TOPOLOGICAL K-THEORY OF QUASI-BPS CATEGORIES OF SYMMETRIC QUIVERS WITH POTENTIAL

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Abstract. In previous work, we studied quasi-BPS categories (of symmetric quivers with potential, of preprojective algebras, of surfaces) and showed they have properties analogous to those of BPS invariants/ cohomologies. For example, quasi-BPS categories are used to formulate categorical analogues of the PBW theorem for cohomological Hall algebras (of Davison–Meinhardt) and of the Donaldson-Thomas/BPS wall-crossing for framed quivers (of Meinhardt–Reineke).

The purpose of this paper is to make the connections between quasi-BPS categories and BPS cohomologies more precise. We compute the topological K-theory of quasi-BPS categories for a large class of symmetric quivers with potential. In particular, we compute the topological K-theory of quasi-BPS categories for a large class of preprojective algebras, which we use (in a different paper) to compute the topological K-theory of quasi-BPS categories of K3 surfaces. A corollary is that there exist quasi-BPS categories with topological K-theory isomorphic to BPS cohomology.

We also compute the topological K-theory of categories of matrix factorizations for smooth affine quotient stacks in terms of the monodromy invariant vanishing cohomology, prove a Grothendieck-Riemann-Roch theorem for matrix factorizations, and check the compatibility between the Koszul equivalence in K-theory and dimensional reduction in cohomology.

1. Introduction

The BPS invariants are integer virtual counts of semistable (compactly supported) coherent sheaves on a smooth complex Calabi-Yau 3-fold. They are fundamental enumerative invariants which determine many other enumerative invariants of interest for Calabi-Yau 3-folds, such as Gromov-Witten, Donaldson-Thomas (DT), or Pandharipande-Thomas invariants [PT14, Section 2 and a half].

Let X be a smooth Calabi-Yau 3-fold, let $v \in H^{\cdot}(X, \mathbb{Z})$, let σ be a stability condition, let $M^{\sigma}_X(v)$ be the good moduli space of σ-semistable (compactly supported) sheaves on X of support v, and let $\Omega^{\sigma}_X(v)$ be the corresponding BPS invariant. An important problem in enumerative algebraic geometry is to define a natural BPS cohomology theory for $M^{\sigma}_X(v)$ which recovers $\Omega^{\sigma}_X(v)$ as its Euler characteristic. Even more, one could attempt to construct a natural dg-category

$$\mathcal{BPS}^{\sigma}_X(v) \tag{1.1}$$

which recovers a 2-periodic version of BPS cohomology (via periodic cyclic homology or topological K-theory [Bla16]), and thus also the BPS invariant $\Omega^{\sigma}_X(v)$. The BPS cohomology, the BPS category (1.1), and the K-theory of (1.1) (BPS K-theory) are alternatives to their classical counterparts for $M^{\sigma}_X(v)$. By dimensional reduction, one also obtains a BPS cohomology/ category/ K-theory for moduli of semistable sheaves on a surface. One may hope that the constructed BPS theories are more tractable than their classical counterparts, and that they will have applications in noncommutative algebraic geometry (by the construction of new spaces of interest, see [PTe, PTd]) and in geometric representation theory (in the study of quantum groups and their representations, see [Soi16, VV]).

Locally, the spaces $M^{\sigma}_X(v)$ can be described as good moduli spaces of representations of certain Jacobi algebras constructed from a symmetric quiver with potential [JS12, Tod18]. One could then attempt to construct the BPS category (1.1) in two steps: first, construct a BPS category for symmetric quivers with potential; second, glue these categories and obtain (1.1).

The purpose of this paper is to complete the first step for a large class of symmetric quivers $Q=(I,E)$ with a potential W. There are natural candidates for the category (1.1): the quasi-BPS categories (introduced in [Păda])

$$\mathbb{S}(d)_v \subset \mathrm{MF}(\mathcal{X}(d), \operatorname{Tr} W) \tag{1.2}$$

which share numerous properties analogous to BPS invariants/ cohomology. Here, $d=(d^i)_{i\in I}\in\mathbb{N}^I$ is a dimension vector, $v\in\mathbb{Z}$ is a weight, $\mathcal{X}(d)=R(d)/G(d)$ is the stack of dimension d representations of Q, and the potential W induces a regular function

$$\operatorname{Tr} W\colon \mathcal{X}(d)\to\mathbb{C}.$$

The BPS cohomology of (Q,W) is the cohomology of a perverse sheaf

$$\mathcal{BPS}_d\in \operatorname{Perv}(X(d)),$$

where $X(d)$ is the good moduli space of $\mathcal{X}(d)$. Let $\underline{d}:=\sum_{i\in I} d^i$ be the total dimension. The following is a corollary of the main theorem of this paper.

Theorem 1.1. (Corollary 6.4) *Let $Q=(I,E)$ be a symmetric quiver, let $d\in\mathbb{N}^I$ be a dimension vector, and let W be a quasi-homogeneous potential of Q. Assume $v\in\mathbb{Z}$ is coprime with $\underline{d}$ and let $i\in\mathbb{Z}$. Then*

$$\dim_{\mathbb{Q}} K^{\mathrm{top}}_i(\mathbb{S}(d)_v)\leqslant\sum_{j\in\mathbb{Z}}\dim_{\mathbb{Q}} H^j(X(d),\mathcal{BPS}_d)^{\mathrm{inv}}. \tag{1.3}$$

If Q has an even number of edges between any two different vertices and an odd number of loops at every vertex, then equality holds in (1.3).

The moduli of sheaves on Calabi-Yau 2-surfaces is locally described by the moduli of representations of a preprojective algebra of a quiver. We show that quasi-BPS categories for preprojective algebras are a categorification of preprojective BPS cohomology. In [PTd], we use local results proved in this paper (about étale covers of moduli of representations of preprojective algebras) to construct natural categorifications (1.1) of the BPS invariants for local Calabi-Yau 3-folds $X=S\times\mathbb{C}$, where S is a K3 surface.

1.1. **Cohomological DT theory.** We briefly review cohomological DT theory. If there are no strictly σ-semistable sheaves of support v, then $\Omega^{\sigma}_X(v)$ equals the Donaldson-Thomas invariant $\mathrm{DT}^{\sigma}_X(v)$ [Tho00]. Joyce-Song [JS12] introduced a perverse sheaf φ_{JS} on $M^{\sigma}_X(v)$ whose Euler characteristic recovers the DT invariant:

$$\sum_{j\in\mathbb{Z}}(-1)^j\dim H^j(M^{\sigma}_X(v),\varphi_{\mathrm{JS}})=\mathrm{DT}^{\sigma}_X(v).$$

The cohomology $H^{\cdot}(M^{\sigma}_X(v),\varphi_{\mathrm{JS}})$ has some advantages over [Sze16, Section 7.3], and is more computable [Sze16, Section 6.2] than singular cohomology of $M^{\sigma}_X(v)$.

If there are strictly σ-semistable sheaves, then $\mathrm{DT}^{\sigma}_X(v)$ may not be an integer. It is thus more natural to search for categorifications of the BPS invariant $\Omega^{\sigma}_X(v)$, such as a BPS cohomology theory which recovers $\Omega^{\sigma}_X(v)$ as its Euler characteristic. Davison–Meinhardt [DM20] defined BPS cohomology for all symmetric quivers with potentials (thus for all local models), and Davison–Hennecart–Schlegel Mejia [DHSMb] defined it for $X = \mathrm{Tot}_S K_S$, where S is a Calabi-Yau surface. For a general CY 3-fold, up to the existence of a certain orientation data, the BPS cohomology is defined in [Tod23c, Definition 2.11].

By dimensional reduction, we also regard BPS cohomology as a cohomology theory for good moduli spaces of objects in categories of dimension 2, for example of the good moduli spaces $P(d)$ of the classical truncation of the quasi-smooth stack $\mathcal{P}(d)$ of dimension d representations of the preprojective algebra of Q°. For a quiver Q°, there is a perverse sheaf

$$\mathcal{BPS}^p_d \in \mathrm{Perv}(P(d))$$

whose cohomology is the BPS cohomology of the preprojective algebra Q°.

1.2. **Categorical DT theory.** We are interested in constructing a category (1.1) which recovers (and has analogous properties to) the BPS invariants/ cohomology. If there are no strictly σ-semistable sheaves of support v, such a category will recover the DT invariants by taking the Euler characteristic of its periodic cyclic homology, see [Tod] for a definition for local surfaces and [HHR] for work in progress addressing the general case.

In previous work, we introduced and studied quasi-BPS categories:

- for symmetric quivers with potential (1.2),
- for preprojective algebras, which we denote by

$$\mathbb{T}(d)_v \subset D^b(\mathcal{P}(d)), \tag{1.4}$$

- for points on smooth surfaces [Păd22, PTc], and
- for semistables sheaves on K3 surfaces [PTd].

These categories have analogous properties to BPS cohomology. Indeed, there are semiorthogonal decompositions of the categorical Hall algebras (of symmetric quivers with potential, and thus also of preprojective algebras, or of K3 surfaces) [Păda, Theorem 1.1] and [Păd23, PTe, PTd], or of Donaldson-Thomas categories (of symmetric quivers with potential) [PTa, PTe] in products of quasi-BPS categories. These semiorthogonal decompositions are analogous to the PBW theorem for cohomological Hall algebras [DM20, DHSMb], or of the DT/ BPS wall-crossing of Meinhardt–Reineke for framed quivers [MR19]. For weights $v \in \mathbb{Z}$ as in Theorem 1.1, we proved categorical versions of the Davison support lemma for BPS sheaves [PTb, PTe, PTd]. However, we observed in [PTa] that quasi-BPS categories do not categorify BPS cohomology for every $v \in \mathbb{Z}$.

1.3. **Matrix factorizations and vanishing cycles.** Locally, BPS sheaves are vanishing cycles of IC sheaves of coarse spaces of smooth quotient stacks, see (1.9). We thus first study vanishing cycles for regular function

$$f\colon \mathcal{X} \to \mathbb{C},$$

where $\mathcal{X} = X/G$ is a smooth quotient stack, where G is a reductive group and X is a smooth affine variety.

It is well-known that the category of matrix factorizations $\mathrm{MF}(\mathcal{X}, f)$ is a categorification of vanishing cohomology $H^\cdot(\mathcal{X}, \varphi_f \mathbb{Q}_{\mathcal{X}})$, see [Efi18, BRTV18]. Let T be the monodromy operator and let $\varphi^{\mathrm{inv}}_f \mathbb{Q}_{\mathcal{X}}$ be the cone of the endomorphism $1 - \mathrm{T}$ on

$\varphi_f\mathbb{Q}_{\mathcal{X}}$. Inspired by [Efi18, BRTV18, BD20], we construct a Chern character map for f quasi-homogeneous:

$$\mathrm{ch}\colon K_i^{\mathrm{top}}(\mathrm{MF}(\mathcal{X},f))\to\bigoplus_{j\in\mathbb{Z}}H^{i+2j}(\mathcal{X},\varphi_f^{\mathrm{inv}}\mathbb{Q}_{\mathcal{X}}),\tag{1.5}$$

which is an isomorphism if $\mathcal{X}$ is a variety, see (4.14). We note that the construction of (1.5) is fairly elementary: by the Koszul equivalence and dimensional reduction, both sides are isomorphic to relative theories; under this identification, (1.5) is the Chern character from relative topological K-theory to relative singular cohomology.

The Chern character map (1.5) induces a cycle map on an associated graded of topological K-theory:

$$\mathrm{c}\colon \mathrm{gr}_\ell K_i^{\mathrm{top}}(\mathrm{MF}(\mathcal{X},f))\to H^{2\dim\mathcal{X}(d)-i-2\ell}(\mathcal{X},\varphi_f^{\mathrm{inv}}\mathbb{Q}_{\mathcal{X}}[-2]).\tag{1.6}$$

In Section 4, we discuss functoriality of (1.5) and (1.6), in particular we prove a Grothendieck-Riemann-Roch theorem, see Theorem 4.7.

1.4. **Quasi-BPS categories for symmetric quivers with potential.** We briefly explain the construction of the quasi-BPS categories (1.2).

Consider a symmetric quiver Q with potential W. For any $v\in\mathbb{Z}$, Špenko–Van den Bergh [ŠVdB17] constructed twisted non-commutative resolutions

$$\mathbb{M}(d)_v\subset D^b(\mathcal{X}(d))_v\tag{1.7}$$

of $X(d)$. The category $\mathbb{M}(d)_v$ is generated by certain vector bundles corresponding to lattice points inside a polytope. Then

$$\mathbb{S}(d)_v:=\mathrm{MF}(\mathbb{M}(d)_v,\mathrm{Tr}\,W)\subset\mathrm{MF}(\mathcal{X}(d),\mathrm{Tr}\,W)$$

is the category of matrix factorizations ($\alpha\colon A\rightleftarrows B\colon\beta$), where A,B are direct sums of the generating vector bundles of $\mathbb{M}(d)_v$, and $\alpha\circ\beta$ and $\beta\circ\alpha$ are multiplication by $\mathrm{Tr}\,W$.

For $d=(d^i)_{i\in I}\in\mathbb{N}^I$ and $v\in\mathbb{Z}$, we define a set S^d_v of partitions of d from the combinatorics of the polytope used to define (1.7). For each partition $A\in S^d_v$, there is a corresponding constructible sheaf $\mathcal{BPS}_A$. Let

$$\mathcal{BPS}_{d,v}:=\bigoplus_{A\in S^d_v}\mathcal{BPS}_A.\tag{1.8}$$

If $\gcd(\underline{d},v)=1$ and Q has an even number of edges between any two different vertices and an odd number of loops at every vertex, then S^d_v consists only of the one term partition of d and then

$$\mathcal{BPS}_{d,v}=\mathcal{BPS}_d:=\begin{cases}\varphi_{\mathrm{Tr}\,W}\mathrm{IC}_{X(d)}[-1], & \text{if } R(d)^{\mathrm{st}}\neq\emptyset,\\ 0, & \text{otherwise.}\end{cases}\tag{1.9}$$

There is thus a monodromy action on the cohomology of $\mathcal{BPS}_{d,v}$ induced from the monodromy of vanishing cycles.

Theorem 1.2. (Theorems 6.2 and 6.3) *Let Q be a symmetric quiver, let W be a quasi-homogeneous potential, let $d\in\mathbb{N}^I$, and let $v\in\mathbb{Z}$. For any $i,\ell\in\mathbb{Z}$, there is an injective cycle map induced from* (1.6)*:*

$$\mathrm{c}\colon \mathrm{gr}_\ell K_i^{\mathrm{top}}(\mathbb{S}(d)_v)\hookrightarrow H^{\dim\mathcal{X}(d)-2\ell-i}(X(d),\mathcal{BPS}^{\mathrm{inv}}_{d,v}[-1]).\tag{1.10}$$

If Q has an even number of edges between any two different vertices and an odd number of loops at every vertex, then the map (1.10) *is an isomorphism.*

A first ingredient in the proof of Theorem 1.2 is the explicit computation of the pushforward $\pi_*\mathrm{IC}_{\mathcal{X}(d)}$ as a sum of shifted perverse sheaves, where $\pi\colon \mathcal{X}(d) \to X(d)$ is the good moduli space map, due to Meinhardt–Reineke [MR19] and Davison–Meinhardt [DM20].

The main ingredient in the proof of Theorem 1.2 is the construction of a cycle map from the topological K-theory of quasi-BPS category to BPS cohomology, see Theorem 6.3. We construct coproduct-like maps on the topological K-theory of the quasi-BPS category $K_i^{\mathrm{top}}(\mathbb{S}(d)_v)$ which we use to restrict the image of $\mathrm{gr}_\ell K_i^{\mathrm{top}}(\mathbb{S}(d)_v) \subset \mathrm{gr}_\ell K_i^{\mathrm{top}}(\mathrm{MF}(\mathcal{X}(d), \mathrm{Tr}\, W))$ under (1.6).

Finally, to show that (1.10) is an isomorphism under the hypothesis above, we use a categorification of the (Meinhardt–Reineke) DT/BPS wall-crossing, which we prove in [PTe].

The case of zero potential of Theorem 1.2 is related to the categorification of intersection cohomology for good moduli spaces of smooth symmetric stacks pursued in [Pădb].

Theorem 1.3. (Theorem 6.6 and Corollary 6.7) *Assume Q has an even number of edges between any two different vertices and an odd number of loops at every vertex. Let $d \in \mathbb{N}^I$ and let $v \in \mathbb{Z}$ such that $\gcd(\underline{d}, v) = 1$. Then $K_1^{\mathrm{top}}(\mathbb{M}(d)_v) = 0$ and there is an isomorphism of $\mathbb{Q}$-vector spaces for all $\ell \in \mathbb{Z}$:*

$$\mathrm{c}\colon \mathrm{gr}_\ell K_0^{\mathrm{top}}(\mathbb{M}(d)_v) \xrightarrow{\sim} \mathrm{IH}^{\dim \mathcal{X}(d) - 2\ell - 1}(X(d)).$$

1.5. **Quasi-BPS categories for preprojective algebras.** Theorem 1.2 can be also used, in conjunction with dimensional reduction, to compute the topological K-theory of quasi-BPS categories for preprojective algebras. For a quiver Q°, consider its tripled quiver with potential (Q, W). The subcategory (1.4) is Koszul equivalent [Isi13] to the subcategory of graded matrix factorizations with summands in $\mathbb{M}(d)_v$:

$$\mathbb{S}^{\mathrm{gr}}(d)_v := \mathrm{MF}^{\mathrm{gr}}\left(\mathcal{X}(d), \mathrm{Tr}\, W\right).$$

We define constructible sheaves $\mathcal{BPS}^p_{d,v}$ on $P(d)$ as in (1.8). There is a cycle map

$$\mathrm{c}\colon \mathrm{gr}_\ell G_0^{\mathrm{top}}(\mathcal{P}(d)) \to H^{\mathrm{BM}}_{2\ell}(\mathcal{P}(d)) \tag{1.11}$$

induced from the Chern character map of $\mathcal{P}(d)$.

Theorem 1.4. (Corollary 7.3 and Theorem 7.6) *Let Q° be a quiver, let $d \in \mathbb{N}^I$, and let $v, \ell \in \mathbb{Z}$. Then $K_1^{\mathrm{top}}(\mathbb{T}(d)_v) = 0$ and the cycle map* (1.11) *induces an injective map:*

$$\mathrm{c}\colon \mathrm{gr}_\ell K_0^{\mathrm{top}}(\mathbb{T}(d)_v) \hookrightarrow H^{-2\ell}(P(d), \mathcal{BPS}^p_{d,v}). \tag{1.12}$$

Next, assume that for any two different vertices of Q°, there is an even number of unoriented edges between them. Then the map (1.12) *is an isomorphism.*

The preprojective algebras which locally model the moduli of semistable sheaves on a K3 surface are of quivers Q° with the property that, for any two different vertices, there is an even number of unoriented edges between them. In Section 9, we prove a version of Theorem 1.4 for étale covers of stacks of representations of preprojective algebra of such quivers, which suffices to compute the topological K-theory of quasi-BPS categories of K3 surfaces [PTd].

1.6. **Weight-independence.** We revisit the discussion from Subsection 1.4, but the same observations apply in the setting of Subsection 1.5. Let $Q = (I, E)$ be a quiver with an even number of edges between any two different vertices and an odd number of loops at every vertex, and let W be a quasi-homogeneous potential of Q. Note that there are equivalences, where $k \in \mathbb{Z}$:

$$\mathbb{S}(d)_v \simeq \mathbb{S}(d)_{v+k\underline{d}},\ \mathbb{S}(d)_v \simeq \mathbb{S}(d)^{\mathrm{op}}_{-v}$$

given by tensoring with the kth power of the determinant line bundle and by taking the derived dual, respectively. There are no obvious other relations between $\mathbb{S}(d)_v$ and $\mathbb{S}(d)_{v'}$ for $v, v' \in \mathbb{Z}$. However, by Theorem 1.4, we obtain:

Corollary 1.5. *Let $v, v' \in \mathbb{Z}$ be such that $\gcd(v, \underline{d}) = \gcd(v', \underline{d})$. Let $i \in \mathbb{Z}$. Then there is an equality of dimensions:*

$$\dim_{\mathbb{Q}} K_i^{\mathrm{top}}(\mathbb{S}(d)_v) = \dim_{\mathbb{Q}} K_i^{\mathrm{top}}(\mathbb{S}(d)_{v'}).$$

Note that the statement is reminiscent to the χ-independence phenomenon [MS23], [KK], see especially [KK, Corollary 1.5]. We observed an analogous statement for quasi-BPS categories of K3 surfaces in [PTd]. We do not know whether a stronger categorical statement, or at the level of algebraic K-theory, should hold for quivers with potential, see [PTd, Conjecture 1.4] for a conjecture in the case of K3 surfaces.

It is natural to ask whether one can use a coproduct to define a primitive part $\mathrm{P}K_i^{\mathrm{top}}(\mathbb{S}(d)_v) \subset K_i^{\mathrm{top}}(\mathbb{S}(d)_v)$ of dimension equal to the dimension of the (total) monodromy invariant BPS cohomology, and thus independent of $v \in \mathbb{Z}$. We defined such spaces in the localized equivariant algebraic K-theory for the tripled quiver with potential in [PTb]. We do not pursue this idea further in this paper.

1.7. **Complements.** In Section 3 we review the Chern character, the cycle map, and the (topological) Grothendieck-Riemann-Roch theorem for quotient stacks.

In Section 5, we compare the Koszul equivalence [Isi13] for dg-categories (and its induced isomorphism in K-theory) with the dimensional reduction theorem in cohomology [Dav17]. In particular, we construct a Chern character map from the topological K-theory of a class of graded matrix factorizations to vanishing cohomology.

In Section 8, we discuss some explicit computations of the topological K-theory of quasi-BPS categories. We mention two examples. First, let $g \geqslant 0$. The coarse space of representations of the $2g+1$ loop quiver is the variety of matrix invariants $X(d) = \mathfrak{gl}(d)^{2g+1} /\!\!/ G(d)$. By Theorem 1.3, we obtain a combinatorial formula for the dimensions of the intersection cohomology $\mathrm{IH}^\bullet(X(d))$, which recovers a formula of Reineke [Rei12]. Second, we compute the topological K-theory for quasi-BPS categories of points in $\mathbb{C}^3$, see Proposition 8.11.

It would be interesting to extend the methods in this paper and obtain computations beyond the case of quivers satisfying Assumption 2.1.

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Notation	Description	Location defined
$\omega_{\mathcal{X}}$	dualizing (constructible) sheaf	Subsection 2.2
ϕ_f, ψ_f	vanishing and nearby fibers	Equation (2.3)
$G^{\mathrm{top}}, K^{\mathrm{top}}$	K-homology, topological K-theory (of a stack)	Equation (2.4)
$l\colon \mathcal{X}^{\mathrm{cl}} \to \mathcal{X}$	inclusion of the classical stack	Equation (2.5)
V_n, U_n, S_n	representations (and open and closed subsets) of a group	Subsection (2.5)
κ	Koszul equivalence	Equation (2.14)
j	inclusion of a Koszul stack in a smooth stack	Diagram (2.13)
η	projection from the total space of a vector bundle	Diagram (2.13)
$\mathcal{X}(d) = R(d)/G(d)$	stack of representations of a quiver Q	Subsection 2.9
$\pi_{X,d}\colon \mathcal{X}(d) \to X(d)$	good moduli space map for $\mathcal{X}(d)$	Subsection 2.9
$T(d) \subset G(d)$	maximal torus	Subsection 2.9
$M(d)$, M, $M(d)_{\mathbb{R}}$, $M_{\mathbb{R}}$	weight spaces	Subsection 2.9
χ	(dominant, integral) weight	Subsection 2.9
1_d	identity cocharacter	Subsection 2.9
$\underline{d}$	sum of components of a dimension vector of a quiver	Subsection 2.9
ρ	half the sum of positive roots	Subsection 2.9
τ_d	Weyl-invariant weight with sum of coefficients 1	Subsection 2.9
$\mathcal{X}^f(d)$, $\mathcal{X}^{\alpha f}(d)$	stack of representations of the framed quiver Q^f, $Q^{\alpha f}$	Subsection 2.10
$\mathcal{Y}(d)$	stack of representations of the doubled quiver $Q^{\circ,d}$	Subsection 2.11
$\mathcal{P}(d)$	stack of representations of the preprojective algebra of Q°	Subsection 2.11
$\pi_{P,d}\colon \mathcal{P}(d)^{\mathrm{cl}} \to P(d)$	good moduli space map of $\mathcal{P}(d)^{\mathrm{cl}}$	Subsection 2.11
(Q, W)	tripled quiver with potential	Subsection 2.12
δ_d	Weyl-invariant real weight	Subsection 2.13
λ	cocharacter (usually antidominant) associated to a partition	Subsection 2.13
p_λ, q_λ	maps used to define the Hall product	Subsection 2.13
$\mathbf{W}(d)$	polytope used to define quasi-BPS categories	Equation (2.16)
$\mathbb{M}(d;\delta_d), \mathbb{M}(d)_v$	magic categories for a quiver with potential	Equation (2.19)
$\mathbb{S}(d;\delta_d), \mathbb{S}(d)_v$	quasi-BPS categories for a quiver with potential	Equation (2.21)
$\mathbb{T}(d;\delta_d), \mathbb{T}(d)_v$	preprojective quasi-BPS categories	Equation (2.22)
$\mathcal{X}(d)^{\mathrm{red}}, \mathbb{M}(d;\delta_d)^{\mathrm{red}}$ etc.	the reduced stack, its magic category etc.	Subsection 2.13
$l'\colon \mathcal{P}(d)^{\mathrm{red}} \to \mathcal{P}(d)$	inclusion of the reduced stack	Subsection 2.13
E_ℓ, gr_ℓ	filtration and the associated graded	Equations 3.5, (3.6)
c	the cycle map	Equation (3.6)
$\iota\colon \mathcal{X}_0 \to \mathcal{X}$	inclusion of the derived zero fiber of a regular function	Section 4
T	monodromy of vanishing cycles	Subsection 4.1
φ_f^{inv}	cone of $1 - \mathrm{T}$ on vanishing cycles	Equations (4.8), (4.6)
Λ	exterior algebra	Lemma 4.12
Θ	forget-the-potential map	Equation (5.10)
$\varepsilon_{\lambda,\delta}$	integer used to measure magic categories	Equation (6.1)
S^d_δ, S^d_v	sets of partitions	Subsection 6.1.2
$\mathbf{d}$	partition of d	Subsection 6.1.1
$\mathcal{BPS}_d, \mathcal{BPS}_A, \mathcal{BPS}_{d,\delta}$	BPS sheaves for a quiver with potential	Subsection 6.1.3
$\pi_{\alpha f,d}$	proper map from the variety of stable framed representations	Subsection (6.2)
$\mathrm{P}_A, \mathrm{Q}_A$	constructible complexes	Equation (6.14)
$Q^{\daleth}$	auxiliary quiver	Subsection 6.4
a_λ	addition map at the level of stacks	Subsection 6.5
$\mathcal{X}(d)'^{\lambda}$	auxiliary stack used to define a coproduct-like map	Subsection 6.5
$w_{\lambda,\delta}$	width of a magic category	Equation (6.28)
Δ_λ	coproduct-like maps	Equation (6.36)
$\mathcal{BPS}^p_d$	preprojective BPS sheaves	Subsection 7.1
L, $\mathcal{L}$	étale covers of $P(d)$, $\mathcal{P}(d)$	Subsection 9.1
F, $\mathcal{F}$	étale covers of $X(d)$, $\mathcal{X}(d)$	Subsection 9.1
$\mathcal{BPS}^L$, $\mathcal{BPS}^L_{d,v}$	BPS sheaves for L	Equation (9.1)
$\mathcal{BPS}^F$, $\mathcal{BPS}^F_{d,v}$	BPS sheaves for F	Subsection 9.3

FIGURE 1. Notation introduced in the paper

2. Preliminaries

For a $\mathbb{Z}$-graded space $V = \bigoplus_{j\in\mathbb{Z}} V^j$, let $\widetilde{V}^i := \prod_{j\in\mathbb{Z}} V^{i+2j}$. For a set S, let $\#S$ be the cardinal of S. We list the main notation used in the paper in Table (1).

2.1. **Stacks and semiorthogonal decompositions.** The spaces $\mathcal{X} = X/G$ considered are quasi-smooth (derived) quotient stacks over $\mathbb{C}$, where G is a reductive group. The classical truncation of $\mathcal{X}$ is denoted by $\mathcal{X}^{\mathrm{cl}} = X^{\mathrm{cl}}/G$. We assume that X^{cl} is quasi-projective. We denote by $\mathbb{L}_{\mathcal{X}}$ the cotangent complex of $\mathcal{X}$. For G a reductive group and X a dg-scheme with an action of G, denote by X/G the corresponding quotient stack. When X is affine, we denote by $X/\!\!/G$ the quotient dg-scheme with dg-ring of regular functions $\mathcal{O}_X^G$.

We will consider semiorthogonal decompositions

$$D^b(\mathcal{X}) = \langle \mathbb{A}_i \mid i \in I \rangle, \tag{2.1}$$

where I is a partially ordered set. Consider a morphism $\pi\colon \mathcal{X} \to S$. We say the semiorthogonal decompositions (2.1) is *S-linear* if $\mathbb{A}_i \otimes \pi^*\mathrm{Perf}(S) \subset \mathbb{A}_i$ for all $i \in I$.

Same as in the papers [PTd, PTe], we use the terminology of *good moduli spaces* of Alper, see [Alp13, Example 8.3].

2.2. **Constructible sheaves.** For $\mathcal{X} = X/G$ a quotient stack, denote by $D^b_{\mathrm{con}}(\mathcal{X})$ the category of bounded complexes of constructible sheaves on $\mathcal{X}$, see [Ols07], and by $\mathrm{Perv}(\mathcal{X}) \subset D^b_{\mathrm{con}}(\mathcal{X})$ the abelian category of perverse sheaves on $\mathcal{X}$, see [LO09]. We denote by

$${}^p\tau^{\leqslant \bullet}\colon D^b_{\mathrm{con}}(\mathcal{X}) \to D^b_{\mathrm{con}}(\mathcal{X})$$

the truncation functors with respect to the perverse t-structure and by

$${}^p\mathcal{H}^\bullet\colon D^b_{\mathrm{con}}(\mathcal{X}) \to \mathrm{Perv}(\mathcal{X})$$

the perverse cohomology sheaves. For $F \in D^b_{\mathrm{con}}(\mathcal{X})$, consider its total perverse cohomology:

$${}^p\mathcal{H}^{\cdot}(F) := \bigoplus_{i\in\mathbb{Z}} {}^p\mathcal{H}^i(F)[-i].$$

We say $F \in D^b_{\mathrm{con}}(\mathcal{X})$ is *a shifted perverse sheaf in degree ℓ* if $F[\ell] \in \mathrm{Perv}(\mathcal{X})$ and *a shifted perverse sheaf* if there exists $\ell \in \mathbb{Z}$ such that $F[\ell] \in \mathrm{Perv}(\mathcal{X})$.

Let $\mathbb{D}$ denote the Verdier duality functor on a stack $\mathcal{X}$. Let $\omega_{\mathcal{X}} := \mathbb{D}\mathbb{Q}_{\mathcal{X}}$. When $\mathcal{X}$ is a smooth stack, equidimensional of dimension d, then $\omega_{\mathcal{X}} = \mathbb{Q}_{\mathcal{X}}[2d]$.

For $\mathcal{X} = X/G$, denote by $H^i(\mathcal{X}) := H^i(\mathcal{X}, \mathbb{Q}) = H^i_G(X, \mathbb{Q})$ the singular cohomology of $\mathcal{X}$ and by $H^{\mathrm{BM}}_i(\mathcal{X}) = H^{\mathrm{BM}}_i(\mathcal{X}, \mathbb{Q}) = H^{\mathrm{BM}}_{i,G}(X, \mathbb{Q})$ the Borel-Moore homology of $\mathcal{X}$ with rational coefficients.

For $F \in D^b_{\mathrm{con}}(\mathcal{X})$, we use the notation $H^\bullet(\mathcal{X}, F)$ for individual cohomology spaces (that is, for $\bullet$ an arbitrary integer) and $H^{\cdot}(\mathcal{X}, F)$ for the total cohomology $H^{\cdot}(\mathcal{X}, F) := \bigoplus_{i\in\mathbb{Z}} H^i(\mathcal{X}, F)$.

2.3. **Nearby and vanishing cycles.** For $\mathcal{X}$ a smooth quotient stack and

$$f\colon \mathcal{X} \to \mathbb{C} \tag{2.2}$$

a regular function, consider the vanishing and nearby cycle functors:

$$\varphi_f, \psi_f\colon D^b_{\mathrm{con}}(\mathcal{X}) \to D^b_{\mathrm{con}}(\mathcal{X}). \tag{2.3}$$

In this paper, we consider regular functions (2.2) such that 0 is the only critical value, equivalently that $\mathrm{Crit}(f) \subset \mathcal{X}_0 := f^{-1}(0)$.

Note that we consider the pushforward along $\iota\colon \mathcal{X}_0 := f^{-1}(0) \hookrightarrow \mathcal{X}$ of the usual vanishing and nearby functors. There is an exact triangle:

$$\iota_*\iota^*\bullet \to \psi_f\bullet \to \varphi_f\bullet \to \iota_*\iota^*\bullet[1].$$

The functors (2.3) restrict to functors

$$\varphi_f[-1], \psi_f[-1] \colon \mathrm{Perv}(\mathcal{X}) \to \mathrm{Perv}(\mathcal{X}).$$

Further, $\varphi_f[-1]$ and $\psi_f[-1]$ commute with $\mathbb{D}$. We will abuse notation and let $\varphi_f := \varphi_f \mathbb{Q}_{\mathcal{X}}$, $\psi_f := \psi_f \mathbb{Q}_{\mathcal{X}}$, $\varphi_f \mathrm{IC} := \varphi_f \mathrm{IC}_{\mathcal{X}}$, $\psi_f \mathrm{IC} := \psi_f \mathrm{IC}_{\mathcal{X}}$. We may drop f from the notation if there is no danger of confusion. For more details on vanishing cycles on quotient stacks, see [Dav17, Subsection 2.2], [DM20, Proposition 2.13].

2.4. **Topological K-theory.** For a dg-category $\mathcal{D}$, Blanc [Bla16] defined the topological K-theory spectrum

$$K^{\mathrm{top}}(\mathcal{D}).$$

For $i \in \mathbb{Z}$, consider its (rational) homotopy groups, which are $\mathbb{Q}$-vector spaces (we drop $\mathbb{Q}$ from the notation):

$$K_i^{\mathrm{top}}(\mathcal{D}) := K_i^{\mathrm{top}}(\mathcal{D}) \otimes_{\mathbb{Z}} \mathbb{Q} := \pi_i(K^{\mathrm{top}}(\mathcal{D})) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

We have that $K_i^{\mathrm{top}}(\mathcal{D}) \cong K_{i+2}^{\mathrm{top}}(\mathcal{D})$ for every $i \in \mathbb{Z}$ by multiplication with a Bott element, see [Bla16, Definition 1.6]. The topological K-theory spectrum sends exact triangles of dg-categories to exact triangles of spectra [Bla16, Theorem 1.1(c)]. We denote the total topological K-theory of $\mathcal{D}$ by:

$$K_{\cdot}^{\mathrm{top}}(\mathcal{D}) = K_0^{\mathrm{top}}(\mathcal{D}) \oplus K_1^{\mathrm{top}}(\mathcal{D}).$$

Given a filtration (indexed by integers) on $K_i^{\mathrm{top}}(\mathcal{D})$ for some $i \in \mathbb{Z}$, we consider the associated graded pieces $\mathrm{gr}_\bullet K_i^{\mathrm{top}}(\mathcal{D})$ for $\bullet \in \mathbb{Z}$ and we let

$$\mathrm{gr}_{\cdot} K_i(\mathcal{D}) := \bigoplus_{j \in \mathbb{Z}} \mathrm{gr}_j K_i(\mathcal{D}).$$

Consider a quotient stack $\mathcal{X} = X/G$ such that G is reductive and X^{cl} is quasi-projective. Let $M \subset G$ be a compact Lie group such that G is the complexification of M. Denote by $K_\bullet^{\mathrm{top}}(\mathcal{X}) := K_{\bullet,M}^{\mathrm{top}}(X)$ the M-equivariant topological K-theory of X (defined by Atiyah and Segal [Seg68]) and by $G_\bullet^{\mathrm{top}}(\mathcal{X}) := G_{\bullet,M}^{\mathrm{top}}(X)$ the M-equivariant K-homology of X (also referred to as the dual of compactly supported equivariant topological K-theory in the literature, defined by Thomason [Tho88]). We refer to [BFM79] for a brief review of properties of topological K-theory, K-homology of varieties, and Grothendieck-Riemann-Roch theorems for varieties. For references on K-homology, see [BD82] for the non-equivariant case and [HLP20, Subsection 2.1.2] for the equivariant case. By [HLP20, Theorem C and the remark following it], we have that:

$$K_\bullet^{\mathrm{top}}(\mathrm{Perf}(\mathcal{X})) \cong K_\bullet^{\mathrm{top}}(\mathcal{X}),\ K_\bullet^{\mathrm{top}}(D^b\mathrm{Coh}(\mathcal{X})) \cong G_\bullet^{\mathrm{top}}(\mathcal{X}). \tag{2.4}$$

Note that $K_\bullet^{\mathrm{top}}(\mathcal{X}) = G_\bullet^{\mathrm{top}}(\mathcal{X})$ if $\mathcal{X}$ is smooth.

For a quotient stack $\mathcal{X}$, there are Chern character maps for $i \in \mathbb{Z}$:

$$\mathrm{ch} \colon K_i^{\mathrm{top}}(\mathcal{X}) \to \prod_{j \in \mathbb{Z}} H^{i+2j}(\mathcal{X}),\ \mathrm{ch} \colon G_i^{\mathrm{top}}(\mathcal{X}) \to \prod_{j \in \mathbb{Z}} H^{\mathrm{BM}}_{i+2j}(\mathcal{X}).$$

If $\mathcal{X}$ is a scheme, then $\mathcal{X}$ is a finite CW complex, and the above Chern character maps are isomorphisms. The first one is the usual Atiyah-Hirzebruch theorem. The second one follows as the dual of the analogous isomorphism for compactly supported topological K-theory, see [BSG11, Section 3.6 and Section 6] or [BD82, Section 11].

The above Chern characters can be also obtained from the Chern character

$$K_i^{\mathrm{top}} \to \mathrm{HP}_i$$

from topological K-theory to periodic cyclic homology applied to the dg-categories $\mathrm{Perf}(\mathcal{X})$ and $D^b\mathrm{Coh}(\mathcal{X})$, respectively, see [Bla16, Section 4.4].

The Chern character maps are not isomorphisms (in general) for $\mathcal{X}$ a quotient stack. In Section 3.1 we review the approximation of the Chern character of quotient stacks by Chern characters for varieties following Edidin–Graham [EG00].

Note that both cohomology (with coefficients in a constructible sheaf F) and topological K-theory depend only on the underlying classical stack. Let

$$l\colon \mathcal{X}^{\mathrm{cl}} \to \mathcal{X} \tag{2.5}$$

The pushforward functor induces isomorphisms:

$$l_*\colon H^{\mathrm{BM}}_\bullet(\mathcal{X}^{\mathrm{cl}}) \xrightarrow{\sim} H^{\mathrm{BM}}_\bullet(\mathcal{X}),\ l_*\colon G^{\mathrm{top}}_\bullet(\mathcal{X}^{\mathrm{cl}}) \xrightarrow{\sim} G^{\mathrm{top}}_\bullet(\mathcal{X}). \tag{2.6}$$

The pullback functor induces isomorphisms:

$$l^*\colon H^\bullet(\mathcal{X}) \xrightarrow{\sim} H^\bullet(\mathcal{X}^{\mathrm{cl}}),\ l^*\colon K^{\mathrm{top}}_\bullet(\mathcal{X}) \xrightarrow{\sim} K^{\mathrm{top}}_\bullet(\mathcal{X}^{\mathrm{cl}}). \tag{2.7}$$

2.5. **Approximation of stacks by varieties.** In the study of cohomology theories for quotient stacks, it is useful to approximate quotient stacks by varieties. We use the method of Totaro [Tot99], Edidin–Graham [EG00]. We exemplify the method for Borel-Moore homology and singular cohomology, but it can be applied in many other situations (such as equivariant Chow groups, see loc. cit., approximation of the algebraic or topological Chern character, see loc. cit. and Subsection 3.1, or vanishing cohomology, see [DM20, Subsection 2.2]).

