

Higher spectral sequences

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Abstract

In this article we construct what we call a higher spectral sequence for any chain complex (or topological space) that is filtered in n compatible ways. For this we extend the previous spectral system construction of the author, and we show that it admits considerably more differentials than what was previously known. As a result, this endows the successive Leray–Serre, Grothendieck, chromatic–Adams–Novikov, and Eilenberg–Moore spectral sequences of the author with the structure of a higher spectral sequence. Another application is a universal coefficient theorem analog for spectral sequences.

1 Introduction

Consider either

- (a) a chain complex (C, d) that is filtered in n different ways over the integers, or
- (b) a topological space (or a spectrum) X that is filtered in n different ways over the integers, together with a generalized homology theory h_* .

In the classical case $n = 1$ we obtain a spectral sequence with limit¹ $H_*(C)$ and $h_*(X)$, respectively. In the general case $n \geq 1$, under a natural compatibility assumption of the n filtrations, we will construct what we call a *higher spectral sequence*. For its construction, we extend the spectral system construction of [2]. While it has the same limit $H_*(C)$ and $h_*(X)$, respectively, it contains considerably more pages than the n naturally associated spectral sequences. The pages considered in this paper are indexed over admissible words, and they can be related via differentials, group extensions, and natural isomorphisms. More precisely, given C or X as in (a) or (b) above, we construct for any admissible word $\omega \in L_a^*$ (see Definition 3.2) over the alphabet $L = \{1, \dots, n, 1^\infty, \dots, n^\infty, x\}$ a so-called ω -page, which is a collection of Abelian groups

$$S(P; \omega).$$

Here P ranges over a quotient $\mathbb{Z}^n/V_\omega \cong \mathbb{Z}^{n-k}$, where k is the number of letters x in ω . In the alphabet L , a letter $j \in [n]$ stands for taking homology with respect to the j 'th differential, j^∞ denotes the same but infinitely often, and x stands for a group extension process. The word ω is read from left to right and it tells how $S(P; \omega)$ is “constructed” from the first page, which itself is indexed by the empty word ε .

In ordinary spectral sequences we have $n = 1$, $L = \{1, 1^\infty, x\}$, and for $\omega = 1^{r-1}$ ($r \geq 1$) the ω -page consists of the columns in E_{**}^r , which are indexed over $P \in \mathbb{Z}$. The letter 1 stands for the connection between any E_{**}^r and E_{**}^{r+1} , 1^∞ stands for the connection between any E_{**}^r and E_{**}^∞ , and x for the connection between E_{**}^∞ and the “limit” of the spectral sequence, e.g. $H_*(C)$ and $h_*(X)$ in the above settings.

The analog of this for higher spectral sequences can be informally summarized as follows.

¹in the sense of [2]

1. The first page $S(P; \varepsilon)$, $P \in \mathbb{Z}^n$, consists of the homologies of the smallest pieces (subquotients of C , and pairs of subsets of X) that can be constructed from the common refinement of the n given filtrations.
2. At most of the pages $S(P; \omega)$, there exist n differentials (unless they are saturated), which are indexed by $[n] = \{1, \dots, n\}$. Taking homology with respect to the j 'th differential yields $S(P; \omega * j)$. Here, “*” denotes concatenation of words.
3. One can take homology in one direction j an infinite number of times to arrive at $S(P; \omega * j^\infty)$. This saturates the differential with index j .
4. After any such step, we perform a group extension process to arrive at $S(P; \omega * j^\infty x)$.
5. If $j^\infty x$ appears in ω for each $j \in [n]$, we call ω final and $S(P; \omega)$ equals $H_*(C)$ or $h_*(X)$, respectively.

This can be used as follows. Step 2 gives a connection between the first page (Step 1) and arbitrary ω -pages for $\omega \in [n]^*$. We can proceed with Step 3 and take homology infinitely often in one direction $j \in [n]$. Note that as with usual spectral sequences, $S(P; \omega * j^\infty)$ may indeed be a proper subquotient of the limit of $S(P; \omega * j^k)$ as $k \rightarrow \infty$, compare with Weibel [4], Boardman [1], McCleary [3]. Then we proceed with Step 4, which connects to $S(P; \omega * j^\infty x)$. As with the extension process in usual spectral sequences, the filtration (F_i) of $S(P; \omega * j^\infty x)$ in Theorem 3.6(e) may be neither Hausdorff nor exhaustive, and even if it is, $S(P; \omega * j^\infty x)$ may not be complete with respect to (F_i) . As usual these two problems in Steps 3 and 4 can be serious obstacles for computations, but they are the standard ones in ordinary spectral sequences. Arriving at $S(P; \omega * j^\infty x)$ we can start again at Step 2 and iterate until ω is final. At that point, Step 5, we arrived at the limit of the higher spectral sequence.

Outline. This paper is organized as follows. In Section 2 we review the notions and basic properties of exact couple systems over $D(\mathbb{Z})$ (“higher exact couples”) and their associated spectral systems (“higher spectral sequences”). This is a common formal framework that includes both major special cases (a) and (b) from above.

In Section 3.1 the ω -pages are defined. In Section 3.2 the main theorem is stated, which makes the above informal description of higher spectral sequences and their basic properties precise. Some examples are given: Higher Leray–Serre, Eilenberg–Moore, and Grothendieck spectral sequences, as well as two universal coefficient higher spectral sequences. The main theorem is proved in Section 3.3. In Section 3.4 we discuss some more properties of the $n = 2$ case.

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2 Preliminaries on spectral systems

We recall the necessary background from [2]. Let $n \geq 1$ and $[n] := \{1, \dots, n\}$. Let e_1, \dots, e_n be the standard basis vectors in \mathbb{Z}^n , and $\mathbb{1} := (1, \dots, 1)^t \in \mathbb{Z}^n$. \mathbb{Z}^n is a poset via $(x_1, \dots, x_n) \leq (x'_1, \dots, x'_n)$ if and only if $x_i \leq x'_i$ for all i .

Throughout the paper, let $I := D(\mathbb{Z}^n)$ denote the lattice of downsets of \mathbb{Z}^n . (Everything in this paper can also be done for filtrations over $D(\overline{\mathbb{Z}}^n)$, where $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{\pm\infty\}$; here we consider only $D(\mathbb{Z}^n)$ because it makes the presentation cleaner.)

I has minimum $-\infty := \emptyset$ and maximum $\infty := \mathbb{Z}^n$. We write $I_k := \{(p_1, \dots, p_k) \in I^k \mid p_1 \geq \dots \geq p_k\}$, which is again a poset via $(p_1, \dots, p_k) \leq (p'_1, \dots, p'_k)$ if and only if $p_i \leq p'_i$ for all i .

For us a chain complex is an Abelian group C together with an endomorphism $d : C \rightarrow C$ with $d \circ d = 0$, and its homology is $H(C, d) := \ker(d)/\text{im}(d)$; the grading is not of importance for us. An I -filtration of C is a family of subchain complexes $(F_p)_{p \in I}$ such that $F_q \subseteq F_p$ whenever $q \leq p$.

