SERRE ALGEBRA, MATRIX FACTORIZATION AND CATEGORICAL TORELLI THEOREM FOR HYPERSURFACES

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ABSTRACT. Let X be a smooth Fano variety. We attach a bi-graded associative algebra $\mathcal{A}_S = \bigoplus_{i,j\in\mathbb{Z}} \mathrm{Hom}(\mathrm{Id}, S^i_{\mathcal{K}u(X)}[j])$ to the Kuznetsov component $\mathcal{K}u(X)$ whenever it is defined. Then we construct a natural sub-algebra of $\mathcal{A}_S$ when X is a Fano hypersurface and establish its relation with Jacobian ring $J(X)$. As an application, we prove a categorical Torelli theorem for Fano hypersurface $X \subset \mathbb{P}^n (n \geq 2)$ of degree d if $\gcd((n+1), d) = 1$. In addition, we give a new proof of the main theorem [Pir22, Theorem 1.2] using a similar idea.

1. INTRODUCTION

Let X be a smooth complex projective variety. Reconstruction of X from its categorical invariant originates from Gabriel's thesis [Gab62], where the author proves X can be recovered from its category of coherent sheaves. Later on, this theorem is generalized to arbitrary quasi-separated scheme in [Ros98]. In the celebrated work [BO01], the authors prove smooth Fano variety X can be reconstructed from its bounded derived category $D^b(X)$ of coherent sheaves. Since the last decades, people are interested in reconstruction of Fano varieties from the non-trivial semi-orthogonal component $\mathcal{K}u(X)$, known as Kuznetsov component, of its bounded derived category of coherent sheaves, called *Categorical Torelli problem*. The first result in this direction is given in [BMMS12], where the authors prove categorical Torelli theorem for smooth cubic threefolds. Since then tremendous work have been carried out along this direction, see [PS23] for a review of known results. In this paper, we focus on Fano hypersurfaces $X \subset \mathbb{P}^n$ of degree $d \leq n$. Our work is inspired by the paper [HR16], where the authors relate a variant of Hochschild cohomology $\mathrm{HH}(\mathcal{K}u(X), (1))$ of Kuznetsov component $\mathcal{K}u(X)$ to the Jacobian ring $J(X)$ of the hypersurface X, in particular they show the Hochschild cohomology ring is isomorphic to the Jacobian ring whenever $\mathcal{K}u(X)$ is a Calabi-Yau category, hence establish a categorical Torelli theorem for cubic fourfolds. In addition, they suggested using the category of graded matrix factorizations $\mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega)$ of a Fano hypersurface defined by a polynomial ω to reconstruct Jacobian ring via Hochschild cohomology $\mathrm{HH}^\bullet(\mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega))$, which motivates our approach to this problem. On the other hand, for any smooth DG category $\mathcal{A}$, there is a natural associative algebra $\mathcal{A}_S$ attached to it, called *Serre algebra*(cf. Definition 3.3), which is a Morita invariant of $\mathcal{A}$. In [BO01], the authors construct a subring of $\mathcal{A}$, i.e. canonical ring to reconstruct smooth

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complex projective variety with canonical bundle ample or anti-ample. It is interesting to ask if certain sub-algebra of the Serre algebra $\mathcal{K}u(X)_S$ of a smooth Fano variety X can be used to determine the isomorphism class of it. In this article, we hope to answer this question.

1.1. **Main results.** Let $X, X' \subset \mathbb{P}^n (n \geq 2)$ be Fano hypersurfaces of degree $d \leq n$. Instead of making additional assumption that the equivalence $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$ is compatible with degree shifting functor (1) and passing the equivalence $\mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega) \simeq \mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega')$ to the equivalence $\mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega)/(1) \simeq \mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega')/(1)$, we only assume that there is an equivalence $\Phi : \mathcal{K}u(X) \simeq \mathcal{K}u(X')$ and note that it commutes with Serre functors of $\mathcal{K}u(X), \mathcal{K}u(X')$ respectively. Then it is not hard to show that the associated *Serre algebra*(cf. Definition 3.3) of $\mathcal{K}u(X)$ and $\mathcal{K}u(X')$ are isomorphic. We construct a subalgebra of Serre algebra $\mathcal{K}u(X)_S$ and establish its relation with Jacobian ring.

Theorem 1.1. *Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d \leq n$ defined by ω. Consider its affine LG model whose underlying stack is $\mathcal{Z} = [\mathbb{A}^{n+1}/\mathbb{Z}_d]$ and associate dg category of matrix factorization* $\mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega) \simeq \mathcal{K}u(X)$. *Assume* $\gcd((n+1), d) = 1$. *There is a sub-algebra* $\bigoplus_{t\geq 0} \mathrm{Hom}(\Delta, \Delta(t))$ *of the Serre algebra* $\mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega)_S$ *such that*

$$\mathrm{Hom}(\Delta, \Delta(t)) = \begin{cases} \mathrm{Jac}(\omega)_t, & n = 2m, m \in \mathbb{Z} \\ \mathrm{Jac}(\omega)_t, & n = 2m+1, m \in \mathbb{Z}, t \neq \frac{(d-2)(n+1)}{2} \\ \mathrm{Jac}(\omega)_t \oplus k^{d-1}, & n = 2m+1, m \in \mathbb{Z}, t = \frac{(d-2)(n+1)}{2}. \end{cases}$$

In particular, if n is odd, the Serre algebra has a sub-graded algebra $\mathrm{Jac}(\omega)$ *and if n is even, the Serre algebra has a degree d graded subspace* $\mathrm{Jac}(\omega)_d$.

