# Integral invariants for framed 3-manifolds associated to trivalent graphs possibly with self-loops 

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#### Abstract

Bott-Cattaneo's theory defines the integral invariants of framed rational homology 3 -spheres with acyclic orthogonal local systems associated to graph cocycles without self-loops. The 2-loop term of their invariants is associated with the Theta graph. Their invariants can be defined when a cohomological condition holds. Cattaneo-Shimizu gave a refinement of the 2-loop term of BottCattaneo invariants by removing this cohomological condition, their 2-loop term is associated with a linear combination of the Theta graph and the dumbbell graph that is the only 2-loop trivalent graph with self-loops. In this article, when an acyclic local system is given by the adjoint representation of a semi-simple Lie group composed with a representation of the fundamental group of a closed 3manifold, we show that the associated integral of dumbbell graph can be vanished by a cohomological reason. Based on this idea, we construct a theory of graph complexes and cocycles, so that higherloop invariants can be defined using both the graph cocycles without self-loop, as by Bott-Cattaneo, and with self-loops, as by Cattaneo-Shimizu. As a consequence, we prove that the generating series from Chern-Simons perturbation theory gives rise to topological invariants for framed 3 -manifolds in our setting, which admits a formula in terms of only trivalent graphs without self-loop.


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## 1 Introduction

The mathematical foundation of Chern-Simons perturbation theory was developed by Axelrod-Singer [AS92, AS94] and Kontsevich [Kon94] around 1990s, after the breakthrough of Witten's work [Wit89] on the Jones polynomials via Chern-Simons theory. Their theories provide a family of topological invariants, parametrized by a linear combination of trivalent graphs. These invariants pertain to closed, oriented, connected smooth 3-manifolds, denoted as $M$, and are associated to a homotopy class of smooth framings of $M$ (namely, smooth trivializations of tangent bundle TM) and an acyclic local system $E_{\rho}$ over $M$. The concerned local system $E_{\rho}:=\pi_{1}(M) \backslash\left(\widetilde{M} \times_{\rho} \mathfrak{g}\right)$ is defined by a representation $\rho: \pi_{1}(M) \rightarrow G \xrightarrow{\operatorname{Ad}} \operatorname{Aut}(\mathfrak{g})$, where $\widetilde{M}$ denotes universal cover of $M, G$ is a (connected) semisimple Lie group with Lie algebra $\mathfrak{g}$, and Ad is the adjoint representation of $G$. Their invariants are defined as the configuration space integrals of the propagators, and the way of defining integrands is encoded by the trivalent graphs. We will always refer to such invariants as the integral invariants for $M$ and $E_{\rho}$.

Inspired by their work, Bott and Cattaneo [BC98, BC99] introduced topological invariants of framed rational homology 3 -spheres with acyclic orthogonal local systems. Their definition is based on the implicit assumption of the vanishing of a cohomology group (denoted by $H_{-}^{2}\left(\Delta ; \pi_{1}^{-1} E_{\rho} \otimes \pi_{2}^{-1} E_{\rho}\right)$ in their article [BC99] and $H_{-}^{2}\left(\Delta ; E_{\rho} \otimes E_{\rho}\right)$ in [CS21] and in the present article, $\Delta \simeq M$ denotes the diagonal of $M \times M$ ). Their invariants (of arbitrary orders) are defined in terms of integral invariants associated to an appropriate graph complex without self-loops (self-loops here are called simple loops in [Kon94]).

In 2021, Cattaneo and Shimizu [CS21] pointed out that there is a gap in the arguments of [BC99, Lemma 1.2] about the vanishing of the cohomology group $H_{-}^{2}\left(\Delta ; E_{\rho} \otimes E_{\rho}\right)$, and then gave a refinement of the 2-loop term of Bott-Cattaneo invariants removing the assumption of the vanishing of $H_{-}^{2}\left(\Delta ; E_{\rho} \otimes E_{\rho}\right)$. The presence of this cohomology group implies that the boundary value of a propagator can have a factor of non-trivial anti-symmetric form $\xi$, analog to the regular part of the propagator constructed in [AS92, AS94] via the Green kernels. This regular form $\xi$ leads to configuration space integrals associated with trivalent graphs with self-loops, and such graphs are not needed in [BC98, BC99] under the assumption $H_{-}^{2}\left(\Delta ; E_{\rho} \otimes E_{\rho}\right)=0$. The refined 2-loop term defined in [CS21] is given by a linear combination of $Z_{\Theta}$ and $Z_{0-0}$, configuration space integrals corresponding to Theta graph and dumbbell graph respectively, where the form $\xi$ will be associated to the self-loops of the dumbbell graph.

Note that, Shimizu [Shi23] also showed that, when $G=\mathrm{SU}(2)$, the cohomology group $H_{-}^{2}\left(\Delta ; E_{\rho} \otimes\right.$ $\left.E_{\rho}\right)$ always vanishes and $Z_{\Theta}$ itself becomes an invariant of closed 3-manifolds with orthogonal local systems. Also note that a class of regular form $\xi$ has recently been studied by Kitano and Shimizu (see [Shi21], [KS23]), motivated by its relation to Reidemeister torsion expected from the viewpoint of quantum Chern-Simons theory.

Therefore, it is important to ask for the existence of examples of a pair of a closed 3-manifold and an acyclic local system via adjoint representation as above that have non-vanishing $H_{-}^{2}\left(\Delta ; E_{\rho} \otimes E_{\rho}\right)$. Such a result assures that the refinement by Cattaneo-Shimizu in [CS21] is meaningful.

In Proposition 3.5.5, we first report a class of examples fulfilling this purpose, which also lies in the framework of Chern-Simons perturbation theory. Following this direction, we investigate further the integral invariants of Cattaneo-Shimizu/Bott-Cattaneo defined from trivalent graphs for a closed

3-manifold and an acyclic local system. In particular, we aim to understand the role of graphs with self-loops.

Our first result follows from the reexamination of the 2-loop invariant introduced in [CS21], we found that, even when $H_{-}^{2}\left(\Delta ; E_{\rho} \otimes E_{\rho}\right) \neq 0$, there is a special choice of propagator by which the integration associated with dumbbell graph vanish, so that, roughly speaking, the essential part of this 2-loop invariant is the term of Theta graph, as in [BC98, BC99].

First, let's introduce the propagators. Let $C_{2}(M)$ denote the compactified 2-point configuration space of $M$, we can think of it as the real blow-up of $M^{2}$ along the diagonal $\Delta \simeq M$. The manifold $C_{2}(M)$ is a smooth manifold with boundary, and the boundary $\partial C_{2}(M)$ can be identified with the sphere tangent bundle $S(T M)$ of $M$. We will denote $\mathfrak{i}_{\partial}: \partial C_{2}(M) \rightarrow C_{2}(M)$ the inclusion.

Let $q: C_{2}(M) \rightarrow M \times M$ denote this blow-up map, and let $q_{\partial}: \partial C_{2}(M) \rightarrow M$ denote its restriction to the boundary, which is a smooth fibration with fibre $\mathbb{S}^{2}$. We always fix an orientation $o(M)$ for $M$, and we also fix a smooth framing $f$ of $M$, i.e., a smooth identification of vector bundles $T M \simeq M \times \mathbb{R}^{3}$ over $M$. This way, we identify $\partial C_{2}(M) \simeq M \times \mathbb{S}^{2}$.

We always fix a connected semi-simple Lie group $G$ with Lie algebra $\mathfrak{g}$. Note that $G$ could be noncompact. As we mentioned, we consider a representation $\rho: \pi_{1}(M) \rightarrow G \xrightarrow{\text { Ad }}$ Aut $(\mathfrak{g})$ and the associated local system $E_{\rho}$ over $M$. Correspondingly, we have the induced tensor bundle $E_{\rho} \boxtimes E_{\rho}$ on $M \times M$, hence after taking its pullback bundle by the blow-up map $q$, we get a flat vector bundle $F_{\rho}:=q^{*}\left(E_{\rho} \boxtimes E_{\rho}\right)$ on $C_{2}(M)$. The restriction of $F_{\rho}$ on $\partial C_{2}(M) \simeq M \times \mathbb{S}^{2}$ is just the pullback of vector bundle $E_{\rho} \otimes E_{\rho} \rightarrow M$. Meanwhile, we also define an involution $T$ acting on $M \times M$ and on $E_{\rho} \boxtimes E_{\rho}$ by swapping two factors in the product or tensor product, this action lifts to $F_{\rho} \rightarrow C_{2}(M)$ and therefore on the sections and cohomology groups.

Assume $E_{\rho}$ to be acyclic, i.e., the corresponding (de Rham) cohomology group $H^{\bullet}\left(M ; E_{\rho}\right)=0$. In this case, a propagator is a closed differential form $\omega \in \Omega^{2}\left(C_{2}(M) ; F_{\rho}\right)$ such that

- it is anti-symmetric, i.e., $T^{*} \omega=-\omega$ or we say $\omega \in \Omega_{-}^{2}\left(C_{2}(M) ; F_{\rho}\right)$, where the subscript corresponds to ( -1 )-eigenvalue of the action of $T$;
- there is a normalized volume form $\eta$ on $\mathbb{S}^{2}$ (i.e., with volume 1) and a closed smooth form $\xi \in \Omega_{-}^{2}\left(M ; E_{\rho} \otimes E_{\rho}\right)$ such that

$$
\begin{equation*}
\mathfrak{i}_{\partial}^{*}(\omega)=\eta \otimes \mathbf{1}+q_{\partial}^{*}(\xi), \tag{1.0.1}
\end{equation*}
$$

where $\eta$ is viewed as a fibrewise vertical volume along the fibration $q_{\partial}: \partial C_{2}(M) \rightarrow M$ (hence depending on the framing $f$ ), and $\mathbf{1}$ is a flat section, called Casimir section(see Lemma 3.4.1), of $E_{\rho} \otimes E_{\rho}$ over $M$.
In (1.0.1), the closed form $\xi$ defines a cohomological class $[\xi] \in H_{-}^{2}\left(M ; E_{\rho} \otimes E_{\rho}\right)$, and generally can not be eliminated when $H_{-}^{2}\left(M ; E_{\rho} \otimes E_{\rho}\right) \neq 0$. This term in the boundary condition for a propagator was missing in [BC99, Lemma 1.2] and then studied by [CS21] to define the 2-loop integral invariant $Z_{\mathrm{O}-\mathrm{O}}(\omega, \xi)$ associated to the dumbbell graph.

Since $E_{\rho}$ is assumed to be acyclic, the existence and certain uniqueness of propagators are already given in [BC99] and [CS21]. In fact, in Proposition 4.3.1, we prove the existence of the propagators for the adjoint local system $E_{\rho}$ which is not necessary to be acyclic. The non-vanishing of $H^{\bullet}\left(M ; E_{\rho}\right)$ requires more constraints and the propagators are no longer closed forms.

In the introduction and in the most part of this article, we will restrict ourselves to the acyclic case. Now we summarize the results for acyclic case proved in Propositions 4.3.1 \& 4.3.4 and Corollary 4.3.2.

Theorem A (Existence and cohomological uniqueness of propagators, cf. [BC99, Lemma 1.2], [CS21, Proposition 2.1]). Given a orientable closed 3-manifold $M$ with a fixed orientation o(M), and a homotopy class $[f]$ of smooth framings of $M$. Let $E_{\rho}$ be an acyclic local system on $M$ associated to a representation as above. Then there always exists a propagator $\omega$, and the cohomological class $[\omega] \in H_{-}^{2}\left(C_{2}(M) ; F_{\rho}\right)$ is uniquely determined by $(M, o(M),[f])$ and $E_{\rho}$; moreover, the cohomological class $[\xi] \in H_{-}^{2}\left(M ; E_{\rho} \otimes E_{\rho}\right)$ is also unique, where $\xi$ is the regular part of the boundary value (or simply, regular form) of $\omega$.

In this article, we notice that the Lie algebra structure of $\mathfrak{g}$, i.e., the Lie bracket $\mathfrak{L}: \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$, lifts naturally to a morphism of vector bundles $\mathfrak{L}: E_{\rho} \otimes E_{\rho} \rightarrow E_{\rho}$, and this extra structure leads to a special kind of propagators for $E_{\rho}$ which will help to understand the role of the trivalent graphs with self-loops. Such terms with self-loops are the necessary corrections for the integral invariants due to the (nontrivial) Jacobi identity on $E_{\rho}$, where the term given by the Lie bracket acting the boundary value of a propagator is needed to complete the whole Jacobi identity to get the invariance of the integral. For the setting of the trivial local system $E_{\rho}=\mathbb{R}$ on $M$, e.g. in [BC98], the Jacobi identity is reduced to be trivial so that such consideration was not touched.

In Definition 4.4.1, we give the following definition: for the acyclic $E_{\rho}$, a propagator $\omega^{\sharp}$ is called adapted (to $E_{\rho}$ ) if the regular part $\xi^{\sharp}$ in its boundary value satisfies the condition

$$
\begin{equation*}
\mathfrak{L}\left(\xi^{\sharp}\right)=0 . \tag{1.0.2}
\end{equation*}
$$

Here we usually put the superscript $\sharp$ to emphasize the propagator being adapted. In Theorem 4.4.2, we prove the existence of adapted propagators for acyclic local systems. Moreover, we can take $\xi^{\sharp} \equiv 0$ when $H_{-}^{2}\left(M ; E_{\rho} \otimes E_{\rho}\right)=0$, which gives the framework in [BC99].

After revisiting the 2-loop invariants introduced in [CS21], we refine their result as follows.
Theorem B (see Theorem 5.2.1). Let $M$ be a closed, connected, orientable smooth 3-manifold. Fix a homotopy class $[f]$ of smooth framings of $M$ and an orientation $o(M)$. Let $E_{\rho}$ be an acyclic local system over $M$ associated with a representation $\rho: \pi_{1}(M) \rightarrow G \xrightarrow{\text { Ad }} \operatorname{Aut}(\mathfrak{g})$. Then, for any adapted propagator $\omega^{\sharp}$ with the regular form $\xi^{\sharp}$, we have $Z_{\mathrm{O}-\mathrm{O}}\left(\omega^{\sharp}, \xi^{\sharp}\right)=0$. In other words, the Theta invariant $Z_{\Theta}\left(\omega^{\sharp}\right)$ gives a 2-loop invariant for a framed closed 3-manifold $M$ and $\rho$.

From this theorem, we can shed some light on trivalent graphs with self-loops. These graphs give the necessary corrections in the configuration space integrals of propagators, since it's possible to connect two different vertices with several edges, leading to the appearance of the $\xi$-term when computing variations of the integral invariants with respect to different choices of propagators. However, we can eliminate these correction terms by taking adapted propagators.

Based on this observation, we generalize the result in Theorem B to higher-loop invariants associated with trivalent graphs with or without self-loops, formulated in terms of certain graph complexes. One consequence is that the Chern-Simons perturbative series fits into this framework so that we prove this series indeed defines integral invariants for framed $(M, o(M))$ and acyclic $E_{\rho}$.

In Section 6, a graph complex (over $\mathbb{Q}$ ) of decorated graphs (see Definition 6.3.1) possibly with self-loops, denoted by $\left(\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}}, \delta\right)$, is defined, which is dedicated to acyclic local systems and only depends on the semi-simple Lie algebra $\mathfrak{g}$. The differential operator $\delta$ is given by contraction on each non-self-loop edge of the decorated graphs. In our convention, each vertex of the graph has at least 3 incident half-edges, so that the degree- 0 subspace of $\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}}$ is exactly spanned by the decorated trivalent graphs, which is isomorphic to the linear space spanned by topological trivalent graphs (see Subsection 6.6). The order of a trivalent graph is defined as half of its total number of vertices.

Then we get two associated graph complexes: one is the subcomplex $\left(\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}}^{\prime}, \delta\right)$ of the above one which consists of decorated graphs with at least one self-loop, and the second complex ( $\mathcal{G}_{\mathrm{ac}, \mathfrak{g}}, \delta^{\sharp}$ ) is the quotient complex $\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}} / \mathcal{G C}_{\mathrm{ac}, \mathfrak{g}}^{\prime}$, which consists of linear combinations of the decorated graphs without self-loops.

The construction of such graph complexes is already sketched in [Kon94, Section 2], [BC99, §4 Discussion], and similar (but not the same) constructions can also be found in [Igu04, Section 1] (without the gauge group $G$ ) and [CV03, §2]. Note that, different from the aforementioned constructions, we do not exclude the graphs with self-loops in our graph complex $\left(\mathcal{G C}_{\text {ac, }}, \delta\right)$. The ideas on the decorated graphs and weight systems are also exploited to define the associated configuration space integrals in [AS92, AS94], [BN95], and [Les04, Les20], etc. In Section 6, a graphic way is employed to explain clearly the construction.

An element in $\operatorname{Ker} \delta \subset \mathcal{G C}_{\mathrm{ac}, \mathfrak{g}}$ or in $\operatorname{Ker} \delta^{\sharp} \subset \mathcal{G}_{\mathrm{ac}, \mathfrak{g}}$ is called a graph cocycle in the respective graph complexes. Then we are mainly concerned with the cocycles of degree 0 and order $n \geq 1$ (hence with $(n+1)$ loops ): $H^{0}\left(\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: n}^{\bullet}, \delta\right), H^{0}\left(\mathcal{G}_{\mathrm{ac}, \mathfrak{g}: n}^{\bullet}, \delta^{\sharp}\right)$. In particular, the 2-loop cocycles $H^{0}\left(\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: 1}^{\bullet}, \delta\right)$ is 1-dimensional and spanned by a certain linear combination of the Theta graph and dumbbell graph, which is implicitly used in [CS21].
Theorem C (For precise statement, see Theorem 7.1.5 \& Theorem 7.1.8). Fix a homotopy class $[f]$ of smooth framings of $M$ and an orientation $o(M)$. Let $E_{\rho}$ be an acyclic local system on $M$ corresponding to $\rho: \pi_{1}(M) \rightarrow G \xrightarrow{\text { Ad }} \operatorname{Aut}(\mathfrak{g})$ as above. Fix an order $n \geq 1$. Any cocycle $H^{0}\left(\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: n}^{\bullet}, \delta\right)$ (i.e., $\delta \Gamma=0$ ), taking the associated configuration space integrals of any propagator for $E_{\rho}$, gives rise to an integral invariant $Z(M, \rho,[f])(\Gamma) \in \mathbb{R}$, which is independent of the choice of the propagator that is used to define it. Similarly, a cocycle $\Gamma^{\prime} \in H^{0}\left(\mathcal{G}_{\mathrm{ac}, \mathfrak{g}: n}^{\bullet}, \delta^{\sharp}\right)$ consisting of connected trivalent graph without self-loops, by using the adapted propagator to define the associated configuration space integral, gives rise to an integral invariant $Z^{\sharp}(M, \rho,[f])\left(\Gamma^{\prime}\right) \in \mathbb{R}$.

Moreover, regarding $Z(M, \rho,[f]), Z^{\sharp}(M, \rho,[f])$ as linear functionals on the cocycles, the following diagram commutes:
where the leftmost vertical map is given by sending graphs with self-loops to zero.
These results imply that the computation of integral invariants associated with trivalent graphs possibly with self-loops can be reduced to those associated with trivalent graphs without self-loops. Note that the reason for the non-necessity of self-loop in the graphs for invariant $Z^{\sharp}(M, \rho,[f])$ is different from trivial local system case, for instance, described in [Kon94, § The graph complex].

Now we explain how Chern-Simons perturbation theory fits into our results. The general idea from Chern-Simons perturbation theory is to define the topological invariants for an oriented 3manifold together with a local system via the configuration space integrals associated to a generating series in terms of the linear combinations of trivalent graphs with given orders (see [Kon94, Section 2], [AS92, AS94], also [Saw06, §3] for an introduction on the finite-dimensional model). For given order $n \geq 1$, the corresponding term in the generating series is given as

$$
\begin{equation*}
\sum_{\substack{\text { connected trivalent } \mathfrak{G} \\ \text { ord }(\mathfrak{G})=n}} \frac{1}{|\operatorname{Aut}(\mathfrak{G})|} \pm \Gamma(\mathfrak{G}) \in \mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: n}^{0}, \tag{1.0.4}
\end{equation*}
$$

where the sum runs over all the connected topological trivalent graph $\mathfrak{G}$ of order $n$, and $\Gamma(\mathfrak{G})$ denotes an arbitrary decorated graph with the underlying topological graph $\mathfrak{G}$ with the sign $\pm$ determined by it.

In Proposition 6.6.6, we prove that the element (1.0.4) is a cocycle in $\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: n}^{0}$; if we remove all the terms with self-loops from (1.0.4), we get a cocycle in $\mathcal{G}_{\mathrm{ac}, \mathfrak{g}: n}^{0}$. Combining with Theorem C, we get the following results.
Corollary D (See Corollary 7.1.9). Let $M$ be a closed, connected, orientable smooth 3-manifold. Fix a homotopy class $[f]$ of smooth framings of $M$ and an orientation $o(M)$. Let $E_{\rho}$ be an acyclic local system over $M$ associated with a representation $\rho: \pi_{1}(M) \rightarrow G \xrightarrow{\text { Ad }} \operatorname{Aut}(\mathfrak{g})$. Then the following formal series

$$
\begin{equation*}
\mathcal{Z}_{\mathrm{CS}}(M, \rho,[f]):=\sum_{\text {connected trivalent } \mathfrak{G}} \frac{\hbar^{\operatorname{ord}(\mathfrak{G})}}{|\operatorname{Aut}(\mathfrak{G})|} Z(M, \rho,[f])( \pm \Gamma(\mathfrak{G})) \in \mathbb{R}[[\hbar]] \tag{1.0.5}
\end{equation*}
$$

is a topological invariants for $(M, o(M),[f])$ and $\rho$. This series $\mathcal{Z}_{\mathrm{CS}}(M, \rho,[f])$ can also be written in terms of $Z^{\sharp}(M, \rho,[f])$ and the connected trivalent topological graphs without self-loop, i.e.,

$$
\mathcal{Z}_{\mathrm{CS}}(M, \rho,[f])=\sum_{\begin{array}{c}
\text { connected trivalent }  \tag{1.0.6}\\
\text { without self-loop } \boldsymbol{G}
\end{array}} \frac{\hbar^{\operatorname{ord}(\mathfrak{G})}}{|\operatorname{Aut}(\mathfrak{G})|} Z^{\sharp}(M, \rho,[f])( \pm \Gamma(\mathfrak{G})) \in \mathbb{R}[[\hbar]] .
$$

Even in most parts of the article we are dealing with the acyclic local system $E_{\rho}$, we are still trying to share some ideas on the case of nonacyclic local system $E_{\rho}$ : in Subsection 5.3, a preliminary computation on the Theta invariant is given under the only assumption $H^{1}\left(M ; E_{\rho}\right)=0$. Moreover, following the framework as in Sections $6 \& 7$, we will formulate a quantum master equation associated to the nonacyclic local system $E_{\rho}$ in a future article from the point of view of the graph complex introduced as in Subsections 6.3 \& 6.4 (cf. Costello [Cos07, Cos11], Iacovino [Iac10], Cattaneo-Mnëv [CM10], Campos-Willwacher [CW23]).

Finally, we give a remark that our construction is applicable to both real and complex semisimple Lie groups. Thus, for simplicity of arguments, this article mainly focuses on the case of real semisimple Lie groups unless otherwise stated.

The organization of this article is as follows. In Section 2, for the convenience of readers, we recall the basis for the compactified configuration spaces via the point-view of smooth manifolds with corners.

In Section 3, we investigate the de Rham cohomology groups of a local system $E_{\rho}$ associated to a representation $\rho: \pi_{1}(M) \rightarrow G \xrightarrow{\text { Ad }} \operatorname{Aut}(\mathfrak{g})$, especially, consequences of the Lie structure on $E_{\rho}$.

In Section 4, we prove the existence of the propagators by explicit construction, and we introduce the notion of adapted propagators for acyclic local system $E_{\rho}$.

In Section 5, we revisit the result of Cattaneo-Shimizu [CS21] for their 2-loop integral invariant, then explain how the use of an adapted propagator kills the dumbbell term.

In Section 6, we introduce our version of graph complexes (which only involve the Lie algebra $\mathfrak{g}$ ) which is used in our theory of integral invariants. In particular, the generating series from ChernSimons perturbation theory is a cocycle.

In Section 7, we prove that for $E_{\rho}$ being acyclic, each cocycle in our complexes defines an integral invariant for the framed 3 -manifold $M$ and the representation $\rho$. The use of adapted propagators reduces the cocycles to the ones without any self-loops.

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## Notation

Let $\mathbb{N}\left(\right.$ resp. $\left.\mathbb{N}_{0}\right)$ denote the set of natural numbers without zero (resp. with zero). We denote the cardinality of a set $S$ by $|S|$. For a non-negative integer $r \geq 1$, let $\mathbb{S}^{r-1}$ denote the unit sphere with the induced metric from the $r$-dimensional Euclidean space $\mathbb{R}^{r}$.

For a graded vector space $V=\bigoplus_{i \in \mathbb{Z}} V^{i}$, the degree of a homogeneous element $v \in V^{i}$ is denoted by $\operatorname{deg}(v)=i$. For two graded algebras $A$ and $B$ over a field, let $A \widehat{\otimes} B$ stand for the graded tensor product of $A$ and $B$ over the field, i.e., its underlying vector space is the tensor product $A \otimes B$ and product structure is given by the linear extension of $(a \otimes b) \cdot\left(a^{\prime} \otimes b^{\prime}\right)=(-1)^{\operatorname{deg}(b) \cdot \operatorname{deg}\left(a^{\prime}\right)}\left(a a^{\prime} \otimes b b^{\prime}\right)$ for homogeneous elements $a, a^{\prime} \in A$ and $b, b^{\prime} \in B$. Depending on the context, $A^{\bullet}$ denotes a graded vector space $A^{\bullet}=\bigoplus_{i \in \mathbb{Z}} A^{i}$, a cochain complex $\left(\bigoplus_{i \in \mathbb{Z}} A^{i}, \partial\right)$, or arbitrary homogeneous component of a graded vector space $\bigoplus_{i \in \mathbb{Z}} A^{i}$, where $\bullet$ plays a role of the placeholder of homogeneous degree.

For an oriented smooth manifold (with corners) $X$ and a local system $E$ of a real or complex vector space on $X$, we denote by $\Omega^{\bullet}(X)$ (resp. $\Omega^{\bullet}(X ; E)$ ) the differential graded commutative algebra of (resp. $E$-valued) smooth differential forms on $X$. For a commutative ring $R$ with a unit and an $R$-module $V$, the constant local system (trivial local system) on $X$ with fiber $V$ is denoted by $V_{X}$ or simply $\underline{V}$.

In this article, we often identify a flat vector bundle and its corresponding local system.

## 2 Preliminary on the geometry of configuration spaces

The compactification of configuration spaces is the ground for the definition of the integral invariants for a 3-manifold. Here, we will follow the constructions given by Axelrod-Singer [AS94], where the compactified configuration spaces are regarded as smooth manifolds with corners.

### 2.1 Manifolds with corners

To describe the geometry of the compactified configuration spaces, we will use the framework of smooth manifolds with corners. For a detailed introduction to the manifold with corners, we refer to [Joy12], a partial note of Melrose [Mel], and the references therein. Set $\mathbb{R}_{+}=[0, \infty[$.

The model spaces to build a manifold with corners of dimension $m$ are

$$
\mathbb{R}_{k}^{m}:=\mathbb{R}_{+}^{k} \times \mathbb{R}^{m-k}, 0 \leq k \leq m
$$

A topological manifold with corners is identical to a topological manifold with boundary, but the smooth structure is built via the above model spaces. Let $X$ be a second-countable Hausdorff topological space. An $m$-dimensional chart with corners in $X$ is a triplet $\left(U, \phi, \phi(U) \subset \mathbb{R}_{k}^{m}\right)$ for some $0 \leq k \leq m$ such that $U$ is an open subset of $M$ and $\phi(U)$ is open in $\mathbb{R}_{k}^{m}$, and the map $\phi: U \rightarrow \phi(U)$ is a homeomorphism. If $\left(U^{\prime}, \phi^{\prime}, \phi^{\prime}\left(U^{\prime}\right) \subset \mathbb{R}_{k^{\prime}}^{m}\right)$ is another such chart in $M$ ( $k^{\prime}$ might be different from $k$ ), then we say it to be compatible with the above $(U, \phi, \phi(U))$ if $U \cap U^{\prime}=\emptyset$ or when $U \cap U^{\prime} \neq \emptyset$, the map $\phi \circ \phi^{\prime-1}: \phi^{\prime}\left(U \cap U^{\prime}\right) \rightarrow \phi\left(U \cap U^{\prime}\right)$ is a (smooth) diffeomorphism from an open subset of $\mathbb{R}_{k^{\prime}}^{m}$ onto an open subset of $\mathbb{R}_{k}^{m}$. Now we give the typical definition of a smooth manifold with corners.
Definition 2.1.1 (manifold with corners). A topological manifold $X$ together with a smooth structure defined by a (maximal) atlas consisting of compatible open $m$-dimensional charts with corners is called a (smooth) manifold with corners of dimension $m$. As usual, the smooth maps between manifolds with corners are in the sense of smooth maps between the local charts with corners.

Note that by the Seeley Extension Theorem, the smooth functions on any open subset in $\mathbb{R}_{k}^{m}$ always extend to its neighbourhood in $\mathbb{R}^{m}$. Then the diffeomorphism $\phi^{\prime} \circ \phi^{-1}: \phi^{\prime}\left(U \cap U^{\prime}\right) \rightarrow$ $\phi\left(U \cap U^{\prime}\right)$ extends to a diffeomorphism between their open neighbourhoods in $\mathbb{R}^{m}$. For a point
$x=\left(x_{1}, \cdots, x_{k}, x_{k+1}, \cdots, x_{m}\right) \in \mathbb{R}_{k}^{m}$, set $\operatorname{depth}(x) \in \mathbb{N}$ to be the number of vanishing coordinates $x_{j}, j=0,1, \cdots, k$. For an open subset $U \subset \mathbb{R}_{k}^{m}$ and $j=0,1, \cdots, k$, set

$$
\begin{equation*}
S^{j}(U):=\left\{x \in U \mid \operatorname{depth}_{U}(x)=j\right\} . \tag{2.1.1}
\end{equation*}
$$

In particular, $S^{0}(U)$ is the interior of $U$ in $\mathbb{R}^{m}$. We also set $\operatorname{depth}(U)=\max \left\{j \mid S^{j}(U) \neq \emptyset\right\}$.
If $\varphi: \mathbb{R}_{k}^{m} \supset U \rightarrow U^{\prime} \subset \mathbb{R}_{k^{\prime}}^{m}$ is a diffeomorphism of open sets, then $\operatorname{depth}(U)=\operatorname{depth}\left(U^{\prime}\right)$, and for $j=0,1, \cdots, \operatorname{depth}(U), f$ identifies $S^{j}(U)$ with $S^{j}\left(U^{\prime}\right)$. As a consequence, if $X$ is a (smooth) manifold with corners, then we can define a canonical function depth ${ }_{X}: X \rightarrow \mathbb{N}_{0}$ which associates a point $x \in X$ with its depth $\operatorname{depth}_{X}(x)$ by taking a compatible chart as above. The depth $j$-stratum of $X$ is the subset

$$
\begin{equation*}
S^{j}(X):=\left\{x \in X \mid \operatorname{depth}_{X}(x)=j\right\} . \tag{2.1.2}
\end{equation*}
$$

We have the following properties:

- $X=\coprod_{j=0}^{m} S^{j}(X)$, and $\overline{S^{k}(X)}=\cup_{j=k}^{m} S^{j}(X)$. In particular, $S^{0}(X)$ is open and dense in $X$, which is called the interior of $X$.
- Each $S^{j}(X)$ has an induced smooth structure as an $(m-j)$-dimensional manifold without boundary.
- $X$ is a manifold without boundary if $S^{j}(X)=\emptyset$ for all $j \geq 1$, so that $S^{0}(X)=X$.
- A smooth function on $X$ restricting to any $S^{k}(X)$ is smooth and extends continously to $\overline{S^{k}(X)}$.

We use $\partial X$ denote the topological boundary of $X$, then as subset of $X$ we have

$$
\partial X=\overline{S^{1}(X)}=\cup_{j \geq 1} S^{j}(X)
$$

In general, $\partial X$, with induced charts from $X$, is not a smooth manifold with corners as defined above, so that it is good to introduce a 'regularized' boundary $\partial^{*} X$ of $X$ which has an induced structure of manifold with corners.

For $x \in X$, a local boundary component $\beta$ of $X$ at $x$ is a local choice of the connected component of $S^{1}(X)$ in a small open neighbourhood of $x$ in $X$. For instance, if $\operatorname{depth}_{X}(x)=k$, let $\left(U, \phi, \phi(U) \subset \mathbb{R}_{k}^{m}\right)$ be a small (contractible) local chart near $x$ such that $\phi(x)=(0,0, \cdots, 0) \in \mathbb{R}_{k}^{m}$, then the choices of a local boundary component $\beta$ are given by following $k$-faces

$$
\left\{z=\left(z_{1}, \ldots, z_{k}, \ldots, z_{m}\right) \in \phi(U) \mid z_{j}=0\right\}, j=1, \ldots, k
$$

In particular, the number of local boundary components at $x$ is exactly $\operatorname{depth}_{X}(x)$.
Set

$$
\partial^{*} X=\{(x, \beta) \mid x \in \partial X, \beta \text { a local boundary component at } x\} .
$$

The following facts are clear by our construction.

- The $m$-dimensional charts with corners of $X$ give canonically the $(m-1)$-dimensional charts with corners on $\partial^{*} X$ so that $\partial^{*} X$ is a smooth manifold with corners of dimension $m-1$.
- We have $S^{0}\left(\partial^{*} X\right)=S^{1}(X)$.
- We have a continuous projection $\partial^{*} X \rightarrow \partial X$ which send $(x, \beta)$ to $x$. In particular, if $X$ is a smooth manifold with boundary (in the usual sense), $\partial^{*} X=\partial X$ is a smooth manifold without boundary.
- Composing the above projection with the inclusion $\mathfrak{i}_{\partial}: \partial X \hookrightarrow X$, we get a map

$$
\mathfrak{i}_{\partial}: \partial^{*} X \rightarrow X
$$

This map $\mathfrak{i}_{2}$ is a smooth map between manifolds with corners, it is not necessary to be injective, for $x \in X$, we have

$$
\left|\mathfrak{i}_{\partial}^{-1}(x)\right|=\operatorname{depth}_{X}(x) .
$$

We can define the tangent bundle $T X$ of $X$ as well as the cotangent bundle $T^{*} X$ as in the manifold case via the local charts. For the points in corners, one may also introduce the notions of inward or outward tangent vectors. Then the smooth vector fields and the differential forms on $X$ are welldefined. In particular, we can talk about the orientable manifold with corners. The usual partition of unity still holds on a manifold with corners, so that if $\alpha$ is a smooth $m$-form on the oriented $X$ with compact support, the integration $\int_{X} \alpha$ is well-defined by considering the integration on local charts. Note that we always have

$$
\int_{X} \alpha=\int_{S^{0}(X)} \alpha
$$

Now let $X$ be a compact oriented (smooth) manifold with corners, and let $o(X)$ denote the orientation of $X$. Let $\mathbf{n}$ denote the outward normal vector field of $\partial^{*} X$ as boundary of $X$, which is defined as the extension of the outward normal vector field of $S^{1}(X)$ in $X$. We orient $\partial^{*} X$ by an orientation $o_{\mathrm{ind}}\left(\partial^{*} X\right)$ such that at all points of $S^{1}(X)$, we have

$$
\begin{equation*}
\left.o(X)\right|_{\partial^{*} X}=\mathbf{n} \wedge o_{\mathrm{ind}}\left(\partial^{*} X\right) \tag{2.1.3}
\end{equation*}
$$

With the above orientation conventions, if $\alpha \in \Omega^{\bullet}(X)$, then $\mathfrak{i}_{\partial}^{*} \alpha$ is a smooth form on $\partial^{*} X$, and we have the Stokes' formula as follows

$$
\begin{equation*}
\int_{X} d \alpha=\int_{S^{1}(X)} \mathfrak{i}_{\partial}^{*} \alpha=\int_{\partial^{*} X} \mathfrak{i}_{\partial}^{*} \alpha . \tag{2.1.4}
\end{equation*}
$$

When there is no confusion, we can simply write the right-hand side of (2.1.4) as $\int_{\partial X} \alpha$.

### 2.2 Submersion and fibrewise boundary

If $f: X \rightarrow B$ is a smooth map between two smooth manifolds with corners, for $x \in X$, the tangent map $d f_{x}: T_{x} X \rightarrow T_{f(x)} B$ is defined via the local charts. Moreover, if $\alpha \in \Omega^{\bullet}(B)$, we have the pull-back $f^{*}(\alpha) \in \Omega^{\bullet}(X)$.
Definition 2.2.1. A smooth map $f: X \rightarrow B$ is called a submersion if for all $x \in X$ with $x \in S^{k}(X)$, $f(x) \in S^{\ell}(B)$, the tangent maps $d f_{x}: T_{x} X \rightarrow T_{f(x)} B$ and $d f_{x}: T_{x} S^{k}(X) \rightarrow T_{f(x)} S^{\ell}(B)$ are surjective.

Analogous to the usual Ehresmann's theorem (also cf. [Joy12, Section 5]), a proper submersion $f: X \rightarrow B$ for manifolds with corners is a locally-trivial fibration on $S^{0}(B)$ where the fibres are compact manifolds with corners. Note that, in general, the locally-trivial fibration could not extend to the corners of $B$, a simple counterexample is as follows: consider the submersion $f: \mathbb{R}_{+} \times \mathbb{R}_{+} \ni$ $(x, y) \mapsto \frac{1}{\sqrt{2}}(x+y) \in \mathbb{R}_{+}$, which is not locally-trivial fibration, since $f^{-1}(0)=\{(0,0)\}$, but $f^{-1}(1)$ is a nontrivial segment of line.

Fix a surjective submersion $p: X \rightarrow B$ of compact orientable manifolds with corners. We suppose that $X$ and $B$ are oriented with orientation $o(X)$ and $o(B)$ respectively. Then there is a unique orientation $o_{\text {fibre }}(p)$ on the fibres of $p$, i.e., orientations on $p^{-1}(b), b \in S^{0}(B)$ such that locally

$$
\begin{equation*}
o(X)=o(B) \wedge o_{\text {fibre }}(p) \tag{2.2.1}
\end{equation*}
$$

Proposition 2.2.2. Let $p: X \rightarrow B$ be a surjective submersion of compact orientable manifolds with corners. Then for any $\alpha \in \Omega^{\bullet}(X)$, there exists a unique smooth form $\hat{\alpha} \in \Omega^{\bullet}(B)$ such that for $\gamma \in \Omega^{\bullet}(B)$, we have

$$
\begin{equation*}
\int_{(X, o(X))} p^{*}(\gamma) \wedge \alpha=\int_{(B, o(B))} \gamma \wedge \hat{\alpha} . \tag{2.2.2}
\end{equation*}
$$

Proof. Set $X_{0}=p^{-1}\left(S^{0}(B)\right)$, it is an open and dense subset of $X$. Then we have the locally-trivial fibration $p: X_{0} \rightarrow S^{0}(B)$. For $\alpha \in \Omega^{\bullet}(X), \hat{\alpha}^{\prime} \in \Omega^{\bullet}\left(S^{0}(B)\right)$ exists uniquely which satisfies the formula (2.2.2). Now we only need to explain $\hat{\alpha}^{\prime}$ extends to a smooth form $\hat{\alpha} \in \Omega^{\bullet}(B)$. In fact, fix $b \in \partial B$ and a small open neighbourhood of $b$ given by a local chart $U$ in $\mathbb{R}_{k}^{m}$. The fibration structure given by $p$ at any point in $S^{0}(B) \cap U$ implies that the first derivatives of $\hat{\alpha}^{\prime}$ on $S^{0}(B) \cap U$ are uniformly bounded, so that $\hat{\alpha}^{\prime}$ extends continuously from $S^{0}(B) \cap U$ to $U$. Then the same arguments for the higher derivatives of $\hat{\alpha}^{\prime}$ infer exactly the smooth extension $\hat{\alpha}$ as we need.

Definition 2.2.3. The linear map $p_{*}: \Omega^{\bullet}(X) \ni \alpha \mapsto p_{*}(\alpha):=\hat{\alpha} \in \Omega^{\bullet}(B)$ is called the fibre integration of the submersion $p: X \rightarrow B$.

