Delta_* \mathcal{O}_{\mathbb{A}^{n+1}} \cong \bigoplus_{g \in \mu_d} \mathcal{O}_{\Gamma_g}$ [BFK14, Lemma 5.31], where Γ_g is the graph $x \mapsto (g \cdot x, x)$. The $\mathbb{C}^* \times_{\mathbb{C}^*} \mathbb{C}^*$ action on $\bigoplus_{g \in \mu_d} \mathcal{O}_{\Gamma_g}$ is defined by $(g_1, g_2) \cdot (gx, \cdot x_1) = (g_1 g \cdot x, g_2 \cdot x)$.

Let $a = (f_1, f_\gamma, \cdots, f_{\gamma^j}, \cdots f_{\gamma^{d-1}}) \in \mathrm{Hom}(\Delta, \Delta(m_1)[t_1]) \cong \bigoplus_{j=0}^{d-1} \mathrm{Jac}(\omega_{\gamma^j})_{m_1 - k_{\gamma^j} + d\frac{t_1 - rkW_{\gamma^j}}{2}}$, and $b = (g_1, g_\gamma, \cdots, g_{\gamma^j}, \cdots, g_{\gamma^{d-1}}) \in \mathrm{Hom}(\Delta, \Delta(m_2)[t_2]) \cong \bigoplus_{j=0}^{d-1} \mathrm{Jac}(\omega_{\gamma^j})_{m_2 - k_{\gamma^j} + d\frac{t_2 - rkW_{\gamma^j}}{2}}$.

Theorem 2.6. *The multiplication map*

$$\Phi : \mathrm{Hom}(\Delta, \Delta(m_1)[t_1]) \times \mathrm{Hom}(\Delta, \Delta(m_2)[t_2]) \to \mathrm{Hom}(\Delta, \Delta(m_1 + m_2)[t_1 + t_2]), (a, b) \mapsto ab.$$

is given by the composition of the following diagram



The left box representing element b is determined by morphism $(g_1, g_{\gamma^2}, \cdots, g_{\gamma^{d-1}})$. *In particular* $g_1 \circ f_1 \in \mathrm{Jac}(\omega)$ *is multiplication of functions.*

Proof. This is essentially the duality of functors $\mathrm{Res}_{\mathbb{C}^*}^{\mathbb{C}^* \times_{\mathbb{C}^*} \mathbb{C}^*}$ and $\mathrm{Ind}_{\mathbb{C}^*}^{\mathbb{C}^* \times_{\mathbb{C}^*} \mathbb{C}^*}$. The element $(\gamma^k, 1) \in \mathbb{C}^* \times_{\mathbb{C}^*} \mathbb{C}^*$ defines an isomorphism

$$\mathrm{Hom}(\mathcal{O}_{\Gamma_{\gamma^i}}, \mathcal{O}_{\Gamma_{\gamma^j}}) \cong \mathrm{Hom}(\mathcal{O}_{\Gamma_{\gamma^{i+k}}}, \mathcal{O}_{\Gamma_{\gamma^{j+k}}}).$$

Since b is a $\mathbb{C}^* \times_{\mathbb{C}^*} \mathbb{C}^*$ invariant morphism, other morphisms except $(g_1, g_\gamma, \cdots, g_{\gamma^{d-1}})$ in the left box are uniquely determined by the $\mathbb{C}^*$ invariant morphisms $g_{\gamma^\bullet}$ via diagonal embedding. After identifying $\mathrm{Hom}(\mathbb{L}\Delta^*\Delta_*\mathcal{O}_{\mathbb{A}^{n+1}}, \mathcal{O}_{\mathbb{A}^{n+1}})$ with certain homogeneous degree of $\mathrm{Jac}(\omega)$, $g_1 \circ f_1$ is the composition of functions, hence multiplication of polynomials. □

Remark 2.7. It is easy to observe that the Hochschild-Serre algebra of the graded matrix factorization is not commutative in general.