Remark 1.2. Consider a smooth hypersuface of degree $d \leq \sum_{j=1}^{n+1} q_j - 1$ in weighted projective space $\mathbb{P}(q_1, q_2 \cdots, q_{n+1})$ with $\gcd(q_1, q_2, \cdots, q_{n+1}) = 1$. One can prove the same statement. Namely if $\gcd(\sum_{j=1}^{n+1} q_j, d) = 1$, then there is a sub-algebra $\bigoplus_{t\geq 0} \mathrm{Hom}(\Delta, \Delta(t))$ of $\mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega)_S$, where $\mathbb{C}^*$-action on $\mathbb{A}^{n+1}$ is of weight $(q_1, q_2, \cdots, q_{n+1})$.

As an application, we establish *Categorical Torelli theorem* for smooth hypersurface $X \subset \mathbb{P}^n$ of degree d whenever $\gcd((n+1), d) = 1$. Namely, we show the Kuznetsov component $\mathcal{K}u(X)$(cf.Proposition 2.2) determines its isomorphism class.

Theorem 1.3. *Let $X, X' \subset \mathbb{P}^n$ be degree d smooth hypersurfaces. Assume* $\gcd(n+1, d) = 1$. *If there is a Fourier-Mukai equivalence $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$, then $X \cong X'$.*

Remark 1.4. Consider a smooth hypersufaces of degree $d \leq \sum_{j=1}^{n+1} q_j - 1$ in weighted projective space $\mathbb{P}(q_1, q_2 \cdots, q_{n+1})$ with $\gcd(q_1, q_2, \cdots, q_{n+1}) = 1$. By similar arguments in the proof of Theorem 1.3(or Theorem 4.1), one is able to obtain *Categorical Torelli theorem* for a series of hypersurfaces. In particular, our method works for a degree 6 hypersurface in weighted projective space $\mathbb{P}(1,1,1,1,3)$, which is isomorphic to an index one degree two Fano threefold(See Section 6).[1]

Let us briefly explain the idea of the proof. We work on *dg* category of graded matrix factorization $\mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1},\mathbb{C}^*,\omega)$. An equivalence $\Phi:\mathcal{K}u(X)\simeq\mathcal{K}u(X')$ in Hqe(dg-cat) induces an equivalence $\mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1},\mathbb{C}^*,\omega)\simeq\mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1},\mathbb{C}^*,\omega')$, commuting with Serre functors on both categories. Whenever n is even, we get graded ring isomorphism $\mathrm{Jac}(\omega)\cong\mathrm{Jac}(\omega')$ coming from a degree one map, then Mather-Yau reconstruction theorem [Don83, Proposition 1.1] gives $X\cong X'$. If n is odd, we get the isomorphism of degree d or $d-1$ components of $\mathrm{Jac}(\omega)$. Then similar arguments shows $X\cong X'$.

As a corollary, we give some interesting examples where *Categorical Torelli theorem* holds.

Corollary 1.5. *Categorical Torelli theorem holds for following Fano varieties:*

1. *Cubic hypersurfaces of dimension $3k-1$ and $3k$ with $k\geq 1$.*
2. *Quintic fourfolds.*

In the paper [Pir22], the author shows in [Pir22, Theorem 1.2] that a class of Fano hypersurfaces X are determined by the Kuznetsov components $\mathcal{K}u(X)$ together with rotation functors $(1):\mathcal{K}u(X)\simeq\mathcal{K}u(X)$, which generalizes a result in [HR16, Corollary 2.10]. Using the framework of matrix factorization, we give a simple proof for [Pir22, Theorem 1.2].

Theorem 1.6. *Let X and X' be smooth hypersurfaces of degree $d<n+1$ in $\mathbb{P}^n(n\geq 2)$. If there is a Fourier-Mukai equivalence of pairs $(\mathcal{K}u(X),(1))\cong(\mathcal{K}u(X'),(1)')$, then $X\cong X'$.*

1.2. **Related Work.** Our work is inspired by the paper [HR16], where the authors suggest using the framework of graded matrix factorization of a smooth hypersurface to relate the extended Hochschild cohomology $\mathrm{HH}(\mathcal{K}u(X),(1))$ to the Jacobian ring $J(X)$, but they do all the work on derived category side. In [Pir22], the author generalized results [HR16, Corollary 2.10] to arbitrary Fano hypersurfaces. In an upcoming paper [Ren23], the author proves if d does not divide $n+1$ and the pair (d,n) is not of the form $(4,4k+1)$, then the Kuznetsov component alone reconstructs X. In upcoming work [LPS23] and [DJR23], the authors prove categorical Torelli theorem for sextic hypersurface in weighted projective space $\mathbb{P}(1,1,1,1,3)$ via completely different methods.

Remark 1.7. While preparing the paper, we learned that the authors of the paper [BFK23] also define and study the Serre algebra under the name Hochschild-Serre cohomology in [BFK23, Definition A.1], where they give a formula for Hochschild cohomology of Hilbert scheme of

[1]In an earlier version of the paper, a weaker version of Theorem 1.3 is obtained. Namely the statement is obtained if $\gcd(2(n+1),d)=1$ for non-weighted case and $\gcd(2\Sigma_{j=1}^{n+1}q_j,d)=1$ for weighted case, while we are informed by Paolo Stellari that in their paper [LPS23], categorical Torelli theorem for sextic hypersurface in $\mathbb{P}(1,1,1,1,3)$ is established. Then we found that our method applies to their case and a stronger statement (as presented in the paper) is obtained and we give a new proof of the sextic weighted hypersurface case in Appendix.

points on a surface in terms of Hochschild-Serre cohomology(as a bi-graded vector space)on the surface.