Let $\mathcal{X} = X/G$ be a quotient stack with G an algebraic group and X quasi-projective scheme with a G-linearized action. Choose representations $V_n \twoheadrightarrow V_{n-1}$ such that

$$\mathrm{codim}(S_n \text{ in } V_n) \geqslant n,$$

where $S_n \subset V_n$ is the closed set of points with non-trivial stabilizer. Further, we may choose V_n such that, for $U_n := V_n \setminus S_n$, the quotient U_n/G is a scheme [EG98, Lemma 9]. Then the quotient $(X \times U_n)/G$ is also a scheme because X is quasi-projective [EG98, Proposition 23]. For ℓ fixed and for n large enough, there are isomorphisms induced by pullback maps:

$$\begin{aligned}
&H^{\mathrm{BM}}_\ell(\mathcal{X}) \xrightarrow{\sim} H^{\mathrm{BM}}_{\ell+2\dim V_n}((X \times V_n)/G) \xrightarrow{\sim} H^{\mathrm{BM}}_{\ell+2\dim V_n}\left((X \times U_n)/G\right),\\
&H^\ell(\mathcal{X}) \xrightarrow{\sim} H^\ell((X \times V_n)/G) \xrightarrow{\sim} H^\ell\left((X \times U_n)/G\right).
\end{aligned}$$

2.6. **The Grothendieck-Riemann-Roch theorem.** We state the (topological) Grothendieck-Riemann-Roch (GRR) theorem for lci morphisms of (classical and possibly singular) varieties of Baum-Fulton-MacPherson [BFM75, BFM79].

Let $\mathcal{X}$ be a classical quotient stack. Recall that there is an intersection product

$$H^i(\mathcal{X}) \otimes H^{\mathrm{BM}}_j(\mathcal{X}) \to H^{\mathrm{BM}}_{j-i}(\mathcal{X})$$

for all $i, j \in \mathbb{Z}$. Further, if $\mathcal{X}' \hookrightarrow \mathcal{X}$ a closed immersion, consider the topological K-theory and the Borel-Moore homology with closed supports $G^{\mathrm{top}}_{\bullet,\mathcal{X}'}(\mathcal{X}) \cong G^{\mathrm{top}}_\bullet(\mathcal{X}')$ and $H^{\mathrm{BM}}_{\bullet,\mathcal{X}'}(\mathcal{X}) \cong H^{\mathrm{BM}}_\bullet(\mathcal{X}')$.

Theorem 2.1. *Assume $\mathcal{X}$ and $\mathcal{Y}$ are classical quotient stacks and let $f\colon \mathcal{X} \to \mathcal{Y}$ be an lci morphism. Let $\mathcal{X}' \subset \mathcal{X}$ and $\mathcal{Y}' \subset \mathcal{Y}$ be closed quotient substacks.*

(a) Assume that $f^{-1}(\mathcal{Y}') \subset \mathcal{X}'$. Then the following diagram commutes:

$$\begin{array}{ccc} G^{\mathrm{top}}_{\bullet,\mathcal{Y}'}(\mathcal{Y}) & \xrightarrow{f^*} & G^{\mathrm{top}}_{\bullet,\mathcal{X}'}(\mathcal{X}) \\ {\scriptstyle \mathrm{ch}_{\mathcal{Y}}}\downarrow & & \downarrow{\scriptstyle \mathrm{ch}_{\mathcal{X}}} \\ H^{\mathrm{BM}}_{\bullet,\mathcal{Y}'}(\mathcal{Y}) & \xrightarrow{f^*} & H^{\mathrm{BM}}_{\bullet,\mathcal{X}'}(\mathcal{X}). \end{array} \tag{2.8}$$

(b) Assume that $f(\mathcal{X}') \subset \mathcal{Y}'$. Let T_f be the virtual tangent bundle of f and consider its Todd class $\mathrm{td}(T_f) \in \widetilde{H}^0(\mathcal{X}) := \prod_{i\geqslant 0} H^{2i}(\mathcal{X})$. Assume f is proper and define $f'_*(-) := f_*(\mathrm{td}(T_f)\cdot(-))$. The following diagrams commute:

$$\begin{array}{ccc} G^{\mathrm{top}}_{\bullet,\mathcal{X}'}(\mathcal{X}) & \xrightarrow{f_*} & G^{\mathrm{top}}_{\bullet,\mathcal{Y}'}(\mathcal{Y}) \\ {\scriptstyle \mathrm{ch}_{\mathcal{X}}}\downarrow & & \downarrow{\scriptstyle \mathrm{ch}_{\mathcal{Y}}} \\ H^{\mathrm{BM}}_{\bullet,\mathcal{X}'}(\mathcal{X}) & \xrightarrow{f'_*} & H^{\mathrm{BM}}_{\bullet,\mathcal{Y}'}(\mathcal{Y}). \end{array} \quad \textit{and} \quad \begin{array}{ccc} K^{\mathrm{top}}_{\bullet}(\mathcal{X}) & \xrightarrow{f_*} & K^{\mathrm{top}}_{\bullet}(\mathcal{Y}) \\ {\scriptstyle \mathrm{ch}_{\mathcal{X}}}\downarrow & & \downarrow{\scriptstyle \mathrm{ch}_{\mathcal{Y}}} \\ H^{\bullet}(\mathcal{X}) & \xrightarrow{f'_*} & H^{\bullet}(\mathcal{Y}). \end{array} \tag{2.9}$$

Proof. (a) There are such pullback (Gysin) functors for any quasi-smooth morphism for Borel-Moore homology [Khab, Construction 3.4] and for derived categories [PS23], and thus for topological K-theory by (2.4). Such maps have also been constructed for lci morphisms of classical schemes in [BFM75, Section 4.4], see also [BFM79, Section 4.2 and 5]. The commutativity of the diagram (2.8) follows from standard properties of the Chern character [BFM79, Section 5].

(b) For f a proper and lci morphism, there are pushforward (Gysin) maps $f_*\colon K^{\mathrm{top}}_{\bullet}(\mathcal{X}) \to K^{\mathrm{top}}_{\bullet}(\mathcal{Y})$ and $f_*\colon H^{\mathrm{BM}}_{\bullet}(\mathcal{X}) \to H^{\mathrm{BM}}_{\bullet}(\mathcal{Y})$, see [BFM75, Section 4.4], [BFM79, Section 5, Remark (2)]. The diagrams commute by the Grothendieck-Riemann-Roch for lci morphisms, see [BFM79, Section 5, Remark (2)] (note that there are typos in the statement of the diagram for $K^{\mathrm{top}}_{\bullet}$, see [Khaa] for a statement in the algebraic case). These are the topological versions of the usual algebraic GRR theorems, see [BFM75, Section 4.3], [Khaa].

Note that Baum-Fulton-MacPherson state the above theorems only for $\bullet = 0$ because they are interested in stating a result for K^{alg}_0 or G^{alg}_0 obtained by composing with the natural map to K^{top}_0 or G^{top}_0, respectively. However, the same proofs (based on deformation to the normal cone and the excess intersection formula) apply for $\bullet = 1$ as well. Finally, note that Baum-Fulton-MacPherson treat the case when $\mathcal{X}$ and $\mathcal{Y}$ are schemes, but the extension to stacks is obtained using the approximation from Subsection 2.5, see also Subsection 3.1. □

2.7. **Matrix factorizations.** We refer to [PTa, Subsection 2.6] for complete definitions and references related to categories of matrix factorizations.

Let $\mathcal{X} = X/G$ be a quotient stack with X affine smooth and G reductive. For a regular function $f\colon \mathcal{X} \to \mathbb{C}$, we denote the corresponding category of matrix factorizations by

$$\mathrm{MF}(\mathcal{X}, f).$$

Its objects are tuples $(\alpha\colon E \rightleftarrows F\colon \beta)$ such that $E, F \in \mathrm{Coh}(\mathcal{X})$ and $\alpha\circ\beta$ and $\beta\circ\alpha$ are multiplication by f. If $\mathbb{M} \subset D^b(\mathcal{X})$ is a subcategory, let

$$\mathrm{MF}(\mathbb{M}, f) \subset \mathrm{MF}(\mathcal{X}, f)$$

the subcategory of totalizations of tuples (E, F, α, β) with $E, F \in \mathbb{M}$. The category $\mathbb{M}$ has a description in terms of categories of singularities [PTa, Subsection 2.6]. In this paper, we consider categories $\mathbb{M}$ generated by a collection $\mathcal{C}$ of vector bundles, then $\mathrm{MF}(\mathbb{M}, f)$ is the category of matrix factorizations with summands E, F which are direct sums of vector bundles in $\mathcal{C}$, see [PTa, Lemma 2.3].

Assume there exists an extra action of $\mathbb{C}^*$ on X which commutes with the action of G on X, and trivial on $\mathbb{Z}/2 \subset \mathbb{C}^*$. Assume that f is weight two with respect to the above $\mathbb{C}^*$-action. Denote by (1) the twist by the character $\mathrm{pr}_2\colon G \times \mathbb{C}^* \to \mathbb{C}^*$. Consider the category of graded matrix factorizations $\mathrm{MF}^{\mathrm{gr}}(\mathcal{X}, f)$. It has objects pairs (P, d_P) with P an equivariant $G\times\mathbb{C}^*$-sheaf on X and $d_P\colon P \to P(1)$ a $G\times\mathbb{C}^*$-equivariant morphism. Note that as the $\mathbb{C}^*$-action is trivial on $\mathbb{Z}/2$, we have the induced action of $\mathbb{C}^\star = \mathbb{C}^*/(\mathbb{Z}/2)$ on X and f is weight one with respect to the above $\mathbb{C}^\star$-action. The objects of $\mathrm{MF}^{\mathrm{gr}}(\mathcal{X}, f)$ can be alternatively described as tuples

$$(E, F, \alpha\colon E \to F(1)', \beta\colon F \to E), \tag{2.10}$$

where E and F are $G \times \mathbb{C}^\star$-equivariant coherent sheaves on X, $(1)'$ is the twist by the character $G \times \mathbb{C}^\star \to \mathbb{C}^\star$, and α and β are $\mathbb{C}^\star$-equivariant morphisms such that $\alpha \circ \beta$ and $\beta \circ \alpha$ are multiplication by f. For a subcategory $\mathbb{M} \subset D^b_{\mathbb{C}^*}(\mathcal{X})$, we define

$$\mathrm{MF}^{\mathrm{gr}}(\mathbb{M}, f) \subset \mathrm{MF}^{\mathrm{gr}}(\mathcal{X}, f)$$

the subcategory of totalizations of tuples (E, F, α, β) with $E, F \in \mathbb{M}$. Alternatively, if $\mathbb{M}$ is generated by a collection of $\mathbb{C}^*$-equivariant vector bundles $\mathcal{C}$, then $\mathrm{MF}^{\mathrm{gr}}(\mathbb{M}, f)$ is the category of matrix factorizations with summands E, F which are direct sums of vector bundles in $\mathcal{C}$.

In this paper, we will consider either ungraded categories of matrix factorizations or graded categories which are Koszul equivalent (see Subsection 2.8) to derived categories of bounded complexes of coherent sheaf on a quasi-smooth stack. When considering the product of two categories of matrix factorizations, which is in the context of the Thom-Sebastiani theorem, we consider the product of dg-categories over $\mathbb{C}(\!(\beta)\!)$ for β of homological degree -2 in the ungraded case, see [Pre, Theorem 4.1.3], and the product of dg-categories over $\mathbb{C}$ in the graded case, see [BFK14, Corollary 5.18] (alternatively in the graded case, one can use the Koszul equivalence).

2.8. **The Koszul equivalence.** Let $\mathcal{X}$ be a smooth quotient stack, let $\eta\colon \mathcal{E} \to \mathcal{X}$ be a rank r vector bundle with a section $s \in \Gamma(\mathcal{X}, \mathcal{E})$, let

$$j\colon \mathcal{K} := s^{-1}(0) \hookrightarrow \mathcal{X} \tag{2.11}$$

be the derived zero locus of s, and let

$$f\colon \mathcal{E}^\vee \to \mathbb{C} \tag{2.12}$$

be the regular function defined by $f(x, v_x) = \langle s(x), v_x\rangle$ for $x \in \mathcal{X}$ and $v_x \in \mathcal{E}^\vee|_x$. Let $\mathcal{E}^\vee_0$ be the derived zero locus of f. We use the following diagram

$$\begin{tikzcd}
\mathcal{E}^\vee|_{\mathcal{K}} \arrow[r, hook, "\tau"] \arrow[d, "\eta'"'] \arrow[rr, hook, bend left, "j'"] & \mathcal{E}^\vee_0 \arrow[r, hook, "\iota"] & \mathcal{E}^\vee \arrow[r, "f"] \arrow[d, "\eta"] & \mathbb{C} \\
\mathcal{K} \arrow[rr, hook, "j"] & & \mathcal{X} &
\end{tikzcd} \tag{2.13}$$

Let $\mathbb{C}^*$ act with weight 2 on the fibers of $\mathcal{E}^\vee$ and consider the corresponding graded category of matrix factorizations $\mathrm{MF}^{\mathrm{gr}}(\mathcal{E}^\vee, f)$. The Koszul equivalence says that [Isi13, Hir17, Tod]:

$$\kappa \colon D^b(\mathcal{K}) \xrightarrow{\sim} \mathrm{MF}^{\mathrm{gr}}(\mathcal{E}^\vee, f). \tag{2.14}$$

Note that κ restricts to an equivalence:

$$\kappa \colon \mathrm{Perf}(\mathcal{K}) \xrightarrow{\sim} \mathrm{MF}^{\mathrm{gr}}_{\mathcal{X}}(\mathcal{E}^\vee, f).$$

Consider the natural closed immersion $l \colon \mathcal{K}^{\mathrm{cl}} \hookrightarrow \mathcal{K}$. The pushforward map induces a weak equivalence $l_* \colon G^{\mathrm{top}}(\mathcal{K}^{\mathrm{cl}}) \xrightarrow{\sim} G^{\mathrm{top}}(\mathcal{K})$. The functor κ has the following explicit description on complexes from the classical stack, see the formula for κ in [Tod, Section 2.3.2].

Proposition 2.2. *The composition*

$$D^b(\mathcal{K}^{\mathrm{cl}}) \xrightarrow{l_*} D^b(\mathcal{K}) \xrightarrow{\kappa} \mathrm{MF}^{\mathrm{gr}}(\mathcal{E}^\vee, f)$$

is isomorphic to the functor $j'_*\eta'^*l_*$ and it induces a weak equivalence

$$j'_*\eta'^*l_* \colon G^{\mathrm{top}}(\mathcal{K}^{\mathrm{cl}}) \xrightarrow{\sim} K^{\mathrm{top}}\left(\mathrm{MF}^{\mathrm{gr}}(\mathcal{E}^\vee, f)\right).$$

There is thus also a weak equivalence:

$$j'_*\eta'^* \colon G^{\mathrm{top}}(\mathcal{K}) \xrightarrow{\sim} K^{\mathrm{top}}\left(\mathrm{MF}^{\mathrm{gr}}(\mathcal{E}^\vee, f)\right).$$

2.9. **Quivers.** Let $Q = (I, E)$ be a quiver and let $d = (d^i)_{i\in I} \in \mathbb{N}^I$ be a dimension vector. Denote by

$$\mathcal{X}(d) = R(d)/G(d)$$

the stack of representations of Q of dimension d, alternatively the stack of representations of dimension d of the path algebra $\mathbb{C}[Q]$. Here $R(d)$, $G(d)$ are given by

$$R(d) = \bigoplus_{(i\to j)\in E} \mathrm{Hom}(V^i, V^j),\ G(d) = \prod_{i\in I} GL(V^i).$$

Consider its good moduli space map:

$$\pi_d := \pi_{X,d} \colon \mathcal{X}(d) \to X(d).$$

We denote by $T(d)$ a maximal torus of $G(d)$, by $M(d)$ the weight lattice of $T(d)$, and by $\mathfrak{g}(d)$ the Lie algebra of $G(d)$. Let $M(d)_{\mathbb{R}} = M(d) \otimes_{\mathbb{Z}} \mathbb{R}$. We drop d from notation when there is no danger of ambiguity.

Let $\mathfrak{S}_a$ be the permutation group on $a \in \mathbb{N}$ letters. Let $W_d := \times_{i\in I}\mathfrak{S}_{d^i}$ be the Weyl group of $G(d)$. For $i \in I$ and $d^i \in \mathbb{N}$, denote by β^i_a for $1 \leqslant a \leqslant d^i$ the weights of the standard representation of $T(d^i)$. Let $M(d)^+_{\mathbb{R}} \subset M(d)_{\mathbb{R}}$ be the dominant chamber consisting of weights

$$\chi = \sum_{i\in I}\sum_{a=1}^{d^i} c^i_a\beta^i_a \text{ such that } c^i_a \geqslant c^i_b \text{ for all } i \in I, d^i \geqslant a \geqslant b \geqslant 1.$$

For $\chi \in M(d)^+$, we denote by $\Gamma_{G(d)}(\chi)$ the irreducible representation of $G(d)$ with highest weight χ. Let ρ_d be half the sum of positive roots of $\mathfrak{g}(d)$. We denote by 1_d the diagonal cocharacter of $T(d)$ (which acts on β^i_a by weight one). For $d = (d^i)_{i\in I}$, denote by $\underline{d} = \sum_{i\in I} d^i$. Define the weights

$$\sigma_d := \sum_{i\in I, 1\leqslant i\leqslant d^i_a} \beta^i_a \in M(d),\ \tau_d := \frac{\sigma_d}{\underline{d}} \in M(d)_{\mathbb{R}}.$$

We denote the cocharacter lattice by $N(d)$. We denote by $\langle\, ,\, \rangle\colon N(d)\times M(d)\to\mathbb{Z}$ the natural pairing, and we use the same notation for its real version. If λ is a cocharacter of $T(d)$ and V is a $T(d)$-representation, we may abuse notation and write

$$\langle\lambda, V\rangle = \langle\lambda, \det(V)\rangle$$

to ease notation.

2.10. **Framed quivers.** Let $Q=(I,E)$ be a quiver. Define the framed quiver $Q^f=(I^f,E^f)$ with vertices $I^f=I\sqcup\{\infty\}$ and edges $E^f=E\sqcup\{e_i\mid i\in I\}$, where e_i is an edge from ∞ to $i\in I$. For $d=(d^i)_{i\in I}\in\mathbb{N}^I$, let $V(d)=\bigoplus_{i\in I}V^i$, where $\dim V^i=d^i$. Denote by

$$R^f(d)=R(d)\oplus V(d)$$

the affine space of representations of Q^f of dimension d and consider the moduli stack of framed representations of Q:

$$\mathcal{X}^f(d):=R^f(d)/G(d).$$

We consider GIT stability on Q^f given by the character $\sigma_{\underline{d}}$. It coincides with the King stability condition on Q^f such that the (semi)stable representations of dimension $(1,d)$ are the representations of Q^f with no subrepresentations of dimension $(1,d')$ for d' different from d, see [Tod, Lemma 5.1.9]. Consider the smooth variety obtained as a GIT quotient, which is a smooth quasi-projective variety:

$$\mathcal{X}^f(d)^{\mathrm{ss}}:=R^f(d)^{\mathrm{ss}}/G(d).$$

2.11. **Double quivers and preprojective algebras.** Let $Q^\circ=(I,E^\circ)$ be a quiver. For an edge e of Q, denote by $\overline{e}$ the edge with opposite orientation. Consider the multiset $E^{\circ,d}=\{e,\overline{e}\mid e\in E\}$. Define *the doubled quiver*

$$Q^{\circ,d}=(I,E^{\circ,d}).$$

Let $\mathcal{I}\subset\mathbb{C}[Q^{\circ,d}]$ be the two-sided ideal generated by $\sum_{e\in E^\circ}[e,\overline{e}]$. The preprojective algebra of Q° is $\Pi_{Q^\circ}:=\mathbb{C}[Q^{\circ,d}]/\mathcal{I}$.

For $d\in\mathbb{N}^I$, recall the stack of representations of dimension d of Q°:

$$\mathcal{X}^\circ(d)=R^\circ(d)/G(d).$$

The stack of representations of $Q^{\circ,d}$ is:

$$\mathcal{Y}(d):=(R^\circ(d)\oplus R^\circ(d)^\vee)/G(d).$$

The stack of representations of the preprojective algebra π_{Q° is:

$$\mathcal{P}(d):=T^*\left(\mathcal{X}^\circ(d)\right):=\mu^{-1}(0)/G(d),$$

where

$$\mu\colon T^*R^\circ(d)=R^\circ(d)\oplus R^\circ(d)^\vee\to\mathfrak{g}(d)^\vee\cong\mathfrak{g}(d)$$

is the moment map and $\mu^{-1}(0)$ is the derived zero of μ. The image of μ lies in the traceless Lie subalgebra $\mathfrak{g}(d)_0\subset\mathfrak{g}(d)$, and thus induces a map $\mu_0\colon T^*R^\circ(d)\to\mathfrak{g}(d)_0$. We define the reduced stack to be

$$\mathcal{P}(d)^{\mathrm{red}}:=\mu_0^{-1}(0)/G(d).$$

Note that $\mathcal{P}(d)^{\mathrm{red,cl}}=\mathcal{P}(d)^{\mathrm{cl}}$. Consider the good moduli space map:

$$\pi_{P,d}\colon\mathcal{P}(d)^{\mathrm{cl}}\to P(d).$$

2.12. **Tripled quivers with potential.** Let $Q^\circ=(I,E^\circ)$ be a quiver and consider its doubled quiver $Q^{\circ,d}=(I,E^{\circ,d})$. *The tripled quiver with potential*

$$(Q,W)$$

is defined as follows. The quiver $Q=(I,E)$ has set of edges $E=E^{\circ,d}\sqcup\{\omega_i\mid i\in I\}$, where ω_i is a loop at the vertex $i\in I$. The potential W is

$$W:=\left(\sum_{i\in I}\omega_i\right)\left(\sum_{e\in E^\circ}[e,\overline{e}]\right)\in\mathbb{C}[Q].$$

We say (Q,W) is a tripled quiver with potential if it is obtained as above for some quiver Q°.

Consider the stack of representations of Q of dimension d:

$$\mathcal{X}(d)=R(d)/G(d)=\left(T^*R^\circ(d)\oplus\mathfrak{g}(d)\right)/G(d)=\big(R^\circ(d)\oplus R^\circ(d)^\vee\oplus\mathfrak{g}(d)\big)/G(d).$$

The potential W induces a regular function:

$$\operatorname{Tr}W\colon\mathcal{X}(d)\to\mathbb{C}.$$

Consider the grading on $\mathcal{X}(d)$ which scales with weight 2 the linear maps corresponding to the loops $\{\omega_i\mid i\in I\}$ and fixes the linear maps in $E^{\circ,d}$. The Koszul equivalence (2.11) says that:

$$\kappa\colon D^b\left(\mathcal{P}(d)\right)\xrightarrow{\sim}\mathrm{MF}^{\mathrm{gr}}\left(\mathcal{X}(d),\operatorname{Tr}W\right). \tag{2.15}$$

2.13. **Quasi-BPS categories.** Consider a symmetric quiver $Q=(I,E)$. Let $d=(d^i)_{i\in I}\in\mathbb{N}^I$ be a dimension vector and let $w\in\mathbb{Z}$ be a weight. Consider the multiset of $T(d)$-weights on $R(d)$:

$$\mathcal{A}:=\{\beta^i_a-\beta^j_b\mid i,j\in I,(i\to j)\in E,1\leqslant a\leqslant d^i,1\leqslant b\leqslant d^j\}.$$

Define the polytope

$$\mathbf{W}(d):=\frac{1}{2}\mathrm{sum}_{\beta\in\mathcal{A}}[0,\beta]\subset M(d)_{\mathbb{R}}, \tag{2.16}$$

where the sums above are Minkowski sums in the space of weights $M(d)_{\mathbb{R}}$. Let λ be an antidominant cocharacter with associated partition $(d_a)_{a=1}^k$ of $d\in\mathbb{N}^I$, meaning that

$$\mathcal{X}(d)^\lambda=\times_{a=1}^k\mathcal{X}(d_a).$$

The multiplication for the categorical Hall algebra of Q, or of (Q,W) for a potential W of Q and a possible grading, is defined as the functor [Păd22]:

$$\begin{aligned}p_{\lambda*}q_\lambda^*\colon&\ \boxtimes_{a=1}^k D^b(\mathcal{X}(d_a))\to D^b(\mathcal{X}(d)),\\ p_{\lambda*}q_\lambda^*\colon&\ \boxtimes_{a=1}^k\mathrm{MF}^\bullet(\mathcal{X}(d_a),\operatorname{Tr}W)\to\mathrm{MF}^\bullet(\mathcal{X}(d),\operatorname{Tr}W),\end{aligned} \tag{2.17}$$

where $\bullet\in\{\emptyset,\mathrm{gr}\}$ and p_λ, q_λ are the maps

$$\mathcal{X}(d)^\lambda=\times_{i=1}^k\mathcal{X}(d_i)\xleftarrow{q_\lambda}\mathcal{X}(d)^{\lambda\geqslant 0}\xrightarrow{p_\lambda}\mathcal{X}(d).$$

Define the sets of weights

$$\begin{aligned}\mathcal{A}_\lambda&:=\{\beta\in\mathcal{A}\mid\langle\lambda,\beta\rangle>0\},\\ \mathfrak{g}_\lambda&:=\{\beta^i_a-\beta^i_b\mid i\in I,1\leqslant a,b\leqslant d^i,\langle\lambda,\beta^i_a-\beta^i_b\rangle>0\}.\end{aligned} \tag{2.18}$$

For a weight $\delta_d\in M(d)_{\mathbb{R}}^{W_d}$, let

$$\mathbb{M}(d;\delta_d)\subset D^b(\mathcal{X}(d)) \tag{2.19}$$

be the full subcategory of $D^b(\mathcal{X}(d))$ generated by vector bundles $\mathcal{O}_{\mathcal{X}(d)} \otimes \Gamma_{G(d)}(\chi)$, where χ is a dominant weight of $G(d)$ such that

$$\chi + \rho - \delta_d \in \mathbf{W}(d).$$

For λ a cocharacter of $T(d)$, define

$$n_\lambda = \left\langle \lambda, \det\left(\mathbb{L}_{\mathcal{X}(d)}|_0^{\lambda>0}\right)\right\rangle = \left\langle \lambda, \det\left((R(d)^\vee)^{\lambda>0}\right)\right\rangle - \left\langle \lambda, \det\left((\mathfrak{g}(d)^\vee)^{\lambda>0}\right)\right\rangle. \tag{2.20}$$

Note that any complex $F \in D^b(B\mathbb{C}^*)$ splits as a direct sum $F = \bigoplus_{w\in\mathbb{Z}} F_w$ such that $\mathbb{C}^*$ acts with weight w on F_w. We say $w \in \mathbb{Z}$ *is a weight of* F if $F_w \neq 0$. The category (2.19) has the following alternative description.

Lemma 2.3. ([PTe, Corollary 3.11]) *The category $\mathbb{M}(d; \delta_d)$ is the subcategory of $D^b(\mathcal{X}(d))$ of objects $F \in D^b(\mathcal{X}(d))$ such that, for any $\nu : B\mathbb{C}^* \to \mathcal{X}(d)$, the weights of ν^*F are contained in the set $\left[-\frac{1}{2}n_\lambda + \langle\lambda, \delta_d\rangle, \frac{1}{2}n_\lambda + \langle\lambda, \delta_d\rangle\right]$. Here ν corresponds to a point $x \in R(d)$ and a cocharacter $\lambda\colon \mathbb{C}^* \to T(d)$ which fixes x.*

Given a potential W for the quiver Q, and possibly a grading as in Subsection 2.7, we define the quasi-BPS categories:

$$\mathbb{S}^\bullet(d; \delta_d) := \mathrm{MF}^\bullet\left(\mathbb{M}(d; \delta_d), \operatorname{Tr} W\right) \text{ for } \bullet \in \{\emptyset, \mathrm{gr}\}. \tag{2.21}$$

If $\delta_d = v\tau_d$, we use the notations:

$$\mathbb{M}(d)_v := \mathbb{M}(d; v\tau_d) \text{ and } \mathbb{S}(d)_v := \mathbb{S}(d; v\tau_d).$$

In the setting of Subsection 2.11, there is a subcategory $\mathbb{T}(d; \delta_d) \subset D^b(\mathcal{P}(d))$ such that, under the Koszul equivalence (2.15), we have that:

$$\kappa\colon \mathbb{T}(d; \delta_d) \xrightarrow{\sim} \mathbb{S}^{\mathrm{gr}}(d; \delta_d), \tag{2.22}$$

see also [PTe, Definition 2.14] for an alternative description of $\mathbb{T}(d; \delta_d)$.

Let $\mathcal{X}(d)^{\mathrm{red}} := (T^*R^\circ(d) \oplus \mathfrak{g}(d)_0)/G(d)$. There is also a Koszul equivalence

$$\kappa'\colon D^b(\mathcal{P}(d)^{\mathrm{red}}) \xrightarrow{\sim} \mathrm{MF}^{\mathrm{gr}}\left(\mathcal{X}(d)^{\mathrm{red}}, \operatorname{Tr} W\right).$$

Define $\mathbb{M}(d; \delta_d)^{\mathrm{red}} \subset D^b\left(\mathcal{X}(d)^{\mathrm{red}}\right)$ as in (2.19), and let $\mathbb{T}(d; \delta_d)^{\mathrm{red}} \subset D^b(\mathcal{P}(d)^{\mathrm{red}})$ be the subcategory such that, under the Koszul equivalence κ', we have that:

$$\kappa'\colon \mathbb{T}(d; \delta_d)^{\mathrm{red}} \xrightarrow{\sim} \mathbb{S}^{\mathrm{gr}}(d; \delta_d)^{\mathrm{red}}.$$

We next discuss the compatibility between reduced and non-reduced quasi-BPS categories. For an isomorphism $\mathfrak{g}(d) \cong \mathfrak{g}(d)_0 \times \mathbb{C}$ of $G(d)$-representation, the projection onto the first factor induces a map $t\colon \mathcal{X}(d) \to \mathcal{X}(d)^{\mathrm{red}}$. We have $t \circ \operatorname{Tr} W = \operatorname{Tr} W$. Let $l'\colon \mathcal{P}(d)^{\mathrm{red}} \hookrightarrow \mathcal{P}(d)$ be the natural closed immersion. The next proposition follows from [Tod, Lemma 2.4.4]:

Proposition 2.4. *The following diagram commutes:*

$$\begin{tikzcd}
D^b(\mathcal{P}(d)^{\mathrm{red}}) \arrow[r, "l'_*"] \arrow[d, "\kappa'", "\wr"'] & D^b(\mathcal{P}(d)) \arrow[d, "\kappa", "\wr"'] \\
\mathrm{MF}^{\mathrm{gr}}(\mathcal{X}(d)^{\mathrm{red}}, \operatorname{Tr} W) \arrow[r, "t^*"] & \mathrm{MF}^{\mathrm{gr}}(\mathcal{X}(d), \operatorname{Tr} W).
\end{tikzcd}$$

It induces a commutative diagram:

$$
\begin{tikzcd}
\mathbb{T}(d;\delta_d)^{\mathrm{red}} \arrow[r, "l'_*"] \arrow[d, "\kappa'", "\wr"'] & \mathbb{T}(d;\delta_d) \arrow[d, "\kappa", "\wr"'] \\
\mathbb{S}^{\mathrm{gr}}(d;\delta_d)^{\mathrm{red}} \arrow[r, "t^*"] & \mathbb{S}^{\mathrm{gr}}(d;\delta_d).
\end{tikzcd}
$$

2.14. **Semiorthogonal decompositions.** We recall several semiorthogonal decompositions from [PTe], see [PTe, Subsection 3.3] for the ordering of summands in all the semiorthogonal decompositions. Recall the convention about the product of categories from Subsection 2.7.

We will consider quivers satisfying the following:

Assumption 2.1. The quiver $Q=(I,E)$ is symmetric and:

- for all $a,b\in I$ different, the number of edges from a to b is even, and
- for all $a\in I$, the number of loops at a is odd.

For $\alpha\in\mathbb{N}$, we define the quiver

$$Q^{\alpha f}=(I^f,E^{\alpha f}),$$

which is a generalization of the framed quiver Q^f. The set of vertices is $I^f=I\sqcup\{\infty\}$, and the set of edges $E^{\alpha f}$ is the disjoint union of E and α edges from ∞ to any vertex of I. Consider the moduli of semistable representations $\mathcal{X}^{\alpha f}(d)^{\mathrm{ss}}$ of the quiver $Q^{\alpha f}$ for the King stability condition σ_d, which is a smooth quasi-projective variety.

Theorem 2.5. ([PTe, Corollary 4.17]) *Let Q be a symmetric quiver satisfying Assumption 2.1. Let $\alpha\in\mathbb{N}$ and $\mu\in\mathbb{R}\setminus\mathbb{Q}$. There is a $X(d)$-linear semiorthogonal decomposition*

$$D^b\left(\mathcal{X}^{\alpha f}(d)^{ss}\right)=\left\langle\bigotimes_{i=1}^k\mathbb{M}(d_i;\theta_i+v_i\tau_{d_i}):\mu\leqslant\frac{v_1}{\underline{d}_1}<\cdots<\frac{v_k}{\underline{d}_k}<\alpha+\mu\right\rangle. \tag{2.23}$$

Here $(d_i)_{i=1}^k$ is a partition of d, $(v_i)_{i=1}^k\in\mathbb{Z}^k$, and $\theta_i\in M(d_i)^{W_{d_i}}$ is defined by

$$\sum_{i=1}^k\theta_i=-\frac{1}{2}R(d)^{\lambda>0}+\frac{1}{2}\mathfrak{g}(d)^{\lambda>0} \tag{2.24}$$

where λ is an antidominant cocharacter corresponding to the partition $(d_i)_{i=1}^k$. The functor from a summand on the right hand side to $D^b\left(\mathcal{X}^f(d)^{ss}\right)$ is the composition of the Hall product with the pullback along the projection map $\mathcal{X}^f(d)\to\mathcal{X}(d)$.

Remark 2.6. Note that there are equivalences

$$\mathbb{M}(d_i)_{v_i}=\mathbb{M}(d_i;v_i\tau_{d_i})\xrightarrow{\sim}\mathbb{M}(d_i;\theta_i+v_i\tau_{d_i}) \tag{2.25}$$

by taking the tensor product with $\theta_i\in M(d_i)^{W_{d_i}}$. Thus the summands in Theorem 2.5 are equivalent to $\bigotimes_{i=1}^k\mathbb{M}(d_i)_{v_i}$.

We next discuss a semiorthogonal decomposition of the stack of representations of Q.

Theorem 2.7. ([PTe, Theorem 4.2]) *Let Q be a symmetric quiver satisfying Assumption 2.1. There is a $X(d)$-linear semiorthogonal decomposition*

$$D^b\left(\mathcal{X}(d)\right)=\left\langle\bigotimes_{i=1}^k\mathbb{M}(d_i)_{v_i}:\frac{v_1}{\underline{d}_1}<\cdots<\frac{v_k}{\underline{d}_k}\right\rangle, \tag{2.26}$$

where $(d_i)_{i=1}^k$ is a partition of d and $(v_i)_{i=1}^k \in \mathbb{Z}^k$. The functor from a summand on the right hand side to $D^b(\mathcal{X}(d))$ is given by the Hall product composed with tensoring with the line bundle $\boxtimes_{i=1}^k \theta_i$, see Remark 2.6.

Using [Păda, Proposition 2.1], [PTa, Proposition 2.5], there are analogous semiorthogonal decompositions in the non-zero potential case constructed from the semiorthogonal decompositions above. The analogue of Theorem 2.5 is the following:

Theorem 2.8. ([PTe, Theorem 4.18]) *Let Q be a symmetric quiver satisfying Assumption 2.1 and let $\alpha \geqslant 1$. Let $\mu \in \mathbb{R}\setminus\mathbb{Q}$. There is a semiorthogonal decomposition*

$$\mathrm{MF}\left(\mathcal{X}^{\alpha f}(d)^{ss}, \operatorname{Tr} W\right) = \left\langle \bigotimes_{i=1}^{k} \mathbb{S}(d_i)_{v_i} : \mu \leqslant \frac{v_1}{\underline{d}_1} < \cdots < \frac{v_k}{\underline{d}_k} < \alpha + \mu \right\rangle,$$

where the right hand side is as in (2.23). If (Q, W) is a tripled quiver with potential, there is an analogous semiorthogonal decomposition of $\mathrm{MF}^{\mathrm{gr}}\left(\mathcal{X}^{\alpha f}(d)^{ss}, \operatorname{Tr} W\right)$ for the grading introduced in Subsection 2.12.

We note the following assumption on a quiver $Q^\circ = (I, E^\circ)$, which says its tripled quiver Q satisfies Assumption 2.1 and thus Theorems 2.7 and 2.8 can be applied for its tripled quiver with potential:

Assumption 2.2. For all $a, b \in I$, we have that

$$\#(a \to b \text{ in } E^\circ) - \#(b \to a \text{ in } E^\circ) \in 2\mathbb{Z}. \tag{2.27}$$

We end with a corollary of [Păda, Theorem 1.1]. We will use it only for quivers Q° satisfying Assumption 2.2, and then the corollary can be also deduced from a version of Theorem 2.7 for an arbitrary $\delta_d \in M(d)_{\mathbb{R}}^{W_d}$ (see [PTe, Theorem 4.2]) using Koszul equivalence and [PTa, Proposition 2.5].

Theorem 2.9. *Let $Q^\circ = (I, E^\circ)$ be a quiver, let $d \in \mathbb{N}^I$, let $\delta_d \in M(d)_{\mathbb{R}}^{W_d}$ with $\langle 1_d, \delta_d \rangle = v$. Recall the quasi-BPS categories from Subsection 2.13. The category $\mathbb{M}(d; \delta_d)$ is right admissible in $D^b(\mathcal{X}(d))_v$, so there is a $X(d)$-linear semiorthogonal decomposition:*

$$D^b(\mathcal{X}(d))_v = \langle \mathbb{B}(d; \delta_d), \mathbb{M}(d; \delta_d) \rangle. \tag{2.28}$$

The category $\mathbb{M}(d; \delta_d)^{\mathrm{red}}$ is right admissible in $D^b(\mathcal{X}(d)^{\mathrm{red}})$.

Applying matrix factorizations and using the Koszul equivalence, the category $\mathbb{T}(d; \delta_d)$ is right admissible in $D^b(\mathcal{P}(d))_v$, so there is a semiorthogonal decomposition:

$$D^b(\mathcal{P}(d))_v = \langle \mathbb{A}(d; \delta_d), \mathbb{T}(d; \delta_d) \rangle. \tag{2.29}$$

The category $\mathbb{T}(d; \delta_d)^{\mathrm{red}}$ is right admissible in $D^b(\mathcal{P}(d)^{\mathrm{red}})_v$.

