POSITROID LINKS AND BRAID VARIETIES

ROGER CASALS, EUGENE GORSKY, MIKHAIL GORSKY, AND JOSÉ SIMENTAL

Abstract. We study braid varieties and their relation to open positroid varieties. We discuss four different types of braids associated to open positroid strata and show that their associated Legendrian links are all Legendrian isotopic. In particular, we prove that each open positroid stratum can be presented as the augmentation variety for four different Legendrian fronts described in terms of either permutations, juggling patterns, cyclic rank matrices or Le diagrams. We also relate braid varieties to open Richardson varieties and brick manifolds, showing that the latter provide projective compactifications of braid varieties, with normal crossing divisors at infinity.

Она любила Ричардсона
Не потому, чтобы прочла
А. С. Пушкин, *Евгений Онегин*[1]

Contents

1. Introduction

This article studies braid varieties [8, 51] and their relation to open positroid varieties [42]. In a nutshell, we study four braids associated to any open positroid variety, and develop new techniques to algebraically study their braid varieties. In addition, this paper brings to bear insight from contact and symplectic topology to explicitly study these braid varieties, with a focus on Legendrian links and their relation to open positroid varieties in Grassmannians.

An open positroid variety Π of the Grassmannian $\mathrm{Gr}(k,n)$ can be indexed by either of the following four pieces of data. First, a pair of permutations $u, w \in S_n$ such that $u \leq w$ in the Bruhat order and w is a k-Grassmannian permutation. Second, a k-bounded affine permutation $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ of size n. Third, a cyclic rank matrix r and, fourth, a Le diagram. The bijections between these objects and the description of their associated positroid varieties are provided in [42, 57]. In this article, we study four braids, one associated to each of these four pieces of data, and introduce and study their associated Legendrian links.

The first result of this manuscript, within the realm of algebraic combinatorics, is showing that these four braids close up to links in $\mathbb{R}^3$ that are smoothly isotopic, up to trivially adding unlinks. For that, we develop new results using the positroid data above: k-Grassmannian permutations, k-bounded affine permutations, cyclic rank matrices and Le diagrams. In particular, this requires addressing the dissonance in the number of strands between these braids, which we address by introducing a Markov-type destabilization move that suits the algebraic combinatorics associated to positroids.

[1] *The reason she loved Richardson was not that she had read him* — A.S. Pushkin, *Eugene Onegin* (tr. V. Nabokov).

The main contact geometric result of this manuscript is then showing that the four associated Legendrian links are all Legendrian isotopic, up to trivially unlinked unknots. In particular, we show that our results and constructions in the smooth case, related to the algebraic combinatorics of positroids, can all be realized by contact isotopies. To our knowledge, the conceptual insight that certain Legendrian links, not just smooth links, underlie each of these four presentations of a positroid variety is also new. It has the important consequence of allowing the description and study of positroid strata in terms of contact topology, which has already been initiated in other works with fruitful consequences, cf. [1, 7, 8, 9, 11]. In particular, [8, Theorem 1.1] established the relationship between braid varieties and augmentation varieties. The contact isotopies between the four Legendrian links above imply that the four braid varieties associated to these Legendrians are isomorphic to the corresponding positroid variety, up to trivially adding frozens.

Finally, the article includes new results relating braid varieties to projective brick manifolds and open Richardson varieties. In particular, we show that brick varieties are good projective compactifications of our affine braid varieties. This also allows us to relate their homology to the top a-degree Khovanov-Rozansky homology of the underlying smooth link and establish the curious Lefschetz property for open Richardson varieties. The article concludes with a brief discussion on conjectural matters regarding cluster structures and Legendrian links.

1.1. **Scientific Context.** Positroid varieties first appeared in the study of total positivity [48, 49, 57, 59] and in the context of Poisson geometry [4]. Let $\Pi_{u,w}$ be the open positroid variety of the Grassmannian $\mathrm{Gr}(k,n)$ indexed by a pair of permutations $u, w \in S_n$, where $u \leq w$ in Bruhat order and w is k-Grassmannian. We consider the bijections between such pairs (u, w), bounded affine permutations $f : \mathbb{Z} \longrightarrow \mathbb{Z}$, cyclic rank matrices r, and Le-diagrams ⅃ established in [42, 57].

For instance, the bounded affine permutation $f(u, w) : \mathbb{Z} \longrightarrow \mathbb{Z}$ corresponding to a pair (u, w) is $f(u, w) := u^{-1} t_k w$, where t_k is the translation by the kth fundamental weight; conversely, f recovers (u, w). Here f is interpreted as a bijection $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(i+n) = f(i) + n$ and $i \leq f(i) \leq i + n$ for all $i \in \mathbb{Z}$. The four pieces of data (u, w), f, r and ⅃ are said to *represent the same positroid type* if they correspond to each other under these bijections. Each piece of data, (u, w), f, r, and ⅃, yields an open stratum $\Pi_{u,v}$, Π_f, Π_r, and $\Pi_{\text{⅃}}$ in $\mathrm{Gr}(k,n)$, and $\Pi_{u,v} = \Pi_f = \Pi_r = \Pi_{\text{⅃}}$ if (u, w), f, r, and ⅃ represent the same positroid type, cf. [42].

In Section 2 we explain how each of these pieces of data, (u, w), f, r and ⅃, also yields a *braid word*. In consequence, we can associate braids and links to each such four types of positroid data. These four braids, which we correspondingly denote $R_n(u, v)$, $J_k(f)$, $M_k(r)$ and $D_k(\text{⅃})$, are studied in detail in this article. Either of these four braids will be referred to as a *positroid braid*. We also connect the results in Section 2 to previous works in the literature, including [29, 42, 61].

In Section 3 we associate a *Legendrian link* to each such positroid braid. This has an important consequence: we can construct the corresponding positroid strata in a contact geometric manner. Namely, for the Legendrian links we construct from positroid braids, the corresponding positroid stratum is recovered as a Legendrian invariant. Specifically, as the spectrum of the 0th homology of the Legendrian contact dg-algebra. Therefore, it becomes a central question whether these Legendrian links are Legendrian isotopic if they are obtained from data representing the same positroid type. This is the content of Section 3, where we develop the necessary results to show that this is the case. Note that these Legendrian links and their connection to positroid strata and their cluster algebras have already featured in the recent preprints [1, 7, 9, 31, 30].

In Section 4 we study the braid varieties associated to these positroid braids. Braid varieties have featured prominently in the series of articles [8, 7, 31, 30], where their cluster structures are studied. The present manuscript is the second part of a trilogy: the first part is [8], where braid varieties were studied through weaves, and third part is [7], which establishes the general existence of cluster algebra structures on braid varieties. The current article studies some relevant algebraic geometric aspects of braid varieties associated to positroid data in Section 4. These include their relation to open Richardson varieties, the construction of smooth projective compactifications with normal crossing divisors, and the computation of their torus-equivariant homology, among others.

1.2. **Main Results.** By definition, two positive braid words β_1, β_2 are said to be equivalent if they represent the same element in the braid group Br_m. By [32], two equivalent β_1, β_2 represent the same element in the braid monoid Br_m^+.

The first result, proven in Section 2, establishes the relation between the four types of positroid braids $R_n(u,v)$, $J_k(f)$, $M_k(r)$ and $D_k(\text{Ⅎ})$.

Theorem 1.1. *Let $u, w \in S_n$ be such that $u \leq w$ in the Bruhat order and w is a k-Grassmannian permutation, f a bounded affine permutation, r a cyclic rank matrix, and Ⅎ a Le-diagram. Suppose that these four pieces of data represent the same positroid type. Then*

(i) *The n-stranded braid $\Delta_n R_n(u,w)$ and the k-stranded braid $\Delta_k J_k(f)$ are equivalent, up to positive Markov stabilizations and adding unlinked disjoint strands.*

(ii) *The k-stranded braid $J_k(f)$ and the k-stranded braid $\Delta_k D_k(\text{Ⅎ})$ are equivalent.*

(iii) *The k-stranded positive braids $\Delta_k J_k(f)$ and $M_k(r)$ are equivalent.* □

Note that Theorem 1.1.(i) relates two braids, $R_n(u,w)$ and $J_k(f)$, on a different number of strands. Section 2.4 develops a Markov-type destabilization which is well-suited for comparing the different types of algebraic combinatorics related to positroids.

The second result, established in Section 3, is a contact geometric counterpart of Theorem 1.1. Section 3.1 introduces four Legendrian links $\Lambda(u,w), \Lambda(f), \Lambda(r)$ and $\Lambda(\text{Ⅎ})$ in $(\mathbb{R}^3, \xi_{\mathrm{st}})$, each one associated to a different type of positroid data.

Theorem 1.2. *Let $u, w \in S_n$ be such that $u \leq w$ in the Bruhat order and w is a k-Grassmannian permutation, f a bounded affine permutation, r a cyclic rank matrix, and Ⅎ a Le-diagram. Suppose that these four pieces of data represent the same positroid type.*

Then the four Legendrian positroid links $\Lambda(u,w), \Lambda(f), \Lambda(r), \Lambda(\text{Ⅎ}) \subseteq (\mathbb{R}^3, \xi_{st})$ are Legendrian isotopic, up to unlinked max-tb Legendrian unknots. □

An important consequence of Theorem 1.2 is that the Legendrian contact dg-algebras associated to each of these Legendrian links are stable tame isomorphic. In particular, the spectra of their 0th homology algebras are isomorphic up to torus factors and they coincide with the corresponding positroid stratum, also up to torus factors. This provides an intrinsic and geometric way to recover positroids from these Legendrian links.

The third result studies braid varieties associated to positroid braids. In particular, it relates braid varieties to open Richardson varieties. Braid varieties $X(\beta; w)$ associated to a braid (word) β and a permutation w were introduced in [8], their definition is recalled in Section 4.1 below. The proof of the following result uses Theorem 1.2 together with [42] to show that any positroid variety $\Pi_{u,w}$ in the Grassmannian $\mathrm{Gr}(k,n)$ can be expressed in terms of braid varieties, either using the n-stranded braid $R_n(u,w)$ or the k-stranded braid $J_k(f)$.

Theorem 1.3. *Let $u, w \in S_n$ with $u \leq w$ in Bruhat order, w a k-Grassmannian permutation, and $f := ut_k w^{-1}$ the corresponding k-bounded affine permutation. Then we have algebraic isomorphisms*

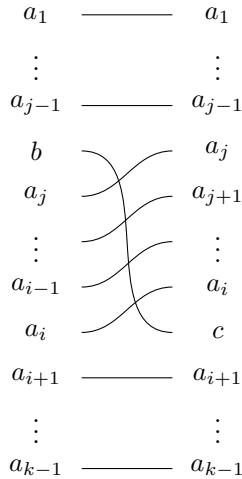(i) $\Pi_{u,w} \cong X(\beta(u^{-1}w_{0,n})\beta(w_{0,n} w w_{0,n}); w_{0,n})$

(ii) $X(\beta(u^{-1}w_{0,n})\beta(w_{0,n} w w_{0,n}); w_{0,n}) \cong X(J_k(f); w_{0,k}) \times (\mathbb{C}^*)^{n-k-\varphi}$

of affine algebraic varieties, where $\varphi := \#\{i \in [1,n] : f(i) = i\}$ is the number of fixed points of f. □

Theorem 1.3 is proven in Section 4.3. Section 4 offers a fourth result as well. Section 4.4 shows that the brick manifolds introduced in [18, Definition 3.2] provide smooth projective compactifications of braid varieties. We denote the brick manifold of β by $\mathrm{brick}(\beta)$ and its maximal open stratum by $\mathrm{brick}^\circ(\beta)$, cf. [18]. The precise relation we establish between braid varieties and brick manifolds is the following:

Theorem 1.4. *Let $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathcal{B}_n$ be a positive braid word, $\text{Ⴑ} \in \mathcal{B}_n$ its opposite, $\delta(\beta)$ the Demazure product of β, and consider the truncations $\beta_j := \sigma_{i_1} \cdots \sigma_{i_j}, j \in [1, \ell]$. The following holds:*

(i) *The algebraic map* $\Theta : \mathbb{C}^\ell \longrightarrow \mathscr{F}\ell_n^{\ell+1}$, $(z_1, \ldots, z_\ell) \mapsto (\mathscr{F}^{st}, \mathscr{F}^1, \ldots, \mathscr{F}^\ell)$, *where* $\mathscr{F}^j$ *is the flag associated to the matrix* $B_{\beta_j}^{-1}(z_{\ell-j+1}, \ldots, z_\ell)$, *restricts to an algebraic isomorphism*

$$\Theta : X(\beta; \delta(\beta)) \xrightarrow{\cong} \mathrm{brick}^\circ(\beta).$$

(ii) *The complement to* $X(\beta; \delta(\beta))$ *in* $\mathrm{brick}(\beta)$ *is a normal crossing divisor. Its components correspond to all possible ways to remove a letter from* β *while preserving its Demazure product.*

In particular, $\mathrm{brick}(\beta)$ *is a smooth projective good compactification of the affine variety* $X(\beta; \delta(\beta))$.

Note that $\mathrm{brick}(\beta)$ depends on the choice of braid word $\beta \in \mathcal{B}^+$, and not just the braid element $[\beta] \in \mathrm{Br}^+$, whereas $X(\beta; w_0)$ only depends on the positive braid $[\beta]$. Therefore, Theorem 1.4.(ii) can be used to construct several smooth projective SNC compactification of the same braid variety.

In addition, Theorem 1.4, in combination with [18], clarifies the connection between braid varieties and the combinatorics of subword complexes. This allows us to translate properties of spherical subword complexes via brick manifolds to braid varieties, and vice versa.

Finally, Section 4.5 explains how to compute the torus-equivariant homology of braid varieties associated to positroids. The article concludes in Section 5 with a brief discussion on conjectural matters.

Notational Conventions: Here are our notational conventions and a comparison to those in the existing literature. As usual, S_n is the symmetric group and $s_i \in S_n$ is the simple transposition that just swaps i and $i+1$. We multiply permutations as we compose functions. For example, $s_2 s_1 \in S_3$ is the permutation $s_2 s_1(1) = 3, s_2 s_1(2) = 1, s_2 s_1(3) = 2$. Our notion of a k-Grassmannian permutation coincides with that of [42] but differs from the one in [29]. A reason to choose this convention is Lemma 4.3 below.