Now we describe Stokes' theorem for the submersion case. We still consider a surjective submersion $p: X \rightarrow B$ of compact oriented manifolds with corners. It defines a surjective submersion $p_{0}: X_{0}:=$ $p^{-1}\left(S^{0}(B)\right) \rightarrow S^{0}(B)$ on smooth manifold $S^{0}(B)$. Here $X_{0}$ is an oriented manifold with corners, for any small open set $V \subset S^{0}(B)$, there exists a compact manifold with corners $Z \simeq p_{0}^{-1}(b)$ for some $b \in V$ such that we have the diffeomorphism of manifolds with corners $p_{0}^{-1}(V) \simeq V \times Z$. Let $X_{0}^{\partial}$ be the manifold with corners by gluing together $V \times \partial^{*} Z$. An equivalent definition is $X_{0}^{\partial}=\partial^{*} X_{0}$. We get a canonical surjective submersion of compact manifolds with corners

$$
p^{\partial}: X_{0}^{\partial} \rightarrow S^{0}(B) .
$$

Note that the orientation $o(X)$ induces an orientation $o_{\mathrm{ind}}\left(X_{0}^{\partial}\right)$, hence induces a fibration orientation $o_{\text {fibre }}\left(p^{\partial}\right)$ for $p^{\partial}$.

The analogous arguments in the proof of Proposition 2.2 .2 give the following result.

Lemma 2.2.4. For $\alpha \in \Omega^{\bullet}(X)$, let $\mathfrak{i}_{\partial}^{*} \alpha$ denote the corresponding smooth form on $X_{0}^{\partial}$, the fibre integration $p_{*}^{\partial}\left(\mathrm{i}_{\partial}^{*} \alpha\right) \in \Omega^{\bullet}\left(S^{0}(B)\right)$ always extends smoothly to $B$, which we denote it by the same notation.

With the above notation, we have the following fibrewise Stokes' formula.
Proposition 2.2.5. For $p \in \mathbb{N}$ and $\alpha \in \Omega^{p}(X)$, we have the following identity of smooth forms on $B$,

$$
\begin{equation*}
d p_{*}(\alpha)=p_{*}(d \alpha)+(-1)^{p+\operatorname{dim} X} p_{*}^{\partial}\left(\mathrm{i}_{\partial}^{*} \alpha\right) \in \Omega^{1+p+\operatorname{dim} B-\operatorname{dim} X}(B) . \tag{2.2.3}
\end{equation*}
$$

Proof. We only need to prove this identity on $S^{0}(B)$ by our constructions of fibre integrations $p_{*}, p_{*}^{\partial}$. Then after taking the locally-trivial fibrations, the proof reduces to the Stokes' formula for fibres as in (2.1.4).

### 2.3 Compactification of configuration spaces

In this subsection, we recall the Fulton-MacPherson-Axelrod-Singer compactification (FMAS compactification in short). For more details see [FM94] and [AS94] (see also [Sin04]). Let $M$ be a closed oriented 3-manifold with a given orientation $o(M)$.

Let $F \rightarrow M$ be a real vector bundle over $M$ with rank $r \geq 2$. The sphere bundle of $F$, denote by $S F \rightarrow M$, is defined as the smooth quotient bundle $\mathbb{R}_{+}^{*} \backslash(F-\{0\}) \rightarrow M$, where the fibres are given by $\mathbb{S}^{r-1}$. If $g^{F}$ is a Euclidean metric on $F$, then we can identify canonically $S F$ with the unit sphere bundle $S_{g^{F}} F \rightarrow M$ of $\left(F, g^{F}\right)$.

Let $S$ be a finite set, and put

$$
\begin{equation*}
M^{S}=\operatorname{Maps}(S, M)=\Pi_{i \in S} M_{i} \tag{2.3.1}
\end{equation*}
$$

where $M_{i}=M$ is just a copy of $M$. If $S=\underline{n}:=\{1,2 \ldots, n\}, n \geq 2$, we simply denote $M^{n}=M^{S}$ to be compatible with the usual notation.

Put $\Delta_{S} \simeq M$ be the subset of $M^{S}$ consisting of constant maps, which is called the principal diagonal of $M^{S}$. Since $M$ is closed, so is $M^{S}$, and $\Delta_{S}$ is a closed (embedding) submanifold of $M^{S}$. Let $B \ell\left(M^{S}, \Delta_{S}\right)$ be the geometric blow-up of $M^{S}$ along $\Delta_{S}$. It can be regarded as replacing $\Delta_{S}$ by its sphere normal bundle $S \nu_{\Delta_{S}}$ in $M^{S}$. If $U_{S}$ is an open small tubular neighbourhood of $\Delta_{S}$ in $M^{S}$, then $B \ell\left(M^{S}, \Delta_{S}\right)$ is diffeomorphic to $M^{S} \backslash U_{S}$ as manifolds with boundary where $\partial B \ell\left(M^{S}, \Delta_{S}\right) \simeq S \nu_{\Delta_{S}}$. Moreover, we have a canonical smooth projection, the blow-down map, $B \ell\left(M^{S}, \Delta_{S}\right) \rightarrow M^{S}$, which, when restricting to the boundary, is given by the projection $S \nu_{\Delta_{S}} \rightarrow \Delta_{S}$.

For a positive integer $n$, we denote by $\operatorname{Conf}_{n}(M)$ the $n$-point configuration space of $M$, i.e.,

$$
\begin{equation*}
\operatorname{Conf}_{n}(M):=\left\{\left(x_{1}, \ldots, x_{n}\right) \in M^{n} \mid x_{i} \neq x_{j}(i \neq j)\right\} . \tag{2.3.2}
\end{equation*}
$$

Equivalently, $\operatorname{Conf}_{n}(M)$ is the open subset of $M^{n}$ consisting of the injective maps from $\underline{n}$ into $M$. We have the following injective smooth map:

$$
\begin{equation*}
\Phi_{n}: \operatorname{Conf}_{n}(M) \rightarrow \mathcal{B}:=M^{n} \times \prod_{S \subset \underline{n},|S| \geq 2} B \ell\left(M^{S}, \Delta_{S}\right) \tag{2.3.3}
\end{equation*}
$$

Note that the target space of $\Phi_{n}$ is a compact manifold with corners as described in Subsection 2.1. The FMAS compactification $C_{n}(M)$ of $\operatorname{Conf}_{n}(M)$ is defined as the closure of the image of $\operatorname{Conf}_{n}(M)$ via $\Phi_{n}$ equipped with the induced smooth structure, i.e.,

$$
\begin{equation*}
C_{n}(M):=\overline{\Phi_{n}\left(\operatorname{Conf}_{n}(M)\right)} \subset \mathcal{B}=M^{n} \times \prod_{S \subset \underline{n},|S| \geq 2} B \ell\left(M^{S}, \Delta_{S}\right) . \tag{2.3.4}
\end{equation*}
$$

By [AS94, Section 5], Axelrod-Singer explained that $C_{n}(M)$, as an embedded submanifold of $\mathcal{B}$, is a compact manifold with corners of dimension $3 n$. In particular, the interior of $C_{n}(M)$ is exactly $\operatorname{Conf}_{n}(M)$. In the next subsection, we will recall in detail the stratification of $C_{n}(M)$ as a manifold with corners described in [AS94, Section 5].

Now we focus on the case of $n=2$, then $C_{2}(M)$ is a compact manifold with boundary. Let $\Delta \subset M \times M$ denotes the diagonal. An elementary argument shows that

$$
\begin{equation*}
C_{2}(M)=(M \times M \backslash \Delta) \cup S \nu_{\Delta}=B \ell\left(M^{2}, \Delta\right) . \tag{2.3.5}
\end{equation*}
$$

The blow-down map $q: C_{2}(M) \rightarrow M^{2}$ satisfies $q\left(S \nu_{\Delta}\right)=\Delta$ and $q=$ Id otherwise.
Note that the sphere normal bundle $S \nu_{\Delta}$ is given by the equivalent classes of the elements $((x, x),(v,-v)), x \in M, v \in T_{x} M$. Then it can be identified with the sphere tangent bundle $S(T M)$ by

$$
\begin{equation*}
S \nu_{\Delta} \xrightarrow{\sim} S(T M), \quad((x, x),(v,-v)) \mapsto(x, v) . \tag{2.3.6}
\end{equation*}
$$

We will always use the identifications $\partial C_{2}(M) \simeq S \nu_{\Delta} \simeq S(T M)$.

Remark 2.3.1. Note that in many other articles, for example, in [Sin04, CW23], etc, the FMAS compactification of $\operatorname{Conf}_{2}(M)$ is defined via en embedding of $M$ into some $\mathbb{R}^{N}$ and taking the closure in $M^{2} \times \mathbb{S}^{N-1}$. This way, we will get the same $C_{2}(M)$ up to diffeomorphism.

### 2.4 Strata of compactified configuration spaces

In this subsection, we recall the main results of [AS94, Section 5], in particular, we will describe the local charts for $C_{n}(M)$ as manifolds with corners.

To understand the structure of $C_{n}(M)$ as a manifold with corners, we need to introduce the following notation. For $S \subset \underline{n}$ with $|S|>1$, for $x=(z, \cdots, z) \in \Delta_{S}$ with $z \in M$, the normal bundle $\nu_{\Delta_{S}, x}$ of $\Delta_{S}$ in $M^{S}$ at $x$ can be identified with the quotient space $\left(T_{z} M\right)^{S} / T_{z} M$. For $u_{S}=$ $\left(u_{i}\right)_{i \in S} \in\left(T_{z} M\right)^{S}$, if all components $u_{i}$ are identical, then $\left[u_{S}\right]=0 \in\left(T_{z} M\right)^{S} / T_{z} M$. For each element $\left[u_{S}\right] \in\left(T_{z} M\right)^{S} / T_{z} M$, there exists a unique representative $u_{S}=\left(u_{i}\right)_{i \in S} \in\left(T_{z} M\right)^{S}$ such that $\sum_{i \in S} u_{i}=0$ in $T_{z} M$. We can also regard such vector $u_{S} \in\left(T_{z} M\right)^{\underline{n}}$ by setting $u_{j}=0$ for $j \notin S$.

Note that $\left.\left.\mathbb{R}_{+}^{*}=\right] 0,+\infty\right]$ acts on $\left(T_{z} M\right)^{S} / T_{z} M$ by the diagonal rescalling on all the components. So that we have the identification $S \nu_{\Delta_{S}} \simeq \mathbb{R}_{+}^{*} \backslash\left(\left(T_{z} M\right)^{S} / T_{z} M-\{0\}\right)$. Then naturally, any nonzero $\left[u_{S}\right] \in\left(T_{z} M\right)^{S} / T_{z} M$ corresponds to an element in $S \nu_{\Delta_{S}}$, which is still denoted by [ $u_{S}$ ] if there is no confusion.

Let $q: B \ell\left(M^{S}, \Delta_{S}\right) \rightarrow M^{S}$ denote the obvious projection. We always use $x_{B, S}$ to denote a point in $B \ell\left(M^{S}, \Delta_{S}\right)$, such that if $x_{S}=q\left(x_{B, S}\right) \notin \Delta_{S}$, then $x_{B, S}=x_{S}$; otherwise, $x_{B, S}=\left(x_{S},\left[u_{S}\right]\right)$ where $x_{S}=(z, \cdots, z) \in \Delta_{S}, 0 \neq\left[u_{S}\right] \in\left(T_{z} M\right)^{S} / T_{z} M$.

As a point set, we have a characterization of $C_{n}(M)$ as a subset of $\mathcal{B}$ : $C_{n}(M)$ are consisting of all points ( $x,\left\{x_{B, S},|S| \geq 2\right\}$ ) in $\mathcal{B}$ which satisfy the following two conditions:

- $x_{S}=q\left(x_{B, S}\right)=\left.x\right|_{S}$, for $S \subset \underline{n},|S|>1$, where $\left.x\right|_{S}=\left(x_{i}\right)_{i \in S} \in M^{S}$ with $x=\left(x_{1}, \cdots, x_{n}\right) \in M^{n}$.
- For any subset $S(|S| \geq 3)$ with $x_{S} \in \Delta_{S}$, write $x_{B, S}=\left(x_{S},\left[u_{S}\right]\right)$, then for each subset $S^{\prime} \subset S$ with $\left|S^{\prime}\right| \geq 2$, if $S^{\prime}$-components of $u_{S}$ are not all equal, we have $x_{B, S^{\prime}}=\left(x_{S^{\prime}},\left[\left.u_{S}\right|_{S^{\prime}}\right]\right)$.
Set $\mathbf{S}_{n}=\{S \subset \underline{n}| | S \mid \geq 2\}$.
Definition 2.4.1. A subset $\mathcal{S} \subset \mathbf{S}_{n}$ is called nested if any two elements $S_{1}, S_{2} \in \mathcal{S}$ are either disjoint or else one contains the other.

For a nested subset $\mathcal{S} \subset \mathbf{S}_{n}$, the open strata $M(\mathcal{S})^{\circ}$ of $C_{n}(M)$ is defined as follows, it consists of the points $\left(x,\left\{x_{B, S},|S| \geq 2\right\}\right) \in C_{n}(M)$ such that

- $\left.x\right|_{S} \in \Delta_{S}$ if and only if $S \subset S^{\prime}$ for some $S^{\prime} \in \mathcal{S}$.
- For $S^{\prime}\left(\left|S^{\prime}\right|>1\right)$ with a minimal element $S \in \mathcal{S}$ such that $S^{\prime} \subset S$, then $\left[u_{S^{\prime}}\right]=\left[\left.u_{S}\right|_{S^{\prime}}\right]$ in $x_{B, S^{\prime}}$.
- For $S_{1}, S_{2} \in \mathcal{S}$ such that $S_{1} \subsetneq S_{2}$, then all $S_{1}$-components of $u_{S_{2}}$ are all equal.

In [AS94, Subsection 5.3 and 5.4], Axelrod and Singer showed the following facts:

- $M(\mathcal{S})^{\circ}$ is a smooth (noncompact) manifold of dimension $3 n-|\mathcal{S}|$, in particular, $M(\emptyset)^{\circ}=$ $\operatorname{Conf}_{n}(M)$.
- The closed strata $M(\mathcal{S})$, defined as the closure of $M(\mathcal{S})^{\circ}$ in $C_{n}(M)$, is given as follows

$$
M(\mathcal{S})=\cup_{\mathcal{T} \supset \mathcal{S}} M(\mathcal{T})^{\circ}
$$

where $\mathcal{T}$ runs over all nested subsets of $\mathbf{S}_{n}$ containing $\mathcal{S}$.

- We have

$$
\begin{equation*}
C_{n}(M)=\cup_{\mathcal{S} \text { nested }} M(\mathcal{S})^{\circ} . \tag{2.4.1}
\end{equation*}
$$

- For the strata of $C_{n}(M)$ as manifold with corners, we have for $k=0,1, \cdots, 3 n$,

$$
\begin{equation*}
S^{k} C_{n}(M)=\cup_{\mathcal{S},|S|=k} M(\mathcal{S})^{\circ} . \tag{2.4.2}
\end{equation*}
$$

This way, we see that the interior of $\partial^{*} C_{n}(M)$ is given by all the single sets $S$ of $\underline{n}$ with $|S| \geq 2$.
The following proposition allows us to define properly the fibre-wise integration for the projections between compactified configuration spaces.
Proposition 2.4.2. For $n \geq 2$, for $1 \leq i \neq j \leq n$, the projection

$$
\begin{equation*}
p_{i j}: \operatorname{Conf}_{n}(M) \ni\left(x_{1}, \ldots, x_{n}\right) \mapsto\left(x_{i}, x_{j}\right) \in \operatorname{Conf}_{2}(M) \tag{2.4.3}
\end{equation*}
$$

induces a surjective submersion of manifolds with corners

$$
\begin{equation*}
C_{n}(M) \rightarrow C_{2}(M) \tag{2.4.4}
\end{equation*}
$$

which is still denoted by $p_{i j}$.

Proof. Without loss of generality, we may assume $(i, j)=(1,2)$. It is clear that the points in $M(\mathcal{S})^{\circ} \subset C_{n}(M)$ with $\mathcal{S}$ containing no $\{1,2\}$ map to the points in $\operatorname{Conf}_{2}(M)$. For such a point, we can verify directly the condition that $p_{i j}$ is submersion. The points in $M(\mathcal{S})^{\circ} \subset C_{n}(M)$ with $\mathcal{S}$ containing $\{1,2\}$ map to the points in $\partial C_{2}(M)$, then the computation for the tangent maps is to consider the projection between the unit spheres of different dimensions, and shows the same conclusion for it being the submersion. This way, we finish the proof.

### 2.5 Framing of closed orientable 3-manifolds

Now we consider a closed orientable 3-manifold $M$. Then the tangent bundle $T M$ of $M$ is always parallelizable, i.e., there always exists a global smooth trivialization (isomorphism of vector bundles) $f: T M \rightarrow M \times \mathbb{R}^{3}$ of $T M$ (cf. [Sti35] and [BZ23]). We call such trivialization $f$ a framing of $M$.

Now we fix a smooth framing $f$ of $M$, then we always identify $T M$ with $M \times \mathbb{R}^{3}$ and the sphere bundle $S(T M)$ with $M \times \mathbb{S}^{2}$. Set $T^{V}(T M), T^{V}(S(T M))$ the vertical tangent bundles for the fibrations $T M \rightarrow M, S(T M) \rightarrow M$ respectively. Then the above identifications induce the splittings

$$
\begin{align*}
& T(T M)=f^{*} T M \oplus f^{*} T \mathbb{R}^{3}=: T_{f}^{H}(T M) \oplus T^{V}(T M) \\
& T(S(T M))=f^{*} T M \oplus f^{*} T \mathbb{S}^{2}=: T_{f}^{H}(S(T M)) \oplus T^{V}(S(T M)) . \tag{2.5.1}
\end{align*}
$$

A differential form $\alpha$ on $T M$ or $S(T M)$ is said to be $f$-vertical (or simply, vertical, when the framing $f$ is fixed) if for all $U \in T_{f}^{H}(T M)$ or $T_{f}^{H}(S(T M))$, we have

$$
\begin{equation*}
\iota_{U} \alpha=0, \tag{2.5.2}
\end{equation*}
$$

where $\iota_{U}$ denotes the contraction of $U$.
Given an orientation $o(M)$, via $f$, we also obtain an orientation $o\left(\mathbb{R}^{3}\right)$ of $\mathbb{R}^{3}$. The standard unit 2 -sphere $\mathbb{S}^{2}$ is boundary of the standard 3 -ball, then let $o_{\text {ind }}\left(\mathbb{S}^{2}\right)$ be the induced orientation. At the same time, the complementary part of open 3-ball in $\mathbb{R}^{3}$ can be identified with the real blow up $B \ell\left(\mathbb{R}^{3}, 0\right)$ of the origin point 0 of $\mathbb{R}^{3}$. Then $\partial B \ell\left(\mathbb{R}^{3}, 0\right)=\mathbb{S}^{2}$. We orient $B \ell\left(\mathbb{R}^{3}, 0\right)$ as for $\mathbb{R}^{3}$. Then

$$
\begin{equation*}
o_{\text {ind }}\left(\partial B \ell\left(\mathbb{R}^{3}, 0\right)\right)=-o_{\text {ind }}\left(\mathbb{S}^{2}\right) \tag{2.5.3}
\end{equation*}
$$

If $N_{1}, N_{2}$ are two oriented manifolds with orientations $o\left(N_{1}\right), o\left(N_{2}\right)$ respectively. Then we denote by $o\left(N_{1} \times N_{2}\right)$ the orientation of $N_{1} \times N_{2}$ given by $o\left(N_{1}\right) \wedge o\left(N_{2}\right)$. This way, we orient product manifolds $M^{n}$ and also $\operatorname{Conf}_{n}(M), C_{k}(M)$. In particular, let $o\left(B \ell\left(M^{2}, \Delta\right)\right)=o\left(C_{2}(M)\right)$ be the induced orientation.

Since $f$ also identifies $S(T M)$ with $M \times \mathbb{S}^{2}$. This way, let $o(S(T M))$ denote the orientation $o(M) \wedge o_{\text {ind }}\left(\mathbb{S}^{2}\right)$.
Lemma 2.5.1. Under the identification $\partial C_{2}(M) \simeq S \nu_{\Delta} \simeq S(T M)$ as explained in (2.3.6), we have

$$
\begin{equation*}
o_{\mathrm{ind}}\left(\partial C_{2}(M)\right)=o(S(T M)) . \tag{2.5.4}
\end{equation*}
$$

In the sequel where we compute the integrals on $C_{2}(M)$, we always fix the orientation $o\left(C_{2}(M)\right)$ induced from $o(M \times M)$, and we fix the induced orientation $o_{\mathrm{ind}}\left(\partial C_{2}(M)\right)$ for $\partial C_{2}(M)$.

## 3 Adjoint local system and diagonal classes

This section recalls several basic facts on local systems $E_{\rho}$ given via adjoint representations of semisimple Lie groups, for example, de Rham cohomology groups of such a local system, the diagonal class, the Killing form and its associated cubic trace form, and Lie structure on $E_{\rho}$. At last, we give an important result in Proposition 3.5 .5 on the nonvanishing of the cohomology group $H_{-}^{2}\left(M ; E_{\rho} \otimes E_{\rho}\right)$, this cohomology group plays an important role in our construction of (adapted) propagators in the next sections.

Now we fix a connected semi-simple Lie group $G$ with Lie algebra $\mathfrak{g}$, and let $B: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ denote the corresponding Killing form. Let $\rho: \pi_{1}(M) \rightarrow G$ be a morphism of groups, composing with the adjoint action $\operatorname{Ad}: G \rightarrow \operatorname{Aut}(\mathfrak{g})$, we get a representation of $\pi_{1}(M)$ with representation space $\mathfrak{g}$, which is still denoted by $\rho$. The representation ( $\rho, \mathfrak{g}$ ) defines canonically a flat vector bundle ( $E_{\rho}, \nabla^{E_{\rho}}$ ) (equivalently, a local system) on $M$ : more precisely, let $\widetilde{M}$ be the universal cover of $M$ on which $\pi_{1}(M)$ acts smoothly and freely, then

$$
E_{\rho}=\pi_{1}(M) \backslash\left(\widetilde{M} \times_{\rho} \mathfrak{g}\right),
$$

and the flat connection $\nabla^{E_{\rho}}$ is induced from the usual differential on $\widetilde{M}$. In the sequel of this section, we use the terminology flat vector bundle for $E_{\rho}$ in order to explain the associated de Rham-Hodge theory, and in the most part of other sections, we will call $E_{\rho}$ the local system (associated to a representation $\rho$ as above).

## 3.1 de Rham cohomology and Poincaré duality

The flat connection $\nabla^{E_{\rho}}$ defines canonically a twisted differential $d$ acting on $\Omega^{\bullet}\left(M ; E_{\rho}\right)$, the differential forms on $M$ valued in $E_{\rho}$. This way, we get the de Rham complex $\left(\Omega^{\bullet}\left(M ; E_{\rho}\right), d\right)$. Let $H^{\bullet}\left(M ; E_{\rho}\right)$ denote the corresponding de Rham cohomology groups.

At the same time, viewing $E_{\rho}$ as a local system, the singular homology groups $H_{\bullet}\left(X ; E_{\rho}\right)$ and singular cohomology groups $H_{\text {sing }}^{\bullet}\left(M, E_{\rho}\right)$ are well-defined, which is also defined via considering the triangulations of $M$. Of course, we have the caonical isomorphism $H_{\text {sing }}^{\bullet}\left(M ; E_{\rho}\right) \simeq H^{\bullet}\left(M ; E_{\rho}\right)$.

Note that we always fix an orientation $o(M)$ on $M$. For the flat vector bundle $E_{\rho}$, we consider its dual bundle $E_{\rho}^{\vee}:=\operatorname{Hom}\left(E_{\rho}, \mathbb{R}\right)$ on $M$, and let $\nabla^{E_{\rho}^{\vee}}$ denote the induced flat connection from $\left(E_{\rho}, \nabla^{E_{\rho}}\right)$.

The classical Poincaré pairing (cf. [MS74, Appendix A], [Spa95, Section 6.2]) is given as the first part of the following isomorphisms for $k=0,1,2,3$,

$$
\begin{equation*}
H_{k}\left(M ; E_{\rho}\right) \simeq H_{\operatorname{sing}}^{3-k}\left(M ; E_{\rho}\right) \simeq H^{3-k}\left(M ; E_{\rho}\right) \tag{3.1.1}
\end{equation*}
$$

The analogous Poincaré pairing in terms of de Rham cohomology groups is given by the following nondegenerate bilinear form (also cf. [MW11, Section 1.4])

$$
\begin{equation*}
H^{k}\left(M ; E_{\rho}^{\vee}\right) \times H^{3-k}\left(M ; E_{\rho}\right) \rightarrow \mathbb{R}, \quad(\alpha, \beta) \mapsto \int_{M}\langle\alpha \wedge \beta\rangle \tag{3.1.2}
\end{equation*}
$$

where $\left\rangle\right.$ denotes the fibrewise paring between $E_{\rho}^{\vee}$ and $E_{\rho}$.
Let $\left(\rho^{*}, \mathfrak{g}^{*}\right)$ be the representation of $\pi_{1}(M)$ adjoint to $(\rho, \mathfrak{g})$. Then actually, we have the identification of flat vector bundles

$$
E_{\rho}^{\vee} \simeq \pi_{1}(M) \backslash\left(\widetilde{M} \times_{\rho^{*}} \mathfrak{g}^{*}\right)
$$

Since $B$ is non-degenerate, it defines an isomorphism

$$
F: \mathfrak{g} \rightarrow \mathfrak{g}^{*}
$$

such that if $a \in \mathfrak{g}$, the element $F(a) \in \mathfrak{g}^{*}$ is defined as follows, if $b \in \mathfrak{g}$,

$$
\langle F(a), b\rangle=B(b, a) .
$$

By the fact that $B$ is $\operatorname{Ad}(G)$-invariant, we get that $F$ is actually the isomorphism between the $\pi_{1}(M)$ representations $(\rho, \mathfrak{g})$ and $\left(\rho^{*}, \mathfrak{g}^{*}\right)$. Hence it induces an identification of flat vector bundle $\left(E_{\rho}, \nabla^{E_{\rho}}\right)$ to its dual bundle $E_{\rho}^{\vee}$ on $M$.

As a consequence of the above consideration, we can reformulate the pairing in (3.1.2) as the following nondegenerate bilinear form

$$
\begin{equation*}
\langle\cdot, \cdot\rangle_{\text {Poinc }}: H^{\bullet}\left(M ; E_{\rho}\right) \times H^{3-\bullet}\left(M ; E_{\rho}\right) \rightarrow \mathbb{R}, \quad(\alpha, \beta) \mapsto \int_{M} B(\alpha \wedge \beta), \tag{3.1.3}
\end{equation*}
$$

where the notation $B(\alpha \wedge \beta)$ means taking wedge product on differential forms and contracting sections of $E_{\rho}$ via $B$. Since $B$ is symmetric, we also conclude that the pairing $\langle\cdot, \cdot\rangle_{\text {Poinc }}$ is symmetric.

Now we give more details on the Poincaré pairing $\langle\cdot, \cdot\rangle_{\text {Poinc }}$ for $H^{0}\left(M ; E_{\rho}\right) \times H^{3}\left(M ; E_{\rho}\right)$, which is useful in Subsection 5.3. Note that $\pi_{1}(M)$ acts on $\mathfrak{g}$ via the representation $\rho$, set

$$
\begin{equation*}
\mathfrak{g}^{\pi_{1}(M)}:=\left\{a \in \mathfrak{g} \mid \rho(\gamma) a=a, \text { for all } \gamma \in \pi_{1}(M)\right\} . \tag{3.1.4}
\end{equation*}
$$

Then we have the natural identification

$$
\begin{equation*}
H^{0}\left(M ; E_{\rho}\right)=\mathfrak{g}^{\pi_{1}(M)}, \tag{3.1.5}
\end{equation*}
$$

where each vector $a \in \mathfrak{g}^{\pi_{1}(M)}$ is a constant section of $E_{\rho}$ on $M$.
Note that $\mathfrak{g}^{\pi_{1}(M)}$ is a Lie subalgebra of $\mathfrak{g}$. We also have the following result.
Lemma 3.1.1. The following two statements are equivalent:

- For any smooth 3 -form $\beta$ on $M$ such that $\int_{M} \beta \neq 0$, the linear map

$$
H^{0}\left(M ; E_{\rho}\right)=\mathfrak{g}^{\pi_{1}(M)} \ni a \mapsto[\beta \otimes a] \in H^{3}\left(M ; E_{\rho}\right)
$$

is an isomorphism, where $[\beta \otimes a]$ denotes the de Rham cohomology class of $\beta \otimes a$;

- the symmetric blinear form $B: \mathfrak{g}^{\pi_{1}(M)} \times \mathfrak{g}^{\pi_{1}(M)} \ni(a, b) \mapsto B(a, b) \in \mathbb{R}$ is nondegenerate.

Remark 3.1.2. When the Lie group $G$ is compact (which is always assumed to be semi-simple) or more generally, ( $\rho, E_{\rho}$ ) is a unitary representation of $\pi_{1}(M), B$ can induce an $\rho\left(\pi_{1}(M)\right.$ )-invariant inner product on $\mathfrak{g}$ so that the two statements in the above lemma always hold.

### 3.2 The diagonal class

For $i \in\{1,2\}$, we denote by $p_{i}: M \times M \rightarrow M$ the $i$-th projection map, i.e., $p_{i}\left(x_{1}, x_{2}\right)=x_{i}$.
Let $H^{\bullet}(M)$ denote the de Rham cohomology of $M$ (valued in $\mathbb{R}$ ). Then

$$
\begin{equation*}
H^{\bullet}(M \times M)=H^{\bullet}(M) \widehat{\otimes} H^{\bullet}(M) . \tag{3.2.1}
\end{equation*}
$$

More precisely, if $\alpha, \beta \in H^{\bullet}(M)$, then $\alpha \otimes \beta=p_{1}^{*}(\alpha) \wedge p_{2}^{*}(\beta) \in H^{\bullet}(M \times M)$.
Let $\mathfrak{i}: \Delta \hookrightarrow M \times M$ denote the inclusion of the diagonal. Then the pull-back map gives the morphism

$$
\begin{equation*}
\mathfrak{i}^{*}: H^{\bullet}(M \times M) \rightarrow H^{\bullet}(\Delta) \simeq H^{\bullet}(M), \tag{3.2.2}
\end{equation*}
$$

so that if $\alpha, \beta \in H^{\bullet}(M)$, then

$$
\begin{equation*}
\mathfrak{i}^{*}\left(p_{1}^{*}(\alpha) \otimes p_{2}^{*}(\beta)\right)=\alpha \wedge \beta \tag{3.2.3}
\end{equation*}
$$

Now we consider the Poincaré dual of $\Delta$ in $H^{3}(M \times M)$, which we still denote by $\Delta$, i.e., for all $\beta \in H^{\bullet}(M \times M)$, we have

$$
\begin{equation*}
\int_{M \times M} \Delta \wedge \beta=\int_{M} \mathfrak{i}^{*}(\beta) . \tag{3.2.4}
\end{equation*}
$$

If $\left\{\gamma_{i}\right\}$ is a homogenous basis of $H^{\bullet}(M)$, and if $\left\{\gamma_{i}^{*}\right\}$ is the corresponding dual basis of $H^{\bullet}(M)$ with respect to the Poincaré pairing, i.e.,

$$
\begin{equation*}
\int_{M} \gamma_{i} \wedge \gamma_{j}^{*}=\delta_{i j} \tag{3.2.5}
\end{equation*}
$$

then we have

$$
\begin{equation*}
\Delta=\sum_{i}(-1)^{\operatorname{deg}\left(\gamma_{i}^{*}\right)} p_{1}^{*}\left(\gamma_{i}\right) \wedge p_{2}^{*}\left(\gamma_{i}^{*}\right) \in H^{3}(M \times M) . \tag{3.2.6}
\end{equation*}
$$

Moreover, we have the following result.

$$
\begin{equation*}
\int_{M} \mathfrak{i}^{*}(\Delta)=\chi(M)=0, \tag{3.2.7}
\end{equation*}
$$

where $\chi(M)$ denotes the Euler characteristic number of $M$, which is 0 since $M$ is odd dimensional.
Now we go back to the case with the flat vector bundle $E_{\rho}$ on $M$. Set

$$
\begin{equation*}
E_{\rho} \boxtimes E_{\rho}:=p_{1}^{*} E_{\rho} \otimes p_{2}^{*} E_{\rho} \rightarrow M \times M \tag{3.2.8}
\end{equation*}
$$

Since $E_{\rho}$ is flat, so is $E_{\rho} \boxtimes E_{\rho}$ and let $\nabla^{E_{\rho} \boxtimes E_{\rho}}$ denote the corresponding induced flat connection. The vector bundle $E_{\rho} \boxtimes E_{\rho}$ on the diagonal $\Delta \subset M \times M$ becomes the tensor bundle $E_{\rho} \otimes E_{\rho}$, which is equipped with respect to the induced flat connection $\nabla^{E_{\rho} \otimes E_{\rho}}$.

Note that we always fix an orientation for $M$. We explain how to define a diagonal class associated with $E_{\rho}$ as in (3.2.6). Recall that the Poincaré pairing $\langle\cdot, \cdot\rangle_{\text {Poinc }}$ given in (3.1.3) between $H^{\bullet}\left(M ; E_{\rho}\right) \times$ $H^{3-\bullet}\left(M ; E_{\rho}\right)$ is nondegenerate. Let $\left\{\alpha_{i}\right\}$ be a homogeneous basis of $H^{\bullet}\left(M ; E_{\rho}\right),\left\{\alpha_{i}^{*}\right\}$ is the dual basis of $\left\{\alpha_{i}\right\}$ with respect to the pairing $\langle\cdot, \cdot\rangle_{\text {Poinc }}$.

The diagonal class $\widetilde{\Delta}$ is defined as

$$
\begin{equation*}
\widetilde{\Delta}=\sum_{i}(-1)^{\operatorname{deg}\left(\alpha_{i}^{*}\right)} \alpha_{i} \otimes \alpha_{i}^{*} \in H^{\bullet}\left(M ; E_{\rho}\right) \widehat{\otimes} H^{\bullet}\left(M ; E_{\rho}\right) . \tag{3.2.9}
\end{equation*}
$$

It is clear that the definition of $\widetilde{\Delta}$ does not depend on the choice of the homogeneous basis $\left\{\alpha_{i}\right\}$.
By choosing (homogeneous) representatives of a basis of $H^{\bullet}\left(M ; E_{\rho}\right)$, we get the corresponding embedding

$$
\begin{equation*}
\iota: H^{\bullet}\left(M ; E_{\rho}\right) \rightarrow \Omega^{\bullet}\left(M ; E_{\rho}\right), \tag{3.2.10}
\end{equation*}
$$

which means that for $\alpha \in H^{j}\left(M ; E_{\rho}\right), j=0, \cdots, 3$, we have $\iota(\alpha) \in \Omega^{j}\left(M ; E_{\rho}\right)$ is a closed form and $\alpha=[\iota(\alpha)] \in H^{j}\left(M ; E_{\rho}\right)$. It extends naturally to

$$
\begin{equation*}
\iota: H^{\bullet}\left(M \times M ; E_{\rho} \boxtimes E_{\rho}\right) \rightarrow \Omega^{\bullet}\left(M \times M ; E_{\rho} \boxtimes E_{\rho}\right) . \tag{3.2.11}
\end{equation*}
$$

In the sequel, we fix $\iota$ once and for all.
Let $T$ denote an involution on $M \times M$ such that $T\left(x_{1}, x_{2}\right)=\left(x_{2}, x_{1}\right)$. Then it lifts to the vector bundle $E_{\rho} \boxtimes E_{\rho}$ via exchanging the two factors of tensors, which we still denote by $T$. By restricting to $\Delta$, the involution $T$ becomes a vector bundle automorphism of $E_{\rho} \otimes E_{\rho}$. In particular, the flat connections $\nabla^{E_{\rho} \boxtimes E_{\rho}}, \nabla^{E_{\rho} \otimes E_{\rho}}$ are equivariant under these $T$-actions.

Let $a, b, c, d$ be sections of $E_{\rho}$ over $M$, and let $a \boxtimes b, c \boxtimes d$ be two sections of $E_{\rho} \boxtimes E_{\rho}$ over $M \times M$, set

$$
\begin{equation*}
B_{1,2}(a \boxtimes b, c \boxtimes d)=p_{1}^{*}(B(a, c)) p_{2}^{*}(B(b, d)) . \tag{3.2.12}
\end{equation*}
$$

It extends to the differential forms valued in $E_{\rho} \boxtimes E_{\rho}$ as in the integrand of (3.1.3). Set

$$
\begin{equation*}
\widetilde{\Delta}_{12}:=\iota(\widetilde{\Delta})=\sum_{i}(-1)^{\operatorname{deg}\left(\alpha_{i}^{*}\right)} p_{1}^{*} \iota\left(\alpha_{i}\right) \wedge p_{2}^{*} \iota\left(\alpha_{i}^{*}\right), \tag{3.2.13}
\end{equation*}
$$

then $\widetilde{\Delta}_{12}$ is a smooth 3-form on $M \times M$ valued in $E_{\rho} \boxtimes E_{\rho}$.
Lemma 3.2.1. The closed 3-form $\widetilde{\Delta}_{12} \in \Omega^{3}\left(M \times M ; E_{\rho} \boxtimes E_{\rho}\right)$ satisfies the following properties:

- the cohomology class $\left[\widetilde{\Delta}_{12}\right]=\widetilde{\Delta} \in H^{3}\left(M \times M ; E_{\rho} \boxtimes E_{\rho}\right)$;
- $T^{*} \widetilde{\Delta}_{12}=-\widetilde{\Delta}_{12}$;
- if $\beta \in H^{\bullet}\left(M \times M ; E_{\rho} \boxtimes E_{\rho}\right)$, then

$$
\begin{equation*}
\int_{M \times M} B_{1,2}\left(\widetilde{\Delta}_{12} \wedge \iota(\beta)\right)=\int_{M} \mathfrak{i}^{*} B(\iota(\beta)) . \tag{3.2.14}
\end{equation*}
$$

Lemma 3.2.2. We have

$$
\begin{equation*}
\int_{\Delta} B\left(\mathfrak{i}^{*}\left(\widetilde{\Delta}_{12}\right)\right)=0 . \tag{3.2.15}
\end{equation*}
$$

Proof. By (3.1.3), (3.2.3) and (3.2.13), we get

$$
\begin{equation*}
\int_{\Delta} B\left(\mathrm{i}^{*}\left(\widetilde{\Delta}_{12}\right)\right)=\chi\left(M, E_{\rho}\right), \tag{3.2.16}
\end{equation*}
$$

where

$$
\begin{equation*}
\chi\left(M, E_{\rho}\right)=\sum_{j}(-1)^{j} \operatorname{dim} H^{j}\left(M, E_{\rho}\right), \tag{3.2.17}
\end{equation*}
$$

is the Euler characteristic number of $M$ associated with the local system $E_{\rho}$. By the Poincaré duality, we have

$$
\begin{equation*}
\chi\left(M, E_{\rho}\right)=-\chi\left(M, E_{\rho}\right)=0 . \tag{3.2.18}
\end{equation*}
$$

This completes the proof of our lemma.
Remark 3.2.3. As in [BC98, p101, Digression], after introducing the Riemannian metric on $M$, the diagonal class $\widetilde{\Delta}$ can be regarded as the integral kernel of the harmonic projection $\pi_{h}: \Omega^{\bullet}\left(M ; E_{\rho}\right) \rightarrow$ $H^{\bullet}\left(M ; E_{\rho}\right)$, that is,

$$
\begin{align*}
\pi_{h}(\beta) & =p_{1 *}\left(B_{2}\left(\widetilde{\Delta}_{12} \wedge p_{2}^{*} \iota(\beta)\right)\right) \\
& =\int_{2} B_{2}\left(\widetilde{\Delta}_{12} \wedge p_{2}^{*} \iota(\beta)\right) \tag{3.2.19}
\end{align*}
$$

### 3.3 Character variety of $\pi_{1}(M)$ and $H^{1}\left(M ; E_{\rho}\right)$

This subsection recalls some basic facts on the $G$-character variety of the fundamental group $\pi_{1}(M)$ of a closed 3-manifold $M$ and its relation with the first cohomology group $H^{1}\left(M ; E_{\rho}\right)$. Fore more details, see [Wei64], [LM85], and [Sav12].