1.3. **Organization of the article.** In Section 2 we introduce the terminology of graded matrix factorization associated with a hypersurface in projective space. Then we describe an important auto-equivalence on category of matrix factorization $\mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega)$. In Section 3, we introduce an associative algebra naturally attached to any smooth and proper differential graded(dg) category, called *Serre algebra* and show it is a Morita invariant. Then we give several examples of Serre algebra for various dg category, in particular, we construct interesting sub-algebra of Serre algebra for category of matrix factorization $\mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega)$ corresponding to a Fano hypersurface in $\mathbb{P}^n$, proving Theorem 1.1. In Section 4.1 we prove Theorem 1.3 and Corollary 1.5. In Section 5 we prove Theorem 1.6. In Section 6, we give a new proof of *Categorical Torelli theorem* for sextic hypersurface in $\mathbb{P}(1,1,1,1,3)$ as recently shown in [LPS23].

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2. DG CATEGORY OF GRADED MATRIX FACTORIZATIONS

In this section, we recall the terminology of dg-category of matrix factorization. We follow the context in [BFK14]. We refer the reader to [Kel06] for the basic of dg categories. Let Hqe(dg-cat) be the localizing of dg-cat with respect to the quasi-equivalences of dg categories. Let (X, G, L, ω) be a quadruple where X is a quasi-projective variety with G action, G is a reductive algebraic group, L is a G-equivariant line bundle and ω is a G-invariant section of L. Our main example is $(\mathbb{A}^{n+1}, \mathbb{C}^*, \mathcal{O}(d), \omega)$. The action of $\lambda \in \mathbb{C}^*$ on $\mathbb{A}^{n+1}$ is given by $\lambda \cdot (x_0, x_1, \cdots, x_n) = (\lambda \cdot x_0, \lambda \cdot x_1, \cdots, \lambda \cdot x_n)$. ω is a $\mathbb{C}^*$-invariant section of $\mathcal{O}(d)$. Namely ω is a degree d polynomial. We always assume ω has only isolated singularity at $0 \in \mathbb{A}^{n+1}$.

We have dg category $\mathrm{Fact}(X, G, L, \omega)$, whose objects are a quadruple $(\mathcal{E}_{-1}, \mathcal{E}_0, \Phi_{-1}, \Phi_0)$, where $\mathcal{E}_{-1}$ and $\mathcal{E}_0$ are G-equivariant quasi-coherent sheaves, $\Phi_{-1} : \mathcal{E}_0 \to \mathcal{E}_{-1} \otimes L$ and $\Phi_0 : \mathcal{E}_{-1} \to \mathcal{E}_0$ are morphism of G-equivariant sheaves such that

$$\Phi_{-1} \circ \Phi_0 = \omega.$$

$$(\Phi_0 \otimes L) \circ \Phi_{-1} = \omega.$$

The space of morphisms in $\mathrm{Fact}(X, G, L, \omega)$ are the internal Hom of G-equivariant sheaves while extending the pairs of morphisms to certain $\mathbb{Z}$-graded complexes. We point out the reference [BFK14] for interested reader. Let $\mathrm{Inj}(X, G, L, \omega) \subset \mathrm{Fact}(X, G, L, \omega)$ be a dg sub-category whose components are G-equivariant injective quasi-coherent sheaves. There is a category $\mathrm{Acycli}(\mathrm{Fact}(X, G, L, \omega))$ which imitates acyclic complexes in category of complexes of sheaves. The absolute derived category $D^{abs}(\mathrm{Fact}(X, G, L, \omega))$ is the homotopy category of *dg* quotient $\frac{\mathrm{Fact}(X,G,L,\omega)}{\mathrm{Acyclic}(\mathrm{Fact}(X,G,L,\omega))}$ in Hqe(dg-cat).

Lemma 2.1. *The natural morphism* $\mathrm{Inj}(X, G, L, \omega) \to D^{abs}(\mathrm{Fact}(X, G, L, \omega))$ *induces isomorphism in homotopic categories.*

Let $\mathrm{Inj}_{\mathrm{coh}}(X, G, L, \omega) \subset \mathrm{Inj}(X, G, L, \omega)$ be a *dg* sub-category whose objects are quasi-isomorphic to objects with coherent components in category $\mathrm{Fact}(X, G, L, \omega)$.

Define shiftting functor

$$[1] : (\mathcal{E}_{-1}, \mathcal{E}_0, \Phi_{-1}, \Phi_0) \mapsto (\mathcal{E}_0, \mathcal{E}_{-1} \otimes L, -\Phi_0, -\Phi_{-1} \otimes L).$$

With cone construction, the homotopic categories $[\mathrm{Inj}_{\mathrm{coh}}(X, G, L, \omega)]$ is a triangulated category which is isomorphic to graded matrix factorization in [Orl09] for $(\mathbb{A}^{n+1}, \mathbb{C}^*, \mathcal{O}(d), \omega)$.

Denote by

$$\{1\} : \mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{\mathrm{n}+1}, \mathbb{C}^*, \mathcal{O}(\mathrm{d}), \omega) \to \mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{\mathrm{n}+1}, \mathbb{C}^*, \mathcal{O}(\mathrm{d}), \omega)$$

the twisting functor which maps

$$\mathcal{E}_{-1} \xrightarrow{\Phi_0} \mathcal{E}_0 \xrightarrow{\Phi_{-1}} \mathcal{E}_{-1}(d)$$

to

$$\mathcal{E}_{-1}(1) \xrightarrow{\Phi_0(1)} \mathcal{E}_0(1) \xrightarrow{\Phi_{-1}(1)} \mathcal{E}_{-1}(d+1)$$

Clearly, we have equality of functors $\{d\} := \{1\}^d = [2]$.

Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d \leq n$ defined by ω. Let

$$\mathcal{K}u(X) := \Big\langle \mathcal{O}_X, \mathcal{O}_X(1), \cdots, \mathcal{O}_X(n-d) \Big\rangle^{\perp}.$$

Consider the natural enhancement $\mathrm{Inj}_{\mathrm{coh}}(X)$, and let $\mathcal{K}u_{dg}(X)$ be a *dg* subcategory that enhance $\mathcal{K}u(X)$. Write (1) as the quasi-endofuntor (Fourier-Mukai type) of $\mathcal{K}u_{dg}(X)$ that define degree shifting functor $(1) : \mathcal{K}u(X) \to \mathcal{K}u(X)$ in the sense of [HR16].