Let Br_n be the braid group in n strands with Artin generators $\sigma_1, \ldots, \sigma_{n-1}$. Let $\mathrm{Br}_n^+ \subseteq \mathrm{Br}_n$ be the monoid of positive braids. Braids are multiplied so that the map $\sigma_i \mapsto s_i$, $\mathrm{Br}_n \to S_n$ is a group homomorphism. When drawing braid diagrams, the convention is that the strands are enumerated by $1, \ldots, n$ from top to bottom. Due to the convention above, we draw the braid diagram of a braid word as follows: we read the crossings (generators) of the braid word right-to-left and we draw them in the braid diagram left-to-right. Thus, the following is a picture of the braid diagram for the braid word $\sigma_2 \sigma_1 \in \mathrm{Br}_3$:

Note that the underlying permutation is indeed $s_2 s_1$. We denote by $\mathcal{B}_n$ the set of braid *words* on in the n Artin generators $\sigma_1, \ldots, \sigma_{n-1}$, of Br_n, and $\mathcal{B}_n^+$ the set of *positive* braid words. Two braids

FIGURE 1. The interval braid $\sigma_i \sigma_{i-1} \cdots \sigma_j$.

are said to be *equivalent* if they represent the same element in the braid group. Equivalently, if they are represented by braid words which are related by a sequence of braid moves. The set of braids equivalent to the braid represented by a braid word $\beta \in \mathcal{B}_n$ is denoted by $[\beta]$. Given a braid word β, read left to right, its *opposite* β is defined to be the braid word β read right to left, i.e. in reverse order. The half-twist word we use will be denoted by

$$\Delta_n := (\sigma_1)(\sigma_2 \sigma_1)(\sigma_3 \sigma_2 \sigma_1) \cdot \ldots \cdot (\sigma_{n-1} \sigma_{n-2} \ldots \sigma_2 \sigma_1) \in \mathcal{B}_n^+,$$

and its Coxeter projection is denoted by $w_{0,n} \in S_n$. If $u, w \in S_n$, we denote $u \leq w$ if u is less than w in the Bruhat order. The tautological braid lift of a reduced expression of a permutaion $w \in S_n$ to a braid word in Br is denoted by $\beta(w) \in$ Br or simply $w \in$ Br. For a permutation $w \in S_n$, we denote by w^* its conjugation by the longest element: $w^* := w_{0,n} w w_{0,n}$. In particular, we have $s_i^* = s_{n-i}$. Finally, the braid matrices we use in Section 4 coincide with those used in [8, 10] but differ from those used in [7]. The two conventions differ by taking inverse matrices.

Finally, sometimes (in particular, in Section 2.3) we will consider permutation braids where the strands are labeled in a non-standard way. In this case, we clearly state the labels and their order *both* on the left and on the right, and how the left labels are connected to the right ones. This determines a permutation braid uniquely up to braid moves.

In particular, we define *interval* braids as follows. An interval braid on k strands is a braid of the form $\sigma_i \sigma_{i-1} \cdots \sigma_j$ for some $1 \leq j \leq i \leq k-1$. When depicting such braids diagrammatically, we label strands on the left and on the right by a pair of k-element subsets of $[1, n]$ differing in precisely one element, in the decreasing order from top to bottom on both sides. Explicitly, assume that $a_1 > \ldots > a_{k-1}$, $a_{j-1} > b > a_j$ for some $1 \leq j \leq k-1$ or $b > a_1$, and $a_i > c > a_{i+1}$ for some $1 \leq j \leq k-2$, $c > a_1$ or $a_{k-1} > c$. The interval braid has labels $a_1, \ldots, a_{j-1}, b, a_j, \ldots, a_{k-1}$ on the left and $a_1, \ldots, a_i, c, a_{i+1}, \ldots, a_{k-1}$ on the right. The labels $a_1, \ldots, a_{k-1}$ are connected with the namesake labels on the right, while b on the left is connected to c on the right. The resulting permutation braid has $i - j$ crossings. See Figure 1 for a visual description. $\square$

2. POSITROID BRAIDS AND EQUIVALENCES

In this section we introduce positroid braids and start discussing equivalences between them. After setting up the necessary combinatorics in Subsection 2.1, we study positroid braids as follows:

(i) The Richardson braid is presented in Subsection 2.2 and the Juggling braid is presented in Subsection 2.3. The former is an n-stranded braid and the latter is a k-stranded braid.

(ii) Subsection 2.4 shows how to relate the Richardson braid and the Juggling braid via a sequence of generalized destabilization moves, which are also discussed in that subsection.

(iii) The Le braid is introduced in Subsection 2.5, which also establishes its relation with the Juggling braid, and the matrix braid is discussed in Subsection 2.6.

2.1. Combinatorial Data.

Let us introduce the combinatorial data used to describe positroid braids. Fix two natural numbers $k, n \in \mathbb{N}$ such that $k \leq n$. There are four equivalent families of combinatorial objects indexing open positroid strata that we employ: certain pairs of permutations $u, w \in S_n$, certain bijections $f : \mathbb{Z} \longrightarrow \mathbb{Z}$, Le diagrams and cyclic rank matrices. These are schematically depicted in Figure 2. The object of this first subsection is to define part of these pieces of combinatorial data and review the bijections we will need.

$u, w \in S_n, \quad u \leq w$ w is k-Grassmannian	$f : \mathbb{Z} \longrightarrow \mathbb{Z}$ affine k-bounded permutation	Le Diagram for λ, λ partition in $k \times (n - k)$ box	Cyclic rank matrix $r = (r_{ij}), \quad i, j \in \mathbb{N}$
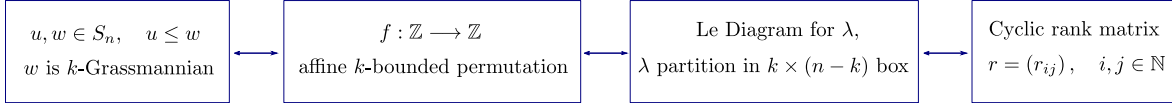
FIGURE 2. The four types of combinatorial data indexing positroid braids.

2.1.1. k-Grassmannian permutations and positroid pairs.

By definition, a permutation $w \in S_n$ is said to be k-Grassmannian if

$$w(1) < w(2) < \cdots < w(k), \text{ and } w(k+1) < \cdots < w(n).$$

Similarly, $\text{III} \in S_n$ is said to be a k-shuffle if III^{-1} is k-Grassmannian, that is, if

$$\text{III}^{-1}(1) < \text{III}^{-1}(2) < \cdots < \text{III}^{-1}(k), \text{ and } \text{III}^{-1}(k+1) < \cdots < \text{III}^{-1}(n).$$

The set of k-Grassmannian permutations in S_n (equivalently, the set of k-shuffles) is in bijection with the set of partitions λ whose Young diagram fits inside a $k \times (n - k)$-rectangle, cf. [57, Section 19]. We do not distinguish between a partition λ and its Young diagram; we draw the latter in French notation. Thus, if the Young diagram of λ fits inside a $k \times (n - k)$-rectangle, we write $\lambda \subseteq (n - k)^k$. Such λ can be written as

$$\lambda = (\lambda_1, \ldots, \lambda_k)$$

where $n - k \geq \lambda_1 \geq \cdots \geq \lambda_k \geq 0$. Note that it is possible that λ has a zero part, e.g. $\lambda_k = 0$ is allowed. Similarly, the transposed partition can be written as

$$\lambda^t = (\lambda_1^t, \ldots, \lambda_{n-k}^t)$$

where $k \geq \lambda_1^t \geq \cdots \geq \lambda_{n-k}^t \geq 0$. Again, it is allowed that $\lambda_{n-k}^t = 0$; this happens if and only if $n - k > \lambda_1$.

For $\lambda \subseteq (n - k)^k$, we denote by $w_\lambda \in S_n$ its associated k-Grassmannian permutation. By using one-line notation, we can write

$$(2.1) \qquad w_\lambda = [1 + \lambda_k, 2 + \lambda_{k-1}, \ldots, k + \lambda_1, k + 1 - \lambda_1^t, k + 2 - \lambda_2^t, \ldots, n - \lambda_{n-k}^t].$$

Note that the length of w_λ is $\ell(w_\lambda) = |\lambda| := \lambda_1 + \ldots + \lambda_k$. In fact, we can obtain reduced decompositions for w_λ as follows:

$$(2.2) \qquad \begin{aligned} w_\lambda &= (s_{\lambda_k} \cdots s_1) \cdots (s_{k+\lambda_2-2} \cdots s_{k-1})(s_{k+\lambda_1-1} s_{k+\lambda_1-2} \cdots s_k) \\ &= (s_{n-\lambda_{n-k}^t} \cdots s_{n-1}) \cdots (s_{k+2-\lambda_2^t} \cdots s_k s_{k+1})(s_{k+1-\lambda_1^t} \cdots s_{k-1} s_k). \end{aligned}$$

This expression can be read pictorially: we draw the Young diagram λ and fill the box in row i and column j with the number $k + j - i$. The first reduced expression in Equation 2.2 above is obtained by reading this diagram by rows. The second reduced expression is obtained by reading it by columns. Throughout the paper, we use the convention that empty rows (with $\lambda_i = 0$) do not contribute to the first product and the empty columns (with $\lambda_i^t = 0$) do not contribute to the second product.

Example 2.1. *Let us consider the values $k = 3$ and $n = 7$ and the Young diagram $\lambda = (4, 3, 1)$. By filling the (i, j)-box of λ with $3 + (j - i)$ we obtain:*

1			
2	3	4	
3	4	5	6

The associated 3-Grassmannian permutation is

$$w_\lambda = (s_1)(s_4 s_3 s_2)(s_6 s_5 s_4 s_3) = (s_6)(s_4 s_5)(s_3 s_4)(s_1 s_2 s_3),$$

and note that the length $\ell(w_\lambda)$ is indeed $|\lambda| = 8$. □

We can also read the permutation w_λ as follows. First, we identify a partition $\lambda \subseteq (n-k)^k$ with a sequence of n vertical and horizontal steps, that start from the northwest corner of the rectangle and follow the shape of the partition until they reach the southeast corner. For example, the partition in Example 2.1 corresponds to the sequence $(H, V, H, H, V, H, V)^2$. Enumerate these steps consecutively along the border of the partition. We will refer to these as the *right labels* of λ. We also enumerate the left border of the rectangle with the numbers $1, \ldots, k$, reading top-to-bottom, and the bottom border with the numbers $k + 1, \ldots, n$, reading left-to-right. We refer to these as the *left labels* of λ.

For each vertical (resp. horizontal) step in the border of λ, draw a horizontal (resp. vertical) ray to the left (resp. down). The resulting diagram is known as the *wiring diagram* of λ, and it gives the permutation w_λ by mapping the left labels to the right labels along the rays. See Figure 3.
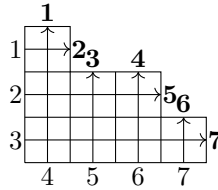
FIGURE 3. In one-line notation, $w_\lambda = [2, 5, 7, 1, 3, 4, 6]$. Right labels shown in bold.

Definition 2.2. A pair of permutations (u, w) with $u, w \in S_n$ is said to be a *k-positroid pair* if $u \leq w$ in the Bruhat order and w is k-Grassmannian. A k-positroid pair will be simply referred to as a *positroid pair* if k is understood from context. □

2.1.2. *Le diagrams.* In order to additionally record the data of u in a positroid pair (u, w), $u, w \in S_n$, one enhances the Young diagram λ for w into a *Le diagram*. Let us recall that the hook of a box in a Young diagram λ consists of all the boxes above it (in the same column) as well as all the boxes to its right (in the same row). By definition, a Le diagram $\beth = (\lambda, P)$ consists of a partition λ together with a collection P of boxes of λ, such that any box belonging to the hooks of two different boxes in P must also be in P. Pictorially, we depict a Le diagram $\beth$ as a partition λ together with dots in some of its boxes, precisely in those that belong to P. We thus refer to boxes in P as boxes *with a dot*.

Fixing a partition $\lambda \subseteq (n-k)^k$, [57, Theorem 19.1] shows that there is a bijection between Le diagrams with underlying partition λ and elements $u \in S_n$ with $u \leq w_\lambda$. To a Le diagram $\beth = (\lambda, P)$, the bijection associates the permutation u that is obtained by deleting the simple transpositions corresponding to boxes in P in either of the reduced decompositions (2.2) of w_λ. See Example 2.3. Thus, for each $k \in [n]$, there is a bijection between k-positroid pairs (u, w) in S_n and Le diagrams whose underlying partition fits into a $k \times (n-k)$-rectangle.

Example 2.3. *Let us consider the values $(k, n) = (4, 6)$ and the Young diagram $\lambda = (2, 2, 2, 2)$. The associated permutation is $w_\lambda = (s_2 s_3 s_4 s_5)(s_1 s_2 s_3 s_4)$. Choose the permutation $u = (s_4)(s_2 s_3) \in S_6$, which satisfies $u \leq w$. The Le diagram associated to this pair (u, w) is drawn on Figure 4.(B). In one-line notation we have*

$$w^{-1} = [5, 6, 1, 2, 3, 4], \; w = [3, 4, 5, 6, 1, 2],$$
$$u^{-1} = [1, 4, 2, 5, 3, 6], \; u = [1, 3, 5, 2, 4, 6]$$

Note that w^{-1} is a 4-shuffle in S_6. □

---

[2]Note that the sequence of steps depends on the size of the box $k \times (n-k)$ and not just on the partition λ. For example, the partition in Example 2.1 considered inside a 4×4 box yields the sequence (V, H, V, H, H, V, H, V).
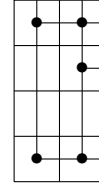
(A) The Young diagram $\lambda = (2,2,2,2)$ associated to $w_\lambda = (s_2s_1)(s_3s_2)(s_4s_3)(s_5s_4) = (s_2s_3s_4s_5)(s_1s_2s_3s_4)$.
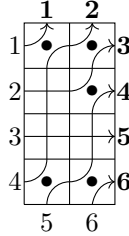
(B) The Le diagram associated to the pair (u, w_λ), for $u = (s_2)(s_4s_3) = (s_4)(s_2s_3)$.

FIGURE 4. Constructing a Le diagram from a pair (u, w).

Note that from a Le diagram $\mathsf{J} = (\lambda, P)$, we can read the permutation u in a similar way to how we read the permutation w from the Young diagram λ. Simply make the following change to any box with a dot:
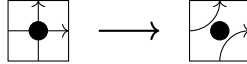
In Example 2.3 we have:

which coincides with $u = [1, 3, 5, 2, 4, 6]$.

2.1.3. *Bounded affine permutations.* Finally, let us discuss k-bounded affine permutations of size n, following [42]. By definition, an affine permutation $f : \mathbb{Z} \to \mathbb{Z}$ of size n is a bijection such that $f(i + n) = f(i) + n$ for all $i \in \mathbb{Z}$; we often denote affine permutations in one-line window notation $f = [f(1) \ldots f(n)]$. By definition, an affine permutation is said to be *k-bounded* if the following conditions are satisfied:

$$i \le f(i) \le i + n, \quad i \in \mathbb{Z} \quad \text{and} \quad \sum_{i=1}^{n}(f(i) - i) = nk.$$

By [42, Prop. 3.15], a k-bounded affine permutation f admits a unique decomposition of the form

(2.3) $f = u_f t_k w_f^{-1}$,   where $t_k := [1 + n, 2 + n, \ldots, k + n, k + 1, k + 2, \ldots, n]$,

with $(u_f, w_f) \in S_n$ a positroid pair. This is to say, any k-bounded affine composition admits a unique decomposition of the form

$$f = \mathcal{Y}^{-1} t_k \mathrm{III}$$

where III is a k-shuffle permutation and $\mathcal{Y} \le \mathrm{III}$.

Let us provide an explicit description of the permutations u_f, w_f appearing in (2.3). For that, we note that there exist exactly k values $i_1 < i_2 < \ldots < i_k$ of $i_r \in [1, n]$ such that $n < f(i_r)$, and exactly $(n - k)$ values $j_1 < j_2 < \cdots < j_{n-k}$ of $j_t \in [1, n]$ such that $f(j_t) \le n$. The permutations u_f, w_f are then described as follows:

(2.4) $w_f := [i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_{n-k}]$,

(2.5) $u_f := [f(i_1) - n, \ldots, f(i_k) - n, f(j_1), \ldots, f(j_{n-k})]$.