Similarly, if X is a topological space then an I -filtration of X is a family of (usually open) subspaces $(X_p)_{p \in I}$ such that $X_q \subseteq X_p$ whenever $q \leq p$.

Whenever we have an I -filtered chain complex (C, d) , or an I -filtered space X together with a generalized homology theory h_* , we can associate a so-called exact couple system via

$$E_q^p := H(F_p/F_q) \quad (1)$$

or

$$E_q^p := h_*(X_p, X_q), \quad (2)$$

respectively, which is defined as follows.

Definition 2.1 (Exact couple system). An *exact couple system over I* is a collection of Abelian groups $(E_q^p)_{(p,q) \in I_2}$ together with homomorphisms $\ell_{p',q'}^{p,q} : E_q^p \rightarrow E_{q'}^{p'}$ for any $(p,q) \leq (p',q')$ and homomorphisms $k_{p,q} : E_q^p \rightarrow E_{-\infty}^q$ for any $(p,q) \in I_2$, such that the following properties are satisfied:

1. $\ell_{p'',q''}^{p',q'} \circ \ell_{p',q'}^{p,q} = \ell_{p'',q''}^{p,q}$.
2. The triangles

$$\begin{array}{ccc} E_{-\infty}^q & \xrightarrow{\ell_{p,-\infty}^{q,-\infty}} & E_{-\infty}^{p'} \\ & \swarrow k_{pq} & \searrow \ell_{p,q}^{p,-\infty} \\ & E_q^p & \end{array}$$

are exact.

3. The diagrams

$$\begin{array}{ccc} E_q^p & \xrightarrow{k_{pq}} & E_{-\infty}^q \\ \ell_{p',q'}^{pq} \downarrow & & \downarrow \ell_{q',-\infty}^{q,-\infty} \\ E_{q'}^{p'} & \xrightarrow{k_{p'q'}} & E_{-\infty}^{q'} \end{array}$$

commute.

Let E be an exact couple system over I . There is a natural differential $d_{pqz} : E_q^p \rightarrow E_z^q$ for any $(p,q,z) \in I_3$ defined by $d_{pqz} := \ell_{q,z}^{q,-\infty} \circ k_{pq}$. With this we define an associated *spectral system over I* via

$$S_{bq}^{pz} := \frac{\ker(d_{pqz} : E_q^p \rightarrow E_z^q)}{\text{im}(d_{bpq} : E_p^b \rightarrow E_q^p)}, \quad (b,p,q,z) \in I_4. \quad (3)$$

In the special case $I = D(\mathbb{Z}^n)$, we call the spectral system S the *higher spectral sequence of E* . At a first glance this is just a collection of Abelian groups, one for each element in I_4 , however there are many connections between them:

First note that the usual goal of computation, $E_{-\infty}^{\infty}$, appears as $S_{\infty,-\infty}^{\infty,-\infty}$. It is called the *limit* of this spectral system (this is just a name; it does not imply any convergence or comparison property). Moreover, terms of the form $S_{pq}^{pq} = E_q^p$ are usually known when p covers q , that is, when $|p \setminus q| = 1$. The collection of these terms is called the *first page* of S .

The following facts are proved in [2]. For any $(b,p,q,z) \leq (b',p',q',z')$ in I_4 , $\ell_{p',q'}^{pq}$ induces maps

$$S_{bq}^{pz} \rightarrow S_{b'q'}^{p'z'}$$

which we call maps induced by inclusion. When there is no confusion, we abbreviate all of them as ℓ .

Lemma 2.2 (Extensions). *For any $z \leq p_1 \leq p_2 \leq p_3 \leq b$ in I , we have a short exact sequence of maps induced by inclusion,*

$$0 \rightarrow S_{b,p_1}^{p_2,z} \rightarrow S_{b,p_1}^{p_3,z} \rightarrow S_{b,p_2}^{p_3,z} \rightarrow 0. \quad (4)$$

Lemma 2.3 (Differentials). *For any $(b, p, q, z), (b', p', q', z') \in I_4$ with $z \leq p'$ and $q \leq b'$ there are natural differentials*

$$d : S_{bq}^{pz} \rightarrow S_{b'q'}^{p'z'}, \quad (5)$$

which commute with ℓ , that is, $\ell \circ d = d \circ \ell$.

Lemma 2.4 (Kernels and cokernels). *For any $(b, p, q, z), (b', p', q', z') \in I_4$ with $z = p'$ and $q = b'$ we have*

$$\ker \left(d : S_{bq}^{pz} \rightarrow S_{b'q'}^{p'z'} \right) = S_{bq}^{p'q'}$$

and

$$\operatorname{coker} \left(d : S_{bq}^{pz} \rightarrow S_{b'q'}^{p'z'} \right) = S_{p'q'}^{p'z'}$$

Lemma 2.5 (∞ -page as filtration quotients). *$E_{-\infty}^{\infty}$ can be I -filtered by*

$$G_p := \operatorname{im}(\ell : E_{-\infty}^p \rightarrow E_{-\infty}^{\infty}) \cong S_{\infty, -\infty}^{p, -\infty}, \quad p \in I.$$

Furthermore the S -terms on the ∞ -page are filtration quotients

$$S_{\infty, q}^{p, -\infty} \cong G_p / G_q.$$

Lemma 2.6 (∞ -page as quotient kernels). *$E_{-\infty}^{\infty}$ has quotients*

$$Q_p := \frac{E_{-\infty}^{\infty}}{\ker(\ell : E_{-\infty}^{\infty} \rightarrow E_p^{\infty})} \cong S_{\infty, p}^{\infty, -\infty}, \quad p \in I.$$

Furthermore the S -terms on the ∞ -page are quotient kernels

$$S_{\infty, q}^{p, -\infty} \cong \ker(Q_q \rightarrow Q_p).$$

Definition 2.7 (Excision). An exact couple system E over I is called *excisive* if for all $a, b \in I$,

$$E_{a \cap b}^a \xrightarrow{\ell} E_b^{a \cup b}$$

is an isomorphism.

The exact couple system (2) is automatically excisive by the excision axiom of h_* if the subspaces X_p are all open. Note however that (1) is in general not excisive, though in many applications it is, for example when $C = \bigoplus_{P \in \mathbb{Z}^n} C_p$ (as Abelian group) and $(F_p)_I$ is the canonical I -filtration given by $F_p = \bigoplus_{P \in p} C_p$. More generally, (1) is excisive if the filtration $(F_p)_I$ is distributive in the sense that $F_{a \cap b} = F_a \cap F_b$ and $F_{a \cup b} = F_a + F_b$ for all $a, b \in I$.