Theorem 2.2. *[BFK14, Theorem 6.13] There is an equivalence in* Hqe(dg-cat)*,*

$$\Phi : \mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega) \cong \mathcal{K}u_{dg}(X).$$

In particular, there is an isomorphism of quasi-funtors

$$\Phi \circ \{1\} \cong (1) \circ \Phi.$$

Proof. Firstly, we have quasi-functor

$$\Phi : \mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \mathcal{O}(d), \omega) \to \mathcal{K}u_{dg}(X),$$

and quasi-functor

$$\Phi^! : \mathrm{Inj}_{\mathrm{coh}}(X) \to \mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \mathcal{O}(d), \omega)$$

such that $\Phi^! \circ \Phi \simeq \mathrm{Id}$. Since Φ defines an equivalence of triangulated category $[\mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \mathcal{O}(d), \omega)] \simeq \mathcal{K}u(X)$, and both $\mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \mathcal{O}(d), \omega)$ and $\mathcal{K}u_{dg}(X)$ are exact *dg* categories, therefore Φ is an isomorphism in Hqe(dg-cat). The equality $\Phi^! \circ \Phi = \mathrm{Id}$ implies $\Phi^!$ is the inverse of Φ when restricting to $\mathcal{K}u_{dg}(X)$. Write $(1)_F : \mathrm{Inj}_{\mathrm{coh}}(X) \to \mathrm{Inj}_{\mathrm{coh}}(X)$ as the Fourier-Mukai functor that restricts to rotation functor (1) on $\mathcal{K}u_{dg}(X)$ in [BFK14, Theorem 6.13]. The isomorphims of quasi-functors $\Phi^! \circ (1)_F \circ \Phi \simeq \{1\}$ implies an isomorphism of quasi-functors $\Phi^{-1} \circ (1) \circ \Phi \simeq \{1\}$. □

3. Serre algebra

Theorem 3.1. *[Toë07] Let $\mathcal{A}$ be a dg category over the field k. In* Hqe(dg-cat)*, we have isomorphism,*

$$\mathcal{D}_{dg}(\mathcal{A}^{op} \otimes \mathcal{A}) \cong \mathcal{R}\mathrm{Hom}_c(\mathcal{D}_{dg}(\mathcal{A}), \mathcal{D}_{dg}(\mathcal{A})),$$

where $\mathcal{R}\mathrm{Hom}_c$ is the quasi-functor preserving coproduct.

If $\mathcal{A}$ is a smooth proper *dg* category, then the bimodules for Serre functor and inverse of Serre functor are constructed explicitly in [Shk07], where the author defines Serre functor(inverse Serre functor) of the triangulated category $\mathrm{Perf}(\mathcal{A})$ in the usual sense. From now on, we interpret those bimoudles by quasi-functors by Theorem 3.1.

Definition 3.2. The Hochschild (co)homology of a smooth proper *dg* category $\mathcal{A}$ are defined as,

$$\mathrm{HH}^m(\mathcal{A}) = \mathrm{Hom}(\mathrm{Id}, \mathrm{Id}[m]),$$

$$\mathrm{HH}_m(\mathcal{A}) = \mathrm{Hom}(\mathrm{Id}, S[m]).$$

The Hochschild cohomology is an algebra, and the homolology is a graded module over the Hochschild homology. We define an algebra which contains Hochschild cohomology and Hochschild homoloy, and encodes the algebra structure of Hochschild cohomology and the module structure of Hochschild homology over the Hochschild cohomology.

Definition 3.3. (Serre algebra) Let $\mathcal{A}$ be a smooth proper *dg* category. Define bi-graded algebra

$$\mathcal{A}_S = \bigoplus_{m,n\in\mathbb{Z}} \mathrm{Hom}(\mathrm{Id}, S^m[n])$$

The multiplication map

$$\mathrm{Hom}(\mathrm{Id}, S^{m_1}[n_1]) \times \mathrm{Hom}(\mathrm{Id}, S^{m_2}[n_2]) \xrightarrow{\times} \mathrm{Hom}(\mathrm{Id}, S^{m_1+m_2}[n_1+n_2])$$

is defined as follows. For element $(a,b) \in \mathrm{Hom}(\mathrm{Id}, S^{m_1}[n_1]) \times \mathrm{Hom}(\mathrm{Id}, S^{m_2}[n_2])$, $a \times b$ is defined as the composition

$$\mathrm{Id} \xrightarrow{\quad b \quad} \mathrm{Id} \circ S^{m_2}[n_2] \xrightarrow{\quad a \circ \mathrm{Id} \quad} S^{m_1}[n_1] \circ S^{m_2}[n_2] = S^{m_1+m_2}[n_1+n_2] \ .$$

We check the associativity. Namely for elements $a \in \mathrm{Hom}(\mathrm{Id}, S^{m_1}[n_1])$, $b \in \mathrm{Hom}(\mathrm{Id}, S^{m_2}[n_2])$, and $c \in \mathrm{Hom}(\mathrm{Id}, S^{m_3}[n_3])$, we have $(ab)c = a(bc) = abc$. For example, $a(bc) = abc$ follows from the commutative diagram,

$$\begin{tikzcd}
\mathrm{Id} \arrow[r, "c"] \arrow[drr, "bc"'] & S^{m_3}[n_3] \arrow[r, "b"] & S^{m_3}[n_3] \circ S^{m_2}[n_2] \arrow[r, "a"] \arrow[d, equal] & S^{m_3}[m_3] \circ S^{m_2}[n_2] \circ S^{m_1}[n_1] \arrow[d, equal] \\
& & S^{m_3+m_2}[n_3+n_2] \arrow[r, "a"] & S^{m_3+m_2+m_1}[n_3+n_2+n_1]
\end{tikzcd}$$

Let Hmo(dg-cat) be the localization of dg-cat with respect to the Morita equivalences of *dg* categories. If $\mathcal{A}$ and $\mathcal{B}$ are smooth and proper, $\mathrm{Hom}_{\mathrm{Hmo(dg\text{-}cat)}}(\mathcal{A}, \mathcal{B})$ is the isomorphism classes of $\mathrm{Perf}(\mathcal{A}^{op} \otimes \mathcal{B})$ [Tab15, Corollary 1.44], and the composition corresponds to tensor product.