Note that w_f is a k-Grassmannian permutation and $u \in S_n$, since $n < f(i_r) \le 2n$ for every $r \in [1, k]$. The permutations u_f, w_f coincide with the permutations in [42, Proposition 3.15]. [3]

Example 2.4. *First, for the trivial n-translation $f = t_k = [1+n, \ldots, k+n, k+1, \ldots, n]$, we have $(i_1, \ldots, i_k) = (1, \ldots, k)$ and thus $w_f = [1, 2, \ldots, n]$. Similarly, $u_f = [1, 2, \ldots, n]$ is also the identity.*

Second, for the k-bounded permutation f defined by $f(i) = i + k$, we obtain that $(i_1, \ldots, i_k) = (n - k + 1, \ldots, n)$ and hence $w_f = [n - k + 1, \ldots, n, 1, \ldots, n - k]$ is the maximal k-Grassmannian permutation. In this second case, the permutation $u_f = [1, 2, \ldots, n]$ is still the identity. $\square$

Next, we record some facts translating the notations between Le diagrams and bounded affine permutations.

Lemma 2.5. *Suppose that $u \le w$ correspond both to the Le diagram of shape λ and to the bounded affine permutation f, so that $f = u t_k w^{-1}$. Let $i_1, \ldots, i_k$ and $j_1, \ldots, j_{n-k}$ be as above. Then:*

a) $i_1 = 1 + \lambda_k, i_2 = 2 + \lambda_{k-1}, \ldots, i_k = k + \lambda_1$ and $j_1 = k + 1 - \lambda_1^t, j_2 = k + 2 - \lambda_2^t \ldots, j_{n-k} = n - \lambda_{n-k}^t$.

b) If the Le diagram for $u \le w$ has no dots in the rightmost column then

$$f(j_{n-k}) = u(n) = w(n) = j_{n-k}.$$

Otherwise $f(j_{n-k}) = u(n) = n - d + 1$, with d the row number of the lowest dot in the last column.

Proof. Part (a) follows by comparing the two formulas (2.1) and (2.4) for w in one-line notation. The first part of Part (b) holds by construction. To prove the second part, observe that

$$w = (s_{n-\lambda_{n-k}^t} \cdots s_{n-1}) \cdots (s_{k+1-\lambda_1^t} \cdots s_k)$$

and that we have $u = (a s_{n-d+1} \cdots s_{n-1}) b$, where b does not contain s_{n-1} and a is a subword of $(s_{n-\lambda_{n-k}^t} \cdots s_{n-d-1})$. Therefore $u(n) = n - d + 1$. $\square$

Lemma 2.5 above gives us a direct way for moving from the Le diagram to the corresponding bounded affine permutation. Let us fix a Le diagram ⅃ with the underlying partition λ. Recall the left and right labels for λ from Section 2.1.1. It follows from identity (2.4) that $i_1, \ldots, i_k$ are the right labels corresponding to the *vertical* steps of λ, while $j_1, \ldots, j_{n-k}$ are the right labels that correspond to the horizontal steps.

Now we consider the wiring diagram of ⅃. The following two facts are a consequence of (2.5).

(a) Consider the wire starting at the bottom of the s-th column of ⅃ with the left label $s + k$. Then, $f(j_s)$ is the right label of the endpoint of this wire.

(b) Consider the wire starting on the left of the ℓ-th row of ⅃ with the left label ℓ. Then, $f(i_\ell) = n + t$, where t is the right label of the endpoint of this wire.

Note that (a) and (b) imply that the dotless columns of ⅃ correspond to the fixed points of f; while the dotless rows correspond to $x \in \{1, \ldots, n\}$ satisfying $f(x) = x + n$. See Figure 5 for an example.

For $1 \le s \le n - k$ we define

(2.6) $$\mathrm{inv}(s) := \sharp\{t > s : f(j_t) < f(j_s)\}.$$

Lemma 2.6. *For all $s \in [1, n - k]$, we have the inequalities*

$$j_s \le f(j_s) - \mathrm{inv}(s) \le k + s$$

The first inequality is sharp unless $f(j_s) = j_s$.

Proof. If $t > s$ then $j_t > j_s$. If $f(j_t) < f(j_s)$ then

$$j_s < j_t \le f(j_t) < f(j_s),$$

and since f is injective all such $f(j_t)$ are distinct. Therefore either $f(j_s) = j_s$ and $\mathrm{inv}(s) = 0$, or $\mathrm{inv}(s) \le f(j_s) - j_s - 1$, so $f(j_s) - \mathrm{inv}(s) \ge j_s + 1$.

For the second inequality, observe that

(2.7) $$\sharp\{t > s : f(j_t) > f(j_s)\} = n - k - s - \mathrm{inv}(s),$$

---

[3]Our notation coincides with that of [42]. Note that what [29] calls a k-Grassmannian permutation is what we call a k-shuffle, so the decomposition in [29, Proposition 4.2] is in fact the decomposition $f = ⅄^{-1} t_k Ⅲ$.
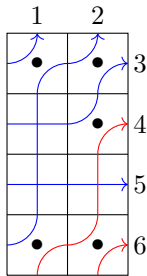
FIGURE 5. The wiring diagram for the Le diagram of Example 2.3.   If f is the corresponding bounded affine permutation, then $f(1), f(2) < n$, while $f(3), f(4), f(5), f(6) > n$. Moreover, to find $f(1)$ we follow the red strand starting opposite to 1, and see that $f(1) = 4$. Similarly, $f(2) = 6$. Likewise, to find $f(3)$ we follow the blue strand starting opposite to 3, and see that $f(3) = 1 + 6$. Similarly, $f(4) = 3 + 6$, $f(5) = 5 + 6$, $f(6) = 2 + 6$. Note that the only dotless row has the right label 5: this corresponds to 5 being the only $x \in \{1, \ldots, 6\}$ satisfying $f(x) = x + 6$.

so we have at least $n - k - s - \operatorname{inv}(s)$ distinct integers between $f(j_s) + 1$ and n. Therefore $f(j_s) + n - k - s - \operatorname{inv}(s) \leq n$ and $f(j_s) - \operatorname{inv}(s) \leq k + s$. $\qquad \square$

2.2. **Richardson Braids.** Given a permutation $v \in S_n$, we will denote by $\beta(v) \in \mathrm{Br}_n^+$ its reduced positive braid lift to the n-stranded braid group. A positive braid word for the braid $\beta(v) \in \mathrm{Br}_n^+$, which we also denote by $\beta(v) \in \mathcal{B}_n^+$, is obtained by considering a reduced expression $v = s_{i_1} \ldots s_{i_{\ell(v)}}$ for $v \in S_n$ in terms of a product of the simple transpositions $s_1, \ldots, s_{n-1} \in S_n$ generating the symmetric group S_n and substituting each s_i by the Artin braid generator σ_i, $i \in [1, n]$, i.e. $\beta(v) = \sigma_{i_1} \ldots \sigma_{i_{\ell(v)}}$. The braid word $\beta(v)$ depends on the choice of a reduced expression of v, but all such words for a given v are related by braid relations $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$, a.k.a. Reidemester III moves, and commutation relations, and thus represent the same braid. Recall the notation $v^* := w_0 v w_0$.

Definition 2.7 (Richardson braid). Let $u, w \in S_n$ be two permutations such that $u \leq w$ in the Bruhat order. The Richardson braid word $R_n(u, w) \in \mathcal{B}_n$ associated to the pair (u, w) is

$$R_n(u, w) := \beta(u^*)^{-1} \beta(w^*).$$

By definition, the smooth Richardson link is the smooth link $\Lambda(u, w) \subseteq \mathbb{R}^3$ given by the 0-framed closure of the n-stranded braid word $R_n(u, w)$. $\qquad \square$

Definition 2.7 is inspired by [29, Section 1.5]. The reason we have to conjugate by w_0 is Corollary 4.6. See also Remark 4.7. Let us also define a version of the Richardson braid with only positive Artin generators, together with its associated smooth link:

Definition 2.8 (Positive Richardson braid). Let $u, w \in S_n$ be two permutations such that $u \leq w$ in the Bruhat order. The positive Richardson braid word $R_n^+(u, w) \in \mathcal{B}_n$ associated to the pair (u, w) is

$$R_n^+(u, w) := \beta(u^{-1} w_0) \beta(w^*).$$

By definition, the positive Richardson link is the smooth link $\Lambda^+(u, w) \subseteq \mathbb{R}^3$ associated to the 0-framed closure of the n-stranded braid word $\Delta_n^{-1} R_n^+(u, w)$. $\qquad \square$

Remark 2.9. We emphasize that the braid words in Definitions 2.7 and 2.8 depend on the choice of reduced expressions of u and w, but the braids $R_n(u, w), R_n^+(u, w)$, and $\Delta_n^{-1} R_n^+(u, w)$ depend only on the pair (u, w). The smooth links $\Lambda(u, w)$ and $\Lambda^+(u, w)$ thus also depend only on the pair (u, w).

By construction, the smooth links $\Lambda(u, w)$ and $\Lambda^+(u, w)$ are smoothly isotopic. Indeed, note that we can find a reduced expression $\Delta_n = \beta(u^{-1} w_0) \beta(u^*)$, so that

$$\Delta_n^{-1} = \beta(u^*)^{-1} \beta(u^{-1} w_0)^{-1}$$

and it follows that, in the braid group Br_n, the braids $\Delta_n^{-1} R_n^+(u, w)$ and $R_n(u, w)$ are equivalent. $\square$

2.3. **Juggling Braid.** Let f be a k-bounded affine permutation. In this section, we construct a braid $J_k(f)$ on k-strands, called the *juggling braid* of f, and provide an explicit braid word for it.

Given a k-bounded affine permutation $f : \mathbb{Z} \longrightarrow \mathbb{Z}$, we picture it as follows. We consider the plane $\mathbb{R}^2$ with Cartesian coordinates (x, y). For each number $i \in \mathbb{Z}$, we join i to $f(i)$ using the upper-circumference arc $A_{f(i)}(f)$[4]:

$$A_{f(i)}(f) = \left\{ (x, y) \in \mathbb{R}^2 : \left(x - \frac{i + f(i)}{2} \right)^2 + y^2 = \frac{(f(i) - i)^2}{4} \right\} \cap \{y \geq 0\} \subseteq \mathbb{R}^2.$$

If $f(i) = i$, then $A_i(f) = A_{f(i)}(f)$ is an arc of radius zero that we represent by a dot at $(i, 0)$. The union of the arcs A_i, $i \in \mathbb{Z}$ is referred to as the *affine juggling diagram* of f, cf. [42].

Example 2.10. *Let us choose* $k = 3, n = 7$ *and* $f = [3, 4, 9, 6, 7, 12, 8]$. *Then the affine juggling diagram has the following form.*
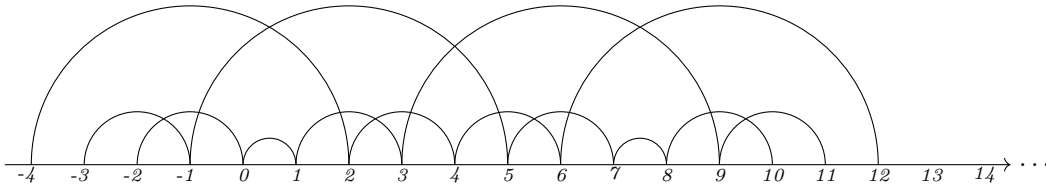
Since the function f is n-periodic, so is its affine juggling diagram. Thus, to recover the affine juggling diagram it is enough to consider the arcs $A_j, j = 1, \ldots, n$, whose *rightmost* points are at $(j, 0)$ with $j = 1, \ldots, n$. The union of all such arcs will be referred to as simply the *juggling diagram* of f. Moreover, we orient the arcs in a *counterclockwise direction*.

Example 2.11. *The juggling diagram of the function* f *from Example 2.10 is*
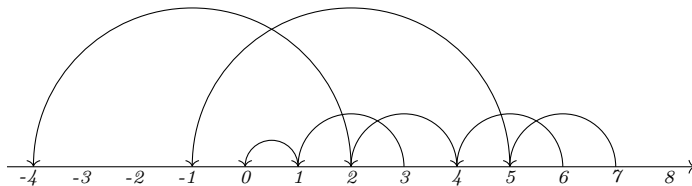
By virtue of f being a k-bounded affine permutation, there exist exactly k values $1 \leq i_1 < \ldots < i_k \leq n$ such that $n < f(i_s) \leq 2n$. Equivalently, $0 < f(i_s) - n \leq n$, and $f^{-1}(f(i_s) - n) = i_s - n \leq 0$. Thus, we obtain k arcs $A_{f(i_1)-n}, \ldots, A_{f(i_k)-n}$ in the juggling diagram of f whose leftmost point is at $(i, 0)$ with $i \leq 0$. We call these upper-circumferences *special*. The juggling braid is defined via a tangle diagram obtained from the juggling diagram, as follows:

Definition 2.12 (Juggling Braid)**.** Let $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ be a k-bounded affine permutation of size n, and consider the juggling diagram for f as defined above, which is the union of all the arcs of non-zero radius among $A_1, \ldots, A_n$ oriented in the counterclockwise direction. By definition, the juggling braid $J_k(f)$ is the braid defined by this tangle, declaring all the crossings between these arcs to be positive and smoothing the intersections with the x-axis according to the local models depicted in Figure 6.
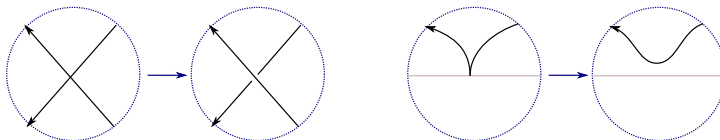

FIGURE 6. Local models constructing the juggling braid from the juggling diagram.

---

[4]For convenience, cf. Definition 2.12, we choose to label the upper-circumference arc $A_{f(i)}(f)$ by its rightmost point.

Remark 2.13. Let us comment on the number of strands of $J_k(f)$, as well as their labeling. Note that the elements $i_1, \ldots, i_k$ satisfying $f(i_k) > n$ are precisely those points which are *not* the leftmost point of an arc in the juggling diagram of f. We thus have k strands and with initial points $i_1, \ldots, i_k$. We label the strands so that the j-th strand is precisely the strand whose initial point is i_{k-j+1}.

To see that $J_k(f)$ has precisely k strands we can alternatively think of a juggler with k balls and think of the arcs in $J_k(f)$ as being the trajectories for these balls while being juggled. Here x denotes the time coordinate and y denotes the height of a ball. This implies the following simple fact:

Proposition 2.14. *Each vertical line with non-integer coordinate intersects $J_k(f)$ in at most k arcs, which correspond to distinct strands of the juggling braid.*   □

Example 2.15. *The following will be our running example. Consider the 4-bounded affine permutation of rank 6, $f = [4, 6, 7, 9, 11, 8]$. The juggling diagram of f is as follows:*
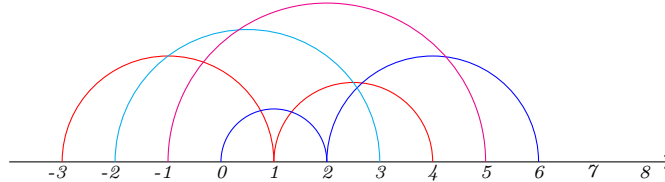
And thus the juggling braid is as follows:
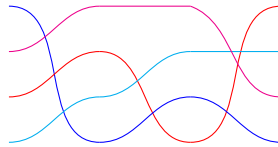
Remark 2.16. One can use parabolas, which correspond to actual juggling trajectories, or other curves instead of circles in the definition of $J_k(f)$. As long as these curves are convex and each pair of curves intersects at most once, the resulting braids are all related by Reidemeister III moves.   □

Thanks to the previous remark, we may assume that all the crossings between special arcs come *after* all other crossings. This means that we have a decomposition

$$J_k(f) = J_k^{(2)}(f) J_k^{(1)}(f)$$

where $J_k^{(2)}(f)$ records the crossings between special arcs, and $J_k^{(1)}(f)$ records all other crossings. Note that by definition, $J_k^{(2)}(f)$ is a reduced k-stranded braid. In our running Example 2.15 we have:

$$J^{(1)}(f) \qquad J^{(2)}(f)$$
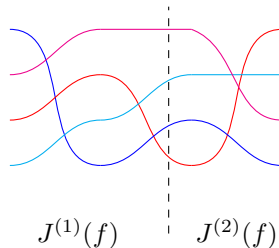
Finally, we define the link associated to the juggling diagram.