In the settings (a) and (b) from the introduction, we refine the n given filtrations to obtain I -filtrations of C and X , respectively, by taking unions of n -wise intersections of the filtration pieces (see [2, §3]). This induces associated exact couple systems (1) and (2), respectively. The natural compatibility assumption on the n filtrations as mentioned in the introduction refers precisely to the excision property of these exact couple systems.

Let us think of $J := \mathbb{Z}^n$ as an undirected graph, whose vertices are the elements of J , and $x, y \in J$ are adjacent if they are related, i.e. $x \geq y$ or $x \leq y$ (coordinate-wise). For $(b, p, q, z) \in I_4$, let $Z(z, q, p, b) \subseteq J$ denote the union of all connected components of $p \setminus z$ that intersect $p \setminus q$, and let $B(z, q, p, b) \subseteq I$ denote the union of all connected components of $b \setminus q$ that intersect $p \setminus q$.

Lemma 2.8 (Natural isomorphisms). *In an excisive exact couple system E over $I = D(J)$, S_{bq}^{pz} is uniquely determined up to natural isomorphism by $Z := Z(z, q, p, b)$ and $B := B(z, q, p, b)$.*

We also write S_B^Z for S_{bq}^{pz} , which is only defined up to natural isomorphisms. A word of warning: This B - Z -description of S_{bq}^{pz} looks quite appealing. However it may be combinatorially non-trivial to check whether some given B and Z come from some (b, p, q, z) , and if so there might be several good choices. Moreover, it can be quite challenging to see whether there is a differential from $S_{B_1}^{Z_1}$ to $S_{B_2}^{Z_2}$ and what the resulting kernels and cokernels are in this case.

3 Higher spectral sequences

Throughout this section let us fix an excisive exact couple system E over $I = D(\mathbb{Z}^n)$.

3.1 Definition of ω -pages

Define an alphabet L ,

$$L := \{1, \dots, n, 1^\infty, \dots, n^\infty, x\}.$$

Remark 3.1 (Some intuition). Here, a letter $j \in [n]$ stands for taking homology with respect to the j 'th differential, j^∞ denotes the same but infinitely often, and x stands for a group extension process. In ordinary spectral sequences, $n = 1$, and the letter 1 stands the connection between some E_{**}^r and E_{**}^{r+1} , 1^∞ stands for the connection between some E_{**}^r and E_{**}^∞ , and x for the connection between E_{**}^∞ and the ‘‘limit’’ of the spectral sequence, e.g. $H(C)$ if the spectral sequence comes from a \mathbb{Z} -filtration of a chain complex C .

Let L^* denote the monoid of words of finite length with letters in L . Denote the empty word by ε , the concatenation of two words ω and ω' by $\omega * \omega'$, $\omega^n := \omega * \dots * \omega$ (n times), and the length of ω by $|\omega|$. L_a^* becomes a poset via $\tau \leq \omega$ if and only if τ is a prefix of ω , that is, a subword that starts from the beginning ($\tau = \varepsilon$ and $\tau = \omega$ are allowed).

Definition 3.2 (Admissible words). Call a finite word $\omega \in L^*$ *admissible* if the following holds:

1. if j^∞ appears, the subsequent subword of ω contains neither j nor j^∞ ,
2. the only letter allowed directly after j^∞ is x ,
3. any x occurring in ω comes directly after some j^∞ .

If furthermore ω contains subwords $j^\infty x$ for all $j \in [n]$ then ω is called *final*.

An exemplary final word for $n = 3$ is $123122^\infty x 133313^\infty x 111^\infty x$ and any prefix of a final word is admissible. Let L_a^* denote the set of all admissible words in L^* . Define $X(\omega) \subseteq [n]$ as the set of $j \in [n]$ such that $j^\infty x$ is a subword of ω , and $Y(\omega) := [n] \setminus X(\omega)$. $X(\omega)$ is so to speak the set of saturated indices along which the extension process has been already made, and $Y(\omega)$ is the set of unsaturated indices along which we still have differentials.

For $\omega \in L_a^*$, $i, j \in [n]$, we inductively define $r_\omega^i, \delta_\omega^i \in \mathbb{Z}^n$ and $B_\omega, Z_\omega \subset \mathbb{Z}^n$ as follows. Put $r_\varepsilon^i := e_i$, $r_{\omega * j^\infty}^i := r_\omega^i$, $r_{\omega * x}^i := r_\omega^i$, and

$$r_{\omega * j}^i := \begin{cases} r_\omega^i & \text{if } i \neq j, \\ r_\omega^i + \delta_\omega^i & \text{if } i = j, \end{cases}$$

where $\delta_\varepsilon^i := e_i$, $\delta_{\omega * j^\infty}^i := \delta_\omega^i$, $\delta_{\omega * x}^i := \delta_\omega^i$, and

$$\delta_{\omega * j}^i := \begin{cases} \delta_\omega^i & \text{if } i \in X(\omega) \cup \{j\}, \\ \delta_\omega^i - \delta_\omega^j & \text{if } i \in Y(\omega) \setminus \{j\}. \end{cases}$$

For $\omega \in [n]^*$, $\delta_\omega^i = \mathbb{1} - \sum_{k \in [n] \setminus i} r_\omega^k$.

Remark 3.3 (Some intuition 2). r_ω^i will be the negated direction of the i 'th differential at the ω -page, and δ_ω^i is the negated change of direction for the i 'th differential that occurs when taking homology with respect to it. In ordinary spectral sequences, $n = 1$, and for $\omega = 1^{r-1}$ the ω -page consists of the columns in E_{**}^r , with $r_\omega^1 = r$, and $\delta_\omega^1 = 1$.

Further put $B_\varepsilon := \{0\}$,

$$B_{\omega*j} := B_\omega + \{0, \delta_\omega^j\}, \quad (6)$$

$$B_{\omega*j^\infty} := B_\omega + \mathbb{Z}_{\geq 0} \cdot \delta_\omega^j, \quad (7)$$

$$B_{\omega*j^\infty_x} := B_\omega + \mathbb{Z} \cdot \delta_\omega^j. \quad (8)$$

Here, plus denotes a Minkowski sum. Thus for $\omega \in [n]^*$, B_ω can be regarded as a discrete zonotope, that is, an affine image of the vertices of an $|\omega|$ -dimensional cube. See Figures 1 and 2. In Section 3.3, several equivalent descriptions of B_ω are given. Define $Z_\omega := -B_\omega$, and for $P \in \mathbb{Z}^n$,

$$S(P; \omega) := S_{P+B_\omega}^{P+Z_\omega}. \quad (9)$$

Below we show that this is indeed a well-defined S -term (only up to natural isomorphism of course) by constructing $(b, p, q, z) \in I_4$ such that S_{bq}^{pz} represents $S(P; \omega)$.