3. Kuznetsov's Fano threefold conjecture for quartic double solids and Gushel-Mukai threefolds

Theorem 3.1. *Let Y be a smooth quartic double solid, whose semi-orthogonal decomposition is given by*

$$D^b(Y) = \langle \mathcal{K}u(Y), \mathcal{O}_Y, \mathcal{O}_Y(1) \rangle,$$

where $\mathcal{K}u(Y)$ is the Kuznetsov component of the quartic double solid Y. The canonical map γ_Y induced by multiplication map (1) of Hochschild-Serre algebra

$$\gamma_Y : \mathrm{HH}^2(\mathcal{K}u(Y)) \longrightarrow \mathrm{Hom}(\mathrm{HH}_{-1}(\mathcal{K}u(Y)), \mathrm{HH}_1(\mathcal{K}u(Y))).$$

has one dimensional kernel.

Proof. We regard Y as a degree 4 smooth hypersurface in weighted projective space $\mathbb{P}(1,1,1,1,2)$. According to Theorem 2.2, $\mathcal{K}u_{dg}(Y) \cong \mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^5, \mathbb{C}^*, \mathcal{O}(d), \omega)$, where ω is the polynomial defining Y, and the $\mathbb{C}^*$-action on $(x_0, x_1, x_2, x_3, x_4)$ is of weight $(1,1,1,1,2)$. Then by Proposition 2.4, we have

$$\mathrm{Hom}(\Delta, \Delta(m)[t]) \cong \Big(\bigoplus_{g\in\mu_4,\ t-\mathrm{rk}\,W_g\ \textbf{is even}} \mathrm{Jac}(\omega_g)(m - k_g + d(\frac{t-\mathrm{rk}\,W_g}{2}))\Big)^{\mathbb{C}^*},$$

where $\mu_4 = \{1, i, -1, -i\}$.

- If $g = 1$, then $(\mathbb{A}^5)^g = \mathbb{A}^5$, $\mathrm{rk}\,W_g = 0, k_g = 0$.
- If $g = i$, then $(\mathbb{A}^5)^g = (0,0,0,0,0)$, $\mathrm{rk}\,W_g = 5, k_g = -6$.
- If $g = -1$, then $(\mathbb{A}^5)^g = (0,0,0,0,x_5)$, $\mathrm{rk}\,W_g = 4, k_g = -4$.
- If $g = -i$, then $(\mathbb{A}^5)^g = (0.0,0,0,0)$, $\mathrm{rk}\,W_g = 5, k_g = -6$.

Note that the Serre functor of the matrix factorization category is $-\otimes\mathcal{O}_{\mathbb{A}^5}(-6)[5]$ by Lemma 2.3, We write $\omega = x_5^2 + f(x_1, x_2, x_3, x_4)$, then

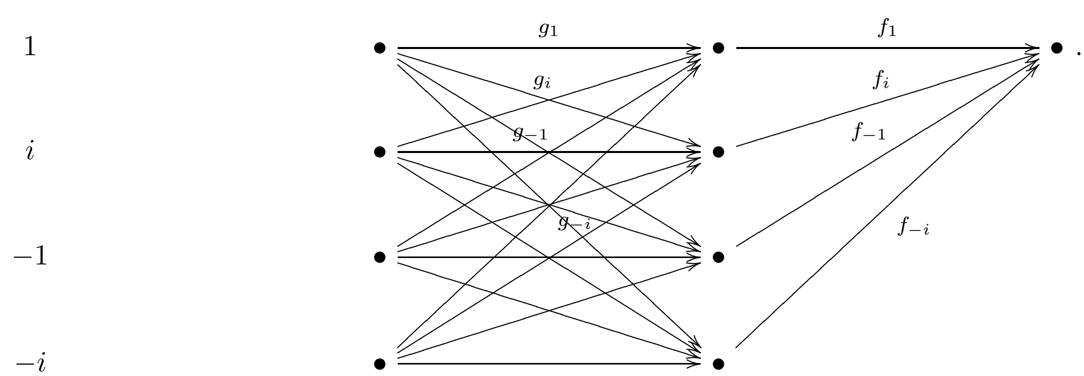
$$\begin{aligned}
\mathrm{HH}_{-1}(\mathcal{K}u(Y)) \cong& \mathrm{Hom}(\Delta, \Delta(-6)[4]) \cong \mathrm{Jac}(\omega)_2 \oplus 0 \oplus \mathrm{Jac}(\omega_{-1})_{-2} \oplus 0 = \mathrm{Jac}(\omega)_2\\
\mathrm{HH}_{1}(\mathcal{K}u(Y)) \cong& \mathrm{Hom}(\Delta, \Delta(-6)[6]) \cong \mathrm{Jac}(\omega)_6 \oplus 0 \oplus \mathrm{Jac}(\omega_{-1})_{2} \oplus 0 = \mathrm{Jac}(\omega)_6\\
\mathrm{HH}^{2}(\mathcal{K}u(Y)) \cong& \mathrm{Hom}(\Delta, \Delta[2]) \cong \mathrm{Jac}(\omega)_4 \oplus 0 \oplus \mathrm{Jac}(\omega_{-1})_{0} \oplus 0 = \mathrm{Jac}(\omega)_4 \oplus k.
\end{aligned}$$