Since $M$ is closed, its fundamental group $\pi_{1}(M)$ admits a finite presentation given as $\pi_{1}(M)=$ $\left\langle x_{1}, \ldots, x_{k} \mid r_{1}, \ldots, r_{m}\right\rangle$. Then, the $G$-representation space $R(M, G)$ of $\pi_{1}(M)$ is defined as $R(M, G)=$ $\operatorname{Hom}\left(\pi_{1}(M), G\right)$. Note that there is an embedding $R(M, G) \hookrightarrow G^{k}$ given by sending each representation $\varphi$ to the $k$-tuple $\left(\varphi\left(x_{1}\right), \ldots, \varphi\left(x_{k}\right)\right) \in G^{k}$. Thus, $R(M, G)$ can be viewed as a real algebraic set (or algebraic variety if $G$ is complex) and it is independent of chosen presentations. Since $G$ acts
on $R(M, G)=\operatorname{Hom}\left(\pi_{1}(M), G\right)$ by diagonal conjugation, the $G$-character variety $\mathcal{X}_{G}(M)$ of $\pi_{1}(M)$ is defined as categorical quotient of $R(M, G)$ by the $G$-conjugate action.

By considering the Zariski tangent space to $R(M, G)=\operatorname{Hom}\left(\pi_{1}(M), G\right)$ at $\varphi \in R(M, G)$, it turns out that there is a canonical identification

$$
\begin{equation*}
T_{\varphi} R(M, G)=Z_{\varphi}^{1}\left(\pi_{1}(M), \mathfrak{g}\right) \tag{3.3.1}
\end{equation*}
$$

where $Z_{\varphi}^{1}\left(\pi_{1}(M), \mathfrak{g}\right)$ is the space of 1-cocycles, i.e., maps $z: \pi_{1}(M) \rightarrow \mathfrak{g}$ satisfying the 1-cocycle condition $z\left(\gamma_{1} \gamma_{2}\right)=z\left(\gamma_{1}\right)+\gamma_{1} \cdot z\left(\gamma_{2}\right)$ for any $\gamma_{1}, \gamma_{2} \in \pi_{1}(M)$. Here, $\pi_{1}(M)$ action on $\mathfrak{g}$ is given by $\gamma \cdot v:=((\operatorname{Ad} \circ \varphi)(\gamma))(v)=\operatorname{Ad}(\varphi(\gamma))(v)$ for any $\gamma \in \pi_{1}(M)$ and $v \in \mathfrak{g}$. Moreover, it gives rise to an isomorphism between the tangent space of the character variety $\mathcal{X}_{G}(M)$ and the first cohomology group of $\pi_{1}(M)$ with coefficient in the $\pi_{1}(M)$-module $\mathfrak{g}$ via $\operatorname{Ad} \circ \varphi: \pi_{1}(M) \rightarrow \mathfrak{g}$

$$
\begin{equation*}
T_{[\varphi]} \mathcal{X}_{G}(M)=H_{\varphi}^{1}\left(\pi_{1}(M), \mathfrak{g}\right)=H^{1}\left(M ; E_{\rho}\right) \tag{3.3.2}
\end{equation*}
$$

for $\rho=\operatorname{Ad\circ } \varphi: \pi_{1}(M) \rightarrow \operatorname{Aut}(\mathfrak{g})$. For the last isomorphism, see for example [Ste43, §20].
Therefore, for $\rho=\operatorname{Ad\circ \varphi }$, if the class [ $\varphi$ ] is isolated point in the $G$-character variety $\mathcal{X}_{G}(M)$ of $\pi_{1}(M)$, then $H^{1}\left(M, E_{\rho}\right)=0$ (but its converse does not always hold).

An element $\varphi \in R(M, G)$ is called irreducible when its centralizer coincides with the center of $G$ (cf. [Sav02, §3.2.1]). Recall that, when $G$ is connected, the kernel of the adjoint action Ad : $G \rightarrow \operatorname{Aut}(\mathfrak{g})$ coincides with the center of $G$. Therefore, one has $H^{0}\left(M ; E_{\rho}\right)=\mathfrak{g}^{\pi_{1}(M)}=0$ if $G$ is connected and $\varphi$ is irreducible.

We end this subsection by giving an important class of examples that satisfy acyclic condition $H^{\bullet}\left(M ; E_{\rho}\right)=0$ for the convenience of the readers.
Example 3.3.1. (Fintushel-Stern ([FS92], see also [Sav12, Lecture 15])) Let $n \geq 3$ be a fixed integer and $a_{1}, \ldots, a_{n}$ be pairwise relatively prime integers with $a_{i} \geq 2$. Let $\Sigma=\Sigma\left(a_{1}, \ldots, a_{n}\right)$ be the Seifert homology sphere determined by $\left(a_{1}, \ldots, a_{n}\right)$ (cf. [Sav12, §6.3]), which is the closed orientable 3 -manifold to be considered, and let $\varphi: \pi_{1}(\Sigma) \rightarrow \mathrm{SU}(2)$ be an irreducible representation. Recall that $\pi_{1}(\Sigma)$ admits a finite presentation as

$$
\begin{equation*}
\pi_{1}(\Sigma)=\left\langle x_{1}, \ldots, x_{n}, h \mid\left[h, x_{i}\right]=1, x_{i}^{a_{i}} h^{b_{i}}=1, x_{1} \cdots x_{n}=1\right\rangle \tag{3.3.3}
\end{equation*}
$$

where $b_{1}, \ldots, b_{n}$ are integers satisfying the equation

$$
\begin{equation*}
a_{1} \cdots a_{n} \cdot \sum_{i=1}^{n} \frac{b_{i}}{a_{i}}=1 \tag{3.3.4}
\end{equation*}
$$

Suppose that, for an integer $m \geq 3, \varphi\left(x_{k}\right) \neq \pm 1$ for $k=1, \ldots, m$ and $\varphi\left(x_{k}\right)= \pm 1$ for $k=m+1, \ldots, n$. Then, it turns out that $H_{\varphi}^{1}\left(\pi_{1}(\Sigma), \mathfrak{s u}(2)\right)=H^{1}\left(\Sigma ; E_{\rho}\right)=\mathbb{R}^{2 m-6}$ with $\rho=\operatorname{Ad} \circ \varphi$. In particular, when $m=3$, one obtains $H^{1}\left(\Sigma ; E_{\rho}\right)=0$. Since $\varphi$ is irreducible, we have $H^{0}\left(\Sigma ; E_{\rho}\right)=0$. Therefore, using Poincaré duality (3.1.3), we conclude $H^{\bullet}\left(\Sigma ; E_{\rho}\right)=0$ and $\rho$ with $m=3$ gives an acyclic local system on $\Sigma$.

### 3.4 Killing form, cubic trace form

Recall that the Killing form $B \in \mathfrak{g}^{*} \otimes \mathfrak{g}^{*}$ is a non-degenerate bilinear form, we have the corresponding Casimir element $1 \in \mathfrak{g} \otimes \mathfrak{g}$. Let $e_{1}, \ldots, e_{\operatorname{dim} \mathfrak{g}}$ be a basis of $\mathfrak{g}$, and let $e_{1}^{*}, \ldots, e_{\operatorname{dim} \mathfrak{g}}^{*} \in \mathfrak{g}$ be dual basis of $\left\{e_{i}\right\}$ with respect to $B$, i.e., $B\left(e_{i}, e_{j}^{*}\right)=\delta_{i j}$. Then $\mathbf{1}$ can be explicitly written as

$$
\begin{equation*}
\mathbf{1}=\sum_{i=1}^{\operatorname{dim} \mathfrak{g}} e_{i} \otimes e_{i}^{*} . \tag{3.4.1}
\end{equation*}
$$

Since $B$ is invariant under the adjoint action of $\pi_{1}(M)$, it induces a fiberwise non-degenerate bilinear form on the vector bundle $E_{\rho} \rightarrow M$. By abuse of notation, we use the same $B$ to denote it, i.e., we have

$$
\begin{equation*}
B: E_{\rho} \otimes E_{\rho} \rightarrow \mathbb{R}, \tag{3.4.2}
\end{equation*}
$$

where $\underline{\mathbb{R}}$ stands for the trivial local system on $M$.
Moreover, the element $\mathbf{1}$ is fixed by the diagonal action of $\pi_{1}(M)$ on $\mathfrak{g} \otimes \mathfrak{g}$. This way, we can view 1 as a smooth section of $E_{\rho} \otimes E_{\rho}$ on $M$. Moreover, we get a well-defined map

$$
\begin{equation*}
I: \mathbb{R} \rightarrow E_{\rho} \otimes E_{\rho}, \quad 1 \mapsto \mathbf{1} \tag{3.4.3}
\end{equation*}
$$

The following result is clear by definition.

Lemma 3.4.1. The section $\mathbf{1}$ is a flat section.
Now we introduce the cubic trace form. If $a, b, c \in \mathfrak{g}$, set

$$
\begin{equation*}
\operatorname{Tr}[a \otimes b \otimes c]=B([a, b], c) \tag{3.4.4}
\end{equation*}
$$

Then $\operatorname{Tr} \in \Lambda^{3}\left(\mathfrak{g}^{*}\right)$, which is $\operatorname{Ad}(G)$-invariant. Then it extends to $\Lambda^{3}\left(E_{\rho}^{*}\right)$, which is flat with respect to $\nabla^{E_{\rho}}$.

### 3.5 Lie bracket on $E_{\rho}$

We set $\mathfrak{L}:=[\cdot, \cdot]: \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}, a \otimes b \mapsto[a, b]$. Let $\mathfrak{h} \subset \mathfrak{g} \otimes \mathfrak{g}$ be the kernel space of $\mathfrak{L}$. Since $\mathfrak{g}$ is semisimple, then $\mathfrak{L}$ is surjective. Then

$$
\begin{equation*}
\operatorname{dim}_{\mathbb{R}} \mathfrak{h}=\left(\operatorname{dim}_{\mathbb{R}} \mathfrak{g}\right)^{2}-\operatorname{dim}_{\mathbb{R}} \mathfrak{g} \tag{3.5.1}
\end{equation*}
$$

In particular, the symmetric tensor space $S^{2} \mathfrak{g}$ is a subspace of $\mathfrak{h}$.
Recall that $\pi_{1}(M)$ acts on $\mathfrak{g} \otimes \mathfrak{g}$ by the diagonal action via $\rho$. The Lie bracket operator $\mathfrak{L}$ is equivariant with respect to the actions of $\pi_{1}(M)$. Then $\pi_{1}(M)$-action preserves the subspace $\mathfrak{h}$. Set the vector bundle

$$
\begin{equation*}
H_{\rho}=\pi_{1}(M) \backslash\left(\widetilde{M} \times_{\rho} \mathfrak{h}\right) . \tag{3.5.2}
\end{equation*}
$$

It is equipped with the canonically induced flat connection $\nabla^{H_{\rho}}$ and is a subbundle of $E_{\rho} \otimes E_{\rho}$.
Moreover, the Lie bracket operator $\mathfrak{L}$ defines canonically a morphism of vector bundles on $M$,

$$
\begin{equation*}
\mathfrak{L}: E_{\rho} \otimes E_{\rho} \rightarrow E_{\rho} . \tag{3.5.3}
\end{equation*}
$$

By construction, it preserves the flat connections, i.e., when acting on smooth sections, $\nabla^{E_{\rho}} \circ \mathfrak{L}=$ $\mathfrak{L} \circ \nabla^{E_{\rho} \otimes E_{\rho}}$. We have the following properties:

- If $s_{i}, i=1,2,3$ are smooth sections of $E_{\rho}$ on $M$, then

$$
\begin{align*}
& \mathfrak{L}\left(s_{1} \otimes s_{2}\right)=-\mathfrak{L}\left(s_{2} \otimes s_{1}\right),  \tag{3.5.4}\\
& \mathfrak{L}\left(\mathfrak{L}\left(s_{1} \otimes s_{2}\right) \otimes s_{3}\right)+\mathfrak{L}\left(\mathfrak{L}\left(s_{2} \otimes s_{3}\right) \otimes s_{1}\right)+\mathfrak{L}\left(\mathfrak{L}\left(s_{3} \otimes s_{1}\right) \otimes s_{2}\right)=0 .
\end{align*}
$$

- We also have

$$
\begin{equation*}
\operatorname{Tr}\left[s_{1} \otimes s_{2} \otimes s_{3}\right]=B\left(\mathfrak{L}\left(s_{1} \otimes s_{2}\right) \otimes s_{3}\right) \in C^{\infty}(M) . \tag{3.5.5}
\end{equation*}
$$

We get the following identification of vector bundles

$$
\begin{equation*}
H_{\rho}=\operatorname{ker}\left(\mathfrak{L}: E_{\rho} \otimes E_{\rho} \rightarrow E_{\rho}\right), \tag{3.5.6}
\end{equation*}
$$

where $\nabla^{H_{\rho}}$ is exactly the one inherited from the flat connection $\nabla^{E_{\rho} \otimes E_{\rho}}$.
We also extend it on $\Omega^{\bullet}\left(M, E_{\rho}\right)$ such that if $\alpha, \beta \in \Omega^{\bullet}(M), s_{1}, s_{2} \in C^{\infty}\left(M, E_{\rho}\right)$,

$$
\begin{equation*}
\mathfrak{L}\left(\alpha s_{1} \otimes \beta s_{2}\right)=\alpha \wedge \beta \mathfrak{L}\left(s_{1} \otimes s_{2}\right) . \tag{3.5.7}
\end{equation*}
$$

Then $\mathfrak{L}$ induces the morphism of de Rham cohomology groups,

$$
\begin{equation*}
\mathfrak{L}: H^{\bullet}\left(M ; E_{\rho} \otimes E_{\rho}\right) \rightarrow H^{\bullet}\left(M ; E_{\rho}\right) . \tag{3.5.8}
\end{equation*}
$$

By our construction, we have the short exact sequence of flat vector bundles:

$$
0 \rightarrow H_{\rho} \hookrightarrow E_{\rho} \otimes E_{\rho} \rightarrow E_{\rho} \rightarrow 0 .
$$

Then we obtain the following long exact sequence of cohomology groups,

$$
\begin{gather*}
\cdots \rightarrow H^{0}\left(M ; E_{\rho}\right) \rightarrow H^{1}\left(M ; H_{\rho}\right) \rightarrow H^{1}\left(M ; E_{\rho} \otimes E_{\rho}\right) \xrightarrow{\mathfrak{L}} H^{1}\left(M ; E_{\rho}\right)  \tag{3.5.9}\\
\rightarrow H^{2}\left(M ; H_{\rho}\right) \rightarrow H^{2}\left(M ; E_{\rho} \otimes E_{\rho}\right) \xrightarrow{\mathfrak{L}} H^{2}\left(M ; E_{\rho}\right) \rightarrow \cdots
\end{gather*}
$$

Note that $T$ acts on $M$ as identity but exchanges the factors of the tensor $E_{\rho} \otimes E_{\rho}$. It induces the involutions $T^{*}$ acting on the de Rham complexes, and hence on the cohomology groups $H^{\bullet}\left(M ; H_{\rho}\right), H^{\bullet}\left(M ; E_{\rho} \otimes E_{\rho}\right)$. Let $\Omega_{ \pm}^{\bullet}\left(M ; H_{\rho}\right), \Omega_{ \pm}^{\bullet}\left(M ; E_{\rho} \otimes E_{\rho}\right), H_{ \pm}^{\bullet}\left(M ; H_{\rho}\right), H_{ \pm}^{\bullet}\left(M ; E_{\rho} \otimes E_{\rho}\right)$ denote the eigenspaces of $T^{*}$ corresponding to the eigenvalues $\pm 1$. One can verify directly that for each class $[\alpha] \in H_{ \pm}^{\bullet}(M ; \cdots)$, it always admits a representative $\alpha \in \Omega_{ \pm}^{\bullet}(M ; \cdots)$. In particular, we have the splitting of the de Rham complexes

$$
\begin{equation*}
\left(\Omega^{\bullet}(M ; \cdots), d\right)=\left(\Omega_{+}^{\bullet}(M ; \cdots), d\right) \oplus\left(\Omega_{-}^{\bullet}(M ; \cdots), d\right), \tag{3.5.10}
\end{equation*}
$$

which also corresponds to the splitting of the associated cohomology groups

$$
\begin{equation*}
H^{\bullet}(M ; \cdots)=H_{+}^{\bullet}(M ; \cdots) \oplus H_{-}^{\bullet}(M ; \cdots) \tag{3.5.11}
\end{equation*}
$$

Example 3.5.1. We consider the flat section 1 of $E_{\rho} \otimes E_{\rho}$ over $M$ defined in previous subsection, i.e., $\mathbf{1} \in H^{0}\left(M ; E_{\rho} \otimes E_{\rho}\right)$. A straightforward computation shows that $\mathfrak{L}(\mathbf{1})=0$, hence we have $\mathbf{1} \in H^{0}\left(M ; H_{\rho}\right)$. Since $B$ is a symmetric bilinear form, we also conclude $T^{*} \mathbf{1}=\mathbf{1}$, i.e., $\mathbf{1} \in H_{+}^{0}\left(M, H_{\rho}\right)$.
Proposition 3.5.2. We have the following isomorphism

$$
\begin{equation*}
H_{+}^{\bullet}\left(M ; H_{\rho}\right) \simeq H_{+}^{\bullet}\left(M ; E_{\rho} \otimes E_{\rho}\right) . \tag{3.5.12}
\end{equation*}
$$

We also have the long exact sequence as follows

$$
\begin{gather*}
\cdots \rightarrow H^{0}\left(M ; E_{\rho}\right) \rightarrow H_{-}^{1}\left(M ; H_{\rho}\right) \rightarrow H_{-}^{1}\left(M ; E_{\rho} \otimes E_{\rho}\right) \xrightarrow{\mathfrak{L}} H^{1}\left(M ; E_{\rho}\right)  \tag{3.5.13}\\
\rightarrow H_{-}^{2}\left(M ; H_{\rho}\right) \rightarrow H_{-}^{2}\left(M ; E_{\rho} \otimes E_{\rho}\right) \xrightarrow{\mathfrak{L}} H^{2}\left(M ; E_{\rho}\right) \rightarrow \cdots
\end{gather*}
$$

Proof. We split $\mathfrak{g} \otimes \mathfrak{g}$ as the anti-symmetric part and symmetric part, then we get

$$
\begin{equation*}
\mathfrak{g} \otimes \mathfrak{g}=\Lambda^{2} \mathfrak{g} \oplus S^{2} \mathfrak{g} \tag{3.5.14}
\end{equation*}
$$

This splitting corresponds to the eigenspace decomposition of the involution $T$ on $\mathfrak{g} \otimes \mathfrak{g}$. Then we get an exact sequence of vector spaces

$$
\begin{equation*}
0 \rightarrow \Lambda^{2} \mathfrak{g} \cap \mathfrak{h} \rightarrow \Lambda^{2} \mathfrak{g} \xrightarrow{\mathfrak{L}} \mathfrak{g} \rightarrow 0 . \tag{3.5.15}
\end{equation*}
$$

This exact sequence is compatible with the action $\pi_{1}(M)$, which lifts to the short exact sequence of associated local systems. Hence we can get the long exact sequence (3.5.13). Then combining it with (3.5.9), we get (3.5.12).

If $E_{\rho}$ is acyclic, then by the above long exact sequence, we get a canonical isomorphism

$$
\begin{equation*}
H_{ \pm}^{\bullet}\left(M ; H_{\rho}\right) \simeq H_{ \pm}^{\bullet}\left(M ; E_{\rho} \otimes E_{\rho}\right) . \tag{3.5.16}
\end{equation*}
$$

As a consequence, for each cohomological class $[\xi] \in H_{ \pm}^{\bullet}\left(M ; E_{\rho} \otimes E_{\rho}\right)$, there exists a closed form $\xi_{0} \in \Omega_{ \pm}^{\bullet}\left(M ; H_{\rho}\right)$, such that $\left[\xi_{0}\right]=[\xi]$ and

$$
\begin{equation*}
\mathfrak{L}\left(\xi_{0}\right)=0 \tag{3.5.17}
\end{equation*}
$$

Remark 3.5.3. By Example 3.5.1, we see that $\mathbf{1} \in H_{+}^{0}\left(M ; E_{\rho} \otimes E_{\rho}\right)$, which means that $E_{\rho} \otimes E_{\rho}$ can never be acyclic.

The following proposition can be viewed as an extension of [Shi23, Lemma 4.6].
Proposition 3.5.4. If $G$ is a real 3-dimensional simple Lie group, then we have

$$
\begin{equation*}
H_{-}^{\bullet}\left(M ; H_{\rho}\right)=0, H_{-}^{\bullet}\left(M ; E_{\rho} \otimes E_{\rho}\right) \simeq H^{\bullet}\left(M ; E_{\rho}\right) \tag{3.5.18}
\end{equation*}
$$

In particular, if in addition $E_{\rho}$ is acyclic, then

$$
H_{-}^{\bullet}\left(M ; E_{\rho} \otimes E_{\rho}\right)=0 .
$$

We need to point out that if $G$ is semisimple and real 3-dimensional, then it has to be a simple Lie group. In fact, for such linear Lie group $G$, if $G$ is compact, then $G=\operatorname{SU}(2)$ or $\mathrm{SO}(3)$; if $G$ is noncompact, then $G=\mathrm{SL}_{2}(\mathbb{R})$ or $\mathrm{SO}(2,1)$.
Proof of Proposition 3.5.4. When $\mathfrak{g}$ is simple with $\operatorname{dim}_{\mathbb{R}} \mathfrak{g}=3$, we conclude directly $\mathfrak{h}=S^{2} \mathfrak{g}$. Then $T^{*}$ acts on $H^{\bullet}\left(M ; H_{\rho}\right)$ as identity, we get

$$
\begin{equation*}
H_{+}^{\bullet}\left(M ; H_{\rho}\right)=H^{\bullet}\left(M ; H_{\rho}\right), \quad H_{-}^{\bullet}\left(M ; H_{\rho}\right)=0 . \tag{3.5.19}
\end{equation*}
$$

Then this proposition follows from Proposition 3.5.2.
At last, we give an example of a pair $(M, \rho)$ such that $H^{\bullet}\left(M ; E_{\rho}\right)=0$, but $H_{-}^{1}\left(M ; E_{\rho} \otimes E_{\rho}\right)=$ $H_{-}^{2}\left(M ; E_{\rho} \otimes E_{\rho}\right) \neq 0$ when $G=\operatorname{SL}(2, \mathbb{C}) \times \operatorname{SL}(2, \mathbb{C})$.
Proposition 3.5.5. Let $M$ be an oriented closed hyperbolic 3-manifold that contains a totally geodesic surface. Let

$$
\rho: \pi_{1}(M) \xrightarrow{\overline{\text { hol }}} \mathrm{SL}(2, \mathbb{C}) \xrightarrow{\text { Id } \times \overline{\mathrm{Id}}} \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C}) \xrightarrow{\text { Ad }} \operatorname{Aut}\left(\mathfrak{s l}_{2}(\mathbb{C}) \oplus \mathfrak{s l}_{2}(\mathbb{C})\right)
$$

where $\pi_{1}(M) \xrightarrow{\widehat{\text { hol }}} \mathrm{SL}(2, \mathbb{C})$ is a lift of the holonomy representation hol : $\pi_{1}(M) \rightarrow \operatorname{PSL}(2, \mathbb{C})$ corresponding to the complete hyperbolic structure of $M$ and $\overline{\mathrm{Id}}: \mathrm{SL}(2, \mathbb{C}) \rightarrow \mathrm{SL}(2, \mathbb{C})$ denotes the complex conjugation of matrix. Then, we have

$$
\begin{equation*}
H^{\bullet}\left(M ; E_{\rho}\right)=0, \quad H_{-}^{1}\left(M ; H_{\rho}\right) \simeq H^{1}\left(M ; \Lambda^{2} E_{\rho}\right) \neq 0 . \tag{3.5.20}
\end{equation*}
$$

Proof. To prove the statement we use several facts about hyperbolic 3-manifolds, for example, summarized in Porti's article [Por13]. Let $V_{2,0}$ denote the space of two variable degree 2 homogeneous polynomials with coefficients in $\mathbb{C}$. Then, $\mathrm{SL}(2, \mathbb{C})$ acts on $V_{2,0}$ by $(A, P) \mapsto P \circ A^{t}$ for $A \in \mathrm{SL}(2, \mathbb{C})$ and $P \in V_{2,0}$, where $A^{t}$ denotes the transposition of $A$. We set $V_{0,2}:=\overline{V_{2,0}}$ the complex conjugate representation to $V_{2,0}$. Then, as is well known $V_{2,0}, V_{0,2}$ and $V_{2,2}:=V_{2,0} \otimes V_{0,2}$ are irreducible representations of $\operatorname{SL}(2, \mathbb{C})$.

Let $\mathfrak{s l}_{2}(\mathbb{C})_{\mathrm{Ad}}$ and $\mathfrak{s l}_{2}(\mathbb{C})_{\overline{\mathrm{Ad}}}$ denote the $\mathfrak{s l}_{2}(\mathbb{C})$ as $\mathrm{SL}(2, \mathbb{C})$-modules via adjoint representation Ad : $\mathrm{SL}(2, \mathbb{C}) \rightarrow \operatorname{Aut}\left(\mathfrak{s l}_{2}(\mathbb{C})\right)$ and its complex conjugate $\overline{\mathrm{Ad}}$ respectively. Then, as $\mathrm{SL}(2, \mathbb{C})$-modules we have $\mathfrak{s l}_{2}(\mathbb{C})_{\mathrm{Ad}} \simeq V_{2,0}$ and $\mathfrak{s l}_{2}(\mathbb{C})_{\overline{\mathrm{Ad}}} \simeq V_{0,2}$. Hence, $H^{\bullet}\left(M ; E_{\rho}\right)=H^{\bullet}\left(M ; V_{2,0}\right) \oplus H^{\bullet}\left(M ; V_{0,2}\right)=0$ by Raghunathan vanishing theorem ([Por13, Theorem 5.1], [Rag65]), irreducibility of representations $V_{2,0}$ and $V_{0,2}$, and Poincaré duality. Here, by abuse of notation, we denote the local systems associated with $\pi_{1}(M) \xrightarrow{\text { hol }} \mathrm{SL}(2, \mathbb{C}) \rightarrow \mathrm{GL}(V)$ for $V=V_{2,0}, V_{0,2}$ by the same symbols $V_{2,0}$ and $V_{0,2}$ respectively. We have isomorphisms as $\operatorname{SL}(2, \mathbb{C})$-modules

$$
\begin{align*}
& \Lambda^{2}\left(\mathfrak{s l}_{2}(\mathbb{C})_{\mathrm{Ad}} \oplus \mathfrak{s l}_{2}(\mathbb{C})_{\overline{\mathrm{Ad}}}\right) \\
& \simeq \Lambda^{2}\left(V_{2,0} \oplus V_{0,2}\right)  \tag{3.5.21}\\
& \simeq \Lambda^{2}\left(V_{2,0}\right) \oplus\left(V_{2,0} \otimes V_{0,2}\right) \oplus \Lambda^{2}\left(V_{0,2}\right) \\
& \simeq V_{2,0} \oplus V_{2,2} \oplus V_{0,2},
\end{align*}
$$

where we refer, for example, [FH91, Excercise 11.35] for the last isomorphism. Again by Raghunathan vanishing theorem, we get

$$
\begin{equation*}
H^{1}\left(M ; \Lambda^{2} E_{\rho}\right) \simeq H^{1}\left(M ; V_{2,2}\right) \tag{3.5.22}
\end{equation*}
$$

Then, by Millson's theorem ([Por13, Proposition 5.4], [Mil85]), we conclude $H^{1}\left(M ; V_{2,2}\right) \neq 0$. The assertion is proved.

Remark 3.5.6. It is known that there are infinitely many hyperbolic rational homology spheres containing closed embedded totally geodesic surfaces [DeB06, Theorem 2]. Therefore, by combining this fact with Proposition 3.5.5, one sees that there are infinitely many examples of a pair ( $M, \rho$ ) which satisfies the condition $H^{\bullet}\left(M ; E_{\rho}\right)=0$ and $H_{-}^{1}\left(M ; E_{\rho} \otimes E_{\rho}\right)=H_{-}^{2}\left(M ; E_{\rho} \otimes E_{\rho}\right) \neq 0$.

## 4 Construction of the propagator on $C_{2}(M)$

This section describes a general recipe to construct propagators, from which one can define configuration space integrals, for non-acyclic local systems inspired by [CS21], [CM10], and [CW23]. Then, we introduce a class of propagators, called adapted propagators, for acyclic local systems. They have a distinguished feature that is crucial for our results in subsequent sections.

Recall that we denote by $p_{i}: M \times M \rightarrow M$ the $i$-th projection map, i.e., $p_{i}\left(x_{1}, x_{2}\right)=x_{i}$. In the sequel, we also denote the $i$-th projection map by the same notation $p_{i}: C_{2}(M) \rightarrow M$, so that the blow-down map $q: C_{2}(M) \rightarrow M \times M$ can be written as $q=\left(p_{1}, p_{2}\right)$. Moreover, for $n \geq 2$, the projection $(i \neq j)$

$$
\begin{equation*}
p_{i j}: \operatorname{Conf}_{n}(M) \ni\left(x_{1}, \ldots, x_{n}\right) \mapsto\left(x_{i}, x_{j}\right) \in \operatorname{Conf}_{2}(M) \tag{4.0.1}
\end{equation*}
$$

induces a smooth map of manifolds with corners

$$
\begin{equation*}
C_{n}(M) \rightarrow C_{2}(M) \tag{4.0.2}
\end{equation*}
$$

which is still denoted by $p_{i j}$. By Proposition 2.4.2, $p_{i j}$ is a submersion for manifolds with corners.
We define the pull-back vector bundle $F_{\rho}:=q^{*}\left(E_{\rho} \boxtimes E_{\rho}\right) \rightarrow C_{2}(M)$. Note that $\left.F_{\rho}\right|_{\partial C_{2}(M)}$ is just the pull-back of $E_{\rho} \otimes E_{\rho} \rightarrow M$ by the projection $q_{\partial}: \partial C_{2}(M) \rightarrow \Delta \simeq M$, which we still denote by $E_{\rho} \otimes E_{\rho}$. Moreover, we have the corresponding induced flat connection $\nabla^{F_{\rho}}$ on $F_{\rho}$.

### 4.1 Regular forms

We recall the notion of regular forms on compactified configuration space, introduced in [CM10, Appendix B]. In some sense, regular forms encode the non-singular part of a form on $M^{n}$ which may diverge along the diagonals but has a smooth extension to $C_{n}(M)$.

Let $q: C_{n}(M) \rightarrow M^{n}$ be the blow-down map. Then, a form in $\Omega^{\bullet}\left(C_{n}(M) ; q^{*}\left(E_{\rho}^{\boxtimes n}\right)\right)$ is called regular if it is the pullback $q^{*}(\xi)$ of a form $\xi \in \Omega^{\bullet}\left(M^{n} ; E_{\rho}^{\boxtimes n}\right)$ via $q$. We denote by $\Omega_{\mathrm{reg}}^{\bullet}\left(C_{n}(M) ; q^{*}\left(E_{\rho}^{\boxtimes n}\right)\right)$ the space of $q^{*}\left(E_{\rho}^{\boxtimes n}\right)$-valued regular forms on $C_{n}(M)$.

For two differential forms $\alpha, \beta \in \Omega^{\bullet}\left(C_{2}(M) ; F_{\rho}\right)$ one of which is regular, their convolution is defined by $\alpha * \beta:=p_{13, *}\left(B_{2}\left(p_{12}^{*} \alpha \wedge p_{23}^{*} \beta\right)\right)$. Then, the convolution $*$ defines bilinear maps as [CM10, Lemma 5],

$$
\begin{align*}
& *: \Omega^{\bullet}\left(C_{2}(M) ; F_{\rho}\right) \times \Omega_{\mathrm{reg}}^{\bullet}\left(C_{2}(M) ; F_{\rho}\right) \rightarrow \Omega_{\mathrm{reg}}^{\bullet}\left(C_{2}(M) ; F_{\rho}\right)  \tag{4.1.1}\\
& *: \Omega_{\mathrm{reg}}^{\bullet}\left(C_{2}(M) ; F_{\rho}\right) \times \Omega^{\bullet}\left(C_{2}(M) ; F_{\rho}\right) \rightarrow \Omega_{\mathrm{reg}}^{\bullet}\left(C_{2}(M) ; F_{\rho}\right) .
\end{align*}
$$

For example, for $\alpha=q^{*} \alpha^{\prime} \in \Omega_{\mathrm{reg}}^{\bullet}\left(C_{2}(M) ; F_{\rho}\right)$ and $\beta \in \Omega^{\bullet}\left(C_{2}(M) ; F_{\rho}\right)$, their convolution is given by $\alpha * \beta=q^{*} \gamma$ where $\gamma=\left(\operatorname{pr}_{1} \times p_{2}\right)_{*} B_{2}\left(\left(\operatorname{pr}_{1} \times p_{1}\right)^{*} \alpha^{\prime} \wedge \operatorname{pr}_{2}^{*} \beta\right), \operatorname{pr}_{1}: M \times C_{2}(M) \rightarrow M$ and $\operatorname{pr}_{2}: M \times C_{2}(M) \rightarrow C_{2}(M)$. Note that here we essentially use the fibration structure of $\mathrm{pr}_{1} \times p_{2}$ : $M \times C_{2}(M) \rightarrow M \times M$ to define this convolution $*$, where the condition of regular forms guarantees the well-definedness.

### 4.2 An element in $H_{-}^{2}\left(\partial C_{2}(M) ; E_{\rho} \otimes E_{\rho}\right)$

The involution $T$ of $M \times M \rightarrow M \times M$ given by $\left(x_{1}, x_{2}\right) \mapsto\left(x_{2}, x_{1}\right)$ extends to an involution on $C_{2}(M)$, which preserves the boundary $\partial C_{2}(M)$. It also lifts to the bundle $F_{\rho}$ by exchanging the factors of the tensor product. Let $\Omega_{ \pm}^{\bullet}\left(C_{2}(M) ; F_{\rho}\right)\left(\operatorname{resp} . \Omega_{ \pm}^{\bullet}\left(\partial C_{2}(M) ; E_{\rho} \otimes E_{\rho}\right)\right)$ denote the ( $\pm 1$ )-eigenspaces of the action of $T^{*}$, and we also use similar convention for the cohomology groups.

We consider the oriented unit sphere $\mathbb{S}^{2}$ in $\mathbb{R}^{3}$. Let $T_{\mathbb{S}^{2}}$ be the involution on $\mathbb{S}^{2}$ given by $T_{\mathbb{S}^{2}}(v)=$ $-v, v \in \mathbb{S}^{2}$. Let $\eta$ denote a smooth normalized volume form on the unit sphere $\mathbb{S}^{2}$ such that $T_{\mathbb{S} 2}^{*} \eta=-\eta$. If $\eta^{\prime}$ is another such volume form, then there exists a 1 -form $\gamma$ on $\mathbb{S}^{2}$ with $T_{\mathbb{S}^{2}}^{*} \gamma=-\gamma$ such that

$$
\eta-\eta^{\prime}=d \gamma
$$

Consider the obvious projection $\pi: M \times \mathbb{S}^{2} \rightarrow \mathbb{S}^{2}$, then $\pi^{*} \eta$ is a closed 2-form on $M \times \mathbb{S}^{2}$.
Note that the sphere normal bundle $S \nu_{\Delta}$ is identified with the sphere tangent bundle $S(T M)$ by

$$
\begin{equation*}
S \nu_{\Delta} \xlongequal[\leftrightarrows]{\cong} S(T M), \quad((x, x),(v,-v)) \mapsto(x, v) . \tag{4.2.1}
\end{equation*}
$$

The involution $T$ on $C_{2}(M)$ restricting to the boundary corresponds to the involution on $S(T M)$ : $(x, v) \mapsto(x,-v)$. We always use $T$ to denote all these involution operators.

The given framing $f$ of $M$ induces a canonical identification $M \times \mathbb{S}^{2} \simeq S(T M) \simeq \partial C_{2}(M)$. This way, we view $\pi^{*} \eta$ as a closed 2-form on $\partial C_{2}(M)$. In particular, we have $T^{*}\left(\pi^{*} \eta\right)=\pi^{*}\left(T_{\mathbb{S} 2}^{*} \eta\right)=-\pi^{*} \eta$. With the above consideration, we have the following natural identification of vector spaces

$$
\begin{equation*}
H_{-}^{2}\left(\partial C_{2}(M) ; E_{\rho} \otimes E_{\rho}\right) \simeq H_{-}^{2}\left(M ; E_{\rho} \otimes E_{\rho}\right) \oplus\left(H^{2}\left(\mathbb{S}^{2} ; \mathbb{R}\right) \otimes H_{+}^{0}\left(M ; E_{\rho} \otimes E_{\rho}\right)\right) \tag{4.2.2}
\end{equation*}
$$

where $H^{2}\left(\mathbb{S}^{2} ; \mathbb{R}\right)=\mathbb{R}[\eta]$ is 1-dimensional, and the direct sum depends on the given framing $f$.
Recall that the element $\mathbf{1}$ is defined in (3.4.1), it is also regarded as a flat section of $E_{\rho} \otimes E_{\rho}$ over $M$ or $S(T M)$, hence on $\partial C_{2}(M)$. We define a 2 -form on $\Omega^{2}\left(\partial C_{2}(M) ; E_{\rho} \otimes E_{\rho}\right)$ by

$$
\begin{equation*}
I(\eta)=\pi^{*} \eta \otimes \mathbf{1} \tag{4.2.3}
\end{equation*}
$$

where the notation $I(\cdot)$ is compatible with the definition given in (3.4.3), and the form $\pi^{*} \eta$ should be viewed as the 2 -form on $\partial C_{2}(M)$ given via the pull-back of $f$, sometimes we also denote it by $f^{*} \eta$ to emphasize the role of the framing $f$.

The following lemma is an analog of [BC98, Proposition 3.1].
Lemma 4.2.1. The 2-form $I(\eta) \in \Omega^{2}\left(\partial C_{2}(M) ; E_{\rho} \otimes E_{\rho}\right)$ satisfies the following properties:
(i) $I(\eta)$ is a closed form,
(ii) $I(\eta)$ is a fiberwise tensor product of volume form and $\mathbf{1}$, i.e., its ( $E_{\rho} \otimes E_{\rho}$-valued) fiber integration is $\mathbf{1}$,
(iii) $I(\eta)$ is antisymmetric under the action of $T^{*}$, i.e., $T^{*}(I(\eta))=-I(\eta)$.

Proof. The first property follows from the closeness of $\eta$ and Lemma 3.4.1. The second and third properties follow from the properties of $\eta$ and the fact that $T^{*}$ acts trivially on $\mathbf{1}$.

The following result was already implied in [CS21, $\S 5.2$ Proof of Theorem 5.1]
Lemma 4.2.2. If $f^{\prime}$ is another smooth framing of $M$ which induces the same orientation as $f$ does, then by taking an oriented normalized volume form $\eta^{\prime}$ on $\mathbb{S}^{2}$, the corresponding closed 2-form $I\left(\eta^{\prime}\right)$ lies in the same de Rham cohomology class $[I(\eta)]$ as of $I(\eta)$ in $H_{-}^{2}\left(\partial C_{2}(M) ; E_{\rho} \otimes E_{\rho}\right)$.