Define lattices $V_\omega \subseteq \mathbb{Z}^n$ for $\omega \in L_a^*$ inductively as follows. Put $V_\varepsilon := \{0\}$, $V_{\omega*j} := V_\omega$, $V_{\omega*j^\infty} := V_\omega$, and

$$V_{\omega*j^\infty_x} := V_\omega + \mathbb{Z} \cdot \delta_\omega^j.$$

Equivalently, $V_\omega = B_\omega \cap Z_\omega = \mathbb{Z}\{\delta_\omega^j \mid j \in X(\omega)\}$. For $P, P' \in \mathbb{Z}^n$ with $P - P' \in V_\omega$, $S(P; \omega) = S(P'; \omega)$. Thus we may also think of $S(P; \omega)$ as being parametrized over $P \in \mathbb{Z}^n / V_\omega$.

Definition 3.4 (ω -page). Let $\omega \in L_a^*$. We call the collection of all $S(P; \omega)$, $P \in \mathbb{Z}^n / V_\omega$, the ω -page.

For $\omega = \varepsilon$ this was called the *first page* in [2], for $\omega = 123 \dots n$ the *second page*, and for $\omega = 1^{q_1} \dots n^{q_n}$ a *generalized second page*, or the Q -page, where $Q = (q_1, \dots, q_n) \in \mathbb{Z}_{\geq 0}^n$.

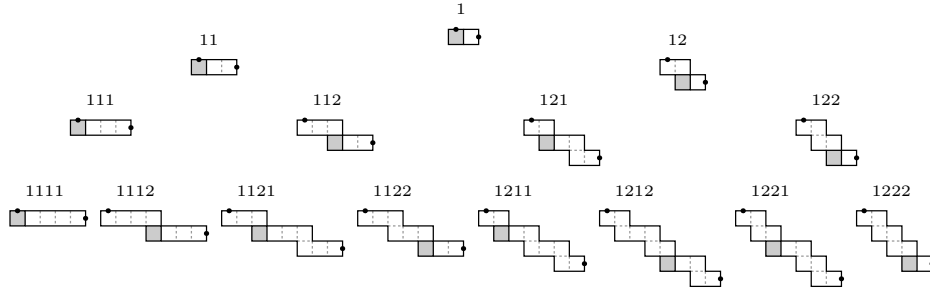


Figure 1: All B_ω with $|\omega| \leq 4$, $\omega_1 = 1$, and $n = 2$. For each B_ω , the origin is marked with a solid square, and the two points $r_\omega^i - e_i/2$ are marked with a black dot.

Remark 3.5 (Relation between jj^∞ and j^∞). Suppose $w \in L_a^*$ contains j^∞ , and let w' be the same word except that j^∞ is replaced by $j^k j^\infty$ for some $k \geq 1$. Then in general, $r_\omega^i \neq r_{\omega'}^i$, and $\delta_\omega^i \neq \delta_{\omega'}^i$, but they always agree modulo $V_\omega = V_{\omega'}$. Also $B_\omega = B_{\omega'}$ and hence $S(P; \omega) = S(P; \omega')$. Moreover one can check that the differentials in the main theorem 3.6 below are the same for ω and ω' . Thus in order to speak about the ω -page it is enough to know the image of ω in the quotient semigroup $L^*/(jj^\infty \sim j^\infty)$.

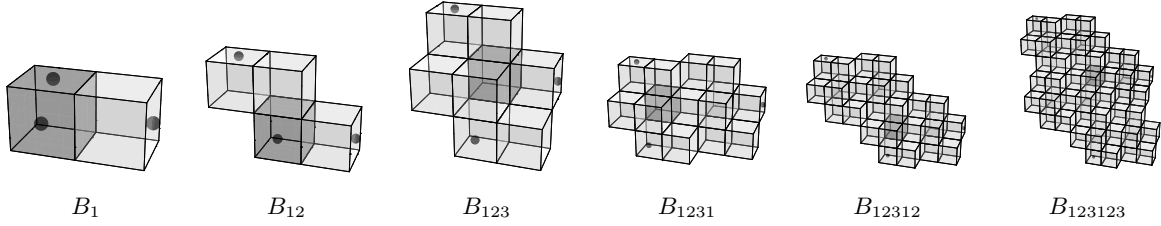


Figure 2: B_ω for $\omega = 1, \dots, 123123$ and $n = 3$. For each B_ω , the origin is marked with a dark cube, and the three points $r_\omega^i - e_i/2$ with a black dot.

3.2 Main theorem

Theorem 3.6 (Main theorem). *Let S denote the higher spectral sequence of a given excisive exact couple system E over $D(\mathbb{Z}^n)$.*

(a) *The first page of S at $P \in \mathbb{Z}^n$ is naturally isomorphic to*

$$S(P; \varepsilon) \cong E_q^p$$

for any two downsets $p, q \subset \mathbb{Z}^n$ that satisfy $p = q \dot{\cup} \{P\}$.

(b) *The limit of S for any final $\omega \in L_a^*$ and $P = \mathbb{Z}^n \in \mathbb{Z}^n / \mathbb{Z}^n$ is naturally isomorphic to*

$$S(P; \omega) \cong E_{-\infty}^\infty.$$

Moreover, for any $\omega \in L_a^*$, $P \in \mathbb{Z}^n / V_\omega$, and $j \in [n]$ such that $\omega * j$ is admissible, the following properties hold.

(c) *There are natural differentials*

$$\dots \xrightarrow{d} S(P + r_\omega^j; \omega) \xrightarrow{d} S(P; \omega) \xrightarrow{d} S(P - r_\omega^j; \omega) \xrightarrow{d} \dots \quad (10)$$

*Taking homology at $S(P; \omega)$ yields $S(P; \omega * j)$.*

(d) *$S(P; \omega * j^\infty)$ is a natural subquotient of $S(P; \omega * j^k)$ for all $k \geq 0$.*

(e) *There exists a natural \mathbb{Z} -filtration $(F_i)_{i \in \mathbb{Z}}$ of $S(P; \omega * j^\infty \mathbf{x})$,*

$$0 \subseteq \dots \subseteq F_i \subseteq F_{i+1} \subseteq \dots \subseteq S(P; \omega * j^\infty \mathbf{x}),$$

*such that $S(P + i \cdot \delta_\omega^j; \omega * j^\infty) \cong F_i / F_{i-1}$, for all $i \in \mathbb{Z}$.*

Remark 3.7 (Multiplicative structure). As usual, under certain assumptions on E there will be a multiplicative structure. The simplest instance is when E comes via (1) from an I -filtered differential algebra C , whose filtration $(F_p)_I$ satisfies $F_p \cdot F_q \subseteq F_{p+q}$, where $p+q$ denotes the Minkowski sum. Then for any $\omega \in L_a^*$ there is a natural product $S(P; \omega) \otimes S(Q; \omega) \rightarrow S(P+Q; \omega)$, which satisfies a Leibniz rule with respect to the differentials (c). Furthermore they are compatible with respect to (d) and (e) in the usual way, and for final ω it coincides with the product on $H(C)$. In other cases, such a natural product will only exist for ω -pages where ω has a certain prefix; examples for this are the higher Leray–Serre spectral sequence (Example 3.8) and the higher Grothendieck spectral sequence below. For details and a more general criterion see [2, §4.4].