Theorem 3.4. *If* $\mathcal{A} \cong \mathcal{B}$ *in* Hmo(dg-cat)*, then* $\mathcal{A}_S \cong \mathcal{B}_S$*.*

Proof. There is an isomorphism of Serre functor $S_{\mathcal{A}} \circ \Phi \cong \Phi \circ S_{\mathcal{B}}$. Hence, Φ induces isomorphism for any integer m, n,

$$\mathrm{Hom}(\mathrm{Id}_{\mathcal{A}}, S^m_{\mathcal{A}}[n]) \cong \mathrm{Hom}(\mathrm{Id}_{\mathcal{B}}, S^m_{\mathcal{B}}[n]).$$

by the following commutative diagram,

$$\begin{tikzcd}
\mathrm{Id}_{\mathcal{B}} \arrow[r] \arrow[d, "\cong"] & S^m_{\mathcal{B}}[n] \arrow[d, "\cong"] \\
\Phi^{-1} \circ \mathrm{Id}_{\mathcal{A}} \circ \Phi \arrow[r] & \Phi^{-1} \circ S^m_{\mathcal{A}}[n] \circ \Phi
\end{tikzcd}$$

The isomorphism is an isomorphism of algebra since both algebra are defined by composition of functors. $\square$

3.1. **Examples of Serre Algebras.** In this section, we give examples of Serre algebra for various categories.

3.1.1. *Orlov's algebra* $\mathrm{HA}(X)$. Let $\mathcal{A} = D^b(X)$ be the bounded derived category of coherent sheaves on a smooth projective variety X. In this case, $S_{\mathcal{A}} = - \otimes \omega_X[l]$, where $l = \dim X$. Thus the Serre algebra $\mathcal{A}_S$ is given by

$$\begin{aligned}
\mathcal{A}_S := \bigoplus_{m,n \in \mathbb{Z}} \mathrm{Hom}(\mathrm{Id}, \mathrm{S}^m_{D^b(X)}[n]) &\cong \bigoplus_{m,n \in \mathbb{Z}} \mathrm{Hom}_{D^b(X \times X)}(\iota_* \mathcal{O}_X, \iota_* \omega_X^{\otimes m}[ml+n]) \\
&\cong \bigoplus_{m,n \in \mathbb{Z}} \mathrm{Ext}^{ml+n}_{X \times X}(\iota_* \mathcal{O}_X, \iota_* \omega_X^{\otimes m}),
\end{aligned}$$

where $\iota : X \hookrightarrow X \times X$ be the diagonal inclusion. It is clear that $\mathcal{A}_S$ is isomorphic to the bi-graded algebra $\mathrm{HA}(X)$ in [Orl03]. In particular, if $ml + n = 0$, then $\mathcal{A}_S = \bigoplus_{m\geq 0} \mathrm{Hom}_{X\times X}(\iota_* \mathcal{O}_X, \iota_*\omega_X^{\otimes m}) \cong \bigoplus_{m\geq 0} \mathrm{Hom}_X(\mathcal{O}_X, \omega_X^{\otimes m})$ is the canonical ring of X.

3.1.2. *dg-category of matrix factorization on affine LG model.* According to Orlov' sigma/LG correspondence, the Kuznetsov components of hypersurfaces $X \subset \mathbb{P}^n$ of degree d are affine LG model. First, the Serre functor of $\mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega)$(Theorem 2.2) is $-\otimes \mathcal{O}_{\mathbb{A}^{n+1}}(-n-1)[n+1]$ [FK18, Theorem 1.2]. According to [BFK14], the natural functors can be reinterpreted as Fourier-Mukai transformation of kernels, and the natural transformation between these functors is morphism of kernels. We write $\Delta(m)$ as the kernel of functor $-\otimes \mathcal{O}_{\mathbb{A}^{n+1}}(m)$. Next we recall a key theorem in [BFK14, Theorem 1.2]. For $g \in \mathbb{C}^*$, we write W_g as the conormal sheaf of $(\mathbb{A}^{n+1})^g$, k_g the character of $\det(W_g)$. We write $H^\bullet(d\omega_g)$ as the Koszul cohomology of the Jacobian ideal of $\omega_g := \omega|_{(\mathbb{A}^{n+1})^g}$.