Proof. Note that $H_{-}^{2}\left(\partial C_{2}(M) ; \mathbb{R}\right) \simeq H_{-}^{2}\left(M \times \mathbb{S}^{2} ; \mathbb{R}\right) \simeq H^{2}\left(\mathbb{S}^{2} ; \mathbb{R}\right)$ is 1-dimensional vector space spanned by $[\eta]$. Hence $\left[f^{*} \eta\right]=\left[\left(f^{\prime}\right)^{*} \eta^{\prime}\right] \in H_{-}^{2}\left(\partial C_{2}(M) ; \mathbb{R}\right)$. Then there exists $\beta^{\prime} \in \Omega_{-}^{1}\left(\partial C_{2}(M) ; \mathbb{R}\right)$ such that

$$
\begin{equation*}
f^{*} \eta-\left(f^{\prime}\right)^{*} \eta^{\prime}=d \beta^{\prime} . \tag{4.2.4}
\end{equation*}
$$

As a consequence, we conclude the identity in $\Omega_{-}^{2}\left(\partial C_{2}(M) ; E_{\rho} \otimes E_{\rho}\right)$,

$$
\begin{equation*}
I(\eta)-I\left(\eta^{\prime}\right)=\left(d \beta^{\prime}\right) \otimes \mathbf{1}=d\left(\beta^{\prime} \otimes \mathbf{1}\right) \tag{4.2.5}
\end{equation*}
$$

This way, we conclude this lemma.
In the above lemma, the framing $f^{\prime}$ is not necessary to be homotopic to $f$. When $f^{\prime}$ is homotopic to $f$, the form $\beta^{\prime}$ in (4.2.4) can be constructed more explicitly as follows.
Lemma 4.2.3. Fix an oriented normalized volume form $\eta$ on $\mathbb{S}^{2}$. If $f^{\prime}$ is another smooth framing of $M$ which is homotopic to $f$, let $I^{\prime}(\eta)$ be the closed 2-form in $\Omega_{-}^{2}\left(\partial C_{2}(M) ; E_{\rho} \otimes E_{\rho}\right)$ defined by $f^{\prime}$. Then there is a $f$-vertical 1 -form $\beta^{\prime} \in \Omega_{-}^{1}\left(\partial C_{2}(M) ; \mathbb{R}\right)$ such that

$$
\begin{equation*}
I^{\prime}(\eta)-I(\eta)=d\left(\beta^{\prime} \otimes \mathbf{1}\right) \tag{4.2.6}
\end{equation*}
$$

Proof. Note that in this case, $f^{\prime} \circ f^{-1}$ is connected to the identity section by a smooth path in $\mathscr{C}^{\infty}\left(M, \operatorname{Diff}\left(\mathbb{S}^{2}\right)\right)$. Let $\psi:[0,1] \ni t \mapsto \psi_{t} \in \mathscr{C}^{\infty}\left(M, \operatorname{Diff}\left(\mathbb{S}^{2}\right)\right)$ denote such a path with $\psi_{0}(x)=\operatorname{Id}_{\mathbb{S}^{2}}$ and $\psi_{1}(x)=\left(f^{\prime} \circ f^{-1}\right)_{x}$. In particular, we view $\psi_{t}$ as a diffeomorphism of $M \times \mathbb{S}^{2}$. Let $\Gamma\left(\mathbb{S}^{2}\right)$ denote the space of smooth vector fields on $\mathbb{S}^{2}$. Set $X_{t}=\frac{\partial}{\partial t} \psi_{t} \in \mathscr{C}^{\infty}\left(M, \Gamma\left(\mathbb{S}^{2}\right)\right)$. Fix an oriented normalized volume form $\eta$ on $\mathbb{S}^{2}$, viewed as a constant form on $M \times \mathbb{S}^{2}$, then

$$
\begin{equation*}
\frac{\partial}{\partial t} \psi_{t}^{*} \eta=d \iota_{X_{t}} \psi_{t}^{*} \eta \tag{4.2.7}
\end{equation*}
$$

where $\iota_{X_{t}}$ denotes the contraction of vector fields $X_{t}$. A direct computation shows that

$$
\begin{equation*}
\psi_{1}^{*} \eta-\eta=d \int_{0}^{1} \iota_{X_{t}} \psi_{t}^{*} \eta d t=: d \beta \tag{4.2.8}
\end{equation*}
$$

Then

$$
\begin{equation*}
\left(f^{\prime}\right)^{*} \eta-f^{*} \eta=d\left(f^{*} \beta\right) . \tag{4.2.9}
\end{equation*}
$$

Note that $\beta$ is a vertical 1-form on $M \times \mathbb{S}^{2}$, hence $\beta^{\prime}:=f^{*} \beta$ is $f$-vertical as desired. This way, we complete the proof.

Now we need to introduce a commutative diagram to understand better all the different cohomology groups $H_{-}^{\bullet}(\cdots ; \cdots)$ that we have seen. Recall that $\mathfrak{i}_{\partial}: \partial C_{2}(M) \hookrightarrow C_{2}(M)$ denotes the inclusion of the boundary of $C_{2}(M)$, and $\mathfrak{i}: \Delta \hookrightarrow M \times M$ denotes the inclusion of the diagonal.

Note that we have the following short exact sequence associated to the relative de Rham complex

$$
0 \rightarrow\left(\Omega^{\bullet}\left(C_{2}(M), \partial C_{2}(M) ; F_{\rho}\right), d\right) \xrightarrow{\mathrm{incl}}\left(\Omega^{\bullet}\left(C_{2}(M) ; F_{\rho}\right), d\right) \xrightarrow{\mathrm{i}_{\mathrm{a}}^{*}}\left(\Omega^{\bullet}\left(\partial C_{2}(M) ; E_{\rho} \otimes E_{\rho}\right), d\right) \rightarrow 0 .
$$

Together with the excision theorem for the pairs $\left(C_{2}(M), \partial C_{2}(M)\right)$ and $(M \times M, \Delta)$, we get the following commutative diagram where the horizontal lines are exact sequences

$$
\begin{align*}
& H_{-}^{2}\left(C_{2}(M) ; \mathbf{F}_{\rho}\right) \xrightarrow{\mathrm{i}_{\partial}^{*}} H_{-}^{2}\left(\partial C_{2}(M) ; \mathbf{F}_{\rho}\right) \xrightarrow{\delta_{C_{2}(M)}^{*}} H_{-}^{3}\left(C_{2}(M), \partial C_{2}(M) ; \mathbf{F}_{\rho}\right) \xrightarrow{\text { incl }} H_{-}^{3}\left(C_{2}(M) ; \mathbf{F}_{\rho}\right) \\
& H_{-}^{2}\left(M \times M ; \mathbf{F}_{\rho}\right) \xrightarrow{q^{*} \uparrow} \begin{array}{c}
q^{*} \uparrow \\
q^{*} \uparrow \\
\left.H_{-}^{2}\left(\Delta ; \mathbf{F}_{\rho}\right) \xrightarrow{q^{*} \uparrow} \xlongequal{\delta_{M^{2}}^{*}} H_{-}^{3}\left(M \times M, \Delta ; \mathbf{F}_{\rho}\right) \xrightarrow{\text { incl }} \begin{array}{c}
q^{*} \uparrow \\
\hline
\end{array} H_{-}^{3}\left(M \times M ; \mathbf{F}_{\rho}\right)\right)
\end{array} \tag{4.2.10}
\end{align*}
$$

where the bundle $\mathbf{F}_{\rho}$ represents $F_{\rho}$ on $C_{2}(M), E_{\rho} \otimes E_{\rho}$ on $\Delta$ or $\partial C_{2}(M), E_{\rho} \boxtimes E_{\rho}$ on $M \times M$ respectively.

The map $\delta_{C_{2}(M)}^{*}$ is given by the zig-zag lemma. More precisely, if $\alpha \in \Omega_{-}^{2}\left(\partial C_{2}(M) ; E_{\rho} \otimes E_{\rho}\right)$ is a closed form, then we have a form $\widetilde{\alpha} \in \Omega_{-}^{2}\left(C_{2}(M) ; F_{\rho}\right)$ such that $i_{\partial}^{*} \widetilde{\alpha}=\alpha$. Then $d \widetilde{\alpha}$ is an exact form in $\Omega_{-}^{3}\left(C_{2}(M) ; F_{\rho}\right)$ with $\mathfrak{i}_{\partial}^{*} d \widetilde{\alpha}=d \alpha=0$, this way $d \widetilde{\alpha}$ is a closed form in $\Omega_{-}^{3}\left(C_{2}(M), \partial C_{2}(M) ; F_{\rho}\right)$ which is not necessary to be exact in this relative de Rham complex. By definition, we have $\delta_{C_{2}(M)}^{*}[\alpha]=$ $[d \widetilde{\alpha}] \in H_{-}^{3}\left(C_{2}(M), \partial C_{2}(M) ; F_{\rho}\right)$. The map $\delta_{M^{2}}^{*}$ is defined in a similar way. In particular, when $E_{\rho}$ is acyclic, $\delta_{M^{2}}^{*}$ is an isomorphism.

### 4.3 Construction of the propagator with a local system

We always fix a map $\iota: H^{\bullet}\left(M ; E_{\rho}\right) \rightarrow \Omega^{\bullet}\left(M ; E_{\rho}\right)$ as in (3.2.10). In this subsection, we prove the following main result, which shows the existence of a propagator.

Proposition 4.3.1. There exists a smooth 2-form

$$
\begin{equation*}
\omega \in \Omega^{2}\left(C_{2}(M) ; F_{\rho}\right) \tag{4.3.1}
\end{equation*}
$$

satisfying the following four properties:
(i) $d \omega=\widetilde{\Delta}_{12}:=\sum_{i}(-1)^{\operatorname{deg}\left(\alpha_{i}^{*}\right)} p_{1}^{*}\left(\iota\left(\alpha_{i}\right)\right) \wedge p_{2}^{*}\left(\iota\left(\alpha_{i}^{*}\right)\right) \in \Omega^{3}\left(C_{2}(M) ; F_{\rho}\right)$
(ii) the restriction of $\omega$ on the boundary $\left.\omega\right|_{\partial C_{2}(M)}$ is a fiberwise tensor product of volume form with 1, i.e., its ( $E_{\rho} \otimes E_{\rho}$-valued) fiber integration is $\mathbf{1}$,
(iii) the form $\omega$ on $C_{2}(M)$ is antisymmetric under the action of $T^{*}$, i.e., $T^{*}(\omega)=-\omega$,
(iv) for any $\alpha \in H^{\bullet}\left(M ; E_{\rho}\right)$, we have

$$
\begin{equation*}
\int_{2} B_{2}\left(\omega_{12} \wedge p_{2}^{*}(\iota(\alpha))=0\right. \tag{4.3.2}
\end{equation*}
$$

Here, the integration $\int_{2}$ means the fiber integration along the fiber of $p_{1}: C_{2}(M) \rightarrow M$ and the subscript of $B_{2}$ means that contraction occurs on the second component of $F_{\rho}=q^{*}\left(E_{\rho} \boxtimes E_{\rho}\right)$.

Proof. The proof is similar to the case of a closed manifold with trivial local systems as in [BC98], [CM10], [CW23]: starting from $I(\eta) \in \Omega^{2}\left(\partial C_{2}(M) ; E_{\rho} \otimes E_{\rho}\right)$, then we extend it to the 2 -form in $\Omega^{2}\left(C_{2}(M) ; F_{\rho}\right)$, after modifying it by certain 2 -forms, we finally get a 2-form $\omega^{\prime}$ satisfying the conditions (i), (ii) and (iii). Finally, adding another correction term $\lambda \in \Omega^{2}\left(C_{2}(M) ; F_{\rho}\right)$ to $\omega^{\prime}$, we obtain a 2 -form $\omega$ which satisfies (i), (ii), (iii), and (iv).

Let $N \supset \partial C_{2}(M)$ be a tubular neighbourhood of $\partial C_{2}(M)$ with a projection $p: N \rightarrow \partial C_{2}(M)$. By pulling back $I(\eta)$ by $p^{*}$, we get the 2 -form

$$
\begin{equation*}
p^{*}(I(\eta)) \in \Omega^{2}\left(N ; p^{*}\left(\left.F_{\rho}\right|_{\Delta}\right)\right) . \tag{4.3.3}
\end{equation*}
$$

Note that, by definition, we have

$$
\begin{align*}
p^{*}\left(\left.F_{\rho}\right|_{\Delta}\right)_{(x, y)} & =\left(\left.F_{\rho}\right|_{\Delta}\right)_{p(x, y)} \\
& \left.=\left(\left.F_{\rho}\right|_{\Delta}\right)_{\left(x_{0}, x_{0}\right)} \quad \text { (here, we set } p(x, y)=\left(x_{0}, x_{0}\right)\right)  \tag{4.3.4}\\
& =E_{\rho, x_{0}} \otimes E_{\rho, x_{0}} .
\end{align*}
$$

Since $E_{\rho}$ is a vector bundle with flat connection $\nabla^{E_{\rho}}$, we have a linear isomorphism

$$
\begin{equation*}
\tau_{(x, y)}:\left(\left.F_{\rho}\right|_{N}\right)_{(x, y)} \xrightarrow{\sim} E_{\rho, x_{0}} \otimes E_{\rho, x_{0}}=p^{*}\left(\left.F_{\rho}\right|_{\Delta}\right)_{(x, y)} \tag{4.3.5}
\end{equation*}
$$

given by parallel transport along the projection $p$ with respect to $\nabla^{E_{\rho}}$. This way, we get an isomorphism of (flat) vector bundles on $N$,

$$
\begin{equation*}
\tau:\left.F_{\rho}\right|_{N} \xrightarrow{\sim} p^{*}\left(\left.F_{\rho}\right|_{\Delta}\right) . \tag{4.3.6}
\end{equation*}
$$

We get a 2-form

$$
\begin{equation*}
\tau^{-1} p^{*}(I(\eta)) \in \Omega^{2}\left(N,\left.F_{\rho}\right|_{N}\right) . \tag{4.3.7}
\end{equation*}
$$

Let us consider a smooth cutoff function $\chi: C_{2}(M) \rightarrow \mathbb{R}$ such that

- $\left.\chi\right|_{C_{2}(M) \backslash N} \equiv 0$,
- $\left.\chi\right|_{U} \equiv 1$,
where $U$ is an open set such that $\partial C_{2}(M) \subset U \subsetneq \operatorname{Inner}(N) \subset C_{2}(M)$.
Now, we can define a 2 -form on $C_{2}(M)$ as follows,

$$
\begin{equation*}
\chi \cdot \tau^{-1} p^{*}(I(\eta)) \in \Omega^{2}\left(C_{2}(M) ; F_{\rho}\right) . \tag{4.3.8}
\end{equation*}
$$

Moreover, if we compute its differential, then

$$
\begin{align*}
d\left(\chi \cdot \tau^{-1} p^{*}(I(\eta))\right) & =d \chi \wedge \tau^{-1} p^{*}(I(\eta))+\chi \cdot d\left(\tau^{-1} p^{*}(I(\eta))\right) \\
& \left.=d \chi \wedge \tau^{-1} p^{*}(I(\eta))+\chi \cdot \tau^{-1} p^{*}(d I(\eta))\right)  \tag{4.3.9}\\
& =d \chi \wedge \tau^{-1} p^{*}(I(\eta)) .
\end{align*}
$$

By the definition of $\chi$, the form $d \chi \wedge \tau^{-1} p^{*}(I(\eta))$ vanishes near $\partial C_{2}(M)$. Then the 3 -form $d\left(\chi \cdot \tau^{-1} p^{*}(I(\eta))\right)$ can be view as a 3 -form on $\Omega^{3}\left(M \times M ; E_{\rho} \boxtimes E_{\rho}\right)$. Note that $d \chi \wedge \tau^{-1} p^{*}(I(\eta))$ is an exact form on $C_{2}(M)$. When we regard it as a differential form on $M \times M$, in general, it is no longer exact, but it is still a closed form.

Hence, for any $\beta \in H^{\bullet}\left(M \times M ; E_{\rho} \boxtimes E_{\rho}\right)$, we have

$$
\begin{align*}
& \int_{M \times M} B_{1,2}\left(d\left(\chi \cdot \tau^{-1} p^{*}(I(\eta))\right) \wedge \iota(\beta)\right) \\
& =\int_{C_{2}(M)} B_{1,2}\left(d\left(\chi \cdot \tau^{-1} p^{*}(I(\eta))\right) \wedge \iota(\beta)\right) \\
& =\int_{\partial C_{2}(M)} B_{1,2}\left(\chi \cdot \tau^{-1} p^{*}(I(\eta)) \wedge \iota(\beta)\right) \\
& =\int_{\partial C_{2}(M)} B_{1,2}(I(\eta) \wedge \iota(\beta))  \tag{4.3.10}\\
& =\int_{M} \mathfrak{i}^{*} B_{1,2}(\mathbf{1} \otimes \iota(\beta)) \\
& =\int_{M} \mathfrak{i}^{*} B(\iota(\beta)) \\
& =\int_{M \times M} B_{1,2}\left(\widetilde{\Delta}_{12} \wedge \iota(\beta)\right) .
\end{align*}
$$

Note that the last equality in (4.3.10) follows from (3.2.14).
By (4.3.10), we see that the cohomology class of $d\left(\chi \cdot \tau^{-1} p^{*}(I(\eta))\right)$ and $\widetilde{\Delta}_{12}$ in $H^{\bullet}\left(M \times M, E_{\rho} \boxtimes E_{\rho}\right)$ are coincide. Then there exists a 2 -form $\psi \in \Omega^{2}\left(M \times M ; E_{\rho} \boxtimes E_{\rho}\right)$ such that

$$
\begin{equation*}
\widetilde{\Delta}_{12}-d\left(\chi \cdot \tau^{-1} p^{*}(I(\eta))\right)=d \psi \tag{4.3.11}
\end{equation*}
$$

This equation also holds if we take its pull-back to $C_{2}(M)$ from $M \times M$.
Now, we set

$$
\begin{equation*}
\omega_{0}=\chi \cdot \tau^{-1} p^{*}(I(\eta))+q^{*} \psi \in \Omega^{2}\left(C_{2}(M) ; F_{\rho}\right) . \tag{4.3.12}
\end{equation*}
$$

Then $\omega_{0}$ satisfies the condition (i) in our proposition. A direct computation shows its ( $E_{\rho} \otimes E_{\rho}$-valued) fiber integration along $\partial C_{2}(M) \rightarrow \Delta$ is $\mathbf{1}$.

For the condition (iii), we need to add more restrictive structures in the above construction of $\omega_{0}$. We can always take the tubular neighbourhood $N$ and the open set $U$ to be invariant by $T$. We also need to take a $T$-invariant cutoff function $\chi$.

Note that on $M \times M$,

$$
\begin{equation*}
T^{*}\left(\widetilde{\Delta}_{12}\right)=-\widetilde{\Delta}_{12} . \tag{4.3.13}
\end{equation*}
$$

Moreover, $T^{*}$ commutes with the differential $d$ on $\Omega^{2}\left(C_{2}(M) ; F_{\rho}\right)$. Therefore, we get a 2-from $\omega_{0}$ as in (4.3.12) such that both $\omega_{0}$ and $-T^{*} \omega_{0}$ satisfy the condition (i). Set

$$
\begin{equation*}
\omega^{\prime}=\frac{1}{2}\left(\omega_{0}-T^{*} \omega_{0}\right) \tag{4.3.14}
\end{equation*}
$$

Then $\omega^{\prime}$ is a 2-form satisfying the conditions (i) and (iii). Moreover, its ( $E_{\rho} \otimes E_{\rho}$-valued) fiber integration along $\partial C_{2}(M) \rightarrow \Delta$ is also 1 .

Now, we verify that $\omega^{\prime}$ satisfies the condition (ii). By (4.3.12), we get

$$
\begin{equation*}
\mathfrak{i}_{\partial}^{*} \omega^{\prime}=I(\eta)+\frac{1}{2}\left(\mathfrak{i}_{\partial}^{*} q^{*} \psi-\mathfrak{i}_{\partial}^{*} T^{*} q^{*} \psi\right) . \tag{4.3.15}
\end{equation*}
$$

Note that the fibre integration along $\partial C_{2}(M) \rightarrow \Delta$ of the second term in the right-hand side of (4.3.15) vanishes. Then $\omega$ also satisfies the condition (ii).

For condition (iv), we use a standard trick for normalizing homotopies as in [CM10] and [CW23] by adding a term $\lambda \in \Omega^{2}\left(C_{2}(M) ; F_{\rho}\right)$ defined as

$$
\begin{equation*}
\lambda:=\int_{3} B_{3}\left(\omega_{13}^{\prime} \widetilde{\Delta}_{23}\right)-\int_{3} B_{3}\left(\omega_{23}^{\prime} \widetilde{\Delta}_{13}\right)-\int_{3,4} B_{3,4}\left(\omega_{34}^{\prime} \widetilde{\Delta}_{13} \widetilde{\Delta}_{24}\right) \tag{4.3.16}
\end{equation*}
$$

Here, integral $\int_{3}$ (resp. $\int_{3,4}$ ) means the fiber integration along the fiber of $p_{12}: C_{3}(M) \rightarrow C_{2}(M)$ (resp. $p_{12}: C_{4}(M) \rightarrow C_{2}(M)$ and $B_{3}$ (resp. $B_{3,4}$ ) means that we contract on the third (resp. third and fourth) component of sections of the flat bundle. Note that all the factors $\widetilde{\Delta}$ in (4.3.16) are regular forms by definition so that $\lambda$ is well-defined as explained in Subsection 4.1.

At first, we shows that $T^{*} \lambda=-\lambda$. In fact, we have

$$
\begin{equation*}
T^{*} \lambda:=\int_{3} B_{3}\left(\omega_{23}^{\prime} \widetilde{\Delta}_{13}\right)-\int_{3} B_{3}\left(\omega_{13}^{\prime} \widetilde{\Delta}_{23}\right)-\int_{3,4} B_{3,4}\left(\omega_{34}^{\prime} \widetilde{\Delta}_{23} \widetilde{\Delta}_{14}\right) . \tag{4.3.17}
\end{equation*}
$$

Note that in the last term, after exchanging the factors 3 and 4 via the involution $T_{34}$ (the orientation changes sign), and using $T_{34}^{*} \omega_{34}^{\prime}=-\omega_{34}^{\prime}$, we get

$$
\begin{equation*}
\int_{3,4} B_{3,4}\left(\omega_{34}^{\prime} \widetilde{\Delta}_{23} \widetilde{\Delta}_{14}\right)=-\int_{3,4} T_{34}^{*}\left(B_{3,4}\left(\omega_{34}^{\prime} \widetilde{\Delta}_{23} \widetilde{\Delta}_{14}\right)\right)=-\int_{3,4} B_{3,4}\left(\omega_{34}^{\prime} \widetilde{\Delta}_{13} \widetilde{\Delta}_{24}\right) \tag{4.3.18}
\end{equation*}
$$

This way, we conclude $T^{*} \lambda=-\lambda$.
By our construction of $\widetilde{\Delta}$, we also have

$$
\begin{equation*}
\int_{2} B_{2}\left(\widetilde{\Delta}_{12} \wedge \widetilde{\Delta}_{23}\right)=\widetilde{\Delta}_{13} \tag{4.3.19}
\end{equation*}
$$

The Stokes' Theorem for the fibration $p_{12}: C_{3}(M) \rightarrow C_{2}(M)$ has the following form,

$$
\begin{equation*}
d \int_{3} B_{3}\left(\omega_{13}^{\prime} \widetilde{\Delta}_{23}\right)=\int_{3} B_{3}\left(\widetilde{\Delta}_{13} \widetilde{\Delta}_{23}\right)+\left.\int_{\partial 3} B_{3}\left(\omega_{13}^{\prime} \widetilde{\Delta}_{23}\right)\right|_{\partial} . \tag{4.3.20}
\end{equation*}
$$

The second fibre integration on right-hand side of (4.3.20) is the integration along the fibrewise boundary of $p_{12}$. One can understand this as follows, when we fix a point ( $x_{1}, x_{2}$ ) in $C_{2}(M)$, we integrate $\left.B_{3}\left(\omega_{13}^{\prime} \widetilde{\Delta}_{23}\right)\right|_{\partial}$ with the third point $x_{3}$ approaches infinitesimally to $x_{1}$ or $x_{2}$ and with a proper induced orientation on the boundary, for instance, we will have

$$
\begin{equation*}
\left.\int_{\partial 3} B_{3}\left(\omega_{13}^{\prime} \widetilde{\Delta}_{23}\right)\right|_{\partial}=-\widetilde{\Delta}_{21} . \tag{4.3.21}
\end{equation*}
$$

Then, by Stokes' Theorem and using (4.3.19) - (4.3.21) and the conditions (ii) (iii), we have

$$
\begin{align*}
d \lambda= & \int_{3} B_{3}\left(\widetilde{\Delta}_{13} \widetilde{\Delta}_{23}\right)-\widetilde{\Delta}_{21} \\
& -\int_{3} B_{3}\left(\widetilde{\Delta}_{23} \widetilde{\Delta}_{13}\right)+\widetilde{\Delta}_{12} \\
& -\int_{3,4} B_{3,4}\left(\widetilde{\Delta}_{34} \widetilde{\Delta}_{13} \widetilde{\Delta}_{24}\right)-\int_{3} B_{3}\left(\widetilde{\Delta}_{13} \widetilde{\Delta}_{23}\right) \\
= & \int_{3} B_{3}\left(\widetilde{\Delta}_{23} \widetilde{\Delta}_{31}\right)-\widetilde{\Delta}_{21} \\
& -\int_{3} B_{3}\left(\widetilde{\Delta}_{13} \widetilde{\Delta}_{32}\right)+\widetilde{\Delta}_{12}  \tag{4.3.22}\\
& -\int_{3,4} B_{3,4}\left(\widetilde{\Delta}_{13} \widetilde{\Delta}_{34} \widetilde{\Delta}_{42}\right)+\int_{3} B_{3}\left(\widetilde{\Delta}_{13} \widetilde{\Delta}_{32}\right) \\
= & \widetilde{\Delta}_{21}-\widetilde{\Delta}_{21} \\
& -\widetilde{\Delta}_{12}+\widetilde{\Delta}_{12} \\
& -\widetilde{\Delta}_{12}+\widetilde{\Delta}_{12}
\end{align*}
$$

We set

$$
\omega:=\omega^{\prime}+\lambda \in \Omega^{2}\left(C_{2}(M) ; F_{\rho}\right) .
$$

$$
=0 .
$$

Then by (4.3.22), $\omega$ satisfies the condition (i). Since $T^{*}(\lambda)=-\lambda, \omega$ still satisfies the condition (iii).
Observe that the 2-form $\lambda$ is horizontal on $C_{2}(M)$ with respect to the projection $q: C_{2}(M) \rightarrow$ $M \times M$. Using $T^{*} \omega^{\prime}=-\omega^{\prime}$ or directly by (4.3.18), then

$$
\begin{equation*}
\mathfrak{i}_{\partial}^{*} \int_{3,4} B_{3,4}\left(\omega_{34}^{\prime} \widetilde{\Delta}_{13} \widetilde{\Delta}_{24}\right)=0 . \tag{4.3.24}
\end{equation*}
$$

Hence $i_{\partial}^{*} \lambda=0$. Then $\omega$ also satisfies the condition (ii).

For the condition (iv), we proceed as follows,

$$
\begin{align*}
\int_{2} B_{2}\left(\lambda \wedge p_{2}^{*}(\iota(\beta))\right) & =-\int_{3} B_{3}\left(\omega_{13}^{\prime} \beta_{3}\right)-\int_{2,3} B_{2,3}\left(\omega_{23}^{\prime} \widetilde{\Delta}_{13} \beta_{2}\right)+\int_{3,4} B_{3,4}\left(\omega_{34}^{\prime} \widetilde{\Delta}_{13} \beta_{4}\right) \\
& =-\int_{2} B_{2}\left(\omega_{12}^{\prime} \beta_{2}\right)-\int_{2,3} B_{2,3}\left(\omega_{23}^{\prime} \widetilde{\Delta}_{13} \beta_{2}\right)+\int_{2,3} B_{2,3}\left(\omega_{23}^{\prime} \widetilde{\Delta}_{12} \beta_{3}\right) \\
& =-\int_{2} B_{2}\left(\omega_{12}^{\prime} \beta_{2}\right)-\int_{2,3} B_{2,3}\left(\omega_{23}^{\prime} \widetilde{\Delta}_{13} \beta_{2}\right)+\int_{3,2} B_{3,2}\left(\omega_{32}^{\prime} \widetilde{\Delta}_{13} \beta_{2}\right)  \tag{4.3.25}\\
& =-\int_{2} B_{2}\left(\omega_{12}^{\prime} \beta_{2}\right)-\int_{2,3} B_{2,3}\left(\omega_{23}^{\prime} \widetilde{\Delta}_{13} \beta_{2}\right)+\int_{2,3} B_{2,3}\left(\omega_{23}^{\prime} \widetilde{\Delta}_{13} \beta_{2}\right) \\
& =-\int_{2} B_{2}\left(\omega_{12}^{\prime} \beta_{2}\right) .
\end{align*}
$$

Here, we denote $p_{i}^{*}(\iota(\beta))$ by $\beta_{i}$ for simplicity. Therefore, we have

$$
\begin{equation*}
\int_{2} B_{2}\left(\omega_{12} \wedge p_{2}^{*}(\iota(\beta))=0\right. \tag{4.3.26}
\end{equation*}
$$

Then $\omega$ is a 2-form with all the desired properties. This completes the proof of our proposition.
Corollary 4.3.2. We can refine the Condition (ii) in Proposition 4.3.1 as follows: there exits $\xi \in$ $\Omega^{2}\left(\Delta ; E_{\rho} \otimes E_{\rho}\right)$, such that $T^{*}(\xi)=-\xi$, and

$$
\begin{equation*}
\mathfrak{i}_{\partial}^{*}(\omega)=I(\eta)+q_{\partial}^{*}(\xi) . \tag{4.3.27}
\end{equation*}
$$

Moreover, if $E_{\rho}$ is acyclic, then $\xi$ is closed and the class $[\xi] \in H_{-}^{2}\left(\Delta ; E_{\rho} \otimes E_{\rho}\right)$ is independent of the choice of the oriented framing $f$ or $\xi$ (which is compatible with the given $o(M)$ ).

Proof. The first part follows directly from our proof of Proposition 4.3.1. For the second part, we use an easy modification of the proof to [CS21, Proposition 2.1].

If $E_{\rho}$ is acyclic, then $\omega$ is closed with (4.3.27). By the definition $\delta_{C_{2}(M)}^{*}$ in (4.2.10), we have

$$
\begin{equation*}
\delta_{C_{2}(M)}^{*}\left[I(\eta)+q_{\partial}^{*}(\xi)\right]=0 . \tag{4.3.28}
\end{equation*}
$$

In this case, $\delta_{M^{2}}^{*}$ is an isomorphism, set

$$
\Phi:=\left(\delta_{M^{2}}^{*}\right)^{-1} \circ\left(q^{*}\right)^{-1} \circ \delta_{C_{2}(M)}^{*}: H_{-}^{2}\left(\partial C_{2}(M) ; E_{\rho} \otimes E_{\rho}\right) \rightarrow H_{-}^{2}\left(\Delta ; E_{\rho} \otimes E_{\rho}\right) .
$$

Then we have $\Phi \circ q_{\partial}^{*}=\operatorname{Id}_{H_{-}^{2}\left(\Delta ; E_{\rho} \otimes E_{\rho}\right)}$, and

$$
\begin{equation*}
[\xi]=-\Phi[I(\eta)] \in H_{-}^{2}\left(\Delta ; E_{\rho} \otimes E_{\rho}\right) . \tag{4.3.29}
\end{equation*}
$$

By Lemma 4.2.2, $[I(\eta)]$ is independent of the choice of oriented framing $f$, we complete our proof.
Definition 4.3.3 (Propagator). For the local system $E_{\rho}$, for given orientation $o(M)$, smooth framing $f$ of $M$ and fix a map $\iota: H^{\bullet}\left(M ; E_{\rho}\right) \rightarrow \Omega^{\bullet}\left(M ; E_{\rho}\right)$ as in (3.2.10), a propagator is a smooth 2-form $\omega \in \Omega^{2}\left(C_{2}(M) ; F_{\rho}\right)$ which satisfies all four conditions in Proposition 4.3.1 and condition (4.3.27) in Corollary 4.3.2.

Note that, in order to emphasize the boundary condition (4.3.27), we use the pair $(\omega, \xi)$ or the triplet $(\omega, \eta, \xi)$ to denote our propagator. In the case of emphasizing the role of the framing $f$, we sometimes also use ( $\omega, f, \eta, \xi$ ) to denote a propagator.

Note that the propagators are generally not unique, but when $E_{\rho}$ is acyclic, the cohomological class of propagators is unique. More precisely, we have the following result.
Proposition 4.3.4 (Uniqueness of propagators for acyclic $E_{\rho}$ ). Fix a homotopy class of the smooth framings $[f]$ of $M$ and an orientation $o(M)$. Assume $E_{\rho}$ to be acyclic, let $\omega \in \Omega_{-}^{2}\left(C_{2}(M) ; F_{\rho}\right)$ be a propagator satisfying (4.3.27). Then the de Rham class $[\omega] \in H_{-}^{2}\left(C_{2}(M) ; F_{\rho}\right)$ is unique (which is independent of the choice of a framing $f \in[f]$ but depends on the homotopy class $[f]$ ).

Proof. We consider the diagram (4.2.10) but for the cohomology groups of degrees 1 and 2. Then from the acyclicness of $E_{\rho}$, we get the map $\delta_{M^{2}}^{*}: H_{-}^{1}\left(\Delta ; E_{\rho} \otimes E_{\rho}\right) \rightarrow H_{-}^{2}\left(M \times M, \Delta ; E_{\rho} \boxtimes E_{\rho}\right)$ is an isomorphism.

Meanwhile, we have the isomorphism: $q^{*}: H_{-}^{2}\left(M \times M, \Delta ; E_{\rho} \boxtimes E_{\rho}\right) \rightarrow H_{-}^{2}\left(C_{2}(M), \partial C_{2}(M) ; F_{\rho}\right)$. As a consequence, we infer that $\delta_{C_{2}(M)}^{*}: H_{-}^{1}\left(\partial C_{2}(M) ; E_{\rho} \otimes E_{\rho}\right) \rightarrow H_{-}^{2}\left(C_{2}(M), \partial C_{2}(M) ; F_{\rho}\right)$ is surjective. Therefore, we conclude that the restriction map

$$
\begin{equation*}
\mathfrak{i}_{\partial}^{*}: H_{-}^{2}\left(C_{2}(M) ; F_{\rho}\right) \rightarrow H_{-}^{2}\left(\partial C_{2}(M) ; E_{\rho} \otimes E_{\rho}\right) \tag{4.3.30}
\end{equation*}
$$

is injective.
Note that $H^{2}\left(I \times \mathbb{S}^{2} ; \mathbb{R}\right) \simeq H^{0}(I ; \mathbb{R}) \otimes H^{2}\left(\mathbb{S}^{2} ; \mathbb{R}\right) \simeq H^{2}\left(\mathbb{S}^{2} ; \mathbb{R}\right)$, then by Corollary 4.3.2, for any propagator $\omega$ defined with a framing $f \in[f]$, the cohomological class $\mathfrak{i}_{\partial}^{*}[\omega]$ is uniquely determined by $(M, o(M),[f], \rho)$. Finally, the uniquenees of $[\omega] \in H_{-}^{2}\left(C_{2}(M) ; F_{\rho}\right)$ follows from the injectivity of $\mathfrak{i}_{\partial}^{*}$ in (4.3.30).

Remark 4.3.5. (1) Our construction is a non-acyclic generalization of so-called framed propagator studied in [CM10, Section 4.4], [CS21] and sketched in [Kon94].
(2) The Proposition 4.3 .1 can be generalized to the case of a closed oriented manifold of dimension $n \geq 2$ as in [CW23], by replacing $\eta$ with its unframed version as in [BC98] if the manifold is not parallelizable.
(3) (With an appropriate modification as above) as a corollary, we can recover the previous results given in [CW23], [CS21]. More concretely, (i) if $E_{\rho}$ is a trivial local system, Proposition 4.3.1 recovers [CW23, Proposition 8] and [CM10, Lemma 2, Lemma 3]; (ii) If $E_{\rho}$ is acyclic $\left(H^{\bullet}\left(M ; E_{\rho}\right)=0\right)$ and we choose a trivialization of $T M$ and equip $M$ with the Riemannian metric compatible with the trivialization. Then, Proposition 4.3.1 and Corollary 4.3.2 recover [CS21, Proposition 2.1].

### 4.4 Adapted propagators for acyclic local systems

Based on our consideration in (3.5.16), Proposition 4.3.1 and Corollary 4.3.2, for an acyclic local system $E_{\rho}$, we can define a propagator $\omega$ which has an extra property with respect to the Lie bracket operator $\mathfrak{L}$, which we call an adapted propagator.

One motivation for definition is to construct the integral invariants for the triplet ( $M, f, \rho$ ) associated to trivalent graphs without self-loops. Note that this definition is cohomologically canonical and in the spirit of Bott-Cattaneo [BC99].
Definition 4.4.1. For an acyclic local system $E_{\rho}$ associated with a representation $\rho: \pi_{1}(M) \rightarrow G \xrightarrow{\text { Ad }}$ $\operatorname{Aut}(\mathfrak{g})$, an adapted propagator is a 2 -form

$$
\begin{equation*}
\omega \in \Omega^{2}\left(C_{2}(M) ; F_{\rho}\right) \tag{4.4.1}
\end{equation*}
$$

satisfying the following four properties:
(i) $d \omega=0$;
(ii) the restriction of $\omega$ on the boundary $\left.\omega\right|_{\partial C_{2}(M)}$ has the form

$$
\begin{equation*}
\mathfrak{i}_{\partial}^{*}(\omega)=I(\eta)+q_{\partial}^{*}(\xi) \tag{4.4.2}
\end{equation*}
$$

where $\eta$ is a normalized (oriented) volume 2 -form on $\mathbb{S}^{2}$, and $\xi \in \Omega^{2}\left(\Delta, E_{\rho} \otimes E_{\rho}\right) ;$
(iii) the form $\omega$ on $C_{2}(M)$ is antisymmetric under the action of $T^{*}$, i.e., $T^{*}(\omega)=-\omega$;
(iv) $\mathfrak{L}(\xi)=0$, or equivalently, $\xi \in \Omega_{-}^{2}\left(\Delta ; H_{\rho}\right)$. It is also equivalent to $\mathfrak{L}\left(\mathrm{i}_{\partial}^{*}(\omega)\right)=0$.

It is clear that an adapted propagator is always a propagator as in Proposition 4.3.1, indeed, an adapted propagator is just a propagator $(\omega, \xi)$ for the acyclic local system $E_{\rho}$ with an extra condition $\mathfrak{L}(\xi)=0$.

Our main results for this subsection are as follows.
Theorem 4.4.2. Given a framing $f$, an oriented normalized volume form $\eta$ on $\mathbb{S}^{2}$ and an acyclic local system $E_{\rho}$ via a representation $\rho: \pi_{1}(M) \rightarrow G \xrightarrow{\text { Ad }}$ Aut $(\mathfrak{g})$, the adapted propagator $\omega \in \Omega^{2}\left(C_{2}(M) ; F_{\rho}\right)$ always exists with the boundary condition (4.4.2).

Proof. Let $\omega^{\prime}$ be a propagator constructed as in Proposition 4.3 .1 which also satisfies (4.3.27) with a closed 2-form $\xi^{\prime} \in \Omega_{-}^{2}\left(\Delta ; E_{\rho} \otimes E_{\rho}\right)$, i.e.,

$$
\mathfrak{i}_{\partial}^{*}\left(\omega^{\prime}\right)=I(\eta)+q_{\partial}^{*}\left(\xi^{\prime}\right) .
$$

Since $E_{\rho}$ is acylic, by Corollary (4.3.2), we have $\left[\xi^{\prime}\right] \in H_{-}^{2}\left(\Delta ; E_{\rho} \otimes E_{\rho}\right)$. In the same time, by (3.5.16), there eixists $\xi_{0} \in \Omega_{-}^{2}\left(\Delta ; H_{\rho}\right)$ (i.e., $\left.\mathfrak{L}\left(\xi_{0}\right)=0\right)$ such that

$$
\begin{equation*}
\xi^{\prime}-\xi_{0}=d \psi \tag{4.4.3}
\end{equation*}
$$

where $\psi \in \Omega_{-}^{1}\left(\Delta ; E_{\rho} \otimes E_{\rho}\right)$.
Now we take a $T$-invariant cut-off function $\chi$ in a small $T$-invariant tubular neighborhood of $\Delta$ in $M \times M$ as in the proof of Proposition 4.3.1 such that we extend $\psi$ to a smooth 1 -from $\widetilde{\psi} \in$ $\Omega_{-}^{1}\left(M \times M, F_{\rho}\right)$ which is supported near $\Delta$ and satisfies

$$
\begin{equation*}
\mathfrak{i}^{*}(\widetilde{\psi})=\psi . \tag{4.4.4}
\end{equation*}
$$

Set $\omega=\omega^{\prime}-d q^{*} \widetilde{\psi} \in \Omega_{-}^{2}\left(C_{2}(M) ; E_{\rho} \otimes E_{\rho}\right)$. Then

$$
\begin{equation*}
\mathfrak{i}_{\partial}^{*} \omega=\mathfrak{i}_{\partial}^{*} \omega^{\prime}-q_{\partial}^{*} d \psi=I(\eta)+q_{\partial}^{*}\left(\xi^{\prime}-d \psi\right)=I(\eta)+q_{\partial}^{*}\left(\xi_{0}\right) . \tag{4.4.5}
\end{equation*}
$$

This closed 2 -form $\omega$ is an adapted propagator as we defined.
A modification of the above proof gives the following statement, which corresponds to the main framework in [BC98, BC99].
Proposition 4.4.3. Given a framing $f$ and an oriented normalized volume form $\eta$ on $\mathbb{S}^{2}$. Assume that $E_{\rho}$ is acyclic and

$$
\begin{equation*}
H_{-}^{1}\left(M ; E_{\rho} \otimes E_{\rho}\right)=0 \tag{4.4.6}
\end{equation*}
$$

then there is an adapted propagator $\omega \in \Omega_{-}^{2}\left(C_{2}(M) ; F_{\rho}\right)$ such that $d \omega=0$ and

$$
\begin{equation*}
\mathfrak{i}_{\partial}^{*}(\omega)=I(\eta) . \tag{4.4.7}
\end{equation*}
$$

Together with Proposition 3.5.4, if $G$ is a real 3-dimensional simple Lie group and $E_{\rho}$ is acyclic, then the conditions in the above proposition are always satisfied. In particular, the above results apply to the case $G=\mathrm{SU}(2)$ or $\mathrm{SL}_{2}(\mathbb{R})$. In general, we cannot always have $H_{-}^{1}\left(M ; E_{\rho} \otimes E_{\rho}\right)=0$. As we saw in Proposition 3.5.5, there are examples of triples of $(M, G, \rho)$ with $H_{-}^{1}\left(M ; E_{\rho} \otimes E_{\rho}\right) \neq 0$.