Definition 2.17. Let f be a k-bounded affine permutation. The juggling link $\Lambda(f) \subseteq \mathbb{R}^3$ is the smooth 0-framed closure of $\Delta_k^{-1} J_k(f)$.

2.3.1. *The braid $J_k^{(1)}(f)$.* In this section, we express the braid $J_k^{(1)}(f)$ as a product of $(n-k)$ interval braids, some of which may be empty, that naturally correspond to the columns of the Le diagram ⅃ corresponding to f. As defined above, a crossing in $J_k(f)$ belongs to the braid $J_k^{(1)}(f)$ if and only if at least one of the arcs involved in the crossing is not special. Here is the construction.

Definition 2.18. Suppose that $s < t < u < f(s) < f(t) < f(u)$, so that the arcs $A_{f(s)}, A_{f(t)}$ and $A_{f(u)}$ pairwise intersect. We say that $A_{f(s)}, A_{f(t)}$ and $A_{f(u)}$ are in *good position* if $A_{f(s)} \cap A_{f(u)}$ is

inside $A_{f(t)}$. Equivalently, $A_{f(s)} \cap A_{f(t)}$ is outside $A_{f(u)}$ and $A_{f(t)} \cap A_{f(u)}$ is outside $A_{f(s)}$ (see Figure 7). Otherwise we say that these three arcs form an *inversion triple*. We say that a juggling diagram is in *good position* if all triples of arcs as above are in good position. $\qquad\square$
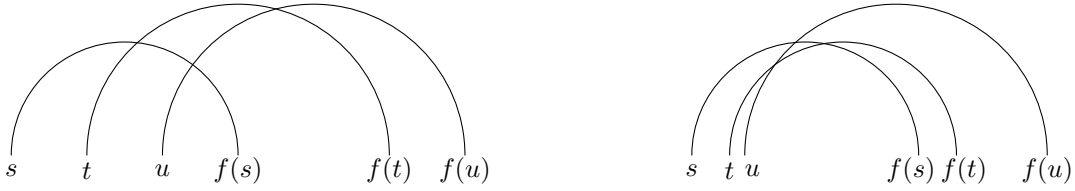

FIGURE 7. Good position (left) and inversion triple (right).

Lemma 2.19. *Up to braid moves, we can draw a juggling diagram for $J_k^{(1)}$ in good position.*

Proof. First, consider a reduced braid with strands labeled on the left. Following [16] (see also [50]), we denote by $(s|t)$ a crossing between the strands labeled by s and t. Three strands labeled by s, t, and u form an inversion triple if $s < t < u$ and the crossings between the respective strands appear from left to right in the order $(t|u), (s|u), (s|t)$ (called the *antilexicograhic order* in [16]). Now [16, Proposition 3.15, Corollary 3.16] state that any reduced braid can be transformed via braid moves to a unique braid word without inversion triples. Furthermore, this can be achieved by braid moves which always reduce the number of inversion triples.

Our braid $J_k^{(1)}$ is not reduced, but any two arcs intersect at most once. Therefore any collection of arcs where each arc belongs to a different strand of $J_k^{(1)}$ is reduced, and we can repeatedly apply the above result and Proposition 2.14 as we scan the braid from right to left. This would ensure that the number of inversion triples to the right of a given vertical line can be decreased to zero, and eventually we will eliminate all inversion triples. $\qquad\square$

To describe the braid $J_k^{(1)}$, we use the Le diagram $\beth$ corresponding to f and the notations $i_1, \ldots, i_k, j_1, \ldots, j_{n-k}$ as above. In particular, $f(i_s) > n$ and $A_{f(i_s)-n}$ are special arcs, while $f(j_t) \leq n$ and $A_{f(j_t)}$ are non-special arcs. Moreover, $A_{i_1}, \ldots, A_{i_k}$ are the initial arcs of the k strands of $J_k^{(1)}$.

Lemma 2.20. *For $t > s \geq 0$, we have $f(j_t) \in \{i_1, \ldots, i_k, j_{n-k}, \ldots, j_{s+1}\}$.*

Proof. Note that $w = [i_1, \ldots, i_k, j_1, \ldots, j_{n-k}]$ is a permutation of $[1, \ldots, n]$, hence
$$\{1, \ldots, n\} = \{j_1, \ldots, j_s\} \sqcup \{i_1, \ldots, i_k, j_{n-k}, \ldots, j_{s+1}\}.$$
For $t > s$, we have $f(j_t) \geq j_t > j_s$, so $f(j_t) \notin \{j_1, \ldots, j_s\}$. $\qquad\square$

We define the sets
$$(2.8) \qquad T_{n-k-s} := \{i_1, \ldots, i_k, j_{n-k}, \ldots, j_{s+1}\} \setminus \{f(j_{n-k}), \ldots, f(j_{s+1})\}, \ \ s = 0, \ldots, n-k.$$
By Lemma 2.20 these have exactly k elements for each s. Note that if j_s is a fixed point of f, then $j_s \notin T_i$ for all i. Also note that
$$T_0 = \{i_1, \ldots, i_k\},$$
$$T_{n-k} = \{i_1, \ldots, i_k, j_{n-k}, \ldots, j_1\} \setminus \{f(j_{n-k}), \ldots, f(j_1)\} = \{f(i_1) - n, \ldots, f(i_k) - n\},$$
and
$$T_{n-k-s+1} = (T_{n-k-s} \cup \{j_s\}) \setminus \{f(j_s)\}$$
so that, if j_s is a fixed point of f, we get $T_{n-k-s+1} = T_{n-k-s}$.

Lemma 2.21. *In the notation above, the following hold:*

(a) *Assume that $J_k^{(1)}$ is in good position. Then it can be written as a product of braids*
$$J_k^{(1)} = I_1 \cdots I_{n-k},$$
where I_s is given by the crossings of the arc $A_{f(j_s)}$ with arcs $A_{f(j_t)}$, $t < s$, as well as all the crossings of $A_{f(j_s)}$ with the special arcs.

(b) If $f(j_s) = j_s$ then the braid I_s is trivial.

(c) If $f(j_s) > j_s$, label the strands by the set T_{n-k-s} on the left, and by $T_{n-k-s+1}$ on the right, in decreasing order from top to bottom. Then the braid I_s is the interval braid which connects $f(j_s)$ on the left to j_s on the right, and all other elements of $T_{n-k-s+1}$ to themselves.

Proof. For Part (a), the definition of good position implies that all crossings between A_{j_t} and $A_{j_{t'}}$ with $t, t' > s$ (resp. $t, t' < s$) are located to the right (resp. left) of A_{j_s}. Furthermore, if $t < s < t'$ then the crossing $A_{j_t} \cap A_{j_s}$ is to the left of the crossing $A_{j_s} \cap A_{j_{t'}}$, so we can indeed sort the crossings as desired. Part (b) is immediate since $f(j_s) = j_s$ is a dot in the juggling diagram which we ignore in the juggling braid.

For Part (c) first note that, since $i_t - n < 0 < j_s \leq f(j_s)$, the arcs $A_{f(j_s)}$ and $A_{f(i_t)-n}$ cross if and only if $j_s < f(i_t) - n < f(j_s)$. It follows that the braid I_s records the crossings of $A_{f(j_s)}$ only with arcs whose right endpoint is to the left of $f(j_s)$, and thus it must be an interval braid. Let $\widetilde{I}_s$ denote the interval braid connecting $T_{n-k-s+1}$ and T_{n-k-s} as above, we need to verify that $I_s = \widetilde{I}_s$. Note that the crossings in $\widetilde{I}_s$ are labeled by $x \in T_{n-k-s+1}$ such that $j_s < x < f(j_s)$. If $x = f(i_t) - n$ for some t, then we have $i_t - n \leq 0 < j_s < f(i_t - n) < f(j_s)$ so the arcs $A_{f(j_s)}$ and $A_{f(i_t)-n}$ intersect once. Now assume that x is *not* of the form $f(i_t) - n$, so it must be of the form $f(j_t)$ for some t. Note that necessarily $t < s$, for otherwise $f(j_t) \notin T_{n-k-s}$. So we have $j_t < j_s < f(j_t) < f(j_s)$ and the arcs $A_{f(j_s)}$ and $A_{f(j_t)}$ intersect once. It follows that both I_s and $\widetilde{I}_s$ involve crossings between the same strands, and since both are interval braids the result follows. $\square$

Example 2.22. *Let us take the bounded affine permutation of Example 2.15. We have $f = [4, 6, 7, 9, 11, 8]$ and, as we have seen, the juggling diagram of f is:*
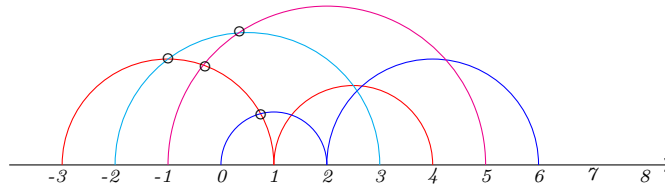
The crossings marked with $\circ$ represent crossings between two special arcs, and thus they will not be represented in the braid $J_k^{(1)}(f)$. Note that $i_1 = 3, i_2 = 4, i_3 = 5, i_4 = 6$, while $j_1 = 1$ and $j_2 = 2$. Thus, we start with the set $T_0 = \{3, 4, 5, 6\}$. Note that $T_1 = (T_0 \cup \{2\}) \setminus \{f(2)\} = \{2, 3, 4, 5\}$, and we find the interval I_2 as follows:
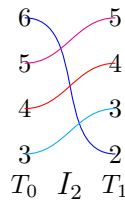
Now, $T_2 = (T_1 \cup \{1\}) \setminus \{f(1)\} = (T_0 \cup \{2, 1\}) \setminus \{f(2), f(1)\} = \{1, 2, 3, 5\}$, and we find the interval I_1 that is concatenated to I_2 on the right:
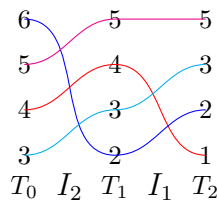
Comparing to Example 2.15, this braid coincides with $J_k^{(1)}(f)$. $\square$

In order to find an explicit description of the interval I_s, we need to find the relative positions of $f(j_s)$ in T_{n-k-s} and of j_s in $T_{n-k-s+1}$. This is achieved in the following lemma.

Lemma 2.23. *Assume $f(j_s) > j_s$. Then:*

    (a) *The set $\{x < f(j_s) \mid x \notin T_{n-k-s}\}$ has precisely $\mathrm{inv}(s) + s$ elements.*

    (b) *The set $\{x < f(j_s) \mid x \in T_{n-k-s}\}$ has precisely $f(j_s) - \mathrm{inv}(s) - s - 1$ elements.*

    (c) *The set $\{x < j_s \mid x \in T_{n-k-s+1}\}$ has precisely $j_s - s = k - \lambda_s^t$ elements.*

Proof. For Part (a), Lemma 2.20 implies

$$\{1, \ldots, n\} \setminus T_{n-k-s} = \{j_1, \ldots, j_s\} \sqcup \{f(j_{s+1}), \ldots, f(j_{n-k})\}.$$

Assume that $x \notin T_{n-k-s}$ and $x < f(j_s)$. We have the following two cases:

    1) $x \in \{j_1, \ldots, j_s\}$, then $x \leq j_s < f(j_s)$ and $x \notin T_{n-k-s}$. This accounts for s elements.

    2) $x = f(j_t)$ for $t > s$ and $f(j_t) < f(j_s)$. This accounts for $\mathrm{inv}(s)$ elements.

Part (b) follows from Part (a). For Part (c), among $j_s - 1$ elements $\{1, \ldots, j_s - 1\}$ there are $s - 1$ elements $j_1, \ldots, j_{s-1}$ and $(j_s - s)$ elements $i_t < j_s$. None of them belongs to $\{f(j_s), \ldots, f(j_{n-k})\}$. The former are all not in $T_{n-k-s+1}$ but the latter all are. The last equation follows from Lemma 2.5. $\square$

Corollary 2.24. *The interval braid I_s is given by*

$$I_s = s_{\lambda_s^t - 1} \cdots s_{k - f(j_s) + \mathrm{inv}(s) + s + 1}.$$

and

$$J_k^{(1)} = I_1 \cdots I_{n-k}.$$

Proof. This follows from Lemma 2.21 and Lemma 2.23. $\square$

2.3.2. *The braid $J^{(2)}(f)$.* The braid $J^{(2)}(f)$ takes care of all the crossings between special arcs and it is reduced. Recall that these arcs are $A_{f(i_1)-n}, \ldots, A_{f(i_k)-n}$, where the special arc $A_{f(i_\ell)-n}$ joins $i_\ell - n$ to $f(i_\ell - n) = f(i_\ell) - n$. If $\ell < s$, then we have $i_\ell - n < i_s - n \leq 0 < f(i_\ell - n)$, so the special arcs $A_{f(i_\ell)-n}$ and $A_{f(i_s)-n}$ will cross precisely when $f(i_\ell - n) < f(i_s - n)$, that is, when $f(i_\ell) < f(i_s)$. This implies the following result:

Proposition 2.25. *Let f be a k-bounded affine permutation. Then the braid $J^{(2)}(f)$ is a positive reduced lift of the inverse of the permutation that sorts $(f(i_1), \ldots, f(i_k))$ in decreasing order.* $\square$

In Example 2.15 we have that $i_1 = 3, i_2 = 4, i_3 = 5, i_4 = 6$, while $f(i_1) = 7, f(i_2) = 9, f(i_3) = 11, f(i_4) = 8$ so that $J^{(2)}(f)$ is
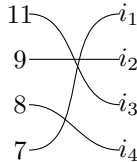
This concludes our presentation of the braid $J_k(f)$ and its properties. Let us now compare the links we obtain from $J_k(f)$ to those obtained from Richardson braids.

2.4. **Generalized destabilizations and comparison of $\Lambda(u, w)$ and $\Lambda(f)$.** The main result of this subsection is to establish that, possibly up to unlinked unknots, the smooth links $\Lambda(u, w)$ and $\Lambda(f)$ are smoothly isotopic.

Theorem 2.26. *Let (u, w) be a positroid pair and $f = ut_kw^{-1}$ its associated bounded affine permutation. Then the smooth link $\Lambda^+(u, w) \subseteq \mathbb{R}^3$ is smoothly isotopic to the smooth link $\Lambda(f) \subseteq \mathbb{R}^3$, up to a possibly empty collection of unlinked unknots.*

Theorem 2.26 is proven by appropriately using the following lemma iteratively.