Theorem 3.5. *[BFK14, Theorem 5.9] Assume ω has isolated singularity exactly at $0 \in \mathbb{A}^{n+1}$.*

$$\begin{aligned}\mathrm{Hom}(\Delta, \Delta(m)[t]) \cong &\Big(\bigoplus_{g\in\mu_d,\ l\geq 0,\ t-\mathrm{rk}\,\omega_g = 2u} H^{2l}(d\omega_g)(m - k_g + d(u-l)) \\ &\oplus \bigoplus_{g\in\mu_d,\ l\geq 0,\ t-\mathrm{rk}\,W_g = 2u+1} H^{2l+1}(d\omega_g)(m - k_g + d(u-l))\Big)^{\mathbb{C}^*}\end{aligned}$$

Furthermore, since $H^\bullet(d\omega_g)$ has only non-trivial cohomology at degree zero, namely $H^\bullet(d\omega_g) = H^0(d\omega_g) = \mathrm{Jac}(\omega_g)$, we have

$$\mathrm{Hom}(\Delta, \Delta(m)[t]) \cong \Big(\bigoplus_{g\in\mu_d,\ t-\mathrm{rk}\,W_g\ \textbf{\textit{is even}}} Jac(d\omega_g)(m - k_g + d(\frac{t-\mathrm{rk}\,W_g}{2}))\Big)^{\mathbb{C}^*}.$$

Remark 3.6. If $\mathrm{Hom}(\Delta, \Delta(t_1)) \cong \mathrm{Jac}(\omega)_{t_1}$ and $\mathrm{Hom}(\Delta, \Delta(t_2)) \cong \mathrm{Jac}(\omega)_{t_2}$, then the multiplication

$$\mathrm{Hom}(\Delta, \Delta(t_1)) \times \mathrm{Hom}(\Delta, \Delta(t_2)) \to \mathrm{Hom}(\Delta, \Delta(t_1 + t_2))$$

is the composition of functions on $\mathbb{A}^{n+1}$ (namely product of polynomials) while identifying with certain graded pieces of Jacobian algebra $\mathrm{Jac}(\omega)$.

Proposition 3.7. *Consider the affine LG model $\mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega)$. Assume $\gcd((n+1), d) = 1$. There is a sub-algebra $\bigoplus_{t\geq 0} \mathrm{Hom}(\Delta, \Delta(t))$ of $\mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega)_S$ such that*

$$\mathrm{Hom}(\Delta, \Delta(t)) = \begin{cases} \mathrm{Jac}(\omega)_t, & n = 2m, m \in \mathbb{Z} \\ \mathrm{Jac}(\omega)_t, & n = 2m+1, m \in \mathbb{Z}, t \neq \frac{(d-2)(n+1)}{2}. \\ \mathrm{Jac}(\omega)_t \oplus k^{d-1}, & n = 2m+1, m \in \mathbb{Z}, t = \frac{(d-2)(n+1)}{2}. \end{cases}$$

In particular, if n is odd, the Serre algebra has a sub-graded algebra $\mathrm{Jac}(\omega)$ and if n is even, the Serre algebra has a degree d graded subspace $\mathrm{Jac}(\omega)_d$.

Proof. Firstly, It is known that the Serre functor $S \cong \Delta(-(n+1))[n+1]$ [FK18, Theorem 1.2], and we have $\Delta(d) = [2]$. If $\gcd(n+1,d) = 1$, then there exists $k_1, k_2 \in \mathbb{Z}$ such that $k_1(n+1) + k_2 d = 1$. Then

$$S^{k_1} \cong \Delta(-k_1(n+1))[k_1(n+1)] \cong \Delta(-1+k_2 d)[k_1(n+1)] \cong \Delta(-1)[2k_2 + k_1(n+1)].$$

Thus for any integer t, $(S^{-k_1}[2k_2 + k_1(n+1)])^t \cong \Delta(t)$. The composition of $-\otimes \mathcal{O}(-)$ is the same as the composition of Serre functors, hence $\bigoplus_{t\geq 1} \mathrm{Hom}(\Delta, \Delta(t))$ is a sub-algebra of $\mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega)_S$. According to Proposition 3.5, we have

$$\mathrm{Hom}(\Delta, \Delta(t)) \cong \Big(\bigoplus_{g\in\mu_d,\quad -\mathrm{rk}\, W_g \textbf{ is even}} Jac(\omega_g)(t - k_g + d(\frac{-\mathrm{rk}\, W_g}{2}))\Big)^{\mathbb{C}^*}.$$

In our case, if $g = 1$, then $\mathrm{rk}\, W_g = 0$, and $k_g = 0$; if $g \neq 1$, then $\mathrm{rk}\, W_g = n+1$, $k_g = -n-1$, and $Jac(\omega_g) = k(0)$. Therefore,

- If $n+1$ is odd, then $\mathrm{Hom}(\Delta, \Delta(t)) \cong (\mathrm{Jac}(\omega)(t))^{\mathbb{C}^*} = \mathrm{Jac}(\omega)_t$.
- If $n+1$ is even,

$$\mathrm{Hom}(\Delta, \Delta(t)) \cong (Jac(\omega)(t) \oplus E)^{\mathbb{C}^*} = \mathrm{Jac}(\omega)_t \oplus E^{\mathbb{C}^*},$$

  where $E = \bigoplus_{g\neq 1} \mathrm{Jac}(\omega_g)(t + n + 1 - d(\frac{n+1}{2})) = \bigoplus_{g\neq 1} \mathrm{Jac}(\omega_g)(t - \frac{(d-2)(n+1)}{2})$. If $t \neq \frac{(d-2)(n+1)}{2}$, then $E^{\mathbb{C}^*} = (\bigoplus^{d-1} k(t - \frac{(d-2)(n+1)}{2}))^{\mathbb{C}^*} = 0$ because $\mathrm{Jac}(\omega_g) = k(0)$ for $g \neq 1$. Thus $\mathrm{Hom}(\Delta, \Delta(t)) \cong \mathrm{Jac}(\omega)_t$.

□

Remark 3.8. In [BO01], the canonical ring as the sub-algebra of Orlov's algebra HA described in section 3.1.1 is used to reconstruct smooth projective varieties with canonical line bundles ample or anti-ample. So it is reasonable to expect the subring of Serre algebra $\mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega)$ can be used to reconstruct the hypersurface defined by ω. We will prove this expectation in Section 4.