We note the following:

Corollary 2.10. *Let $Q^\circ = (I, E^\circ)$ be a quiver, let $d \in \mathbb{N}^I$, and let $\delta_d \in M(d)_{\mathbb{R}}^{W_d}$. The closed immersion $l' \colon \mathcal{P}(d)^{\mathrm{red}} \hookrightarrow \mathcal{P}(d)$ induces a weak equivalence of spectra:*

$$l'_* \colon K^{\mathrm{top}}(\mathbb{T}(d; \delta)^{\mathrm{red}}) \xrightarrow{\sim} K^{\mathrm{top}}(\mathbb{T}(d; \delta)).$$

Proof. There is an equivalence of spectra $l'_* \colon G^{\mathrm{top}}(\mathcal{P}(d)^{\mathrm{red}}) \xrightarrow{\sim} G^{\mathrm{top}}(\mathcal{P}(d))$, see the isomorphism (2.6). The claim follows from Proposition 2.4 and Theorem 2.9. □

2.15. **Base-change and semiorthogonal decompositions.** In Section 9, we need to construct semiorthogonal decompositions for étale covers of moduli of representations of a quiver, or for moduli of representations of preprojective algebras. It will be convenient to use the following base-change result for semiorthogonal decompositions, see [Kuz11] for the case of derived categories of varieties.

For a pretriangulated dg-category $\mathcal{D}$ and a subcategory $\mathcal{C} \subset \mathcal{D}$, we say that $\mathcal{D}$ *is classically generated by* $\mathcal{C}$ if the smallest pretriangulated subcategory of $\mathcal{D}$ which contains $\mathcal{C}$ and is closed under direct summands is $\mathcal{D}$.

Proposition 2.11. *Let $\mathcal{X}$ be a QCA (quasi-compact with affine stabilizers) derived stack with a morphism $\pi\colon \mathcal{X} \to S$ to a scheme S. Let*

$$D^b(\mathcal{X}) = \langle \mathcal{C}_i \mid i \in I \rangle$$

be a S-linear semiorthogonal decomposition. Then, for any étale map $f\colon T \to S$ and $f_T\colon \mathcal{X}_T = \mathcal{X} \times_S T \to \mathcal{X}$, there is a semiorthogonal decomposition

$$D^b(\mathcal{X}_T) = \langle \mathcal{C}_{i,T} \mid i \in I \rangle,$$

where $\mathcal{C}_{i,T} \subset D^b(\mathcal{X}_T)$ is the subcategory classically generated by $f_T^*\mathcal{C}_i$.

Proof. The image of $f_T^*\colon \operatorname{Ind} D^b(\mathcal{X}) \to \operatorname{Ind} D^b(\mathcal{X}_T)$ classically generates $\operatorname{Ind} D^b(\mathcal{X}_T)$, as any $A \in \operatorname{Ind} D^b(\mathcal{X}_T)$ is a direct summand of $f_T^* f_{T*} A$. Indeed, consider the diagram:

$$\begin{tikzcd}
\mathcal{X}' := \mathcal{X}_T \times_{\mathcal{X}} \mathcal{X}_T \arrow[r, "g_T"] \arrow[d, "g_T"] & \mathcal{X}_T \arrow[d, "f_T"] \\
\mathcal{X}_T \arrow[r, "f_T"] & \mathcal{X}.
\end{tikzcd}$$

Then $f_T^* f_{T*} A = g_{T*} g_T^* A = A \otimes g_{T*}\mathcal{O}_{\mathcal{X}'}$. The map g_T has a section given by the diagonal map $\Delta\colon \mathcal{X}_T \to \mathcal{X}'$, thus $g_{T*}\mathcal{O}_{\mathcal{X}'}$ has $\mathcal{O}_{\mathcal{X}_T}$ as a direct summand, and so A is indeed a direct summand of $f_T^* f_{T*} A$.

By the QCA assumption, objects in $D^b(\mathcal{X}_T) \subset \operatorname{Ind} D^b(\mathcal{X}_T)$ are compact, see [DG13]. Therefore $D^b(\mathcal{X}_T)$ is classically generated by $f_T^* D^b(\mathcal{X})$, thus by $\mathcal{C}_{i,T}$ for $i \in I$.

To show semiorthogonality, consider $i, j \in I$ such that $\operatorname{Hom}(A_i, A_j) = 0$ for all $A_i \in \mathcal{C}_i$ and $A_j \in \mathcal{C}_j$. We have

$$\begin{aligned}
\operatorname{Hom}_{D^b(\mathcal{X}_T)}(f_T^* A_i, f_T^* A_j) &= \operatorname{Hom}_{\operatorname{Ind} D^b(\mathcal{X})}(A_i, f_{T*} f_T^* A_j) \\
&= \operatorname{Hom}_{\operatorname{Ind} D^b(\mathcal{X})}(A_i, A_j \otimes_{\mathcal{O}_S} f_*\mathcal{O}_T).
\end{aligned} \tag{2.30}$$

Here $f_*\mathcal{O}_S \in D_{\mathrm{qc}}(S) = \operatorname{Ind} \operatorname{Perf}(S)$, and the S-linearity of $\mathcal{C}_j$ implies $A_j \otimes_{\mathcal{O}_S} f_*\mathcal{O}_T \in \operatorname{Ind} \mathcal{C}_j$. Then the vanishing of (2.30) follows from the compactness of A_i (see the end of the proof of [PTd, Lemma 5.5] for how compactness is used). □

3. Topological K-theory of quotient stacks

In this section, we recall the definition of the Chern character maps for quotient stacks and we prove versions of the Atiyah-Hirzebruch theorem for quotient stacks. The main tool we use is the approximation of cohomology theories of quotient stacks by varieties [Tot99, EG00]. We also construct a Chern character map for quasi-smooth quotient stacks and discuss versions of the GRR and Atiyah-Hirzebruch theorems for quasi-smooth morphisms. The results are most probably well known to the experts, but we do not know a reference for them.

3.1. **The Chern character map for a classical quotient stack.** Consider a quotient stack

$$\mathcal{X} = X/G,$$

where G is a connected reductive group and X is a classical quasi-projective scheme with an action of G. Let M be a compact Lie group such that G is the complexification of M. Let EM be a contractible CW complex with a free action of M. For $i \in \mathbb{Z}$, consider the Chern character map of the CW complex $EM \times_M X$:

$$\begin{aligned} \mathrm{ch}\colon G_i^{\mathrm{top}}(\mathcal{X}) &= G_i^{\mathrm{top}}(EM \times_M X) \\ &\to \widetilde{H}_i^{\mathrm{BM}}(EM \times_M X) = \widetilde{H}_i^{\mathrm{BM}}(\mathcal{X}) := \prod_{j\in\mathbb{Z}} H_{i+2j}^{\mathrm{BM}}(\mathcal{X}). \end{aligned} \tag{3.1}$$

The above Chern character map is, in general, neither injective nor surjective. It becomes an isomorphism when the K-theory is completed with respect to an augmentation ideal by theorems of Atiyah-Segal [AS69], and Edidin-Graham [EG00] in the algebraic case.

The Chern character map for a stack can be approximated by the Chern character map for varieties as follows [EG00], see also Subsection 2.5. For V a representation of G, denote by $S \subset V$ the closed set of points with non-trivial stabilizer. Let $U := V \setminus S$. We may choose V such that U/G and $(X \times U)/G$ are schemes. Then the following diagram commutes, where the vertical maps are pullback maps and the bottom map is an isomorphism by the Atiyah-Hirzebruch theorem:

$$\begin{tikzcd} G_i^{\mathrm{top}}(\mathcal{X}) \arrow[r, "\mathrm{ch}"] \arrow[d, "\wr"'] \arrow[dd, bend right=60, "\mathrm{res}_V"'] & \widetilde{H}_i^{\mathrm{BM}}(\mathcal{X}) \arrow[d, "\wr"'] \arrow[dd, bend left=60, "\mathrm{res}_V"] \\ G_i^{\mathrm{top}}((X\times V)/G) \arrow[r, "\mathrm{ch}"] \arrow[d, "\mathrm{res}"] & \widetilde{H}_i^{\mathrm{BM}}((X\times V)/G) \arrow[d, "\mathrm{res}"] \\ G_i^{\mathrm{top}}\left((X\times U)/G\right) \arrow[r, "\mathrm{ch}_V", "\sim"'] & \widetilde{H}_i^{\mathrm{BM}}\left((X\times U)/G\right). \end{tikzcd}$$

Choose representations $V_n \twoheadrightarrow V_{n-1}$ and closed subsets $S_n \subset V_n$ as in Subsection 2.5. For ℓ fixed and for n large enough, recall that we have isomorphisms induced by pullbacks:

$$H_\ell^{\mathrm{BM}}(\mathcal{X}) \xrightarrow{\sim} H_{\ell+2\dim V_n}^{\mathrm{BM}}((X\times V_n)/G) \xrightarrow{\sim} H_{\ell+2\dim V_n}^{\mathrm{BM}}\left((X\times U_n)/G\right).$$

Then $\mathrm{ch}(y)$ for $y \in G_i^{\mathrm{top}}(\mathcal{X})$ equals the limit of $\mathrm{ch}_{V_n}(\mathrm{res}_{V_n}(y))$. Note that, in the algebraic case, Edidin–Graham show in [EG00, Proposition 3.1] that the limit of $\mathrm{ch}_{V_n}(\mathrm{res}_{V_n}(y))$ is well-defined and use it to define the Chern character.

Let $\mathcal{X}' \subset \mathcal{X}$ be a closed quotient stack. There are also Chern character maps with closed supports:

$$\mathrm{ch}_{\mathcal{X}',\mathcal{X}}\colon G_{i,\mathcal{X}'}^{\mathrm{top}}(\mathcal{X}) \to H_{i,\mathcal{X}'}^{\mathrm{BM}}(\mathcal{X}). \tag{3.2}$$

There is also a Chern character map:

$$\mathrm{ch}\colon K_i^{\mathrm{top}}(\mathcal{X}) \to \widetilde{H}^i(\mathcal{X}) := \prod_{j\in\mathbb{Z}} H^{i+2j}(\mathcal{X}), \tag{3.3}$$

where $i \in \mathbb{Z}$. As above, the Chern character map (3.3) can be approximated by Chern character maps of varieties.

The Chern character maps (3.1) and (3.3) are compatible as follows, where ε and ε' are the maps induced by intersecting with the fundamental class, see [BFM79, Section 5 and Property 2 from Section 4.1]:

$$\begin{tikzcd} K_i^{\mathrm{top}}(\mathcal{X}) \arrow[r, "\varepsilon'"] \arrow[d, "\mathrm{ch}"'] & G_i^{\mathrm{top}}(\mathcal{X}) \arrow[d, "\mathrm{ch}"] \\ \widetilde{H}^i(\mathcal{X}) \arrow[r, "\varepsilon"] & \widetilde{H}_i^{\mathrm{BM}}(\mathcal{X}). \end{tikzcd} \tag{3.4}$$

3.2. **An Atiyah-Hirzeburch type theorem for quotient stacks.** We assume $\mathcal{X}$ is a classical quotient stack as in the previous subsection. Let $i \in \mathbb{Z}$. Consider the increasing filtration

$$E_\ell G_i^{\mathrm{top}}(\mathcal{X}) := \mathrm{ch}^{-1}\left(H^{\mathrm{BM}}_{\leqslant i+2\ell}(\mathcal{X})\right) \subset G_i^{\mathrm{top}}(\mathcal{X}). \tag{3.5}$$

Note that $E_\ell G_i^{\mathrm{top}}(\mathcal{X}) = G_i^{\mathrm{top}}(\mathcal{X})$ for ℓ large enough. Denote by

$$\mathrm{gr}_\ell G_i^{\mathrm{top}}(\mathcal{X}) := E_\ell G_i^{\mathrm{top}}(\mathcal{X})/E_{\ell-1}G_i^{\mathrm{top}}(\mathcal{X}).$$

The Chern character induces a map, which we call *the cycle map*:

$$\mathrm{c}\colon \mathrm{gr}_\ell G_i^{\mathrm{top}}(\mathcal{X}) \to H^{\mathrm{BM}}_{i+2\ell}(\mathcal{X}). \tag{3.6}$$

Note that the cycle map is injective by construction. We prove the following version of the Atiyah-Hirzebruch theorem for quotient stacks.

Proposition 3.1. *For $i, \ell \in \mathbb{Z}$, the map* (3.6) *is an isomorphism.*

Proof. Let $i, \ell \in \mathbb{Z}$ and let $a = i + 2\ell$. Let $x \in H_a^{\mathrm{BM}}(\mathcal{X})$. Let V be a representation of G such that $S \subset X \times V$, the locus of points with non-trivial stabilizer, satisfies:

$$\mathrm{codim}\,(S \text{ in } X \times V) > \dim X - \frac{a}{2} + 1.$$

Let $b := \dim V$, $U := V \setminus S$ and let $t\colon (X \times V)/G \to X/G$ be the natural projection. The map t induces an isomorphism

$$t^*\colon H^{\mathrm{BM}}_{i+2\ell}(X/G) \xrightarrow{\sim} H^{\mathrm{BM}}_{i+2\ell+2b}((X \times V)/G).$$

Next, the restriction map α is an isomorphism for $\delta \geqslant \ell + b$.

$$\alpha\colon H^{\mathrm{BM}}_{i+2\delta}((X \times V)/G) \xrightarrow{\sim} H^{\mathrm{BM}}_{i+2\delta}\left((X \times U)/G\right). \tag{3.7}$$

It suffices to check that $H^{\mathrm{BM}}_{i+2\delta-\eta, S/G}((X \times V)/G) \cong H^{\mathrm{BM}}_{i+2\delta-\eta}(S/G) = 0$ for $\eta \in \{0, 1\}$. This is true because $i + 2\delta - \eta > 2\dim S$. Indeed, it suffices to check that $\frac{1}{2}a + b > \dim S + 1$, alternatively that $\mathrm{codim}(S \text{ in } X \times V) > \dim X - \frac{1}{2}a + 1$, which is true by our assumption on V. There is a commutative diagram with rows exact sequences:

$$\begin{tikzcd} G_i^{\mathrm{top}}((X \times V)/G) \arrow[r] \arrow[d, "\mathrm{ch}"] & G_i^{\mathrm{top}}\left((X \times U)/G\right) \arrow[r] \arrow[d, "\mathrm{ch}"] & G^{\mathrm{top}}_{i-1, S/G}((X \times V)/G) \arrow[d, "\mathrm{ch}"] \\ \widetilde{H}_i^{\mathrm{BM}}((X \times V)/G) \arrow[r] \arrow[d, "\beta"] & \widetilde{H}_i^{\mathrm{BM}}\left((X \times U)/G\right) \arrow[r] \arrow[d, "\beta"] & \widetilde{H}^{\mathrm{BM}}_{i-1, S/G}((X \times V)/G) \arrow[d, "\beta"] \\ \prod H^{\mathrm{BM}}_{i+2\delta}((X \times V)/G) \arrow[r, "\alpha", "\sim"'] & \prod H^{\mathrm{BM}}_{i+2\delta}\left((X \times U)/G\right) \arrow[r] & 0. \end{tikzcd}$$

In the above, the products are after $\delta > \ell + b$ and and the maps β are natural projections. The kernels of the maps $\beta \circ \mathrm{ch}$ lie in exact sequences for ℓ and $\ell - 1$, and by their taking their quotient we obtain a diagram:

$$\begin{tikzcd}
\mathrm{gr}_{\ell+b} G_i^{\mathrm{top}}((X \times V)/G) \arrow[r, "\sim"] \arrow[d, hook, "\mathrm{c}"] & \mathrm{gr}_{\ell+b} G_i^{\mathrm{top}}((X \times U)/G) \arrow[d, "\mathrm{c}'", "\wr"'] \\
H^{\mathrm{BM}}_{i+2\ell+2b}((X \times V)/G) \arrow[r, "\alpha", "\sim"'] & H^{\mathrm{BM}}_{i+2\ell+2b}((X \times U)/G).
\end{tikzcd}$$

The map c' is an isomorphism by the Atiyah-Hirzebruch theorem, thus the map c is also an isomorphism, and the claim follows.

□

For $\mathcal{X}$ a quotient stack as above, recall that there is an intersection product $H_i^{\mathrm{BM}}(\mathcal{X}) \otimes H^j(\mathcal{X}) \to H_{i+j}^{\mathrm{BM}}(\mathcal{X})$ for $i, j \in \mathbb{Z}$. Note the following immediate statement.

Proposition 3.2. *Let $\alpha \in \prod_{i \geqslant 0} H^i(\mathcal{X})$ such that $\alpha = 1 + \alpha'$ for $\alpha' \in \prod_{i \geqslant 1} H^i(\mathcal{X})$. Define $\mathrm{ch}'(-) := \mathrm{ch}(-) \cdot \alpha$. Then ch' induces a function on the associated graded pieces $\mathrm{gr}_\ell G_i^{\mathrm{top}}(\mathcal{X})$, and this function equals the cycle map* (3.6).

3.3. **The Chern character map for quasi-smooth stacks.** Assume $\mathcal{X} = X/G$ is a quotient stack with X a quasi-smooth scheme. One can define a Chern character map for $\mathcal{X}$ using the the Chern character map for $\mathcal{X}^{\mathrm{cl}}$ and the isomorphisms (2.6). However, we define a topological Chern character map which takes into account the derived structure.

For a closed immersion $\mathcal{X} \hookrightarrow \mathcal{Y}$, consider the topological K-theory with closed supports $G^{\mathrm{top}}_{\bullet,\mathcal{X}}(\mathcal{Y}) \cong G^{\mathrm{top}}_\bullet(\mathcal{X})$ and the Borel-Moore homology with closed supports $H^{\mathrm{BM}}_{\bullet,\mathcal{X}}(\mathcal{Y}) \cong H^{\mathrm{BM}}_\bullet(\mathcal{X})$.

Let $\mathcal{X}$ be a quasi-smooth quotient stack. Consider a closed immersion $i\colon \mathcal{X} \hookrightarrow \mathcal{Y}$, where $\mathcal{Y}$ is a smooth classical quotient stack. Let N be the normal bundle of i, which is a vector bundle on $\mathcal{X}$ and thus has a Todd class $\mathrm{td}(N) \in \widetilde{H}^0(\mathcal{X})$. Consider the local Chern character map

$$\mathrm{ch}_{\mathcal{X},\mathcal{Y}}\colon G^{\mathrm{top}}_{\bullet,\mathcal{X}}(\mathcal{Y}) \to \widetilde{H}^{\mathrm{BM}}_{\bullet,\mathcal{X}}(\mathcal{Y}).$$

Define

$$\mathrm{ch}_{\mathcal{X}} := \mathrm{ch}_{\mathcal{X},\mathcal{Y}} \cdot \mathrm{td}(N)\colon G^{\mathrm{top}}_\bullet(\mathcal{X}) \to \widetilde{H}^{\mathrm{BM}}_\bullet(\mathcal{X}). \tag{3.8}$$

Lemma 3.3. *The map $\mathrm{ch}_{\mathcal{X}}$ is independent of a choice of $\mathcal{Y}$ as above.*

Proof. Let $i'\colon \mathcal{X} \hookrightarrow \mathcal{Y}'$ be a different closed immersion. Choose $\mathcal{Y}''$ and closed immersions $j\colon \mathcal{Y} \hookrightarrow \mathcal{Y}''$ and $j'\colon \mathcal{Y}' \hookrightarrow \mathcal{Y}''$. Note that the Todd classes for the normal bundles of ji and $j'i'$ are the same. The statement then follows from the GRR theorem for the closed immersions j and j', see Theorem 2.1. □

Remark 3.4. If $\mathcal{X}$ is a classical stack, then the Chern character constructed above coincides with the usual Chern character (3.2), as one can see using the GRR theorem for a closed immersion of $\mathcal{X}$ in a smooth ambient stack.

Similarly, one proves a topological GRR theorem for quasi-smooth morphisms using Theorem 2.1.

Proposition 3.5. *(i) Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a quasi-smooth proper map of quasi-smooth quotient stacks. Let T_f be the virtual tangent bundle and consider the Todd class*

$\mathrm{td}(T_f) \in \widetilde{H}^0(\mathcal{X})$. *Define* $f'_*(-) := f_*(\mathrm{td}(T_f)\cdot(-))$. *Then the following diagram commutes:*

$$\begin{tikzcd} G^{\mathrm{top}}_\bullet(\mathcal{X}) \arrow[r, "f_*"] \arrow[d, "\mathrm{ch}_{\mathcal{X}}"'] & G^{\mathrm{top}}_\bullet(\mathcal{Y}) \arrow[d, "\mathrm{ch}_{\mathcal{Y}}"] \\ \widetilde{H}^{\mathrm{BM}}_\bullet(\mathcal{X}) \arrow[r, "f'_*"] & \widetilde{H}^{\mathrm{BM}}_\bullet(\mathcal{Y}). \end{tikzcd} \tag{3.9}$$

(ii) Further, for any smooth morphism $f\colon \mathcal{X}\to\mathcal{Y}$ *between quasi-smooth quotient stacks, the following diagram commutes:*

$$\begin{tikzcd} G^{\mathrm{top}}_\bullet(\mathcal{Y}) \arrow[r, "f^*"] \arrow[d, "\mathrm{ch}_{\mathcal{Y}}"'] & G^{\mathrm{top}}_\bullet(\mathcal{X}) \arrow[d, "\mathrm{ch}_{\mathcal{X}}"] \\ \widetilde{H}^{\mathrm{BM}}_\bullet(\mathcal{Y}) \arrow[r, "f^*"] & \widetilde{H}^{\mathrm{BM}}_\bullet(\mathcal{X}). \end{tikzcd} \tag{3.10}$$

Define a filtration on $G^{\mathrm{top}}_\bullet(\mathcal{X})$ as in (3.5), the associated graded, and a cycle map as in (3.6), which is also an isomorphism by a relative version of Proposition 3.1 and Proposition 3.2:

$$\mathrm{c}\colon \mathrm{gr}_\ell G^{\mathrm{top}}_i(\mathcal{X}) \xrightarrow{\sim} H^{\mathrm{BM}}_{i+2\ell}(\mathcal{X}). \tag{3.11}$$

We have that $\mathrm{td}(T_f) = 1 + x \in \widetilde{H}^0(\mathcal{X})$ for $x \in \prod_{i\geqslant 2} H^i(\mathcal{X})$. We record the following corollary of the diagrams (3.9) and (3.10), see also Proposition 3.2.

Corollary 3.6. *Let* $f\colon \mathcal{X}\to\mathcal{Y}$ *be a quasi-smooth morphism of quasi-smooth quotient stacks of relative dimension* d. *Let* $i, \ell \in \mathbb{Z}$. *If* f *is smooth, then it induces a pullback map:*

$$f^*\colon \mathrm{gr}_\ell G^{\mathrm{top}}_i(\mathcal{Y}_0) \to \mathrm{gr}_{\ell+d} G^{\mathrm{top}}_i(\mathcal{X}_0).$$

If f *is proper, then it induces a pushforward map:*

$$f_*\colon \mathrm{gr}_\ell G^{\mathrm{top}}_i(\mathcal{X}_0) \to \mathrm{gr}_\ell G^{\mathrm{top}}_i(\mathcal{Y}_0).$$

4. Topological K-theory of categories of singularities

In this section, we compute the topological K-theory of categories of matrix factorizations in terms of the monodromy invariant cohomology of vanishing cycles. The results and approach are inspired by work of Efimov [Efi18], Blanc–Robalo–Toën–Vesozzi [BRTV18], and Brown–Dyckerhoff [BD20].

Let $\mathcal{X}$ be a smooth quotient stack, let $f\colon \mathcal{X}\to\mathbb{C}$ be a regular function with 0 the only singular value, let $\iota\colon \mathcal{X}_0 \hookrightarrow \mathcal{X}$ be the (derived) fiber over 0. Let $d := \dim_{\mathbb{C}} \mathcal{X}$. The category of singularities

$$D_{\mathrm{sg}}(\mathcal{X}_0) := D^b\mathrm{Coh}(\mathcal{X}_0)/\mathrm{Perf}(\mathcal{X}_0) \tag{4.1}$$

is equivalent to the category of matrix factorizations [Orl04, EP15]:

$$\mathrm{MF}(\mathcal{X}, f) \xrightarrow{\sim} D_{\mathrm{sg}}(\mathcal{X}_0). \tag{4.2}$$

We denote by $K^{\mathrm{sg}}_\bullet(\mathcal{X}_0)$ the topological K-theory of $D_{\mathrm{sg}}(\mathcal{X}_0)$. From (4.1), there is a long exact sequence of $\mathbb{Q}$-vector spaces:

$$\ldots \to K^{\mathrm{top}}_i(\mathcal{X}_0) \to G^{\mathrm{top}}_i(\mathcal{X}_0) \to K^{\mathrm{sg}}_i(\mathcal{X}_0) \to K^{\mathrm{top}}_{i-1}(\mathcal{X}_0) \to G^{\mathrm{top}}_{i-1}(\mathcal{X}_0) \to \ldots. \tag{4.3}$$

We assume throughout the section that f is *quasi-homogeneous*, that is, that there exists an action of $\mathbb{C}^*$ on $\mathcal{X}$ contracting $\mathcal{X}$ onto $\mathcal{X}_0$ such that f is of weight

$d > 0$ with respect to the action of $\mathbb{C}^*$, or $f = 0$. Note that the function (2.12) is quasi-homogeneous of weight 1 with respect to the weight 1 scaling action on the fibers. Then 0 is the only singular value of f. Further, there is a weak equivalence induced by restriction:

$$K^{\mathrm{top}}(\mathcal{X}) \xrightarrow{\sim} K^{\mathrm{top}}(\mathcal{X}_0).$$

Note that actually all the results in this section hold as long as the isomorphism $K^{\mathrm{top}}(\mathcal{X}) \xrightarrow{\sim} K^{\mathrm{top}}(\mathcal{X}_0)$ holds, and this is used only in the proof of Proposition 4.2.

4.1. **Vanishing cycle cohomology.** We begin by recalling two distinguished triangles relating the vanishing and cycle functors applied to the constant sheaf. A reference is [Mas, Chapter 3], especially [Mas, pages 24-28]. The results in loc. cit. are stated for varieties, but they also hold for quotient stacks as in [DM20, Subsection 2.2]. There is an exact triangle in $D^b_{\mathrm{con}}(\mathcal{X})$:

$$\iota_*\mathbb{Q}_{\mathcal{X}_0}[-1] \to \psi_f[-1] := \psi_f\mathbb{Q}_{\mathcal{X}}[-1] \xrightarrow{\mathrm{can}} \varphi_f[-1] := \varphi_f\mathbb{Q}_{\mathcal{X}}[-1] \to \iota_*\mathbb{Q}_{\mathcal{X}_0}. \tag{4.4}$$

By taking the dual of the above triangle, we obtain the distinguished triangle:

$$\varphi_f[-1] \xrightarrow{\mathrm{var}} \psi_f[-1] \to \iota_*\iota^!\mathbb{Q}_{\mathcal{X}}[1] \to \varphi_f[1]. \tag{4.5}$$

We have that $\mathrm{var}\circ\mathrm{can} = 1 - \mathrm{T}$, where T is the monodromy operator. Consider the map

$$\alpha\colon \mathbb{Q}_{\mathcal{X}_0} = \iota^*\mathbb{Q}_{\mathcal{X}} \to \iota^!\mathbb{Q}_{\mathcal{X}}[2]$$

given by capping with the fundamental class of the quasi-smooth variety $\mathcal{X}_0$. If f is not the zero map, then this is the usual construction. If f is the zero map, then $\mathcal{X}_0 \cong \mathcal{X} \times r$, where $r = \operatorname{Spec}\mathbb{C}[\epsilon]$ for ϵ in homological degree 1. The map α is then the zero map.

Let φ_f^{inv} be the cone of $1 - \mathrm{T}$:

$$\varphi_f \xrightarrow{1-\mathrm{T}} \varphi_f \to \varphi_f^{\mathrm{inv}} \to \varphi_f[1]. \tag{4.6}$$

Consider the diagram, where the rows and the columns are distinguished triangles:

$$\begin{tikzcd}
\iota_*\mathbb{Q}_{\mathcal{X}_0} \arrow[r, "\alpha"] & \iota_*\iota^!\mathbb{Q}_{\mathcal{X}}[2] \arrow[r] & \varphi_f^{\mathrm{inv}} \\
\iota_*\mathbb{Q}_{\mathcal{X}_0} \arrow[r] \arrow[u, "\mathrm{id}"] & \psi_f \arrow[r, "\mathrm{can}"] \arrow[u] & \varphi_f \arrow[u] \\
0 \arrow[r] \arrow[u] & \varphi_f \arrow[r, "\sim"] \arrow[u, "\mathrm{var}"] & \varphi_f. \arrow[u, "1-\mathrm{T}"']
\end{tikzcd} \tag{4.7}$$

In the above diagram, the second row is (4.4), the second column is (4.5), and the third column is (4.6). We obtain that the first row is also a distinguished triangle:

$$\iota_*\mathbb{Q}_{\mathcal{X}_0} \xrightarrow{\alpha} \iota_*\iota^!\mathbb{Q}_{\mathcal{X}}[2] \to \varphi_f^{\mathrm{inv}} \to \iota_*\mathbb{Q}_{\mathcal{X}_0}[1]. \tag{4.8}$$

We will also use later the notations $\varphi_f^{\mathrm{inv}}\,\mathbb{Q}_{\mathcal{X}}$ and $\varphi_f^{\mathrm{inv}}\,\mathrm{IC}_{\mathcal{X}}$ when it is convenient to indicate the ambient space. Denote by

$$\begin{aligned}
H^\bullet(\mathcal{X}, \varphi_f)^{\mathrm{inv}} &:= \ker(1-\mathrm{T}) \subset H^\bullet(\mathcal{X}, \varphi_f),\\
H^\bullet(\mathcal{X}, \varphi_f)_{\mathrm{inv}} &:= H^\bullet(\mathcal{X}, \varphi_f)/\mathrm{image}(1-\mathrm{T}).
\end{aligned}$$

There is a long exact sequence:

$$\begin{aligned}(4.9)\quad \ldots \to H^{2d-i-2}(\mathcal{X}_0) \xrightarrow{\alpha} H^{\mathrm{BM}}_i(\mathcal{X}_0) \to H^{2d-i}(\mathcal{X}, \varphi_f^{\mathrm{inv}}[-2]) = H^{2d-i-2}(\mathcal{X}, \varphi_f^{\mathrm{inv}}) \\ \to H^{2d-i-1}(\mathcal{X}_0) \to H^{\mathrm{BM}}_{i-1}(\mathcal{X}_0) \to \ldots\end{aligned}$$

and there are short exact sequences:

$$(4.10)\qquad 0 \to H^i(\mathcal{X}, \varphi_f)_{\mathrm{inv}} \to H^i(\mathcal{X}, \varphi_f^{\mathrm{inv}}) \to H^{i+1}(\mathcal{X}, \varphi_f)^{\mathrm{inv}} \to 0.$$

We note the following compatibility between K-theory and cohomology. Let $\alpha' \colon \mathrm{Perf}(\mathcal{X}_0) \hookrightarrow D^b(\mathcal{X}_0)$ be the inclusion.

Proposition 4.1. *The following diagram commutes:*

$$\begin{tikzcd} K_i^{\mathrm{top}}(\mathcal{X}) \arrow[r, "\alpha'"] \arrow[d, "\mathrm{ch}"'] & G_i^{\mathrm{top}}(\mathcal{X}) \arrow[d, "\mathrm{ch}"] \\ \widetilde{H}^i(\mathcal{X}) \arrow[r, "\alpha"] & \widetilde{H}^{\mathrm{BM}}_i(\mathcal{X}). \end{tikzcd}$$

Proof. If f is not the zero map, then the diagram above is the same as the diagram (3.4).

If f is zero, then α is zero. We show that α' is also the zero map on topological K-theory. Let $\mathcal{X}_0$ be the derived zero locus of $0 \colon \mathcal{X} \to \mathbb{C}$. Let $r = \mathrm{Spec}\,\mathbb{C}[\epsilon]$ for ϵ of homological degree 1, then $\mathcal{X}_0 \cong \mathcal{X} \times r$. Consider the natural projection $\pi \colon \mathcal{X}_0 = \mathcal{X} \times r \to \mathcal{X}$ and let $l \colon \mathcal{X}_0^{\mathrm{cl}} \cong \mathcal{X} \to \mathcal{X}_0$. Then $\pi^* \colon K_\bullet^{\mathrm{top}}(\mathcal{X}) \xrightarrow{\sim} K_\bullet^{\mathrm{top}}(\mathcal{X}_0)$ and $l_* \colon G_\bullet^{\mathrm{top}}(\mathcal{X}_0) \xrightarrow{\sim} G_\bullet^{\mathrm{top}}(\mathcal{X})$. For any topological vector bundle E on $\mathcal{X}$, there is an isomorphism:

$$\pi^*(E) \cong l_*(E) \oplus l_*(E)[1] \in G_0^{\mathrm{top}}(\mathcal{X}_0),$$

so the conclusion for $i = 0$ holds. A similar computation holds for the suspension of $\mathcal{X}$, so the conclusion also holds for $i = 1$. □

4.2. **Chern character maps for matrix factorizations.** Let X be a smooth affine variety with an action of a reductive group G. Consider the quotient stack $\mathcal{X} = X/G$. Let $f \colon \mathcal{X} \to \mathbb{C}$ be a regular function. The main result of this subsection is the construction of a Chern character map:

$$\mathrm{ch} \colon K_i^{\mathrm{top}}(\mathrm{MF}(\mathcal{X}, f)) \to \widetilde{H}^i(\mathcal{X}, \varphi_f^{\mathrm{inv}}).$$

We may assume that $\mathrm{Crit}(f) \subset \mathcal{X}_0 := f^{-1}(0)$. Further, replacing $\mathcal{X}$ with an open neighborhood of $\mathcal{X}_0$, we may also assume that the pull-back gives a weak equivalence of spectra $K^{\mathrm{top}}(\mathcal{X}) \xrightarrow{\sim} K^{\mathrm{top}}(\mathcal{X}_0)$.

Consider the regular function $\widetilde{f} \colon \mathcal{X} \times \mathbb{C} \to \mathbb{C}$ defined by $\widetilde{f}(x, t) = t \cdot f(x)$ and set

$$F_{\widetilde{f}} = (\widetilde{f})^{-1}(1) \subset \mathcal{X} \times \mathbb{C}^*.$$

For a closed substack $\mathcal{Y} \subset \mathcal{X}$, we denote by $K^{\mathrm{top}}(\mathcal{X}/\mathcal{Y})$ the relative topological K-theory spectra, i.e. the fiber of the map $K^{\mathrm{top}}(\mathcal{X}) \to K^{\mathrm{top}}(\mathcal{Y})$.

Proposition 4.2. *There is a canonical weak equivalence of spectra:*

$$K^{\mathrm{top}}(\mathrm{MF}(\mathcal{X}, f)) \xrightarrow{\sim} K^{\mathrm{top}}(\mathcal{X} \times \mathbb{C}^*/F_{\widetilde{f}}).$$

Proof. We consider graded categories of matrix factorizations of $\mathcal{X}\times\mathbb{C}$, where the grading is given by the $\mathbb{C}^*$-action with weight $(0,2)$. By the Koszul equivalence (2.14) and (4.2), there are equivalences:

$$\begin{tikzcd} \mathrm{Perf}(\mathcal{X}_0) \arrow[r, hook] \arrow[d, "\kappa"', "\simeq"] & D^b(\mathcal{X}_0) \arrow[r] \arrow[d, "\kappa"', "\simeq"] & \mathrm{MF}(\mathcal{X}, f) \arrow[d, "\simeq"] \\ \mathrm{MF}^{\mathrm{gr}}_{\mathcal{X}\times\{0\}}(\mathcal{X}\times\mathbb{C}, \widetilde{f}) \arrow[r, hook] & \mathrm{MF}^{\mathrm{gr}}(\mathcal{X}\times\mathbb{C}, \widetilde{f}) \arrow[r] & \mathrm{MF}^{\mathrm{gr}}(\mathcal{X}\times\mathbb{C}^*, \widetilde{f}). \end{tikzcd} \tag{4.11}$$

In the above diagram, the horizontal sequences are exact sequences of dg-categories and the vertical arrows are equivalences induced by (2.14).