Lemma 2.27. *Let $\beta_1 \in \mathrm{Br}_n$ be a positive braid on n strands in the generators $\sigma_1, \ldots, \sigma_{n-1}$ and $\beta_2 \in \mathrm{Br}_{n+1}$ the lift of a permutation $w \in S_{n+1}$ on $(n+1)$ strands. Set $j := w^{-1}(n+1)$, let $\tilde{\beta}_1 \in \mathrm{Br}_{n+1}$ be the braid obtained from β_1 by shifting the indices of the Artin generators up by 1, and let $\beta_2' \in \mathrm{Br}_n$ be the braid obtained from β_2 by removing the strand that ends at the bottom on the right of β_2.*
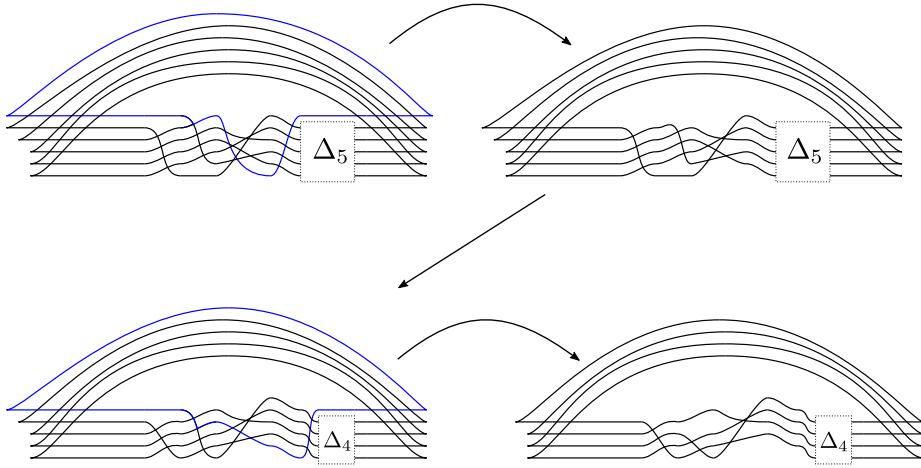
FIGURE 8. The sequence of moves starting with $R(u, w)$, on the upper left corner, and finishing with $J_k(f)$ on the lower right corner. We have marked in blue the occasions where we use Lemma 2.27, and we have chosen to draw the link diagrams as fronts, as in Lemma 3.7, for easier reference. (So all crossings in these link diagrams are over-crossings.)

(1) *Consider the braid words* $\eta_1 := \Delta_{n+1}\beta_2(\sigma_\ell \cdot \ldots \cdot \sigma_{j+1}\sigma_j \cdot \ldots \cdot \sigma_2\sigma_1)\tilde{\beta}_1 \in \mathrm{Br}_{n+1}$, *an* $(n+1)$-*stranded positive braid, and* $\eta_2 := \Delta_n\beta_2'(\sigma_{\ell-1}\cdot \ldots \cdot\sigma_{j+1})\beta_1 \in \mathrm{Br}_n$, *an n-stranded positive braid.*

   Then the 0-framed smooth closure of η_1 is smoothly isotopic to that of η_2.

(2) *Consider the braid words* $\eta_1 := \Delta_{n+1}\beta_2(\sigma_j\sigma_{j-1} \cdot \ldots \cdot \sigma_1)\tilde{\beta}_1 \in \mathrm{Br}_{n+1}$, *an* $(n+1)$-*stranded positive braid, and* $\eta_2 := \Delta_n\beta_2'\beta_1 \in \mathrm{Br}_n$, *an n-stranded positive braid.*

   Then the 0-framed smooth closure of η_1 is smoothly isotopic to the unlinked union of the 0-framed smooth closure of η_2 and an unknot.

   $\square$

For now, Lemma 2.27 will be left unproven and we just directly use it. In the next section we will independently prove Lemma 3.7, which is a stronger version that implies Lemma 2.27.

Example 2.28. *Before proving Theorem 2.26, we verify it in Example 2.15 as follows. Recall that* $f = [4, 6, 7, 9, 11, 8]$, *so that* $w = [3, 4, 5, 6, 1, 2]$ *and* $u = [1, 3, 5, 2, 4, 6]$. *From Example 2.3 we obtain:*

$$\beta(w) = (\sigma_2\sigma_3\sigma_4\sigma_5)(\sigma_1\sigma_2\sigma_3\sigma_4) \Rightarrow \beta(w^*) = (\sigma_4\sigma_3\sigma_2\sigma_1)(\sigma_5\sigma_4\sigma_3\sigma_2),$$

and $u^{-1}w_0 = [1, 4, 2, 5, 3, 6]w_0 = [6, 3, 5, 2, 4, 1]$. *Figure 8 then shows the sequence of moves, using Lemma 2.27, that starts with the Richardson link and ends with the Juggling link.* $\square$

Now, Subsection 2.2 defines the Richardson link $\Lambda(u, w)$ to be smooth 0-framed closure of the braid $\Delta_n^{-1}\beta(u^{-1}w_0)\beta(w^*)$, where (u, w) is a positroid pair. It will be more convenient to write this braid word as $\Delta_n^{-2}\Delta_n\beta(u^{-1}w_0)\beta(w^*)$ and focus on the piece $\Delta_n\beta(u^{-1}w_0)\beta(w^*)$, where all our changes will take place, see e.g. Figure 8. Let $f = f_{u,w} = ut_kw^{-1}$ be the associated bounded affine permutation, as in Subsection 2.1.3. By the identity (2.5),

$$u = [f(i_1) - n, \ldots, f(i_k) - n, f(j_1), \ldots, f(j_{n-k})]$$

and therefore

$$uw_0 = [f(j_{n-k}), \ldots, f(j_1), f(i_k) - n, \ldots, f(i_1) - n].$$

Now note that

(2.9) $$u^{-1}w_0 = (w_0(uw_0)w_0)^{-1}.$$

In particular, up to conjugation by the longest element w_0, the inverse of the restriction of the permutation $u^{-1}w_0$ to $\{n-k+1, \ldots, n\}$ is the permutation $J^{(2)}(f)$ which sorts $f(i_1), \ldots, f(i_k)$ in decreasing order, see Proposition 2.25.

The iterative applications of Lemma 2.27 that will lead to a destabilization procedure from $R_n(u, w)$ to $J_k(f)$ are indexed by the columns of the corresponding Le diagram. The two basic steps we need are as follows:

(1) First, suppose that the Le diagram for (u, w) has dots in the rightmost column, with the lowest dot in row d. Then we schematically draw the braid as follows:
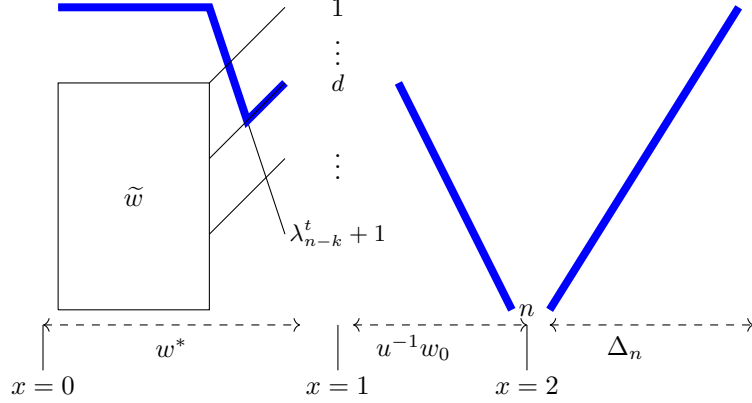

Write $w = (s_{n-\lambda^t_{n-k}} \cdots s_{n-1})w'$, so that $w^* = (s_{\lambda^t_{k-t}} \cdots s_1)\widetilde{w}$, and note that, by (2.9), $u^{-1}w_0$ is a permutation braid that connects n on the right with $w_0u(n) = w_0(f(j_{n-k})) = w_0(n-d+1) = d$ on the left, where the equation $f(j_{n-k}) = n-d+1$ follows from Lemma 2.5. Then, the thick blue strand in the figure above can be removed by Lemma 2.27, leaving a braid which can be described as follows. On the side of $w_0ww_0 = w^*$, we get

$$\left(s_{\lambda^t_{n-k}} \cdots s_d\right) \cdot \widetilde{w} = \widetilde{w} \cdot \left(s_{\lambda^t_{n-k}+n-k-1} \cdots s_{d+n-k-1}\right).$$

On the side of $u^{-1}w_0$, we just remove the strand connecting d to n and leave the rest unchanged. On the side of Δ_n, we remove the strand connecting n to 1, and get Δ_{n-1}.

(2) Second, suppose that the last column of the Le diagram is empty. Then by Lemma 2.5 $u(n) = f(j_{n-k}) = n - \lambda^t_{n-k}$, so by (2.9) we get $u^{-1}w_0(\lambda^t_{n-k}+1) = n$, and thus the diagram has the following form:
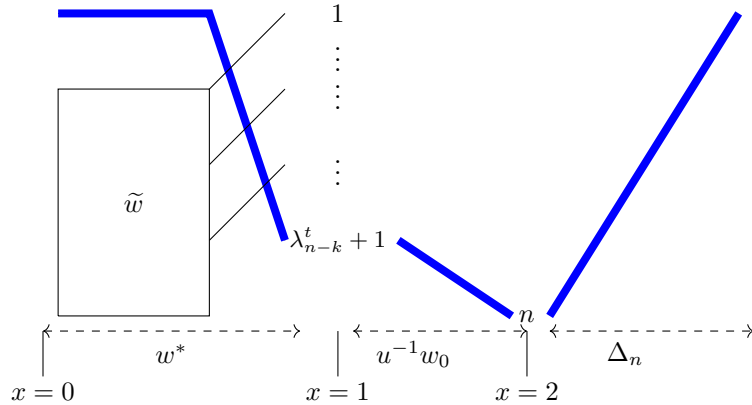
In this case the thick line closes up to an unknot. By Lemma 2.27 this unknot is unlinked from the rest of the link diagram. After moving it away, we are left with $\widetilde{w}$ instead of w.

We refer to either of the two operations above as a (generalized) destabilization. Let us now prove Theorem 2.26 by appropriately applying it $(n-k)$ times, as follows.

Lemma 2.29. *At the m-th step, $1 \le m \le n-k+1$, of the above destabilization procedure we get an $(n+1-m)$-strand braid that can be decomposed into the following three parts:*

(a) *The first part $\beta_m^{(1)}$ is the product of interval braids*

$$\prod_{h=m}^{n-k} \left(s_{\lambda_{n-k+1-h}^t + h-1} \cdots s_h \right) \times \prod_{\ell=0}^{m-2} \left(s_{\lambda_{n-k-m-2+\ell}^t + n-k-1} \cdots s_{2n-k-m-\ell+3+\mathrm{inv}(n-k-m+2-\ell)-f(j_{n-k-m+2-\ell})} \right),$$

where for $m = 1$ the second product is assumed to equal 1 and for $m = n - k + 1$ the first product is assumed to equal 1.

(b) *The second part $\beta_m^{(2)}$ is a permutation braid obtained as a part of the wiring diagram for $u^{-1}w_0$ connecting $1, \ldots, n - m + 1$ on the right with $u^{-1}(n), \ldots, u^{-1}(m)$ on the left.*

(c) *The third part is Δ_{n-m+1}.*

Proof. We prove this by induction in m. For notational convenience, we refer to the left end of $\beta_m^{(1)}$ as $x = 0$, to the right end of $\beta_m^{(1)}$ which coincides the left end of $\beta_m^{(2)}$ as $x = 1$ and to the right end of $\beta_m^{(2)}$ as $x = 2$ (see figures above). For $m = 1$ we get

$$(2.10) \qquad \beta_1^{(1)} = w^* = \left(s_{\lambda_{n-k}^t} \cdots s_1 \right) \cdots \left(s_{\lambda_1^t + n-k-1} \cdots s_{n-k} \right).$$

and $\beta_1^{(2)} = u^{-1}w_0$.

Assume that the statement holds for all $1 \le j \le m$. To prove the step of induction, we mark in blue the bottom-most strand at $x = 2$. It is labeled by $n - m + 1$ at $x = 2$ and by $(u^{-1}w_0)^{-1}(n - m + 1) = w_0 u(n - m + 1) = w_0(f(j_{n-k-m+1})) = n + 1 - f(j_{n-k-m+1})$ at $x = 1$. For brevity, denote $m' := n - k - m + 1$.

By assumption, the rightmost interval braid in $\beta_m^{(1)}$ is

$$(2.11) \qquad s_{m-1+\lambda_{n-k-m+1}^t} \cdots s_m = s_{m-1+\lambda_{m'}^t} \cdots s_m.$$

We need to verify that we can apply Lemma 2.27. Let us count the number of strands *above* the blue strand $n+1-f(j_{m'})$ we have removed at $x = 1$. Note that such strands correspond to $1 \le \ell < m$ such that $n+1-f(j_{\ell'}) < n+1-f(j_{m'})$ where $\ell' := n-k-\ell+1$, that is, they correspond to elements $n - k \ge \ell' > m'$ such that $f(j_{\ell'}) > f(j_{m'})$. Thanks to (2.7), there are precisely $n - k - m' - \mathrm{inv}(m')$ such strands. We conclude that the blue strand occupies the position

$$n + 1 - f(j_{m'}) - (n - k - m' - \mathrm{inv}(m')) = k + 1 + m' + \mathrm{inv}(m') - f(j_{m'})$$

and, in order to verify we can apply Lemma 2.27, we need to verify that

$$k + m' + \mathrm{inv}(m') - f(j_{m'}) \le \lambda_{m'}^t = k + m' - j_{m'} \Leftrightarrow j_{m'} \le f(j_{m'}) - \mathrm{inv}(m')$$

which holds thanks to Lemma 2.6. So we can indeed use Lemma 2.27 in order to destabilize. We have cases:

- If $j_{m'} = f(j_{m'}) - \mathrm{inv}(m')$, then we remove a disjoint blue unknot and erase the interval braid completely.
- If $j_{m'} = f(j_{m'}) - \mathrm{inv}(m') - 1$, then we also erase the interval braid completely.
- Else, we delete the part $(s_{m+k+m'+\mathrm{inv}(m')-f(j_{m'})} \cdots s_m)$ intersecting the blue strands out of the interval braid (2.11) and are left with

$$s_{m-1+\lambda_{m'}^t} \cdots s_{m+k+m'+\mathrm{inv}(m')-f(j_{m'})+1} = s_{m-1+\lambda_{m'}^t} \cdots s_{n+\mathrm{inv}(m')-f(j_{m'})+2}.$$

Now we push this interval through the $m' - 1$ remaining interval braids and get

$$(2.12) \qquad s_{m-1+\lambda_{m'}^t + m'-1} \cdots s_{n+\mathrm{inv}(m')-f(j_{m'})+2+m'-1} = s_{n-k-1+\lambda_{m'}^t} \cdots s_{n+\mathrm{inv}(m')-f(j_{m'})+m'+1}.$$

This agrees with the interval braid in $\beta_{m+1}^{(1)}$.