4. Categorical Torelli theorem for Fano hypersurfaces

Theorem 4.1. *Let X and X' be degree $d \leq n$ smooth hypersurfaces in $\mathbb{P}^n (n \geq 2)$ defined by ω and ω' respectively. Assume* $\gcd(n+1, d) = 1$. *If there is a Fourier-Mukai equivalence $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$, then we have isomorphism $X \cong X'$.*

Proof. First, if $n+1$ is odd, then the Serre algebra has a natural sub-algebra isomorphic to $\bigoplus_{t\geq 0} \mathrm{Hom}(\Delta, \Delta(t))$ by Proposition 3.7. By the calculation in [BFK14, Theorem 5.39], the composition of $\bigoplus_{t\geq 0} \mathrm{Hom}(\Delta, \Delta(t))$ is the composition of functions under isomorphism $\bigoplus_{t\geq 0} \mathrm{Hom}(\Delta, \Delta(t)) \cong \mathrm{Jac}(\omega)$, see also Remark 3.6. Therefore, $\mathcal{K}u(X) \cong \mathcal{K}u(X')$ implies an isomorphism of graded algebra $\mathrm{Jac}(\omega) \cong \mathrm{Jac}(\omega')$ by Theorem 3.4. Note that in degree one, it is a linear map A. Therefore, we have equality of ideal $\langle \partial_i(A\omega)\rangle = \langle \partial_i \omega'\rangle$. Then by Mather-Yau's reconstruction theorem [Don83, Propsition 1.1], $A\omega$ is projective equivalent to ω'. Thus, ω is projective equivalent to ω', which implies $X \cong X'$.

Next, if $n+1$ is even, though we don't have $\mathrm{Jac}(\omega)$ as a natural sub-algebra, still we have natural graded piece $\mathrm{Jac}(\omega)_d$ since $d \neq \frac{(d-2)(n+1)}{2}$ by Proposition 3.7, otherwise

$$d = \frac{(d-2)(n+1)}{2} > \frac{(d-2)d}{2}.$$

Then $d \leq 3$. If $d = 3$, then $n+1 = 6$, contradicts that $\gcd(n+1, d) = 1$. It is clear that $d \neq 1$ and $d \neq 2$. Thus $\mathcal{K}u(X) \cong \mathcal{K}u(X')$ implies isomorphism $\mathrm{Jac}(\omega)_d \cong \mathrm{Jac}(\omega')_d$ which is induced by a linear transformation of degree one polynomials. Then similar argument as above implies $X \cong X'$. □

Corollary 4.2. *Categorical Torelli theorem holds for following Fano varieties:*

(1) *Cubic hypersurfaces of dimension $3k-1$ and $3k$ for $k \geq 1$.*
(2) *Quintic fourfolds.*

Proof.

(1) Assume $d = 3$, then $\gcd(3, n+1) = 1$ implies $3 \nmid n+1$, this means that dimension of X is $n = 3k-1$ or $3k$ with $k \geq 1$.
(2) If $d = 5$, and $n = 5$, then $\gcd(6, 5) = 1$.

Then the statement follows from Theorem 4.1. □

5. Categorical Torelli Theorem with rotation functor

In this section, we give a very simple proof of [Pir22, Theorem 1.2] via matrix factorizations.

Theorem 5.1. *Let X and X' be smooth hypersurfaces of degree $d < n+1$ in $\mathbb{P}^n (n \geq 2)$. If there is a Fourier-Mukai equivalence of pairs $(\mathcal{K}u(X), (1)) \cong (\mathcal{K}u(X'), (1)')$, then $X \cong X'$.*

Proof. Let ω and ω' define X and X' respectively. According to Theorem 2.2, there are isomorphisms of pairs in Hqe(dg-cat)

$$\begin{aligned}(\mathcal{K}u_{dg}(X), (1)) \cong& (\mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega), \{1\}) \\ (\mathcal{K}u_{dg}(X'), (1)') \cong& (\mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega'), \{1\}')\end{aligned}$$

Which induces an isomorphism of pairs by diagram chasing

$$(\mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega), \{1\}) \cong (\mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega'), \{1\}').$$

If $n+1$ is odd, then we have isomorphism of graded algebra by Proposition 3.7and Theorem 3.4,

$$\bigoplus_{t \geq 0}^{(n+1)(d-2)} \mathrm{Hom}(\Delta, \Delta(t)) \cong \bigoplus_{t \geq 0}^{(n+1)(d-2)} \mathrm{Hom}(\Delta', \Delta'(t)).$$

That is, we have isomorphism of graded algebra $\mathrm{Jac}(\omega) \cong \mathrm{Jac}(\omega')$. Thus, ω is projective equivalent to ω'.

If $n+1$ is even, the case for $d = \frac{(d-2)(n+1)}{2}$ is $(d,n) = (3,5)$. But then $d-1 \neq \frac{(d-2)(n+1)}{2}$. So in this case we have isomorphism $\mathrm{Jac}(\omega)_{d-1} \cong \mathrm{Jac}(\omega')_{d-1}$ induced by linear map of degree one polynomials by Proposition 3.7 and Theorem 3.4. Thus $X \cong X'$. Similarly, the case for $d-1 = \frac{(d-2)(n+1)}{2}$ is $(d,n) = (3,3)$. So there is an isomorphism $\mathrm{Jac}(\omega)_d \cong \mathrm{Jac}(\omega')_d$ induced by linear map of degree one polynomials by Proposition 3.7 and Theorem 3.4, which implies $X \cong X'$. □