## 5 Two-loop invariant of framed closed 3-manifolds with acyclic local systems

In this section, 2-loop invariants of framed closed 3-manifolds equipped with acyclic local systems are revisited. After recalling its definition and independence of the choice of propagators following [CS21], we observe that a particular choice of propagators, i.e., adapted propagators, gives vanishing of integration associated with the dumbbell graph.

We will continue using the notation introduced at the beginning of Section 4.

### 5.1 Cattaneo-Shimizu's result on 2-loop invariant of framed closed 3-manifolds

Recall that $T$ acts on $M^{2}\left(\right.$ resp. $C_{2}(M)$ or $\left.\partial C_{2}(M)\right)$, which lifts to $E_{\rho} \boxtimes E_{\rho}\left(\right.$ resp. $F_{\rho}$ or $\left.\left.F_{\rho}\right|_{\partial C_{2}(M)}\right)$. On $C_{2}(M)$, we have the canonical identification of flat vector bundles

$$
\begin{equation*}
F_{\rho}^{\otimes 3}=q^{*}\left(E_{\rho}^{\otimes 3} \boxtimes E_{\rho}^{\otimes 3}\right)=q^{*}\left(\left(E_{\rho}^{\otimes 2} \boxtimes \mathbb{R}\right) \otimes\left(\mathbb{R} \boxtimes E_{\rho}^{\otimes 2}\right)\right) \otimes F_{\rho} . \tag{5.1.1}
\end{equation*}
$$

Then we have the linear form $\operatorname{Tr}^{\boxtimes 2}: F_{\rho}^{\otimes 3} \rightarrow \underline{\mathbb{R}}$.
As in [BC98, BC99] and in [CS21], we now consider the integral invariants, which are known as 2-loops terms in Chern-Simons perturbation theory. Theta graph and dumbbell graph are the only two connected topological trivalent graphs with 2-loop, the precise figures are given in Fig. 4. For each graph, we can define a configuration space integral as our potential invariant. Besides the Theta-invariant and dumbbell invariant via integrations on $C_{2}(M)$, we also introduce an integral invariant $Z_{1}(\cdots)$ following the work of Cattaneo-Shimizu in [CS21]. We always fix a framing $f$ and an orientation $o(M)$ of $M$.

Definition 5.1.1. Provided a propagator $\omega$ as in Definition 4.3.3: it satisfies all four conditions in Proposition 4.3.1 and there exits $\xi \in \Omega_{-}^{2}\left(\Delta, E_{\rho} \otimes E_{\rho}\right)$, such that

$$
\begin{equation*}
\mathfrak{i}_{\partial}^{*}(\omega)=I(\eta)+q_{\partial}^{*}(\xi) \tag{5.1.2}
\end{equation*}
$$

We define the following integrals,

$$
\begin{equation*}
Z_{\Theta}(\omega)=\int_{C_{2}(M)} \operatorname{Tr}^{\boxtimes 2}\left[\omega^{3}\right], Z_{\mathrm{O}-\mathrm{O}}(\omega, \xi)=\int_{C_{2}(M)} \operatorname{Tr}^{\boxtimes 2}\left[\left(p_{1}^{*} \xi\right)\left(p_{2}^{*} \xi\right) \omega\right], \tag{5.1.3}
\end{equation*}
$$

and set

$$
\begin{equation*}
Z_{1}(\rho ; \omega, \xi)=Z_{\Theta}(\omega)-\frac{3}{2} Z_{\circ-\bigcirc}(\omega, \xi) . \tag{5.1.4}
\end{equation*}
$$

For acyclic local systems, we recall one of the main results of Cattaneo-Shimizu [CS21]. Note that the framing $f$ and $o(M)$ are given.
Theorem 5.1.2 ([CS21, Theorem 2.3]). If $E_{\rho}$ is acyclic, then $Z_{1}(\rho ; \omega, \xi)$ is independent of the choice of the triplet $(\omega, \eta, \xi)$ satisfying the conditions in Proposition 4.3.1 and in Corollary 4.3.2, so that it is an invariant for $\left(M, E_{\rho}\right)$, which we denote simply by $Z_{1}(M, \rho)$. In particular, $Z_{1}(M, \rho)$ is an invariant depending only on $M, \rho$ and the homotopy class of framing $f$.

Note that if we take generally a propagator $\omega$ given as in Proposition 4.3.1, the Theta-term $Z_{\Theta}(\omega)$ will depend on the choice of $\omega$. Cattaneo-Shimizu [CS21] introduced the correction by the dumbeell term $Z_{\mathrm{O}-\mathrm{O}}(\omega, \xi)$ to finally obtain a two-loop integral invariant $Z_{1}(M, \rho)$.
Remark 5.1.3. Note that we put the coefficient $\frac{3}{2}$ instead of 3 (the coefficient originally given in [CS21]) in front of the dumbbell term, this difference follows from our convention of the computations (comparing (6.4.8) with [CS21, §4.2. Proof of Proposition 4.2]), more details are referred to Examples 6.5.4 \& 6.6.7.

### 5.2 Vanishing of the dumbbell graph with an adapted propagator

Using our construction of an adapted propagator, we can refine Cattaneo-Shimizu's result (Theorem 5.1.2) as follows.

Theorem 5.2.1. Assume $E_{\rho}$ to be acyclic. Let $\omega^{\sharp} \in \Omega_{-}^{2}\left(C_{2}(M) ; F_{\rho}\right)$ be an adapted propagator with $\mathfrak{i}_{\partial}^{*}\left(\omega^{\sharp}\right)=I(\eta)+q_{\partial}^{*}\left(\xi^{\sharp}\right)$ as in (4.4.2), then

$$
\begin{equation*}
Z_{\mathrm{O}-\mathrm{O}}\left(\omega^{\sharp}, \xi^{\sharp}\right)=0, Z_{1}(M, \rho)=Z_{\Theta}\left(\omega^{\sharp}\right) . \tag{5.2.1}
\end{equation*}
$$

Equivalently, $Z_{\Theta}\left(\omega^{\sharp}\right)$ itself defined via an adapted propagator gives the 2-loop invariant for $(M, \rho,[f])$.
Proof. We only need to prove that

$$
\begin{equation*}
Z_{\mathrm{O}-\mathrm{O}}\left(\omega^{\sharp}, \xi^{\sharp}\right)=0 . \tag{5.2.2}
\end{equation*}
$$

Note that for an adapted propagator, we have $\mathfrak{L}\left(\xi^{\sharp}\right)=0$.
By (3.5.5), (5.1.1), we have

$$
\begin{align*}
& \operatorname{Tr}^{\boxtimes 2}\left[\left(p_{1}^{*} \xi^{\sharp}\right)\left(p_{2}^{*} \xi^{\sharp}\right) \omega^{\sharp}\right] \\
& =B_{1,2}\left(\mathfrak{L}\left(\xi^{\sharp}\right) \boxtimes \mathfrak{L}\left(\xi^{\sharp}\right), \omega^{\sharp}\right)  \tag{5.2.3}\\
& =0 .
\end{align*}
$$

This implies exactly (5.2.2). Then the proof of our proposition is completed.
The result of Proposition 5.2 .1 shows that, for an acyclic local system $E_{\rho}$, the use of an adapted propagator defined in Definition 4.4 . can reduce the computation of $Z_{1}(M, \rho)$ to compute only the Theta-invariant, hence $Z_{1}(M, \rho)$ is essentially the Theta-invariant. Note that the dumbbell invariant corresponds to the dumbbell graph, which is a two-loop trivalent graph with self-loops, the proof of the above proposition indicates that the extra condition $\mathfrak{L}\left(\xi^{\sharp}\right)=0$ in Definition 4.4.1 is the key point to vanish the self-loops. Such an idea will be exploited further for the integral invariants associated to higher loop terms in subsequent sections.

### 5.3 A preliminary result for certain nonacyclic local systems

In this subsection, we show a preliminary result on the 2-loop integral invariants for a local system $E_{\rho}$ with the condition $H^{1}\left(M ; E_{\rho}\right)=0$, which is an attempt to extend [CS21, Theorem 2.3](cf. Theorem 5.1.2). The results presented in this subsection are related to Subsection 4.3, but independent from the rest of this article.

In this subsection, we do not assume $E_{\rho}$ to be acyclic anymore, but we assume that $H^{1}\left(M ; E_{\rho}\right)=0$. By the Porincaré duality with respect to $B$, we have $H^{2}\left(M ; E_{\rho}\right)=0$. By the interpretation given in Subsection 3.3, this assumption holds true when $\rho$ is an isolated point in the character variety $\mathcal{X}_{G}(M)$ of $\pi_{1}(M)$. On $H^{0}$, $H^{3}$-parts, we do not make any assumption. We always fix a map $\iota: H^{\bullet}\left(M ; E_{\rho}\right) \rightarrow \Omega^{\bullet}\left(M ; E_{\rho}\right)$ as in (3.2.10).

We have the following observation

$$
\begin{equation*}
H^{1}\left(M \times M ; E_{\rho} \boxtimes E_{\rho}\right)=H^{2}\left(M \times M ; E_{\rho} \boxtimes E_{\rho}\right)=0 . \tag{5.3.1}
\end{equation*}
$$

Moreover, by the long exact sequence (3.5.13), we also have

$$
\begin{equation*}
H_{-}^{2}\left(M ; H_{\rho}\right) \simeq H_{-}^{2}\left(M ; E_{\rho} \otimes E_{\rho}\right) . \tag{5.3.2}
\end{equation*}
$$

The following result is an analog of [CS21, Lemma 4.3], and it partially extends [CS21, Proposition 4.1] and Theorem 5.2.1 for nonacyclic case.

Proposition 5.3.1. Assume that $H^{1}\left(M ; E_{\rho}\right)=0$. Let $\omega$, $\omega^{\prime}$ be two propagators as in Definition 4.3.3 with the same framing $f$, the same $\eta$ and the same $\xi$. Then there exists a 1 -form $\psi \in \Omega_{-}^{1}\left(M \times M ; F_{\rho}\right)$ such that $\mathfrak{L}\left(\mathrm{i}^{*}(\psi)\right)=0$, and

$$
\begin{equation*}
\omega-\omega^{\prime}=d\left(q^{*} \psi\right) . \tag{5.3.3}
\end{equation*}
$$

As a consequence, we have

$$
\begin{equation*}
Z_{\Theta}(\omega)=Z_{\Theta}\left(\omega^{\prime}\right) . \tag{5.3.4}
\end{equation*}
$$

Proof. By the assumption, we have

$$
\begin{equation*}
d\left(\omega-\omega^{\prime}\right)=0, \mathfrak{i}_{\partial}^{*}\left(\omega-\omega^{\prime}\right)=0 . \tag{5.3.5}
\end{equation*}
$$

By the excision Theorem, we have the following canonical isomorphism

$$
\begin{equation*}
q^{*}: H^{2}\left(M \times M, \Delta ; E_{\rho} \boxtimes E_{\rho}\right) \xrightarrow{\sim} H^{2}\left(C_{2}(M), \partial C_{2}(M) ; F_{\rho}\right) . \tag{5.3.6}
\end{equation*}
$$

Since $H^{2}\left(M \times M ; E_{\rho} \boxtimes E_{\rho}\right)=0$, by (5.3.5), there exists $\psi^{\prime} \in \Omega_{-}^{1}\left(M \times M ; E_{\rho} \boxtimes E_{\rho}\right)$ such that

$$
\begin{equation*}
\left(q^{*}\right)^{-1}\left(\omega-\omega^{\prime}\right)=d \psi^{\prime} . \tag{5.3.7}
\end{equation*}
$$

Combining together (5.3.5) and (5.3.7), we also have

$$
\begin{equation*}
d\left(\mathfrak{i}^{*}\left(\psi^{\prime}\right)\right)=0 . \tag{5.3.8}
\end{equation*}
$$

Thus $\mathfrak{i}^{*}\left(\psi^{\prime}\right)$ represents a cohomology class in $H_{-}^{1}\left(M ; E_{\rho} \otimes E_{\rho}\right)$. Note that since $H^{1}\left(M ; E_{\rho}\right)=0$, then $H^{1}\left(M ; H_{\rho}\right) \rightarrow H^{1}\left(M, E_{\rho} \otimes E_{\rho}\right)$ is surjective. As a consequence, there exists a closed form $\psi_{1} \in \Omega_{-}^{1}\left(M ; H_{\rho}\right)$ and a section $\Phi_{1} \in \Omega_{-}^{0}\left(M ; E_{\rho} \otimes E_{\rho}\right)$ such that

$$
\begin{equation*}
\psi_{1}-\mathfrak{i}^{*}\left(\psi^{\prime}\right)=d \Phi_{1} . \tag{5.3.9}
\end{equation*}
$$

As explained in the proof to Proposition 5.2.1, we can extend $\Phi_{1}$ to a section $\widetilde{\Phi}_{1}$ of $E_{\rho} \boxtimes E_{\rho}$ on $M \times M$, which is also a ( -1 )-eigensection of $T^{*}$. Now we set

$$
\begin{equation*}
\psi=\psi^{\prime}+d \widetilde{\Phi}_{1} \in \Omega^{1}\left(M \times M ; E_{\rho} \boxtimes E_{\rho}\right) . \tag{5.3.10}
\end{equation*}
$$

By (5.3.7) and (5.3.10), we get

$$
\begin{equation*}
\omega-\omega^{\prime}=d\left(q^{*} \psi\right) . \tag{5.3.11}
\end{equation*}
$$

By (5.3.9) and the definition of $H_{\rho}$, we have

$$
\begin{equation*}
\mathfrak{L}\left(\mathrm{i}^{*}(\psi)\right)=0 . \tag{5.3.12}
\end{equation*}
$$

This proves the first part of our proposition. Now we prove (5.3.4).
By (5.1.3) and (5.3.3) and using Stokes' Theorem, we have

$$
\begin{equation*}
Z_{\Theta}(\omega)-Z_{\Theta}\left(\omega^{\prime}\right)=6 \int_{\Delta} \operatorname{Tr}^{\boxtimes 2}\left[i^{*}(\psi) \xi \mathbf{1}\right]+3 \int_{C_{2}(M)} \operatorname{Tr}^{\boxtimes 2}\left[q^{*}(\psi) \widetilde{\Delta}_{12}\left(\omega+\omega^{\prime}\right)\right] . \tag{5.3.13}
\end{equation*}
$$

An elementary computation shows that

$$
\begin{equation*}
\operatorname{Tr}^{\boxtimes 2}\left[\mathfrak{i}^{*}(\psi) \xi \mathbf{1}\right]=\frac{1}{2} B\left(\mathfrak{L}\left(\mathrm{i}^{*}(\psi)\right), \mathfrak{L}(\xi)\right) . \tag{5.3.14}
\end{equation*}
$$

By (5.3.12), the function in (5.3.14) vanishes identically.
Since $H^{1}\left(M ; E_{\rho}\right)=H^{2}\left(M ; E_{\rho}\right)=0$, we can write

$$
\begin{equation*}
\widetilde{\Delta}_{12}=\Delta^{\prime}+\Delta^{\prime \prime} \tag{5.3.15}
\end{equation*}
$$

where $\Delta^{\prime}$ is a $(0,3)$-form, and $\Delta^{\prime \prime}$ is a (3, 0)-form, with respect to the two factors of $M \times M$. Similarly, we write

$$
\begin{equation*}
\psi^{\prime}=\psi^{(0,1)}, \psi^{\prime \prime}=\psi^{(1,0)}, \kappa^{\prime}=\left(\omega+\omega^{\prime}\right)^{(0,2)}, \kappa^{\prime \prime}=\left(\omega+\omega^{\prime}\right)^{(2,0)} \tag{5.3.16}
\end{equation*}
$$

Then

$$
\begin{align*}
& \int_{C_{2}(M)} \operatorname{Tr}^{\boxtimes 2}\left[q^{*}(\psi) \widetilde{\Delta}_{12}\left(\omega+\omega^{\prime}\right)\right] \\
& =\int_{C_{2}(M)} \operatorname{Tr}^{\boxtimes 2}\left[q^{*}\left(\psi^{\prime \prime}\right) \kappa^{\prime \prime} \Delta^{\prime}\right]+\int_{C_{2}(M)} \operatorname{Tr}^{\boxtimes 2}\left[\Delta^{\prime \prime} q^{*}\left(\psi^{\prime}\right) \kappa^{\prime}\right] \tag{5.3.17}
\end{align*}
$$

Using essentially that $T^{*}$ acts on $\psi, \widetilde{\Delta}_{12}, \omega, \omega^{\prime}$ as -1 and its action on $M \times M$ exchanges the two factors. We get that

$$
\begin{equation*}
\int_{C_{2}(M)} \operatorname{Tr}^{\boxtimes 2}\left[q^{*}(\psi) \widetilde{\Delta}_{12}\left(\omega+\omega^{\prime}\right)\right]=0 . \tag{5.3.18}
\end{equation*}
$$

By (5.3.14) and (5.3.18), we get

$$
\begin{equation*}
Z_{\Theta}(\omega)-Z_{\Theta}\left(\omega^{\prime}\right)=0 . \tag{5.3.19}
\end{equation*}
$$

This way, we complete the proof to this proposition.
Note that Proposition 5.3.1 is still far from being an extension of [CS21, Theorem 2.3](see Theorem 5.1.2), since we have to fix the boundary conditions such as $\eta$ and $\xi$ for the propagator $\omega$.

Using the same arguments in the proof of Proposition 4.3.4, we get the following result.
Lemma 5.3.2. Assume $H^{1}\left(M ; E_{\rho}\right)=0$. Let $f$ and $f^{\prime}$ be two homotopic framings of $M$. Let $(\omega, f, \eta, \xi),\left(\omega^{\prime}, f^{\prime}, \eta^{\prime}, \xi^{\prime}\right)$ be two set of propagators as in Definition 4.3.3 such that $\xi-\xi^{\prime}$ is exact in $\Omega_{-}^{2}\left(M ; E_{\rho} \otimes E_{\rho}\right)$. Then $\omega-\omega^{\prime}$ is an exact form in $\Omega_{-}^{2}\left(C_{2}(M) ; F_{\rho}\right)$.

## 6 Graph complex associated to acyclic adjoint local systems

This section introduces a graph complex associated with an acyclic local system which corresponds to $\rho: \pi_{1}(M) \rightarrow G \xrightarrow{\text { Ad }} \operatorname{Aut}(\mathfrak{g})$. The construction is an analogous version of the one defined by BottCattaneo [BC99] specialized so that $\rho$ is given as above, equivariant homomorphisms associated with (internal) vertices are defined from Tr , and the Killing form $B$. Different from Bott-Cattaneo [BC99], we include the graphs with self-loops in our graph complex.

In this section, only $\mathfrak{g}$ is involved, information from $M$ or $\rho$ is not needed.

### 6.1 Preliminary on graphs

Here we always consider the finite graph (i.e., with finite number of vertices and edges).
Definition 6.1.1. (1) A self-loop of a graph is an edge that connects the same vertex.
(2) If two distinct vertices of a graph are connected by exactly one edge, then this edge is said to be regular. A graph is said to be connected if it is path connected (every two vertices can be connected by a path of edges).
(3) Let $\Gamma$ be a graph whose edges are directed. Let $v(\Gamma)$ and $e(\Gamma)$ denote the sets of vertices and edges of $\Gamma$ respectively. For an directed edge $e$ of $\Gamma$ connecting the vertex $i$ to $j$, we define a map $s: e(\Gamma) \rightarrow v(\Gamma)$ and $t: e(\Gamma) \rightarrow v(\Gamma)$ by $s(e)=i$ and $t(e)=j$. Then, a half edge of a graph $\Gamma$ is defined as an element of the form

$$
(s(e), e,+1) \text { or }(t(e), e,-1) \in v(\Gamma) \times e(\Gamma) \times\{ \pm 1\}
$$

for $e \in e(\Gamma)$. We call the number of half-edges at a vertex $i$ valency of the vertex $i$. Usually, we use $h(\Gamma)$ to denote the set of all half-edges of $\Gamma$.
(4) A graph $\Gamma$ is said to be trivalent (resp. uni-trivalent) if the valencies for vertices all are 3 (resp. 1 or 3 ).
(5) A univalent vertex of a graph $\Gamma$ is called external vertex and a vertex with valency $\geq 2$ of $\Gamma$ is called internal vertex. Similarly, an edge of $\Gamma$ which connects two internal vertices is called internal edge and called external edge otherwise.


Fig. 1: A self-loop of a graph, vertex with valency 4 as displayed
In the sequel, a connected graph always means a connected graph whose internal vertices have valency $\geq 3$. Next, we define several orientations of a connected graph.
Definition 6.1.2. Let $\Gamma$ be a connected uni-trivalent graph. A vertex-wise orientation of $\Gamma$ is cyclic orders at internal vertices of $\Gamma$, i.e., collection of a cyclic order of half-edges connecting to an internal vertex.
Remark 6.1.3. A topological trivalent graph together with the information of cyclic orders on the half-edges incident to each vertex is known as a ribbon graph or fat graph (cf. [Igu04, Section 1]). Here we will not emphasize this terminology.
Definition 6.1.4 (vertex orientation and edge orientation of half-edges). Let $\Gamma$ be a connected (uni)trivalent graph $\Gamma$. Let $h^{\text {int }}(\Gamma)$ be the set of internal half-edges (i.e., half-edges attached to internal vertices). An orientation of $h^{\text {int }}(\Gamma)$ is a numbering on $h^{\text {int }}(\Gamma)$ up to even permutations, i.e., a bijection $h^{\text {int }}(\Gamma) \simeq\left\{1,2, \ldots,\left|h^{\text {int }}(\Gamma)\right|\right\}$ where two such bijections are identified if they are related by even permutations. We introduce the following two orientations for $h^{\mathrm{int}}(\Gamma)$ :
(1) Suppose that $\Gamma$ is vertex-wise oriented and the set $v^{\text {int }}(\Gamma)$ is ordered. Then, a vertex orientation of $h^{\mathrm{int}}(\Gamma)$ is defined as follows: according to the order of $v^{\mathrm{int}}(\Gamma)$, take an internal vertex $v$ and order the set of half-edges at $v$.
(2) Suppose that all of edges of $\Gamma$ are directed and the set $e(\Gamma)$ is ordered. Then, an edge orientation of $h^{\mathrm{int}}(\Gamma)$ is defined as the induced orientation from that of $e(\Gamma)$ and directions of edges. Here, for an oriented self-loop $e$ connecting the vertex $v$, the order of two half-edges $(v, e,+1)$ and $(v, e,-1)$ is defined so that $(v, e,+1)$ is putted just before $(v, e,-1)$.

### 6.2 Weight systems associated with uni-trivalent trees

Now let $G$ be a connected semi-simple Lie group with Lie algebra $\mathfrak{g}$ and the Killing form $B$ as considered in Section 3. Here, we describe a way to obtain $\operatorname{Ad}(G)$-invariant multi-linear map $\mathfrak{g}^{\otimes n} \rightarrow \mathbb{R}$ obtained from the Killing form $B$ and the corresponding Casimir element 1 and uni-trivalent trees, as in [BN95].

Let $T$ be a vertex-wise oriented uni-trivalent tree diagram with $n$ external vertices (hence it has no loops). Suppose that the set of $n$ external vertices are ordered. Then, associated with $T$, the weight system

$$
\begin{equation*}
W_{T}: \mathfrak{g}^{\otimes n} \rightarrow \mathbb{R} \tag{6.2.1}
\end{equation*}
$$

is defined as follows. For each external vertex, we associate it with $n$ inputs of elements of $\mathfrak{g}$ according to the order on the set of external vertices. For each trivalent vertex, we assign the cubic trace form Tr, defined in Subsection 3.1, according to the cyclic order of half-edges at the vertex, and for each internal edges, we assign the Casimir element 1. Then, taking contraction with respect to $B$ associated with internal edges, we obtain the desired multilinear form $W_{T}$. By construction, the $W_{T}$ is $\operatorname{Ad}(G)$-invariant and independent of the order of contractions.

The cubic trace form Tr is considered as the weight system $W_{Y}$ associated with the uni-trivalent tree $Y$ with exactly one trivalent vertex as in Fig. 2. Then, by definition, we have the identity we call anti-symmetric identity

$$
\begin{equation*}
W_{Y}+W_{\bar{Y}}=0 \tag{6.2.2}
\end{equation*}
$$

More generally, for uni-trivalent trees $T_{1}, T_{2}$ which are the same except for some small regions where $Y, \bar{Y}$ diagrams are inserted respectively. Then, we have

$$
\begin{equation*}
W_{T_{1}}+W_{T_{2}}=0 . \tag{6.2.3}
\end{equation*}
$$



Fig. 2: Anti-symmetric identity for weight systems. The graphs are denoted by $Y$ and $\bar{Y}$ respectively. In this figure, the blue curved arrows indicate the cyclic ordering for each trivalent vertex (i.e., vertex-wise orientation), and the number labels $1,2,3$ exhibit an ordering for external edges or possibly for the half-edges.


Fig. 3: Jacobi identity for weight systems. The graphs are denoted by $I, H, X$ respectively.

## Lemma 6.2.1. (Jacobi identity)

(1) For uni-trivalent trees I, H, X given as in Fig. 3, we have the following identity on the associated weight systems

$$
\begin{equation*}
W_{I}+W_{H}+W_{X}=0 \tag{6.2.4}
\end{equation*}
$$

(2) More generally, for uni-trivalent trees $T_{1}, T_{2}, T_{3}$ which are the same except for some region where $I, H$ and $X$ diagrams are inserted respectively. Then, their associated weight systems satisfy the identity

$$
\begin{equation*}
W_{T_{1}}+W_{T_{2}}+W_{T_{3}}=0 . \tag{6.2.5}
\end{equation*}
$$

Proof. (1) The proof is given by direct computation as follows. For $v_{1}, v_{2}, v_{3}, v_{4} \in \mathfrak{g}$, we have

$$
\begin{equation*}
W_{I}\left[v_{1}, v_{2}, v_{3}, v_{4}\right]=B\left(\left[v_{1}, v_{2}\right],\left[v_{3}, v_{4}\right]\right)=B\left(v_{1},\left[v_{2},\left[v_{3}, v_{4}\right]\right]\right) . \tag{6.2.6}
\end{equation*}
$$

Similarly,

$$
\begin{align*}
& W_{H}\left[v_{1}, v_{2}, v_{3}, v_{4}\right]=B\left(\left[v_{1}, v_{4}\right],\left[v_{2}, v_{3}\right]\right)=B\left(v_{1},\left[v_{4},\left[v_{2}, v_{3}\right]\right]\right), \\
& W_{X}\left[v_{1}, v_{2}, v_{3}, v_{4}\right]=B\left(\left[v_{1}, v_{3}\right],\left[v_{4}, v_{2}\right]\right)=B\left(v_{1},\left[v_{3},\left[v_{4}, v_{2}\right]\right]\right) . \tag{6.2.7}
\end{align*}
$$

Therefore,

$$
\begin{align*}
& \left(W_{I}+W_{H}+W_{X}\right)\left[v_{1}, v_{2}, v_{3}, v_{4}\right] \\
= & B\left(v_{1},\left[v_{2},\left[v_{3}, v_{4}\right]\right]\right)+B\left(v_{1},\left[v_{4},\left[v_{2}, v_{3}\right]\right]\right)+B\left(v_{1},\left[v_{3},\left[v_{4}, v_{2}\right]\right]\right)  \tag{6.2.8}\\
= & B\left(v_{1},\left[v_{2},\left[v_{3}, v_{4}\right]\right]+\left[v_{4},\left[v_{2}, v_{3}\right]\right]+\left[v_{3},\left[v_{4}, v_{2}\right]\right]\right) \\
= & 0 .
\end{align*}
$$

(2) follows immediately from (1), since the computation of a weight system can be decomposed into that of several pieces of weight systems by construction.

Remark 6.2.2. We can also consider weight systems associated with trivalent graphs which have loops. However, in the present article, we only consider weight systems associated with trees, because these tensors appear by iterated contractions of internal edges of trivalent graphs as we will see in the subsequent subsections.

### 6.3 Decorated graphs

In this subsection, we define a decorated graph which is a variant of one defined in [BC99]. Let $H=\oplus_{i \in \mathbb{Z}} H^{i}$ be a $\mathbb{Z}$-graded finite dimensional vector space. Now we introduce the definition of a decorated graph associated to a given $H$ and the connected semi-simple Lie group $G$ considered in the previous subsection. Moreover, in the potential applications of such graphs to integral invariants, the vector space $H$ will be taken as the cohomology group $H^{\bullet}\left(M ; E_{\rho}\right)$ of the local system $E_{\rho}$ discussed before.

Definition 6.3.1. (Decorated graph) Let $\Gamma$ be a connected graph whose internal vertices have valency $\geq 3$. Also suppose that when $H=0, \Gamma$ has no univalent vertex. A decorated graph is a graph $\Gamma$ endowed with the following data:

- enumerations on the set of edges $e(\Gamma)$ and the set of internal vertices $v(\Gamma)^{\text {int }}$, i.e., $\Gamma$ is endowed with fixed bijections $e(\Gamma) \simeq\{1,2, \ldots,|e(\Gamma)|\}$ and $v(\Gamma)^{\text {int }} \simeq\left\{1,2, \ldots,\left|v(\Gamma)^{\text {int }}\right|\right\}$;
- directions on internal edges;
- induced order on the set of $h_{\Gamma}(i)$ of half-edges at each vertex $i \in v(\Gamma)$ from the order of $e(\Gamma)$. Here, for a self-loop $e$ connecting the vertex $i$, the order of two half-edges $(i, e,+1)$ and $(i, e,-1)$ is defined so that $(i, e,+1)$ is putted just before $(i, e,-1)$; Note that this order on $h_{\Gamma}(i)$ defines the vertex-wise orientation at vertex $i$;
- for each internal vertex $i$ with valency $n \geq 4$, information of an insertion of oriented uni-trivalent tree $T_{i}$ with exactly $(2 n-3)$ edges and $n$ of them are ordered external vertices corresponding to the $n$ incident half-edges at this vertex $i$. More precisely, we consider a small ball centered at $v$ which intersects on the boundary with half-edges at $n$ distinct points. Then, we endow it with the information of embedding of $T_{i}$ into the ball so that the $n$-external vertices are put on the intersection points disjoint way on the boundary. Here, we also require that the embedding of $T_{i}$ is done so that
(i) the order of the external vertices of $T_{i}$ are given by the order of their corresponding halfedges of $\Gamma$ attached to $i$;
(ii) the cyclic order of $T_{i}$ at a trivalent vertex which is connected to more than one external vertices are compatible with the order of the half-edges of $\Gamma$ attached to $i$;
- for each internal vertex $i$, we equip it with the weight system defined as in (6.2.1) associated to $T_{i}$, which is a $\pi_{1}(M)$-equivariant homomorphism, $W_{i}:=W_{T_{i}}: \otimes_{h \in h_{\Gamma}(i)} \mathfrak{g}_{h} \rightarrow \mathbb{R}$ which, sometimes, is also regarded as the map

$$
\begin{equation*}
W_{i}: \mathbb{R} \rightarrow \otimes_{h \in h_{\Gamma}(i)} \mathfrak{g}_{h}^{*}, 1 \mapsto W_{T_{i}} \tag{6.3.1}
\end{equation*}
$$

where $\mathfrak{g}_{h}$ (resp. $\mathfrak{g}_{h}^{*}$ ) is a copy of $\mathfrak{g}$ (resp. $\mathfrak{g}^{*}$ ). To unify the notation, when $i$ is a trivalent vertex, then we set $T_{i}$ to be the $Y$-shape uni-trivalent tree and $W_{i}:=\operatorname{Tr}_{i}$;

- external vertices of $\Gamma$ are decorated by homogeneous elements of $H$. If $H=0$, then only the graph without external edges is concerned.

Remark 6.3.2. Note that for a vertex $i$ of valency $n \geq 4$, the inserted uni-trivalent tree $T_{i}$ is required to have exactly $(2 n-3)$ edges with $n$ of them being external, this condition forces the choices of such tree to lie in a finite list of uni-trivalent trees. This way, if we fix the numbers of edges and vertices, we only have finitely many different decorated graphs satisfying our definition.

In this article, we depict decorated graphs by dashed curves as Fig. 4, where the trivalent graphs with two loops are presented. As long as we have the ordering on the half-edges, the decoration Tr (or the corresponding $\pi_{1}(M)$-equivariant homomorphism) for each vertex is determined uniquely by our above conventions. Note that in the sequel, we sometimes omit the numberings of vertices, edges, and half-edges or the equivariant homomorphisms for simplicity when depicting decorated graphs.


Fig. 4: An example of decorated trivalent graphs, whose underlying topological graphs are called Theta graph and dumbbell graph respectively. Here, $h_{i, a}$ denotes the $i$-th half-edge at the vertex $v_{a}$, and the associated equivariant homomorphisms are omitted.

Example 6.3.3. For a vertex with valency 4, there are only three ways (by inserting only one edge) to insert uni-trivalent trees as Fig. 5. Each tree in the small balls carries a vertex-wise orientation uniquely induced from the order of these four half-edges and corresponds to one of the cases presented in Fig. 3 up to a sign (also cf. (6.4.7) and (6.4.8)).


Fig. 5: Three ways to insert a uni-trivalent tree in place of a 4 -valent vertex. These graphs are denoted by $I, H$, and $X$ respectively.

Generally, one can also decorate a vertex with valency 4 with a more complicated weight system, for example, inserting a uni-trivalent connected graph as follows, but we exclude this case from our definition.


Fig. 6: A more complicated weight system, which is not allowed in our definition.

### 6.4 Graph complex for an acyclic local system

Now let $G$ be a connected semi-simple Lie group with Lie algebra $\mathfrak{g}$ and the Killing form $B$ as considered in Section 3. In this subsection, we introduce a graph complex for the acyclic local systems associated to $\mathfrak{g}$ and $B$, more precisely, the trace form Tr. The acyclic local system corresponds to the case $H=0$ in the definition of decorated graphs, so that, by our convention, only the decorated graphs without external edges are concerned here. In the sequel, we do not distinguish internal/external edges and vertices.

Let $\widetilde{\mathcal{G C}}_{\text {ac }, \mathfrak{g}}$ be the vector space spanned by all the decorated graphs without external edges (by our convention the valency at each vertex $\geq 3$ ) over $\mathbb{Q}$. These spaces of decorated graphs are bigraded by the following order and degree:

$$
\begin{align*}
\operatorname{ord}(\Gamma) & =|e(\Gamma)|-|v(\Gamma)|,  \tag{6.4.1}\\
\operatorname{deg}(\Gamma) & =2|e(\Gamma)|-3|v(\Gamma)| .
\end{align*}
$$

Note that by our convention on the valencies, the decorated graphs that we consider here always have $\operatorname{deg} \geq 0$.

In some context, we also like to talk about the loops for a connected graph $\Gamma$. Viewing the (topological) graph as a $C W$-complex, then the Euler characteristic number is

$$
\begin{equation*}
\chi(\Gamma)=|v(\Gamma)|-|e(\Gamma)|=-\operatorname{ord}(\Gamma)=1-\ell, \tag{6.4.2}
\end{equation*}
$$

where $\ell$ corresponds to the first Betti number of $\Gamma$ hence the number of loops in $\Gamma$. Note that the number of loops as above only makes sense for a connected graph, if the graph is not connected, we also need to consider the number of the connected components to conculde the number of loops in a topological sense.
Remark 6.4.1. - For any trivalent graph $\Gamma$, i.e., a graph whose all the (internal) vertices have valency 3 , is of $\operatorname{deg}(\Gamma)=0$. Since we assume that the valencies for the internal vertices are at least 3 , so that such a finite graph without any external edges and of degree 0 has to be a trivalent graph. A (nonempty) trivalent graph has at least 2 vertices and 3 edges, so the least order is 1. The trivalent graphs of order 1 have only two possibilities: Theta-graph and dumbbell graph, both are connected.

- For a trivalent graph $\Gamma$, its order $\operatorname{ord}(\Gamma)=\frac{1}{2}|v(\Gamma)|$ defined in (6.4.1) agrees with its degree customarily used in the theory of finite-type (Vassiliev) invariants.
Next, we define an equivalence relation on $\widetilde{\mathcal{G C}}_{\mathrm{ac}, \mathfrak{g}}$ as follows: if two decorated graphs $\Gamma$ and $\Gamma^{\prime}$ differ by only
(1) permutation of numberings for all edges which induces $k$ times change in total of cyclic orders of associated trees at vertices, where the changes on the cyclic orders are forced by the compatibility of cyclic orders on the associated trees with the new numberings on their external edge,
(2) edge (including self-loop edges and non-self-loop edges) direction reversals of times $m$; let $(-1)^{m^{\prime}}$ denote the total sign change of the cyclic orders of the associated trees induced by the direction changes on the self-loop edges (since the cyclic orders are not affected by direction reversals on non-self-loop edges),
(3) permutation of numberings of vertices, let $(-1)^{d}$ denote the sign,
then we set $\operatorname{sign}\left(\Gamma, \Gamma^{\prime}\right)=(-1)^{k+m+m^{\prime}+d}$, and

$$
\begin{equation*}
\Gamma=\operatorname{sign}\left(\Gamma, \Gamma^{\prime}\right) \cdot \Gamma^{\prime} \tag{6.4.3}
\end{equation*}
$$

We also introduce a relation, called internal vertex-wise $A S$ relation, of connected decorated graphs induced from relations on equivariant homomorphisms attached at an internal vertex as follows. Let $\Gamma_{Y}, \Gamma_{\bar{Y}}$ be two connected decorated graphs with the same underlying topological graph and the same decoration except that, at one fixed internal vertex, embedded uni-trivalent trees are different but related by anti-symmetric identity as (6.2.2). Then, we set

$$
\begin{equation*}
\Gamma_{Y}+\Gamma_{\bar{Y}}=0 . \tag{6.4.4}
\end{equation*}
$$

Similarly, we define internal vertex-wise IHX relation. Let $\Gamma_{I}, \Gamma_{H}, \Gamma_{X}$ be three connected decorated graphs with the same underlying topological graph and the same decoration except that, at one fixed internal vertex, embedded uni-trivalent trees are different but related by Jacobi identity as (6.2.4). Then, we set

$$
\begin{equation*}
\Gamma_{I}+\Gamma_{H}+\Gamma_{X}=0 . \tag{6.4.5}
\end{equation*}
$$

For a self-loop edge, we now give more details to clarify the equivalence relation under the change of direction. Let $\Gamma$ be a decorated graph, and let $v$ be a vertex in $\Gamma$ of valency 3 and attached by a self-loop edge, let $\Gamma^{\prime}$ be the decorated graph obtained by reverse the direction of this self-loop edge attached to $v$, then our equivalence relation shows

$$
\begin{equation*}
\Gamma=\Gamma^{\prime} \tag{6.4.6}
\end{equation*}
$$

i.e., direction reversal for a self-loop edge attached to trivalent vertices does change the equivalent class of the given graph.

However, this situation might be different for the vertex with higher valency. For a vertex with valency 4 , this relation is presented by the following figure:
where black filled circles denote the small balls centered at an internal vertex which contains embedding information of uni-trivalent trees, and the figures for $T_{v}$ or $T_{v}^{\prime}$ mean that the associated uni-trivalent trees are the respective $-X$ and $H$ defined in Fig. 3. Note that by our conditions for $T_{v}$ or $T_{v}^{\prime}$ given in Definition 6.3.1, the vertex-wise orientations of $T_{v}$ and $T_{v}^{\prime}$ are uniquely determined by the ordering of the external vertices, so that we do not emphasize the cyclic order for each vertex in the figures of (6.4.7), that's also why we need to put minus sign in front of $X$. Note that the cyclic orders in $T_{v}$ remain the same after exchanging labels 3 and 4 , then we have to put a minus sign on the right-hand side of (6.4.7).