□

According to Theorem 2.6, the composition

$$\mathrm{HH}^2(\mathcal{K}u(Y)) \times \mathrm{HH}_{-1}(\mathcal{K}u(Y)) \longrightarrow \mathrm{HH}_1(\mathcal{K}u(Y))\ .$$

is represented by



Other morphisms except $(g_1, g_i, g_{-1}, g_{-i})$ in the first box are unqiuely determined by $(g_1, g_i, g_{-1}, g_{-i}) \in \mathrm{Jac}(\omega) \oplus \mathrm{Jac}(\omega_i) \oplus \mathrm{Jac}(\omega_{-1}) \oplus \mathrm{Jac}(\omega_{-i})$. Consider element $a = (0, 0, f_{-1}, 0) \in \mathrm{HH}^2(\mathcal{K}u(Y)) \cong \mathrm{Jac}(\omega)_4 \oplus 0 \oplus \mathrm{Jac}(\omega_{-1})_0 \oplus 0 = \mathrm{Jac}(\omega)_4 \oplus k$ and $b = (g_1, 0, 0, 0) \in \mathrm{HH}_{-1}(\mathcal{K}u(Y)) \cong \mathrm{Jac}(\omega)_2 \oplus 0 \oplus 0 \oplus 0$. Then,

$$(0, 0, f_{-1}, 0) \cdot (g_1, 0, 0, 0) = (g_1 \circ 0 + 0 \circ f_{-1}, 0, g_1 \circ f_{-1}, 0) = 0 \in \mathrm{HH}_1(\mathcal{K}u(Y)) \cong \mathrm{Jac}(\omega)_6.$$

Simple computation shows $\mathrm{Jac}(\omega)_4 = \mathrm{Jac}(f)_4$, $\mathrm{Jac}(\omega)_2 = \mathrm{Jac}(f)_2$, and $\mathrm{Jac}(\omega)_6 = \mathrm{Jac}(f)_6$. The map

$$\mathrm{Jac}(\omega)_4 \to \mathrm{Hom}(\mathrm{Jac}(\omega)_2, \mathrm{Jac}(\omega)_6).$$

is injective since it is induced by the non degenerate multiplication [Don83, Theorem 2.6],

$$\mathrm{Jac}(f)_4 \times \mathrm{Jac}(f)_2 \to \mathrm{Jac}(f)_6.$$

Hence the canonical map

$$\gamma_Y : \mathrm{HH}^2(\mathcal{K}u(Y)) \to \mathrm{Hom}(\mathrm{HH}_{-1}(\mathcal{K}u(Y)), \mathrm{HH}_1(\mathcal{K}u(Y))).$$

has one dimensional kernel.