Example 3.8 (Higher Leray–Serre spectral sequence). Consider a vertical tower of Serre-fibrations $F_{i-1} \hookrightarrow E_{i-1} \rightarrow E_i$ ($1 \leq i \leq n$) and a generalized homology theory h_* , such that each E_i has the

homotopy type of a CW-complex. Define $F_n := E_n$ as if we append the trivial fibration $F_n \xrightarrow{\text{id}} E_n \rightarrow \text{pt}$ to the tower. In [2, §5], we associated to the fibration tower a spectral system over $D(\overline{\mathbb{Z}}^n)$. Obviously we can restrict it to a spectral system over $D(\mathbb{Z}^n)$, and it becomes a higher spectral sequence in the sense of the present article. Its limit is $h_*(E_0)$. According to [2, Thm 5.1], its “second page” $S(P, 123 \dots n)$ ($P = (p_1, \dots, p_n) \in \mathbb{Z}^n$) can be naturally identified with

$$S(P, 123 \dots n) \cong H_{p_n}(F_n; H_{p_{n-1}}(F_{n-1}; \dots H_{p_1}(F_1; h_*(F_0))))), \quad (11)$$

where the coefficients for the homology of each F_i are the natural local coefficient systems over $\pi_1(F_i)$.

There is an obvious cohomological version of this higher Leray–Serre spectral sequence for the same fibration tower and for any generalized cohomology theory h^* . If h^* is moreover multiplicative, we obtain a product structure $S(P; \omega) \otimes S(Q; \omega) \rightarrow S(P + Q; \omega)$ for every $\omega \in L_a^*$, where it has the properties mentioned in Remark 3.7. This product structure agrees with the natural cup product of $H^*(F_n; \dots H^*(F_1; h^*(F_0)))$ for $\omega = 123 \dots n$, and it is natural with respect to maps of fibration towers (up to homotopies) for the $\omega \in L_a^*$ with prefix $123 \dots n$. The proof for this multiplicative structure is analogous to the arguments in [2, §5.3] once we have the description of the ω -pages $S(P; \omega)$ as $S_{b_\omega q_\omega}^{p_\omega z_\omega}$ with σ -lexicographic downsets $z_\omega, q_\omega, p_\omega, b_\omega$ (see the proof of Theorem 3.6).

Example 3.9 (Higher Eilenberg–Moore spectral sequence). Consider n fibrations $f_i : E_i \rightarrow B$ ($1 \leq i \leq n$) over the same base space with $\pi_1(B) = 0$ (as usual with Eilenberg–Moore spectral sequences, this assumption can be weakened). Let X denote the pullback of f_1, \dots, f_n . In [2, §8] we constructed a spectral system over $D(\overline{\mathbb{Z}}^n)$ from the cube of pullbacks of f_1, \dots, f_n , which becomes a higher spectral sequence in the sense of the present article. Its limit is $H_*(X)$. For any $\sigma \in \mathfrak{S}_n$, let $\omega_\sigma = \sigma_1 * \dots * \sigma_n$. Then the ω_σ -page at $P = (p_1, \dots, p_n)$ can be naturally identified with

$$S(P; \omega_\sigma) \cong \text{Tor}_{H_*(B)}^{p_{\sigma_n}} \left(\dots \text{Tor}_{H_*(B)}^{p_{\sigma_2}} \left(\text{Tor}_{H_*(B)}^{p_{\sigma_1}} (H_*(B), H_*(E_{\sigma_1})), H_*(E_{\sigma_2}) \right) \dots, H_*(E_{\sigma_n}) \right).$$

Here, $\text{Tor}_{H_*(B)}^{p_{\sigma_1}} (H_*(B), H_*(E_{\sigma_1}))$ simplifies to $H_*(E_{\sigma_1})$ if $p_{\sigma_1} = 0$ and to 0 otherwise. Note that in this example, we have $n!$ meaningful “second pages”, in comparison to just one in Example 3.8.

Example 3.10 (Higher Grothendieck spectral sequence). Let $\mathcal{A}_0 \xrightarrow{F_1} \mathcal{A}_1 \xrightarrow{F_2} \dots \xrightarrow{F_n} \mathcal{A}_n$ be a sequence of n right exact functors between Abelian categories with enough projectives. Write $F_{ij} = F_j \circ \dots \circ F_{i+1} : \mathcal{A}_i \rightarrow \mathcal{A}_j$ and assume that for any $i < j < k$, F_{ij} sends projective objects to F_{jk} -acyclic objects. In [2, §6] we constructed for each object $X \in \mathcal{A}_0$ a spectral system over $D(\overline{\mathbb{Z}}^n)$, which becomes a higher spectral sequence in the sense of the present article. Its limit is $L_* F_{0n}(X)$. Its “second page” $S(P; 123 \dots n)$ ($P = (p_1, \dots, p_n) \in \mathbb{Z}^n$) can be naturally identified with

$$S(P; 123 \dots n) \cong (L_{p_n} F_n) \circ \dots \circ (L_{p_1} F_1)(X).$$

Furthermore, in the setting of [2, §6.3] this higher Grothendieck spectral sequence admits a natural product structure $S(P; \omega) \otimes S(Q; \omega) \rightarrow S(P + Q; \omega)$ for every $\omega \in L_a^*$ with prefix $123 \dots n$; and for such ω it has the properties mentioned in Remark 3.7.

Similarly, the spectral system obtained from the chromatic spectral sequence followed by the Adams–Novikov spectral sequence [2, §7] becomes a higher spectral sequence in the sense of the present article.

Example 3.11 (Universal coefficients for spectral sequences). Let C be a free graded chain complex over a ring R and let M be an R -module. There is a universal coefficient spectral sequence that relates $H_*(C)$ with $H_*(C \otimes_R M)$, which simplifies to the usual universal coefficient theorem for $_ \otimes_R M$ in case R is a principal ideal domain. This motivates the question, whether there exists a universal coefficient theorem that relates the spectral sequence of a \mathbb{Z} -filtration $(F_n)_n$ of C with the spectral sequence of the induced filtration $(F_n \otimes_R M)_n$ of $C \otimes_R M$. Let us assume that the subchain complexes F_n are free. Then a reasonable positive answer to this is the following “universal coefficient higher

spectral sequence for $_ \otimes_R M^n$. Let $P_* \rightarrow M$ be a projective resolution of M . Define the (total) chain complex $D := C \otimes_R P_*$, which is filtered in two ways, $(F_n \otimes_R P_*)_n$ and $(C \otimes_R P_{\leq m})_m$. They refine to an I -filtration ($I = D(\mathbb{Z}^2)$), which is distributive and hence the associated exact couple system is excisive. The induced higher spectral sequence has limit naturally isomorphic to $H_*(C \otimes_R M)$. Various of its ω -pages can be naturally identified, such as $S((n, m); 1^{r-1}2) = \text{Tor}_m(E_n^r(C), M)$ ($1 \leq r \leq \infty$) and $S([(0, m)]; 1^\infty x 2) = \text{Tor}_m(H_*(C), M)$, where $E_n^r(C)$ denotes the n 'th column of the r 'th page of the spectral sequence of C with filtration $(F_n)_n$. Analogously there is a ‘‘universal coefficient higher spectral sequence for $\text{Hom}_R(_, M)$ ’’.