Remark 5.2. Let $\{i\} : \mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega) \simeq \mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega)$ be the degree shift functor with corresponding Fourier-Mukai kernel Q_i. Define $L_{\mathrm{MF}}(X) := \bigoplus \mathrm{Hom}(Q_0, Q_i)$. On the other hand, the degree shift auto-equivalence $(i) : \mathcal{K}u(X) \simeq \mathcal{K}u(X)$ is represented by Fourier-Mukai kernel P_i. Then we define another ring $L(X) := \bigoplus_i \mathrm{Hom}(P_0, P_i)$. In [HR16], the authors conjecture that $L_{\mathrm{MF}}(X) \cong L(X)$. Indeed, by [Pir22], $\mathrm{Hom}(P_0, P_i) \cong \mathrm{Hom}(\mathrm{Id}, (1)^i)$. Now since we have equivalence of the pair $\langle \mathcal{K}u(X), (1)\rangle \simeq^{\phi} \langle \mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, G, \omega), \{1\}\rangle$, namely $(1) \cong \phi^{-1} \circ \{1\} \circ \phi$ by Theorem 2.2. We get

$$\mathrm{Hom}(\mathrm{Id}, (1)^i) \cong \mathrm{Hom}(\mathrm{Id}, \{1\}^i).$$

Then

$$L(X) := \bigoplus \mathrm{Hom}(P_0, P_i) \cong \bigoplus \mathrm{Hom}(Q_0, Q_i) \cong L_{MF}(X).$$

6. Appendix: Categorical Torelli theorem for weighted hypersurfaces

In this section, we illustrate the method used in proof of Theorem 4.1 for a degree 6 hypersurface in weighted projective space $\mathbb{P}(1,1,1,1,3)$, which is isomorphic to an index one prime Fano threefold of genus 2. It is constructed as a double cover of $\mathbb{P}^3$ with branch divisor a sextic hypersurface. *Categorical Torelli theorem* for this case was already established in [LPS23] and [DJR23] via completely different methods. We give a new proof.

Theorem 6.1. *Let X and X' be smooth sextic hypersurfaces in weighted projective space $\mathbb{P}(1,1,1,1,3)$. Assume there is a Fourier-Mukai equivalence $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$, then $X \cong X'$.*

Proof. Consider Matrix Factorization $\mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^5, \mathbb{C}^*, \omega)$, the weight of $\mathbb{C}^*$-action is $(1,1,1,1,3)$. According to [BFK14, Theorem 6.13], we have $\mathcal{K}u(X) \cong \mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^5, \mathbb{C}^*, \omega)$ and $\mathcal{K}u(X') \cong Inj_{coh}(\mathbb{A}^5, \mathbb{C}^*, \omega')$, where ω and ω' are degree 6 polynomial defining X and X' respectively. Then a Fourier-MuKai equivalence $\mathcal{K}u(X) \cong \mathcal{K}u(X')$ induces an equivalence $\mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^5, \mathbb{C}^*, \omega) \cong \mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^5, \mathbb{C}^*, \omega)$ in Hqe(dg-cat). Since $\gcd(\sum_j^{n+1} q_j, d) = \gcd(7,6) = 1$, according to Proposition 3.4 and the same proof in Proposition 3.7, we have isomorphism of algebra,

$$\bigoplus_{t\geq 0} \mathrm{Hom}(\Delta, \Delta(t)) \cong \bigoplus_{t\geq 0} \mathrm{Hom}(\Delta', \Delta'(t)). \tag{1}$$

Then by [BFK14, Theorem 1.2],

$$\mathrm{Hom}(\Delta, \Delta(t)) \cong \Big(\bigoplus_{g\in\mu_6,\ -\mathrm{rk}\, W_g \ \textbf{is even}} \mathrm{Jac}(\omega_g)(t - k_g + 6(\frac{-\mathrm{rk}\, W_g}{2})))\Big)^{\mathbb{C}^*}.$$

Write $\mu_6 = \langle\lambda\rangle$. Then

$$(\mathbb{A}^5)^{\lambda^i} = \begin{cases} (0,0,0,0,0); k_{\lambda^i} = -7; \mathrm{rk}(W_{\lambda^i}) = 5, & if\ i = 1,3,5, \\ (0,0,0,0,x_5); k_{\lambda^i} = -4; \mathrm{rk}(W_{\lambda^i}) = 4, & if\ i = 2,4 \\ \mathbb{A}^5; k_{\lambda^i} = 0; \mathrm{rk}\, W_{\lambda^i} = 0, & if\ i = 6 \end{cases}$$

Write $\omega = x_5^2 + f(x_1,x_2,x_3,x_4)$. Then $\mathrm{Jac}(\omega_{\lambda^4}) = \mathrm{Jac}(\omega_{\lambda^2}) = k[x_5]/\partial x_5^2 = k(0)$, and $\mathrm{Jac}(\omega_{\lambda^1}) = \mathrm{Jac}(\omega_{\lambda^3}) = \mathrm{Jac}(\omega_{\lambda^5}) = \mathrm{Jac}(\omega_{\lambda^6}) = k(0)$. Therefore,

$$\begin{aligned} \mathrm{Hom}(\Delta, \Delta(t)) &\cong \bigoplus_i (\mathrm{Jac}(\omega_{\lambda^i})(t - k_{\lambda^i} + 6(\frac{-\mathrm{rk}\, W_{\lambda^i}}{2})))^{\mathbb{C}^*} \\ &= \mathrm{Jac}(\omega)_t \oplus k(t-8) \oplus k(t-8). \end{aligned}$$

Thus $\mathrm{Hom}(\Delta, \Delta(t)) \cong \mathrm{Jac}(\omega)_t$ for $t \neq 8$. The same for $\mathrm{Hom}(\Delta', \Delta'(t))$, $\omega' = x_5^2 + f'(x_1,x_2,x_3,x_4)$. According to isomorphism (1), we have $k[x_1,x_2,x_3,x_4]_6/\langle\frac{\partial f}{\partial x_i}\rangle_k = \mathrm{Jac}(\omega)_6 \cong \mathrm{Jac}(\omega')_6 = k[x_1,x_2,x_3,x_4]_6/\langle\frac{\partial f'}{\partial x_i}\rangle_k$ induced by autormorphism of degree one polynomials, which implies f is projective equivalent to f'. Thus $X \cong X'$. $\square$

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