Lemma 3.2. *[JLLZ22, Theorem 4.6] Let X be an ordinary GM threefold. Then the natural map*

$$\gamma_X : \mathrm{HH}^2(\mathcal{K}u(X)) \to \mathrm{Hom}(\mathrm{HH}_{-1}(\mathcal{K}u(X)), \mathrm{HH}_1(\mathcal{K}u(X))).$$

is injective.

Proof. The map γ in [JLLZ22, Theorem 4.6] is related to γ_X as

$$\begin{array}{ccc} \mathrm{HH}^2(\mathcal{K}u(X)) & \xrightarrow{\gamma_X} & \mathrm{Hom}(\mathrm{HH}_{-1}(\mathcal{K}u_X), \mathrm{HH}_1(X)) \; . \\ & \searrow^{\gamma} & \downarrow \simeq \\ & & \mathrm{Hom}(H^{2,1}(X), H^{1,2}(X)) \end{array}$$

Since γ is injective, γ_X is injective. □

Corollary 3.3. *For any Gushel-Mukai threefold X and quartic double solid Y, there is no Fourier-Mukai type equivalence between the category $\mathcal{K}u(X)$ and $\mathcal{K}u(Y)$.*

Proof. Assume there is a Fourier-Mukai type equivalence $\Phi : \mathcal{K}u(Y) \simeq \mathcal{K}u(X)$ for any quartic double solid Y and ordinary Gushel-Mukai threefold X. Then [JLLZ22, Theorem 4.8] tells us the morphism γ_X is injective if and only if γ_Y is injective. By Remark 1.4, we can assume X is an ordinary Gushel-Mukai threefold, then γ_X is injective by Lemma 3.2. Thus γ_Y is also injective, which contradicts Theorem 3.1. □

Remark 3.4. In this paper, we work with dg-enhanced Kuznetsov categories, so any equivalence between them amounts to a Fourier-Mukai type equivalence. But in the cases of interest in this paper, all the equivalences between triangulated categories $\mathcal{K}u(X)$ and $\mathcal{K}u(Y)$ are proved to be of Fourier-Mukai type in [LPZ22], so there is no harm to work with enhanced Kuznetsov components.

4. Appendix: Infinitesimal Torelli theorem for Gushel-Mukai threefolds

In [DIM08, Theorem 7.1], the authors show the differential $d\mathcal{P}$ of the period map $\mathcal{X} \to \mathcal{A}_{10}$ of ordinary Gushel-Mukai threefold X has two-dimensional kernel.

Proposition 4.1. *The kernel of* $d\mathcal{P} : H^1(X, T_X) \to \mathrm{Hom}(H^{2,1}(X), H^{1,2}(X))$ *is two dimensional.*

In this section, we prove Proposition 4.1 for *special* Gushel-Mukai threefold using categorical methods.

Lemma 4.2. *Let X be a special Gushel-Mukai threefold, then the morphism*

$$\gamma_X : \mathrm{HH}^2(\mathcal{K}u(X)) \to \mathrm{Hom}(\mathrm{HH}_{-1}(\mathcal{K}u(X)), \mathrm{HH}_1(\mathcal{K}u(X)))$$

is injective.

Proof. By Remark 1.4, there is an ordinary Gushel-Mukai threefold X' such that $\Phi\colon \mathcal{K}u(X') \simeq \mathcal{K}u(X)$ is a Fourier-Mukai type equivalence. Then by [JLLZ22, Theorem 4.8], injectivity of γ_X is equivalent to injectivity of $\gamma_{X'}$. By Lemma 3.2, $\gamma_{X'}$ is injective. Thus γ_X is injective. □

By [JLLZ22, Corollary 3.16], there is a commutative diagram:

$$\begin{array}{ccc} \mathrm{HH}^2(\mathcal{K}u(X)) & \xrightarrow{\quad\gamma'\quad} & \mathrm{Hom}(\mathrm{H}^{2,1}(X), \mathrm{H}^{1,2}(X) \\ \uparrow{\scriptstyle\eta} & \nearrow{\scriptstyle \mathrm{d}\mathcal{P}} & \\ H^1(X, T_X) & & \end{array} .$$

Injectivity of γ' is equivalent to injectivity of γ_X by the proof of Lemma 3.2. By Lemma 4.2, γ_X is injective, hence γ' is injective. Note that $d\mathcal{P} = \gamma' \circ \eta$, then $\mathrm{Ker}(d\mathcal{P}) = \mathrm{Ker}\eta$.