More generally, every higher spectral sequence of a graded chain complex C with distributive $D(\mathbb{Z}^n)$ -filtration $(F_p)_p$ yields analogously two universal coefficient higher spectral sequences over $D(\mathbb{Z}^{n+1})$ that relate to the higher spectral sequences over $D(\mathbb{Z}^n)$ of $(F_p \otimes_R M)_p$ and $(\text{Hom}_R(F_p, M))_p$, respectively. Another or additional way of generalizing is to replace M in the above by another free graded chain complex, possibly with a \mathbb{Z} - or I -filtration of its own, and we arrive at various higher spectral sequences of K nneth type.

3.3 Proof of the main theorem

Before proving Theorem 3.6 we need some preparation. In particular it will be convenient to move to another basis spanned by the δ_ω^i , which depends on ω .

For $\omega \in L_a^*$, define

$$M_\omega := (\delta_\omega^1 \quad \dots \quad \delta_\omega^n)^{-1} \in \mathbb{Q}^{n \times n}.$$

For $k \in [n]$ and $\omega \in L_a^*$, let T_ω^k denote the matrix

$$T_\omega^k := \text{id}_{\mathbb{Z}^n} + e_k(\mathbb{1}_{Y(\omega) \setminus \{k\}})^t = (\delta_{i=j} \text{ or } (i=k \text{ and } j \in Y(\omega)))_{ij} \in \text{SL}(n, \mathbb{Z}).$$

A quick calculation shows that $M_\varepsilon = \text{id}_n$, $M_{\omega * j^\infty} = M_\omega$, $M_{\omega * x} = M_\omega$, and

$$M_{\omega * j} = T_\omega^j M_\omega.$$

Therefore, $M_\omega \in \text{SL}(n, \mathbb{Z})$, and all its entries are non-negative. It will be very convenient to transform B_ω and V_ω by M_ω , so we define

$$B_\omega^M := M_\omega \cdot B_\omega, \quad V_\omega^M := M_\omega \cdot V_\omega \subseteq \mathbb{Z}^n.$$

In particular,

$$B_{\omega * j}^M = T_\omega^j(B_\omega^M + \{0, e_j\}), \tag{12}$$

$$B_{\omega * j^\infty}^M = T_\omega^j(B_\omega^M + \mathbb{Z}_{\geq 0} \cdot e_j), \tag{13}$$

$$B_{\omega * j^\infty x}^M = T_\omega^j(B_\omega^M + \mathbb{Z} \cdot e_j), \tag{14}$$

and $V_\omega^M = \mathbb{Z} \cdot \{e_i \mid i \in X(\omega)\}$. In particular, $\mathbb{Z}^n / V_\omega^M$ can be naturally identified with $\mathbb{Z}^{Y(\omega)}$.

Define $u_\omega \in \mathbb{Z}_{\geq 0}^n$ inductively via $u_\varepsilon := 0$, $u_{\omega * j^\infty} := u_\omega$, $u_{\omega * x} := u_\omega$, and

$$u_{\omega * j} := e_j + T_\omega^j u_\omega.$$

For $n \geq 2$ and $\omega \in [n]^*$, a more explicit formula is $u_\omega = \frac{1}{n-1}(M_\omega \mathbb{1} - \mathbb{1})$.

The following lemmas about B_ω are stated and proved only in the special case when $\omega \in [n]^*$, since this simplifies the notation considerably. When working modulo V_ω and V_ω^M , respectively, analogous statements still hold for arbitrary $\omega \in L_a^*$, as long as ω does not end on some j^∞ . It should be rather clear how to state and prove them.

Lemma 3.12. *For any $\omega \in [n]^*$, B_ω^M is a path in the unit-distance graph of \mathbb{Z}^n from 0 to u_ω , which is monotone with respect to all coordinates.*

We remark that this path is in general not simply a discretized line segment (i.e., the set of all lattice points with ℓ^∞ -distance at most $1/2$ from some line segment in \mathbb{R}^n).

Proof. We proceed by induction. Assume that the lemma holds for B_ω^M , and we want to prove it for $B_{\omega^*j}^M$ using (12). Suppose $x, x + e_k \in B_\omega^M$ are the vertices of an edge in B_ω^M . If $k \neq j$ then this edge gives rise to a path of length 2 in $B_{\omega^*j}^M$ along the vertices $T_\omega^j x, T_\omega^j(x + e_j) = T_\omega^j x + e_j$, and $T_\omega^j(x + e_k) = T_\omega^j x + e_j + e_k$. If $k = j$ then this edge gives rise to an edge in $B_{\omega^*j}^M$ whose vertices are $T_\omega^j x$ and $T_\omega^j(x + e_k) = T_\omega^j x + e_j$. Moreover $u_{\omega^*j} = T_\omega^j(u_\omega + e_j)$. \square

Corollary 3.13. *For any $\omega \in [n]^*$, $B_\omega^M \subseteq \{x \in \mathbb{Z}^n \mid 0 \leq x \leq u_\omega\}$. Moreover, $B_\omega^M = u_\omega - B_\omega^M$.*

Lemma 3.14. *For any $\omega \in [n]^*$, $i, j \in [n]$, $x \in B_\omega^M$, the following holds:*

1. *Either $x - M_\omega e_i \in B_\omega^M$, or $x - M_\omega e_i \leq 0$, or both.*
2. *Either $x + M_\omega e_i \in B_\omega^M$, or $x + M_\omega e_i \geq u_\omega$, or both.*

Proof. Suppose the lemma holds for B_ω^M , and we want to prove it for $B_{\omega^*j}^M$. By Corollary 3.13 it is enough to prove the first statement. By (12), any element of $B_{\omega^*j}^M$ is of the form $T_\omega^j x$ or $T_\omega^j(x + e_j) = T_\omega^j x + e_j$ for some $x \in B_\omega^M$. If now $z := x - M_\omega e_i \in B_\omega^M$, then also $z' := T_\omega^j x - M_{\omega^*j} e_i = T_\omega^j z$ and $z'' := T_\omega^j(x + e_j) - M_{\omega^*j} e_i = T_\omega^j(z + e_j)$ lie in B_ω^M .