In the two sides of (6.4.7), the different directions of the self-loop yield a minus sign. As a consequence, we conclude an equivalence between the weight systems $X$ and $H$ at this vertex with valency 4 and attached by a self-loop. Then combining it with the internal vertex-wise IHX relation (6.4.5), we can conclude the following nontrivial identity (where we assume that the two decorated graphs are exactly the same except that, the weight systems for this depicted vertex are respectively $-X$ and $I$ ):

Remark 6.4.2. When considering trivial local systems, as is well known, graphs with self-loop edges are zero by AS relation (i.e., Antisymmetry of internal vertices given in [BN95, Theorem 6 (1)]) by
using arguments in [BN95, Section 2.4]. On the other hand, in a non-trivial local system case, this is not the case. This can be explained as follows. For trivial local system case, (if we only consider the Lie algebra factor of associated integrations) every internal edge are associated with $\mathbf{1}$ which lies in the symmetric part of $\mathfrak{g} \otimes \mathfrak{g}$, whereas for a non-trivial case we associate self-loop edges anti-symmetric element of $E_{\rho} \otimes E_{\rho}$ so that AS relation does not imply vanishings of graphs with self-loops. See also [AS94, Page 180].

Similarly to (6.4.8), if a given vertex $v$ of valency 4 has two attached edges connecting to the same trivalent vertex $v^{\prime}$, we can also conclude an analogous nontrivial identity. More precisely, we consider the following part of a connected decorated graph $\Gamma$ (without external edges) described in Fig. 7.


Fig. 7: An example of two non-self-loop edges with the same ending vertices, we assume $e<e^{\prime}$ in the given numberings on the edges of $\Gamma$.

Now we exchange the numberings $e$ and $e^{\prime}$ for these two edges described in Fig. 7, and we get a decorated graph $\Gamma^{\prime}$. As a consequence, the number labels 3 and 4 in $T_{v}$ are exchanged, but the induced cyclic orders at each vertex in $T_{v}$ remain the same so that there is no sign produced for $v$. However, since $v^{\prime}$ is trivalent, exchanging $e$ and $e^{\prime}$ produces a factor $(-1)$ for the equivalence relation between $\Gamma$ and $\Gamma^{\prime}$. Then, combining the internal IHX relation, we conclude an identity in Fig. 8 (assume the two terms are given by the same graph $\Gamma$ with the same decorations and numberings except for the vertex $v$ ).


Fig. 8: A special case of internal IHX relation.
Another situation for a vertex $v$ with valency 4 is given in Fig. 9, where the decorated graph is always identified to be zero by the internal IHX relations.


Fig. 9: Vertex $v$ with valency 4 , vertex $v^{\prime}$ with valency $3, T_{v}$ is one of the cases $\{I, H, X\}$, the above decorated graph is identified to be 0 by the internal IHX relation and sign relation.

Remark 6.4.3. In (6.4.8) and in Fig. 8 \& Fig. 9, the identifications among $I, H, X$ for a vertex of valency 4 follow from the symmetry of the underlying topological graph $\Gamma$, for instance, the self-loop edge has a natural symmetry by flipping out, the two edges $e, e^{\prime}$ in Fig. 7 are invariant under the swapping themselves, the three edges $e_{1}, e_{2}, e_{3}$ in Fig. 9 carry the natural permutation symmetries. Definition 6.4.4. We define $\mathcal{G} \mathcal{C}_{\mathrm{ac}, \mathfrak{g}}:=\widetilde{\mathcal{G}}_{\mathrm{ac}, \mathfrak{g}} / \sim$ as the graded commutative algebra over $\mathbb{Q}$ generated by equivalent classes of decorated connected graphs without external edges, subject to

- the sign relation (6.4.3),
- internal vertex-wise AS relation (6.4.4),
- internal vertex-wise IHX relation (6.4.5),
the (graded commutative) algebra structure on $\mathcal{G} \mathcal{C}_{\mathrm{ac}, \mathfrak{g}}$ is given by disjoint union (denoted by $\cup$ ), which is defined as follows, the numberings on the edges and internal vertices of $\Gamma \cup \Gamma^{\prime}$ are given as keeping the same for $\Gamma$ and shifting the numberings for $\Gamma^{\prime}$ by adding $|e(\Gamma)|,|v(\Gamma)|$ respectively. Note that the disjoint union of two decorated connected graphs, viewed as a newly decorated graph, leads to the summation of their respective orders and degrees so that a multiple of a connected decorated graph with order $n$ and degree $t$ is still considered as having the same order and degree, which is a different object from the disjoint union of multiple copies of this graph (see Fig. 10). We have the following commutative relation

$$
\begin{equation*}
\Gamma \cup \Gamma^{\prime}=(-1)^{\operatorname{deg}(\Gamma) \operatorname{deg}\left(\Gamma^{\prime}\right)} \Gamma^{\prime} \cup \Gamma \tag{6.4.9}
\end{equation*}
$$

In particular, if $\Gamma$ has an odd degree (equivalently, has odd number of vertices), then we have (in $\left.\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}}\right)$

$$
\begin{equation*}
\Gamma \cup \Gamma=0 . \tag{6.4.10}
\end{equation*}
$$



Fig. 10: Multiple of a connected graph considered different from the disjoint union of multiple copies of the graph.

Now we introduce an operator $\delta$ on $\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}}$ as follows. Set

$$
\begin{equation*}
\delta \Gamma=\sum_{e=(i j): \text { non-self-loop internal edge }} \sigma(i, j) \cdot \Gamma / e, \tag{6.4.11}
\end{equation*}
$$

where $e=(i j)$ denotes a non-self-loop edge connecting the vertex $i$ to the vertex $j, \Gamma / e$ means the decorated graph obtained from $\Gamma$ by contracting $e=(i j)$ to the original vertex $i$ then equipped with the consistent renumbering of edges and vertices and with the obvious information of insertion of one edge in place of the resulting vertex, and the sign $\sigma(i, j)$ is defined as follows:

$$
\sigma(i, j)= \begin{cases}(-1)^{j} & \text { if } j>i  \tag{6.4.12}\\ (-1)^{i+1} & \text { if } j<i\end{cases}
$$

More concretely, the renumbering of $\Gamma / e$ is defined as follows. If $e=(i j)$ is the $k$-th edge of $\Gamma$, we renumber edges $e_{l}$ with $k<l$ by letting them decrease by one. We renumber the vertices $v_{l}$ with $\max \{i, j\} \leq l$ by letting decrease by one and label the resulting vertex where the contraction has happened by $\min \{i, j\}$.

For a non-self-loop edge $e=(i j)$ connecting the vertex $i$ to the vertex $j$, the resulting vertex $i^{\prime}:=\min \{i, j\}$ by contracting $e=(i j)$ is attached with the equivariant homomorphism

$$
\begin{equation*}
W_{i^{\prime}}: \otimes_{h \in h_{\Gamma / e}\left(i^{\prime}\right)} \mathfrak{g}_{h} \rightarrow \mathbb{R} \tag{6.4.13}
\end{equation*}
$$

defined as follows. For defining $W_{i^{\prime}}$, it is enough to define the corresponding oriented uni-trivalent tree $T_{i^{\prime}}$ inserted at vertex $i^{\prime}$ in $\Gamma / e$. Assume $T_{i}, T_{j}$ to be the inserted oriented uni-trivalent trees attached to vertex $i$ and $j$ respectively, then the inserted tree $T_{i^{\prime}}$ is defined as the tree given by connecting $T_{i}$ and $T_{j}$ via the external edges corresponding to the edge $e=(i j)$. Note that the external edges of $T_{i^{\prime}}$ are ordered according to the numberings on the edges and the directions of self-loops (if attached to $i^{\prime}$ ), whose ordering is compatible with the ones of $T_{i}$ and $T_{j}$. The vertexwise orientation on $T_{i^{\prime}}$ is the one inherited from $T_{i}$ and $T_{j}$. Let $n_{i}, n_{j}$ denote the valencies of $i, j$ respectively, then the valency for this vertex $i^{\prime}$ in $\Gamma / e$ is $n_{i^{\prime}}:=n_{i}+n_{j}-2$, and the total edge number of $T_{i^{\prime}}$ is $\left(2 n_{i}-3\right)+\left(2 n_{j}-3\right)-1=2 n_{i^{\prime}}-3$, this way, we confirm that $\Gamma / e$ with the above weight system $T_{i^{\prime}}$ at $i^{\prime}$ satisfies Definition 6.3.1, i.e., $\Gamma / e \in \mathcal{G C}_{\text {ac }, \mathfrak{g}}$.

Remark 6.4.5. Assuming that $e=(i j)$ connecting $k$-th half-edge of $\left|h_{\Gamma}(i)\right|$ half-edges at $i$ and $l$-th half-edge of $\left|h_{\Gamma}(j)\right|$ half-edges at $j$, then the weight system $W_{i^{\prime}}$ can be computed by (after re-order the tensor factors according to the ordering of half-edges)

$$
\begin{equation*}
W_{i^{\prime}}=B_{k,\left|h_{\Gamma}(i)\right|+l}\left(W_{i} \otimes W_{j}\right) \tag{6.4.14}
\end{equation*}
$$

where $B_{r, s}$ denotes the bilinear form $B$ acting on $r$-th and $s$-th components of tensor products $\left(\otimes_{h \in h_{\Gamma}(i)} \mathfrak{g}_{h}\right) \otimes\left(\otimes_{h \in h_{\Gamma}(j)} \mathfrak{g}_{h}\right)$.

One simple example of the above contraction of one edge is illustrated as Fig. 11


Fig. 11: The map $\delta$ for an internal edge $e$

Then, we have the following proposition analogous to [BC99, Proposition 3.4].
Proposition 6.4.6. The operator $\delta$ is a well-defined linear operator on $\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}}$ and satisfies $\delta^{2}=0$. Moreover, for each $t \in \mathbb{Z}$, denoting by $\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}}^{t}$ the subspace of $\mathcal{G} \mathcal{C}_{\mathrm{ac}, \mathfrak{g}}$ spanned by the decorated graphs of degree $t$, we have

$$
\begin{equation*}
\delta: \mathcal{G C}_{\mathrm{ac}, \mathfrak{g}}^{t} \rightarrow \mathcal{G C}_{\mathrm{ac}, \mathfrak{g}}^{t+1} . \tag{6.4.15}
\end{equation*}
$$

That is, the pair $\left(\bigoplus_{t} \mathcal{G C}_{\mathrm{ac}, \mathfrak{g}}^{t}, \delta\right)$ forms a complex.
If $\Gamma$ and $\Gamma^{\prime}$ are two connected decorated graphs, then we have

$$
\begin{equation*}
\delta\left(\Gamma \cup \Gamma^{\prime}\right)=(\delta \Gamma) \cup \Gamma^{\prime}+(-1)^{\operatorname{deg}(\Gamma)} \Gamma \cup\left(\delta \Gamma^{\prime}\right) . \tag{6.4.16}
\end{equation*}
$$

Proof. By (6.4.12) and Remark 6.4.5, the well-definedness of $\delta$ and $\delta^{2}=0$ follows from the same arguments as in the proof of [BC99, Proposition 3.4], the existence of self-loops in the graphs does not produce any new obstacles. The identity (6.4.16) follows from the explicit computations for the operations $\delta$ and $\cup$.

### 6.5 Graph complexes of decorated graphs without self-loops for an acyclic local system

The self-loops of the graph concerned here play a different role from the regular edges, so that we will investigate separately the subspaces of $\mathcal{G} \mathcal{C}_{\mathrm{a}, \mathfrak{g}}$ consisting of graphs without any self-loop and with at least one self-loop.
Definition 6.5.1. Let $\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}}=\left(\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}}, \delta\right)$ be the graph complex defined in Subsection 6.4. Then, we similarly let $\mathcal{G}_{\mathrm{ac}, \mathfrak{g}}$ and $\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}}^{\prime}$ be the $\mathbb{Q}$-vector subspaces of $\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}}$ spanned by the equivalent classes of decorated graphs, respectively, without self-loops and with at least one self-loop.

These spaces of decorated graphs are bigraded by their order and degree. For $n, t \in \mathbb{Z}$, let $\mathcal{G}_{\mathrm{ac}, \mathfrak{g}: n}^{t}, \mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: n}^{t}, \mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: n}^{\prime, t}$ denote the subspaces of $\mathcal{G}_{\mathrm{ac}, \mathfrak{g}}, \mathcal{G C}_{\mathrm{ac}, \mathfrak{g}}, \mathcal{G C}_{\mathrm{ac}, \mathfrak{g}}^{\prime}$ respectively spanned by all the equivalent classes of decorated graphs with order $n$ and degree $t$.

Note that $\delta$-action yields a self-loop when $\delta$ acts on non-self-loop edge which is non-regular. Therefore, $\delta$ does not preserve the $\mathbb{Q}$-subspace of decorated graphs without self-loops. For example, we consider the decoration defined in 6.4 .12 , the $\Theta$-graph with $\operatorname{Tr}$ at each vertex is sent to 3 figureeight graphs decorated at the unique vertex as in Fig. 12.


Fig. 12: The action of the operator $\delta$ on $\Theta$-graph

On the other hand, $\delta$ preserves the $\mathbb{Q}$-subspace of decorated graphs with self-loops since $\delta$ does not act on self-loops. In Fig. 13, following Example 6.3.3, the dumbbell graph with Tr at each vertex


Fig. 13: The action of the operator $\delta$ on dumbbell graph
is mapped to figure-eight graph with the weight system at the unique vertex $v$ as given in Fig. 13. Note that this decoration is different from the one from $\Theta$-graph in Fig. 12.

Then, noting that $\delta$ preserves the order of a connected decorated graph, by Proprositin 6.4.6, we conclude the following results.
Proposition 6.5.2. (1) For each $n \in \mathbb{Z}$, the pair $\left(\bigoplus_{t} \mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: n}^{t}, \delta\right)$ forms a complex.
(2) Taking the graphs always with at least one self-loop, $\left(\bigoplus_{t} \mathcal{G C}_{\mathrm{ac}, \mathfrak{\mathfrak { g }},}^{\prime,}, \delta\right)$ form a subcomplex of $\left(\bigoplus_{t} \mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: n}^{t}, \delta\right)$.
By Proposition 6.5.2 (2), we can define a complex by

$$
\begin{equation*}
\left(\bigoplus_{t} \mathcal{G}_{\mathrm{ac}, \mathfrak{g}: n}^{t}, \delta^{\sharp}\right):=\left(\bigoplus_{t}\left(\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: n}^{t} / \mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: n}^{\prime, t}\right), \delta\right) . \tag{6.5.1}
\end{equation*}
$$

Then, the quotient complex $\left(\bigoplus_{t} \mathcal{G}_{\mathrm{ac}, \mathfrak{g}: n}^{t}, \delta^{\sharp}\right)$ is the direct analog of the graph complex defined in [BC99, Proposition 3.4], where the differential $\delta^{\sharp}$ acts only on a decorated graph $\Gamma$ without self-loops by

$$
\begin{equation*}
\delta^{\sharp} \Gamma=\sum_{e=(i j): \text { regular edge }} \sigma(i, j) \cdot \Gamma / e \tag{6.5.2}
\end{equation*}
$$

where $e=(i j)$ denotes a regular edge connecting the vertex $i$ to the vertex $j$, and the sum is set to be zero if there is no regular edge in $\Gamma$.

Note that for $\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: n}^{t}$ being nonzero, we only consider $t \geq 0, n \geq 1$. Fix such a $n$, for $i \in \mathbb{N}$, let $H^{i}\left(\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: n}^{\bullet}, \delta\right)\left(\right.$ resp. $\left.H^{i}\left(\mathcal{G}_{\mathrm{ac}, \mathfrak{g}: n}^{\bullet}, \delta^{\sharp}\right), H^{i}\left(\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: n}^{\prime,}, \delta\right)\right)$ denote the $\bar{i}$-th cohomology group of the complex $\left(\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: n}^{\bullet}, \delta\right)\left(\right.$ resp. $\left.\left(\mathcal{G}_{\mathrm{ac}, \mathfrak{g}: n}^{\bullet}, \delta^{\sharp}\right),\left(\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: n}^{\prime, \bullet}, \delta\right)\right)$. Moreover, we have the following exact sequence

$$
\begin{equation*}
0 \rightarrow H^{0}\left(\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: n}^{\prime \cdot \bullet}, \delta\right) \rightarrow H^{0}\left(\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: n}^{\bullet}, \delta\right) \rightarrow H^{0}\left(\mathcal{G}_{\mathrm{ac}, \mathfrak{g}: n}^{\bullet}, \delta^{\sharp}\right) \rightarrow H^{1}\left(\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: n}^{\prime \bullet}, \delta\right) \rightarrow \cdots \tag{6.5.3}
\end{equation*}
$$

Definition 6.5.3. An element in $\operatorname{Ker} \delta \subset \mathcal{G C} \mathcal{C}_{\mathrm{ac}, \mathfrak{g}}$ or in $\operatorname{Ker} \delta^{\sharp} \subset \mathcal{G}_{\text {ac, } \mathfrak{g}}$ is called a graph cocycle in the respective graph complexes. In particluar, the graph cocycles of degree 0 are exactly the elements in $H^{0}\left(\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: n}^{\bullet}, \delta\right), H^{0}\left(\mathcal{G}_{\mathrm{ac}, \mathfrak{g}: n}^{\bullet}, \delta^{\sharp}\right)$.

In fact, we will be mainly concerned with the graph cocycles (or simply cocycles) of degree 0 . Before we proceed to see some examples, we give several easy facts on the cocycles.

- For $n=1$, we have $\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: 1}^{\prime, 0}=\mathbb{Q} \cdot$ dumbbell, $\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: 1}^{\prime, 1}=\mathbb{Q} \cdot$ figure-eight, which are 1-dimensional. We can conclude

$$
\begin{equation*}
H^{0}\left(\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: 1}^{\prime, \bullet}, \delta\right)=H^{1}\left(\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: 1}^{\prime}, \delta\right)=0 \tag{6.5.4}
\end{equation*}
$$

So that $H^{0}\left(\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: 1}^{\bullet}, \delta\right)=H^{0}\left(\mathcal{G}_{\mathrm{ac}, \mathfrak{g}: 1}^{\bullet}, \delta^{\sharp}\right)$, and they are also 1-dimensional (over $\mathbb{Q}$ ).

- Any cocycle in $\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}}$, by taking its quotient class or equivalently, by removing all the terms including self-loops, gives a cocycle in $\mathcal{G}_{\mathrm{ac}, \mathfrak{g}}$.
- If $\Gamma, \Gamma^{\prime}$ are two cocycles, then so is $\Gamma \cup \Gamma^{\prime}$. So that the spaces of cocycles carry the induced structure of graded commutative algebra.
Example 6.5.4 (2-loop cocycles). (1) In the graph complex $\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}}$, the following linear combination gives a cocycle of degree 0 with 2-loops:

$$
\begin{equation*}
\Theta-\frac{3}{2} \mathrm{O}-\mathrm{O} \tag{6.5.5}
\end{equation*}
$$

where $\Theta$ and $\mathrm{O}-\mathrm{O}$ decorated as in Fig. 4. In fact, $H^{0}\left(\mathcal{G C}_{\mathbf{a c}, \mathfrak{g}: 1}^{\bullet}, \delta\right)$ is exactly spanned by the above cocycle over $\mathbb{Q}$. Here we need to put coefficient $\frac{3}{2}$ instead of 3 because the factor 2 appeared in (6.4.8), following from the internal IHX relation.
(2) In the graph complex $\mathcal{G}_{\mathrm{ac}, \mathfrak{g}}$, the $\Theta$-graph itself gives a cocycle since $\delta$-action on it yields graph with self-loops which is defined to be zero in the quotient space $\mathcal{G}_{\text {ac, } \mathfrak{g}} \simeq \mathcal{G} \mathcal{C}_{\text {ac, } \mathfrak{g}} / \mathcal{G C}_{\text {ac, } \mathfrak{g}}^{\prime}$. So that $H^{0}\left(\mathcal{G}_{\mathrm{ac}, \mathfrak{g}: 1}^{\bullet}, \delta^{\sharp}\right)$ is spanned by the $\Theta$-graph.
(3) The decorated graph $\Theta \cup \Theta$ is a nontrivial cocycle in $\mathcal{G}_{\mathrm{ac}, \mathfrak{g}: 2}^{0}$, but it has 4 loops in topological sense.
Example 6.5.5 (3-loop cocycles). In [BC98, Example 4.6], for the trivial local system on a framed homology 3-sphere, Bott-Cattaneo gave an example of cocycle with degree 0 and order 2 (hence with 3 loops) by the following linear combination:

$$
\begin{equation*}
\Gamma^{\prime}=\frac{1}{12} \Gamma_{1}+\frac{1}{4} \Gamma_{2} \tag{6.5.6}
\end{equation*}
$$

where $\Gamma_{1}$ and $\Gamma_{2}$ are given in Fig. 14 (without the weight systems induced from Lie algebra $\mathfrak{g}$ or the numbering on the edges).

As an element in $\mathcal{G}_{\mathrm{ac}, \mathfrak{g}: 2}^{0}$, the following linear combination is a cocycle (the coefficient of $\Gamma_{2}$ has been changed to $-\frac{1}{8}$ ):

$$
\begin{equation*}
\Gamma=\frac{1}{12} \Gamma_{1}-\frac{1}{8} \Gamma_{2}, \tag{6.5.7}
\end{equation*}
$$

where $\Gamma_{1}$ and $\Gamma_{2}$ are given in Fig. 14 with all the decorations.

$\Gamma_{1}$

$\Gamma_{2}$

Fig. 14: Two examples of decorated trivalent 3-loop graphs without self-loop


Fig. 15: Graph $\Gamma_{I}$ in the computation of $\delta^{\sharp} \Gamma$


Fig. 16: Three examples of decorated trivalent 3-loop graphs with self-loops.
To show $\Gamma$ being a cocycle in $\mathcal{G}_{\mathrm{ac}, \mathfrak{g}: 2}^{0}$, we need to use the identity in Fig. 8 and the following equations in $\mathcal{G}_{\mathrm{ac}, \mathfrak{g}: 2}^{0}$,

$$
\begin{equation*}
\delta^{\sharp} \Gamma_{1}=3 \Gamma_{I}, \delta^{\sharp} \Gamma_{2}=2 \Gamma_{I} . \tag{6.5.8}
\end{equation*}
$$

However, if we view $\Gamma$ as an element in $\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: 2}^{0}$, then it is no longer a cocycle since the contraction map on $\Gamma_{2}$ produces the terms with self-loops which do not cancel out each other.

In fact, we have three more connected trivalent graphs of order 2, which are displayed in Fig. 16 (one needs to assign numberings on their vertices or edges to make a decorated graph). We can add these graphs with self-loops into $\Gamma$ to obtain a cocycle $\widetilde{\Gamma}$ in $\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: 2}^{0}$ :

$$
\begin{equation*}
\widetilde{\Gamma}=\Gamma+\mathbb{Q} \text {-linear combination of } \Gamma_{3}, \Gamma_{4} \text { and } \Gamma_{5} . \tag{6.5.9}
\end{equation*}
$$

A precise formula for $\widetilde{\Gamma}$ can be deduced from Example 6.6.8.
Remark 6.5.6. As we already saw in Section 5, the graph cocycle in Example 6.5.4 (1) is implicitly considered in [CS21] to define their $Z_{1}$-invariants. That's also one motivation that we need to include the graphs with self-loops into our consideration in the first place, which makes differences from the settings in [BC98, BC99]. Then we will explain in the next subsection the use of adapted propagators for an acyclic $E_{\rho}$ will naturally exclude such graphs with self-loops.

### 6.6 Topological trivalent graph and Chern-Simons perturbation theory

Now we focus on the connected trivalent graphs. Let $\Gamma$ be a connected trivalent graph, for $\Gamma$ being a decorated trivalent graph as in Definition 6.3.1, it is enough to fix numberings on the edges and vertices as well as directions on the edges. Then the weight system at each vertex of $\Gamma$ is uniquely determined by the induced cyclic order of the incident half-edges.
Definition 6.6.1 (Relative orientation). Suppose that $\Gamma$ is a connected decorated trivalent graph, and let $h(\Gamma)$ denote the set of all half-edges of $\Gamma$. Then we have the induced vertex orientation and the induced edge orientation on the same set $h(\Gamma)$, the relative orientation of $\Gamma$, denoted by or $_{\Gamma} \in\{ \pm 1\}$, is defined as the sign of the permutation which maps the edge orientation to the vertex orientation.

The following Lemma 6.6.2 is an analog of [CV03, Corollary 1] and [AS92, §3].
Lemma 6.6.2. Let $\Gamma$ be a connected decorated trivalent graph, let $\Gamma^{\prime}$ be the decorated trivalent graph given by the same underlying topological graph as of $\Gamma$ but with different numberings on the edges and vertices, then we have the identity in $\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}}^{0}$

$$
\begin{equation*}
\text { or }_{\Gamma} \cdot \Gamma=\text { or }_{\Gamma^{\prime}} \cdot \Gamma^{\prime} \tag{6.6.1}
\end{equation*}
$$

Let $\Gamma_{1}, \Gamma_{2}$ be two connected decorated trivalent graphs, then we have

$$
\begin{equation*}
\mathrm{or}_{\Gamma_{1} \cup \Gamma_{2}}=\operatorname{or}_{\Gamma_{1}} \cdot \mathrm{or}_{\Gamma_{2}} . \tag{6.6.2}
\end{equation*}
$$

Proof. It is enough to check that the term or ${ }_{\Gamma}$ cancels the sign change coming from the numbering change of vertices, edges, and direction reversals of edges. If we permute the numbering on $v(\Gamma)$ by a permutation of order $p$, then edge orientation of $h(\Gamma)$ is fixed but vertex orientation of $h(\Gamma)$ gives sign change of $(-1)^{p}$ so that or ${ }_{\Gamma}$ also change sign by $(-1)^{p}$. If we change the direction of a non-self-loop edge $e=(i j)$, then the edge orientation of $h(\Gamma)$ differs by $(-1)$ under this direction reversal whereas the vertex orientation of $h(\Gamma)$ is fixed. If we change the direction of a self-loop edge $e=(i i)$ incident to vertex $i$, then both edge orientation and vertex orientation of $h(\Gamma)$ differ by $(-1)$ under this change, hence the total change is 1 . Similarly, if numbering change of $e(\Gamma)$ gives rise to $k$-times changes of cyclic ordering at trivalent vertices, then edge orientation of $h(\Gamma)$ is fixed but vertex orientation of $h(\Gamma)$ changes by the same manner.

Therefore, or ${ }_{\Gamma}$ cancels any sign change from permutation of numberings of edges and vertices, and direction reversals of edges.

The above lemma indicates that for the decorated trivalent graphs, the underlying topological graph determines uniquely its equivalence class up to a sign.

For integer $n \geq 1$, let $\mathfrak{G}$ be a topological trivalent graph with $2 n$ vertices and $3 n$ edges, or equivalently, consider a set $h(\mathfrak{G})$ of $6 n$ (abstract) half-edges, then a trivalent graph means the couple of partitions of $h(\mathfrak{G})$ :

- A partition into pairs of half-edges which we call edges.
- A partition into sets of cardinality (=valency) 3 which we call vertices.

If any two vertices can be connected by a consecutive path of edges (any neighboring edges have only one common half-edge), then we call the graph to be connected. If $\mathfrak{G}_{1}, \mathfrak{G}_{2}$ are two topological trivalent graphs, they are called equivalent to each other if there is a bijection between $h\left(\mathfrak{G}_{1}\right)$ and $h\left(\mathfrak{G}_{2}\right)$ which maps the couple of partitions of $h\left(\mathfrak{G}_{1}\right)$ to the ones of $h\left(\mathfrak{G}_{2}\right)$. We will always identify the equivalent graphs as the same one. Note that the order of $\mathfrak{G}$ is defined by the same formula as in (6.4.1) (since $\mathfrak{G}$ is trivalent, it always has degree 0 ).

Definition 6.6.3 (Automorphism group of topological trivalent graph). let $\mathfrak{G}$ be a topological trivalent graph with $2 n$ vertices and $3 n$ edges, then an automorphism of $\mathfrak{G}$ is an element of the permutation group of $h(\mathfrak{G})$ which preserves both partitions of $h(\mathfrak{G})$ for the edges and vertices of $\mathfrak{G}$, we denote the group of all automorphism of $\mathfrak{G}$ by $\operatorname{Aut}(\mathfrak{G})$.
Remark 6.6.4. If $\mathfrak{G}$ is a topological trivalent graph that is not connected, then by our definition of $\operatorname{Aut}(\mathfrak{G})$, its action always preserves the non-equivalent connected components of $\mathfrak{G}$. For example, suppose that $\mathfrak{G}_{1}, \mathfrak{G}_{2}$ are two connected topological trivalent graphs, then

$$
\operatorname{Aut}\left(\mathfrak{G}_{1} \cup \mathfrak{G}_{2}\right)= \begin{cases}\operatorname{Aut}\left(\mathfrak{G}_{1}\right) \times \operatorname{Aut}\left(\mathfrak{G}_{2}\right) \rtimes \mathbb{Z}_{2}, & \text { if } \mathfrak{G}_{1}=\mathfrak{G}_{2} \neq \emptyset  \tag{6.6.3}\\ \operatorname{Aut}\left(\mathfrak{G}_{1}\right) \times \operatorname{Aut}\left(\mathfrak{G}_{2}\right), & \text { if else }\end{cases}
$$

Let $\mathcal{T} \mathcal{G}_{n}$ denote the vector space spanned by all the topological trivalent graphs with $2 n$ vertices over $\mathbb{Q}$. We consider the linear map

$$
\begin{equation*}
\Psi_{n}: \mathcal{T G}_{n} \rightarrow \mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: n}^{0}, \quad \mathfrak{G} \mapsto \operatorname{or}_{\Gamma(\mathfrak{G})} \cdot \Gamma(\mathfrak{G}) \tag{6.6.4}
\end{equation*}
$$

where $\Gamma(\mathfrak{G})$ is any decorated trivalent graph whose underlying topological graph is $\mathfrak{G}$.
As a consequence of Lemma 6.6.2, we have the following result.
Corollary 6.6.5. For each $n \geq 1$, the linear map $\Psi_{n}$ is an isomorphism of finite dimensional vector spaces.

Let's consider the generating series of perturbative invariants for a framed closed 3-manifold defined from the perturbative Chern-Simons theory (see [Kon94, Section 2], [AS92, AS94], [Saw06, §3]), which, in terms of the trivalent graph, is formally given by

$$
\sum_{n \geq 0} \hbar^{n} \sum_{\substack{\text { trivalent graph }  \tag{6.6.5}\\
\text { of order } n}} \frac{1}{|\operatorname{Aut}(\mathfrak{G})|} \mathfrak{G}=\exp \left(\sum_{n \geq 1} \hbar^{n} \sum_{\begin{array}{c}
\text { connected trivalent graph } \mathfrak{G} \\
\text { of order } n
\end{array}} \frac{1}{|\operatorname{Aut}(\mathfrak{G})|} \mathfrak{G}\right),
$$

where for $n=0$ we take $\mathfrak{G}=\emptyset$ viewed as a unit element, $|\operatorname{Aut}(\mathfrak{G})|=1$, and the multiplication of the topological graphs is given by the disjoint union $\cup$ (it is commutative).
Proposition 6.6.6. - For each integer $n$ with $n \geq 1$, there is a cocycle of order $n$ in $\mathcal{G C}_{\mathrm{a}, \mathfrak{g}: n}^{0}$ given as the form

$$
\begin{equation*}
\sum_{\text {connected } \mathfrak{G}} \frac{1}{|\operatorname{Aut}(\mathfrak{G})|} \Psi_{n}(\mathfrak{G}) \in H^{0}\left(\mathcal{G C}_{\mathbf{a c}, \mathfrak{g}: n}^{\bullet}, \delta\right) \tag{6.6.6}
\end{equation*}
$$

where the sum runs over all the connected topological trivalent graph $\mathfrak{G}$ of order $n$.

- For each integer $n$ with $n \geq 1$, there is a cocycle of order $n$ in $\mathcal{G}_{\mathrm{ac}, \mathfrak{g}: n}^{0}$ given as the form

$$
\sum_{\begin{array}{c}
\text { connected } \mathfrak{G}  \tag{6.6.7}\\
\text { without self-loops }
\end{array}} \frac{1}{|\operatorname{Aut}(\mathfrak{G})|} \Psi_{n}(\mathfrak{G}) \in H^{0}\left(\mathcal{G}_{\mathrm{ac}, \mathfrak{g}: n}^{\bullet}, \delta^{\sharp}\right),
$$

where the sum runs over all the connected topological trivalent graph $\mathfrak{G}$ without self-loops and of order $n$.

Proof. Fix a partition $V$ of ( $6 n$ ) half-edges into the sets of cardinality 3 (viewed as vertices), let $P_{n}$ denote the set of partitions of this set of ( $6 n$ ) half-edges into pairs. Then, consider a surjective map

$$
\begin{equation*}
\pi_{n}: P_{n} \rightarrow \mathcal{T} \mathcal{G}_{n} \tag{6.6.8}
\end{equation*}
$$

which sends a partition $E$ in $P_{n}$ to the topological graph $\mathfrak{G}(E, V)$ given by the equivalent class of the couple of partitions $(E, V)$.

For a topological graph $\mathfrak{G} \in \mathcal{T} \mathcal{G}_{n}$, let $G_{h(\mathfrak{G})}$ be the permutation group of the set $h(\mathfrak{G})$ of half-edges of $\mathfrak{G}$, and let $G_{v(\mathfrak{G})}, G_{e(\mathfrak{G})} \subset G_{h(\mathfrak{G})}$ denote the subgroups preserving the partitions of $h(\mathfrak{G})$ for the vertices and edges of $\mathfrak{G}$ respectively. With these notations, we get

$$
\begin{equation*}
\left|\pi_{n}^{-1}(\mathfrak{G})\right|=\frac{\left|G_{v(\mathfrak{G})}\right|}{|\operatorname{Aut}(\mathfrak{G})|} \tag{6.6.9}
\end{equation*}
$$

Then the sum (6.6.6) can be written as

$$
\begin{align*}
& \sum_{\text {connected } \mathfrak{G}} \frac{1}{|\operatorname{Aut}(\mathfrak{G})|} \Psi_{n}(\mathfrak{G}) \\
= & \sum_{\text {connected } \mathfrak{G}} \frac{\left|\pi_{n}^{-1}(\mathfrak{G})\right|}{\left|G_{v(\mathfrak{G})}\right|} \Psi_{n}(\mathfrak{G}) \\
= & \sum_{\text {connected } \mathfrak{G}} \frac{\left|\pi_{n}^{-1}(\mathfrak{G})\right|}{(3!)^{|v(\mathfrak{G})|}|v(\mathfrak{G})|!} \Psi_{n}(\mathfrak{G})  \tag{6.6.10}\\
= & \frac{1}{(3!)^{2 n}(2 n)!} \sum_{E \in P_{n} ; \text { connected }} \Psi_{n}(\mathfrak{G}(E, V))
\end{align*}
$$

where in the last summation $E$ runs over all the partitions in $P_{n}$ which give connected trivalent graphs. Since we consider all the possible partitions giving connected trivalent graphs, all the resulting terms after the map $\delta$ vanish by IHX relation at the vertex with valency 4 . Indeed, we focus on a vertex with valency 4 . Then, there are only 3 possible ways to insert weight systems at the vertex, i.e., IHX type graphs as Fig. 5. We know that such insertions happen at each vertex with valency 4 with the same coefficient from the last equation. Finally, we note that the sign of the resulting weight systems is compatible with Jacobi identity (6.2.4), since $X$ graph in Fig. 3 has opposite cyclic ordering from our convention but one sees that this sign emerges from or ${ }_{\Gamma}$ in $\Psi_{n}$ by direct computation for local graphs as Fig. 3. (2) is immediate from (1) by removing graphs with self-loops. This completes the proof.

Example 6.6.7. In particular, considering the order-1 part in Proposition 6.6.6, we can recover Cattaneo-Shimizu's 2-loop term (5.1.4). Indeed, let $\Theta$ and $\mathrm{O}-\mathrm{O}$ be the theta graph and dumbbell graph decorated as Fig. 4 (they are the only connected trivalent graphs of order 1), then or ${ }_{\Theta}=-1$ and or $_{O-O}=1$. For their underlying topological graphs, we have $|\operatorname{Aut}(\Theta)|=12,|\operatorname{Aut}(O-O)|=8$ (cf. [Saw06, §3]). Thus, we get a cocycle in $\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: 1}^{0}$,

$$
\begin{equation*}
\Gamma=\frac{1}{12} \text { or }_{\Theta} \cdot \Theta+\frac{1}{8} \text { or }_{O-O} \cdot \mathrm{O}-\mathrm{O}=-\frac{1}{12} \Theta+\frac{1}{8} \mathrm{O}-\mathrm{O} . \tag{6.6.11}
\end{equation*}
$$

Applying the linear map $(-12) Z_{\Gamma}(\omega)$, we obtain $Z_{1}(\rho ; \omega, \xi)$ in (5.1.4).
Example 6.6.8. For the connected topological trivalent graph of order 2, there are two cases without self-loops as given in Fig. 14, and there are 3 other cases with at least one self-loop edge given in Fig. 16. Let $\mathfrak{G}_{j}, j=1, \cdots, 5$, denote the respective underlying topological graphs of $\Gamma_{j}, j=1, \cdots, 5$ in Fig. 14 \& Fig. 16, then we have

$$
\begin{align*}
& \left|\operatorname{Aut}\left(\mathfrak{G}_{1}\right)\right|=24,\left|\operatorname{Aut}\left(\mathfrak{G}_{2}\right)\right|=16, \\
& \left|\operatorname{Aut}\left(\mathfrak{G}_{3}\right)\right|=16,\left|\operatorname{Aut}\left(\mathfrak{G}_{4}\right)\right|=8,\left|\operatorname{Aut}\left(\mathfrak{G}_{5}\right)\right|=48 . \tag{6.6.12}
\end{align*}
$$

Meanwhile, we have or $_{\Gamma_{1}}=-1$ and or $_{\Gamma_{2}}=1$, this way, from (6.6.7) we get a cocycle (without selfloops) in $\mathcal{G}_{\mathrm{ac}, \mathfrak{g}: 2}^{0}$, which is proportional to the one defined in (6.5.7). If we include the other 3 cases with self-loops ( $\Gamma_{3}, \Gamma_{4}, \Gamma_{5}$ ), we can work out explicitly a cocycle $\Gamma^{\prime}$ mentioned in (6.5.9).

## 7 Higher-loop invariants of framed closed 3-manifolds with acyclic local systems

This section studies the higher integral invariants associated with graph cocycles in the complex of decorated graphs. In general, the refined situation as in Cattane-Shimizu [CS21] requires a graph complex allowing self-loops which generalizes that of Bott-Cattaneo [BC99] which consists of graphs without self-loops. By using the graph complex introduced in Section 6 and the adapted propagators as in Definition 4.4.1, we show that the integral map $Z$ factors through a quotient graph complex which coincides with that of Bott-Cattaneo [BC99] without self-loops. This means that when an acyclic local system is given by $\rho: \pi_{1}(M) \rightarrow G \xrightarrow{\text { Ad }} \operatorname{Aut}(\mathfrak{g})$, graph complex without self-loops are enough to define the integral invariants associated to the higher-loop terms.

In this section, a framing $f$ of $M$ and an orientation $o(M)$ are always fixed, and we always assume the local system $E_{\rho}$ to be acyclic.

### 7.1 Integral invariants of higher order associated to acyclic local systems

In this subsection, we study graph cocycle invariants of a framed closed 3-manifold with acyclic local system associated with a representation $\pi_{1}(M) \rightarrow G \xrightarrow{\text { Ad }} \operatorname{Aut}(\mathfrak{g})$.

In the sequel, we mainly consider degree 0 part and degree 1 part of graph complexes considered in subsection 6.5 , hence internal vertices are decorated by the cubic trace form $\operatorname{Tr}$ or $W_{T}$ with $T=I, H, X$ as given in Example 6.3.3.

Assume the local system $E_{\rho}$ to be acyclic. For a fixed propagator $\omega$ as defined in Subsection 4.3 which satisfying conditions in Proposition 4.3 .1 and Corollary 4.3.2, let us define a $\mathbb{Q}$-linear map $Z_{-}(\omega)$ on $\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}}^{0}$.