Recall the semi-orthogonal decomposition for (ordinary)Gushel-Mukai threefold X' is given by

$$D^b(X') = \langle \mathcal{K}u(X'), \mathcal{O}_{X'}, \mathcal{U}^\vee_{X'} \rangle.$$

Then by [Kuz15, Theorem 3.3], there is a long exact sequence

$$\ldots \to \mathrm{HH}^1(\mathcal{K}u(X')) \to \mathrm{NHH}^2(\langle \mathcal{O}_{X'}, \mathcal{U}^\vee_{X'} \rangle, X') \to \mathrm{HH}^2(X') \to \mathrm{HH}^2(\mathcal{K}u(X')) \to \ldots$$

where $\mathrm{NHH}^2(\langle \mathcal{O}_{X'}, \mathcal{U}^\vee_{X'} \rangle, X')$ is the normal Hochschild cohomology of $\langle \mathcal{O}_{X'}, \mathcal{U}^\vee_{X'} \rangle$ in $D^b(X')$ defined in [Kuz15, Definition 3.2]. Note that $\mathrm{HH}^2(X') \cong \mathrm{H}^2(X', \mathcal{O}_{X'}) \bigoplus \mathrm{H}^1(X', T_{X'}) \bigoplus \mathrm{H}^0(X', \bigwedge^2 T_{X'}) = \mathrm{H}^1(X', T_{X'})$. On the other hand, by [KP18, Proposition 2.12], $\mathrm{HH}^1(\mathcal{K}u(X')) = 0$. Then the long exact sequence above becomes

$$\ldots 0 \to \mathrm{NHH}^2(\langle \mathcal{O}_{X'}, \mathcal{U}^\vee_{X'} \rangle, X') \xrightarrow{i} H^1(X', T_{X'}) \xrightarrow{\eta'} \mathrm{HH}^2(\mathcal{K}u(X')) \to \ldots.$$

Thus $\mathrm{Ker}(\eta') = \mathrm{Im}(i) \cong \mathrm{NHH}^2(\langle \mathcal{O}_{X'}, \mathcal{U}^\vee_{X'} \rangle, X')$. Then by [JLLZ22, Remark 4.7], $\mathrm{NHH}^2(\langle \mathcal{O}_{X'}, \mathcal{U}^\vee_{X'} \rangle, X') \cong \mathrm{Ker}(\eta') = k^2$. From the computation in [Kuz15, Section 3.3] and in particular [Kuz15, Proposition 3.7], the normal Hochschild cohomology $\mathrm{NHH}^2(\langle \mathcal{O}_{X'}, \mathcal{U}^\vee_{X'} \rangle, X') \cong k^2$ by replacing X' by a special Gushel-Mukai threefold X. Thus $\mathrm{Ker}\eta \cong k^2$. Then $\mathrm{Ker}(d\mathcal{P}) \cong k^2$.

Remark 4.3. The key to computation of normal Hochschild cohomology $\mathrm{NHH}^2(\langle\mathcal{O}_X,\mathcal{U}_X^\vee\rangle, X)$ for either ordinary Gushel-Mukai threefold or special Gushel-Mukai threefold X is surjectivity of the map $\mathrm{Hom}(\mathcal{O}_X,\mathcal{U}_X^\vee)^{\otimes 2}\xrightarrow{p}\mathrm{Hom}(\mathcal{O}_X,\mathcal{U}_X^\vee\otimes\mathcal{U}_X^\vee)$ and the spectral sequence (30) in [Kuz15, Section 3.3]. We leave the details to interested reader.

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