Thus it remains to check the case when $z \leq 0$. Then clearly $z' = T_\omega^j z \leq 0$ since all entries of T_ω^j are non-negative. Similarly $z'' = T_\omega^j z + e_j \leq e_j$. If $z'' \leq 0$ does not hold, then $1 = z''_j = 1 + \sum_k z_k$, hence $z = 0$, thus $z'' = e_j$, which lies in $B_{\omega^*j}^M$ since $0 \in B_\omega^M$. \square

Let us regard any subset of \mathbb{Z}^n as a graph by connecting any two elements with distance 1 by an edge. In particular we can then talk about connected components of such subsets.

Lemma 3.15. *For any $\omega \in [n]^*$, B_ω is connected.*

Proof. By induction one immediately sees that

$$r_\omega^j - e_j \in B_\omega. \quad (15)$$

Also by induction,

$$M_\omega r_\omega^j = u_\omega + e_j \in e_j + B_\omega^M, \quad (16)$$

from which we get

$$r_\omega^j \in \delta_\omega^j + B_\omega. \quad (17)$$

Thus if B_ω is connected then so is B_{ω^*j} by (6), (15), and (17). \square

For any subset $X \subseteq \mathbb{Z}^n$, let $\text{Comp}_0(X)$ denote the connected component that contains 0.

Corollary 3.16. *For any $\omega \in [n]^*$, B_ω is $\text{Comp}_0(X)$ where X is the intersection of the n “discrete hyperplanes”*

$$\{x \in \mathbb{Z}^n \mid 0 \leq e_i^t M_\omega x \leq e_i^t u_\omega\}, \quad 1 \leq i \leq n.$$

Proof. This follows from Lemmas 3.14 and 3.15. \square

For our purposes the following similar description of B_ω is more useful. The symmetric group \mathfrak{S}_n acts on \mathbb{Z}^n by permutation of the coordinates, $\sigma \cdot x := (x_{\sigma^{-1}(i)})_{i \in [n]}$. Let \leq_{lex} denote the lexicographic relation on \mathbb{Z}^n . For $\sigma \in \mathfrak{S}_n$, define a new relation $\leq_{\sigma\text{-lex}}$ by setting $x \leq_{\sigma\text{-lex}} y$ if and only if $\sigma x \leq_{\text{lex}} \sigma y$.

For any $P \in (\mathbb{Z} \cup \{\infty\})^n$, $M \in \mathbb{Z}_{\geq 0}^{n \times n}$ with $\det M = 1$, and $\sigma \in \mathfrak{S}_n$, define

$$D(P; M, \sigma) := \{x \in \mathbb{Z}^n \mid Mx \leq_{\sigma\text{-lex}} MP\}$$

and

$$D^\circ(P; M, \sigma) := \{x \in \mathbb{Z}^n \mid Mx <_{\sigma\text{-lex}} MP\}.$$

Lemma 3.17. For any $\omega \in [n]^*$, $j \in [n]$, and $\sigma \in \mathfrak{S}_n$ with $\sigma(j) = n$, the following two equations hold.

1. $B_\omega = \text{Comp}_0(D^\circ(r_\omega^j; M_\omega, \sigma) \setminus D^\circ(0; M_\omega, \sigma)).$

2. $B_{\omega*j} = \text{Comp}_0(D(r_\omega^j; M_\omega, \sigma) \setminus D^\circ(0; M_\omega, \sigma)).$

Of course analogous formulas also hold for Z_ω and $Z_{\omega*j}$.

Proof. Since $\sigma(j) = n$, the only $z \in \mathbb{Z}^n$ with $u_\omega <_{\sigma\text{-lex}} M_\omega z \leq_{\sigma\text{-lex}} M_\omega r_\omega^j$ is r_ω^j . Now the first equation follows readily from Lemmas 3.14 and 3.15.

As for the second equation, “ \subseteq ” follows from $M_\omega B_{\omega*j} = M_\omega B_\omega + \{0, e_j\}$ and the connectivity of $B_{\omega*j}$. It remains to check “ \supseteq ”. By Lemma 3.14 and (16) the only neighbor y of $x \in B_\omega$ such that $y \notin B_\omega$ and $y \in D(r_\omega^j; M_\omega, \sigma) \setminus D^\circ(0; M_\omega, \sigma)$ is $y = r_\omega^j$. Similarly (or by symmetry), the only neighbor y of $x \in B_\omega + \delta_\omega^j$ such that $y \notin B_\omega + \delta_\omega^j$ and $y \in D(r_\omega^j; M_\omega, \sigma) \setminus D^\circ(0; M_\omega, \sigma)$ is $y = 0$. Both, 0 and r_ω^j , lie in $B_{\omega*j}$, which proves the claimed equality. \square

Proof of Theorem 3.6. We may assume $P = 0$, otherwise translate everything. (a) and (b) hold by definition and the excision property.

(c) We first consider the case $\omega \in [n]^*$. Let $\sigma \in \mathfrak{S}_n$ be any permutation with $\sigma(j) = n$. Define

$$\begin{aligned} p_\omega &:= D(0; M_\omega, \sigma), \\ q_\omega &:= D^\circ(0; M_\omega, \sigma) = p_\omega \setminus 0, \\ b_\omega &:= D^\circ(r_\omega^j; M_\omega, \sigma), \\ z_\omega &:= D(-r_\omega^j; M_\omega, \sigma). \end{aligned}$$

Then Lemma 3.17 implies that $S_{b_\omega q_\omega}^{p_\omega z_\omega}$ represents $S_{B_\omega}^{Z_\omega}$, which by definition is $S(0; \omega)$. One can describe $S(r_\omega^j; \omega)$ and $S(-r_\omega^j; \omega)$ similarly by translating all downsets by $\pm r_\omega^j$. Now Lemma 2.3 implies the claimed differentials in (10). Lemma 2.4 shows that taking homology in (10) at $S(0; \omega)$ yields $S_{b_\omega^* q_\omega^*}^{p_\omega z_\omega^*}$ where

$$\begin{aligned} b_\omega^* &:= D(r_\omega^j; M_\omega, \sigma) = b_\omega \cup \{r_\omega^j\}, \\ z_\omega^* &:= D^\circ(-r_\omega^j; M_\omega, \sigma) = z_\omega \setminus \{-r_\omega^j\}. \end{aligned}$$

By Lemma 3.17, $S_{b_\omega^* q_\omega^*}^{p_\omega z_\omega^*}$ represents $S_{B_{\omega*j}}^{Z_{\omega*j}} = S(0; \omega * j)$.