At first, associated to each edge $e \in e(\Gamma)$ of a decorated trivalent graph $\Gamma$ (hence of degree 0 ), we define the 2 -form $\omega_{e}$ on $C_{2 n}(M)$ as follows:

$$
\omega_{e}:= \begin{cases}p_{i j}^{*} \omega & \text { if } e=(i j) \text { with } i \neq j,  \tag{7.1.1}\\ q^{*} p_{i}^{*} \xi & \text { if } e=(i i) \text { is a directed self-loop }\end{cases}
$$

where $p_{i j}: C_{2 n}(M) \rightarrow C_{2}(M)$ is the natural projection map induced by $M^{2 n} \rightarrow M \times M$ which sends $\left(x_{1}, \ldots, x_{2 n}\right) \mapsto\left(x_{i}, x_{j}\right), q: C_{2 n}(M) \rightarrow M^{2 n}$ is the blow-down map (by abuse of notation), and $p_{i}: M^{2 n} \rightarrow M$ is the natural $i$-th projection map. Note that when $e=(i j), i \neq j$, the coefficient of form $\omega_{e}$ is in $p_{i}^{*} E_{\rho} \otimes p_{j}^{*} E_{\rho}$; when $e=(i i)$ is a self-loop with the orientation given by the ordered half-edges $h_{+}=(i, e,+1)<h_{-}=(i, e,-1)$, then the form $\omega_{(i i)}$ is valued in $p_{i}^{*}\left(E_{\rho, h_{+}} \otimes E_{\rho, h_{-}}\right)$. In particular, since $T^{*} \omega=-\omega$, we conclude for $i \neq j$,

$$
\begin{equation*}
\omega_{(i j)}=-\omega_{(j i)} . \tag{7.1.2}
\end{equation*}
$$

Take a decorated trivalent graph $\Gamma$ of order $n$, then $2 n=2 \operatorname{ord}(\Gamma)=|v(\Gamma)|$ is the number of vertices of $\Gamma$. At each vertex, the weight system $\operatorname{Tr}$ is a $\operatorname{Ad}(G)$-invariant linear form on $\mathfrak{g}^{\otimes 3}$, so that it descends to a morphism of vector bundles $E_{\rho}^{\otimes 3} \rightarrow \mathbb{R}$ over $M$, equivalently, we view the weight system at each vertex of $\Gamma$ as a flat skew-symmetric section of $\left(E_{\rho}^{\vee}\right)^{\otimes 3} \rightarrow M$ (also cf. Subsection 3.4).

The order of the product manifold $M^{2 n}$ (which has an induced orientation from $o(M)$ ) coincides with the given numbering on $v(\Gamma)$, or equivalently, we may write $\left(x_{1}, \ldots, x_{2 n}\right)=\left(x_{i}\right)_{i \in v(\Gamma)} \in M^{2 n}$. At each point $\left(x_{1}, \ldots, x_{2 n}\right) \in M^{2 n}$, we have the tensor product of vector bundles:

$$
\begin{equation*}
\left(E_{\rho}^{\vee}\right)_{x_{1}}^{\otimes 3} \otimes\left(E_{\rho}^{\vee}\right)_{x_{2}}^{\otimes 3} \otimes \cdots \otimes\left(E_{\rho}^{\vee}\right)_{x_{2 n}}^{\otimes 3}, \tag{7.1.3}
\end{equation*}
$$

then by considering the set of half-edges $h(\Gamma)$ of $\Gamma$, each factor $E_{\rho, x_{j}}^{\vee}$ in (7.1.3) can be regard as a copy of $E_{\rho}^{\vee}$ indexed by a half-edge $h$ attached to vertex $j$. Then we consider the pull-back of $E_{\rho}^{\vee} \boxtimes E_{\rho}^{\vee} \rightarrow M \times M$ by $p_{i j}$ for a non-self-loop edge $e=(i j)$ and the pull-back $E_{\rho}^{\vee} \otimes E_{\rho}^{\vee} \rightarrow M$ by $p_{i}$ for a self-loop edge $e=(i i)$, then each copy of $E_{\rho}^{\vee}$ is clearly index by the half-edges of $e$, therefore we get again the tensor product of vector bundles as in (7.1.3). We always identify these two perspectives for the vector bundle $p_{1}^{*}\left(E_{\rho}^{\vee}\right)^{\otimes 3} \otimes p_{2}^{*}\left(E_{\rho}^{\vee}\right)^{\otimes 3} \otimes \cdots \otimes p_{2 n}^{*}\left(E_{\rho}^{\vee}\right)^{\otimes 3}$ over $M^{2 n}$ or $C_{2 n}(M)$.

Set

$$
\begin{equation*}
Z_{\Gamma}(\omega):=\int_{C_{2 n}(M)}\left(\bigotimes_{i \in v(\Gamma)} \operatorname{Tr}_{i}\right) \bigwedge_{e \in e(\Gamma)} \omega_{e}, \tag{7.1.4}
\end{equation*}
$$

where $e=(i j)$ means the edge connecting the vertex $i$ to the vertex $j$ (which always carry an orientation when $i=j$ ). Note that in (7.1.4), the factor $\bigotimes_{i \in v(\Gamma)} \operatorname{Tr}_{i}$ corresponds to the tensor product of the decoration Tr at each vertex given in Definition 6.3.1. Note that by our convention, to apply $\otimes_{i \in v(\Gamma)} \operatorname{Tr}_{i}$ on $\bigwedge_{e \in e(\Gamma)} \omega_{e}$, we need to pair the factor of $E_{\rho}^{\vee}$ in $\bigotimes_{i \in v(\Gamma)} \operatorname{Tr}_{i}$ corresponding to a half-edge $h \in h(\Gamma)$ with the factor $E_{\rho}$ in $\bigwedge_{e \in e(\Gamma)} \omega_{e}$ that corresponds to the same half-edge $h$.
Proposition 7.1.1 (Definition of $\left.Z_{-}(\omega)\right)$. Let $\Gamma$ be a decorated trivalent graph of order n. If $\Gamma^{\prime}$ is another decorated trivalent graph of order $n$ which is equivalent to $\Gamma$ via the equivalence relation of (6.4.3), then $\operatorname{sign}\left(\Gamma, \Gamma^{\prime}\right) Z_{\Gamma^{\prime}}(\omega)=Z_{\Gamma}(\omega)$.

Therefore, the following linear map is well-defined:

$$
\begin{equation*}
Z_{-}(\omega): \mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: n}^{0} \rightarrow \mathbb{R}, \quad \Gamma \mapsto Z_{\Gamma}(\omega) \tag{7.1.5}
\end{equation*}
$$

Proof. Note that if we permute the numbering on $v(\Gamma)$ by a permutation of order $p$, this gives an auto-identification of $C_{2 n}(M)$ with the orientation change by $(-1)^{p}$. If we change an orientation of a non-self-loop edge $e=(i j)$, it is equivalent to change $\omega_{(i j)}$ to $\omega_{(j i)}$, we obtain a factor ( -1 ) by (7.1.2), let $(-1)^{m}$ denoete the total change by this kind of operation. If we change the orientation of
a self-loop edge at vertex $i$, we obtain the same term by our convention and the property $T^{*} \xi=-\xi$. If we permute the numbering on $e(\Gamma)$ which implies $k$-times change cyclic orders at trivalent vertices, we obtain a factor $(-1)^{k}$ by corresponding sign change on associated cubic traces. This way, we obtain the sign $(-1)^{p+m+k}$ when we compare $Z_{\Gamma^{\prime}}(\omega)$ with $Z_{\Gamma}(\omega)$, it completes the proof of our proposition.

Lemma 7.1.2. Fix a propagator $\omega$. If $\Gamma_{1}, \Gamma_{2}$ are two decorated trivalent graphs, then

$$
\begin{equation*}
Z_{\Gamma_{1} \cup \Gamma_{2}}(\omega)=Z_{\Gamma_{1}}(\omega) Z_{\Gamma_{2}}(\omega) . \tag{7.1.6}
\end{equation*}
$$

Proof. Set $n_{1}=\operatorname{ord}\left(\Gamma_{1}\right), n_{2}=\operatorname{ord}\left(\Gamma_{2}\right)$, then $\operatorname{ord}\left(\Gamma_{1} \cup \Gamma_{2}\right)=n_{1}+n_{2}$. Consider the smooth map

$$
\begin{equation*}
\Psi: \operatorname{Conf}_{2 n_{1}+2 n_{2}}(M) \rightarrow \operatorname{Conf}_{2 n_{1}}(M) \times \operatorname{Conf}_{2 n_{2}}(M) \tag{7.1.7}
\end{equation*}
$$

It induces a diffeomorphism between $\operatorname{Conf}_{2 n_{1}+2 n_{2}}(M)$ and Image $(\Psi)$, and Image( $\Psi$ ) has full measure in $\operatorname{Conf}_{2 n_{1}}(M) \times \operatorname{Conf}_{2 n_{2}}(M)$, i.e., $\left(\operatorname{Conf}_{2 n_{1}}(M) \times \operatorname{Conf}_{2 n_{2}}(M)\right) \backslash \operatorname{Image}(\Psi)$ has Lebesgue measure zero. Moreover, the tangent map of $\Psi$ acts as identity map on each copy of $T M$.

In the same time, on $\operatorname{Conf}_{2 n_{1}+2 n_{2}}(M) \simeq \operatorname{Image}(\Psi)$, we have the identity

$$
\begin{equation*}
\left(\bigotimes_{i \in v\left(\Gamma_{1} \cup \Gamma_{2}\right)} \operatorname{Tr}_{i}\right) \bigwedge_{e \in e\left(\Gamma_{1} \cup \Gamma_{2}\right)} \omega_{e}=\left(\bigotimes_{i \in v\left(\Gamma_{1}\right)} \operatorname{Tr}_{i}\right) \bigwedge_{e \in e\left(\Gamma_{1}\right)} \omega_{e} \wedge\left(\bigotimes_{i \in v\left(\Gamma_{2}\right)} \operatorname{Tr}_{i}\right) \bigwedge_{e \in e\left(\Gamma_{2}\right)} \omega_{e} \tag{7.1.8}
\end{equation*}
$$

In the definition (7.1.4), we can replace the integrals on $C_{2 n}(M)$ by the integrals on $\operatorname{Conf}_{2 n}(M)$ or on an open dense subset with full measure. Therefore, our lemma follows from the relation (7.1.8).

In the above definition, $Z_{\Gamma}(\omega)$ depends on the decorations of connected graphs. Following Lemma 6.6.2, for a fixed propagator, we can get configuration integrals depending only on the underlying topological graph as follows.

Lemma 7.1.3. Fix a propagator $\omega$. Let $\Gamma$ be a decorated trivalent graph, and let $\mathrm{or}_{\Gamma}$ be the relative orientation as in Definition 6.6.1. Then, the quantity

$$
\begin{equation*}
\mathrm{or}_{\Gamma} \cdot Z_{\Gamma}(\omega) \tag{7.1.9}
\end{equation*}
$$

is independent of the choice of numbering of $v(\Gamma), e(\Gamma)$ and orientations of edges. In other words, it only depends on the underlying topological graph of $\Gamma$.
Remark 7.1.4. Definition in (7.1.9) is essentially the same as [AS94] where they use super propagator to define their integral invariants. The key difference is that we are allowed to permute freely each factor $E_{\rho}^{\vee}$ in $p_{1}^{*}\left(E_{\rho}^{\vee}\right)^{\otimes 3} \otimes p_{2}^{*}\left(E_{\rho}^{\vee}\right)^{\otimes 3} \otimes \cdots \otimes p_{2 n}^{*}\left(E_{\rho}^{\vee}\right)^{\otimes 3}$ to make $\left(\otimes_{i \in v(\Gamma)} \operatorname{Tr}_{i}\right)$ pair with $\bigwedge_{e \in e\left(\Gamma_{2}\right)} \omega_{e}$, while permutations produce nontrivail signs in the formalism of super propagators of [AS94]. Our definition here is inspired by that of [Les04, Les20] for the integral invariants of rational homology 3 -spheres.

We first give the following theorem which can be viewed as a direct higher-order extension of the $Z_{1}$-invariants introduced by Cattaneo-Shimizu [CS21], where we use the general propagators instead of adapted propagators to define the integral invariants for the cocycles in $\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: n}^{0}$. The $Z_{1}$-invariants are associated to the cocycle Example 6.5.4 (1) in $\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: 1}^{0}$ (with order $n=1$ or with 2 loops), and here we show that for the cocycles with higher orders (or loops), we also have the well-defined invariants associated to the acyclic local system $E_{\rho}$.
Theorem 7.1.5. Fix a homotopy class $[f]$ of framing of $M$ and an orientation $o(M)$. Let $E_{\rho}$ be an acyclic local system over $M$ associated with a representation $\rho: \pi_{1}(M) \rightarrow G \xrightarrow{\text { Ad }} \operatorname{Aut}(\mathfrak{g})$. Let $\Gamma \in \mathcal{G C} \mathcal{a}_{\mathrm{ac}, \mathfrak{g}: n}^{0}$ be a cocycle (i.e., $\delta \Gamma=0$ ). Then the number $Z_{\Gamma}(\omega) \in \mathbb{R}$ is independent of the choice of the propagator $\omega$ or the framing $f \in[f]$, which is called the integral invariant associated to the cocycle $\Gamma$.

Therefore, the linear functional

$$
Z(M, \rho,[f]): \operatorname{ker}\left(\left.\delta\right|_{\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: n}^{0}}\right)=H^{0}\left(\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: n}^{\bullet}, \delta\right) \rightarrow \mathbb{R},
$$

given by $Z(M, \rho,[f])(\Gamma):=Z_{\Gamma}(\omega)$ with any propagator $\omega$ constructed from a given framing $f \in[f]$, is an invariant of $(M, o(M),[f])$ and of the acyclic local system $E_{\rho}$.

The proofs of Theorem 7.1.5 will be given in Subsection 7.2.
Next, we connect the integral invariants associated with graph complex $\mathcal{G C} \mathcal{C}_{\text {ac, }}$ possibly with selfloops and those associated with $\mathcal{G}_{\text {ac, } \mathfrak{g}}$ without self-loops. This extends the idea in Theorem 5.2.1, where the introduction of an adapted propagator is the key step.

Recall that $\mathcal{G C}^{\prime, 0}, \mathfrak{G c}, \mathcal{G C}_{\mathrm{ac}, \mathfrak{g}}^{0}$ is the subspace consisting of all decorated trivalent graphs always with self-loops.
Proposition 7.1.6. Let $\omega^{\sharp}$ be an adapted propagator with $\mathfrak{i}_{\partial}^{*}\left(\omega^{\sharp}\right)=I(\eta)+q_{\partial}^{*}\left(\xi^{\sharp}\right)$ as in Definition 4.4.1, the map $Z_{-}\left(\omega^{\sharp}\right)$ restricts to zero on $\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}}^{\prime, 0}$.

Proof. The vanishing argument is almost the same as the case of dumbbell graph. We focus only on integrand $q^{*} p_{i}^{*} \xi^{\sharp}$ associated with a self-loop (ii). Note that by our definition of an adapted propagator $\omega^{\sharp}$, we have $\mathfrak{L}\left(\xi^{\sharp}\right)=0$.

Suppose that the vertex $i$ is connected by an edge $(i j)$ with $j \neq i$. By (3.5.5) and (5.1.1), the integrand associated with the vertex $i$ and the edges $(i i)$ and ( $i j$ ) becomes

$$
\begin{align*}
& \operatorname{Tr}_{i}\left(\omega_{(i j)}^{\sharp} q^{*} p_{i}^{*} \xi^{\sharp}\right) \\
= & B_{i}\left(\mathfrak{L}\left(\xi^{\sharp}\right), \omega_{(i j)}^{\sharp}\right)  \tag{7.1.10}\\
= & 0
\end{align*}
$$

where we suppress other forms associated with edges connecting the vertex $j$ and the associated cubic trace for simplicity. Therefore, it means that, if a decorated trivalent graph $\Gamma$ has at least one self-loop, $Z_{\Gamma}\left(\omega^{\sharp}\right)=0$. This completes the proof.

Recall $\mathcal{G}_{\text {ac, } \mathfrak{g}}$ denote the subspace of $\mathcal{G C} \mathcal{C}_{\text {ac, }}$ spanned by the decorated graphs without self-loops. For each order $n$, we have the quotient graph complex ( $\mathcal{G}_{\mathrm{ac}, \mathfrak{g}: n}^{\bullet}, \delta^{\sharp}$ ) defined in (6.5.1).
Theorem 7.1.7. Assume $E_{\rho}$ to be acyclic. For any adapted propagator $\omega^{\sharp}$ in Definition 4.4.1, the map $Z_{-}\left(\omega^{\sharp}\right)$ factors through the quotient $\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: n}^{0} / \mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: n}^{\prime, 0} \simeq \mathcal{G}_{\mathrm{ac}, \mathfrak{g}: n}^{0}$ :

$$
\begin{equation*}
\left.\underset{\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: n}^{0} / \mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: n}^{\prime, 0}}{\downarrow} \simeq \mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: n}^{0} \xrightarrow{\mathcal{G}_{-, \mathfrak{g}: n}^{0}} \mathbb{R}^{\sharp}\right) \tag{7.1.11}
\end{equation*}
$$

This way, we have a linear map

$$
\begin{equation*}
Z_{-}\left(\omega^{\sharp}\right): \mathcal{G}_{\mathrm{ac}, \mathfrak{g}: n}^{0} \rightarrow \mathbb{R} \tag{7.1.12}
\end{equation*}
$$

Proof. This follows from (6.5.1) and Propositions 7.1.1 \& 7.1.6.
Our main results of this section are as follows. The first part is a refinement of [BC99, Theorem 1.1] for acyclic representation via adjoint representation of $\pi_{1}(M)$, which asserts that, for any cocycle without self-loops (i.e. in $H^{0}\left(\mathcal{G}_{\mathrm{ac}, \mathfrak{g}: n}^{\bullet}, \delta^{\sharp}\right)$ ), we have the associated integral invariant of framed closed 3-manifolds with acyclic local systems. In the second part, we connect the integral invariants defined by graph cocycles with self-loops to those without self-loops. Recall that the adapted propagator is defined in Definition 4.4.1.
Theorem 7.1.8. Fix a homotopy class $[f]$ of framing of $M$ and an orientation $o(M)$. Let $E_{\rho}$ be an acyclic local system over $M$ associated with a representation $\rho: \pi_{1}(M) \rightarrow G \xrightarrow{\operatorname{Ad}} \operatorname{Aut}(\mathfrak{g})$.
(1) Let $\Gamma \in \mathcal{G}_{\mathrm{ac}, \mathfrak{g}: n}^{0}$ be a cocycle of order $n$ (i.e., $\delta^{\sharp} \Gamma=0$ ). Then the number $Z_{\Gamma}\left(\omega^{\sharp}\right) \in \mathbb{R}$ is independent of the choice of the adapted propagator $\omega^{\sharp}$ or the framing $f \in[f]$, which is called the integral invariant associated to the cocycle $\Gamma$.
Therefore, the linear functional

$$
Z^{\sharp}(M, \rho,[f]): \operatorname{ker}\left(\left.\delta^{\sharp}\right|_{\mathcal{G}_{\mathrm{ac}, \mathfrak{g}: n}^{0}}\right)=H^{0}\left(\mathcal{G}_{\mathrm{ac}, \mathfrak{g}: n}^{\bullet}, \delta^{\sharp}\right) \rightarrow \mathbb{R},
$$

given by $Z^{\sharp}(M, \rho,[f])(\Gamma):=Z_{\Gamma}\left(\omega^{\sharp}\right)$ with any adapted propagator $\omega^{\sharp}$ defined with a framing $f \in[f]$, is an invariant of $(M, o(M),[f])$ and local system $E_{\rho}$.
(2) With the notation in (1), we have the following commutative diagram:

where the left vertical map is induced by the quotient map $\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: n}^{0} \rightarrow \mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: n}^{0} / \mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: n}^{\prime, 0} \simeq \mathcal{G}_{\mathrm{ac}, \mathfrak{g}: n}^{0}$, which is already explained in (6.5.3).
More precisely, let $\Gamma \in \mathcal{G} \mathcal{C}_{\mathrm{ac}, \mathfrak{g}: n}^{0}$ be a cocycle, and let $\omega$ be any propagator as in Definition 4.3.3 which is not necessary to be adapted. Let $\Gamma^{\prime} \in \mathcal{G}_{\mathrm{ac}, \mathrm{g}: n}^{0}$ be the cocycle given by removing the terms with self-loops from $\Gamma$. Then we have, for any adapted propagator $\omega^{\sharp}$,

$$
\begin{equation*}
Z_{\Gamma}(\omega)=Z(M, \rho,[f])(\Gamma)=Z^{\sharp}(M, \rho,[f])\left(\Gamma^{\prime}\right)=Z_{\Gamma^{\prime}}\left(\omega^{\sharp}\right) . \tag{7.1.14}
\end{equation*}
$$

The proof of Theorem 7.1.8 will be given in Subsection 7.3.
By Theorem 7.1.5 and Theorem 7.1.8, we conclude that, for acyclic local system associated with $\rho: \pi_{1}(M) \rightarrow G \xrightarrow{\text { Ad }} \operatorname{Aut}(\mathfrak{g})$, any graph cocycle invariant can be computed via a graph cocycle without self-loops. However, it is interesting to ask about the explicit relation between these two spaces $H^{0}\left(\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}: n}^{\bullet}, \delta\right)$ and $H^{0}\left(\mathcal{G}_{\mathrm{ac}, \mathfrak{g}: n}^{\bullet}, \delta^{\sharp}\right)$ : when $n=1$, they are the isomorphic and 1-dimensional (cf. Example 6.5.4), so that $Z(M, \rho,[f])=Z^{\sharp}(M, \rho,[f])$, how about the cases of higher orders?

Finally, combining Theorems $7.1 .5 \& 7.1 .8$ with Proposition 6.6.6, we obtain a generating series of perturbative invariants of a closed 3-manifold associated with acyclic representation $\rho: \pi_{1}(M) \rightarrow$ $G \xrightarrow{\mathrm{Ad}} \operatorname{Aut}(\mathfrak{g})$.
Corollary 7.1.9. Fix a homotopy class $[f]$ of smooth framing of $M$ and an orientation o(M). Let $E_{\rho}$ be an acyclic local system over $M$ associated with a representation $\rho: \pi_{1}(M) \rightarrow G \xrightarrow{\text { Ad }} \operatorname{Aut}(\mathfrak{g})$.
(1) Let $\omega$ be a propagator. Consider the formal sum

$$
\begin{equation*}
\mathcal{Z}_{\mathrm{CS}}(M, \rho,[f]):=\sum_{\text {connected } \mathfrak{G}} \frac{\hbar^{\operatorname{ord}(\mathfrak{G})}}{|\operatorname{Aut}(\mathfrak{G})|} Z_{\Psi_{n}(\mathfrak{G})}(\omega) \in \mathbb{R}[[\hbar]], \tag{7.1.15}
\end{equation*}
$$

where the sum runs over all the connected topological trivalent graph $\mathfrak{G}$. Then, it is independent of the choice of propagator $\omega$. Therefore, $\mathcal{Z}_{\mathrm{CS}}(M, \rho,[f])$ is an invariant of $(M, o(M), \rho,[f])$.
(2) Let $\omega^{\sharp}$ be any adapted propagator, then the formal sum in (7.1.15) satisfies the following identity

$$
\mathcal{Z}_{\mathrm{CS}}(M, \rho,[f])=\sum_{\begin{array}{c}
\text { connected }  \tag{7.1.16}\\
\text { without self-loops }
\end{array}} \frac{\hbar^{\operatorname{ord}(\mathfrak{G})}}{|\operatorname{Aut}(\mathfrak{G})|} Z_{\Psi_{n}(\mathfrak{G})}\left(\omega^{\sharp}\right) \in \mathbb{R}[[\hbar]],
$$

where the sum runs over all the connected topological trivalent graph $\mathfrak{G}$ without self-loops.
Remark 7.1.10. Corollary 7.1.9 (1) is our version for the analogous result given in [AS92] and [AS94].

### 7.2 A variation formula and proof of Theorem 7.1.5

We will adopt some ideas from the proofs of [BC98, Theorem 4.7] and of [BC99, Theorem 1.1] to achieve our proofs of Theorem 7.1.5 and Theorem 7.1.8, we also provide the necessary details for completeness. One of the differences from theirs is that we compute the graphs obtained by contracting non-regular edges in detail, which involves self-loops. For the case of 2-loop trivalent graphs, such computation was already explained in [CS21, §4.2].

Proof of Theorem 7.1.5 is given in the similar way as in [BC98, Theorem 4.7]. Here is the outline of the proof: we consider a smooth one parameter family of propagators over the unit interval $I=[0,1]$ as a parameter space. Here the unit interval $I$ parametrizes information used to define a propagator, that is, representatives of the class of framing $f, \eta, \omega$ and $\xi$ in Proposition 4.3.1 and Corollary 4.3.2. Then, this family of propagators gives rise to a family of integrals associated to a given decorated trivalent graph. To prove the independence of the choice of propagators, it suffices to show this family of integrals being constant on $I$, or equivalently, its differential on $I$ is identically 0 . Stokes' formula and Kontsevich's lemma (Lemma 7.2.3) tell that there are non-vanishing boundary contributions, but they can be made zero by graph cocycle relation. This way, we finally obtain Theorem 7.1.5. In fact, we at first will prove a result analogous to [BC98, Corollary 4.12] from which Theorem 7.1.5 follows
clearly. This result gives us a formula for the variations of $Z_{\Gamma}(\omega)$ as $\omega$ varies smoothly and for any decorated trivalent graph $\Gamma$ which is not necessary to be a cocycle.

Note that in our construction the map $Z(M, \rho,[f])$ exactly gives rise to an invariant of framed 3 -manifold with acyclic representation $\rho$ associated to a graph cocycle $\Gamma$. This is different from [BC99] where, to obtain a graph cocycle invariant, a modification of $Z(M, \rho,[f])$ is required to cancel a boundary contribution by adding correction terms.

### 7.2.1 A variation formula for a family of propagators

Let $I=[0,1]$ denote the unit interval with the standard coordinate $\tau \in[0,1]$. The vector bundles on $M, C_{2}(M)$, etc, are viewed naturally as vector bundles on $I \times M, I \times C_{2}(M)$, etc, respectively, and so do the differential forms. We also extend the action of $T$ on $I \times \cdots$ by trivial action on the factor $I$. Let $d^{\text {tot }}=d \tau \wedge \frac{\partial}{\partial \tau}+d^{M}$ denote the total differential on the product space $I \times M$. We will use the same notation for the spaces $I \times C_{2}(M), I \times \partial C_{2}(M), I \times M \times M$, etc.

If $\Gamma$ is a connected decorated graph with degree 1 and without any external edges, due to our convention that the minimal valency at each vertex is at least 3 , we can conclude that $\Gamma$ has exactly one vertex with valency 4 and all other vertices are trivalent. Set $m=|v(\Gamma)|$. Then $m$ has to be an odd integer, and we have

$$
\begin{equation*}
\operatorname{ord}(\Gamma)=\frac{1}{2}(m+1) \tag{7.2.1}
\end{equation*}
$$

We consider a pair of differential 2-forms $(\widetilde{\omega}, \widetilde{\xi}) \in \Omega^{2}\left(I \times C_{2}(M) ; F_{\rho}\right) \times \Omega^{2}\left(I \times M ; E_{\rho} \otimes E_{\rho}\right)$ such that

$$
\begin{equation*}
T^{*} \widetilde{\omega}=-\widetilde{\omega}, T^{*} \widetilde{\xi}=-\widetilde{\xi} \tag{7.2.2}
\end{equation*}
$$

Analogous to (7.1.1), we associate a 2-form on $I \times C_{m}(M)$ to each $e=(i j) \in e(\Gamma)$ as follows

$$
\widetilde{\omega}_{e}:= \begin{cases}p_{i j}^{*} \widetilde{\omega} & \text { if } e=(i j) \text { with } i \neq j,  \tag{7.2.3}\\ q^{*} p_{i}^{*} \widetilde{\xi} & \text { if } e=(i i) \text { is a directed self-loop, }\end{cases}
$$

Let $\sigma: I \times C_{m}(M) \rightarrow I$ denote the obvious projection, and let $\sigma_{*}: \Omega^{\bullet+3 m}\left(I \times C_{m}(M)\right) \rightarrow \Omega^{\bullet}(I)$ be the fiber integration (Definition 2.2.3). Similar to (7.1.4), we define

$$
\begin{equation*}
Z_{\Gamma}(\widetilde{\omega}, \widetilde{\xi}):=\sigma_{*}\left[\left(\bigotimes_{i \in v(\Gamma)} W_{i}\right) \bigwedge_{e \in e(\Gamma)} \widetilde{\omega}_{e}\right] \in \Omega^{1}(I), \tag{7.2.4}
\end{equation*}
$$

where $W_{i}$ denotes the weight system at vertex $i$, it is $\operatorname{Tr}_{i}$ when vertex $i$ is trivalent and is $\pm W_{I}$, $\pm W_{H}, \pm W_{X}$ at the only vertex of valency 4 .

If $\Gamma$ is a decorated trivalent graph, then $Z_{\Gamma}(\widetilde{\omega}, \widetilde{\xi}) \in \Omega^{0}(I)$ can also be defined by considering a smooth family of the integrations as in (7.1.4). In summary, we have the following result.
Lemma 7.2.1. The following linear map is well-defined for $j=0,1$,

$$
\begin{equation*}
Z_{-}(\widetilde{\omega}, \widetilde{\xi}): \mathcal{G C}_{\mathrm{ac}, \mathfrak{g}}^{j} \rightarrow \Omega^{j}(I), \quad \Gamma \mapsto Z_{\Gamma}(\widetilde{\omega}, \widetilde{\xi}) . \tag{7.2.5}
\end{equation*}
$$

Proof. If $j=0$, this is exactly a family version of Proposition 7.1.1, which follows from the same proof since the boundary condition (4.3.27) for $\left\{\left.\widetilde{\omega}\right|_{\{\tau\} \times C_{2}(M)}\right\}_{\tau \in I}$ is not needed.

For the case of $j=1$, by (7.2.2) and (7.2.3), the same arguments in the proof of Proposition 7.1.1 shows that the definition (7.2.4) is compatible with the sign convention on the decorated graphs of degree 1. For internal IHX relation, we can consider a decorated graph to be the sum of three decorated graphs $\Gamma_{j}, j=1,2,3$, which have exactly the same underlying topological graph and the decorations on the edges and vertices except for the different weight systems (I, H, X, respectively) at the vertices of valency 4 , then by (6.2.4), the sum of $\left(\otimes_{i \in v\left(\Gamma_{j}\right)} W_{i}\right) \bigwedge_{e \in e\left(\Gamma_{j}\right)} \widetilde{\omega}_{e}$ vanishes identically. This way, we complete our proof.

Now we can state our result to compute the variations of the integrals $Z_{\Gamma}(\omega)$ defined in (7.1.4), when $\omega$ varies smoothly, which is an analog of the second part of [BC98, Corollary 4.12].
Proposition 7.2.2. Let the pair $(\widetilde{\omega}, \widetilde{\xi}) \in \Omega^{2}\left(I \times C_{2}(M) ; F_{\rho}\right) \times \Omega^{2}\left(I \times M ; E_{\rho} \otimes E_{\rho}\right)$ be such that

- $d^{\mathrm{tot}} \widetilde{\omega}=0, d^{\mathrm{tot}} \widetilde{\xi}=0$;
- $T^{*} \widetilde{\omega}=-\widetilde{\omega}, T^{*} \widetilde{\xi}=-\widetilde{\xi}$;
- there exists a closed smooth 2 -form $\widetilde{\mu} \in \Omega^{2}\left(I \times \partial C_{2}(M) ; \mathbb{R}\right)$ such that
- $\widetilde{\mu}$ is vertical 2-form with respect to the submersion $I \times \partial C_{2}(M) \simeq I \times M \times \mathbb{S}^{2} \rightarrow M$;
- $\widetilde{q_{\partial}}{ }_{*} \widetilde{\mu}=1$ on $I \times M$, where $\widetilde{q_{\partial}}:=\left(\operatorname{Id}_{I}, q_{\partial}\right): I \times \partial C_{2}(M) \simeq I \times M \times \mathbb{S}^{2} \rightarrow I \times M$;
- let $\widetilde{\mathfrak{i}_{\partial}}$ denote the inclusion $I \times \partial C_{2}(M) \rightarrow I \times C_{2}(M)$, then analogous to (4.3.27), we have

$$
\begin{equation*}
{\widetilde{\mathfrak{i}_{\partial}}}^{*}(\widetilde{\omega})=\widetilde{\mu} \otimes \mathbf{1}+{\widetilde{q_{\partial}}}^{*}(\widetilde{\xi}), \tag{7.2.6}
\end{equation*}
$$

where $\mathbf{1}$ is the flat section in Lemma 3.4.1.
Then we have the following identity for any $\Gamma \in \mathcal{G C}_{\mathrm{ac}, \mathfrak{g}}^{0}$,

$$
\begin{equation*}
d Z_{\Gamma}(\widetilde{\omega}, \widetilde{\xi})=-Z_{\delta \Gamma}(\widetilde{\omega}, \widetilde{\xi}) \in \Omega^{1}(I) . \tag{7.2.7}
\end{equation*}
$$

Let's do some preparations before proving the above proposition, we will always take the pair $(\widetilde{\omega}, \widetilde{\xi})$ as given in the proposition. Let $\Gamma$ is a connected decorated trivalent graph with order $n$, then $\delta \Gamma$ is a linear combination of connected decorated graphs in $\mathcal{G} \mathcal{a}_{\mathrm{ac}, \mathfrak{g}: n}^{1}$. For simplicity, set

$$
\begin{equation*}
\operatorname{Tr}^{\Gamma}(\widetilde{\omega}, \widetilde{\xi})=\left(\left(\bigotimes_{i \in v(\Gamma)} \operatorname{Tr}_{i}\right) \bigwedge_{e \in e(\Gamma)} \widetilde{\omega}_{e}\right) \in \Omega^{6 n}\left(I \times C_{2 n}(M)\right) . \tag{7.2.8}
\end{equation*}
$$

It is clear that $\operatorname{Tr}^{\Gamma}(\widetilde{\omega}, \widetilde{\xi})$ is $d^{\text {tot }}$-closed form. Applying Stokes' formula (Proposition 2.2.5), we get

$$
\begin{align*}
d Z_{\Gamma}(\widetilde{\omega}, \widetilde{\xi}) & =d \sigma_{*} \operatorname{Tr}^{\Gamma}(\widetilde{\omega}, \widetilde{\xi}) \\
& =\sigma_{*}\left(d^{\operatorname{tot}} \operatorname{Tr}^{\Gamma}(\widetilde{\omega}, \widetilde{\xi})\right)-\sigma_{*}^{\partial}\left({\widetilde{\mathfrak{i}_{\partial}}}^{*} \operatorname{Tr}^{\Gamma}(\widetilde{\omega}, \widetilde{\xi})\right)  \tag{7.2.9}\\
& =-\sigma_{*}^{\partial}\left({\widetilde{i_{\partial}}}^{*} \operatorname{Tr}^{\Gamma}(\widetilde{\omega}, \widetilde{\xi})\right),
\end{align*}
$$

where $\tilde{\mathfrak{i}_{\partial}}$ denotes the inclusion $I \times \partial C_{2 n}(M) \rightarrow I \times C_{2 n}(M)$, and $\sigma^{\partial}: I \times \partial C_{2 n}(M) \rightarrow I$ denotes the obvious projection.

Therefore, our computation reduces to that of $\sigma_{*}^{\partial}\left(\widetilde{\mathfrak{i}}^{*}{ }^{*} \operatorname{Tr}^{\Gamma}(\widetilde{\omega}, \widetilde{\xi})\right)$. So that we need to investigate the geometry of 1-codimensional boundary $\partial^{*} C_{2 n}(M)$, more precisely, $S^{1}\left(C_{2 n}(M)\right)$. Let $S$ be a subset of $\{1,2, \cdots, 2 n\}$ or $v(\Gamma)$ with $\ell=|S| \geq 2$. Let $\partial_{S} C_{2 n}(M)$ denote the component of $\partial^{*} C_{2}(M)$ corresponding to $M(\{S\})$ in the notation of Subsection 2.4, they are defined by collapsing points $\left\{\mathbf{x}_{i}\right\}_{i \in S} \in M^{\ell}$ into the same point.

Note that a point in the open strata $M(\{S\})^{0}$ can be represented by
$\left(\mathbf{x}_{S}=(z, \ldots, z) \in \Delta_{S} \simeq M, u_{S} \in \mathbb{R}_{+}^{*} \backslash\left(\left(T_{z} M\right)^{S} / T_{z} M-\{0\}\right) ;\left\{\mathbf{x}_{j}\right\}_{j \notin S} \in \operatorname{Conf}_{2 n-\ell}(M)\right.$ with $\left.\mathbf{x}_{j} \neq z\right)$.
By the results recalled in Subsection 2.4, the normal vector $u_{S}$ do not have any two components which are equal. Let $\operatorname{Conf}_{S}\left(T_{z} M\right)$ denote the configuration space of vectors in $T_{z} M$ indexed by $S$, let $T_{z} M$ act on $\operatorname{Conf}_{S}\left(T_{z} M\right)$ by on-diagonal translations and let $\mathbb{R}_{+}^{*}$ act on $\operatorname{Conf}_{S}\left(T_{z} M\right)$ by rescalings. Then we can rewrite the above requirements on $u_{S}$ as $u_{S} \in \operatorname{Conf}_{S}\left(T_{z} M\right) / T_{z} M \rtimes \mathbb{R}_{+}^{*}$. Consider the smooth projection $\operatorname{Pr}_{S}: M(\{S\})^{0} \rightarrow \operatorname{Conf}_{2 n-\ell+1}(M)$ which sends the above point in $M(\{S\})^{0}$ to the point $\left(z, \mathbf{x}_{j}, j \notin S\right) \in \operatorname{Conf}_{2 n-\ell+1}(M)$, then the fibre of this projection is given by $\operatorname{Conf}_{S}\left(T_{z} M\right) / T_{z} M \rtimes \mathbb{R}_{+}^{*} \simeq \operatorname{Conf}_{\ell}\left(\mathbb{R}^{3}\right) / \mathbb{R}^{3} \rtimes \mathbb{R}_{+}^{*}$.

Then we extend it smoothly to the projection, denoted by the same notation, $\operatorname{Pr}_{S}: \partial_{S} C_{2 n}(M) \rightarrow$ $C_{2 n-\ell+1}(M)$. The generic fibre of $\operatorname{Pr}_{S}$ is given by $F_{S} \simeq C_{\ell}\left(\mathbb{R}^{3}\right) / \mathbb{R}^{3} \rtimes \mathbb{R}_{+}^{*}$. In particular, $\operatorname{dim}_{\mathbb{R}} F_{S}=$ $3 \ell-4$.

To compute the contribution of $\partial_{S} C_{2 n}(M)$ in $\sigma_{*}^{\partial}\left(\widetilde{\mathfrak{i}}^{*}{ }^{*} \operatorname{Tr}^{\Gamma}(\widetilde{\omega}, \widetilde{\xi})\right)$, we need the following lemma.
Lemma 7.2.3 (Kontsevich's vanishing lemma [Kon94, Lemma 2.1], [BC98, Lemma 4.9]). Let $F_{S}$ be the fiber of the face $\partial_{S} C_{2 n}(M)$ corresponding to the collapse of $\ell$ points with coordinate $\mathbf{x}_{j}, j \in S$, i.e., $F_{S}$ denote the generic fibre of $\operatorname{Pr}_{S}: \partial_{S} C_{2 n}(M) \rightarrow C_{2 n-\ell+1}(M)$. Fix a smooth framing $f: T M \rightarrow$ $M \times \mathbb{R}^{3}$, then it induces an identification $F_{S}=f^{*}\left(C_{\ell}\left(\mathbb{R}^{3}\right) / \mathbb{R}^{3} \rtimes \mathbb{R}_{+}^{*}\right)$. Let $\eta \in \Omega^{2}\left(\mathbb{S}^{2} ; \mathbb{R}\right)$ be any volume form of $\mathbb{S}^{2}$ with $T^{*} \eta=-\eta$. For $i, j \in S$, $i \neq j$, let $\pi_{i j}: F_{S} \rightarrow \mathbb{S}^{2}$ be the projection defined as

$$
\begin{equation*}
\pi_{i j}: F_{S} \rightarrow \mathbb{S}^{2} ; \quad\left(\mathbf{x}_{j}\right)_{j \in S} \mapsto \frac{\mathbf{x}_{j}-\mathbf{x}_{i}}{\left|\mathbf{x}_{j}-\mathbf{x}_{i}\right|} \quad(i \neq j) \tag{7.2.11}
\end{equation*}
$$

and $\pi_{i j}^{*} \eta$ be the pullback $\eta$ via $\pi_{i j}$. Then, any triple of indices $i, j, k$ in $S$ with $i \neq j$ and $i \neq k$, the integral vanishes:

$$
\begin{equation*}
\int_{\mathbf{x}_{i}} \pi_{i j}^{*} \eta \wedge \pi_{i k}^{*} \eta=0, \tag{7.2.12}
\end{equation*}
$$

where $\int_{\mathbf{x}_{i}} \cdots$ denotes the fibre integration for the projection of forgetting $\mathbf{x}_{i}$-coordinate: $C_{\ell}\left(\mathbb{R}^{3}\right) / \mathbb{R}^{3} \rtimes$ $\mathbb{R}_{+}^{*} \rightarrow C_{\ell-1}\left(\mathbb{R}^{3}\right) / \mathbb{R}^{3} \rtimes \mathbb{R}_{+}^{*}$ (provided $\ell \geq 3$ ).