$\square$

Proof of Theorem 2.26. We apply $n - k$ destabilization steps in Lemma 2.29 and obtain a k-stranded braid at $m = n - k + 1$, which is described by the following three pieces. First,

$$\beta_m^{(1)} = \left(s_{\lambda_1^t + n-k-1} \cdots s_{2n-f(j_1)+\mathrm{inv}(1)-n+2} \right) \cdots \left(s_{\lambda_{n-k}^t + n-k-1} \cdots s_{2n-k-f(j_{n-k})+\mathrm{inv}(n-k)+1} \right).$$

Since we did not relabel the strands yet, we need to shift all the subindices of the Coxeter generators down by $n - k$ and obtain

$$\left(s_{\lambda_1^t - 1} \cdots s_{k - f(j_1) + \mathrm{inv}(1) + 1 + 1}\right) \cdots \left(s_{\lambda_{n-k}^t - 1} \cdots s_{k - f(j_{n-k}) + \mathrm{inv}(n-k) + n - k + 1}\right).$$

The second part $\beta_m^{(2)}$ is a permutation braid obtained as a part of the wiring diagram for $u^{-1} w_0$ connecting $1, \ldots, k$ on the right with $2n - f(i_1) + 1, \ldots, 2n - f(i_k) + 1$ on the left. Finally, the third part is Δ_k. By Corollary 2.24 and Proposition 2.25, this implies that we have obtained $J_k(f)$. $\square$

2.5. **Le Braid.** In this section, we define the Le braid $D_k(\mathrm{J})$ associated to a Le diagram J and compare it to $J_k(f)$, where f is the bounded affine permutation associated to J. See Theorem 2.34 below. The Le braid will be a concatenation of two braids in k-strands: $D_k(\mathrm{J}) = D_k^{(2)}(\mathrm{J}) D_k^{(1)}(\mathrm{J})$, in such a way that $D_k^{(1)}(\mathrm{J})$ is positive and (typically) not reduced, while $D_k^{(2)}(\mathrm{J})$ is negative and reduced.

Let us start with the braid $D_k^{(2)}(\mathrm{J})$, since it is easier to define. In order to do this, recall the wiring diagram of J: there are exactly k wires starting on the left border of the Young diagram λ. We obtain a k-stranded braid by:

- Reading these wires in the northeast to southwest direction. Note that this direction is opposite to that in Section 2.1.
- Delcaring all crossings to be *negative*.

The obtained braid is by definition the braid $D^{(2)}(\mathrm{J})$. Clearly, it is reduced. Note that $D^{(2)}(\mathrm{J})$ is the inverse of the braid obtained by joining i_s to $f(i_s)$, $s = 1, \ldots, k$. See Figure 9 for an example.
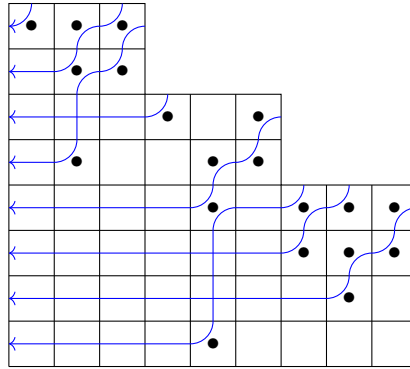

FIGURE 9. The braid $D_k^{(2)}(\mathrm{J})$. All of the crossings are negative.

Lemma 2.30. *Let J be a Le diagram and f its associated bounded affine permutation. Then, the k-stranded braids $D_k^{(2)}(\mathrm{J})$ and $\Delta_k^{-1} J^{(2)}(f)$ are braid equivalent.*

Proof. The statement is equivalent to showing that the positive braids $\Delta_k(f)$ and $J^{(2)}(f)(D^{(2)}(\mathrm{J}))^{-1}$ are braid equivalent. This is equivalent to showing that in the positive braid $J^{(2)}(f)(D^{(2)}(\mathrm{J}))^{-1}$ each pair of strings cross exactly once. For this, we picture this braid as follows:
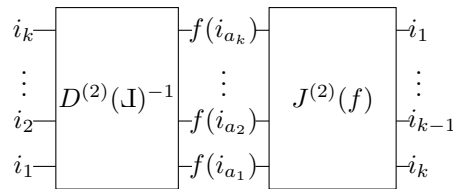
Here $f(i_{a_k}), \ldots, f(i_{a_1})$ in the middle column are ordered in decreasing order when reading top to bottom, the braid $D^{(2)}(\mathrm{J})^{-1}$ joins i_s to $f(i_s)$, and the braid $J^{(2)}(f)$ joins $f(i_s)$ to i_s.

Let $1 \leq s < t \leq k$. If $f(i_s) < f(i_t)$, the strands whose left labels are i_s and i_t will not meet in the $D^{(2)}(\mathrm{J})^{-1}$ part of the braid, but they will meet exactly once in the $J^{(2)}(f)$ part of it. If, on the

contrary, $f(i_t) < f(i_s)$, then the same strands will meet exactly once in the $D^{(2)}(\lrcorner)^{-1}$ part of the braid, and they will not meet in the $J^{(2)}(f)$ part of it. The result follows. $\square$

Let us now define $D^{(1)}(\lrcorner)$. We need an auxiliary diagram, close in spirit to the wiring diagram, that essentially keeps track of all the crossings between non-special circumferences in the juggling diagram of the associated bounded affine permutation.

Definition 2.31. The *bounded wiring diagram* of a Le diagram $\lrcorner$ is defined as follows. For each non-empty column of $\lrcorner$, draw a wire from the top of the column to the lowest dot on it, always passing through the left of every dot of the column. Right after the lowest dot on the column take a right U-turn (going under that lowest dot), and then proceed with the usual wiring rules. This wire connects the top of the starting column with either the rightmost edge of a row, or the top of a different column. If the latter situation occurs, then further connect the wire with the wire starting at the top of this second column. $\square$

See Figure 10 for an instance of such a bounded wiring diagram from Definition 2.31.
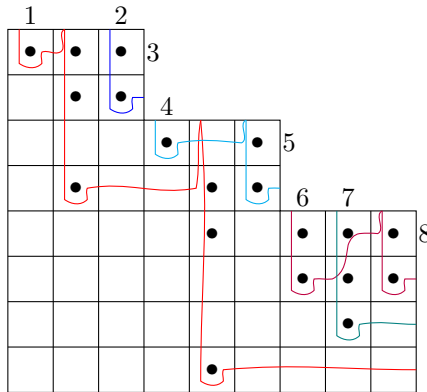

FIGURE 10. The labeled diagram $\widetilde{\mathcal{W}(\lrcorner)}$.

Definition 2.31 creates a (typically) *non-reduced* braid with k' strands, $k' \leq k$. Indeed, note that we have an injective map from the set of wires to the rows of $\lrcorner$. In fact, it follows from identities (2.4) and (2.5) that these wires correspond to chains of non-special circumferences in the juggling braid of the associated bounded affine permutation f. Moreover, the crossings in the bounded wiring diagram correspond precisely to the crossings between these non-special circumferences.

Let us enhance the bounded wiring diagram by enumerating both the right ends of the rows which do not have a wire ending on them and the top ends of those columns where a wire starts, as follows. Reading the steps of the diagram from northwest to southeast (both vertical *and* horizontal steps), we enumerate the following two types of steps in the order in which these steps are found:

-   A horizontal step at the top of a column which is the initial step of a wire. (In particular, there is no wire starting in a column to its left that goes up to the top of that column.)

-   A vertical step at the right of a row which is not the endpoint of a wire.

In this manner, we have labeled exactly k steps. Let us denote by $\mathcal{W}(\lrcorner)$ the bounded wiring diagram of $\lrcorner$ and by $\widetilde{\mathcal{W}(\lrcorner)}$ its labeled version. We are ready to define the braid $D^{(1)}(\lrcorner)$.

Definition 2.32. The k-stranded braid $D_k^{(1)}(\lrcorner)$ of a $\lrcorner$ diagram with at most k rows is defined inductively on the columns of $\lrcorner$ as follows.

(1) If $\lrcorner$ is the empty Young diagram, $D_k^{(1)}(\lrcorner)$ is the trivial braid on k strands.

(2) Assume $D_k^{(1)}(\lrcorner)$ has been defined and let $\lrcorner'$ be a Le diagram obtained from $\lrcorner$ by attaching a column of height h to the left of $\lrcorner$. Then:

(i) If this column has no dots, then we define $D_k^{(1)}(\text{J}') := D_k^{(1)}(\text{J})$.

(ii) Else, we first draw the bounded wiring diagram $\mathcal{W}(\text{J}')$ of J'. The wire associated to the first column of J', i.e. to the column that does not belong to J, ends in a labeled step of $\widetilde{\mathcal{W}(\text{J})}$; say with label t. Then we define

$$D_k^{(1)}(\text{J}') := (\sigma_{h-1} \cdots \sigma_{k-t+1}) \cdot D_k^{(1)}(\text{J}).$$
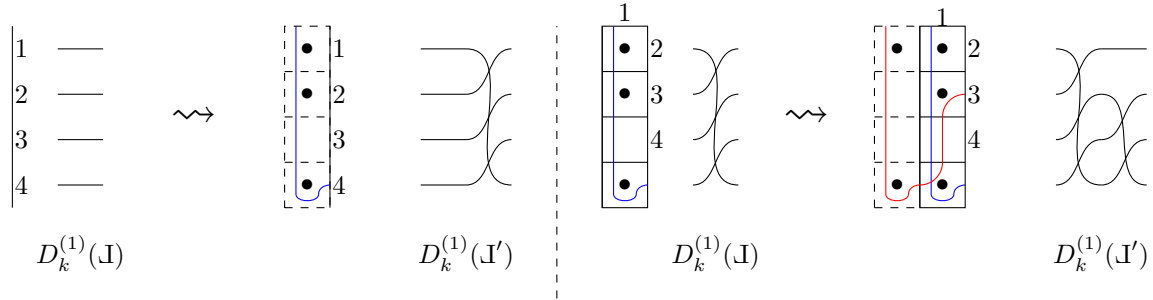
$\square$

See Figure 11 for an example.

FIGURE 11. Adding a column to the left of a Le diagram, and its effect on the braid $D_k^{(1)}(\text{J})$. Here $k = 4$. On the left-hand side of the figure, we add a column to a Le diagram associated to the empty partition. Since we are adding a column of height $h = 4$ and the endpoint of the new strand is $t = 4$, we have to multiply by the interval braid $\sigma_{h-1} \ldots \sigma_{k-t+1} = \sigma_3 \sigma_2 \sigma_1$. On the right-hand side of the figure, we add a column to a nonempty Le diagram. On this side, $h = 4$ and $t = 3$, so we have $D_k^{(1)}(\text{J}') = (\sigma_3 \sigma_2) D_k^{(1)}(\text{J})$.

The comparison to $J_k(f)$ needs the analogue of Lemma 2.30, which reads as follows:

Lemma 2.33. *Let J be a Le diagram and f its associated bounded affine permutation. Then the k-stranded braids $D_k^{(1)}(\text{J})$ and $J^{(1)}(f)$ are braid equivalent.*

Proof. Let us work by induction on the number of columns of J. If J has no columns, i.e. if it is associated to the empty Young diagram, then $D_k^{(1)}(\text{J})$ is the trivial braid. In that case, all the arcs in the juggling diagram for f are special, which implies that $J^{(1)}(f)$ is the trivial braid as well. Let us now assume the result for the Le diagram J, with associated bounded affine permutation f, and let J' be a Le diagram (with permutation f') obtained from J by adjoining an extra column to the left. We let n be the width of the Le diagram J, so that $n + 1$ is the width of J'.

If this added column is empty then $D_k^{(1)}(\text{J}) = D_k^{(1)}(\text{J}')$ by definition. Since $f'(1) = 1$ and $f'(i) = f(i-1) + 1$ for $i > 1$, the juggling braids for f and f' coincide, and thus $J^{(1)}(f) = J^{(1)}(f')$.

If this added column is non-empty, let us study first how the bounded affine permutation f' is obtained from the bounded affine permutation f. For this, let h be the height of the added column. Note that, if $i < k - h + 1$ then $f'(i) = n + 1 + i = f(i) + 1$. On the other hand, if $i > k - h + 1$ then $f'(i) = f(i-1) + 1$, this follows from (2.5), see e.g. Figure 5. Finally, the juggling diagram of f' is then obtained from that of f by inserting a new non-special arc, which is precisely the arc joining $f'(k - h + 1)$ to $k - h + 1$. Recall now that t is the label in $\widetilde{\mathcal{W}(\text{J})}$ of the strand starting in this new column of J'. Then, $k - t + 1$ is the position of the strand in $J^{(1)}(f)$ that we join to this new arc. The end position of this strand is $k - (k - h + 1) + 1 = h$. Thus, $J^{(1)}(f') = \sigma_{h-1} \cdots \sigma_{k-t+1} J^{(1)}(f)$ and the result follows. $\square$

Lemmas 2.30 and 2.33 thus imply the following result:

Theorem 2.34. *Let J be a Le diagram with corresponding bounded affine permutation f. Then, the braids $\Delta_k^2 D_k(\text{J})$ and $\Delta_k J_k(f)$ are braid equivalent.*

Proof. By Lemma 2.33, $\Delta_k^2 D_k(\mathbb{I}) = \Delta_k^2 D_k^{(2)}(\mathbb{I})D_k^{(1)}(\mathbb{I}) = \Delta_k^2 D_k^{(2)}(\mathbb{I})J_k^{(1)}(f)$. By Lemma 2.30, this also equals $\Delta_k^2 D_k^{(2)}(\mathbb{I})J_k^{(1)}(f) = \Delta_k^2 \Delta_k^{-1} J_k^{(2)}(f)J_k^{(1)}(f) = \Delta_k J_k(f)$. □

This concludes our discussion of braids directly associated to Le-diagrams. A Legendrian link associated to $D_k(\mathbb{I})$ will be discussed in Section 3.

2.6. **Matrix Braids.** Let us finally introduce cyclic rank matrices $r = (r_{ij})$, their associated matrix braids $M_k(r) \in \mathrm{Br}_k$, and conclude Theorem 1.1.(ii).

Definition 2.35. A cyclic rank matrix of type (k, n) is an array $r = (r_{ij})$ indexed by $(i, j) \in \mathbb{Z}^2$ satisfying the following conditions:

(i) $r_{ij} = 0$ if $j < i$ and $r_{ij} = k$ if $i + n - 1 \le j$.
(ii) $r_{ij} - r_{(i+1)j} \in \{0, 1\}$, $r_{ij} - r_{i(j-1)} \in \{0, 1\}$, and $r_{ij} = r_{(i+n)(j+n)}$, for all $i, j \in \mathbb{Z}$.
(iii) If $r_{(i+1)(j-1)} = r_{(i+1)j} = r_{i(j-1)}$ then $r_{ij} = r_{(i+1)(j-1)}$.

Note that we can restrict to the grid $j \in [1, n]$ due to the condition $r_{ij} = r_{(i+n)(j+n)}$, for all $i, j \in \mathbb{Z}$. □

Given a cyclic rank matrix $r = (r_{ij})$, and each $i \in \mathbb{Z}$, there is a unique index $f(i)$ such that

$$r_{i\ f(i)} = r_{(i+1)\ f(i)} = r_{i\ (f(i)-1)} = r_{(i+1)\ (f(i)-1)} + 1.$$

Then the map $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by $f(i) = j$ if and only if

$$r_{ij} = r_{(i+1)j} = r_{i(j-1)} = r_{(i+1)(j-1)} + 1,$$

defines a bounded affine permutation. In fact, this establishes a bijection between cyclic rank matrices and bounded affine permutations, as explained in [42, Section 3.3].