Consider the inclusion $\iota\colon \mathcal{X}_0 \hookrightarrow \mathcal{X}$ and the projection $p\colon \mathcal{X}\times\mathbb{C}\to\mathcal{X}$. Note that $p|_{F_{\widetilde{f}}}\colon F_{\widetilde{f}} \to \mathcal{X}\backslash\mathcal{X}_0$ is an isomorphism. We have the commutative diagram of spectra:

$$\begin{tikzcd} G^{\mathrm{top}}(\mathcal{X}_0) \arrow[r, "\iota_*"] \arrow[d, "\kappa"', "\simeq"] & K^{\mathrm{top}}(\mathcal{X}) \arrow[r] \arrow[d, "p^*"', "\simeq"] & K^{\mathrm{top}}(\mathcal{X}\setminus\mathcal{X}_0) \arrow[d, "p|^*_{F_{\widetilde{f}}}"', "\simeq"] \\ K^{\mathrm{top}}(\mathrm{MF}^{\mathrm{gr}}(\mathcal{X}\times\mathbb{C}, \widetilde{f})) \arrow[r] & K^{\mathrm{top}}(\mathcal{X}\times\mathbb{C}) \arrow[r] & K^{\mathrm{top}}(F_{\widetilde{f}}). \end{tikzcd}$$

The horizontal sequences are exact triangles of spectra, and the vertical arrows are equivalences. Let $i\colon \mathcal{X}\hookrightarrow\mathcal{X}\times\mathbb{C}$ be the inclusion into $\mathcal{X}\times\{0\}$. By Lemma 4.3 below together with the isomorphism $K^{\mathrm{top}}(\mathcal{X})\xrightarrow{\sim} K^{\mathrm{top}}(\mathcal{X}_0)$ (this is the only place where we use that f is quasi-homogeneous), we have the equivalence

$$i_*\colon K^{\mathrm{top}}(\mathcal{X}) \xrightarrow{\sim} K^{\mathrm{top}}(\mathrm{MF}^{\mathrm{gr}}_{\mathcal{X}\times\{0\}}(\mathcal{X}\times\mathbb{C}, \widetilde{f})).$$

Therefore by taking the cofibers of

$$\begin{tikzcd} K^{\mathrm{top}}(\mathrm{MF}^{\mathrm{gr}}_{\mathcal{X}\times\{0\}}(\mathcal{X}\times\mathbb{C}, \widetilde{f})) \arrow[r, "(i_*)^{-1}", "\simeq"'] \arrow[d] & K^{\mathrm{top}}(\mathcal{X}) \arrow[r] \arrow[d, "i_*"'] & 0 \arrow[d] \\ K^{\mathrm{top}}(\mathrm{MF}^{\mathrm{gr}}(\mathcal{X}\times\mathbb{C}, \widetilde{f})) \arrow[r] & K^{\mathrm{top}}(\mathcal{X}\times\mathbb{C}) \arrow[r] & K^{\mathrm{top}}(F_{\widetilde{f}}) \end{tikzcd}$$

we obtain the equivalence

$$K^{\mathrm{top}}(\mathrm{MF}^{\mathrm{gr}}(\mathcal{X}\times\mathbb{C}^*, \widetilde{f})) \xrightarrow{\sim} \mathrm{fib}\big(K^{\mathrm{top}}(\mathcal{X}\times\mathbb{C}^*) \to K^{\mathrm{top}}(F_{\widetilde{f}})\big).$$

Therefore the desired equivalence follows from the right vertical equivalence in (4.11). $\square$

We have used the following lemma:

Lemma 4.3. *The following diagram commutes*

$$\begin{tikzcd} D^b(\mathcal{X}) \arrow[r, "\iota^*"'] \arrow[rr, bend left, "i_*[1]"] & \mathrm{Perf}(\mathcal{X}_0) \arrow[r, "\simeq", "\kappa"'] & \mathrm{MF}^{\mathrm{gr}}_{\mathcal{X}\times\{0\}}(\mathcal{X}\times\mathbb{C}, \widetilde{f}). \end{tikzcd}$$

Proof. The equivalence κ is given by $(-)\otimes_{\mathcal{O}_{\mathcal{X}_0}}\mathcal{K}$ for the Koszul factorization $\mathcal{K}$, see [Tod, Section 2.3.3]:

$$\mathcal{K} = \mathcal{O}_{\mathcal{X}_0}\otimes_{\mathcal{O}_{\mathcal{X}}}\mathcal{O}_{\mathcal{X}\times\mathbb{C}} = \mathcal{O}_{\mathcal{X}}[\varepsilon, t]$$

where $\deg \varepsilon = -1$, $\deg t = 2$, with differential $d_{\mathcal{K}}(\alpha(t)+\beta(t)\varepsilon) = f\beta(t)+t\alpha(t)\varepsilon$. By construction, it commutes with tensor product from $D^b(\mathcal{X})$. Moreover, as an object of $\mathrm{MF}^{\mathrm{gr}}(\mathcal{X}\times\mathbb{C}, \widetilde{f})$, the object $\mathcal{K}$ is isomorphic to $i_*\mathcal{O}_{\mathcal{X}}[1]$, see [BFK14, Proposition 3.20] or [Tod, Equation (2.3.6)]. Therefore the lemma holds. □

We next relate the relative cohomology to the monodromy invariant cohomology of vanishing cycles:

Proposition 4.4. *There are canonical isomorphisms:*

$$H^\bullet(\mathcal{X}\times\mathbb{C}^*/F_{\widetilde{f}}) \cong H^\bullet(\mathcal{X}, \varphi_f^{\mathrm{inv}}[-2]).$$

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\iota_*\mathbb{Q}_{\mathcal{X}_0} \arrow[r] \arrow[d, "0"'] & \psi_f \arrow[r] \arrow[d, "1-\mathrm{T}"'] & \varphi_f \arrow[r] \arrow[d, "1-\mathrm{T}"'] & \iota_*\mathbb{Q}_{\mathcal{X}_0}[1] \arrow[d, "0"'] \\
\iota_*\mathbb{Q}_{\mathcal{X}_0} \arrow[r] & \psi_f \arrow[r] & \varphi_f \arrow[r] & \iota_*\mathbb{Q}_{\mathcal{X}_0}[1],
\end{tikzcd}$$

where horizontal sequences are exact triangles. By taking the fibers of the vertical maps, we obtain the exact triangle

$$\iota_*\mathbb{Q}_{\mathcal{X}_0}\oplus\iota_*\mathbb{Q}_{\mathcal{X}_0}[-1] \to \psi_f^{\mathrm{inv}}[-1] \to \varphi_f^{\mathrm{inv}}[-1] \to \iota_*\mathbb{Q}_{\mathcal{X}_0}[1]\oplus\iota_*\mathbb{Q}_{\mathcal{X}_0}. \tag{4.12}$$

Let $u\colon \mathcal{X}\setminus\mathcal{X}_0 \hookrightarrow \mathcal{X}$. Then $\psi_f^{\mathrm{inv}}[-1] = \iota_*\iota^*u_*u^*\mathbb{Q}_{\mathcal{X}}$, see [Max20, Equation (17)]. We then have that:

$$\mathbb{Q}_{\mathcal{X}_0}\oplus\mathbb{Q}_{\mathcal{X}_0}[-1] = \iota^*p_*\mathbb{Q}_{\mathcal{X}\times\mathbb{C}^*},\ \psi_f^{\mathrm{inv}}[-1] = \iota_*\iota^*u_*u^*\mathbb{Q}_{\mathcal{X}} = \iota_*\iota^*p_*\mathbb{Q}_{F_{\widetilde{f}}}.$$

The first map in (4.12) is identified with $\iota_*\iota^*p_*$ of the natural map $\mathbb{Q}_{\mathcal{X}\times\mathbb{C}^*}\to\mathbb{Q}_{F_{\widetilde{f}}}$. Therefore we obtain the desired isomorphism. □

Consider the Chern character map of relative K-theories:

$$\mathrm{ch}\colon K_i^{\mathrm{top}}(\mathcal{X}\times\mathbb{C}^*/F_{\widetilde{f}}) \to \widetilde{H}^i(\mathcal{X}\times\mathbb{C}^*/F_{\widetilde{f}}). \tag{4.13}$$

Define the Chern character map

$$\mathrm{ch}\colon K_i^{\mathrm{top}}(\mathrm{MF}(\mathcal{X},f)) \to \widetilde{H}^i(\mathcal{X},\varphi_f^{\mathrm{inv}}). \tag{4.14}$$

such that the following diagram commutes, where the horizontal maps are isomorphisms by Propositions 4.2 and 4.4:

$$\begin{tikzcd}
K_i^{\mathrm{top}}(\mathrm{MF}(\mathcal{X},f)) \arrow[r, "\sim"] \arrow[d, "\mathrm{ch}"] & K_i^{\mathrm{top}}(\mathcal{X}\times\mathbb{C}^*/F_{\widetilde{f}}) \arrow[d, "\mathrm{ch}"] \\
\widetilde{H}^i(\mathcal{X},\varphi_f^{\mathrm{inv}}[-2]) \arrow[r, "\sim"] & \widetilde{H}^i(\mathcal{X}\times\mathbb{C}^*/F_{\widetilde{f}}).
\end{tikzcd}$$

Recall that $d := \dim_{\mathbb{C}}\mathcal{X}$. Define the filtration

$$E_\ell K_i^{\mathrm{top}}(\mathrm{MF}(\mathcal{X},f)) := \mathrm{ch}^{-1}\left(H^{\geqslant 2d-i-2\ell}(\mathcal{X},\varphi_f^{\mathrm{inv}}[-2])\right). \tag{4.15}$$

We obtain cycle maps on the associated graded pieces:

$$\mathrm{c}\colon \mathrm{gr}_\ell K_i^{\mathrm{top}}(\mathrm{MF}(\mathcal{X},f)) \to H^{2d-i-2\ell}(\mathcal{X},\varphi_f^{\mathrm{inv}}[-2]). \tag{4.16}$$

Proposition 4.5. *The maps* (4.16) *are isomorphisms for all $i,\ell\in\mathbb{Z}$, and the map* (4.14) *is an isomorphism if $\mathcal{X}$ is a variety.*

Proof. Define a filtration:

$$E_\ell K_i^{\mathrm{top}}(\mathcal{X}\times\mathbb{C}^*/F_{\widetilde{f}}) := \mathrm{ch}^{-1}\left(H^{\geqslant 2d-i-2\ell}(\mathcal{X}\times\mathbb{C}^*/F_{\widetilde{f}})\right)$$

and the cycle maps on the associated graded pieces, which are isomorphisms using the long exact sequence for relative K-theory and Proposition 3.1:

$$\mathrm{c}\colon \mathrm{gr}_\ell K_i^{\mathrm{top}}(\mathcal{X}\times\mathbb{C}^*/F_{\widetilde{f}}) \xrightarrow{\sim} H^{2\dim\mathcal{X}-i-2\ell}(\mathcal{X}\times\mathbb{C}^*/F_{\widetilde{f}}).$$

The conclusions then follow. □

Composing with the inverse of the equivalence (4.2), we also obtain a Chern character:

$$\mathrm{ch}\colon K_i^{\mathrm{sg}}(\mathcal{X}_0)\to \widetilde{H}^i(\mathcal{X},\varphi_f^{\mathrm{inv}}). \tag{4.17}$$

Note the following compatibility of the Chern character maps.

Proposition 4.6. *The following diagram commutes, where the top sequence is* (4.3) *and the bottom sequence is* (4.9)*:*

$$\begin{tikzcd}
\cdots \arrow[r] & G_i^{\mathrm{top}}(\mathcal{X}_0) \arrow[r]\arrow[d,"\mathrm{ch}"] & K_i^{\mathrm{sg}}(\mathcal{X}_0) \arrow[r]\arrow[d,"\mathrm{ch}"] & K_{i-1}^{\mathrm{top}}(\mathcal{X}_0) \arrow[r]\arrow[d,"\mathrm{ch}"] & \cdots \\
\cdots \arrow[r] & \widetilde{H}_i^{\mathrm{BM}}(\mathcal{X}_0) \arrow[r] & \widetilde{H}^i(\mathcal{X},\varphi_f^{\mathrm{inv}}) \arrow[r] & \widetilde{H}^{i-1}(\mathcal{X}_0) \arrow[r] & \cdots .
\end{tikzcd} \tag{4.18}$$

Proof. By the construction of the Chern character (4.17) and the GRR theorem, it suffices to show the following diagram commutes, which is indeed the case:

$$\begin{tikzcd}
K_i^{\mathrm{top}}(\mathcal{X}\times\mathbb{C}/F_{\widetilde{f}}) \arrow[r]\arrow[d,"\mathrm{ch}"] & K_i^{\mathrm{top}}(\mathcal{X}\times\mathbb{C}^*/F_{\widetilde{f}}) \arrow[r]\arrow[d,"\mathrm{ch}"] & K_{i-1,\mathcal{X}}^{\mathrm{top}}(\mathcal{X}\times\mathbb{C}) \arrow[d,"\mathrm{ch}"] \\
\widetilde{H}^i(\mathcal{X}\times\mathbb{C}/F_{\widetilde{f}}) \arrow[r] & \widetilde{H}^i(\mathcal{X}\times\mathbb{C}^*/F_{\widetilde{f}}) \arrow[r] & \widetilde{H}^{i-1}_{\mathcal{X}}(\mathcal{X}\times\mathbb{C}).
\end{tikzcd}$$

□

4.3. **The Grothendieck-Riemann-Roch theorem for matrix factorizations.**

The Grothendieck-Riemann-Roch theorem for relative topological K-theory and cohomology implies the following.

Theorem 4.7. *Let $h\colon \mathcal{X}\to\mathcal{Y}$ be a morphism of smooth quotient stacks. Consider a regular function $f\colon \mathcal{Y}\to\mathbb{C}$, let $g := f\circ h$, and assume that f and g are quasi-homogeneous. Let $i\in\mathbb{Z}$.*

(a) The following diagram commutes:

$$\begin{tikzcd}
K_i^{\mathrm{top}}(\mathrm{MF}(\mathcal{Y},f)) \arrow[r,"h^*"]\arrow[d,"\mathrm{ch}"'] & K_i^{\mathrm{top}}(\mathrm{MF}(\mathcal{X},g)) \arrow[d,"\mathrm{ch}"] \\
\widetilde{H}^i(\mathcal{Y},\varphi_f^{\mathrm{inv}}) \arrow[r,"h^*"] & \widetilde{H}^i(\mathcal{X},\varphi_g^{\mathrm{inv}}).
\end{tikzcd}$$

(b) Assume h is proper. Let $\mathrm{td}(T_h)\in\widetilde{H}^0(\mathcal{X}_0)$ be the Todd class of the virtual tangent bundle T_h of h and let $h'_*(-) := h_*(\mathrm{td}(T_h)\cdot(-))$. Then the following diagram

commutes:

$$\begin{array}{ccc} K_i^{\mathrm{top}}(\mathrm{MF}(\mathcal{X},g)) & \xrightarrow{h_*} & K_i^{\mathrm{top}}(\mathrm{MF}(\mathcal{Y},f)) \\ \Big\downarrow{\scriptstyle \mathrm{ch}} & & \Big\downarrow{\scriptstyle \mathrm{ch}} \\ \widetilde{H}^i(\mathcal{X},\varphi_g^{\mathrm{inv}}) & \xrightarrow{h'_*} & \widetilde{H}^i(\mathcal{Y},\varphi_f^{\mathrm{inv}}). \end{array}$$

Proof. We may assume that f and g have only 0 as a critical value. The equivalence from Proposition 4.2 and the isomorphism from Proposition 4.4 commutes with both h_* and h^*. The Chern character (4.13) commutes with h^*, so part (a) follows. Finally, the topological Grothendieck-Riemann-Roch theorem implies that the following diagram commutes, so part (b) follows as well:

$$\begin{array}{ccc} K_i^{\mathrm{top}}(\mathcal{X}\times\mathbb{C}^*/F_{\widetilde{g}}) & \xrightarrow{h_*} & K_i^{\mathrm{top}}(\mathcal{Y}\times\mathbb{C}^*/F_{\widetilde{f}}) \\ \Big\downarrow{\scriptstyle \mathrm{ch}} & & \Big\downarrow{\scriptstyle \mathrm{ch}} \\ \widetilde{H}^i(\mathcal{X}\times\mathbb{C}^*/F_{\widetilde{g}}) & \xrightarrow{h'_*} & \widetilde{H}^i(\mathcal{Y}\times\mathbb{C}^*/F_{\widetilde{f}}). \end{array}$$

□

We note the following functoriality of graded topological K-theory of categories of singularities.

Proposition 4.8. *Let $h\colon \mathcal{X}\to\mathcal{Y}$ be a morphism of smooth quotient stacks of relative dimension d, let $f\colon \mathcal{Y}\to\mathbb{C}$ be a regular function, let $g:=f\circ h$, and assume that f and g are quasi-homogeneous. Let $\mathcal{X}_0$ and $\mathcal{Y}_0$ be the (derived) zero loci of g and f, respectively. Let $i,\ell\in\mathbb{Z}$. Then h induces a pullback map:*

$$h^*\colon \mathrm{gr}_\ell K_i^{\mathrm{top}}(\mathrm{MF}(\mathcal{Y},f))\to \mathrm{gr}_{\ell+d}K_i^{\mathrm{top}}(\mathrm{MF}(\mathcal{X},g)).$$

If h is proper, then there is a pushforward map:

$$h_*\colon \mathrm{gr}_\ell K_i^{\mathrm{top}}(\mathrm{MF}(\mathcal{X},g))\to \mathrm{gr}_\ell K_i^{\mathrm{top}}(\mathrm{MF}(\mathcal{Y},f)).$$

Proof. The claim follows from Theorem 4.7 and Proposition 3.2. □

For future reference, we also state explicitly the compatibility of the Chern character maps with Knörrer periodicity, which is a particular case of Theorem 4.7.

Corollary 4.9. *Let X be a smooth affine variety with an action of a reductive group, let $\mathcal{X}:=X/G$ and consider a regular function $f\colon\mathcal{X}\to\mathbb{C}$ with only 0 as a critical value. Let U be a finite dimensional representation of G and consider the natural pairing $w\colon U\times U^\vee\to\mathbb{C}$. Let $\mathcal{Y}:=(X\times U\times U^\vee)/G$ and consider the regular function $f+w\colon\mathcal{Y}\to\mathbb{C}$, where f and w are pulled-back from X and $U\times U^\vee$, respectively. Consider the natural maps:*

$$X\xleftarrow{v} X\times U\xhookrightarrow{s} X\times U\times U^\vee$$

where v is the projection and $s(x,u)=(x,u,0)$. Let $\mathrm{ch}':=\mathrm{ch}\cdot\mathrm{td}(T_s)$, where T_s is the relative tangent complex of s. The following diagram commutes:

$$\begin{array}{ccc} K_i^{\mathrm{top}}(\mathrm{MF}(\mathcal{X},f)) & \xrightarrow{s_*v^*} & K_i^{\mathrm{top}}(\mathrm{MF}(\mathcal{Y},f+w)) \\ \Big\downarrow{\scriptstyle \mathrm{ch}'} & & \Big\downarrow{\scriptstyle \mathrm{ch}} \\ \widetilde{H}^i(\mathcal{X},\varphi_f^{\mathrm{inv}}) & \xrightarrow{s_*v^*} & \widetilde{H}^i(\mathcal{Y},\varphi_{f+w}^{\mathrm{inv}}). \end{array}$$

Note that the horizontal maps are isomorphisms by the Thom-Sebastiani theorem, see the proofs of Propositions 6.11 and 6.13. The top horizontal map is called Knörrer periodicity [Orl06, Hir17]

4.4. **Complements.**

4.4.1. *Injectivity of the cycle map.* The Chern characters (3.1), (3.3), or (4.14) may not be injective when $\mathcal{X}$ is a stack. However, they are all isomorphism when $\mathcal{X}$ is a variety. In some cases of interest, we can show that (4.14) is injective for $\mathcal{X}$ a stack using the following propositions.

Proposition 4.10. *Let $\mathcal{X}$ be a smooth quotient stack and let $f\colon \mathcal{X} \to \mathbb{C}$ be a regular function. Let $\mathbb{S}$ be a subcategory of $\mathrm{MF}(\mathcal{X}, f)$. Assume there exists a smooth variety Y and a morphism $r\colon Y \to \mathcal{X}$ such that $r^*\colon \mathbb{S} \to \mathrm{MF}(Y, g)$ is (left or right) admissible, where $g := f \circ r$. Let $i \in \mathbb{Z}$. Then the Chern character*

$$\mathrm{ch}\colon K_i^{\mathrm{top}}(\mathbb{S}) \to K_i^{\mathrm{top}}(\mathrm{MF}(\mathcal{X}, f)) \to \widetilde{H}^i(\mathcal{X}, \varphi_f^{\mathrm{inv}})$$

is injective.

Proof. The pullback map $r^*\colon K_i^{\mathrm{top}}(\mathbb{S}) \hookrightarrow K_i^{\mathrm{top}}(\mathrm{MF}(Y, g))$ is injective. The claim then follows from the diagram:

$$\begin{tikzcd}
K_i^{\mathrm{top}}(\mathbb{S}) \arrow[r] \arrow[rr, hook, bend left] \arrow[dr] & K_i^{\mathrm{top}}(\mathrm{MF}(\mathcal{X}, f)) \arrow[r, "r^*"] \arrow[d, "\mathrm{ch}"'] & K_i^{\mathrm{top}}(\mathrm{MF}(Y, g)) \arrow[d, "\mathrm{ch}", "\wr"'] \\
& \widetilde{H}^i(\mathcal{X}, \varphi_f^{\mathrm{inv}}) \arrow[r, "r^*"] & \widetilde{H}^i(Y, \varphi_g^{\mathrm{inv}}).
\end{tikzcd}$$

□

Proposition 4.11. *Let $\mathcal{X}$ be a smooth quotient stack and let $f\colon \mathcal{X} \to \mathbb{C}$ be a regular function. Assume there is a semiorthogonal decomposition $\mathrm{MF}(\mathcal{X}, f) = \langle \mathbb{B}_i \mid i \in I\rangle$ and a collection of finite subsets $I_n \subset I$ for $n \in \mathbb{N}$ with the following two properties:*

- *for any finite subset $S \subset I$, there exists $n \in \mathbb{N}$ such that $S \subset I_n$,*
- *for all $n \in \mathbb{N}$, there exists a smooth variety Y_n and a morphism $r_n\colon Y_n \to \mathcal{X}$ such that the category $\mathbb{B}^n := \langle \mathbb{B}_i \mid i \in I_n\rangle$ is (left or right) admissible in $\mathrm{MF}(Y_n, f \circ r_n)$ via r_n^*.*

Let $i \in \mathbb{Z}$. Then the Chern character

$$\mathrm{ch}\colon K_i^{\mathrm{top}}(\mathrm{MF}(\mathcal{X}, f)) \to \widetilde{H}^i(\mathcal{X}, \varphi_f^{\mathrm{inv}})$$

is injective.

Proof. Let $x \in K_i^{\mathrm{top}}(\mathrm{MF}(\mathcal{X}, f)) = \bigoplus_{j \in I} K_i^{\mathrm{top}}(\mathbb{B}_j)$. Let $S \subset I$ be a finite set such that $x \in \bigoplus_{j \in S} K_i^{\mathrm{top}}(\mathbb{B}_j)$. Then there exists n such that $x \in K^{\mathrm{top}}(\mathbb{B}^n)$. The Chern character

$$\mathrm{ch}\colon K_i^{\mathrm{top}}(\mathbb{B}^n) \to \widetilde{H}^i(\mathcal{X}, \varphi_f^{\mathrm{inv}})$$

is injective by Proposition 4.10, and the claim follows. □

4.4.2. *Action of exterior algebra on the K-theory of matrix factorizations.* Denote by $p := \operatorname{Spec}\mathbb{C}$. The following computation follows as in Proposition 4.1.

Lemma 4.12. *As a $\mathbb{Z}/2$-algebra, we have*

$$K^{\mathrm{top}}_{\cdot}(\mathrm{MF}(p,0)) = \Lambda := \mathbb{Q}[\epsilon]$$

where ϵ has degree one.

Note that, for any regular function on a smooth stack $h\colon \mathcal{Y} \to \mathbb{C}$, the category $\mathrm{MF}(\mathcal{Y},h)$ is a module over $\mathrm{MF}(p,0)$, so $K^{\mathrm{top}}_{\cdot}(\mathrm{MF}(\mathcal{Y},h))$ is a $\mathbb{Z}/2$-graded Λ-module by Lemma 4.12.

Proposition 4.13. *Let $\mathcal{X}$ be a smooth stack. Then*

$$K^{\mathrm{top}}_{\cdot}(\mathrm{MF}(\mathcal{X},0)) \cong K^{\mathrm{top}}_{\cdot}(\mathcal{X}) \otimes_{\mathbb{Q}} \Lambda$$

as Λ-modules. Then, if $\mathbb{M} \subset D^b(\mathcal{X})$ is an admissible subcategory of $D^b(\mathcal{X})$, there is an isomorphism of Λ-modules:

$$K^{\mathrm{top}}_{\cdot}(\mathrm{MF}(\mathbb{M},0)) \cong K^{\mathrm{top}}_{\cdot}(\mathbb{M}) \otimes_{\mathbb{Q}} \Lambda.$$

Proof. It suffices to prove the first isomorphism. Let $\mathcal{X}_0$ be the derived zero locus of $0\colon \mathcal{X} \to \mathbb{C}$. By the long exact sequence (4.3), it suffices to show that the map $\alpha'\colon \mathrm{Perf}(\mathcal{X}_0) \to D^b(\mathcal{X}_0)$ induces the zero map:

$$\alpha'\colon K^{\mathrm{top}}_{\bullet}(\mathcal{X}_0) \to G^{\mathrm{top}}_{\bullet}(\mathcal{X}_0),$$

which we showed in the proof of Proposition 4.1. $\square$

4.4.3. *The Chern character for the algebraic K-theory of matrix factorizations.* Consider the natural transformation

$$\gamma\colon K_0^{\mathrm{alg}} := K_0 \to K_0^{\mathrm{top}}$$

from algebraic K-theory to topological K-theory [Bla16, Remark 4.14]. For a quotient stack $\mathcal{X} = X/G$, where G is a reductive group acting on a smooth affine scheme X, there is a Chern character:

$$\mathrm{ch}^{\mathrm{alg}}\colon K_0^{\mathrm{alg}}(\mathrm{MF}(\mathcal{X},f)) \xrightarrow{\gamma} K_0^{\mathrm{top}}(\mathrm{MF}(\mathcal{X},f)) \xrightarrow{\mathrm{ch}} \widetilde{H}^0(\mathcal{X},\varphi_f^{\mathrm{inv}}).$$

We next state an algebraic version of the GRR theorem 4.7.

Theorem 4.14. *Let $h\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of smooth quotient stacks. Consider a regular function $f\colon \mathcal{Y} \to \mathbb{C}$, let $g := f \circ h$, and assume that f and g are quasi-homogeneous.*

(a) The following diagram commutes:

$$\begin{tikzcd} K_0^{\mathrm{alg}}(\mathrm{MF}(\mathcal{Y},f)) \arrow[r, "h^*"] \arrow[d, "\mathrm{ch}^{\mathrm{alg}}"'] & K_0^{\mathrm{alg}}(\mathrm{MF}(\mathcal{X},g)) \arrow[d, "\mathrm{ch}^{\mathrm{alg}}"] \\ \widetilde{H}^0(\mathcal{Y},\varphi_f^{\mathrm{inv}}) \arrow[r, "h^*"] & \widetilde{H}^0(\mathcal{X},\varphi_g^{\mathrm{inv}}). \end{tikzcd}$$

(b) Assume h is proper. Let $\mathrm{td}(T_h) \in \widetilde{H}^0(\mathcal{X}_0)$ be the Todd class of the virtual tangent bundle T_h of h, and let $h'_*(-) := h_*(\mathrm{td}(T_h)\cdot(-))$. Then the following

diagram commutes:

$$\begin{array}{ccc} K_0^{\mathrm{alg}}(\mathrm{MF}(\mathcal{X},g)) & \xrightarrow{h_*} & K_0^{\mathrm{alg}}(\mathrm{MF}(\mathcal{Y},f)) \\ {\scriptstyle \mathrm{ch}^{\mathrm{alg}}}\downarrow & & \downarrow{\scriptstyle \mathrm{ch}^{\mathrm{alg}}} \\ \widetilde{H}^0(\mathcal{X},\varphi_g^{\mathrm{inv}}) & \xrightarrow{h'_*} & \widetilde{H}^0(\mathcal{Y},\varphi_f^{\mathrm{inv}}). \end{array}$$

Proof. Both claims follow from Theorem 4.7 and the commutativity of γ with h^* and h_*. □

4.4.4. *Graded and ungraded matrix factorizations.* One can define graded categories of matrix factorizations in more generality than the one used in Subsection 2.7, see below for one example. It is natural to ask for an analogue of Proposition 4.5 for categories of graded matrix factorizations. We do not know how to answer this question for general graded categories, but we study some examples in Section 5.

We mention a theorem of Brown–Dyckerhoff in [BD20, Theorem 1.3] which computes the topological K-theory for a class of graded matrix factorizations not covered by our methods. Let $f\colon \mathbb{C}^n \to \mathbb{C}$ be a homogeneous polynomial of degree d. Let $\mathbb{C}^*$ act on $\mathbb{C}^n$ with weight 1. Consider category $\mathrm{MF}^{\mathrm{gr}}(\mathbb{C}^n, f)$ with objects of the form

$$(\alpha\colon \mathcal{F} \rightleftarrows \mathcal{G}\colon \beta),\ \alpha\circ\beta = \beta\circ\alpha = \times f,$$

where α is homogeneous of degree d and β is homogeneous of degree zero. For each $i \in \mathbb{Z}$, there are isomorphisms:

$$K^{\mathrm{top}}_{\mu_d,i}(\mathbb{C}^n, f^{-1}(1)) \xrightarrow{\sim} K_i^{\mathrm{top}}(\mathrm{MF}^{\mathrm{gr}}(\mathbb{C}^n, f)),$$

where the left hand side is the μ_d-equivariant relative topological K-theory space, see loc. cit. for more details. Note that, for a homogeneous polynomial, the vanishing cycle cohomology can be computed in terms of relative cohomology [DP22, Proposition 6.4].

We do not have an alternative proof of [BD20, Theorem 1.3]. However, we note the following relation between graded and ungraded matrix factorizations, that may be used in conjunction with excision arguments for computation, but which we do not use later in the paper.

Proposition 4.15. *Let $\mathbb{C}^*$ act on $\mathbb{C}^{n+1}$ with weight 1, consider the grading with respect to this weight, and by abuse of notation denote by $f\colon \mathbb{C}^n\times\mathbb{C}^* \xrightarrow{\pi_1} \mathbb{C}^n \xrightarrow{f} \mathbb{C}$. There is an equivalence*

$$\mathrm{MF}(\mathbb{C}^n, f) \xrightarrow{\sim} \mathrm{MF}^{\mathrm{gr}}(\mathbb{C}^n\times\mathbb{C}^*, f). \tag{4.19}$$

Proof. We have the isomorphism of stacks

$$p\colon (\mathbb{C}^n\times\mathbb{C}^*)/\mathbb{C}^* \xrightarrow{\sim} \mathbb{C}^n,\ (x_1,\ldots,x_n,t)\mapsto (t^{-1}x_1,\ldots,t^{-1}x_n).$$

For an object $(\alpha\colon \mathcal{F} \rightleftarrows \mathcal{G}\colon \beta)$ in $\mathrm{MF}(\mathbb{C}^n, f)$, we associate the object

$$(\alpha'\colon p^*\mathcal{F} \rightleftarrows p^*\mathcal{G}\colon \beta'),\ \alpha' = t^d p^*\alpha,\ \beta' = p^*\beta$$

in $\mathrm{MF}^{\mathrm{gr}}(\mathbb{C}^n\times\mathbb{C}^*, f)$. Note that α' is degree d and β' is degree zero. Since $\alpha\circ\beta = \beta\circ\alpha = \times f$ and $p^*f = t^{-d}f$, we have $\alpha'\circ\beta' = \beta'\circ\alpha' = \times f$, so it determines an object in $\mathrm{MF}^{\mathrm{gr}}(\mathbb{C}^n\times\mathbb{C}^*, f)$.

Conversely, given an object $(\gamma\colon \mathcal{P} \rightleftarrows \mathcal{Q}\colon \delta)$ in $\mathrm{MF}^{\mathrm{gr}}(\mathbb{C}^n\times\mathbb{C}^*, f)$, we associate the object

$$(\gamma'\colon \mathcal{P}_0 \rightleftarrows \mathcal{Q}_0\colon \delta'),\ \gamma' = t^{-d}\gamma_0,\ \delta' = \delta_0$$

in $\mathrm{MF}(\mathbb{C}^n, f)$. In the above, the subscript 0 means taking the degree zero part and the morphism γ' is $\mathcal{P}_0 \stackrel{\gamma_0}{\to} \mathcal{Q}_d \stackrel{t^{-d}}{\to} \mathcal{Q}_0$. It is easy to see that the above correspondences give mutually inverse functors, giving the equivalence (4.19). $\square$

5. Dimensional reduction

In this section, we show that the Koszul equivalence (2.11) and the dimensional reduction in cohomology are compatible via the Chern character map. We will use these compatibilities in Subsection 7 to compute the topological K-theory of preprojective quasi-BPS categories from the topological K-theory of quasi-BPS categories of tripled quivers with potential.

5.1. **Dimensional reduction.** Recall the setting of the Koszul equivalence from Subsection 2.8. We will use the notations of the various maps from the diagram (2.13). In this subsection, we review the dimensional reduction theorem in cohomology due to Davison [Dav17, Theorem A.1] (note that, to obtain the maps in loc. cit., one needs to precompose all the following maps by l_*, see the isomorphism (2.6)).

For $\bullet \in D^b_{\mathrm{con}}(\mathcal{E}^\vee_0)$, there is a natural isomorphism:

$$\varphi_f[-1]\iota_*\bullet \xrightarrow{\sim} \iota_* \bullet\,. \tag{5.1}$$

For $\bullet \in D^b_{\mathrm{con}}(\mathcal{X})$, there is a natural transformation

$$\eta^*\bullet \to \eta^* j_* j^* \bullet \tag{5.2}$$

The natural transformations (5.2) and (5.1) induce a natural transformation for $\bullet \in D^b_{\mathrm{con}}(\mathcal{X})$:

$$\varphi_f[-1]\eta^*\bullet \to \varphi_f[-1](\eta^* j_* j^* \bullet) = \eta^* j_* j^* \bullet\,.$$

The dimensional reduction isomorphism in cohomology [Dav17, Theorem A.1] is the following natural isomorphism for $\bullet \in D^b_{\mathrm{con}}(\mathcal{X})$:

$$\eta_!\varphi_f[-1]\eta^*\bullet \xrightarrow{\sim} \eta_!\eta^* j_* j^* \bullet\,.$$

By taking the Verdier dual of the above natural isomorphism, we obtain:

$$\eta_* j'_* j'^! \eta^! \bullet = \eta_* \eta^! j_* j^! \bullet \xrightarrow{\sim} \eta_*\varphi_f[-1]\eta^! \bullet, \tag{5.3}$$

which alternatively can be described as applying the functor $\eta_*\varphi_f[-1]$ to the natural transformation $j'_* j'^! \eta^! \bullet \to \eta^! \bullet$ for $\bullet \in D^b_{\mathrm{con}}(\mathcal{X})$. By taking the cohomology of the two sides in (5.3), one obtains the *dimensional reduction* isomorphism:

$$j'_*\eta'^* \colon H^{\mathrm{BM}}_i(\mathcal{K}) \xrightarrow{\sim} H^{\mathrm{BM}}_{i+2r}(\mathcal{E}^\vee|_{\mathcal{K}}) \xrightarrow{\sim} H^{2\dim \mathcal{E}-2r-i}(\mathcal{E}^\vee, \varphi_f \mathbb{Q}[-1]). \tag{5.4}$$

Further, the monodromy on the left hand side is trivial.

The isomorphism (5.3) factors through:

$$\eta_* j'_* j'^! \eta^! \bullet \to \eta_*\iota_*\iota^!\eta^!\bullet = \eta_*\varphi_f[-1]\iota_*\iota^!\eta^!\bullet \to \eta_*\varphi_f[-1]\eta^! \bullet\,.$$

Recall that φ^{inv}_f is the cone of the map $\alpha \colon \iota_*\mathbb{Q}_{\mathcal{E}^\vee_0} \to \iota_*\iota^!\mathbb{Q}_{\mathcal{E}^\vee}[2]$, see (4.8). From the diagram (4.7), the map $\eta_*\iota_*\iota^!\mathbb{Q}_{\mathcal{E}^\vee}[2] \to \eta_*\varphi_f\mathbb{Q}_{\mathcal{E}^\vee}[1]$ factors through $\eta_*\varphi^{\mathrm{inv}}_f$, and thus there are maps whose composition is an isomorphism:

$$\beta' \colon \eta_* j'_*\omega_{\mathcal{E}^\vee|_{\mathcal{K}}} \to \eta_*\iota_*\iota^!\omega_{\mathcal{E}^\vee} \to \eta_*\varphi^{\mathrm{inv}}_f[2\dim \mathcal{E}^\vee - 2] \xrightarrow{\beta} \eta_*\varphi_f\omega_{\mathcal{E}^\vee}[-1]. \tag{5.5}$$

We let $\beta^{\diamond}\colon \eta_* j'_*\omega_{\mathcal{E}^\vee|_{\mathcal{K}}} \to \eta_*\iota_*\iota^!\omega_{\mathcal{E}^\vee} \to \eta_*\varphi_f^{\mathrm{inv}}[2\dim\mathcal{E}^\vee-2]$. The map $\beta^{\diamond}$ provides a splitting of the map β, thus the triangle (4.6) becomes the natural isomorphism:

$$\eta_*\varphi_f^{\mathrm{inv}}[-2] \cong \eta_*\varphi_f[-2] \oplus \eta_*\varphi_f[-1]. \tag{5.6}$$

By taking global sections of this isomorphism, there is a natural injective map:

$$\gamma\colon H^\bullet(\mathcal{E}^\vee, \varphi_f[-1]) \hookrightarrow H^\bullet(\mathcal{E}^\vee, \varphi_f^{\mathrm{inv}}[-2]). \tag{5.7}$$

We also note that, by taking global sections of the complexes in (5.5), we obtain an isomorphism:

$$\begin{aligned}\beta'\colon H_i^{\mathrm{BM}}(\mathcal{K}) \xrightarrow{\sim} H_{i+2r}^{\mathrm{BM}}(\mathcal{E}^\vee|_{\mathcal{K}}) \to H_{i+2r}^{\mathrm{BM}}(\mathcal{E}_0^\vee) \to H^{2\dim\mathcal{E}-2r-i}(\mathcal{E}^\vee, \varphi_f^{\mathrm{inv}}[-2]) \\ \to H^{2\dim\mathcal{E}-2r-i}(\mathcal{E}^\vee, \varphi_f\mathbb{Q}[-1]) = H^{2\dim\mathcal{X}-i}(\mathcal{E}^\vee, \varphi_f\mathbb{Q}[-1]).\end{aligned} \tag{5.8}$$

Note that the composition of the maps on the top row on (5.8) is given by $\beta^{\diamond}$.