Remark 7.2.4. With the same notation as above, note that Lemma 7.2 .3 immediately implies the following (original) statement. For any two sequences $s_{i}, t_{i}(i=1, \ldots, L)$ of integers with $s_{i} \neq t_{i}$ $\left(1 \leq s_{i}, t_{i} \leq \ell\right)$, the integral vanishes:

$$
\begin{equation*}
\int_{\left(\mathbf{x}_{1}, \ldots, \mathbf{x}_{\ell}\right) \in F_{S}} \bigwedge_{i=1}^{L} \pi_{s_{i} t_{i}}^{*} \eta=0 . \tag{7.2.13}
\end{equation*}
$$

Proof of Proposition 7.2.2. We use the above notation, and we consider the face $\partial_{S} C_{2 n}(M)$. Note that each component in the coordinate $\left(\mathbf{x}_{1}, \ldots, \mathbf{x}_{2 n}\right)$ corresponds to one labeled vertex of the decorated trivalent graph $\Gamma$. We will regard the vertices in $S$ as the collapsing vertices.

Using the projection $\operatorname{Pr}_{S}: \partial_{S} C_{2 n}(M) \rightarrow C_{2 n-\ell+1}(M)$, and for a differential form $\alpha$ on $\partial_{S} C_{2 n}(M)$, we can decompose $\alpha$ into two parts: the vertical direction and the basic direction. For the integral $\int_{\partial_{S} C_{2 n}(M)} \alpha$ being nonzero, the degree of $\alpha$ shall be $6 n-1$ with the vertical degree of $\alpha$ being $3 \ell-4$. Moreover, we have

$$
\begin{equation*}
\int_{\partial_{S} C_{2 n}(M)} \alpha=\int_{C_{2 n-\ell+1}(M)}\left(\operatorname{Pr}_{S}\right)_{*} \alpha . \tag{7.2.14}
\end{equation*}
$$

Let's consider the differential form $\left.\operatorname{Tr}^{\Gamma}(\widetilde{\omega}, \widetilde{\xi})\right|_{\partial_{S} C_{2 n}(M)}$, which can be obtained by taking the product of $\left.\widetilde{\omega}_{e}\right|_{\partial_{S} C_{2 n}(M)}$. Note that by (7.2.3), if $e=(i i)$ is a self-loop edge, then $\left.\widetilde{\omega}_{e}\right|_{\partial_{S} C_{2 n}(M)}$ is always basic differential form (with respect to the projection $\operatorname{Pr} r_{S}$ ); if $e=(i j)$ is non-self-loop edge such that $i$ or $j$ does not lie in $S$, then $\left.\widetilde{\omega}_{e}\right|_{\partial_{S} C_{2 n}(M)}$ is also basic. To have the vertical directions in $\left.\widetilde{\omega}_{e}\right|_{\partial_{S} C_{2 n}(M)}$, we need $e=(i j)$ with $i, j \in S, i \neq j$, and in this case the vertical form contributed by $\left.\widetilde{\omega}_{e}\right|_{\partial_{S} C_{2 n}(M)}$ is $\widetilde{\mu}$ in (7.2.6). Our assumptions on $\widetilde{\mu}$ implies that its contribution in $\left.\widetilde{\omega}_{e}\right|_{\partial_{S} C_{2 n}(M)}$ can be written as follows, for $\tau \in I$,

$$
\begin{equation*}
\widetilde{\mu}=\pi_{i j}^{*} \eta_{\tau}+d \tau \wedge \pi_{i j}^{*} \beta_{\tau}, \tag{7.2.15}
\end{equation*}
$$

where $\eta_{\tau}$ is a volume form on $\mathbb{S}^{2}$ (depending smoothly on $\tau$ ), and $\beta_{\tau}$ is a 1 -form on $\mathbb{S}^{2}$.
Let $e_{v}$ be the total number of edges connecting two distinct collapsing vertices (in $S$ ) and let $e_{h}$ be the total number of self-loop edges incident to the collapsing vertices. Let $e_{0}$ be the number of edges connecting a collapsing vertex in $S$ with a non-collapsing one. Since we consider trivalent graphs, we have the relation $2\left(e_{v}+e_{h}\right)+e_{0}=3 \ell$. Then, the maximal degree of vertical form in $\operatorname{Tr}^{\Gamma}(\widetilde{\omega}, \widetilde{\xi})| |_{\partial_{S} C_{2 n}(M)}$ is $2 e_{v}$. Considering $\left.\left(\operatorname{Pr}_{S}\right)_{*} \operatorname{Tr}^{\Gamma}(\widetilde{\omega}, \widetilde{\xi})\right|_{\partial_{S} C_{2 n}(M)}$, it is nonzero only if $2 e_{v}-(3 \ell-4)=4-e_{0}-2 e_{h} \geq 0$.

Let us first consider the case $\ell=|S| \geq 3$. By (7.2.15), the integrand form along the vertical direction of $\operatorname{Pr}_{S}$ is given by a product of $\pi_{i j}^{*} \eta_{\tau}$ and $d \tau \wedge \pi_{i j}^{*} \beta_{\tau}$. Since $\ell \geq 3, \operatorname{dim}_{\mathbb{R}} F_{S}=3 \ell-4 \geq 5$, so that we shall have at least two non-self-loop edges attached to the collapsing vertices to reach this vertical degree, then by Kontsevich's vanishing lemma (Lemma 7.2.3), we get $\left.\left(\operatorname{Pr}_{S}\right)_{*} \operatorname{Tr}^{\Gamma}(\widetilde{\omega}, \widetilde{\xi})\right|_{\partial_{S} C_{2 n}(M)}=0$.

The remaining case is that $\ell=2$ and $S=\{i, j\}(i \neq j)$ with $e=(i j)$ or $(j i)$ is an edge of $\Gamma$. If $e_{1}, e_{2}$ are two different non-self-loop edges in $\Gamma$ connecting the same vertices $S=\{i, j\}$, then $\left.\widetilde{\omega}_{e_{1}} \wedge \widetilde{\omega}_{e_{2}}\right|_{\partial_{S} C_{2 n}(M)}$ has two nontrivial terms

$$
\widetilde{u} \otimes \mathbf{1} \wedge \widetilde{\xi}+\widetilde{\xi} \wedge \widetilde{u} \otimes \mathbf{1}
$$

The first term, in the computation of $\sigma_{*}^{\partial}\left(\widetilde{\mathfrak{i}}^{*}{ }^{*} \operatorname{Tr}^{\Gamma}(\widetilde{\omega}, \widetilde{\xi})\right)$, corresponds to the contraction operation on $e_{1}$, i.e., the term $\Gamma / e_{1}$ in $\delta \Gamma$, and the second term corresponds to $\Gamma / e_{2}$. It is similar to the case where we have three different non-self-loop edges with the same ending vertices. Note that in the definition of the weight system $W_{i}$ at a vertex of valency 4 , it is the same as decorating the contracted edge by the Casimir element $\mathbf{1}$ then applying the cubic trace Tr. This way, we conclude from (7.2.9) and the assumption $\widetilde{q_{\partial}}{ }_{*} \widetilde{\mu}=1$ that

$$
\begin{equation*}
d Z_{\Gamma}(\widetilde{\omega}, \widetilde{\xi})=\sum_{\substack{e \in e(\Gamma) \\ \text { non-self-loop edge }}} \pm Z_{\Gamma / e}(\widetilde{\omega}, \widetilde{\xi}) \tag{7.2.16}
\end{equation*}
$$

The last step is to calculate precisely the sign $\pm$ in front of each term and then check the compatibility with the sign convention (6.4.12) in the definition of $\delta \Gamma$ in (6.4.11).

By Proposition 6.4.6, the map $\delta$ is well-defined under the sign relation (6.4.3), so that we can assume that $S=\{1,2\}, e=(12)$ is the edge numbered as 1 . Then (7.2.16) can be written as

$$
d Z_{\Gamma}(\widetilde{\omega}, \widetilde{\xi})=-Z_{\Gamma / e}(\widetilde{\omega}, \widetilde{\xi})+\sum_{\begin{array}{c}
\text { other } e^{\prime} \in e(\Gamma)  \tag{7.2.17}\\
\text { non-self-loop edge }
\end{array}} \pm Z_{\Gamma / e^{\prime}}(\widetilde{\omega}, \widetilde{\xi}),
$$

while we have $\delta \Gamma=\Gamma / e+\sum_{\text {other } e^{\prime} \in e(\Gamma)} \pm \Gamma / e^{\prime}$. This way, we get exactly (7.2.7) for a connected decorated trivalent graph $\Gamma$. Then combing this result with (6.4.16) and Lemma 7.1.2, we complete the proof for general $\Gamma$ in $\mathcal{G C}_{\mathrm{ac}, \mathfrak{g}}^{0}$.

### 7.2.2 Proof of Theorem 7.1.5

Note that we always fix an orientation $o(M)$ of $M$. Let $f$ and $f^{\prime}$ be two smooth framings of $M$ which are homotopic, and let $\eta, \eta^{\prime}$ be two normalized volume forms on $\mathbb{S}^{2}$. Let $(\omega, f, \eta, \xi),\left(\omega^{\prime}, f^{\prime}, \eta^{\prime}, \xi^{\prime}\right)$ be two propagators defined for the acyclic local sytem $E_{\rho}$ as in Definition 4.3.3.

Since $E_{\rho}$ is assumed to be acyclic, by Proposition 4.3.4, the cohomological class $[\omega]=\left[\omega^{\prime}\right]$ is unique. But we need a more explicit relation between $\omega, \omega^{\prime}$ with which we can apply Proposition 7.2.2.

Recall that $H^{2}\left(I \times \mathbb{S}^{2} ; \mathbb{R}\right) \simeq H^{2}\left(\mathbb{S}^{2} ; \mathbb{R}\right)$. For two $T$-asymmetric normalized volume form $\eta, \eta^{\prime}$ on $\mathbb{S}^{2}$, there exists a closed 2-form $\widetilde{\eta} \in \Omega^{2}\left(I \times \mathbb{S}^{2} ; \mathbb{R}\right)$ such that

$$
\begin{equation*}
\widetilde{\eta}_{\tau=0}=\eta, \widetilde{\eta}_{\tau=1}=\eta^{\prime} \tag{7.2.18}
\end{equation*}
$$

The closedness of $\widetilde{\eta}$ implies that for each $\tau \in I$,

$$
\begin{equation*}
\int_{\mathbb{S}^{2}} \widetilde{\eta}_{\tau}=1 \tag{7.2.19}
\end{equation*}
$$

We also require that $T_{\mathbb{S}^{2}}^{*} \widetilde{\eta}=-\widetilde{\eta}$.
Since $f$ and $f^{\prime}$ are homotopic, let $\tilde{f}: I \times S(T M) \rightarrow I \times M \times \mathbb{S}^{2}$ denote the smooth path of framings which connects $f(\tau=0)$ and $f^{\prime}(\tau=1)$. Set

$$
\begin{equation*}
I(\widetilde{\eta})=\widetilde{f}^{*}(\widetilde{\eta}) \otimes \mathbf{1} \in \Omega_{-}^{2}\left(I \times \partial C_{2}(M) ; E_{\rho} \otimes E_{\rho}\right) \tag{7.2.20}
\end{equation*}
$$

At the same time, we have

$$
\begin{equation*}
H^{2}\left(I \times M \times M ; E_{\rho} \boxtimes E_{\rho}\right) \simeq H^{2}\left(M \times M ; E_{\rho} \boxtimes E_{\rho}\right) \oplus \mathbb{R} d \tau \wedge H^{1}\left(M \times M ; E_{\rho} \boxtimes E_{\rho}\right)=0 \tag{7.2.21}
\end{equation*}
$$

Note that by Corollary 4.3.2, the cohomology class $[\xi]$ is uniquely determined for any propagator $\omega$ associated to this acyclic local system $E_{\rho}$.

Now we can follow the arguments as the proof of Theorem 4.3.1 to construct a propagator on $I \times C_{2}(M)$ with the analogous properties. More precisely, there exists a closed 2 -form $\widetilde{\omega} \in \Omega_{-}^{2}(I \times$ $\left.C_{2}(M) ; F_{\rho}\right)$ (i.e., $\left.d^{\text {tot }} \widetilde{\omega}=0\right)$ and closed 2-form $\widetilde{\xi} \in \Omega_{-}^{2}\left(I \times \Delta ; E_{\rho} \otimes E_{\rho}\right)$ such that

- $\widetilde{\omega}_{\{0\} \times C_{2}(M)}=\omega, \widetilde{\omega}_{\{1\} \times C_{2}(M)}=\omega^{\prime}$, or equivalently, $\widetilde{\xi}_{\{0\} \times C_{2}(M)}=\xi, \widetilde{\xi}_{\{1\} \times C_{2}(M)}=\xi^{\prime}$;
- let $\tilde{\mathfrak{i}_{\partial}}$ denote the inclusion $I \times \partial C_{2}(M) \rightarrow I \times C_{2}(M)$, then analogous to (4.3.27), we have

$$
\begin{equation*}
{\widetilde{\mathfrak{i}_{\partial}}}^{*}(\widetilde{\omega})=I(\widetilde{\eta})+{\widetilde{q_{\partial}}}^{*}(\widetilde{\xi}) \tag{7.2.22}
\end{equation*}
$$

where $\widetilde{q_{\partial}}=\left(\operatorname{Id}_{I}, q_{\partial}\right): I \times \partial C_{2}(M) \rightarrow I \times \Delta \simeq I \times M$;
Now we can take $\widetilde{\mu}$ to be $\widetilde{f} * \widetilde{\eta}$, and then $\widetilde{\omega}, \widetilde{\xi}$ constructed above satisfy the conditions in Proposition 7.2.2. Let $\Gamma \in \mathcal{G C}_{\mathrm{ac}, \mathfrak{g}}^{0}$ be a cocycle, i.e., $\delta \Gamma=0$, then by (7.2.7), we get

$$
\begin{equation*}
d Z_{\Gamma}(\widetilde{\omega}, \widetilde{\xi})=-Z_{\delta \Gamma}(\widetilde{\omega}, \widetilde{\xi})=0 \tag{7.2.23}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
\left.Z_{\Gamma}(\widetilde{\omega}, \widetilde{\xi})\right|_{\tau=0}=\left.Z_{\Gamma}(\widetilde{\omega}, \widetilde{\xi})\right|_{\tau=1} \tag{7.2.24}
\end{equation*}
$$

Then by (7.1.4) and (7.2.8), we get $Z_{\Gamma}(\omega)=Z_{\Gamma}\left(\omega^{\prime}\right)$. This way we prove Theorem 7.1.5.

### 7.3 Proof of Theorem 7.1.8

The proof of Theorem 7.1 .8 (1) goes along the same line as that of Theorem 7.1 .5 , but, to prove that $Z_{\Gamma}\left(\omega^{\sharp}\right)$ is an invariant defined from the graph complex $\left(\mathcal{G}_{\mathrm{ac}, \mathfrak{g}: n}^{\bullet}, \delta^{\sharp}\right)$, we need to check the followings in addition to the above.
(i) For an adapted propagator $\omega^{\sharp}$, we need to show that the map $Z_{-}\left(\omega^{\sharp}\right)$ factors through the space of decorated trivalent graphs without self-loops.
(ii) (Analogous to Subsection 7.2.2) For two different adapted propagators $\omega^{\sharp}, \omega^{\sharp, \prime}$, a closed form $\widetilde{\omega}^{\sharp}$ on $I \times C_{2}(M)$ can be constructed to connect smoothly $\omega^{\sharp}$ and $\omega^{\sharp, \prime}$, and that for $\tau \in I$, $\left.\widetilde{\omega}^{\sharp}\right|_{\{\tau\} \times C_{2}(M)}$ is an adapted propagator.
(iii) Applying Stokes' theorem to one parameter family $Z_{-}\left(\widetilde{\omega}^{\sharp}, \widetilde{\xi}^{\sharp}\right)$ as defined in (7.2.5), we need to show that the vanishing of the contributions of the boundary terms corresponding to the collapse of two distinct vertices connected by a non-regular edge, so that $Z_{-}\left(\omega^{\sharp}\right)$ is invariant associated with the cocylces in $\mathcal{G}_{\mathrm{ac}, \mathfrak{g}: n}^{0}$.

For (i), it follows immediately from Theorem 7.1.7. For (ii), combining the proof of Theorem 4.4.2 with the proof in Subsection 7.2.2, we can construct a closed 2-form $\widetilde{\omega}^{\sharp} \in \Omega_{-}^{2}\left(I \times C_{2}(M) ; F_{\rho}\right)$ and closed 2-form $\widetilde{\xi}^{\sharp} \in \Omega_{-}^{2}\left(I \times \Delta ; E_{\rho} \otimes E_{\rho}\right)$ such that

- $\widetilde{\omega}_{\{0\} \times C_{2}(M)}^{\sharp}=\omega^{\sharp}, \widetilde{\omega}_{\{1\} \times C_{2}(M)}^{\sharp}=\omega^{\sharp, \prime}$, or equivalently, $\widetilde{\xi}_{\{0\} \times C_{2}(M)}^{\sharp}=\xi^{\sharp}, \widetilde{\xi}_{\{1\} \times C_{2}(M)}^{\sharp, \prime}=\xi^{\sharp, \prime}$;
- Analogous to (4.4.2), we have

$$
\begin{equation*}
{\tilde{\mathfrak{i}_{\partial}}}^{*}\left(\widetilde{\omega}^{\sharp}\right)=I(\widetilde{\eta})+{\widetilde{q_{\partial}}}^{*}\left(\widetilde{\xi}^{\sharp}\right) \tag{7.3.1}
\end{equation*}
$$

with $\mathfrak{L}\left(\widetilde{\xi}^{\sharp}\right)=0$.
Therefore, it suffices to show (iii). For this, under the same arguments in the proof of Proposition 7.2.2, we investigate in a more detailed manner the case that $\ell=|S|=2$ and $S$ corresponds exactly to an edge in the graph. Note that a variation formula like (7.2.7) can be deduced for $Z_{\Gamma}\left(\widetilde{\omega}^{\sharp}, \widetilde{\xi}^{\sharp}\right)$ with arbitrary $\Gamma \in \mathcal{G}_{\mathrm{ac}, \mathfrak{g}: n}^{0}$, but we now focus on the proof of Theorem 7.1.8, so that we will assume in the sequel that $\Gamma \in \mathcal{G}_{\mathrm{ac}, \mathfrak{g}: n}^{0}$ is a cocycle, i.e., $\delta^{\sharp} \Gamma=0$.

There are following cases (a) and (b), where we use the notation introduced after (7.2.15),
(a) one of the edges connecting two collapsing vertices $\{i, j\}$ is regular; in this case, there are three types of local graphs corresponding to $\left(e_{0}, e_{v}, e_{h}\right)=(4,1,0),(2,1,1),(0,1,2)$ as in Fig. 17. For


Fig. 17: Parts of trivalent graphs with regular edge (ij) connecting two collapsing vertices corresponding to $\left(e_{0}, e_{v}, e_{h}\right)=(4,1,0),(2,1,1),(0,1,2)$ respectively. Here, such regular edges are depicted as solid lines.
$\left(e_{0}, e_{v}, e_{h}\right)=(2,1,1),(0,1,2)$, the corresponding graphs must have at least one self-loop edge, so there is nothing to show. For the remaining case $\left(e_{0}, e_{v}, e_{h}\right)=(4,1,0)$, the graph cocycle condition $\left(\delta^{\sharp} \Gamma=0\right)$ gives a cancellation of these boundary contributions;
(b) the edge connecting two collapsing vertices $i$ and $j$ is not regular; This case is further divided into two cases (b-1) and (b-2):
(b-1) the number of such non-regular edges is 2 as Fig. 18 (case that $e_{0}=2, e_{v}=2, e_{h}=0$ ); in


Fig. 18: Part of trivalent graph with two non-regular edge $e$ and $e^{\prime}$ which connect the vertices $i$ and $j$. The contraction of the edge $e$ or $e^{\prime}$ yields one self-loop edge. Here, such regular edges are depicted as solid lines.
this case integrand $\operatorname{Tr}_{i} \otimes \operatorname{Tr}_{j}\left(\widetilde{\omega}_{(\bullet i)}^{\sharp} \wedge\left(\widetilde{\omega}_{(i j)}^{\sharp}\right)^{2} \wedge \widetilde{\omega}_{(j \bullet)}^{\sharp}\right)$ associated to edges connecting $i$ and $j$ (we may assume $i<j$ ) restricts to

$$
\begin{align*}
& \operatorname{Tr}_{i} \otimes \operatorname{Tr}_{i}\left(\widetilde{\omega}_{(\cdot i)}^{\sharp} \wedge\left(I(\widetilde{\eta})_{(i i)}+\widetilde{q}^{*} \widetilde{p}_{i}^{*} \widetilde{\xi}^{\sharp}\right)^{2} \wedge \widetilde{\omega}_{(i \bullet)}^{\sharp}\right)  \tag{7.3.2}\\
= & \operatorname{Tr}_{i} \otimes \operatorname{Tr}_{i}\left(\widetilde{\omega}_{(\cdot i)}^{\sharp} \wedge\left(I(\widetilde{\eta})_{(i i)}^{2}+2 \widetilde{p}^{*} I(\widetilde{\eta})_{(i i)} \wedge \widetilde{q}^{*} \widetilde{p}_{i}^{*} \widetilde{\xi}^{\sharp}+\widetilde{q}^{*} \widetilde{p}_{i}^{*}\left(\widetilde{\xi}^{\sharp}\right)^{2}\right) \wedge \widetilde{\omega}_{(\bullet \bullet)}^{\sharp}\right) \tag{7.3.3}
\end{align*}
$$

on the boundary component.
Then, performing fiber integration along $\mathbb{S}^{2}$ fiber, we get $2 \operatorname{Tr}_{i} \otimes \operatorname{Tr}_{i}\left(\widetilde{\omega}_{(\bullet i)}^{\sharp} \wedge \mathbf{1} \widetilde{q}^{*} \widetilde{p}_{i}^{*} \widetilde{\xi}^{\sharp} \wedge \widetilde{\omega}_{(i \bullet)}\right)$ since $I(\widetilde{\eta})^{2}=0$ on $\mathbb{S}^{2}$ and $\widetilde{q}^{*} \widetilde{p}_{i}^{*} \widetilde{\xi}^{2}$ is degree 0 along the fiber. We need the following Lemma which is a variant of [CS21, Lemma 4.4].
Lemma 7.3.1. With the same notations as above, we have the following equation.

$$
\begin{equation*}
\operatorname{Tr}_{i} \otimes \operatorname{Tr}_{i}\left(\widetilde{\omega}_{(\bullet i)}^{\sharp} \wedge \mathbf{1} \widetilde{q}^{*} \widetilde{p}_{i}^{*} \widetilde{\xi}^{\sharp} \wedge \widetilde{\omega}_{(i \bullet)}^{\sharp}\right)=\frac{1}{2} B_{i}\left(\mathfrak{L}_{i}\left(\widetilde{q}^{*} \widetilde{p}_{i}^{*} \widetilde{\xi}^{\sharp}\right), \mathfrak{L}_{i}\left(\widetilde{\omega}_{(\bullet i)}^{\sharp} \wedge \widetilde{\omega}_{(i \bullet)}^{\sharp}\right)\right) \tag{7.3.4}
\end{equation*}
$$

where $B_{i}, \mathfrak{L}_{i}$ means that such operations occur at the vertex $i$.

Proof of Lemma 7.3.1. It suffices to show the claim fiberwise, that is, for $\mathfrak{g}^{\otimes 3} \otimes \mathfrak{g}^{\otimes 3}$. Since $T^{*}$ acts on the diagonal subspace $\Delta$ by identity, we have $\Omega_{-}^{\bullet}\left(\Delta ; E_{\rho} \otimes E_{\rho}\right)=\Omega^{\bullet}\left(\Delta ;\left(E_{\rho} \otimes E_{\rho}\right)-\right)=\Omega^{\bullet}\left(\Delta ; \Lambda^{2} E_{\rho}\right)$. Let $e_{1}, \ldots, e_{\operatorname{dim} \mathfrak{g}}$ be basis of $\mathfrak{g}$ which is normalized with the condition $B\left(e_{i}, e_{j}\right)=\varepsilon_{i} \delta_{i j}, \varepsilon_{i} \in\{1,-1\}$. Then the Casimir element is given as

$$
\mathbf{1}=\sum_{i} \varepsilon_{i} e_{i} \otimes e_{i} .
$$

Then, for each fiber at $x \in M, \Lambda^{2} E_{\rho, x}=\Lambda^{2} \mathfrak{g}$ has a basis $\left\{e_{i} \otimes e_{j}-e_{j} \otimes e_{i} \mid 1 \leq i<j \leq n\right\}$. Thus, in terms of this basis, one obtains, for some $e_{a}$ and $e_{b}$ corresponding to the $i$-components of coefficients of $\widetilde{\omega}_{(\cdot i)}^{\sharp}$ and $\widetilde{\omega}_{(\bullet)}^{\sharp}$ respectively,

$$
\begin{align*}
& \operatorname{Tr} \otimes \operatorname{Tr}\left(\left(e_{i} \otimes e_{j}-e_{j} \otimes e_{i}\right) \otimes\left(e_{a} \otimes e_{b}\right) \otimes\left(\sum_{n=1}^{\operatorname{dim} \mathfrak{g}} \varepsilon_{n} e_{n} \otimes e_{n}\right)\right) \\
= & \sum_{i=1}^{\operatorname{dim} \mathfrak{g}} B\left(\left[e_{i}, e_{a}\right], \varepsilon_{n} e_{n}\right) B\left(e_{n},\left[e_{j}, e_{b}\right]\right)-\sum_{i=1}^{\operatorname{dim} \mathfrak{g}} B\left(\left[e_{j}, e_{a}\right], \varepsilon_{n} e_{n}\right) B\left(e_{n},\left[e_{i}, e_{b}\right]\right) \\
= & B\left(\left[e_{i}, e_{a}\right],\left[e_{j}, e_{b}\right]\right)-B\left(\left[e_{j}, e_{a}\right],\left[e_{i}, e_{b}\right]\right) \\
= & B\left(e_{i},\left[e_{a},\left[e_{j}, e_{b}\right]\right]\right)-B\left(\left[e_{j}, e_{a}\right],\left[e_{i}, e_{b}\right]\right)  \tag{7.3.5}\\
= & -B\left(e_{i},\left[e_{j},\left[e_{b}, e_{a}\right]\right]\right)-B\left(e_{i},\left[e_{b},\left[e_{a}, e_{j}\right]\right]\right)-B\left(\left[e_{j}, e_{a}\right],\left[e_{i}, e_{b}\right]\right) \\
= & -B\left(\left[e_{i}, e_{j}\right],\left[e_{b}, e_{a}\right]\right)-B\left(\left[e_{i}, e_{b}\right],\left[e_{a}, e_{j}\right]\right)-B\left(\left[e_{j}, e_{a}\right],\left[e_{i}, e_{b}\right]\right) \\
= & B\left(\left[e_{i}, e_{j}\right],\left[e_{a}, e_{b}\right]\right) \\
= & \frac{1}{2} B\left(\mathfrak{L}\left(e_{i} \otimes e_{j}-e_{j} \otimes e_{i}\right), \mathfrak{L}\left(e_{a} \otimes e_{b}\right)\right)
\end{align*}
$$

where in the third and fifth equality use the property $B([x, y], z)=B(x,[y, z])$ for $x, y, z \in \mathfrak{g}$ and the fourth equality follows from the Jacobi identity.

By Lemma 7.3.1 and $\mathfrak{L}\left(\widetilde{\xi}^{\sharp}\right)=0$, we conclude that the factor

$$
\begin{equation*}
2 \operatorname{Tr}_{i} \otimes \operatorname{Tr}_{i}\left(\widetilde{\omega}_{(\cdot i)}^{\sharp} \wedge \mathbf{1} \widetilde{q}^{*} \widetilde{p}_{i}^{*} \widetilde{\xi}^{\sharp} \wedge \widetilde{\omega}_{(i \bullet)}^{\sharp}\right)=B_{i}\left(\mathfrak{L}_{i}\left(\widetilde{q}^{*} \widetilde{p}_{i}^{*} \widetilde{\xi}^{\sharp}\right), \mathfrak{L}_{i}\left(\widetilde{\omega}_{(\cdot i)}^{\sharp} \wedge \widetilde{\omega}_{(i \bullet)}^{\sharp}\right)\right)=0 . \tag{7.3.6}
\end{equation*}
$$

(b-2) the number of such non-regular edges is 3 (case that $e_{0}=0, e_{v}=3, e_{h}=0$ ); note that this case occurs only when the given connected trivalent graph is the Theta graph. The integrand associated with edges connecting $i$ and $j$ becomes

$$
\begin{align*}
& \operatorname{Tr}_{i} \otimes \operatorname{Tr}_{i}\left(I(\widetilde{\eta})_{(i i)}+\widetilde{q}^{*} \widetilde{p}_{i}^{*} \widetilde{\xi}^{\sharp}\right)^{3} \\
= & \operatorname{Tr}_{i} \otimes \operatorname{Tr}_{i}\left(I(\widetilde{\eta})_{(i i)}^{3}+3 I(\widetilde{\eta})_{(i i)} \wedge q^{*} \pi_{i}^{*}\left(\widetilde{\xi}^{\sharp}\right)^{2}+3 I(\widetilde{\eta})_{(i i)}^{2} \wedge \widetilde{q}^{*} \widetilde{p}_{i}^{*} \widetilde{\xi}^{\sharp}+\widetilde{q}^{*} \widetilde{p}_{i}^{*}\left(\widetilde{\xi}^{\sharp}\right)^{3}\right) . \tag{7.3.7}
\end{align*}
$$

Similarly, after integrating along the fiber, we get $3 \operatorname{Tr}_{i} \otimes \operatorname{Tr}_{i}\left(1 \widetilde{q}^{*} \widetilde{p}_{i}^{*}\left(\widetilde{\xi}^{\sharp}\right)^{2}\right)$ and it vanishes since

$$
\begin{equation*}
\operatorname{Tr}_{i} \otimes \operatorname{Tr}_{i}\left(\mathbf{1} \widetilde{q}^{*} \widetilde{p}_{i}^{*}\left(\widetilde{\xi}^{\sharp}\right)^{2}\right)=\frac{1}{2} B_{i}\left(\mathfrak{L}\left(\widetilde{q}^{*} \widetilde{p}_{i}^{*} \widetilde{\xi}^{\sharp}\right), \mathfrak{L}\left(\widetilde{q}^{*} \widetilde{p}_{i}^{*} \widetilde{\xi}^{\sharp}\right)\right)=0 \tag{7.3.8}
\end{equation*}
$$

by [CS21, Lemma 4.4] and $\mathfrak{L}\left(\widetilde{\xi}^{\sharp}\right)=0$. This completes the proof of Theorem 7.1.8 (1).
Next, we show Theorem 7.1.8 (2). By Theorem 7.1.5, $Z(M, \rho,[f])(\Gamma)$ is independent of the choice of propagators. Hence, by using adapted propagator instead of a general propagator, we obtained the same invariant as $Z(M, \rho,[f])(\Gamma)$, but associated with $\Gamma^{\prime}$ obtained by removing graphs with self-loops. As we have shown in Theorem 7.1.8 (1), $Z^{\sharp}(M, \rho,[f])\left(\Gamma^{\prime}\right)$ is an invariant associated with cocycle $\Gamma^{\prime}$ in $\mathcal{G}_{\mathrm{ac}, \mathfrak{g}}^{0}$. Thus, we obtain the commutative diagram (7.1.13). This completes the proof.

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## References

[AS92] Scott Axelrod and I. M. Singer, Chern-Simons perturbation theory, Proceedings of the XXth International Conference on Differential Geometric Methods in Theoretical Physics, Vol. 1, 2 (New York, 1991), World Sci. Publ., River Edge, NJ, 1992, pp. 3-45. 1, 1, 1, 6.6, 6.6, 7.1.10
[AS94] , Chern-Simons perturbation theory. II, J. Differential Geom. 39 (1994), no. 1, 173213. 1, 1, 1, 2, 2.3, 2.3, 2.4, 2.4, 6.4.2, 6.6, 7.1.4, 7.1.10
[BC98] Raoul Bott and Alberto S. Cattaneo, Integral invariants of 3-manifolds, J. Differential Geom. 48 (1998), no. 1, 91-133. 1, 1, 3.2.3, 4.2, 4.3, 2, 4.4, 5.1, 6.5.5, 6.5.6, 7.2, 7.2.1, 7.2.3
[BC99] , Integral invariants of 3-manifolds. II, J. Differential Geom. 53 (1999), no. 1, 1-13. $1,1, \mathrm{~A}, 1,1,4.4,4.4,5.1,6,6.3,6.4,6.4,6.5,6.5 .6,7,7.1,7.2$
[BN95] Dror Bar-Natan, On the Vassiliev knot invariants, Topology 34 (1995), no. 2, 423-472. 1, 6.2, 6.4.2
[BZ23] Valentina Bais and Daniele Zuddas, On Stiefel's parallelizability of 3-manifolds, Expo. Math. 41 (2023), no. 1, 238-243. 2.5
[CM10] Alberto S. Cattaneo and Pavel Mnëv, Remarks on Chern-Simons invariants, Comm. Math. Phys. 293 (2010), no. 3, 803-836. 1, 4, 4.1, 4.3, 4.3, 1, 3
[Cos07] Kevin Costello, Renormalisation and the Batalin-Vilkovisky formalism, arXiv (2007), 88 pages. 1
[Cos11] , Renormalization and effective field theory, Mathematical Surveys and Monographs, vol. 170, American Mathematical Society, Providence, RI, 2011. MR 27785581
[CS21] Alberto S. Cattaneo and Tatsuro Shimizu, A note on the $\Theta$-invariant of 3-manifolds, Quantum Topol. 12 (2021), no. 1, 111-127. 1, 1, A, 1, 1, 1, 4, 4.2, 4.3, 1, 3, 5, 5.1, 5.1, 5.1.2, 5.1, $5.1 .3,5.3,5.3,5.3,6.5 .6,7,7.1,7.2,7.3,7.3$
[CV03] James Conant and Karen Vogtmann, On a theorem of Kontsevich, Algebr. Geom. Topol. 3 (2003), 1167-1224. 1, 6.6
[CW23] Ricardo Campos and Thomas Willwacher, A model for configuration spaces of points, Algebr. Geom. Topol. 23 (2023), no. 5, 2029-2106. 1, 2.3.1, 4, 4.3, 4.3, 2, 3
[DeB06] Jason DeBlois, Totally geodesic surfaces and homology, Algebr. Geom. Topol. 6 (2006), 1413-1428. 3.5.6
[FH91] William Fulton and Joe Harris, Representation theory, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991, A first course, Readings in Mathematics. 3.5
[FM94] William Fulton and Robert MacPherson, A compactification of configuration spaces, Ann. of Math. (2) 139 (1994), no. 1, 183-225. 2.3
[FS92] Ronald Fintushel and Ronald J. Stern, Integer graded instanton homology groups for homology three-spheres, Topology 31 (1992), no. 3, 589-604. 3.3.1
[Iac10] Vito Iacovino, Master equation and perturbative Chern-Simons theory, preprint, 2010, arXiv:0811.2181v2. 1
[Igu04] Kiyoshi Igusa, Graph cohomology and Kontsevich cycles, Topology 43 (2004), no. 6, 14691510. 1, 6.1.3
[Joy12] Dominic Joyce, On manifolds with corners, Advances in geometric analysis, Adv. Lect. Math. (ALM), vol. 21, Int. Press, Somerville, MA, 2012, pp. 225-258. 2.1, 2.2
[Kon94] Maxim Kontsevich, Feynman diagrams and low-dimensional topology, First European Congress of Mathematics, Vol. II (Paris, 1992), Progr. Math., vol. 120, Birkhäuser, Basel, 1994, pp. 97-121. 1, 1, 1, 1, 6.6, 7.2.3
[KS23] Teruaki Kitano and Tatsuro Shimizu, Gluing formula for an invariant related to the ChernSimons perturbation theory, preprint, 2023, arXi:2306.09717. 1
[Les04] Christine Lescop, On the Kontsevich-Kuperberg-Thurston construction of a configurationspace invariant for rational homology 3-spheres, math.GT/0411088 Prepublication Institut Fourier 655 (2004), 71 pages. 1, 7.1.4
[Les20] , Invariants of links and 3-manifolds that count graph configurations, Winter Braids Lect. Notes 7 (2020), no. Winter Braids X (Pisa, 2020), Exp. No. 1, 35. 1, 7.1.4
[LM85] Alexander Lubotzky and Andy R. Magid, Varieties of representations of finitely generated groups, Mem. Amer. Math. Soc. 58 (1985), no. 336, xi+117. 3.3
[Mel] Richard Melrose, Differential analysis on manifolds with corners (in preparation), partially available at http://www-math.mit.edu/~rbm/book.html. 2.1
[Mil85] John J. Millson, A remark on Raghunathan's vanishing theorem, Topology 24 (1985), no. 4, 495-498. 3.5
[MS74] John W. Milnor and James D. Stasheff, Characteristic classes, vol. No. 76., Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974. 3.1
[MW11] Varghese Mathai and Siye Wu, Analytic torsion for twisted de Rham complexes, J. Differential Geom. 88 (2011), no. 2, 297-332. 3.1
[Por13] Joan Porti, Local and infinitesimal rigidity of representations of hyperbolic three manifolds, 2013, p. 154-177. 3.5, 3.5
[Rag65] M. S. Raghunathan, On the first cohomology of discrete subgroups of semisimple Lie groups, Amer. J. Math. 87 (1965), 103-139. 3.5
[Sav02] Nikolai Saveliev, Invariants for homology 3-spheres, Encyclopaedia of Mathematical Sciences, vol. 140, Springer-Verlag, Berlin, 2002, Low-Dimensional Topology, I. 3.3
[Sav12] , Lectures on the topology of 3-manifolds, revised ed., De Gruyter Textbook, Walter de Gruyter \& Co., Berlin, 2012, An introduction to the Casson invariant. 3.3, 3.3.1
[Saw06] Justin Sawon, Perturbative expansion of Chern-Simons theory, The interaction of finite-type and Gromov-Witten invariants (BIRS 2003), Geom. Topol. Monogr., vol. 8, Geom. Topol. Publ., Coventry, 2006, pp. 145-166. 1, 6.6, 6.6.7
[Shi21] Tatsuro Shimizu, A geometric description of the Reidemeister-Turaev torsion of 3-manifolds, RIMS preprint (2021), 34 pages. 1
[Shi23] , Morse homotopy for the SU(2)-Chern-Simons perturbation theory, J. Differential Geom. 123 (2023), no. 2, 363-390. 1, 3.5
[Sin04] Dev P. Sinha, Manifold-theoretic compactifications of configuration spaces, Selecta Math. (N.S.) 10 (2004), no. 3, 391-428. 2.3, 2.3.1
[Spa95] Edwin H. Spanier, Algebraic topology, Berlin: Springer-Verlag, 1995. 3.1
[Ste43] Norman E. Steenrod, Homology with local coefficients, Ann. of Math. (2) 44 (1943), 610-627. 3.3
[Sti35] Eduard Stiefel, Richtungsfelder und Fernparallelismus in n-dimensionalen Mannigfaltigkeiten, Comment. Math. Helv. 8 (1935), no. 1, 305-353. 2.5
[Wei64] André Weil, Remarks on the cohomology of groups, Ann. of Math. (2) 80 (1964), 149-157. 3.3
[Wit89] Edward Witten, Quantum field theory and the Jones polynomial, Comm. Math. Phys. 121 (1989), no. 3, 351-399. 1
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