Remark 2.36. Note that $r_{i(j+1)} = r_{ij} + 1 = r_{(i+1)j} + 1 = r_{(i+1)(j+1)} + 1$ can only happen if $i = j + 1$ and $r_{ij} = r_{(i+1)j} = r_{(i+1)(j+1)} = 0$, see e.g. [42, Corollary 3.12].
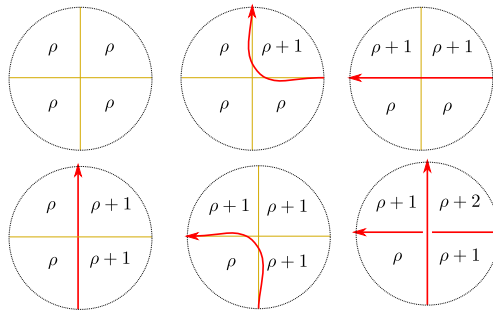

FIGURE 12. The local models for the matrix braid $M_k(r)$ associated to a cyclic rank matrix $r = (r_{ij})$, drawn near each four entries of the matrix. The value of a given entry r_{ij} is denoted by $r_{ij} = \rho$ and the braid is depicted in red strands. The yellow lines are used to separate the matrix entries of r. Note that, by Remark 2.36, the middle model of the top row can only happen if $\rho = 0$.

Definition 2.37. Let $r = (r_{ij})$ be a cyclic rank matrix of type (k, n), $(i, j) \in \mathbb{Z}^2$. By definition, the infinite matrix braid $M_k^\infty(r)$ is given by the tangle diagram obtained by drawing in $\mathbb{R}^2$ the six local tangles according to Figure 12. By definition, the matrix braid $M_k(r) \in \mathcal{B}_k$ is obtained from the infinite matrix braid $M_k^\infty(r) \in \mathcal{B}_k$ by restricting its diagram to the grid $j \in [1, n]$. We define the matrix link $\Lambda(r) \subseteq \mathbb{R}^3$ to be the smooth 0-framed closure of $M_k(r)$. □

Thanks to Remark 2.36, the following gives a procedure for drawing the infinite matrix braid $M_k^\infty(r)$ for a cyclic rank matrix associated to a bounded affine permutation f.

- For each i such that $i \ne f(i)$, connect $(i, f(i))$ to (i, i) using a horizontal line.
- For each i such that $i \ne f(i)$, connect (i, i) to $(f^{-1}(i), i)$ using a vertical line.

See Figure 13 for an example of a matrix braid $M_k(r)$. We remark that we are using matrix notation, so (i, j) is the ij-th entry of a matrix: the coordinate i increases down, and the coordinate j increases to the right.

Remark 2.38. In Definition 2.37, we orient the strands so that they point northwest. □
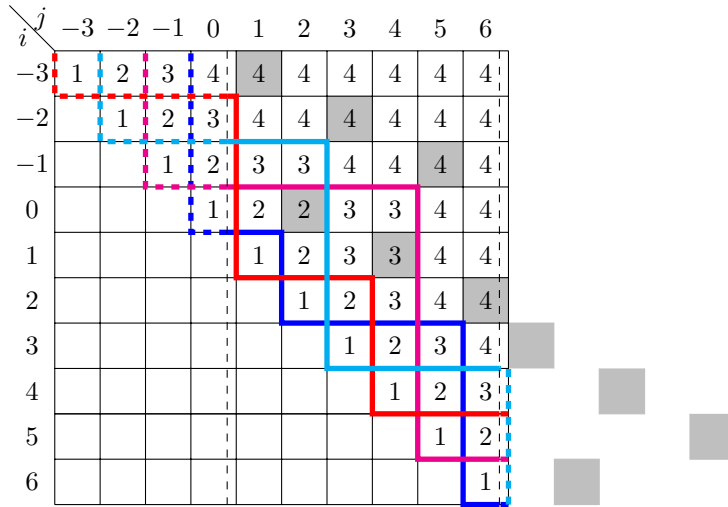

FIGURE 13. Cyclic rank matrix and matrix braid for the bounded affine permutation $f = [4, 6, 7, 9, 11, 8]$ of Example 2.15. The dashed lines indicate that we restrict to $1 \leq j \leq 6$, and we have marked in gray the boxes $(i, f(i))$.

The conditions in Definition 2.35 imply that the matrix braid $M_k(r) \in \mathcal{B}_k$ is a k-stranded tangle. These braids were introduced in [61, Section 3.2]. The following result compares the braids $M_k(r)$ to the juggling braids $J_k(f)$.

Theorem 2.39. *Let $r = (r_{ij})$ be a cyclic rank matrix of type (k, n), $(i, j) \in \mathbb{Z}^2$, and f its associated bounded affine permutation. The matrix braid $M_k(r) \in \mathcal{B}_k$ is equivalent to the braid $J_k(f)\Delta_k \in \mathcal{B}_k$.*

Proof. Let us first restrict to the case $i \in \{f^{-1}(1), \ldots, f^{-1}(n)\}$ and let us look at the horizontal segment separating the rows i and $i + 1$. If $f(i) = i$, then there is no horizontal segment of the braid separating these rows. Else, we have a horizontal segment of the braid connecting $(i, f(i))$ to (i, i) (or to $(i, 1)$, if $i \leq 0$), which corresponds to the arc $A_{f(i)}$ in the juggling diagram of f, connecting $f(i)$ to i.

Assume now $i \notin \{f^{-1}(1), \ldots, f^{-1}(n)\}$. If $f(i) < 0$, and $i \neq f(i)$, then the horizontal segment of the infinite braid $M_k^\infty(r)$ separating the rows i and $i + 1$ is contained entirely outside (to the left) of the strip $1 \leq j \leq n$. The same holds if $f(i) > n$ and $i \notin \{1, \ldots, n\}$. It remains to see the case $i \in \{1, \ldots, n\}$ and $f(i) > n$. Note that there are exactly k such values of i. In this case, we will have a horizontal segment in the braid $M_k(r)$ connecting (i, n) to (i, i). This horizontal segment will intersect exactly once with any strand in the braid that contains a vertical segment separating j and $j + 1$ for $j > i$. Thus, up to braid moves pulling the strands to the right, the braid $M_k(r)$ is of the form $J_k(f)\Delta_k$, as needed.

□

3. Legendrian Links and positroid data

The goal of this section is to associate Legendrian links to positroid data and show that equivalent positroid data yield Legendrian isotopic links, up to trivially adding unlinked unknots. These Legendrian links are introduced in Subsection 3.1. Theorem 3.6 establishes the necessary Legendrian isotopies. It is proven in Subsection 3.3, after the main technical lemma is proven in Subsection 3.2.

3.1. **Legendrian links associated to positroid data.** In Section 2 we introduced the following braid words associated to positroid data:

(1) For $u, w \in S_n$ permutations and w k-Grassmannian, the positive Richardson braid word $R_n^+(u, w) = \beta(u^{-1}w_0)\beta(w^*)$, in Subsection 2.2. It is a positive n-stranded braid word.

(2) For a bounded affine permutation f of size n, the juggling braid $J_k(f) = J_k^{(2)}(f)J_k^{(1)}(f)$, in Subsection 2.3. It is a positive k-stranded braid word. By Corollary 2.24, $J_k^{(1)}$ is a product of interval braids, and Proposition 2.25 shows that $J_k^{(2)}(f)$ is a permutation braid.

(3) For a Le diagram ⅃, the Le braid $D_k(⅃) = D_k^{(2)}(⅃)D_k^{(1)}(⅃)$, in Subsection 2.5. It is a k-stranded braid word. By construction, $D_k^{(1)}(⅃)$ is a positive braid word and $D_k^{(2)}(⅃)$ is a permutation braid with all its crossings being negative.

(4) For a cyclic rank matrix r, the matrix braid $M_k(r)$, in Subsection 2.6. It is a k-stranded positive braid word.

Let us use [10, Section 2.2] to associate a Legendrian link in $(\mathbb{R}^3, \xi_{st})$ to a positive braid word. Recall that the front projection of a Legendrian link in $(\mathbb{R}^3, \xi_{st})$ with the standard contact form $\xi_{st} = \ker\{dz - ydx\}$ is its projection onto the (xz)-plane $\mathbb{R}^2_{x,z}$. The front of a Legendrian link recovers the link, cf. [33, Section 3.2].

Definition 3.1 ([10]). Let β be a positive braid word. By definition, the (-1)-closure of β is the Legendrian link $\Lambda_\beta \subseteq (\mathbb{R}^3, \xi_{st})$ whose front projection is drawn in Figure 14 (left).   □
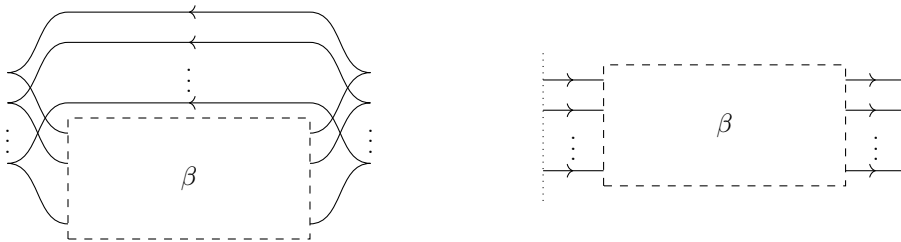

FIGURE 14. (Left) (-1)-closure of a positive braid word β. (Right) A front in $S^1 \times \mathbb{R}$ associated to β: its satellite along the standard Legendrian unknot yields the (-1)-closure on the left.

Let β' be a positive braid word obtained from β by applying braid moves. The Legendrian Reidemeister moves can then be used the show that $\Lambda_{\beta'}$ is Legendrian isotopic to Λ_β, cf. [19, Section 2.3] and [39]. In particular, the Legendrian isotopy type of Λ_β is independent of the choice of braid word representing the element inside the positive monoid.

Definition 3.1 can now be applied to the positive Richardson and juggling braids, both of which are positive braids. This leads to the following definitions:

Definition 3.2 (Richardson Legendrians). Let $u, w \in S_n$ be two permutations such that $u \leq w$ in the Bruhat order and w is k-Grassmannian. By definition, the Legendrian Richardson link $\Lambda^+(u, w) \subseteq (\mathbb{R}^3, \xi_{st})$ is the (-1)-closure of $\Delta_n R_n^+(u, w)$.   □

To ease notation, we often denote the Legendrian link $\Lambda^+(u, w) \subseteq (\mathbb{R}^3, \xi_{st})$ in Definition 3.2 by $\Lambda(u, w) \subseteq (\mathbb{R}^3, \xi_{st})$. See Subsection 3.4 for results making this abuse of notation justified.

Definition 3.3 (Juggling Legendrians). Let $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ be bounded affine permutation of size n. By definition, the Legendrian juggling link $\Lambda(f) \subseteq (\mathbb{R}^3, \xi_{st})$ is the (-1)-closure of $\Delta_k J_k(f)$.   □

Definition 3.4 (Matrix Legendrians). Let r be a cyclic rank matrix of type (k, n). By definition, the Legendrian matrix link $\Lambda(r) \subseteq (\mathbb{R}^3, \xi_{st})$ is the (-1)-closure of $M_k(r)$.   □

Note that Theorem 2.39 implies that the matrix braid $M_k(r) \in \mathcal{B}_k$ is directly equivalent to $\Delta_k J_k(f) \in \mathcal{B}_k$ and they are both positive and k-stranded.

By construction, the smooth links underlying the Legendrian links $\Lambda(u,w)$ and $\Lambda(f)$ in Definitions 3.2, 3.3 and 3.4 above coincide with the homonymous links introduced in Section 2. This justifies using the same notation. Since the Le braid $D_k(\beth)$ is not a positive braid, Definition 3.1 cannot be used directly. An appropriate modification will suffice, as follows.

Definition 3.5 (Le Legendrians)**.** Let $\beth$ be a Le diagram. By definition, the (-1)-closure of a positive braid of the form $\Delta_k \beta(w_0 D_k^{(2)}(\beth)) D_k^{(1)}(\beth)$ is said to be a Legendrian Le link $\Lambda(\beth) \subseteq (\mathbb{R}^3, \xi_{st})$. $\square$

First, $\beta(w_0 D_k^{(2)}(\beth))$ denotes a positive braid lift of the permutation $w_0 D_k^{(2)}(\beth)$, e.g. as in Definition 2.7. Second, any two different choices of such lifts lead to equivalent braid words and it follows that the Legendrian links they define are Legendrian isotopic. Thus our use of the notation $\Lambda(\beth)$ for any such Legendrian links, which does not explicitly refer to the choice of lift. By construction, the smooth link underlying any such $\Lambda(\beth)$ in Definitions 3.5 coincides with the homonymous link introduced in Section 2.

The main goal of the rest of this section is to prove the following result:

Theorem 3.6. *Let $u, w \in S_n$ be two permutations with w is k-Grassmannian, $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ be bounded affine permutation of size n and $\beth$ a Le diagram all representing the same positroid data. Then the Legendrian links $\Lambda(u,w), \Lambda(f), \Lambda(\beth)$ and $\Lambda(r)$ are all Legendrian isotopic in $(\mathbb{R}^3, \xi_{st})$, up to adding unlinked max-tb Legendrian unknots.*

Theorem 3.6 is proven in Subsection 3.3 below, once we have established Lemma 3.7. In fact, the proof shows that $\Lambda(f), \Lambda(\beth) \subseteq (\mathbb{R}^3, \xi_{st})$ are Legendrian isotopic, without adding any max-tb Legendrian unknots. Here max-tb stands for maximal Thurston-Bennequin number, cf. [19, Section 2.6] for details on the Thurston-Bennequin invariant.

3.2. **Variations on the Markov move.** The Legendrian links that we are comparing are associated to braids with different number of strands. For instance, the Richardson braids are n-stranded and the juggling braids are k-stranded for a positroid in $\mathrm{Gr}(k,n)$. In order to show that these Legendrian links are Legendrian isotopic, we must therefore apply a type of (de)stabilization. This is also the reason to work in $(\mathbb{R}^3, \xi_{st})$ instead of the 1-jet space $(J^1 S^1, \xi_{st})$. The following destabilization lemma, which implies Lemma 2.27, will suffice for our purposes:
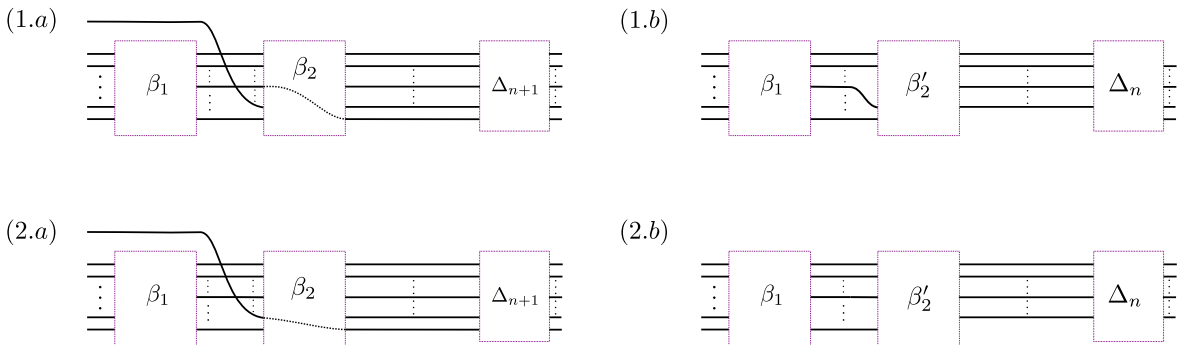
FIGURE 15. Lemma 3.7 states that two fronts $(1.a)$ and $(1.b)$, resp. $(2.a)$ and $(2.b)$ have front homotopic (-1)-closures, resp. front homotopic (-1)-closures up to an unlinked max-tb unknotted component. Note that these fronts, if to be understood cyclically in $S^1_\theta \times \mathbb{R}_z$, do *not* yields Legendrian (or even smoothly) isotopic links. This forces considering (-1)-closures so as to find the Legendrian isotopy in $(\mathbb{R}^3, \xi_{st})$.