5.2. **The Chern character for graded matrix factorizations.** The purpose of this subsection is to construct a Chern character map:

$$\mathrm{ch}\colon K_i^{\mathrm{top}}(\mathrm{MF}^{\mathrm{gr}}(\mathcal{E}^\vee, f)) \to \widetilde{H}^i(\mathcal{E}^\vee, \varphi_f[-1]) \tag{5.9}$$

compatible with the Chern character map (3.8) for $\mathcal{K}$ and the Chern character map (4.14) for $\mathrm{MF}(\mathcal{E}^\vee, f)$, see Proposition 5.1. We begin with a few preliminaries. Recall the forget-the-grading functor

$$\Theta\colon \mathrm{MF}^{\mathrm{gr}}(\mathcal{E}^\vee, f) \to \mathrm{MF}(\mathcal{E}^\vee, f) \tag{5.10}$$

and the equivalence between matrix factorizations and categories of singularities [Orl04, EP15]:

$$\mathrm{MF}(\mathcal{E}^\vee, f) \xrightarrow{\sim} D_{\mathrm{sg}}(\mathcal{E}_0^\vee).$$

The following diagram commutes:

$$\begin{tikzcd} D^b(\mathcal{K}^{\mathrm{cl}}) \arrow[r, "j'_*\eta'^*l_*"] \arrow[d, "\tau_*\eta'^*l_*"] & \mathrm{MF}^{\mathrm{gr}}(\mathcal{E}^\vee, f) \arrow[r, "\Theta"] & \mathrm{MF}(\mathcal{E}^\vee, f) \arrow[d, "\wr"'] \\ D^b(\mathcal{E}_0^\vee) \arrow[rr] & & D_{\mathrm{sg}}(\mathcal{E}_0^\vee). \end{tikzcd}$$

By the isomorphism (2.6), we obtain a commutative diagram, and we define Ψ as the resulting map:

$$\begin{tikzcd} G_\bullet^{\mathrm{top}}(\mathcal{K}) \arrow[r, "j'_*\eta'^*"] \arrow[d, "\tau_*\eta'^*"'] \arrow[drr, "\Psi"] & K_\bullet^{\mathrm{top}}(\mathrm{MF}^{\mathrm{gr}}(\mathcal{E}^\vee, f)) \arrow[r, "\Theta"] & K_\bullet^{\mathrm{top}}(\mathrm{MF}(\mathcal{E}^\vee, f)) \arrow[d] \\ G_\bullet^{\mathrm{top}}(\mathcal{E}_0^\vee) \arrow[rr] & & K_\bullet^{\mathrm{sg}}(\mathcal{E}_0^\vee). \end{tikzcd} \tag{5.11}$$

Recall the Chern character (4.17) and the splitting (5.8). Let N be the normal bundle of $\mathcal{K} \hookrightarrow \mathcal{E}^\vee$ and let M be the normal bundle of $\mathcal{E}_0^\vee \hookrightarrow \mathcal{E}^\vee$. Let $\mathrm{ch}' := \mathrm{ch}\cdot\mathrm{td}(N)$

and $\mathrm{ch}'' := \mathrm{ch} \cdot \mathrm{td}(M)$. Then the following diagram commutes by Proposition 3.5:

$$\begin{tikzcd}
G_i^{\mathrm{top}}(\mathcal{K}) \arrow[r, "\tau_*\eta'^*"] \arrow[d, "\mathrm{ch}'"] \arrow[rr, bend left, "\Psi"] & G_i^{\mathrm{top}}(\mathcal{E}_0^\vee) \arrow[r] \arrow[d, "\mathrm{ch}''"] & K_i^{\mathrm{top}}(\mathrm{MF}(\mathcal{E}^\vee, f)) \arrow[d, "\mathrm{ch}''"] \\
\widetilde{H}_i^{\mathrm{BM}}(\mathcal{K}) \arrow[r, "\tau_*\eta'^*"] \arrow[rr, bend right, "\beta^\diamond"'] & \widetilde{H}_i^{\mathrm{BM}}(\mathcal{E}_0^\vee) \arrow[r] & \widetilde{H}^i(\mathcal{E}^\vee, \varphi_f^{\mathrm{inv}}).
\end{tikzcd} \tag{5.12}$$

Proposition 5.1. *There is an injective Chern character* (5.9) *such that, in the following commutative diagram, the horizontal maps are injective:*

$$\begin{tikzcd}
K_i^{\mathrm{top}}(\mathrm{MF}^{\mathrm{gr}}(\mathcal{E}^\vee, f)) \arrow[r, hook, "\Theta"] \arrow[d, "\mathrm{ch}"] & K_i^{\mathrm{top}}(\mathrm{MF}(\mathcal{E}^\vee, f)) \arrow[d, "\mathrm{ch}"] \\
\widetilde{H}^i(\mathcal{E}^\vee, \varphi_f[-1]) \arrow[r, hook, "\gamma"] & \widetilde{H}^i(\mathcal{E}^\vee, \varphi_f^{\mathrm{inv}}[-2]),
\end{tikzcd} \tag{5.13}$$

and such that the following diagram commutes as well for the modified Chern character for the immersions of $\mathcal{E}^\vee|_{\mathcal{K}}$ and $\mathcal{E}_0^\vee$ in $\mathcal{E}^\vee$:

$$\begin{tikzcd}
G_i^{\mathrm{top}}(\mathcal{K}) \arrow[r, "j'_*\eta'^*", "\sim"'] \arrow[d, "\mathrm{ch}'"] & K_i^{\mathrm{top}}(\mathrm{MF}^{\mathrm{gr}}(\mathcal{E}^\vee, f)) \arrow[d, "\mathrm{ch}''"] \\
\widetilde{H}_i^{\mathrm{BM}}(\mathcal{K}) \arrow[r, "j'_*\eta'^*", "\sim"'] & \widetilde{H}^i(\mathcal{E}^\vee, \varphi_f[-1]).
\end{tikzcd} \tag{5.14}$$

Proof. Define (5.9) such that the diagram (5.14) commutes. We have that $\gamma \circ j'_*\eta'^* = \beta^\diamond$ and $\Theta \circ j'_*\eta'^* = \Psi$, so the diagram (5.13) commutes as well. It remains to show that Θ is injective. The map $\beta^\diamond$ is injective by (5.8). Then Ψ is also injective by the commutativity of the diagram (5.12). By the factorization $\Theta \circ j'_*\eta'^* = \Psi$, the map Θ is indeed injective. $\square$

We define an increasing filtration $E_\ell K_i^{\mathrm{top}}(\mathrm{MF}^{\mathrm{gr}}(\mathcal{E}^\vee, f)) \subset K_i^{\mathrm{top}}(\mathrm{MF}^{\mathrm{gr}}(\mathcal{E}^\vee, f))$ by

$$E_\ell K_i^{\mathrm{top}}(\mathrm{MF}^{\mathrm{gr}}(\mathcal{E}^\vee, f)) := \mathrm{ch}^{-1}\left(H^{\geqslant 2\dim \mathcal{E}^\vee - i - 2\ell}(\mathcal{E}^\vee, \varphi_f[-1])\right).$$

We obtain cycle maps:

$$\mathrm{c} \colon \mathrm{gr}_\ell K_i^{\mathrm{top}}(\mathrm{MF}^{\mathrm{gr}}(\mathcal{E}^\vee, f)) \xrightarrow{\sim} H^{2\dim \mathcal{E}^\vee - i - 2\ell}(\mathcal{E}^\vee, \varphi_f[-1]).$$

The above cycle maps are isomorphisms by the isomoprhim (3.11) together with the commutative diagram (5.14). The following is a corollary of Propositions 4.10, 4.5, and 5.1:

Proposition 5.2. *The following diagram commutes, where all the cycle maps are isomorphisms:*

$$\begin{tikzcd}
\mathrm{gr}_\ell G_i^{\mathrm{top}}(\mathcal{K}) \arrow[r, "j'_*\eta'^*", "\sim"'] \arrow[d, "\mathrm{c}", "\wr"'] & \mathrm{gr}_{\ell+r} K_i^{\mathrm{top}}(\mathrm{MF}^{\mathrm{gr}}(\mathcal{E}^\vee, f)) \arrow[r, hook] \arrow[d, "\mathrm{c}", "\wr"'] & \mathrm{gr}_{\ell+r} K_i^{\mathrm{top}}(\mathrm{MF}(\mathcal{E}^\vee, f)) \arrow[d, "\mathrm{c}", "\wr"'] \\
H_{i+2\ell}^{\mathrm{BM}}(\mathcal{K}) \arrow[r, "j'_*\eta'^*", "\sim"'] & H^{2\dim \mathcal{X} - i - 2\ell}(\mathcal{E}^\vee, \varphi_f[-1]) \arrow[r, hook, "\gamma"] & H^{2\dim \mathcal{X} - i - 2\ell}(\mathcal{E}^\vee, \varphi_f^{\mathrm{inv}}[-2]).
\end{tikzcd}$$

Proof. The modified Chern characters ch' and ch'' induce the cycle maps c on the associated graded, see Proposition 3.2. □

Remark 5.3. Recall that $K^{\mathrm{top}}_{\cdot}(\mathcal{E}^{\vee}, f)$ is a $\Lambda = \mathbb{Q}[\epsilon]$-module, where ϵ has degree 1. We include the following computation of a Λ-module structure on the topological K-theory of a category of matrix factorizations, see also Proposition 4.13, but note that we do not use it later in the paper.

Proposition 5.4. *The forget-the-potential functor induces an isomorphism*

$$K^{\mathrm{top}}_{\cdot}(\mathrm{MF}^{\mathrm{gr}}(\mathcal{E}^{\vee}, f)) \otimes_{\mathbb{Q}} \Lambda \xrightarrow{\sim} K^{\mathrm{top}}_{\cdot}(\mathrm{MF}(\mathcal{E}^{\vee}, f)) \tag{5.15}$$

of Λ-modules. Thus, if $\mathbb{M}$ is admissible in $D^b(\mathcal{E}^{\vee})$, there is an isomorphism of Λ-modules:

$$K^{\mathrm{top}}_{\cdot}(\mathrm{MF}^{\mathrm{gr}}(\mathbb{M}, f)) \otimes_{\mathbb{Q}} \Lambda \xrightarrow{\sim} K^{\mathrm{top}}_{\cdot}(\mathrm{MF}(\mathbb{M}, f)).$$

Proof. It is enough to prove (5.15). Let $p := \operatorname{Spec} \mathbb{C}$ and let $r := \operatorname{Spec} \Lambda$. Recall the Koszul equivalence (2.14). Using [HLP20, Proposition 3.24] (also see [Tod23a, Proposition 3.9] and note that $\mathrm{MF}(p, 0) \simeq D^b(r)/\mathrm{Perf}(r)$), the equivalence (2.14) induces an equivalence:

$$\kappa' \colon D^b(\mathcal{K}) \otimes_{D^b(p)} \mathrm{MF}(p, 0) \xrightarrow{\sim} \mathrm{MF}(\mathcal{E}^{\vee}, f).$$

Let $\mathcal{K}_0 := \mathcal{K} \times r$ and let $\pi \colon \mathcal{K}_0 \to \mathcal{K}$ and $t \colon r \to p$ be the natural projections. We have that $\mathrm{MF}(p, 0) \cong D^b(r)/t^*(D^b(p))$. Then

$$D^b(\mathcal{K}) \otimes_{D^b(p)} \mathrm{MF}(p, 0) \cong D^b(\mathcal{K}_0)/\pi^*(D^b(\mathcal{K})).$$

It suffices to show that the map

$$\pi^* \colon G^{\mathrm{top}}_i(\mathcal{K}) \to G^{\mathrm{top}}_i(\mathcal{K}_0) \cong G^{\mathrm{top}}_i(\mathcal{K})$$

is zero, which follows as in the proof of Proposition 4.1. □

6. Topological K-theory of quasi-BPS categories for quivers with potential

In this section, we compute the topological K-theory of quasi-BPS categories for symmetric quivers satisfying Assumption 2.1 with a quasi-homogeneous potential in terms of BPS cohomology, see Theorem 6.2. The main step in the proof of Theorem 6.2 is the construction of the cycle map from topological K-theory of quasi-BPS categories to BPS cohomology, see Theorem 6.3 (which holds for all symmetric quivers). The conclusion then follows by comparing the decomposition of DT invariants in BPS invariants of Meinhardt–Reineke (which also holds for all symmetric quivers) and and the semiorthogonal decomposition of the variety of framed representations from Theorem 2.8. We note that there is a version of Theorem 2.8 for all symmetric quivers, see [PTe]. However, under Assumption 2.1, all quasi-BPS categories appearing in the semiorthogonal decomposition are of the form $\mathbb{S}(d)_v$, which is used crucially in the computation in Subsection 6.3.

The construction of the cycle map from Theorem 6.3 holds for all quasi-BPS categories $\mathbb{S}(d; \delta)$ for $\delta \in M(d)^{W_d}_{\mathbb{R}}$. Theorem 6.3 is proved in Subsections 6.6 and is based on the fact that the weight conditions for complexes in $\mathbb{S}(d; \delta)$ restrict the possible perverse degree of their image under the cycle map, see Proposition 6.15 and Corollaries 6.18 and 6.19.

In view of the assumptions in Section 4, we assume throughout this section that the potential W of $Q = (I, E)$ is *quasi-homogeneous*, that is, there exists a weight function $w\colon E \to \mathbb{Z}_{\geqslant 0}$ such that W is homogeneous of weight $d > 0$ with respect to the function w.

6.1. **Statement of the main theorem.** Before we state Theorem 6.2, we introduce notation related to quasi-BPS categories and BPS sheaves.

6.1.1. *Quasi-BPS categories.* Let $\delta \in M(d)_{\mathbb{R}}^{W_d}$. Consider the category $\mathbb{M}(d;\delta)$ defined in (2.19) and recall the definition of quasi-BPS categories $\mathbb{S}(d;\delta)$ from (2.21). For λ a cocharacter of $T(d)$, recall the definition of n_λ from (2.20). For λ a dominant cocharacter of $T(d)$, define

$$\varepsilon_{\lambda,\delta} = \begin{cases} 1, \text{ if } \frac{1}{2}n_\lambda + \langle \lambda, \delta \rangle \in \mathbb{Z}, \\ 0, \text{ otherwise.} \end{cases} \tag{6.1}$$

For a partition $\mathbf{d} = (d_i)_{i=1}^k$ of d, let $\varepsilon_{\mathbf{d},\delta} = 1$ if $\varepsilon_{\lambda,\delta} = 1$ for all cocharacters λ with associated partition $\mathbf{d}$ and let $\varepsilon_{\mathbf{d},\delta} = 0$ otherwise.

6.1.2. *Sets of partitions.* For a dimension vector $d = (d^j)_{j\in I} \in \mathbb{N}^I$, recall that $\underline{d} := \sum_{j\in I} d^j$. Let $\delta \in M(d)_{\mathbb{R}}^{W_d}$. Denote by S_δ^d the set of partitions $\mathbf{d} = (d_i)_{i=1}^k$ of d such that $\varepsilon_{\mathbf{d},\delta} = 1$, where λ is any antidominant cocharacter with associated partition $(d_i)_{i=1}^k$. If $\delta = v\tau_d$, we use the notation S_v^d instead of $S_{v\tau_d}^d$.

Consider $(d_i)_{i=1}^k \in S_\delta^d$, and an antidominant cocharacter with associated partition $(d_i)_{i=1}^k$. Define $\theta_i \in \frac{1}{2}M(d_i)$ with

$$\sum_{i=1}^k \theta_i = -\frac{1}{2}R(d)^{\lambda>0} + \frac{1}{2}\mathfrak{g}(d)^{\lambda>0}.$$

Let $\delta_{d_i} \in M(d_i)_{\mathbb{R}}$ such that $\sum_{i=1}^k \delta_{d_i} = \delta$. Then the Hall product induces a functor

$$m = m_\lambda\colon \bigotimes_{i=1}^k \mathbb{M}(d_i;\theta_i + \delta_{d_i}) \to \mathbb{M}(d;\delta)$$

and similarly for categories of matrix factorizations, see [Păda, Propositions 3.5 and 3.6] (in loc. cit. and using the notations used there, [Păda, Proposition 3.6] is stated that $r > \frac{1}{2}$, but for $r = \frac{1}{2}$ it is still true that $\chi - \sigma_I \in \frac{1}{2}\mathbb{W}$). If we assume that Q satisfies Assumption 2.1, then $\theta_i \in M(d_i)^{W_{d_i}}$, and so there are functors, see Remark 2.6:

$$\bigotimes_{i=1}^k \mathbb{M}(d_i;\delta_{d_i}) \xrightarrow{\sim} \bigotimes_{i=1}^k \mathbb{M}(d_i;\theta_i + \delta_{d_i}) \xrightarrow{m} \mathbb{M}(d;\delta_d). \tag{6.2}$$

Assume that $\delta = v\tau_d$ and write $\delta_i = v_i\tau_{d_i}$ for $1 \leqslant i \leqslant k$. Then

$$\frac{v}{\underline{d}} = \frac{v_i}{\underline{d}_i} \tag{6.3}$$

for any $1 \leqslant i \leqslant k$. If we assume that Q satisfies Assumption 2.1, the Hall product then induces functors, see (6.2):

$$\bigotimes_{i=1}^k \mathbb{M}(d_i)_{v_i} \xrightarrow{\sim} \bigotimes_{i=1}^k \mathbb{M}(d_i;\theta_i + \delta_{d_i}) \xrightarrow{m} \mathbb{M}(d)_v.$$

We end this subsection with the following computation:

Proposition 6.1. *Let $Q = (I, E)$ be a quiver satisfying Assumption 2.1. Let $(d, v) \in \mathbb{N}^I \times \mathbb{Z}$. The set S^d_v contains all partitions $\boldsymbol{d} = (d_i)_{i=1}^k$ such that*

$$\underline{d} | v \cdot \gcd(\underline{d}_1, \dots, \underline{d}_k).$$

In particular, if $\gcd(v, \underline{d}) = 1$, then S^d_v contains only the one term partition of d.

Proof. Let $d = (d^a)_{a \in I} \in \mathbb{N}^I$. Note that $n_\lambda = \langle \lambda, \mathbb{L}^{\lambda>0}_{\mathcal{X}(d)}|_0 \rangle \in 2\mathbb{Z}$ because Q satisfies Assumption 2.1. Then $\varepsilon_{\mathbf{d}, v\tau_d} = 1$ if and only if $\langle \lambda, v\tau_d \rangle \in \mathbb{Z}$ for all cocharacters λ with associated partition $\mathbf{d}$.

Write $\lambda = (\lambda^a)_{a \in I}$, where $\lambda^a \colon \mathbb{C}^* \to T(d^a)$ is a cocharacter

$$\lambda(t) = (t^{m_1}, \dots, t^{m_1}, t^{m_2}, \dots, t^{m_2}, \dots, t^{m_k}),$$

where m_i appears d^a_i-times, and $m_i \neq m_j$ for $1 \leqslant i \neq j \leqslant k$. Then the condition $\langle \lambda, v\tau_d \rangle \in \mathbb{Z}$ is equivalent to that

$$v/\underline{d} \cdot \sum_{i=1}^k m_i \underline{d}_i \in \mathbb{Z}$$

for all tuples of pairwise distinct integers $(m_i)_{i=1}^k \in \mathbb{Z}^k$, which implies the desired conclusion. □

6.1.3. *BPS sheaves and cohomologies.* Let $Q = (I, E)$ be a symmetric quiver, let W be a potential of Q, and let $d \in \mathbb{N}^I$. Consider the stack of dimension d representations of Q and its good moduli space:

$$\pi_d \colon \mathcal{X}(d) := R(d)/G(d) \to X(d) := R(d) /\!\!/ G(d).$$

We denote by $\mathrm{IC} := \mathrm{IC}_{\mathcal{X}(d)} = \mathbb{Q}_{\mathcal{X}(d)}[\dim \mathcal{X}(d)]$ and we may drop $\mathrm{Tr}\, W$ from the notation of the vanishing cycle functor. Recall that $\varphi := \varphi_{\mathrm{Tr}\, W} \mathbb{Q}_{\mathcal{X}(d)}$. Following [DM20], define the BPS sheaf

$$\mathcal{BPS}_d := \begin{cases} \varphi_{\mathrm{Tr}\, W} \mathrm{IC}_{X(d)}[-1], \text{ if } X(d)^{\mathrm{st}} \neq \emptyset, \\ 0, \text{ if } X(d)^{\mathrm{st}} = \emptyset. \end{cases}$$

Note that $\mathcal{BPS}_d \in \mathrm{Perv}(X(d))$.

Consider a partition $A = (d_i)_{i=1}^k$ of d. Let $\ell(A) := k$. Assume the set $\{d_1, \dots, d_k\} = \{e_1, \dots, e_s\}$ has cardinality s and that, for each $1 \leqslant i \leqslant s$, there are m_i elements in $\{d_1, \dots, d_k\}$ equal to e_i. Define the addition maps $\oplus_i \colon X(e_i)^{\times m_i} \to X(m_i e_i)$ for $1 \leqslant i \leqslant s$ and $\oplus' \colon \times_{i=1}^s X(m_i e_i) \to X(d)$, which are finite. Define the sheaves:

$$\begin{aligned} \mathrm{Sym}^{m_i}\left(\mathcal{BPS}_{e_i}\right) &:= \oplus_{i,*}\left(\mathcal{BPS}_{e_i}^{\boxtimes m_i}\right)^{\mathfrak{S}_{m_i}} \in \mathrm{Perv}(X(m_i e_i)), \\ \mathcal{BPS}_A &:= \oplus'_*\left(\boxtimes_{i=1}^s \mathrm{Sym}^{m_i}(\mathcal{BPS}_{e_i})\right) \in \mathrm{Perv}(X(d)). \end{aligned} \tag{6.4}$$

Alternatively, by the Thom-Sebastiani theorem, the sheaf $\mathcal{BPS}_A$ has the following description. Let $\mathcal{BPS}^0_A$ be the sheaf defined above for $W = 0$. Then $\mathcal{BPS}_A = \varphi_{\mathrm{Tr}\, W} \mathcal{BPS}^0_A[-1]$.

Define the complexes

$$\begin{aligned} \mathcal{BPS}_{d,\delta} &:= \bigoplus_{A \in S^d_\delta} \mathcal{BPS}_A[-\ell(A)] \in D^b_{\mathrm{con}}(X(d)), \\ \mathcal{BPS}_{d,v} &:= \mathcal{BPS}_{d, v\tau_d} \in D^b_{\mathrm{con}}(X(d)). \end{aligned} \tag{6.5}$$

As we will see in (6.18), the complexes $\mathcal{BPS}_A$ and $\mathcal{BPS}_{d,\delta}$ are direct summands of $\pi_*\varphi\mathrm{IC}[-1]$ preserved by $1-\mathrm{T}$. Define $\mathcal{BPS}_A^{\mathrm{inv}}, \mathcal{BPS}_{d,\delta}^{\mathrm{inv}} \in D^b_{\mathrm{con}}(X(d))$ by the exact triangles:

$$\mathcal{BPS}_A^{\mathrm{inv}}[-1] \to \mathcal{BPS}_A \xrightarrow{1-\mathrm{T}} \mathcal{BPS}_A \to \mathcal{BPS}_A^{\mathrm{inv}},$$
$$\mathcal{BPS}_{d,\delta}^{\mathrm{inv}}[-1] \to \mathcal{BPS}_{d,\delta} \xrightarrow{1-\mathrm{T}} \mathcal{BPS}_{d,\delta} \to \mathcal{BPS}_{d,\delta}^{\mathrm{inv}}.$$

6.1.4. *Statement of the main theorem.* Let $Q=(I,E)$ be a symmetric quiver and let W be a quasi-homogeneous potential. Consider the Chern character map (4.14):

$$\mathrm{ch}\colon K_i^{\mathrm{top}}(\mathbb{S}(d)_v) \hookrightarrow K_i^{\mathrm{top}}(\mathrm{MF}(\mathcal{X}(d), \mathrm{Tr}\, W)) \to \widetilde{H}^i(\mathcal{X}(d), \varphi^{\mathrm{inv}}_{\mathrm{Tr}\, W}). \tag{6.6}$$

Recall (4.15) and define the filtration:

$$E_\ell K_i^{\mathrm{top}}(\mathbb{S}(d)_v) := K_i^{\mathrm{top}}(\mathbb{S}(d)_v) \cap E_\ell K_i^{\mathrm{top}}(\mathrm{MF}(\mathcal{X}(d), \mathrm{Tr}\, W)) \subset K_i^{\mathrm{top}}(\mathbb{S}(d)_v).$$

There is an injective cycle map on the associated graded pieces:

$$\begin{aligned}\mathrm{c}\colon \mathrm{gr}_\ell K_i^{\mathrm{top}}(\mathbb{S}(d)_v) &\to H^{2\dim \mathcal{X}(d)-2\ell-i}(\mathcal{X}(d), \varphi^{\mathrm{inv}}_{\mathrm{Tr}\, W}[-2]) \\ &\xrightarrow{\sim} H^{\dim \mathcal{X}(d)-2\ell-i}(\mathcal{X}(d), \varphi^{\mathrm{inv}}_{\mathrm{Tr}\, W}\, \mathrm{IC}_{\mathcal{X}(d)}[-2]),\end{aligned} \tag{6.7}$$

where we used $\varphi_{\mathrm{Tr}\, W}\mathrm{IC}_{\mathcal{X}(d)} = \varphi_{\mathrm{Tr}\, W}[\dim \mathcal{X}(d)]$ for computing the cohomological degree. The following is the main result of this section:

Theorem 6.2. *Assume the quiver Q satisfies Assumption 2.1 and let W be a quasi-homogeneous potential of Q. Then the cycle map* (6.7) *induces an isomorphisms for $i,\ell \in \mathbb{Z}$:*

$$\mathrm{c}\colon \mathrm{gr}_\ell K_i^{\mathrm{top}}\left(\mathbb{S}(d)_v\right) \xrightarrow{\sim} H^{\dim \mathcal{X}(d)-2\ell-i}(X(d), \mathcal{BPS}^{\mathrm{inv}}_{d,v}[-1]). \tag{6.8}$$

The main part of proving Theorem 6.2 is the construction of a cycle map from the topological K-theory of quasi-BPS categories to BPS cohomology, which applies to all symmetric quivers Q.

Theorem 6.3. *Let Q be an arbitrary symmetric quiver and let W be a quasi-homogeneous potential of Q. Let $d \in \mathbb{N}^I$, $\delta \in M(d)^{W_d}_{\mathbb{R}}$, and $i, \ell \in \mathbb{Z}$. The cycle map* (6.7) *induces a map:*

$$\mathrm{c}\colon \mathrm{gr}_\ell K_i^{\mathrm{top}}\left(\mathbb{S}(d;\delta)\right) \to H^{\dim \mathcal{X}(d)-2\ell-i}\left(X(d), \mathcal{BPS}^{\mathrm{inv}}_{d,\delta}[-1]\right). \tag{6.9}$$

We mention the following numerical corollary of Theorems 6.2 and 6.3.

Corollary 6.4. *Let Q be an arbitrary symmetric quiver and let W be a quasi-homogeneous potential. Let $(d,v) \in \mathbb{N}^I \times \mathbb{Z}$ and let $i \in \mathbb{Z}$. Then:*

$$\dim_{\mathbb{Q}} K_i^{\mathrm{top}}(\mathbb{S}(d)_v) \leqslant \dim_{\mathbb{Q}} H^{\cdot}(X(d), \mathcal{BPS}_{d,v})^{\mathrm{inv}}. \tag{6.10}$$

If Q satisfies Assumption 2.1, then equality holds in (6.10).

When $\gcd(\underline{d}, v)=1$ and Q satisfies Assumption 2.1, we regard $\mathbb{S}(d)_v$ as a categorification of the monodromy invariant BPS cohomology of (Q,W). Before we prove Corollary 6.4, note the following:

Proposition 6.5. *Let Q be a quiver satisfying Assumption 2.1. The Chern character map* (6.6) *is injective.*

Proof. It follows from Proposition 4.10 and Theorem 2.8. $\square$

Proof of Corollary 6.4 from Theorem 6.3. Note that there is a (non-canonical) isomorphism

$$H^\bullet(X(d), \mathcal{BPS}_{d,v})^{\mathrm{inv}} \cong H^\bullet(X(d), \mathcal{BPS}_{d,v})_{\mathrm{inv}}. \tag{6.11}$$

The cycle map (6.9) is injective because (6.8) is injective. Then, by Theorem 6.3, we have that:

$$\dim_{\mathbb{Q}} \mathrm{gr}_\cdot K_i^{\mathrm{top}}(\mathbb{S}(d)_v) \leqslant \dim_{\mathbb{Q}} H^\cdot(X(d), \mathcal{BPS}_{d,v})^{\mathrm{inv}}. \tag{6.12}$$

If Q satisfies Assumption 2.1, then (6.12) is an equality. It suffices to show that $\dim_{\mathbb{Q}} K_i^{\mathrm{top}}(\mathbb{S}(d)_v) = \dim_{\mathbb{Q}} \mathrm{gr}_\cdot K_i^{\mathrm{top}}(\mathbb{S}(d)_v)$, equivalently that (6.6) is injective, which is Proposition 6.5. □

Corollary 1.5 follows easily from Theorem 6.2.

Proof of Corollary 1.5. Note that Proposition 6.1 implies that $\mathbf{d} = (d_i)_{i=1}^k \in S_v^d$ if and only if $\underline{d}/\gcd(\underline{d}, v)$ divides $\underline{d}_i$ for $1 \leqslant i \leqslant k$. Then $S_v^d = S_{v'}^d$ for $v, v' \in \mathbb{Z}$ such that $\gcd(\underline{d}, v) = \gcd(\underline{d}, v')$. The statement then follows from Theorem 6.2. □

In Section 7, we compute the topological K-theory of quasi-BPS categories of preprojective algebras of quivers satisfying Assumption 2.2 using Theorem 6.2, see Theorem 7.6. In [PTd], we further use Theorem 7.6 to compute the topological K-theory of quasi-BPS categories of K3 surfaces. In particular, we obtain categorifications of the BPS cohomology of a large class of preprojective algebras and of K3 surfaces.

We end this subsection by discussing the zero potential case of Theorem 6.2. Then $\mathcal{BPS}_d = \mathrm{IC}_{X(d)}$. Denote by $\mathrm{IH}^\bullet(X(d)) := H^\bullet\big(X(d), \mathrm{IC}_{X(d)}\big)$. Note that $H^{\mathrm{odd}}(\mathcal{X}(d)) = H^{\mathrm{odd}}(\mathcal{X}(d), \mathrm{IC}_{\mathcal{X}(d)}) = 0$. We then have that $\mathrm{IH}^{\mathrm{even}}(X(d)) = 0$ because $\mathrm{IC}_{X(d)}[-1]$ is a direct summand of $R\pi_*\mathrm{IC}_{\mathcal{X}(d)}$, see (6.16); alternatively, the vanishing $\mathrm{IH}^{\mathrm{even}}(X(d)) = 0$ follows from Kirwan surjectivity. By Theorem 6.2 and Proposition 4.13, we obtain the following:

Theorem 6.6. *Let Q be a quiver satisfying Assumption 2.1, let $d \in \mathbb{N}^I$, and let $v \in \mathbb{Z}$ such that* $\gcd(\underline{d}, v) = 1$. *For $\ell \in \mathbb{Z}$, the cycle map induces an isomorphism:*

$$\mathrm{c} \colon \mathrm{gr}_\ell K_0^{\mathrm{top}}(\mathbb{M}(d)_v) \xrightarrow{\sim} \mathrm{IH}^{\dim \mathcal{X}(d) - 2\ell - 1}(X(d)).$$

We note a consequence of Corollary 6.4, alternatively a numerical corollary of Theorem 6.6.

Corollary 6.7. *Let Q be a quiver satisfying Assumption 2.1, let $d \in \mathbb{N}^I$, and let $v \in \mathbb{Z}$ such that* $\gcd(\underline{d}, v) = 1$. *Then*

$$\dim_{\mathbb{Q}} K_0^{\mathrm{top}}(\mathbb{M}(d)_v) = \dim_{\mathbb{Q}} \mathrm{IH}^\cdot(X(d))$$

for any $i \in \mathbb{Z}$.

For $(d, v) \in \mathbb{N}^I \times \mathbb{Z}$ and Q as in the statement of Corollary (6.7), we regard $\mathbb{M}(d)_v$ as a categorification of the intersection cohomology of $X(d)$. Note that, in general, $X(d)$ is a singular scheme.

6.2. **The decomposition theorem.** Let $\alpha \in \mathbb{N}$ and recall the construction of framed quivers $Q^{\alpha f}$ from Subsection 2.14.

We review the explicit computation of summands in the BBDG decomposition theorem [BBD82] for the pushforwards of the constant sheaves along the maps:

$$\pi_{\alpha f,d}\colon \mathcal{X}^{\alpha f}(d)^{\mathrm{ss}} \to X(d),\ \pi_d\colon \mathcal{X}(d) \to X(d)$$

due to Meinhardt–Reineke [MR19] and Davison–Meinhardt [DM20]. The maps $\pi_{\alpha f,d}$ "approximate" the map π_d, see [DM20, Subsection 4.1]. The computation of $\pi_{d*}\mathrm{IC}_{\mathcal{X}(d)}$ is deduced from the computation of $\pi_{\alpha f,d*}\mathbb{Q}_{\mathcal{X}^{\alpha f}(d)^{\mathrm{ss}}}[\dim \mathcal{X}(d)]$.

We introduce some constructible sheaves on $X(d)$. Let A be a tuplet $(e_i, m_{i,a})$ for $1 \leqslant i \leqslant s$ and for $a \geqslant 0$, with $(e_i)_{i=1}^s \in \mathbb{Z}_{\geqslant 1}^s$ pairwise distinct and $m_{i,a} \geqslant 0$ such that $\sum_{i=1}^s \sum_{a\geqslant 0} e_i m_{i,a} = d$. Let $\mathcal{P}$ be the set of all such tuplets A and let $\mathcal{P}_\alpha \subset \mathcal{P}$ be the subset of such tuplets with $m_{i,a} = 0$ for $a \geqslant \alpha e_i$. Note that each A has a corresponding partition with terms e_i with multiplicity $\sum_{a\geqslant 0} m_{i,a}$ for $1 \leqslant i \leqslant s$.

Consider the addition maps:

$$\oplus_{i,a}\colon X(e_i)^{\times m_{i,a}} \to X(m_{i,a}e_i),\ \oplus'\colon \times_{i=1}^s \times_{a\geqslant 0} X(m_{i,a}e_i) \to X(d). \tag{6.13}$$

Define the constructible complexes:

$$\begin{aligned}\mathrm{Sym}^{m_{i,a}}\left(\mathrm{IC}_{X(e_i)}[-2a-1]\right) &:= \oplus_{i,a*}\left(\left(\mathrm{IC}_{X(e_i)}[-2a-1]\right)^{\boxtimes m_{i,a}}\right)^{\mathfrak{S}_{m_{i,a}}},\\ \mathrm{P}_A &:= \oplus'_*\left(\boxtimes_{1\leqslant i\leqslant s, a\geqslant 0}\mathrm{Sym}^{m_{i,a}}\left(\mathrm{IC}_{X(e_i)}[-2a-1]\right)\right).\end{aligned}$$

Then P_A is supported on the image of $\oplus'$ and is a shifted perverse sheaf of degree

$$p_A := \sum_{i=1}^k \sum_{a\geqslant 0} m_{i,a}(2a+1),$$

meaning that $\mathrm{P}_A[p_A] \in \mathrm{Perv}(X(d))$. Define analogously

$$\mathrm{Q}_A := \oplus'_*\left(\boxtimes_{1\leqslant i\leqslant s, a\geqslant 0}\mathrm{Sym}^{m_{i,a}}\left(\varphi_{\mathrm{Tr}\,W}\mathrm{IC}_{X(e_i)}[-2a-2]\right)\right). \tag{6.14}$$

Then one can show, using the Thom-Sebastiani theorem, that

$$\mathrm{Q}_A = \varphi_{\mathrm{Tr}\,W}\mathrm{P}_A[-1].$$

Let α be an even positive natural number. The following explicit form of the BBDG decomposition theorem for $\pi_{\alpha f,d}$ was determined by Meinhardt–Reineke [MR19, Proposition 4.3]:

$$\pi_{\alpha f,d*}\left(\mathbb{Q}_{\mathcal{X}^{\alpha f}(d)^{\mathrm{ss}}}[\dim \mathcal{X}(d)]\right) = \bigoplus_{A\in\mathcal{P}_\alpha} P_A. \tag{6.15}$$

The result in loc. cit. is stated as an equality in the Grothendieck group of constructible sheaves, but the above stronger statement holds by the argument in [DM20, Proof of Theorem 4.10]. Using the above, one can obtain, see [DM20, Theorem C], the following decomposition:

$$\pi_{d*}\mathrm{IC}_{\mathcal{X}(d)} = \bigoplus_{A\in\mathcal{P}} \mathrm{P}_A. \tag{6.16}$$

The proper pushforward commutes with the vanishing cycle functor, so applying the vanishing cycle functor to (6.15) one obtains the following decomposition, which is also called the DT/ BPS wall-crossing:

$$\pi_{\alpha f,d*}\varphi_{\mathrm{Tr}\,W}\left(\mathbb{Q}_{\mathcal{X}^{\alpha f}(d)^{\mathrm{ss}}}[\dim \mathcal{X}(d)-1]\right) = \bigoplus_{A\in\mathcal{P}_\alpha} Q_A. \tag{6.17}$$

The map π_d can be approximated by the proper maps $\pi_{\alpha f,d}$, thus π_{d*} also commutes with the vanishing cycle functor. From (6.16), we obtain:

$$\pi_{d*}\varphi_{\mathrm{Tr}\,W}\mathrm{IC}_{\mathcal{X}(d)}[-1]=\bigoplus_{A\in\mathcal{P}}Q_A. \tag{6.18}$$

The summands in all the above decompositions are induced via the Hall product.

We now state a corollary of (6.17).

Proposition 6.8. *Let α be an even positive integer and let $i\in\mathbb{Z}$. Then there is an isomorphism of $\mathbb{N}^I\times\mathbb{N}$-graded vector spaces, where the second grading is the cohomological grading:*

$$\bigoplus_{d\in\mathbb{N}^I}H^\bullet\left(\mathcal{X}^{\alpha f}(d)^{\mathrm{ss}},\varphi\left[\dim\mathcal{X}(d)-1\right]\right)^{\mathrm{inv}}\cong \left(\mathrm{Sym}\left(\bigoplus_{d\in\mathbb{N}^I}H^\bullet\left(X(d),\mathcal{BPS}_d[-1]\right)\otimes H^\bullet(\mathbb{P}^{\alpha\underline{d}-1})\right)\right)^{\mathrm{inv}}. \tag{6.19}$$

By forgetting the cohomological grading, there is an isomorphism of $\mathbb{N}^I$-graded vector spaces:

$$\bigoplus_{d\in\mathbb{N}^I}H^{\cdot}\left(\mathcal{X}^{\alpha f}(d)^{\mathrm{ss}},\varphi\right)^{\mathrm{inv}}\cong\left(\mathrm{Sym}\left(\bigoplus_{d\in\mathbb{N}^I}H^{\cdot}\left(X(d),\mathcal{BPS}_d\right)^{\oplus\alpha\underline{d}}\right)\right)^{\mathrm{inv}}.$$

Proof. By taking global sections of the two sides of (6.17), we obtain an isomorphism:

$$\bigoplus_{d\in\mathbb{N}^I}H^\bullet\left(\mathcal{X}^{\alpha f}(d)^{\mathrm{ss}},\varphi\left[\dim\mathcal{X}(d)-1\right]\right)\cong \mathrm{Sym}\left(\bigoplus_{d\in\mathbb{N}^I}H^\bullet\left(X(d),\mathcal{BPS}_d[-1]\right)\otimes H^\bullet(\mathbb{P}^{\alpha\underline{d}-1})\right). \tag{6.20}$$

The isomorphism (6.19) follows by taking the monodromy invariant parts on the two sides of the isomorphism (6.20). □

6.3. **Semiorthogonal decompositions and the BBDG decomposition theorem.** In this section, we prove Theorem 6.2 assuming Theorem 6.3. The proof follows from a comparison of the pieces in the semiorthogonal decomposition from Theorem 2.8 with the summands of the DT/ BPS wall-crossing (6.17). Actually, the proof is based on a comparison of dimensions of certain vector spaces. In the rest of this subsection, we will use certain non-canonical maps, but they suffice for comparing dimensions of vector spaces.

We rewrite the Chern character isomorphism (4.17) for $\mathcal{X}$ a smooth variety with a regular function $f\colon\mathcal{X}\to\mathbb{C}$. Observe that there is a (non-canonical) isomorphism $H^i(\mathcal{X},\varphi_f)^{\mathrm{inv}}\cong H^i(\mathcal{X},\varphi_f)_{\mathrm{inv}}$ of $\mathbb{Q}$-vector spaces. Rewrite (4.17) as the following (non-canonical) isomorphism of $\mathbb{Q}$-vector spaces for every $i\in\mathbb{Z}$:

$$\mathrm{ch}\colon K_i^{\mathrm{sg}}(\mathcal{X}_0)\xrightarrow{\sim}H^{\cdot}(\mathcal{X},\varphi_f)^{\mathrm{inv}}. \tag{6.21}$$

Recall the notations $\mathrm{gr}_{\cdot}K_i^{\mathrm{top}}:=\bigoplus_{a\in\mathbb{Z}}\mathrm{gr}_aK_i^{\mathrm{top}}$ and $H^{\cdot}:=\bigoplus_{a\in\mathbb{Z}}H^a$. Given a vector space V with a linear map $T\colon V\to V$, we denote by V^{inv} the kernel of $(1-T)$. For a set A of pairs $(V_a,T_a)_{a\in A}$ we denote by $(\otimes_{a\in A}V_a)^{\mathrm{inv}}$ the kernel of

$1-\otimes_{a\in A}T_a$. Note that $\otimes_{a\in A}V_a^{\mathrm{inv}}\subset(\otimes_{a\in A}V_a)^{\mathrm{inv}}$. The same notation is also used for symmetric products. We will apply the above notation when T_a are monodromy operators on vanishing cycle cohomologies.