The general case $\omega \in L_a^*$ only needs minor modifications: In particular, the downsets b_ω , p_ω , q_ω , z_ω need to be replaced by the sum of V_ω with the analogous downsets in \mathbb{Z}^n/V_ω . Explicitly, let $k := |X(\omega)|$, and choose a $\sigma \in \mathfrak{S}_n$ that satisfies $\sigma(j) = n - k$ and $\sigma(i) > n - k$ for all $i \in X(\omega)$. Then p_ω can be defined as

$$p_\omega := \{x \in \mathbb{Z}^n \mid M_\omega x \leq_{\sigma\text{-lex}} (0^{n-k}, \infty^k)\},$$

and put $q_\omega := p_\omega \setminus V_\omega$, $b_\omega := q_\omega + r_\omega^j$, $z_\omega := p_\omega - r_\omega^j$, $b_\omega^* := p_\omega + r_\omega^j = b_\omega \cup (r_\omega^j + V_\omega)$ and $z_\omega^* := q_\omega - r_\omega^j = z_\omega \setminus (-r_\omega^j + V_\omega)$. Note that $V_\omega^M = \mathbb{Z}^{X(\omega)}$. Now one can repeat the previous argument in the quotient space \mathbb{Z}^n/V_ω .

(d) In M_ω -coordinates,

$$M_\omega B_{\omega*j^i} = B_\omega^M + \{0, \dots, i\}.$$

Hence as in the proof of Lemma 3.17 one can show that

$$B_{\omega*j^i} = \text{Comp}_0((b_\omega + i \cdot \delta_\omega^i) \setminus q_\omega),$$

where b_ω and q_ω are as above. Therefore $S(0; \omega * j^i)$ can be represented by $S_{b_\omega^i q_\omega^i}^{p_\omega z_\omega^i}$, where

$$\begin{aligned} b_\omega^i &:= b_\omega + i \cdot \delta_\omega^i = \{x \in \mathbb{Z}^n \mid M_\omega x \leq_{\sigma\text{-lex}} u_\omega + (0^{n-k-1}, i, \infty^k)^t\}, \\ z_\omega^i &:= z_\omega - i \cdot \delta_\omega^i = \{x \in \mathbb{Z}^n \mid M_\omega x \leq_{\sigma\text{-lex}} -u_\omega + (0^{n-k-1}, -i - 1, \infty^k)^t\}. \end{aligned}$$

Let $b_\omega^\infty := \bigcup_i b_\omega^i$ and $z_\omega^\infty := \bigcap_i z_\omega^i$. Then $S(0; \omega * j^\infty) = S_{b_\omega^\infty, z_\omega^\infty}^{p_\omega}$.

(e) Let $p_\omega, b_\omega^\infty, z_\omega^\infty$ as above. Put $p_\omega^i := i \cdot \delta_\omega^j + p_\omega$, $p_\omega^{-\infty} := \bigcap_{i \in \mathbb{Z}} p_\omega^i$, and $p_\omega^\infty := \bigcup_{i \in \mathbb{Z}} p_\omega^i$. Define

$$F_i := \text{im} \left(\ell : S_{b_\omega^\infty, p_\omega^\infty}^{p_\omega^i, z_\omega^\infty} \rightarrow S_{b_\omega^\infty, p_\omega^\infty}^{p_\omega, z_\omega^\infty} \right).$$

Then the assertion follows from Lemma 2.2 (or from the proof of Lemma 2.5). \square

3.4 The 2-dimensional case

The probably most frequent case (apart from the classical one, $n = 1$) is $n = 2$. A few more things can be said about this case:

Every final $\omega \in L_a^*$ is of the form

$$\omega = \tau * j_1^\infty x j_2^k j_2^\infty x, \quad (18)$$

for some $\tau \in [2]^*$, $\{j_1, j_2\} = [2]$, and $k \geq 0$. Any such ω gives a recipe to connect the first page to the limit of the spectral system. This recipe is therefore already determined by τ and j_1 .

Note that for all prefixes $\tau \leq \omega' \leq \omega$, $M_\tau = M_{\omega'} = M_\omega$. Let's define

$$N_\omega := (e_{j_2})^t \cdot M_\omega,$$

which is the ‘‘normal vector’’ along which the downsets b, p, q, z grow respectively shrink during $j_2^k j_2^\infty x$. Clearly $N_\omega \geq 0$ and it is primitive (i.e. its entries are coprime), and $N = (e_i)^t$ can happen only if $i = j_2$. Also, N_ω is invariant under the relation $jj^\infty \sim j^\infty$, compare with Remark 3.5.

Observation 3.18. *Modulo $jj^\infty \sim j^\infty$, ω is uniquely determined by N_ω and j_1 . Conversely, for any primitive $N^t \in \mathbb{Z}_{\geq 0}^2$ and $j_1 \in [2]$ with $N^t \neq e_{j_1}$ there is a final $\omega \in L_a^*$ of the form (18) such that $N = N_\omega$.*

Thus the connection determined by ω can be equivalently described by the pair (N_ω, j_1) .

Proof. In fact there is a simple algorithm that determines all possible τ from N (respectively N_ω) and $j_2 = 3 - j_1$. If $j_2 = 1$, choose $(N')^t \in \mathbb{Z}_{\geq 0}^2$ such that $M := \begin{pmatrix} N \\ N' \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \cap \mathbb{Z}_{\geq 0}^{2 \times 2}$. If $j_2 = 2$, choose $(N')^t \in \mathbb{Z}_{\geq 0}^2$ such that $M := \begin{pmatrix} N \\ N' \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \cap \mathbb{Z}_{\geq 0}^{2 \times 2}$. In any case, N' is well-defined up to adding an integral multiple of N . If N'_0 is the smallest choice, then all others are of the form $N'_k := N'_0 + kN$, $k \in \mathbb{Z}_{\geq 0}$. Now one can repetitively take one of the two rows of M and subtract it from the other one such that all entries stay non-negative until one arrives at $\text{id}_{\mathbb{Z}^2}$, and there is a unique way to do that. Let $q_i \in [2]$ denote the index of the column from which the other column was subtracted during the i 'th round. And say there were ℓ rounds. Then $M = M_\omega$ for $\omega = \tau j_1^\infty x j_2^\infty x$ and $\tau := q_\ell * \dots * q_1$. The choice of N' correspond to how often j_1 appears at the end of τ , namely k times if $N' = N'_k$. \square

The algorithm has similarities to the extended Euclidean algorithm applied to the first column of M . The only difference is that in the extended Euclidean algorithm, one subtracts *multiples* of one number from the other.

Example 3.19. Consider an excisive exact couple system E over $I(\mathbb{Z}^2)$. Suppose we want to determine both ω for which $N_\omega = (3, 5)$. For $j_1 = 1$, the algorithm runs as follows:

$$\begin{pmatrix} N \\ N' \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} \xrightarrow{q_1=2} \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \xrightarrow{q_2=1} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \xrightarrow{q_3=2} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \xrightarrow{q_4=1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

Thus $\omega = 12121^\infty x 2^\infty x$ does it. Similarly, for $j_1 = 2$,

$$\begin{pmatrix} N \\ N' \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} \xrightarrow{q_1=1} \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \xrightarrow{q_2=1} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \xrightarrow{q_3=2} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \xrightarrow{q_4=1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and thus $\omega = 12112^\infty x 1^\infty x$ does it too. See Figure 3.