Note the following corollary of Theorem 6.3, which follows because the cycle map (6.8) is injective:

Corollary 6.9. *Assume Theorem 6.3 holds. Then the cycle map* (6.9) *is injective.*

Proof of Theorem 6.2 assuming Theorem 6.3. Let α be an even positive integer and fix $i\in\mathbb{Z}$. By Theorem 2.8, there is a semiorthogonal decomposition:

$$\mathrm{MF}\left(\mathcal{X}^{\alpha f}(d)^{\mathrm{ss}},\mathrm{Tr}\,W\right)=\left\langle\bigotimes_{j=1}^{k}\mathbb{S}(d_j)_{v_j}\right\rangle,$$

where the right hand side is after all partitions $\sum_{j=1}^k d_j=d$ and all weights $v_j\in\mathbb{Z}$ with $0\leqslant v_1/\underline{d}_1<\ldots<v_k/\underline{d}_k<\alpha$. There is thus an isomorphism of $\mathbb{N}^I$-graded vector spaces:

$$\bigoplus_{d\in\mathbb{N}^I}\mathrm{gr}_{\cdot}K_i^{\mathrm{top}}\left(\mathrm{MF}\left(\mathcal{X}^{\alpha f}(d)^{\mathrm{ss}},\mathrm{Tr}\,W\right)\right)\cong\bigoplus_{0\leqslant v_1/\underline{d}_1<\cdots<v_k/\underline{d}_k<\alpha}\mathrm{gr}_{\cdot}K_i^{\mathrm{top}}\left(\bigotimes_{j=1}^{k}\mathbb{S}(d_j)_{v_j}\right).$$

Recall the isomorphism of $\mathbb{Q}$-vector spaces (6.11). By Corollary 6.9, there is an injective (non-canonical) map:

$$\mathrm{gr}_{\cdot}K_i^{\mathrm{top}}(\mathbb{S}(d)_v)\hookrightarrow H^{\cdot}(X(d),\mathcal{BPS}_{d,v})^{\mathrm{inv}}. \tag{6.22}$$

Then we have injective maps

$$\begin{aligned}&\bigoplus_{0\leqslant v_1/\underline{d}_1<\cdots<v_k/\underline{d}_k<\alpha}\mathrm{gr}_{\cdot}K_i^{\mathrm{top}}\left(\bigotimes_{j=1}^{k}\mathbb{S}(d_j)_{v_j}\right)\\
&\hookrightarrow\bigoplus_{0\leqslant v_1/\underline{d}_1<\cdots<v_k/\underline{d}_k<\alpha}H^{\cdot}\left(\times_{j=1}^{k}X(d_j),\boxtimes_{j=1}^{k}\mathcal{BPS}_{d,v}\right)^{\mathrm{inv}}\\
&\hookrightarrow\left(\bigotimes_{\substack{\mu\in\mathbb{Q}\\0\leqslant\mu<\alpha}}\left(\mathrm{Sym}\left(\bigoplus_{\substack{d\in\mathbb{N}^I\\\exists\, v\text{ s.t. }\mu=v/\underline{d}}}H^{\cdot}(X(d),\mathcal{BPS}_d)\right)\right)\right)^{\mathrm{inv}}\\
&\xrightarrow{\sim}\left(\mathrm{Sym}\left(\bigoplus_{d\in\mathbb{N}^I}H^{\cdot}(X(d),\mathcal{BPS}_d)^{\oplus\alpha\underline{d}}\right)\right)^{\mathrm{inv}}\xrightarrow{\sim}\bigoplus_{d\in\mathbb{N}^I}H^{\cdot}\left(\mathcal{X}^{\alpha f}(d)^{\mathrm{ss}},\varphi\right)^{\mathrm{inv}}\end{aligned}$$

where the first inclusion follows from Corollary 6.9 (applied to disjoint union of k-copies of Q, the $k=1$ case is (6.22)), the second inclusion follows from the definition of $\mathcal{BPS}_{d,v}$, Proposition 6.1, and the fact that the Thom-Sebastiani isomorphism is natural with respect to the monodromy actions, the first isomorphism follows from a combinatorial observation, and the second isomorphism follows from Proposition 6.8. We thus obtain an injective map of $\mathbb{N}^I$-graded vector spaces:

$$\bigoplus_{d\in\mathbb{N}^I}\mathrm{gr}_{\cdot}K_i^{\mathrm{top}}\left(\mathrm{MF}^{\mathrm{gr}}\left(\mathcal{X}^{\alpha f}(d)^{\mathrm{ss}},\mathrm{Tr}\,W\right)\right)\hookrightarrow\bigoplus_{d\in\mathbb{N}^I}H^{\cdot}\left(\mathcal{X}^{\alpha f}(d)^{\mathrm{ss}},\varphi\right)^{\mathrm{inv}}.$$

By the isomorphism (6.21) together with the exact sequence (4.10), the $\mathbb{N}^I$-graded piece of both sides of the above map has the same (finite) dimension, hence the map above is an isomorphism. The map (6.22) is then also an isomorphism, thus also the maps (6.8) are isomorphisms. □

It thus remains to prove Theorem 6.3. In Subsection 6.10, we reduce the proof for a general symmetric quiver to that of a quiver with at least two loops at every vertex. In Subsection 6.5, we prove a restriction statement of the image under the cycle map of an object in a quasi-BPS category. In Subsection 6.6, we combine the above restriction with the decomposition theorems (6.16) to prove Theorem 6.3.

6.4. **Reduction to quivers with enough edges.** Consider an arbitrary symmetric quiver with potential (Q, W). Let $Q = (I, E)$. For $i \in I$, let ω_i, ω_i' be two loops at i. Let $E^{\daleth} := E \sqcup \{\omega_i, \omega_i' \mid i \in I\}$ and consider the quadratic potential $W^q := \sum_{I\in I} \omega_i\omega_i'$. Define the quiver with potential:

$$Q^{\daleth} := (I, E^{\daleth}),\ W^{\daleth} := W + W^q.$$

Proposition 6.10. *Assume Theorem 6.3 holds for $(Q^{\daleth}, W^{\daleth})$. Then Theorem 6.3 holds for (Q, W).*

Recall the stack of representations $\mathcal{X}(d) = R(d)/G(d)$ of Q. For the quiver $Q^{\daleth}$ and for $d \in \mathbb{N}^I$, we consider the following: the stack of representations with its good moduli space

$$\pi_d^{\daleth} \colon \mathcal{X}^{\daleth}(d) := \left(R(d) \oplus \mathfrak{g}(d)^{\oplus 2}\right)/G(d) \to X^{\daleth}(d),$$

the BPS sheaves $\mathcal{BPS}^{\daleth}_{d,v}$ as defined in (6.5), the polytope $\mathbb{W}^{\daleth}(d)$ as in (2.16), the integers $n^{\daleth}_{\lambda}$ as in (2.20), the quasi-BPS categories $\mathbb{M}^{\daleth}(d;\delta)$ from (2.19) and $\mathbb{S}^{\daleth}(d;\delta)$ from (2.21). Let

$$\mathcal{S}(d) := (R(d) \oplus \mathfrak{g}(d))/G(d)$$

and consider the maps, where v, t are the natural projections and s is the natural inclusion:

$$\begin{tikzcd} \mathcal{X}(d) & \mathcal{S}(d) \arrow[l, "v"'] \arrow[r, hook, "s"] & \mathcal{X}^{\daleth}(d). \arrow[ll, bend right, "t"'] \end{tikzcd}$$

Let $G := G(d)$ and $\mathfrak{g} := \mathfrak{g}(d)$. We discuss two preliminary propositions.

Proposition 6.11. *Let $\delta \in M(d)^{W_d}_{\mathbb{R}}$ and let $i \in \mathbb{Z}$. There is an isomorphism:*

$$s_*v^* \colon H^i(\mathcal{X}(d), \varphi_{\mathrm{Tr}\, W}\mathrm{IC}_{\mathcal{X}(d)}[-1]) \xrightarrow{\sim} H^i(\mathcal{X}^{\daleth}(d), \varphi_{\mathrm{Tr}\, W^{\daleth}}\mathrm{IC}_{\mathcal{X}^{\daleth}(d)}[-1]),$$
$$H^i(X(d), \mathcal{BPS}_{d,\delta}) \xrightarrow{\sim} H^i(X^{\daleth}(d), \mathcal{BPS}^{\daleth}_{d,\delta}).$$

Proof. Consider the diagram:

$$(6.23)\qquad \begin{tikzcd} \mathcal{X}(d) \arrow[d, "\pi_d"] & \mathcal{S}(d) \arrow[l, "v"'] \arrow[r, "s"] & \mathcal{X}^{\daleth}(d) \arrow[d, "\pi_d^{\daleth}"] \\ X(d) \arrow[rr, "u"] & & X^{\daleth}(d). \end{tikzcd}$$

We first show there is an isomorphism of sheaves on $D^b_{\mathrm{con}}(X^{\daleth}(d))$:

$$(6.24)\qquad s_*v^* \colon u_*\pi_{d*}\varphi_{\mathrm{Tr}\, W}\mathrm{IC}_{\mathcal{X}(d)}[-1] \xrightarrow{\sim} \pi^{\daleth}_{d*}\varphi_{\mathrm{Tr}\, W^{\daleth}}\mathrm{IC}_{\mathcal{X}^{\daleth}(d)}[-1].$$

First, there is an isomorphism of sheaves on $D^b_{\mathrm{con}}(X^{\daleth}(d))$:

$$(6.25)\qquad s_*v^* \colon u_*\pi_{d*}\mathrm{IC}_{\mathcal{X}(d)} \xrightarrow{\sim} \pi^{\daleth}_{d*}\varphi_{\mathrm{Tr}\, W^q}\mathrm{IC}_{\mathcal{X}^{\daleth}(d)}[-1].$$

The above map is obtained from base-change from the the map for $\mathcal{X}(d)=\mathrm{pt}$, that is, from the map

$$s_{0*}v_0^*\colon \pi_{d,0*}u_{0*}\mathrm{IC}_{BG}\xrightarrow{\sim}\pi_{d,0*}\varphi_{\mathrm{Tr}\,W^q}\mathrm{IC}_{\mathfrak{g}^{\oplus 2}/G}[-1], \tag{6.26}$$

where s_0, v_0, u_0 are the maps as in (6.23) for $\mathcal{X}(d)$ replaced by $\mathrm{Spec}\,\mathbb{C}=\mathrm{pt}$, and where $\pi_{d,0}\colon \mathfrak{g}^{\oplus 2}/G\to\mathfrak{g}^{\oplus 2}/\!\!/G$. By a direct computation, we have that

$$\varphi_{\mathrm{Tr}\,W^q}\mathrm{IC}_{\mathfrak{g}^{\oplus 2}/G}[-1]=\mathrm{IC}_{BG}$$

because W^q is a Morse function with critical locus BG, the origin in $\mathfrak{g}^{\oplus 2}/G$. Further, (6.26) is an isomorphism for global sections by dimensional reduction, see Subsection 5.1, so (6.26) is an isomorphism. Then (6.25) is also an isomorphism.

Abuse notation and write $\mathrm{Tr}\,W\colon \mathcal{X}^{\daleth}(d)\xrightarrow{\mathrm{proj}}\mathcal{X}(d)\xrightarrow{\mathrm{Tr}\,W}\mathbb{C}$. Note that π_{d*} commutes with $\varphi_{\mathrm{Tr}\,W}$ because π_d can be approximated with the proper maps $\pi_{\alpha f,d}$, see Subsection 6.2. Further, $\varphi_{\mathrm{Tr}\,W}$ commutes with proper pushforward and smooth pullback. Apply $\varphi_{\mathrm{Tr}\,W}$ to both sides of (6.25) and use the Thom-Sebastiani theorem for vanishing cycles to obtain:

$$\begin{aligned}s_*v^*\colon u_*\pi_{d*}\varphi_{\mathrm{Tr}\,W}\mathrm{IC}_{\mathcal{X}(d)}[-1]&\xrightarrow{\sim}\pi^{\daleth}_{d*}\varphi_{\mathrm{Tr}\,W}\left(\varphi_{\mathrm{Tr}\,W^q}\mathrm{IC}_{\mathcal{X}^{\daleth}(d)}[-1]\right)\\&\cong\pi^{\daleth}_{d*}\varphi_{\mathrm{Tr}\,W^{\daleth}}\mathrm{IC}_{\mathcal{X}^{\daleth}(d)}[-1].\end{aligned}$$

We now explain that the isomorphism (6.24) induces an isomorphism of sheaves in $D^b_{\mathrm{con}}(X^{\daleth}(d))$:

$$u_*\mathcal{BPS}_{d,\delta}\xrightarrow{\sim}\mathcal{BPS}^{\daleth}_{d,\delta}.$$

First, we explain that $S^d_\delta(Q)=S^d_\delta(Q^{\daleth})$. Let λ be a cocharacter of $T(d)$. Let n_λ and $n^{\daleth}_\lambda$ be the integers (2.20) for Q and $Q^{\daleth}$, respectively. Let $\varepsilon_{\lambda,\delta}$ and $\varepsilon^{\daleth}_{\lambda,\delta}$ be the integers (6.1) for Q and $Q^{\daleth}$, respectively. Then

$$n^{\daleth}_\lambda-n_\lambda=2\langle\lambda,\mathfrak{g}(d)^{\lambda>0}\rangle,$$

thus $\varepsilon_{\lambda,\delta}=\varepsilon^{\daleth}_{\lambda,\delta}$, so indeed $S^d_\delta(Q)=S^d_\delta(Q^{\daleth})=:S^d_\delta$.

It suffices to check that (6.24) induces isomorphisms:

$$u_*\mathcal{BPS}_A\xrightarrow{\sim}\mathcal{BPS}^{\daleth}_A \tag{6.27}$$

for any $A\in S^d_\delta$. The isomorphism (6.24) is obtained by applying the functor $\varphi_{\mathrm{Tr}\,W}$ to the isomorphism (6.24) for $W=0$, that is, from the isomorphism (6.25). Therefore it suffices to check (6.27) when $W=0$, so we assume that $W=0$ in the rest of the proof. Assume A has a corresponding partition $(d_i)_{i=1}^k$ of d. Let X_A be the image of $\oplus\colon \times_{i=1}^k X(d_i)\to X(d)$. There is an isomorphism:

$$u_*{}^p\mathcal{H}^k(\pi_{d*}\mathrm{IC}_{\mathcal{X}(d)})\xrightarrow{\sim}{}^p\mathcal{H}^k(\pi^{\daleth}_{d*}\varphi_{\mathrm{Tr}\,W^q}\mathrm{IC}_{\mathcal{X}^{\daleth}(d)}[-1]).$$

There are either no summands of support X_A on both sides, case in which both $u_*\mathcal{BPS}_A$ and $\mathcal{BPS}^{\daleth}_A$ are zero, or there are unique summands of support X_A on both sides, namely $u_*\mathcal{BPS}_A$ and $\mathcal{BPS}^{\daleth}_A$, and thus (6.27) follows. $\square$

We note the following corollary of Proposition 6.11.

Corollary 6.12. *Let $\delta\in M(d)^{W_d}_{\mathbb{R}}$ and let $i\in\mathbb{Z}$. There is an isomorphism:*

$$\begin{aligned}s_*v^*\colon H^i(\mathcal{X}(d),\varphi^{\mathrm{inv}}_{\mathrm{Tr}\,W}\mathrm{IC}_{\mathcal{X}(d)}[-1])&\xrightarrow{\sim}H^i(\mathcal{X}^{\daleth}(d),\varphi^{\mathrm{inv}}_{\mathrm{Tr}\,W^{\daleth}}\mathrm{IC}_{\mathcal{X}^{\daleth}(d)}[-1]),\\H^i(X(d),\mathcal{BPS}^{\mathrm{inv}}_{d,\delta})&\xrightarrow{\sim}H^i(X^{\daleth}(d),\mathcal{BPS}^{\daleth,\mathrm{inv}}_{d,\delta}).\end{aligned}$$

We also relate quasi-BPS categories under Knorrer periodicity:

Proposition 6.13. *There is an equivalence:*

$$s_*v^*\colon \mathrm{MF}(\mathcal{X}(d), \operatorname{Tr} W) \xrightarrow{\sim} \mathrm{MF}(\mathcal{X}^{\daleth}(d), \operatorname{Tr} W^{\daleth}),$$
$$\mathbb{S}(d;\delta) \xrightarrow{\sim} \mathbb{S}^{\daleth}(d;\delta).$$

Proof. (cf. [PTe, Proposition 2.14]) Consider the Koszul complex

$$\mathcal{K} := s_*v^*\mathcal{O}_{\mathcal{X}} \in \mathrm{MF}(\mathcal{X}^{\daleth}(d), \operatorname{Tr} W^q),$$

where $s_*v^*\colon \mathrm{MF}(\mathcal{X}(d), 0) \xrightarrow{\sim} \mathrm{MF}(\mathcal{X}^{\daleth}(d), \operatorname{Tr} W^q)$ is an equivalence by Knorrer periodicity. By the Thom-Sebastiani theorem for matrix factorizations [Pre], there is then an equivalence:

$$t^*(-)\otimes \mathcal{K}\colon \mathrm{MF}(\mathcal{X}(d), \operatorname{Tr} W) \xrightarrow{\sim} \mathrm{MF}(\mathcal{X}^{\daleth}(d), \operatorname{Tr} W^{\daleth}).$$

Note that $t^*(-)\otimes\mathcal{K} = s_*v^*(-)$. It remains to show that

$$t^*(-)\otimes \mathcal{K}\colon \mathbb{S}(d;\delta) \xrightarrow{\sim} \mathbb{S}^{\daleth}(d;\delta).$$

It suffices to show that, for $F \in D^b(\mathcal{X}(d))$, we have that $F \in \mathbb{M}(d;\delta)$ if and only if $t^*(F)\otimes\mathcal{K} \in \mathbb{M}^{\daleth}(d;\delta)$.

We use Lemma 2.3. Let $\nu\colon B\mathbb{C}^* \to \mathcal{X}(d)$, let $F \in D^b(\mathcal{X}(d))$, and let $A_F \subset \mathbb{Z}$ be the set of weights of $\nu^*(F)$. Note that for any $\nu^{\daleth}\colon B\mathbb{C}^* \to \mathcal{X}^{\daleth}(d)$ such that $t\circ\nu^{\daleth} = \nu$, we have that the weights of $(\nu^{\daleth})^*(t^*(F)\otimes\mathcal{K})$ are the Minkowski sum $A_F + [-\langle \nu, \mathfrak{g}\rangle, \langle \nu, \mathfrak{g}\rangle]$. We have that

$$A_F \subset \left[-\frac{1}{2}n_\lambda + \langle \lambda, \delta_d\rangle, \frac{1}{2}n_\lambda + \langle\lambda,\delta_d\rangle\right]$$

if and only if

$$A_F + [-\langle \nu, \mathfrak{g}\rangle, \langle \nu, \mathfrak{g}\rangle] \subset \left[-\frac{1}{2}n^{\daleth}_\lambda + \langle \lambda, \delta_d\rangle, \frac{1}{2}n^{\daleth}_\lambda + \langle\lambda,\delta_d\rangle\right].$$

The conclusion then follows. □

Proof of Proposition 6.10. Let $i, \ell \in \mathbb{Z}$. By Corollary 4.9, there is a commutative diagram, where $b = \dim \mathcal{X}(d) - i - 2\ell = \dim \mathcal{X}^{\daleth}(d) - i - 2(\ell + \dim \mathfrak{g})$:

$$\begin{tikzcd}
\mathrm{gr}_\ell K_i^{\mathrm{top}}(\mathrm{MF}(\mathcal{X}(d), \operatorname{Tr} W)) \arrow[r, "s_*v^*", "\sim"'] \arrow[d, "\mathrm{c}", "\wr"'] & \mathrm{gr}_{\ell+\dim\mathfrak{g}} K_i^{\mathrm{top}}(\mathrm{MF}(\mathcal{X}^{\daleth}(d), \operatorname{Tr} W^{\daleth})) \arrow[d, "\mathrm{c}", "\wr"'] \\
H^b(\mathcal{X}(d), \varphi^{\mathrm{inv}}_{\operatorname{Tr} W}\mathrm{IC}_{\mathcal{X}(d)}[-2]) \arrow[r, "s_*v^*", "\sim"'] & H^b(\mathcal{X}^{\daleth}(d), \varphi^{\mathrm{inv}}_{\operatorname{Tr} W^{\daleth}}\mathrm{IC}_{\mathcal{X}^{\daleth}(d)}[-2]).
\end{tikzcd}$$

The conclusion follows from Corollary 6.12 and Proposition 6.13. □

It will be convenient to make the following assumption on a quiver:

Assumption 6.1. Assume the quiver Q is symmetric and has at least two loops at any vertex.

We introduce some notation. For any cocharacter λ with associated partition $\mathbf{d}$, define $c_{\mathbf{d}} := c_\lambda := \dim \mathcal{X}(d) - \dim \mathcal{X}(d)^{\lambda\geqslant 0}$.

Lemma 6.14. *Let $Q = (I, E)$ be a quiver which satisfies Assumption 6.1 and let $d \in \mathbb{N}^I$ be non-zero.*

(a) For any cocharacter λ of $T(d)$, we have $c_\lambda \geqslant 0$, and the inequality is strict if λ has an associated partition with at least two terms. Moreover, we have

$$\dim \mathcal{X}(d)^{\lambda\geqslant 0} - \dim \mathcal{X}(d)^\lambda = c_\lambda.$$

(b) The map $\pi_d\colon \mathcal{X}(d)\to X(d)$ is generically a $\mathbb{C}^*$-gerbe, in particular there exists a stable representation of dimension d.

Proof. We only discuss the first claim of part (a), the second claim and part (b) are similar. Let $S(d)$ be the affine space of dimension d representations of the quiver obtained from Q by deleting one loop at every vertex in I. Then

$$\mathcal{X}(d)=\left(S(d)\oplus\mathfrak{g}(d)\right)/G(d).$$

We have that

$$\dim\mathcal{X}(d)=\dim S(d)\text{ and }\dim\mathcal{X}(d)^{\lambda\geqslant 0}=\dim\left(S(d)\right)^{\lambda\geqslant 0},$$

so $c_\lambda=\dim\left(S(d)\right)^{\lambda<0}$, and the first claim follows. $\square$

6.5. **Coproduct-like maps in K-theory.** In this subsection,we assume that Q satisfies Assumption 6.1. Consider an antidominant cocharacter λ of $T(d)$ and let $a_\lambda\colon\mathcal{X}(d)^\lambda\to\mathcal{X}(d)$ be the natural morphism inducing pullback maps for any $i,\ell\in\mathbb{Z}$:

$$\begin{aligned}
&a_\lambda^*\colon K_i^{\mathrm{top}}\left(\mathrm{MF}\left(\mathcal{X}(d),\mathrm{Tr}\,W\right)\right)\to K_i^{\mathrm{top}}\left(\mathrm{MF}\left(\mathcal{X}(d)^\lambda,\mathrm{Tr}\,W\right)\right),\\
&a_\lambda^*\colon \mathrm{gr}_\ell K_i^{\mathrm{top}}(\mathrm{MF}\left(\mathcal{X}(d),\mathrm{Tr}\,W\right))\to \mathrm{gr}_{\ell+2d_\lambda}K_i^{\mathrm{top}}\left(\mathrm{MF}\left(\mathcal{X}(d)^\lambda,\mathrm{Tr}\,W\right)\right),
\end{aligned}$$

where $d_\lambda:=\dim\mathcal{X}(d)^\lambda-\dim\mathcal{X}(d)$ is the relative dimension of a_λ. Consider the quotient $G(d)':=G(d)^\lambda/\mathrm{image}(\lambda)$ and the stack $\mathcal{X}(d)'^\lambda:=R(d)^\lambda/G(d)'$. There is an isomorphism:

$$K_i^{\mathrm{top}}\left(\mathrm{MF}\left(\mathcal{X}(d)^\lambda,\mathrm{Tr}\,W\right)\right)\cong K_i^{\mathrm{top}}\left(\mathrm{MF}\left(\mathcal{X}(d)'^\lambda,\mathrm{Tr}\,W\right)\right)[q^{\pm1}].$$

There is an analogous isomorphism for graded K-theory. There are also maps in cohomology:

$$\begin{aligned}
&a_\lambda^*\colon H^\cdot\left(\mathcal{X}(d),\varphi_{\mathrm{Tr}\,W}\right)\to H^\cdot\left(\mathcal{X}(d)'^\lambda,\varphi_{\mathrm{Tr}\,W}\right)[h],\\
&a_\lambda^*\colon H^\cdot\left(\mathcal{X}(d),\varphi^{\mathrm{inv}}_{\mathrm{Tr}\,W}\right)\to H^\cdot\left(\mathcal{X}(d)'^\lambda,\varphi^{\mathrm{inv}}_{\mathrm{Tr}\,W}\right)[h].
\end{aligned}$$

Assume the associated partition of λ is $\mathbf{d}=(d_i)_{i=1}^k$. Recall that $c_\lambda:=c_{\mathbf{d}}:=\dim\mathcal{X}(d)-\dim\mathcal{X}(d)^{\lambda\geqslant 0}$. We define the following integers (which we call *widths* of magic or quasi-BPS categories in this paper):

$$c_{\lambda,\delta}:=c_\lambda+\varepsilon_{\lambda,\delta},\ c_{\mathbf{d},\delta}:=c_{\mathbf{d}}+\varepsilon_{\mathbf{d},\delta}. \tag{6.28}$$

Proposition 6.15. *Let λ be an antidominant cocharacter of $G(d)$ and let $i,\ell\in\mathbb{Z}$. Consider the diagram:*

$$\begin{tikzcd}
\mathrm{gr}_\ell K_i^{\mathrm{top}}(\mathbb{S}(d;\delta)) \arrow[r, "a_\lambda^*"] \arrow[d, hook, "\mathrm{c}"] & \mathrm{gr}_{\ell+2d_\lambda}K_i^{\mathrm{top}}\left(\mathrm{MF}\left(\mathcal{X}(d)^\lambda,\mathrm{Tr}\,W\right)\right) \arrow[d, hook, "\mathrm{c}"] \\
H^{2\dim\mathcal{X}(d)-2\ell-i}\left(\mathcal{X}(d),\varphi^{\mathrm{inv}}_{\mathrm{Tr}\,W}[-2]\right) \arrow[r, "a_\lambda^*"] & H^\cdot\left(\mathcal{X}(d)'^\lambda,\varphi^{\mathrm{inv}}_{\mathrm{Tr}\,W}[-2]\right)[h].
\end{tikzcd}$$

Then the image of $\mathrm{c}\,a_\lambda^*\mathrm{gr}_\bullet K_\bullet^{\mathrm{top}}\left(\mathbb{S}(d;\delta)\right)$ lies in the subspace

$$\bigoplus_{j=0}^{c_{\lambda,\delta}-1}H^{2\dim\mathcal{X}(d)-2\ell-i-2j}\left(\mathcal{X}(d)'^\lambda,\varphi^{\mathrm{inv}}[-2]\right)h^j\subset H^\cdot\left(\mathcal{X}(d)'^\lambda,\varphi^{\mathrm{inv}}[-2]\right)[h].$$

Note that a_λ only depends on the partition $\mathbf{d}$, so we obtain that the image of $\mathrm{c}\, a_\lambda^* \mathrm{gr}_\ell K_i^{\mathrm{top}}\left(\mathbb{S}(d;\delta)\right)$ *lies in the subspace*

$$\bigoplus_{j=0}^{c_{d,\delta}-1} H^{2\dim \mathcal{X}(d)-2\ell-i-2j}\left(\mathcal{X}(d)'^{\lambda}, \varphi^{\mathrm{inv}}[-2]\right) h^j \subset H^{\cdot}\left(\mathcal{X}(d)'^{\lambda}, \varphi^{\mathrm{inv}}[-2]\right)[h].$$

Proof. Consider a complex A in $\mathbb{S}(d;\delta)$. Then $a_\lambda^*(A)$ is in the subcategory of $\mathrm{MF}(\mathcal{X}(d)^\lambda, \mathrm{Tr}\, W)$ generated by $\mathrm{MF}(\mathcal{X}(d)'^\lambda, \mathrm{Tr}\, W)_v$ for

$$v \in S_{\lambda,\delta} := \left[-\frac{1}{2}\langle \lambda, \mathbb{L}_{\mathcal{X}(d)}^{\lambda>0}\rangle + \langle \lambda, \delta\rangle, \frac{1}{2}\langle \lambda, \mathbb{L}_{\mathcal{X}(d)}^{\lambda>0}\rangle + \langle \lambda, \delta\rangle\right] \cap \mathbb{Z}.$$

Thus

$$a_\lambda^* K_i^{\mathrm{top}}\left(\mathbb{S}(d;\delta)\right) \subset K_i^{\mathrm{top}}\left(\mathrm{MF}\left(\mathcal{X}(d)'^\lambda, \mathrm{Tr}\, W\right)\right) \otimes \mathcal{A}, \tag{6.29}$$

where $\mathcal{A} := \bigoplus_{j \in S_{\lambda,\delta}} \mathbb{Q} \cdot q^j$. There are filtrations pulled back from cohomology by the Chern character for both $K_i^{\mathrm{top}}\left(\mathrm{MF}(\mathcal{X}(d)'^\lambda, \mathrm{Tr}\, W)\right)$ and $K_0^{\mathrm{top}}(B\mathbb{C}^*)$, and there is an isomorphism obtained by the Kunneth formula:

$$\mathrm{gr}_\ell K_i^{\mathrm{top}}\left(\mathrm{MF}(\mathcal{X}(d)^\lambda, \mathrm{Tr}\, W)\right) \cong \bigoplus_{a+b=\ell} \mathrm{gr}_a K_i^{\mathrm{top}}\left(\mathrm{MF}(\mathcal{X}(d)'^\lambda, \mathrm{Tr}\, W)\right) \otimes \mathrm{gr}_b K_0^{\mathrm{top}}\left(B\mathbb{C}^*\right). \tag{6.30}$$

The filtration $E_b K_0^{\mathrm{top}}(B\mathbb{C}^*)$ on $K_0^{\mathrm{top}}(B\mathbb{C}^*)$ induces a filtration

$$E_b \mathcal{A} := \mathcal{A} \cap E_b K_0^{\mathrm{top}}(B\mathbb{C}^*)$$

on $\mathcal{A}$. There are natural inclusions $\mathrm{gr}_b \mathcal{A} \hookrightarrow \mathrm{gr}_b K_0^{\mathrm{top}}(B\mathbb{C}^*)$. We obtain a Kunneth formula:

$$\mathrm{gr}_\ell\left(K_i^{\mathrm{top}}\left(\mathrm{MF}(\mathcal{X}(d)'^\lambda, \mathrm{Tr}\, W)\right) \otimes \mathcal{A}\right) \cong \bigoplus_{a+b=\ell} \mathrm{gr}_a K_i^{\mathrm{top}}\left(\mathrm{MF}(\mathcal{X}(d)'^\lambda, \mathrm{Tr}\, W)\right) \otimes \mathrm{gr}_b \mathcal{A}. \tag{6.31}$$

By (6.29) and (6.31), we have

$$a_\lambda^* \mathrm{gr}_\ell K_i^{\mathrm{top}}\left(\mathbb{S}(d;\delta)\right) \subset \mathrm{gr}_{\cdot} K_i^{\mathrm{top}}\left(\mathrm{MF}\left(\mathcal{X}(d)'^\lambda, \mathrm{Tr}\, W\right)\right) \otimes \mathrm{gr}_{\cdot} \mathcal{A}.$$

It suffices to show that:

$$\mathrm{c}\left(\mathrm{gr}_{\cdot} \mathcal{A}\right) \subset \bigoplus_{i=0}^{c_{\lambda,\delta}-1} \mathbb{Q} \cdot h^i. \tag{6.32}$$

For any $1 \leqslant i \leqslant k$, let F_i be a stable representation of dimension d_i (which exists by Lemma 6.14, and note that this is the only place where we use that Q satisfies Assumption 6.1) and let $F := \bigoplus_{i=1}^k F_i$. Let $V/(\mathbb{C}^*)^k$ be the moduli stack of representations of the Ext quiver of F and dimension vector $(1, \ldots, 1) \in \mathbb{N}^k$. Note that there is an étale map

$$V/(\mathbb{C}^*)^k \to \mathcal{X}(d),\ 0 \mapsto F.$$

We have the equality of sets

$$S_{\lambda,\delta} = \Big[-\frac{1}{2}\langle \lambda, V^{\lambda>0}\rangle + \langle \lambda, \delta\rangle, \frac{1}{2}\langle \lambda, V^{\lambda>0}\rangle + \langle \lambda, \delta\rangle\Big] \cap \mathbb{Z}.$$

We denote the image of λ in $(\mathbb{C}^*)^k$ by $\mathbb{C}^*$. Consider the maps:

$$V^\lambda \xleftarrow{q'} V^{\lambda \geqslant 0} \xrightarrow{p'} V.$$

Let ℓ be a generic linearization of $\mathbb{C}^*$. By [HL12, Theorem 2.10], see also [HLS16, Equation (3)], the subcategory of $D^b(V/\mathbb{C}^*)$ generated by $\mathcal{O}_V(v)$ for weights $v \in S_{\lambda,\delta}$ is equivalent to $D^b(V^{\ell\text{-ss}}/\mathbb{C}^*)$ if $\varepsilon_{\lambda,\delta}=0$, and has a semiorthogonal decomposition with pieces $D^b(V^{\ell\text{-ss}}/\mathbb{C}^*)$ and $p'_*q'^*D^b(V^\lambda)$ if $\varepsilon_{\lambda,\delta}=1$. Define the map

$$\mathrm{s}\colon K_0^{\mathrm{top}}(B\mathbb{C}^*) \cong K_0^{\mathrm{top}}(V/\mathbb{C}^*) \to K_0^{\mathrm{top}}(V^{\ell\text{-ss}}/\mathbb{C}^*) \oplus p'_*q'^*K_0^{\mathrm{top}}(V^\lambda)^{\oplus \varepsilon_{\lambda,\delta}}$$

as the direct sum of the restriction onto $V^{\ell\text{-ss}}/\mathbb{C}^*$ and the inverse of the inclusion:

$$p'_*q'^*\colon K_0^{\mathrm{top}}\left(D^b(V^\lambda)_a\right)^{\oplus \varepsilon_{\lambda,\delta}} \cong K_0^{\mathrm{top}}(V^\lambda)^{\oplus \varepsilon_{\lambda,\delta}} \to K_0^{\mathrm{top}}(V/\mathbb{C}^*)$$

for a weight $a = \left\lfloor \frac{1}{2}\langle \lambda, V^{\lambda>0}\rangle + \langle \lambda, \delta\rangle \right\rfloor \in \mathbb{Z}$ of λ, constructed by the semiorthogonal decomposition [HL12]. The following composition is an isomorphism:

$$\mathcal{A} \hookrightarrow K_0^{\mathrm{top}}(B\mathbb{C}^*) \cong K_0^{\mathrm{top}}(V/\mathbb{C}^*) \xrightarrow{\mathrm{s}} K_0^{\mathrm{top}}(V^{\ell\text{-ss}}/\mathbb{C}^*) \oplus p'_*q'^*K_0^{\mathrm{top}}(V^\lambda)^{\oplus \varepsilon_{\lambda,\delta}}.$$

Note that the Hall product $p'_*q'^*\colon H^\cdot(V^\lambda) \to H^\cdot(V/\mathbb{C}^*)$ has image $\mathbb{Q}\cdot h^{c_\lambda}$, and thus it has a natural inverse $H^\cdot(V/\mathbb{C}^*) \to p'_*q'^*H^\cdot(V^\lambda)$. Let t be the direct sum of this inverse and the restriction map:

$$\mathrm{t}\colon H^{\mathrm{BM}}_\cdot(V/\mathbb{C}^*) \to H^{\mathrm{BM}}_\cdot(V^{\ell\text{-ss}}/\mathbb{C}^*) \oplus p'_*q'^*H^{\mathrm{BM}}_\cdot(V^\lambda)^{\oplus \varepsilon_{\lambda,\delta}}.$$

Recall that the Hall products in K-theory and cohomology are compatible via the cycle map, see Proposition 3.6. There is a commutative diagram:
(6.33)

$$\begin{tikzcd}
\mathrm{gr}_\cdot\mathcal{A} \arrow[r] \arrow[dr, "\mathrm{c}"'] & \mathrm{gr}_\cdot K_0^{\mathrm{top}}(V/\mathbb{C}^*) \arrow[r, "\mathrm{s}"] \arrow[d, "\mathrm{c}"] & \mathrm{gr}_\cdot K_0^{\mathrm{top}}(V^{\ell\text{-ss}}/\mathbb{C}^*) \oplus p'_*q'^*\mathrm{gr}_\cdot K^{\mathrm{top}}(V^\lambda)^{\oplus \varepsilon_{\lambda,\delta}} \arrow[d, "\mathrm{c}"] \\
& H^{\mathrm{BM}}_\cdot(V/\mathbb{C}^*) \arrow[r, "\mathrm{t}"] & H^{\mathrm{BM}}_\cdot(V^{\ell\text{-ss}}/\mathbb{C}^*) \oplus p'_*q'^*H^{\mathrm{BM}}_\cdot(V^\lambda)^{\oplus \varepsilon_{\lambda,\delta}}.
\end{tikzcd}$$

Note that $V^{\ell\text{-ss}}/\mathbb{C}^* = V^\lambda \times V'^{\ell\text{-ss}}/\mathbb{C}^*$, where $V' \subset V$ is the subspace spanned by non-zero λ-weights, and thus

$$H^\cdot(V^{\ell\text{-ss}}/\mathbb{C}^*) \cong H^\cdot(V'^{\ell\text{-ss}}/\mathbb{C}^*) \cong \bigoplus_{i=0}^{c_\lambda-1}\mathbb{Q}\cdot h^i.$$

Then the map t is the truncation of polynomials:

$$\begin{tikzcd}
H^{\mathrm{BM}}_\cdot(V/\mathbb{C}^*) \arrow[r, "\mathrm{t}"] \arrow[d, "\wr"] & H^{\mathrm{BM}}_\cdot(V^{\ell\text{-ss}}/\mathbb{C}^*) \oplus p'_*q'^*H^{\mathrm{BM}}_\cdot(V^\lambda)^{\oplus \varepsilon_{\lambda,\delta}} \arrow[d, "\wr"] \\
\mathbb{Q}[h] \arrow[r] & \bigoplus_{i=0}^{c_{\lambda,\delta}-1}\mathbb{Q}\cdot h^i.
\end{tikzcd} \tag{6.34}$$

The conclusion then follows. ☐

6.6. **Coproduct-like maps in cohomology.** In this section, we assume that $Q=(I,E)$ satisfies Assumption 6.1. By Proposition 6.10, we obtain Theorem 6.3 for general symmetric quivers Q.

For λ an antidominant cocharacter with associated partition $(d_i)_{i=1}^k$, consider the good moduli space map

$$\pi_\lambda := \times_{i=1}^k \pi_{d_i}\colon \mathcal{X}(d)^\lambda \to X(d)^\lambda := \times_{i=1}^k X(d_i).$$

Consider the projection map $t_\lambda\colon \mathcal{X}(d)^\lambda \to \mathcal{X}(d)'^\lambda$. The maps introduced fit in the following diagram:

$$\begin{tikzcd}
\mathcal{X}(d)^\lambda \arrow[r, "a_\lambda"] \arrow[d, "t_\lambda"] \arrow[dd, bend right, "\pi_\lambda"'] & \mathcal{X}(d) \arrow[dd, "\pi_d"] \\
\mathcal{X}(d)'^\lambda \arrow[d] & \\
X(d)^\lambda \arrow[r, "i_\lambda"] & X(d).
\end{tikzcd}$$

Consider the following perverse truncation

$$\begin{aligned}
\text{(6.35)} \qquad \mathrm{S}\colon t_{\lambda*}\mathrm{IC}_{\mathcal{X}(d)^\lambda}[c_\lambda] &\cong \mathrm{IC}_{\mathcal{X}(d)'^\lambda}[c_\lambda - 1]\otimes \mathbb{Q}[h] \\
&\to {}^p\tau^{\geqslant c_\lambda+1}\left(\mathrm{IC}_{\mathcal{X}(d)'^\lambda}[c_\lambda-1]\otimes\mathbb{Q}[h]\right) \\
&\cong \bigoplus_{j\geqslant 0}\mathrm{IC}_{\mathcal{X}(d)'^\lambda}[-c_\lambda-1-2j]\cong t_{\lambda*}\mathrm{IC}_{\mathcal{X}(d)^\lambda}[-c_\lambda].
\end{aligned}$$

Define the map Δ_λ as the composition:

$$\text{(6.36)} \quad \Delta_\lambda\colon \pi_{d*}\mathrm{IC}_{\mathcal{X}(d)} \to \pi_{d*}a_{\lambda*}\mathrm{IC}_{\mathcal{X}(d)^\lambda}[2c_\lambda] = i_{\lambda*}\pi_{\lambda*}\mathrm{IC}_{\mathcal{X}(d)^\lambda}[2c_\lambda] \xrightarrow{\mathrm{S}} i_{\lambda*}\pi_{\lambda*}\mathrm{IC}_{\mathcal{X}(d)^\lambda}.$$

Recall the notations from Subsection 6.2 and the decomposition theorem (6.16). Consider the total perverse cohomology

$$\mathcal{H}\left(\pi_{d*}\mathrm{IC}_{\mathcal{X}(d)}\right) := \bigoplus_{i\in\mathbb{Z}} {}^p\mathcal{H}^i\left(\pi_{d*}\mathrm{IC}_{\mathcal{X}(d)}\right)[-i].$$

For $A\in\mathcal{P}$ as in Subsection 6.2, there are then natural maps

$$\mathrm{P}_A \to \mathcal{H}\left(\pi_{d*}\mathrm{IC}_{\mathcal{X}(d)}\right)\to \mathrm{P}_A.$$

Proposition 6.16. *Let $A, B\in\mathcal{P}$ with corresponding sheaves P_A and P_B of different support. Assume that $p_B\leqslant p_A$.*

(a) The map (6.36) *induces an isomorphism*

$$\text{(6.37)} \qquad \Delta_\lambda\colon \mathrm{P}_A \xrightarrow{\sim} \mathrm{P}_A.$$

(b) The map $\Delta_\lambda\colon \mathrm{P}_B\to\mathrm{P}_A$ is zero.

Proof. (a) Assume λ has associated partition $(d_i)_{i=1}^k$. Assume further that the set $\{d_1,\ldots,d_k\} = \{e_1,\ldots,e_s\}$ has cardinality s and that each e_i appears m_i times among the d_j for $1\leqslant j\leqslant k$. Let $A^\circ\in\mathcal{P}$ be the tuplet $(e_i, m_{i,a})$ with $m_{i,0}=m_i$ and $m_{i,a}=0$ for $a\geqslant 1$.

For $d\in\mathbb{N}^I$, let $\hbar_d := c_1(\mathcal{O}(\sigma_d))\in H^2(\mathcal{X}(d))$. By [DM20, Theorem C], the summand P_A of $\mathcal{H}\left(\pi_{d*}\mathrm{IC}_{\mathcal{X}(d)}\right)$ is obtained from $\boxtimes_{i=1}^k(\mathrm{IC}_{X(d_i)}[-1])$ by multiplication with the equivariant parameters $\hbar_{d_i}$ for $1\leqslant i\leqslant k$. The map $\Delta_\lambda\colon \mathrm{P}_A\to\mathrm{P}_A$ is thus obtained by multiplication by equivariant parameters $\hbar_{d_i}$ for $1\leqslant i\leqslant k$ from the map

$$\Delta_\lambda m_\lambda\colon i_{\lambda*}\boxtimes_{i=1}^k\left(\mathrm{IC}_{X(d_i)}[-1]\right)\to i_{\lambda*}\boxtimes_{i=1}^k\left(\mathrm{IC}_{X(d_i)}[-1]\right).$$

By [DM20, Theorem C], the image of $m_\lambda\left(i_{\lambda*}\boxtimes_{i=1}^k\left(\mathrm{IC}_{X(d_i)}[-1]\right)\right)$ in $\mathcal{H}\left(\pi_{d*}\mathrm{IC}_{\mathcal{X}(d)}\right)$ is P_{A°. Thus the map (6.37) is obtained by multiplication by equivariant parameters from the map

$$\Delta_\lambda\colon \mathrm{P}_{A^\circ}\to\mathrm{P}_{A^\circ}.$$

We may thus assume that $A = A^{\circ}$. The Hall product is induced by a map

$$m_\lambda \colon i_{\lambda*}\pi_{\lambda*}\mathrm{IC}_{\mathcal{X}(d)^\lambda} \to \pi_{d*}\mathrm{IC}_{\mathcal{X}(d)}.$$

The lowest non-zero piece of the perverse filtration on $i_{\lambda*}\pi_{\lambda*}\mathrm{IC}_{\mathcal{X}(d)^\lambda}$ is given by

$${}^p\tau^{\leqslant k} i_{\lambda*}\pi_{\lambda*}\mathrm{IC}_{\mathcal{X}(d)^\lambda} = i_{\lambda*} \boxtimes_{i=1}^{s} \left(\mathrm{IC}_{X(e_i)}[-1]\right)^{\boxtimes m_i}.$$

The (shifted) perverse sheaf $i_{\lambda*} \boxtimes_{i=1}^{s} \left(\mathrm{IC}_{X(e_i)}[-1]\right)^{\boxtimes m_i}$ splits as a direct sum of simple sheaves, and one such sheaf is P_A. There is thus a natural inclusion $\mathrm{P}_A \subset i_{\lambda*}\pi_{\lambda*}\mathrm{IC}_{\mathcal{X}(d)^\lambda}$. The map

$$\Delta_\lambda m_\lambda \colon \mathrm{P}_A \to i_{\lambda*}\pi_{\lambda*}\mathrm{IC}_{\mathcal{X}(d)^\lambda} \tag{6.38}$$

has image in the lowest non-zero perverse truncation of $i_{\lambda*}\pi_{\lambda*}\mathrm{IC}_{\mathcal{X}(d)^\lambda}$, and thus (6.38) induces a map:

$$\Delta_\lambda m_\lambda \colon \mathrm{P}_A \to {}^p\tau^{\leqslant s} i_{\lambda*}\pi_{\lambda*}\mathrm{IC}_{\mathcal{X}(d)^\lambda} = \boxtimes_{i=1}^{s} \left(\mathrm{IC}_{X(e_i)}[-1]\right)^{\boxtimes m_i}. \tag{6.39}$$

The (shifted) perverse sheaf $i_{\lambda*} \boxtimes_{i=1}^{s} \left(\mathrm{IC}_{X(e_i)}[-1]\right)^{\boxtimes m_i}$ has only one summand isomorphic to P_A, which is a simple (shifted) perverse sheaf. Thus the map (6.39) restricts to a map

$$\mathrm{P}_A \to \mathrm{P}_{\mathrm{A}} \tag{6.40}$$

All such maps are given by multiplication by scalars. It is thus an isomorphism if it is not the zero map. It suffices to show that the maps (6.38) or (6.39) are not zero. We will show this after passing to a convenient open set of $X(d)^\lambda$.

For any non-zero $e \in \mathbb{N}^I$, by the same argument used to prove that (6.14), there exists a stable point in $R(e)$, equivalently the map $\pi_e \colon \mathcal{X}(e) \to X(e)$ is generically a $\mathbb{C}^*$-gerbe. For $1 \leqslant i \leqslant k$, let R_i be a simple representation of Q of dimension d_i such that R_i and R_j are not isomorphic for $1 \leqslant i < j \leqslant k$. Let $R := \bigoplus_{i=1}^{k} R_i$. Note that the stabilizer of R is $T = (\mathbb{C}^*)^k$. By the etale slice theorem, there is an analytic smooth open substack $R \in \mathcal{U}/T \subset \mathcal{X}(d)$ such that

$$\mathcal{U}/\!\!/T \to X(d) \text{ and } \mathcal{U}^\lambda \to X(d)^\lambda$$

are an analytic neighborhoods of $\pi_d(R)$ and $\times_{i=1}^{k}\pi_{d_i}(R_i) = \pi_\lambda(R)$, respectively. After possibly shrinking $\mathcal{U}$, we may assume that $\mathcal{U}$ and $\mathcal{U}^\lambda$ are contractible. The maps

$$\begin{tikzcd} \mathcal{X}(d)^\lambda & \mathcal{X}(d)^{\lambda\geqslant 0} \arrow[l, "q_\lambda"'] \arrow[r, "p_\lambda"] & \mathcal{X}(d) \arrow[from=1-1, to=1-3, bend right, "a_\lambda"'] \end{tikzcd}$$

are, analytically locally over $\pi_d(R) \in X(d)$, isomorphic to the following:

$$\begin{tikzcd} \mathcal{U}^\lambda/T & \mathcal{U}^{\lambda\geqslant 0}/T \arrow[l, "q_\lambda"'] \arrow[r, hook, "p_\lambda"] & \mathcal{U}/T. \arrow[from=1-1, to=1-3, hook, bend right, "a_\lambda"'] \end{tikzcd} \tag{6.41}$$

Note that the maps p_λ and a_λ in (6.41) are closed immersions. To show that the map (6.40) is non-zero, it suffices to check that the map

$$\Delta_\lambda m_\lambda|_{\mathcal{U}^\lambda} \colon \mathrm{P}_A|_{\mathcal{U}^\lambda} \to \mathrm{P}_A|_{\mathcal{U}^\lambda} \tag{6.42}$$

is non-zero. It suffices to check that the map is non-zero after passing to global sections. We drop the restriction to $\mathcal{U}^\lambda$ from the notation from now on. The element $1 \in H^0(\mathcal{U}^\lambda/T)$ is in $P^{\leqslant s}H^\cdot(\mathcal{U}^\lambda/T)$. We check by a direct computation that

$$\Delta_\lambda m_\lambda(1) = 1 \in H^0(\mathcal{U}^\lambda/T). \tag{6.43}$$

Note that the computation (6.43) shows that the map (6.42) is non-zero, and thus the conclusion follows. It suffices to check the computation in (6.43) for $\mathcal{U}^\lambda/\mathbb{C}^*$, where by $\mathbb{C}^*$ we denote the image of λ, because $H^0(\mathcal{U}^\lambda/\mathbb{C}^*) \cong H^0(\mathcal{U}^T/T) \cong \mathbb{Q}$. Observe that $H^\cdot(\mathcal{U}^\lambda/\mathbb{C}^*) \cong \mathbb{Q}[h]$ and that

$$m_\lambda(1) = p_{\lambda*}q_\lambda^*(1) = h^{c_\lambda}$$

because p_λ has relative dimension $-c_\lambda$. Note that $\Delta_\lambda(h^{c_\lambda}) = 1$ from the construction of Δ_λ, and thus the conclusion follows.

(b) If $p_B < p_A$, the map $\Delta_\lambda \colon \mathrm{P}_B \to \mathrm{P}_A$ is zero by considering the perverse degree. If $p_B = p_A$, then the map is zero because, after a shift, it is a map of simple perverse sheaves with different support. $\square$

We next prove the analogue of Proposition 6.16 for a non-zero potential. Let W be an arbitrary potential of Q. Recall the sheaves Q_A from Subsection 6.2. Let $\mathcal{H}(\pi_*\varphi_{\mathrm{Tr}\,W}\mathrm{IC}_{\mathcal{X}(d)}[-1])$ be the total perverse cohomology of $\pi_*\varphi_{\mathrm{Tr}\,W}\mathrm{IC}_{\mathcal{X}(d)}[-1]$. There are natural maps:

$$\mathrm{Q}_A \to \mathcal{H}(\pi_*\varphi_{\mathrm{Tr}\,W}\mathrm{IC}_{\mathcal{X}(d)}[-1]) \to \mathrm{Q}_A.$$

Apply the vanishing cycle functor to the maps (6.36) to obtain:

$$\begin{aligned} \Delta_\lambda \colon \pi_{d*}\varphi_{\mathrm{Tr}\,W}\mathrm{IC}_{\mathcal{X}(d)}[-1] &\to i_{\lambda*}\pi_{\lambda*}\varphi_{\mathrm{Tr}\,W}\mathrm{IC}_{\mathcal{X}(d)^\lambda}[-1] \\ &= i_{\lambda*}\boxtimes_{i=1}^k \left(\pi_{d_i*}\varphi_{\mathrm{Tr}\,W}\mathrm{IC}_{\mathcal{X}(d_i)}[-1]\right). \end{aligned} \tag{6.44}$$

Let $\mathrm{Q}_A^{\mathrm{inv}}$ be defined by the exact triangle

$$\mathrm{Q}_A^{\mathrm{inv}}[-1] \to \mathrm{Q}_A \xrightarrow{1-\mathrm{T}} \mathrm{Q}_A \to \mathrm{Q}_A^{\mathrm{inv}}.$$

Proposition 6.17. *Let $A, B \in \mathcal{P}$ with corresponding sheaves Q_A and Q_B of different support. Assume that $p_B \leqslant p_A$.*

(a) The map (6.44) induces isomorphisms

$$\Delta_\lambda \colon \mathrm{Q}_A \xrightarrow{\sim} \mathrm{Q}_A,\ \Delta_\lambda \colon \mathrm{Q}_A^{\mathrm{inv}} \xrightarrow{\sim} \mathrm{Q}_A^{\mathrm{inv}}. \tag{6.45}$$

(b) The maps $\Delta_\lambda \colon \mathrm{Q}_B \to \mathrm{Q}_A$ and $\Delta_\lambda \colon \mathrm{Q}_B^{\mathrm{inv}} \to \mathrm{Q}_A^{\mathrm{inv}}$ are zero.

Proof. The maps above are induced from the map (6.36), thus the conclusion follows from Proposition 6.16. $\square$

We now record corollaries to be used in the proof of Theorem 6.3. Fix a splitting

$$H^\bullet\left(\mathcal{X}(d), \varphi^{\mathrm{inv}}_{\mathrm{Tr}\,W}\mathrm{IC}_{\mathcal{X}(d)}[-1]\right) = \bigoplus_{A\in\mathcal{P}} H^\bullet(X(d), \mathrm{Q}_A^{\mathrm{inv}}). \tag{6.46}$$

Let $x \in H^\bullet\left(\mathcal{X}(d), \varphi^{\mathrm{inv}}_{\mathrm{Tr}\,W}\mathrm{IC}_{\mathcal{X}(d)}[-1]\right)$. Use the decomposition above to write

$$x = \sum_{A\in\mathcal{P}} x_A \tag{6.47}$$

with $x_A \in H^\bullet(X(d), \mathrm{Q}_A^{\mathrm{inv}})$.

Corollary 6.18. *Let $\delta \in M(d)^{W_d}_{\mathbb{R}}$. Let λ be an antidominant cocharacter of $T(d)$ with associated partition $\mathbf{d}$ such that $\varepsilon_{\lambda,\delta} = \varepsilon_{\mathbf{d},\delta}$. Let $x \in H^i(\mathcal{X}(d), \varphi^{\mathrm{inv}}_{\mathrm{Tr}\,W}\mathrm{IC}_{\mathcal{X}(d)}[-1])$ and assume that*

$$a^*_\lambda(x) \in \bigoplus_{j=0}^{c_{\mathbf{d},\delta}-1} H^{i-2j}(\mathcal{X}(d)'^\lambda, \varphi^{\mathrm{inv}}_{\mathrm{Tr}\,W}\mathrm{IC}_{\mathcal{X}(d)'^\lambda}[-2])h^j.$$

(a) If $\varepsilon_{\mathbf{d},\delta} = 1$, then $\Delta_\lambda(x) = 0$.
(b) If $\varepsilon_{\mathbf{d},\delta} = 0$, then $\Delta_\lambda(x)$ is in the image of

$$H^i(\mathcal{X}(d)'^\lambda, \varphi^{\mathrm{inv}}_{\mathrm{Tr}\,W}\mathrm{IC}_{\mathcal{X}(d)'^\lambda}[-2]) \hookrightarrow H^i(\mathcal{X}(d)^\lambda, \varphi^{\mathrm{inv}}_{\mathrm{Tr}\,W}\mathrm{IC}_{\mathcal{X}(d)^\lambda}[-1]).$$

Proof. Recall the definition of S from (6.35).

(a) If $\varepsilon_{\mathbf{d},\delta} = 1$, then $\mathrm{S}p^*_\lambda(x) = 0$, so $\Delta_\lambda(x) = 0$.

(b) If $\varepsilon_{\mathbf{d},\delta} = 0$, then $\mathrm{S}p^*_\lambda(x) \in H^\cdot\left(X(d)^\lambda, \varphi^{\mathrm{inv}}_{\mathrm{Tr}\,W}\mathrm{IC}_{\mathcal{X}(d)'^\lambda}[-c_\lambda - 2]\right)$. The conclusion follows from the definition of Δ_λ in (6.44). $\square$

Corollary 6.19. *Let $\delta \in M(d)^{W_d}_{\mathbb{R}}$. Let λ be an antidominant cocharacter of $T(d)$ with associated partition $\mathbf{d}$ such that $\varepsilon_{\lambda,\delta} = \varepsilon_{\mathbf{d},\delta}$. Let $x \in H^i(\mathcal{X}(d), \varphi^{\mathrm{inv}}_{\mathrm{Tr}\,W}\mathrm{IC}_{\mathcal{X}(d)}[-1])$ and assume that*

$$a^*_\lambda(x) \in \bigoplus_{j=0}^{c_{\mathbf{d},\delta}-1} H^{i-2j}(\mathcal{X}(d)'^\lambda, \varphi^{\mathrm{inv}}_{\mathrm{Tr}\,W}\mathrm{IC}_{\mathcal{X}(d)'^\lambda}[-2])h^j.$$

Recall the decomposition (6.47).

(a) If $\varepsilon_{\mathbf{d},\delta} = 1$, then $x_A = 0$ for all tuples $A \in \mathcal{P}$ with corresponding cocharacter λ.

(b) If $\varepsilon_{\mathbf{d},\delta} = 0$, then $x_A = 0$ for all tuples $A \in \mathcal{P}$ with corresponding cocharacter λ and different from A°.

Proof. Both claims follow from Proposition 6.17 and Corollary 6.18. $\square$

Proof of Theorem 6.3. Recall the cycle map in (6.7)

$$\mathrm{c}\colon \mathrm{gr}_a K^{\mathrm{top}}_i(\mathbb{S}(d;\delta)) \to H^{\dim \mathcal{X}(d)-2a-i}(\mathcal{X}(d), \varphi^{\mathrm{inv}}_{\mathrm{Tr}\,W}\mathrm{IC}_{\mathcal{X}(d)}[-2]).$$

By Proposition 6.10, we may assume that Q has at least two loops at every vertex. Let y be in the image of the above map. By Proposition 6.15 and Corollary 6.19, we have that $y_A = 0$ unless $A = A^\circ$ for some partition $\mathbf{d} = (d_i, m_i)^k_{i=1}$ of d with $m_i \geqslant 1$ and d_i pairwise distinct with $\varepsilon_{\mathbf{d},\delta} = 0$. The statement thus follows. $\square$

7. Topological K-theory of quasi-BPS categories for preprojective algebras

In this section, we use the results of Sections 5 and 6 to compute the topological K-theory of preprojective algebras of quivers satisfying Assumption 2.1 in terms of BPS cohomology, see Theorem 7.6.

7.1. **The preprojective BPS sheaf.** Let $Q^\circ = (I, E^\circ)$ be a quiver. Recall the moduli stack of dimension d representations of the tripled quiver Q of Q° of dimension d and its good moduli space:

$$\pi_{X,d} := \pi_d \colon \mathcal{X}(d) \to X(d).$$

Recall also the moduli stack of dimension d representation of the preprojective algebra of Q° and its good moduli space:

$$\pi_{P,d}\colon \mathcal{P}(d)^{\mathrm{cl}} \to P(d).$$

Consider the moduli stack of dimension d representations of the double quiver of Q° and its good moduli space:

$$\pi_{Y,d}\colon \mathcal{Y}(d) := (R^\circ(d) \oplus R^\circ(d)^\vee)/G(d) \to Y(d).$$

Consider the diagram:

$$\begin{tikzcd}
\mathcal{P}(d)^{\mathrm{cl}} \arrow[r, hook, "j"] \arrow[d, "\pi_{P,d}"] & \mathcal{Y}(d) \arrow[d, "\pi_{Y,d}"] & \mathcal{X}(d) \arrow[l, "\eta"'] \arrow[d, "\pi_{X,d}"] \\
P(d) \arrow[r, hook, "j"] & Y(d) & X(d). \arrow[l, "\eta"']
\end{tikzcd}$$

Here $\eta\colon \mathcal{X}(d) \to \mathcal{Y}(d)$ is the projection which forgets the $\mathfrak{g}(d)$-component and the bottom horizontal arrows are induced maps on good moduli spaces. Let $\mathbb{C} \hookrightarrow \mathfrak{g}(d)$ be the diagonal embedding, which induces the closed immersion

$$\gamma\colon X'(d) := (R^\circ(d) \oplus R^\circ(d)^\vee \oplus \mathbb{C})/\!\!/G(d) \hookrightarrow X(d).$$

Let $\eta' := \eta|_{X'(d)}$. By [Dava, Theorem/ Definition 4.1], there exists a *preprojective BPS sheaf*

$$\mathcal{BPS}^p_d \in \mathrm{Perv}(P(d))$$

such that the BPS sheaf of the tripled quiver with potential (Q, W) associated to Q° is

$$\mathcal{BPS}_d = \gamma_*\eta'^* j_*(\mathcal{BPS}^p_d)[1] \in \mathrm{Perv}(X(d)). \tag{7.1}$$

For a partition $A = (d_i)_{i=1}^k$ of d, define $\mathcal{BPS}^p_A \in \mathrm{Perv}(P(d))$ as in (6.4). For $\delta \in M(d)^{W_d}_{\mathbb{R}}$, define the following perverse sheaves on $P(d)$:

$$\mathcal{BPS}^p_\delta := \bigoplus_{A \in S^d_\delta} \mathcal{BPS}^p_A,\ \mathcal{BPS}^p_{d,v} := \mathcal{BPS}^p_{v\tau_d}, \tag{7.2}$$

where the set of partitions S^d_δ is defined from the tripled quiver Q, see Subsection 6.1.2. Then $\mathcal{BPS}^p_{d,v}$ is a direct summand of $\pi_{P,d*}\omega_{\mathcal{P}(d)^{\mathrm{cl}}}$, see [Dava, Theorem A], and so $H^{-a}(P(d), \mathcal{BPS}^p_{d,v})$ is a direct summand of

$$H^{\mathrm{BM}}_a(\mathcal{P}(d)^{\mathrm{cl}}) = H^{-a}\big(P(d), \pi_{P,d*}\omega_{\mathcal{P}(d)^{\mathrm{cl}}}\big).$$

Recall the maps

$$\mathcal{P}(d) \xleftarrow{\eta'} \eta^{-1}(\mathcal{P}(d)) \xrightarrow{j'} \mathcal{X}(d).$$

The dimension of $\mathcal{P}(d)$ as a quasi-smooth stack is $\dim \mathcal{P}(d) := \dim \mathcal{Y}(d) - \dim \mathfrak{g}(d)$. Recall the dimensional reduction isomorphism from Subsection 5.1:

$$\begin{aligned} j'_*\eta'^*\colon H^{\mathrm{BM}}_a(\mathcal{P}(d)^{\mathrm{cl}}) \cong H^{\mathrm{BM}}_a(\mathcal{P}(d)) &\xrightarrow{\sim} H^{2\dim \mathcal{Y}(d)-a}(\mathcal{X}(d), \varphi_{\mathrm{Tr}\,W}\mathbb{Q}_{\mathcal{X}(d)}[-1]) \\ &= H^{\dim \mathcal{P}(d)-a}(\mathcal{X}(d), \varphi_{\mathrm{Tr}\,W}\mathrm{IC}_{\mathcal{X}(d)}[-1]). \end{aligned}$$

By the construction of the PBW isomorphism for preprojective Hall algebras [Dava, Equation (31)], the above isomorphism preserves the BPS cohomologies:

$$j'_*\eta'^*\colon H^{-a}(P(d), \mathcal{BPS}^p_{d,v}) \xrightarrow{\sim} H^{\dim \mathcal{P}(d)-a}(X(d), \mathcal{BPS}_{d,v}). \tag{7.3}$$

7.2. **Computations.** Recall the categories

$$\mathbb{T}(d)_v \subset D^b(\mathcal{P}(d)) \text{ and } \mathbb{T}(d)_v^{\mathrm{red}} \subset D^b(\mathcal{P}(d)^{\mathrm{red}})$$

from Subsection 2.13. Consider the natural closed immersion $l' \colon \mathcal{P}(d)^{\mathrm{red}} \hookrightarrow \mathcal{P}(d)$. The closed immersion $l \colon \mathcal{P}(d)^{\mathrm{cl}} \hookrightarrow \mathcal{P}(d)$ factors through $\mathcal{P}(d)^{\mathrm{cl}} \hookrightarrow \mathcal{P}(d)^{\mathrm{red}} \xhookrightarrow{l'} \mathcal{P}(d)$.

Proposition 7.1. *Let Q be a symmetric quiver. Then there is a weak equivalence of spectra $l'_* \colon K^{\mathrm{top}}(\mathbb{T}(d)_v^{\mathrm{red}}) \to K^{\mathrm{top}}(\mathbb{T}(d)_v)$.*

Proof. There is a weak equivalence of spectra $l'_* \colon G^{\mathrm{top}}(\mathcal{P}(d)^{\mathrm{red}}) \xrightarrow{\sim} G^{\mathrm{top}}(\mathcal{P}(d))$. The claim then follows from Theorem 2.9. $\square$

For $i \in \mathbb{Z}$, consider the Chern character map (3.8) for the quasi-smooth stack $\mathcal{P}(d)$:

$$\mathrm{ch} \colon G_i^{\mathrm{top}}(\mathcal{P}(d)) \to \widetilde{H}_i^{\mathrm{BM}}(\mathcal{P}(d)). \tag{7.4}$$

It induces a Chern character map

$$\mathrm{ch} \colon K_i^{\mathrm{top}}(\mathbb{T}(d)_v) \hookrightarrow G_i^{\mathrm{top}}(\mathcal{P}(d)) \to \widetilde{H}_i^{\mathrm{BM}}(\mathcal{P}(d)). \tag{7.5}$$

Corollary 7.2. *The maps* (7.4) *and* (7.5) *are injective.*

Proof. It suffices to check that (7.4) is injective. This follows from Proposition 4.10, Theorem 2.8 (applied to a fixed μ and all $\alpha \in \mathbb{Z}_{\geqslant 1}$), and the Koszul equivalence (2.14). $\square$

Corollary 7.3. *We have that $G_1^{\mathrm{top}}(\mathcal{P}(d)) = 0$. Thus also $K_1^{\mathrm{top}}(\mathbb{T}(d)_v) = 0$.*

Proof. We have that $H_{\mathrm{odd}}^{\mathrm{BM}}(\mathcal{P}(d)^{\mathrm{cl}}) = 0$ by [Davb, Theorem A]. The conclusion follows by Proposition 7.2. $\square$

Recall the filtration $E_\ell G_0^{\mathrm{top}}(\mathcal{P}(d))$ of $G_0^{\mathrm{top}}(\mathcal{P}(d))$ from Subsection 3.3. Define the filtration:

$$E_\ell K_0^{\mathrm{top}}(\mathbb{T}(d)_v) := E_\ell G_0^{\mathrm{top}}(\mathcal{P}(d)) \cap K_0^{\mathrm{top}}(\mathbb{T}(d)_v) \subset K_0^{\mathrm{top}}(\mathbb{T}(d)_v).$$

We denote by $\mathrm{gr}_\ell K_0^{\mathrm{top}}(\mathbb{T}(d)_v)$ the associated graded piece, and note that it is a direct summand of $\mathrm{gr}_\ell G_0^{\mathrm{top}}(\mathcal{P}(d))$ by Theorem 2.9. Define similarly a filtration $E_\ell G_0^{\mathrm{top}}(\mathcal{P}(d)^{\mathrm{red}}) \subset G_0^{\mathrm{top}}(\mathcal{P}(d)^{\mathrm{red}})$ and a filtration $E_\ell K_0^{\mathrm{top}}(\mathbb{T}(d)_v^{\mathrm{red}}) \subset K_0^{\mathrm{top}}(\mathbb{T}(d)_v^{\mathrm{red}})$.

Corollary 7.4. *The forget-the-potential functor Θ induces an isomorphism:*

$$\mathrm{gr}_\ell K_0^{\mathrm{top}}\left(\mathrm{MF}^{\mathrm{gr}}(\mathcal{X}(d), \mathrm{Tr}\, W)\right) \xrightarrow{\sim} \mathrm{gr}_\ell K_0^{\mathrm{top}}\left(\mathrm{MF}(\mathcal{X}(d), \mathrm{Tr}\, W)\right). \tag{7.6}$$

There are thus also isomorphisms:

$$\mathrm{gr}_\ell K_0^{\mathrm{top}}\left(\mathbb{T}(d)_v\right) \xrightarrow{\sim} \mathrm{gr}_{\ell+\dim \mathfrak{g}(d)} K_0^{\mathrm{top}}\left(\mathbb{S}^{\mathrm{gr}}(d)_v\right) \xrightarrow{\sim} \mathrm{gr}_{\ell+\dim \mathfrak{g}(d)} K_0^{\mathrm{top}}\left(\mathbb{S}(d)_v\right).$$

Proof. The isomorphism (7.6) follows from Corollaries 5.2 and 7.3. The other isomorphism follow from the Koszul equivalence, see Proposition 5.2 for an explanation of the degree of the graded pieces. $\square$

Corollary 7.5. *There is a commutative diagram, where the vertical maps are cycle maps and the left horizontal maps are the dimensional reduction maps $i'_* p'^*$.*

$$\begin{tikzcd}
\mathrm{gr}_\cdot G_0^{\mathrm{top}}(\mathcal{P}(d)) \arrow[r, "\sim"] \arrow[d, "\mathrm{c}", "\wr"'] & \mathrm{gr}_\cdot K_0^{\mathrm{top}}(\mathrm{MF}^{\mathrm{gr}}(\mathcal{X}(d), \mathrm{Tr}\, W)) \arrow[r, "\sim"] \arrow[d, "\mathrm{c}", "\wr"'] & \mathrm{gr}_\cdot K_0^{\mathrm{top}}(\mathrm{MF}(\mathcal{X}(d), \mathrm{Tr}\, W)) \arrow[d, "\mathrm{c}", "\wr"'] \\
\widetilde{H}_0^{\mathrm{BM}}(\mathcal{P}(d)) \arrow[r, "\sim"] & \widetilde{H}^0(\mathcal{X}(d), \varphi_{\mathrm{Tr}\, W}[-1]) \arrow[r, "\sim"] & \widetilde{H}^0(\mathcal{X}(d), \varphi^{\mathrm{inv}}_{\mathrm{Tr}\, W}[-2]).
\end{tikzcd}$$

Here we have suppressed the cohomological degrees to make the diagram simpler.

Proof. The claim follows from Proposition 5.2 and Corollary 7.4. ☐

Theorem 7.6. *For an arbitrary quiver Q°, the cycle map* (7.4) *for $\mathcal{P}(d)$ induces a cycle map*

$$\mathrm{c}\colon \mathrm{gr}_\ell K_0^{\mathrm{top}}(\mathbb{T}(d)_v) \cong \mathrm{gr}_\ell K_0^{\mathrm{top}}(\mathbb{T}(d)_v^{\mathrm{red}}) \to H^{-2\ell}(P(d), \mathcal{BPS}^p_{d,v}). \tag{7.7}$$

If Q° satisfies Assumption 2.2, then (7.7) *is an isomorphism.*

Proof. The isomorphism $\mathrm{gr}_\ell K_0^{\mathrm{top}}(\mathbb{T}(d)_v) \cong \mathrm{gr}_\ell K_0^{\mathrm{top}}(\mathbb{T}(d)_v^{\mathrm{red}})$ follows from Proposition 7.1. Consider the diagram, whose lower square commutes from Corollary 7.5 and the top horizontal map is an isomorphism by Corollary 7.4:

$$\begin{tikzcd}
\mathrm{gr}_\ell K_0^{\mathrm{top}}(\mathbb{T}(d)_v) \arrow[r, "\sim"] \arrow[d, hook] \arrow[dd, bend right=60, "\alpha"'] & \mathrm{gr}_{\ell+\dim \mathfrak{g}(d)} K_0^{\mathrm{top}}(\mathbb{S}(d)_v) \arrow[d, hook] \arrow[dd, bend left=60, "\beta"] \\
\mathrm{gr}_\ell G_0^{\mathrm{top}}(\mathcal{P}(d)) \arrow[r, "\sim"] \arrow[d, "\mathrm{c}", "\wr"'] & \mathrm{gr}_{\ell+\dim \mathfrak{g}(d)} K_0^{\mathrm{top}}(\mathrm{MF}(\mathcal{X}(d), \mathrm{Tr}\, W)) \arrow[d, "\mathrm{c}", "\wr"'] \\
H^{\mathrm{BM}}_{2\ell}(\mathcal{P}(d)) \arrow[r, "\sim"] & H^{2\dim \mathcal{Y}(d)-2\ell}(\mathcal{X}(d), \varphi_f[-1]).
\end{tikzcd}$$

By Theorem 6.3, the map β has image in

$$H^{\dim \mathcal{P}(d)-2\ell}(\mathcal{X}(d), \mathcal{BPS}_{d,v}) \subset H^{2\dim \mathcal{Y}(d)-2\ell}(\mathcal{X}(d), \varphi_{\mathrm{Tr}\, W}[-1]).$$

If Q° satisfies Assumption 2.2, it is an isomorphism onto $H^{\dim \mathcal{P}(d)-2\ell}(\mathcal{X}(d), \mathcal{BPS}_{d,v})$ by Theorem 6.2. By (7.3), the map α has image in $H^{-2\ell}(P(d), \mathcal{BPS}^p_{d,v})$, and, if Q° satisfies Assumption 2.2, it is an isomorphism onto $H^{-2\ell}(P(d), \mathcal{BPS}^p_{d,v})$. ☐

Remark 7.7. There are two perverse filtrations on $H^{\mathrm{BM}}_{\cdot}(\mathcal{P}(d))$ for any quiver Q°. One of them is induced from the tripled quiver with potential (Q, W) and studied in [DM20]; the first non-zero piece in the perverse filtration is ${}^p\tau^{\leqslant 1}\pi_{d*}\varphi_{\mathrm{Tr}\, W}\mathrm{IC}_{\mathcal{X}(d)} = \mathcal{BPS}_d$. Another filtration is induced from the map $\pi_{P,d}$ and studied in [Dava], where it is called the "less perverse filtration"; the first non-zero piece in the perverse filtration is ${}^p\tau^{\leqslant 0}\pi_{P,d*}\omega_{\mathcal{P}(d)^{\mathrm{cl}}}$. Note that, for any $v \in \mathbb{Z}$, ${}^p\tau^{\leqslant 1}\pi_{d*}\varphi_{\mathrm{Tr}\, W}\mathrm{IC}_{\mathcal{X}(d)}$ is a direct summand of $\mathcal{BPS}_{d,v}$, which itself is a direct summand of ${}^p\tau^{\leqslant 0}\pi_{P,d*}\omega_{\mathcal{P}(d)^{\mathrm{cl}}}$. Thus the topological K-theory of quasi-BPS categories (for Q° satisfying Assumption 2.2, and for any $v \in \mathbb{Z}$) lies between the first non-zero pieces of these two perverse filtrations.

Remark 7.8. Davison–Hennecart–Schlegel Mejia [DHSMb, DHSMa] computed the preprojective BPS sheaves in terms of the intersection complexes of the varieties $P(d)$.

We note the following numerical corollary of Theorem 7.6.

Corollary 7.9. *Let Q° be a quiver satisfying Assumption 2.2 and let $(d, v) \in \mathbb{N}^I \times \mathbb{Z}$. Then*

$$\dim_{\mathbb{Q}} K_0^{\mathrm{top}}(\mathbb{T}(d)_v) = \dim_{\mathbb{Q}} H^{\cdot}(P(d), \mathcal{BPS}_{d,v}).$$

Proof. The map (7.5) is injective by Proposition 7.2. The conclusion then follows from Theorem 7.6. ☐

8. Examples

In this section, we discuss some explicit examples of computations of the topological K-theory of quasi-BPS categories. All vector spaces considered in this section are $\mathbb{Q}$-vector spaces. We first note a preliminary proposition.

Proposition 8.1. *Let $Q=(I,E)$ be a symmetric quiver, let $d\in\mathbb{N}^I$, and let $v\in\mathbb{Z}$. Then*

$$\dim K_0^{\mathrm{top}}(\mathbb{M}(d)_v)=\#\left(M(d)^+\cap(\mathbf{W}(d)+v\tau_d-\rho)\right).$$

Proof. There is a natural isomorphism

$$K_0(\mathcal{X}(d))\xrightarrow{\sim}K_0^{\mathrm{top}}(\mathcal{X}(d))\cong K_0(BG(d)).$$

The category $\mathbb{M}(d)_v$ is admissible in $D^b(\mathcal{X}(d))$, so the above isomorphism restricts to the isomorphism

$$K_0(\mathbb{M}(d)_v)\xrightarrow{\sim}K_0^{\mathrm{top}}(\mathbb{M}(d)_v).$$

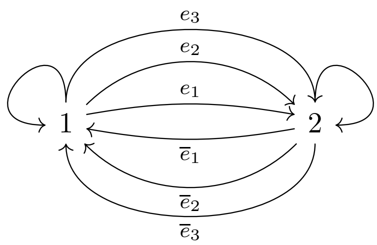
The generators of $K_0(\mathcal{X}(d))$ are the classes of the vector bundles $\mathcal{O}_{\mathcal{X}(d)}\otimes\Gamma_{G(d)}(\chi)$, where χ is a dominant weight of $G(d)$ and $\Gamma_{G(d)}(\chi)$ is the irreducible representation of $G(d)$ of highest weight χ. The computation

$$\dim K_0(\mathbb{M}(d)_v)=\#\left(M(d)^+\cap(\mathbf{W}(d)+v\tau_d-\rho)\right)$$

follows then from the definition of $\mathbb{M}(d)_v$. □

Remark 8.2. In view of Proposition 8.1 and Theorem 6.6, the total intersection cohomology of the spaces $X(d)$ can be determined by counting lattice points inside the polytope $(\mathbf{W}(d)+v\tau_d-\rho)$.

8.1. **Toric examples.** Let $g\in\mathbb{N}$. Consider the quiver $Q=(I,E)$, where $I=\{1,2\}$ and E has one loop at 1, one loop at 2, $2g+1$ edges $\{e_1,\ldots,e_{2g+1}\}$ from 0 to 1 and $2g+1$ edges $\{\overline{e}_1,\ldots,\overline{e}_{2g+1}\}$ from 1 to 0. The following is a figure for $g=1$.



Fix $d=(1,1)\in\mathbb{N}^I$. Then

$$\mathcal{X}(d)=\left(\mathbb{C}^2\oplus\mathbb{C}^{2(2g+1)}\right)/(\mathbb{C}^*)^2.$$

The diagonal $\mathbb{C}^*\hookrightarrow(\mathbb{C}^*)^2$ acts trivially on $\mathbb{C}^2\oplus\mathbb{C}^{2(2g+1)}$. The factor $\mathbb{C}^*$ corresponding to the vertex 1 acts with weight 0 on $\mathbb{C}^2$, weight 1 on $\mathbb{C}^{2g+1}$, and weight -1 on $\mathbb{C}^{2g+1}$. We consider the stack, which is the $\mathbb{C}^*$-rigidification of $\mathcal{X}(d)$:

$$\mathcal{X}'(d)=\left(\mathbb{C}^2_0\oplus\mathbb{C}^{2g+1}_1\oplus\mathbb{C}^{2g+1}_{-1}\right)/\mathbb{C}^*.$$

The GIT quotient for any non-trivial stability condition provides a small resolution of singularities:

$$\tau\colon Y:=\left(\mathbb{C}^2_0\oplus\mathbb{C}^{2g+1}_1\oplus\mathbb{C}^{2g+1}_{-1}\right)^{\mathrm{ss}}/\mathbb{C}^*=\mathbb{C}^2\times\mathrm{Tot}_{\mathbb{P}^{2g}}\left(\mathcal{O}(-1)^{2g+1}\right)\to X(d).$$

Here, *small* means that $\dim Y\times_{X(d)}Y=\dim X(d)$ and $Y\times_{X(d)}Y$ has a unique irreducible component of maximal dimension. Then, by the BBDG decomposition theorem, we have that $\tau_*\mathrm{IC}_Y = \mathrm{IC}_{X(d)}$. We decorate the BPS sheaves with a superscript zero to indicate that the potential is zero. We obtain that:

$$\mathcal{BPS}^0_d = \tau_*\mathrm{IC}_Y = \mathrm{IC}_{X(d)} \text{ and } \mathcal{BPS}^0_{(1,0)} = \mathcal{BPS}^0_{(0,1)} = \mathrm{IC}_{\mathbb{C}}. \tag{8.1}$$

Proposition 8.3. *If v is odd, then $\mathbb{M}(d)_v \cong D^b(Y)$ and $\mathcal{BPS}^0_{d,v} = \mathcal{BPS}^0_d$.*

If v is even, then $\mathbb{M}(d)_v$ has a semiorthogonal decomposition with summands equivalent to $D^b(Y)$ and $D^b(\mathbb{C}^2)$, and $\mathcal{BPS}^0_{d,v} = \mathcal{BPS}^0_d \oplus \mathcal{BPS}^0_{(1,0)} \boxtimes \mathcal{BPS}^0_{(0,1)}$.

Proof. The category $\mathbb{M}(d)_v$ is the subcategory of $D^b(\mathcal{X}(d))$ generated by the line bundles $\mathcal{O}_{\mathcal{X}(d)}(w\beta_2 + (v-w)\beta_1)$ for $w \in \mathbb{Z}$ such that

$$\frac{v}{2} \leqslant w \leqslant 2g+1+\frac{v}{2}. \tag{8.2}$$

One can show that $\mathbb{M}(d)_v$ is equivalent to the "window subcategory" (in the sense of [HL15]) of $D^b(\mathcal{X}'(d))$ containing objects F such that the weights of $\mathbb{C}^*$ on $F|_0$ are in $\left[\frac{v}{2}, \frac{v}{2}+2g+1\right] \cap \mathbb{Z}$.

If v is odd, then $\mathbb{M}(d)_v \cong D^b(Y)$ by [HL15, Theorem 2.10]. The boundary points $\frac{v}{2}$ and $\frac{v}{2}+2g+1$ are not integers, so $\mathcal{BPS}^0_{d,v} = \mathcal{BPS}^0_d$.

If v is even, then $\mathcal{BPS}^0_{d,v} = \mathcal{BPS}^0_d \oplus \mathcal{BPS}^0_{(1,0)} \boxtimes \mathcal{BPS}^0_{(0,1)}$. The fixed locus of the unique Kempf-Ness locus in the construction of Y is $(\mathbb{C}^2_0 \oplus \mathbb{C}^{2g+1}_1 \oplus \mathbb{C}^{2g+1}_{-1})^{\mathbb{C}^*} = \mathbb{C}^2$. As a corollary of [HL15, Theorem 2.10], see the remark in [HLS16, Equation (3)], the category $\mathbb{M}(d)_v$ has a semiorthogonal decomposition with summands $D^b(Y)$ and $D^b(\mathbb{C}^2)$. □

As a corollary of the above proposition and of the computations (8.1), we obtain the following:

$$\dim K_0^{\mathrm{top}}(\mathbb{M}(d)_v) \overset{(*)}{=} \dim H^{\cdot}(X(d), \mathcal{BPS}^0_{d,v}) = \begin{cases} 2g+1, & \text{if } v \text{ is odd,} \\ 2g+2, & \text{if } v \text{ is even.} \end{cases}$$

The equality $(*)$ is also the consequence (6.4) of Theorem 6.2. Note that the dimensions of $K^{\mathrm{top}}(\mathbb{M}(d)_v)$ can be computed immediately using Proposition 8.1 and (8.2), and then $(*)$ can be checked without using window categories. However, by Proposition 8.3, the equality $(*)$ is obtained as a corollary of the Atiyah-Hirzebruch theorem for the smooth varieties Y and $\mathbb{C}^2$.

Further, Proposition 8.3 is useful when considering a non-zero potential for Q. For example, consider the potential

$$W := \sum_{i=1}^{2g+1} e_i \overline{e}_i.$$

Note that $W \colon Y \to \mathbb{C}$ is smooth. The computation (8.1) implies that:

$$\mathcal{BPS}_d = \varphi_W \mathrm{IC}_{X(d)} = \tau_*\varphi_W \mathrm{IC}_Y = 0 \text{ and } \mathcal{BPS}^0_{(1,0)} = \mathcal{BPS}^0_{(0,1)} = \mathrm{IC}_{\mathbb{C}}. \tag{8.3}$$

The BPS sheaves have trivial monodromy. Further, Proposition 8.3 implies that:

$$\mathbb{S}(d)_v \simeq \mathrm{MF}(Y, W) = 0 \text{ if } v \text{ is odd,}$$
$$\mathbb{S}(d)_v = \langle \mathrm{MF}(Y, W), \mathrm{MF}(\mathbb{C}^2, 0)\rangle \simeq \mathrm{MF}(\mathbb{C}^2, 0) \text{ if } v \text{ is even.}$$

Let $i \in \mathbb{Z}$. The following equality (which also follows by (6.4)) holds by a direct computation:

$$\dim K_i^{\mathrm{top}}(\mathbb{S}(d)_v) = \dim H^{\cdot}(X(d), \mathcal{BPS}_{d,v}) = \begin{cases} 0, & \text{if } v \text{ is odd,} \\ 1, & \text{if } v \text{ is even.} \end{cases}$$

Remark 8.4. A similar analysis can be done for any symmetric quiver $Q = (I, E)$ (not necessarily satisfying Assumption 2.1) and a dimension vector $d = (d^i)_{i\in I} \in \mathbb{N}^I$ such that $d^i \in \{0, 1\}$ for every $i \in I$. We do not give the details for the proofs. Let $v \in \mathbb{Z}$ such that $\gcd(\underline{d}, v) = 1$. Assume $W = 0$.

One can show that, for a generic GIT stability $\ell \in M(d)_{\mathbb{R}}^{W_d} \cong M(d)_{\mathbb{R}}$, the GIT quotient $Y := R(d)^{\ell\text{-ss}}/G(d) \cong R(d)^{\ell\text{-ss}} /\!\!/ G(d)$ is a small resolution

$$\tau \colon Y \to X(d).$$

Then $\mathcal{BPS}_d^0 = \tau_*\mathrm{IC}_Y$. By [HLS20], there is an equivalence:

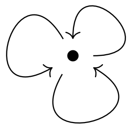
$$\mathbb{M}(d)_v \cong D^b(Y).$$

The following equality (which is a corollary of Theorem 6.2) follows then by the Atiyah-Hirzebruch theorem for the smooth variety Y:

$$\dim K_0^{\mathrm{top}}(\mathbb{M}(d)_v) = \dim K_0^{\mathrm{top}}(Y) = \dim H^{\cdot}(Y) = \dim H^{\cdot}(X(d), \mathcal{BPS}_d^0).$$

Similar computations can be done also for a general $v \in \mathbb{Z}$.

8.2. **Quivers with one vertex and an odd number of loops.** Let $g \in \mathbb{N}$. Consider Q the quiver with one vertex and $2g + 1$ loops. The following is a picture for $g = 1$.

(8.4) 

For $d \in \mathbb{N}$, recall the good moduli space map:

$$\mathcal{X}(d) := \mathfrak{gl}(d)^{\oplus(2g+1)}/GL(d) \to X(d) := \mathfrak{gl}(d)^{\oplus(2g+1)} /\!\!/ GL(d).$$

For $g > 0$, the variety $X(d)$ is singular. For every stability condition $\ell \in M(d)_{\mathbb{R}}^{W_d}$, we have that $\mathcal{X}(d)^{\ell\text{-ss}} = \mathcal{X}(d)$, so we do not obtain resolutions of singularities of $X(d)$ as in the previous example. There are no known crepant geometric resolutions (in particular, small resolutions) of $X(d)$. For $\gcd(d, v) = 1$, Špenko–Van den Bergh [ŠVdB17] proved that $\mathbb{M}(d)_v$ is a twisted noncommutative crepant resolution of $X(d)$. In view of Theorem 6.6, we regard $\mathbb{M}(d)_v$ as the categorical analogue of a small resolution of $X(d)$.

Reineke [Rei12] and Meinhardt–Reineke [MR19] provided multiple combinatorial formulas for the dimensions of the individual intersection cohomology vector spaces $\mathrm{IH}^{\bullet}(X(d))$. As noted in Remark 8.2, Theorem 6.6 also provides combinatorial formulas for the total intersection cohomology of $X(d)$. We explain that our formula recovers a formula already appearing in the work of Reineke [Rei12, Theorem 7.1].

Fix $v \in \mathbb{Z}$. By Proposition 8.1, we need to determine the number of (integral, dominant) weights $\chi = \sum_{i=1}^d c_i\beta_i \in M(d)^+$ with $\sum_{i=1}^d c_i = v$ and $c_i \geqslant c_{i-1}$ for every $2 \leqslant i \leqslant d$, such that

$$\chi + \rho - v\tau_d \in \frac{2g+1}{2}\mathrm{sum}[0, \beta_i - \beta_j], \tag{8.5}$$

where the Minkowski sum is after all $1 \leqslant i, j \leqslant d$. Define $\widetilde{\chi} \in M(d)$ and $\widetilde{c}_i \in \mathbb{Z}$ for $1 \leqslant i \leqslant d$ as follows:

$$\widetilde{\chi} := \chi - g \cdot (2\rho) = \sum_{i=1}^{d} \widetilde{c}_i \beta_i.$$

Note that, for every $2 \leqslant i \leqslant d$, the inequality $c_i \geqslant c_{i-1}$ becomes:

$$\widetilde{c}_i - \widetilde{c}_{i-1} + 2g \geqslant 0. \tag{8.6}$$

A dominant weight χ satisfies (8.5) if and only if, for all dominant cocharacters λ of $T(d) \subset GL(d)$, we have:

$$\langle \lambda, \chi + \rho - v\tau_d \rangle \leqslant \frac{2g+1}{2}\langle \lambda, \mathfrak{g}^{\lambda>0} \rangle = \frac{2g+1}{2}\langle \lambda, \rho \rangle. \tag{8.7}$$

Proposition 8.5. *The inequalities* (8.7) *hold for all dominant cocharacters* λ *if and only they hold for the cocharacters* $\lambda_k(z) = (\overbrace{1 \ldots, 1}^{d-k}, \overbrace{z, \ldots, z}^{k}) \in T(d)$ *for* $1 \leqslant k \leqslant d$.

Proof. In the cocharacter lattice, any dominant cocharacter λ is a linear combination with nonnegative coefficients of λ_k for $1 \leqslant k \leqslant d$. Then, if (8.7) holds for all λ_k, it also holds for all dominant λ. □

We rewrite the conditions (8.7) for λ_k using the weight $\widetilde{\chi}$:

$$\langle \lambda_k, \widetilde{\chi} \rangle \leqslant \langle \lambda_k, v\tau_d \rangle.$$

Alternatively, the condition above can be written as:

$$\sum_{i=d-k+1}^{d} \widetilde{c}_i \leqslant \frac{vk}{d}. \tag{8.8}$$

Definition 8.6. Let $\mathcal{H}^{2g+1}_{d,v}$ be the set of tuplets of integers $(\widetilde{c}_i)_{i=1}^d \in \mathbb{Z}^d$ satisfying the inequality (8.6) and (8.8) for every $2 \leqslant k \leqslant d$ and such that $\sum_{i=1}^d \widetilde{c}_i = v$. Let $H^{2g+1}_{d,v} := \#\mathcal{H}^{2g+1}_{d,v}$.

Remark 8.7. The numbers $H^{2g+1}_{d,0}$ appear in combinatorics as "score sequences of complete tournaments", and in the study of certain $\mathbb{C}^*$-fixed points in the moduli of $SL(n)$-Higgs bundles, see [Rei12, Section 7]. By Proposition 8.1, we have that:

$$\dim K_0^{\mathrm{top}}(\mathbb{M}(d)_v) = H^{2g+1}_{d,v}.$$

By Theorem 6.6, for any $v \in \mathbb{Z}$ such that $\gcd(d, v) = 1$, we obtain that:

$$\dim \mathrm{IH}^{\cdot}(X(d)) = H^{2g+1}_{d,v}. \tag{8.9}$$

The above statement was already proved (by different methods) by Reineke and Meinhardt–Reineke by combining [Rei12, Theorem 7.1] and [MR19, Theorem 4.6], see also [MR19, Section 4.3]. Note that we assume that the number of loops is odd in order to apply Theorem 6.2. In loc. cit., Reineke also provided combinatorial formulas for m-loop quivers for m even.

Remark 8.8. Note that, as a corollary of (8.9), we obtain that $H^{2g+1}_{d,v} = H^{2g+1}_{d,v'}$ if $\gcd(d,v) = \gcd(d,v') = 1$. There are natural bijections $\mathcal{H}^{2g+1}_{d,v} \xrightarrow{\sim} \mathcal{H}^{2g+1}_{d,v'}$ for $d|v-v'$ or for $v' = -v$, but we do not know such natural bijections for general v, v' coprime with d.

For $\gcd(d,v) = 1$ and $n \in \mathbb{Z}_{\geqslant 1}$, the topological K-theory $K^{\mathrm{top}}_i(\mathbb{M}(nd)_{nv})$ is computed from the intersection cohomology of $X(e)$ for $e \in \mathbb{N}$, and the set S^{nd}_{nv}. The following is a corollary of Proposition 6.1:

Corollary 8.9. *For $\gcd(d,v) = 1$ and $n \in \mathbb{Z}_{\geqslant 1}$, the set S^{nd}_{nv} consists of partitions $(d_i)^k_{i=1}$ of nd such that $d_i = n_i d$ for $(n_i)^k_{i=1} \in \mathbb{N}^k$ a partition of n.*

Example 8.10. Suppose that $g = 0$. In this case, the variety $X(d)$ is smooth:

$$X(d) = \mathfrak{gl}(d)/\!\!/GL(d) \xrightarrow{\sim} \mathrm{Sym}^d(\mathbb{C}) \cong \mathbb{C}^d.$$

The above isomorphism is given by sending an element of $\mathfrak{gl}(d)$ to the set of generalized eigenvalues. However $X(d)^{\mathrm{st}} = \emptyset$ if $d > 1$, thus $\mathcal{BPS}_d = \mathrm{IC}_{\mathbb{C}}$ if $d = 1$, and $\mathcal{BPS}_d = 0$ for $d > 1$. Then by Corollary 8.9, we have $\mathcal{BPS}_{d,v} = 0$ unless $d|v$, case in which $\mathcal{BPS}_{d,v} = \mathrm{Sym}^d(\mathcal{BPS}_1) = \mathrm{IC}_{X(d)}$. Thus for $g = 0$, we have

$$\dim H^{\cdot}(X(d), \mathcal{BPS}_{d,v}) = \begin{cases} 1, & \text{if } d|v, \\ 0, & \text{otherwise.} \end{cases} \tag{8.10}$$

On the other hand, by [Tod23b, Lemma 3.2] we have that $\mathbb{M}(d)_v = 0$ unless $d|v$, case in which it is the subcategory of $D^b(\mathcal{X}(d))$ generated by $\mathcal{O}_{\mathcal{X}(d)}(v\tau_d)$, and thus equivalent to $D^b(X(d))$, see [Tod23b, Lemma 3.3]. Then:

$$\dim K^{\mathrm{top}}_0(\mathbb{M}(d)_v) = \begin{cases} 1, & \text{if } d|v, \\ 0, & \text{otherwise.} \end{cases} \tag{8.11}$$

For $g = 0$, we can thus verify (6.4) by the direct computations (8.10), (8.11).

8.3. **The three loop quiver.** In this subsection, we make explicit the corollary of Theorem 6.2 for the three loop quiver (8.4) with loops $\{x, y, z\}$ and with potential $W = x[y, z]$. The quasi-BPS categories $\mathbb{S}(d)_v$ are the quasi-BPS categories of $\mathbb{C}^3$ and are admissible in the DT category $\mathcal{DT}(d)$ studied in [PTa]. The quasi-BPS categories $\mathbb{T}(d)_v$ are the quasi-BPS categories of $\mathbb{C}^2$ and are admissible in $D^b(\mathcal{C}(d))$, where $\mathcal{C}(d)$ is the commuting stack of matrices of size d. For $n \in \mathbb{N}$, we denote by $p_2(n)$ the number of partitions of n.

Proposition 8.11. *Let $(d,v) \in \mathbb{N} \times \mathbb{Z}$ be coprime, let $n \in \mathbb{N}$ and $i \in \mathbb{Z}$. Then:*

$$\dim K^{\mathrm{top}}_i(\mathbb{S}(nd)_{nv}) = \dim K^{\mathrm{top}}_0(\mathbb{T}(nd)_{nv}) = p_2(n).$$

Proof. By a theorem of Davison [Davb, Theorem 5.1], we have that

$$\mathcal{BPS}_e = \mathrm{IC}_{\mathbb{C}^3}$$

for every $e \in \mathbb{N}$, where $\mathbb{C}^3 \hookrightarrow X(e)$ is the subvariety parameterizing three diagonal matrices. Then $\dim H^{\cdot}(X(e), \mathcal{BPS}_e) = 1$, and so $\mathrm{Sym}^k\left(H^{\cdot}(X(e), \mathcal{BPS}_e)\right) = 1$ for every positive integers e and k. Then $H^{\cdot}(X(nd), \mathcal{BPS}_A)$ is also one dimensional for every $A \in S^{nd}_{nv}$. Note that $\#S^{nd}_{nv} = p_2(n)$ by Corollary 8.9. Then

$$H^{\cdot}(X(nd), \mathcal{BPS}_{nd,nv}) = \bigoplus_{A \in S^{nv}_{nd}} H^{\cdot}(X(nd), \mathcal{BPS}_A) = \mathbb{Q}^{\oplus p_2(n)}.$$

The monodromy is trivial on $H^{\cdot}(X(nd), \mathcal{BPS}_{nd,nv})$. By Theorems 6.2 and 7.6, we obtain the desired computations. $\square$

Remark 8.12. By Theorem 6.2, the topological K-theory of quasi-BPS categories may be determined whenever one can compute the BPS cohomology and the set S^v_d. Proposition 8.11 is an example of such a computation. We mention two other computations for the three loop quiver with potentials $W' := x[y, z] + z^a$ (for $a \geqslant 2$) and $W'' := x[y, z] + yz^2$.

Let $\mathcal{BPS}'_d$ and $\mathbb{S}'(d)_v$ be the BPS sheaves and the quasi-BPS categories of (Q, W'). Denote similarly the BPS sheaves and the quasi-BPS categories of (Q, W''). By [DP22, Theorem 1.5], we have that $H^{\cdot}(X(d), \mathcal{BPS}'_d)^{\mathrm{inv}} = 0$ because $H^{\cdot}(\mathbb{C}, \varphi_{t^a})^{\mathrm{inv}} = 0$. Then Theorem 6.2 implies that, for every $i, v \in \mathbb{Z}$ with $\gcd(d, v) = 1$:

$$K^{\mathrm{top}}_i(\mathbb{S}'(d)_v) = 0.$$

By [DP22, Corollary 7.2], we have that $H^{\cdot}(X(d), \mathcal{BPS}''_d)^{\mathrm{inv}} = H^{\cdot}(X(d), \mathcal{BPS}''_d)$ is one dimensional. As in Proposition 8.11, we have that, for every $i, v \in \mathbb{Z}$:

$$\dim K^{\mathrm{top}}_i(\mathbb{S}''(d)_v) = p_2(\gcd(d, v)).$$

9. Étale covers of preprojective algebras

In this section, we prove an extension of Theorem 7.6 to étale covers of preprojective stacks which we use to compute the topological K-theory of quasi-BPS categories of K3 surfaces in [PTd].

We first define quasi-BPS categories and BPS cohomology for étale covers of preprojective stacks. BPS sheaves are defined by base-change from the BPS sheaves of preprojective algebras. Quasi-BPS categories are defined via graded matrix factorizations and the Koszul equivalence, see Subsection 9.2.

Recall that Theorem 7.6 follows, via dimensional reduction and the Koszul equivalence, from Theorem 6.2. The two main ingredients in the proof of Theorem 6.2 are the semiorthogonal decomposition from Theorem 2.5 and the construction of the cycle map (6.9) from Theorem 6.3. The analogous statements for étale covers are Propositions 9.5 and 9.6, respectively.

We will use the notations and constructions from Subsection 7.1. Throughout this section, we fix a quiver Q° satisfying Assumption 2.2 and a dimension vector $d \in \mathbb{N}^I$.

We begin by discussing the setting and by stating Theorem 9.2, the main result of this section.

9.1. **Preliminaries.** Let E be an affine variety with an action of $G := G(d)$ and with a G-equivariant étale map

$$e \colon E \to R^\circ(d) \oplus R^\circ(d)^\vee.$$

Consider the quotient stacks with good moduli spaces

$$\pi_F \colon \mathcal{F} := (E \oplus \mathfrak{g})/G \to F := (E \oplus \mathfrak{g})/\!\!/G.$$

Consider the moment map $\mu \colon E \to \mathfrak{g}^\vee$ and the induced function $f \colon \mathcal{F} \to \mathbb{C}$, where $f(x, v) = \langle \mu(x), v \rangle$ for $x \in E$ and $v \in \mathfrak{g}$. Consider the quotient stack with good moduli space

$$\pi_L \colon \mathcal{L} := \mu^{-1}(0)/G \to L := \mu^{-1}(0)/\!\!/G.$$

There are maps:

$$\begin{array}{ccc} \mathcal{L}^{\mathrm{cl}} & \xrightarrow{e} & \mathcal{P}(d)^{\mathrm{cl}} \\ \Big\downarrow{\scriptstyle \pi_L} & & \Big\downarrow{\scriptstyle \pi_{P,d}} \\ L & \xrightarrow{e} & P(d). \end{array}$$

Throughout this section, we assume that both horizontal maps in the above diagram are étale.

Note that the moment map μ has image in the traceless subalgebra $\mathfrak{g}_0^\vee \cong \mathfrak{g}_0 \subset \mathfrak{g}$. Let $\mu_0 \colon E \to \mathfrak{g}_0^\vee$ and let $\mathcal{L}^{\mathrm{red}} := \mu_0^{-1}(0)/G$.

Definition 9.1. Let $\mathcal{BPS}_d^p \in \mathrm{Perv}(P(d))$ be the preprojective BPS sheaf and let

$$\mathcal{BPS}^L := e^*(\mathcal{BPS}_d^p) \in \mathrm{Perv}(L). \tag{9.1}$$

One defines $\mathcal{BPS}^L_\delta$ for $\delta \in M(d)^{W_d}_{\mathbb{R}}$ and $\mathcal{BPS}^L_{d,v} := \mathcal{BPS}^L_{v\tau_d}$ as in (7.2).

By Theorem 2.9, there is a semiorthogonal decomposition:

$$D^b\left(\mathcal{P}(d)\right)_v = \left\langle \mathbb{A}(d)_v, \mathbb{T}(d)_v \right\rangle$$

for any $v \in \mathbb{Z}$. The purpose of this section is to prove the following:

Theorem 9.2. *Let $v \in \mathbb{Z}$. There are subcategories $\mathbb{T} = \mathbb{T}(L)_v$ and $\mathbb{A} = \mathbb{A}(L)_v$ of $D^b(\mathcal{L})_v$ such that:*

1. *there is a semiorthogonal decomposition $D^b(\mathcal{L})_v = \langle \mathbb{A}, \mathbb{T} \rangle$,*
2. *if e is the identity, then $\mathbb{T} = \mathbb{T}(d)_v$ and $\mathbb{A} = \mathbb{A}(d)_v$,*
3. *if $h \colon E' \to E$ is an étale map inducing $e' := e \circ h \colon E' \to R^\circ(d) \oplus R^\circ(d)^\vee$, and if we consider $\pi_{L'} \colon \mathcal{L}' \to L'$ and the categories $\mathbb{A}(L'), \mathbb{T}(L') \subset D^b(\mathcal{L}')$ for E', then h induces functors $h^* \colon \mathbb{T}(L)_v \to \mathbb{T}(L')_v$ and $h^* \colon \mathbb{A}(L)_v \to \mathbb{A}(L')_v$,*
4. *for any $i, \ell \in \mathbb{Z}$, the cycle map (3.6) for $\mathcal{L}$ induces isomorphisms*

$$\mathrm{c} \colon \mathrm{gr}_\ell K_i^{\mathrm{top}}(\mathbb{T}) \xrightarrow{\sim} H^{-2\ell-i}(L, \mathcal{BPS}^L_{d,v}).$$

Further, one can also define categories $\mathbb{T}^{\mathrm{red}}, \mathbb{A}^{\mathrm{red}} \subset D^b(\mathcal{L}^{\mathrm{red}})_v$ which satisfy the analogous conditions to (1)-(4) above. In particular, the map $l' \colon \mathcal{L}^{\mathrm{red}} \to \mathcal{L}$ induces an isomorphism

$$\mathrm{c} \circ l'_* \colon \mathrm{gr}_\ell K_i^{\mathrm{top}}(\mathbb{T}^{\mathrm{red}}) \xrightarrow{\sim} \mathrm{gr}_\ell K_i^{\mathrm{top}}(\mathbb{T}) \xrightarrow{\sim} H^{-2\ell-i}(L, \mathcal{BPS}^L_{d,v}). \tag{9.2}$$

We will only explain the constructions for $\mathcal{L}$, the case of $\mathcal{L}^{\mathrm{red}}$ is similar. In Subsection 9.2, we define the categories $\mathbb{T}$ and $\mathbb{A}$ using graded matrix factorizations and the Koszul equivalence. In Subsection 9.3, we prove the third claim in Theorem 9.2.

9.2. **Quasi-BPS categories for étale covers.** We will use the setting from Subsection 9.1. There is a Cartesian diagram, where the maps e are étale maps:

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{e} & \mathcal{X}(d) \\ {\scriptstyle \pi_F}\Big\downarrow & \square & \Big\downarrow{\scriptstyle \pi_{X,d}} \\ F & \xrightarrow{e} & X(d). \end{array}$$

By Theorem 2.9, there is a semiorthogonal decomposition

$$D^b\big(\mathcal{X}(d)\big)_v = \big\langle \mathbb{B}(d)_v, \mathbb{M}(d)_v \big\rangle. \tag{9.3}$$

Define subcategories $\mathbb{B} = \mathbb{B}(F)$, $\mathbb{M} = \mathbb{M}(F)$ of $D^b(\mathcal{F})_v$ to be classically generated (see Subsection 2.15) by $e^*\mathbb{B}(d)_v$, $e^*\mathbb{M}(d)_v$ respectively. Note that, for $\delta \in M(d)_{\mathbb{R}}^{W_d}$, we can define analogously

$$\mathbb{M}(\delta) = \mathbb{M}(F;\delta) \subset D^b(\mathcal{F}). \tag{9.4}$$

Lemma 2.11 implies that:

Corollary 9.3. *There is a semiorthogonal decomposition*

$$D^b(\mathcal{F})_v = \langle \mathbb{B}, \mathbb{M}\rangle.$$

If e is the identity, then $\mathbb{M}(d) = \mathbb{M}(d)_v$ and $\mathbb{B}(d) = \mathbb{B}(d)_v$.

Consider the category of graded matrix factorizations $\mathrm{MF}^{\mathrm{gr}}(\mathcal{F}, f)$, where the grading is of weight 2 for the summand $\mathfrak{g}$ and is of weight 0 on E. By the Koszul equivalence, we have that:

$$\kappa_L \colon D^b(\mathcal{L}) \xrightarrow{\sim} \mathrm{MF}^{\mathrm{gr}}(\mathcal{F}, f). \tag{9.5}$$

Define the subcategories of $D^b(\mathcal{L})$:

$$\mathbb{T} = \mathbb{T}(L) := \kappa_L^{-1}\left(\mathrm{MF}^{\mathrm{gr}}(\mathbb{M}, f)\right),\ \mathbb{A} = \mathbb{A}(L) := \kappa_L^{-1}\left(\mathrm{MF}^{\mathrm{gr}}(\mathbb{B}, f)\right).$$

By [PTa, Proposition 2.5], we obtain:

Corollary 9.4. *The properties (1), (2), and (3) in the statement of Theorem 9.2 hold for the categories $\mathbb{A}$ and $\mathbb{T}$ of $D^b(\mathcal{L})$.*

We also need a version of Theorem 2.8 for étale covers. Consider the forget-the-framing map $\tau_\alpha \colon \mathcal{X}^{\alpha f}(d)^{\mathrm{ss}} \to \mathcal{X}(d)$. Let $\mathcal{F}^{\alpha f}$ be such that the following diagram is Cartesian:

$$\begin{tikzcd}
\mathcal{F}^{\alpha f} \arrow[r, "e"] \arrow[d, "\tau_{\alpha,F}"'] \arrow[dr, phantom, "\square"] & \mathcal{X}^{\alpha f}(d)^{\mathrm{ss}} \arrow[d, "\tau_\alpha"] \\
\mathcal{F} \arrow[r, "e"] \arrow[d, "\pi_F"'] & \mathcal{X}(d) \arrow[d, "\pi_{X,d}"] \\
F \arrow[r, "e"] & X(d).
\end{tikzcd} \tag{9.6}$$

Recall the semiorthogonal decomposition of Theorem 2.5:

$$D^b\left(\mathcal{X}^{\alpha f}(d)^{\mathrm{ss}}\right) = \left\langle \tau_\alpha^*\left(\otimes_{i=1}^k \mathbb{M}(d_i)_{v_i}\right)\right\rangle.$$

For a partition $\underline{d} := (d_i)_{i=1}^k$ of $d \in \mathbb{N}^I$ and for integer weights $\underline{v} := (v_i)_{i=1}^k$, define $\mathbb{M}(\underline{d}, \underline{v}) \subset D^b\left(\mathcal{F}^{\alpha f}\right)$ to be classically generated by $e^*\tau_\alpha^*\left(\otimes_{i=1}^k \mathbb{M}(d_i)_{v_i}\right)$. By Lemma 2.11, we obtain that:

Proposition 9.5. *There is a semiorthogonal decomposition*

$$D^b\left(\mathcal{F}^{\alpha f}\right) = \left\langle \mathbb{M}(\underline{d}, \underline{v})\right\rangle,$$

where the left hand side is as in Theorem 2.5.

The category $\mathbb{M}(\underline{d}, \underline{v})$ can be described via the Hall product, same as in Theorem 2.5. Let λ be an antidominant cocharacter associated to $(d_i)_{i=1}^k$. Consider the diagram:

$$\mathcal{F}^\lambda \xleftarrow{q_F} \mathcal{F}^{\lambda \geqslant 0} \xrightarrow{p_F} \mathcal{F}.$$

There is an étale map $e\colon \mathcal{F}^{\lambda} \to \mathcal{X}(d)^{\lambda} \cong \times_{i=1}^{k} \mathcal{X}(d_i)$. Then the Hall product

$$*_F = p_{F*}q_F^*\colon D^b(\mathcal{F}^{\lambda}) \to D^b(\mathcal{F})$$

is base-change of the categorical Hall product $D^b\big(\times_{i=1}^{k} \mathcal{X}(d_i)\big) \cong \otimes_{i=1}^{k} D^b(\mathcal{X}(d_i)) \to D^b(\mathcal{X}(d))$. Let $\widetilde{\mathbb{M}}(\underline{d}, \underline{v}) \subset D^b\left(\mathcal{F}(\underline{d})\right)$ be the subcategory classically generated by $e^*\left(\otimes_{i=1}^{k} \mathbb{M}(d_i)_{v_i}\right)$. There is then an equivalence:

$$\tau_{\alpha,F}^* \circ *_F\colon \widetilde{\mathbb{M}}(\underline{d}, \underline{v}) \xrightarrow{\sim} \mathbb{M}(\underline{d}, \underline{v}). \tag{9.7}$$

9.3. **Comparison with BPS cohomology.** Recall the notation from Subsection 7.1. Consider the commutative diagram:

$$\begin{tikzcd}
 & & \mathcal{L}^{\mathrm{cl}} \arrow[ld, hook] \arrow[dd] \arrow[rr] & & \mathcal{P}(d)^{\mathrm{cl}} \arrow[ld, hook] \arrow[dd] \\
\mathcal{F} \arrow[r, "\eta"] \arrow[dd, "\pi_F"'] & E/G \arrow[rr] \arrow[dd, "\pi_E"] & & \mathcal{Y}(d) \arrow[dd] & \\
 & & L \arrow[rr] \arrow[ld, hook] & & P(d) \arrow[ld, hook] \\
F \arrow[r, "\eta"] & E/\!\!/G \arrow[rr] & & Y(d) &
\end{tikzcd}$$

Recall the sheaf $\mathcal{BPS}^L \in \mathrm{Perv}(L)$ defined in (9.1) and consider the BPS sheaf $\mathcal{BPS}_d \in \mathrm{Perv}(X(d))$ for the tripled quiver with potential (Q, W) associated to Q°. Define the BPS sheaf:

$$\mathcal{BPS}^F = e^*(\mathcal{BPS}_d) \in \mathrm{Perv}(F).$$

For $\delta \in M(d)_{\mathbb{R}}^{W_d}$, define

$$\mathcal{BPS}_{\delta}^F \in D^b_{\mathrm{con}}(F),\ \mathcal{BPS}_{d,v}^F := \mathcal{BPS}_{v\tau_d}^F$$

as in (6.5). Note that, by base-change of the decomposition (6.18), we obtain the analogous decomposition for $\pi_F\colon \mathcal{F} \to F$:

$$\pi_{F*}\varphi_f \mathrm{IC}_{\mathcal{F}}[-1] = \bigoplus_{A \in \mathcal{P}} e^*(\mathrm{Q}_A). \tag{9.8}$$

By base-change of (6.17), we obtain the following decomposition for the map $\pi_{\alpha,F} := \pi_F \circ \tau_{\alpha,F}\colon \mathcal{F}^{\alpha f} \to F$:

$$\pi_{\alpha,F*}\varphi_f \mathbb{Q}_{\mathcal{F}^{\alpha f}}[\dim \mathcal{F} - 1] = \bigoplus_{A \in \mathcal{P}_{\alpha}} e^*(\mathrm{Q}_A). \tag{9.9}$$

The monodromy on $H^{\bullet}(\mathcal{F}, \varphi_f)$ is trivial, so there is a cycle map:

(9.10)
$$\mathrm{c}\colon \mathrm{gr}_a K_i^{\mathrm{top}}\left(\mathrm{MF}(\mathcal{F}, f)\right) \xrightarrow{\sim} H^{2\dim \mathcal{F} - i - 2a}(\mathcal{F}, \varphi_f \mathbb{Q}_{\mathcal{F}}[-1]) \oplus H^{2\dim \mathcal{F} - i - 2a}(\mathcal{F}, \varphi_f \mathbb{Q}_{\mathcal{F}}[-2]).$$

We now define a cycle map from topological K-theory of quasi-BPS categories to BPS cohomology, which is the analogue of Theorem 6.2.

Proposition 9.6. *Let $\delta \in M(d)_{\mathbb{R}}^{W_d}$ and recall the categories $\mathbb{M}(\delta)$ from (9.4). The cycle map (9.10) has image in*

(9.11)
$$\mathrm{c}\colon \mathrm{gr}_a K_i^{\mathrm{top}}\left(\mathrm{MF}(\mathbb{M}(\delta), f)\right) \to H^{\dim \mathcal{F} - i - 2a}(F, \mathcal{BPS}_{\delta}^F) \oplus H^{\dim \mathcal{F} - i - 2a}(F, \mathcal{BPS}_{\delta}^F[-1]).$$

Thus, for $\delta = v\tau_d$ and $\mathbb{M} = \mathbb{M}(v\tau_d)$, the cycle map (9.10) *has image in*

$$\mathrm{c}\colon \mathrm{gr}_a K_i^{\mathrm{top}}\left(\mathrm{MF}(\mathbb{M}, f)\right) \to H^{\dim \mathcal{F}-i-2a}(F, \mathcal{BPS}^F_{d,v}) \oplus H^{\dim \mathcal{F}-i-2a}(F, \mathcal{BPS}^F_{d,v}[-1]). \tag{9.12}$$

Proof. The same argument used in the proof of Theorem 6.3 applies here. The λ-widths (see (6.28)) of the category $\mathbb{M}(\delta)$ are equal to the λ-widths of the category $\mathbb{M}(d;\delta)$ for all cocharacters λ. The analogue of Proposition 6.15 then holds for $\mathrm{MF}(\mathbb{M}(\delta), f)$.

There is an explicit decomposition of $\pi_{F*}\mathrm{IC}_{\mathcal{F}}$ obtained by base-change from (6.16), and where the summands are in the image of (the base-change of the) Hall product. In Subsection 6.6, we constructed the map (6.36), proved Proposition 6.16, and noted corollaries of Proposition 6.16. There are versions of the map (6.36) and of Proposition 6.16 by F by base-change, and the results in Subsection 6.6 also apply for $\pi_{F*}\mathrm{IC}_{\mathcal{F}}$, and thus for $\pi_{F*}\varphi_f\mathrm{IC}_{\mathcal{F}}[-1]$. □

We next prove the analogue of Theorem 6.2.

Proposition 9.7. *The cycle map* (9.12) *is an isomorphism.*

Proof. The same argument used to prove Theorem 6.2 applies here, see Subsection 6.3, that is, the statement follows from comparing summands in the semiorthogonal decomposition (9.5) with summands in the decomposition (9.9). The cycle map (9.12) is injective by (9.10) and the admissibility of $\mathrm{MF}^{\mathrm{gr}}(\mathbb{M}, f)$ in $\mathrm{MF}^{\mathrm{gr}}(\mathcal{F}, f)$.

Consider a partition $\underline{d} = (d_i)_{i=1}^k$ of d and weights $\underline{v} = (v_i)_{i=1}^k \in \mathbb{Z}^k$. Consider the perverse sheaf

$$\mathcal{BPS}_{\underline{d},\underline{v}} := \oplus_* \left(\boxtimes_{i=1}^k \mathcal{BPS}_{d_i,v_i}\right) \in \mathrm{Perv}(X(d)),$$

where $\oplus\colon \times_{i=1}^k X(d_i) \to X(d)$ is the direct sum map. By Proposition 9.6 for the disjoint union of k copies of Q, dimension vector $(d_i)_{i=1}^k \in \left(\mathbb{N}^I\right)^k$, and $\delta = \sum_{i=1}^k v_i\tau_{d_i}$, there is an injective map for any $i \in \mathbb{Z}$:

$$\mathrm{gr}_\cdot K_i^{\mathrm{top}}\left(\mathbb{M}(\underline{d},\underline{v})\right) \hookrightarrow H^\cdot\left(F, e^*\mathcal{BPS}_{\underline{d},\underline{v}}\right).$$

The claim now follows as in the proof of Theorem 6.2. □

We prove the analogue of Theorem 7.6.

Proposition 9.8. *The cycle map obtained by composing* (9.10) *with the forget-the-potential map is an isomorphism:*

$$\mathrm{c}\colon \mathrm{gr}_a K_i^{\mathrm{top}}\left(\mathrm{MF}^{\mathrm{gr}}(\mathcal{F}, f)\right) \xrightarrow{\sim} H^{2\dim \mathcal{F}-i-2a}(\mathcal{F}, \varphi_f[-1]). \tag{9.13}$$

Thus there is an isomorphism:

$$\mathrm{c}\colon \mathrm{gr}_a K_i^{\mathrm{top}}\left(\mathrm{MF}^{\mathrm{gr}}(\mathbb{M}, f)\right) \to H^{\dim \mathcal{F}-i-2a}(F, \mathcal{BPS}^F_{d,v}). \tag{9.14}$$

Further, there is an isomorphism:

$$\mathrm{gr}_a K_i^{\mathrm{top}}(\mathbb{T}) \xrightarrow{\sim} H^{-2a-i}(L, \mathcal{BPS}^L_{d,v})$$

Proof. The isomorphism (9.13) follows from Proposition 5.2. The isomorphism (9.14) follows then from Proposition 9.6. The last isomorphism follows from (9.14) and the compatibility between dimensional reduction and the Koszul equivalence from Proposition 5.2. □

Proof of Theorem 9.2. The first three properties hold by Corollary 9.4. The fourth property follows from Proposition 9.8. The statement for reduced stacks follows similarly. The isomorphism (9.2) also follows directly from Proposition 3.5. □

We also note the following analogue of Corollary 7.2.

Proposition 9.9. *The Chern character map*

$$\mathrm{ch}\colon K_i^{\mathrm{top}}(\mathbb{T}) \hookrightarrow G_i^{\mathrm{top}}(\mathcal{L}) \to \bigoplus_{j\in\mathbb{Z}} H^{\mathrm{BM}}_{i+2j}(\mathcal{L})$$

is injective.

Proof. The proof is analogous to that of Corollary 7.2. The claim follows from Proposition 4.10, a version of Proposition 9.5 involving potential, and the Koszul equivalence. □

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