Lemma 3.7 (Legendrian destabilizations)**.** *Let $\beta_1 \in \mathrm{Br}_n$ be a positive braid on n strands in the Artin generators $\sigma_1, \ldots, \sigma_{n-1}$ and $\beta_2 \in \mathrm{Br}_{n+1}$ a reduced positive lift of a permutation $w \in S_{n+1}$ on $(n+1)$ strands. Set $j := w^{-1}(n+1)$, let $\tilde{\beta}_1 \in \mathrm{Br}_{n+1}$ the braid obtained from β_1 by shifting all the indices of*

the Artin generators up by 1, and let $\beta_2' \in \mathrm{Br}_n$ be the braid obtained from β_2 by removing the strand that ends at the bottom on the right of β_2. Then the following two statements hold:

(1) *Consider the following two positive braid words:*

    (a) $\eta_1 := \Delta_{n+1}\beta_2(\sigma_\ell \cdot \ldots \cdot \sigma_{j+1}\sigma_j \cdot \ldots \cdot \sigma_2\sigma_1)\tilde{\beta}_1 \in \mathrm{Br}_{n+1}$, *the* $(n+1)$-*stranded positive braid depicted in Figure 15.(1.a),*

    (b) $\eta_2 := \Delta_n\beta_2'(\sigma_{\ell-1} \cdot \ldots \cdot \sigma_{j+1})\beta_1 \in \mathrm{Br}_n$, *the* n-*stranded positive braid depicted in Figure 15.(1.b).*

Then the Legendrian link Λ_{η_1} is Legendrian isotopic to Λ_{η_2}.

(2) *Consider the following two positive braid words:*

    (a) $\eta_1 := \Delta_{n+1}\beta_2(\sigma_j\sigma_{j-1} \cdot \ldots \cdot \sigma_1)\tilde{\beta}_1 \in \mathrm{Br}_{n+1}$, *the* $(n+1)$-*stranded positive braid depicted in Figure 15.(2.a),*

    (b) $\eta_2 := \Delta_n\beta_2'\beta_1 \in \mathrm{Br}_n$, *the* n-*stranded positive braid depicted in Figure 15.(2.b).*

Then the Legendrian link Λ_{η_1} is Legendrian isotopic to the Legendrian link given by a max-tb Legendrian unknot unlinked union with Λ_{η_2}.
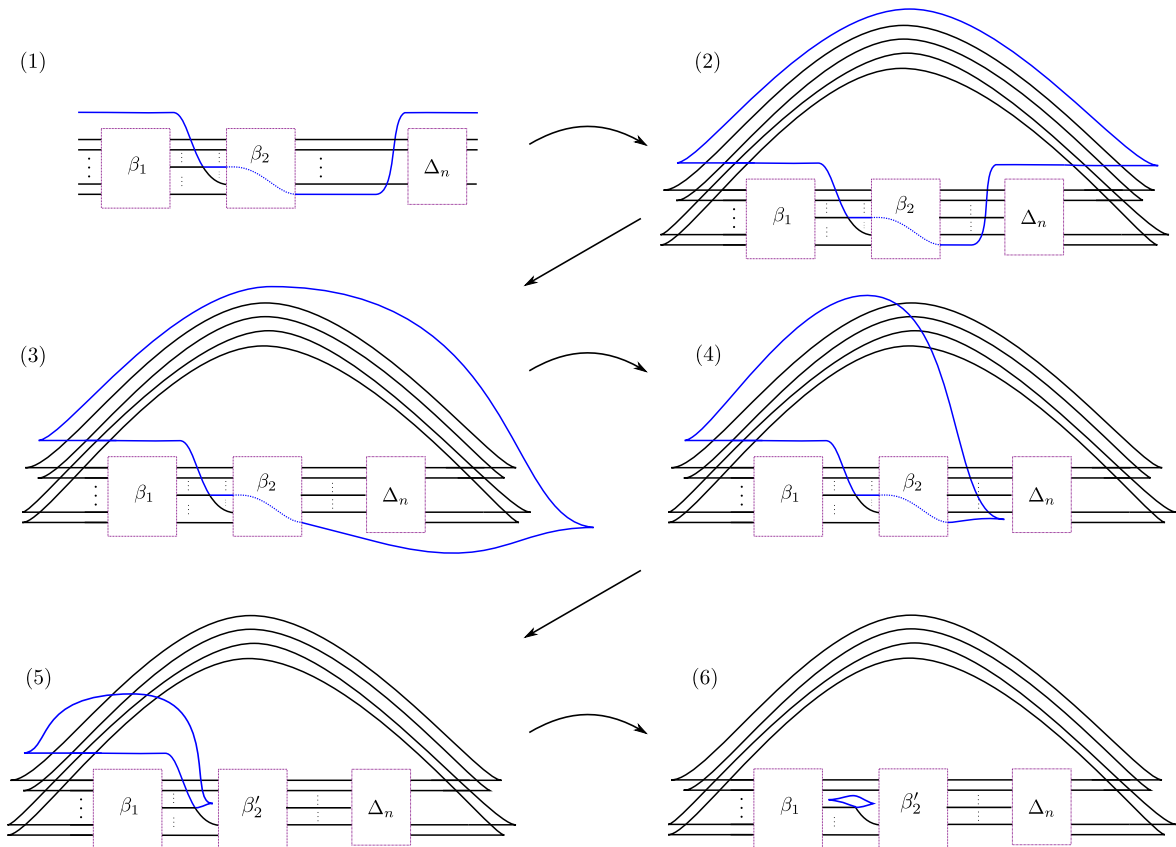

FIGURE 16. In (1) the starting piece of a front. From (2)-(6), the Legendrian isotopy that proves Lemma 3.7 for its (-1)-closure.

Proof. The proof is contained in Figure 16, for item (1), and 17, for item (2), which we now explain. First item (1). Choose a braid word for the half-twist Δ_{n+1} which is of the form $\Delta_n(\sigma_n\sigma_{n-1}\ldots\sigma_2\sigma_1)$. The resulting braid is then as in Figure 16.(1) and its (-1)-closure is depicted in Figure 16.(2).

The fronts in Figure 16.(2)-(6) are then front homotopic, i.e. realized by Legendrian isotopies, as follows. Front (2) to (3) consists of a series of Reidemeister III moves followed by a sequence of Reidemeister II moves that pull the top (blue) strand to the right of the front past Δ_n and n right cusps. Front (3) to (4) is the reverse of that sequence applied to the piece of the (blue) strand above the rightmost (blue) cusp: first a sequence of Reidemeister II moves and then a sequence of Reidemeister III moves. Front (4) to (5) is a sequence of Reiedemeister III and Reidemeister II moves that pull the right blue cusp pointing down up to the center region. The sequence of Reidemeister III moves necessary to realize this Legendrian isotopy from (4) to (5) exists because β_2 is (the lift of) a permutation braid. In particular, the blue strand inside the β_2-box in Front (4) always goes above the other strands. Then the isotopy from (4) to (5) is as follows: first pull the blue strand exiting the right blue cusp from above across part of the β_2-box via Reidemeister III moves; then use Reidemeister II moves to pull the right blue cusp through the β_2-box so as to reach Front (5). Finally, Front (5) to (6) is a sequence of Reidemeister II moves that pulls the top leftmost blue cusp to the region containing the right blue cusp. Front (6) is homotopic to the (-1)-closure of η_2 via a Reidemeister I move applied to the crossing between β_1 and β_2, which proves item (1) of the lemma.

For item (2), proceed as in item (1) by choosing a braid word for the half-twist Δ_{n+1} of the form

$$\Delta_n(\sigma_n\sigma_{n-1}\cdot\ldots\cdot\sigma_2\sigma_1).$$

The resulting braid is drawn in Figure 17.(1) and its (-1)-closure is in Figure 17.(2). From Front (2) to Front (3) we perform a sequence of Reidemeister III and then Reidemeister II moves moving the right piece of the blue strand (under the cusps). This is the same first step as for item (1). From Front (3) to Front (4) we perform a movie similar to the previous one but to the left piece, using the left cusps. The only difference is that we must pull the strand through a piece of the β_2-box, as indicated by the red arrows in Figure 17.(3). This is indeed possible because β_2 is a permutation braid. Once at Front (4), the blue component is a max-tb Legendrian unknot which is unlinked from the other components of the Legendrian link, thus item (2) follows.
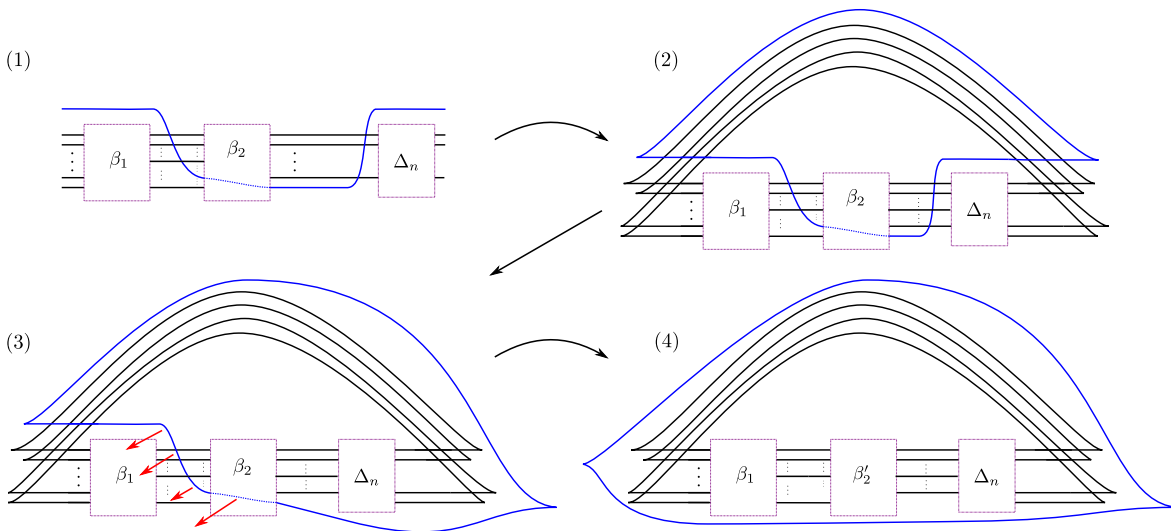
$\square$

FIGURE 17. In (1) the starting piece of a front. From (2)-(4), the Legendrian isotopy that proves Lemma 3.7 for its (-1)-closure.

Remark 3.8. Lemma 3.7 does not hold for general positive braid words $\beta_2 \in \mathrm{Br}_{n+1}$. Fortunately, the hypothesis of β_2 being a permutation braid is met for the positroid Richardson braids. $\square$

3.3. **Proof of Theorem 3.6.** Let us conclude Theorem 3.6 from the results established so far. Let us first show that $\Lambda(u, w)$ and $\Lambda(f)$ are Legendrian isotopic in $(\mathbb{R}^3, \xi_{st})$, up to adding unlinked max-tb Legendrian unknots. For that we follow the proof of Theorem 2.26 in Subsection 2.4. Thanks to Lemma 3.7, we claim that the same argument gives a Legendrian isotopy, not just a smooth one. Indeed, it suffices to note the following two facts:

(i) All braids being used in the proof of Theorem 2.26 are positive braids. Therefore, at any stage, we can consider the (-1)-closure and have the argument be about Legendrian links.

(ii) The only result that is used in the proof of Theorem 2.26 is Lemma 2.27. Since Lemma 3.7 is precisely a Legendrian realization of Lemma 2.27, we can apply the argument in the proof of Theorem 2.26 to Legendrian links and use instead Lemma 2.27 each time that Lemma 2.27 is invoked in that smooth proof.

This concludes the desired statement about $\Lambda(u, w)$ and $\Lambda(f)$. The Legendrian links $\Lambda(f)$ and $\Lambda(r)$ are Legendrian isotopic because they are (-1)-closures of equivalent positive k-stranded braids, thanks to Theorem 2.39.

Finally, let us now show that $\Lambda(f)$ and $\Lambda(\unicode{x0197})$ are Legendrian isotopic. Sections 2.3 and 2.5 provided the decompositions $J_k(f) = J_k^{(2)}(f) J_k^{(1)}(f)$ and $D_k(\unicode{x0197}) = D_k^{(2)}(\unicode{x0197}) D_k^{(1)}(\unicode{x0197})$. Here, $J_k^{(1)}(f)$ and $D_k^{(1)}(\unicode{x0197})$ are k-stranded positive braids and $J_k^{(2)}(f)$ and $D_k^{(2)}(\unicode{x0197})$ are reduced, but the latter has negative crossings. We claim that the two positive braid words $\Delta_k J_k^{(2)}(f) J_k^{(1)}(f)$ and $\Delta_k \beta(w_0 D_k^{(2)}(\unicode{x0197})) D_k^{(1)}(\unicode{x0197})$ are equivalent through positive braid words. For ease of notation, we denote $J_i := J_k^{(i)}(f)$ and $D_i := D_k^{(i)}(\unicode{x0197})$, $i = 1, 2$, and write $\beta_1 \overset{+}{\sim} \beta_2$ if two positive braid words are equivalent through positive braid words.

Let us prove the claim. First, Lemma 2.33 and its proof imply that $J_1 \overset{+}{\sim} D_1$. Second, we now need $J_2 \overset{+}{\sim} \beta(w_0 D_2)$. Lemma 2.30 and its proof show that $J_2 D_2^{-1} \overset{+}{\sim} \Delta_k$, where D_2^{-1} is a positive braid word. By construction, $\Delta_k \overset{+}{\sim} \beta(w_0 D_2) D_2^{-1}$ and thus $J_2 D_2^{-1} \overset{+}{\sim} \beta(w_0 D_2) D_2^{-1}$. By [13, Prop. II.4.7], the positive braid monoid is right-cancellative and thus $J_2 D_2^{-1} \overset{+}{\sim} \beta(w_0 D_2) D_2^{-1}$ implies $J_2 \overset{+}{\sim} \beta(w_0 D_2)$. Therefore, we have $J_1 \overset{+}{\sim} D_1$ and $J_2 \overset{+}{\sim} \beta(w_0 D_2)$. Combined, these imply $\Delta_k J_2 J_1 \overset{+}{\sim} \Delta_k \beta(w_0 D_2) D_1$. Hence, their (-1)-closures $\Lambda(f)$ and $\Lambda(\unicode{x0197})$ are Legendrian isotopic. $\square$

3.4. **Lagrangian projections and negative crossings.** This subsection is not logically needed for the rest of the results in this manuscript, but we include this brief discussion for completeness. Both in Section 2 above and the literature, see e.g. [29, Section 3], braids with negative crossings are discussed in relation to positroids. In the former instance, both the Richardson braid word, in Definition 2.7 in Subsection 2.2, and the Le braid $D_k(\unicode{x0197})$, in Subsection 2.5, typically have negative crossings. This raises the question of whether there are Legendrian links in $(\mathbb{R}^3, \xi_{st})$ that can be naturally associated to them. In particular, this would allow us to describe certain types of varieties and their algebraic combinatorics, associated to such braid words (with some negative crossings), intrinsically in terms of Legendrian links and their invariants, cf. [8, 10].

The answer is affirmative in this context, as the Lagrangian projection allows for the introduction of negative crossings in the following case. It is possible to introduce a cancelling pair vv^{-1} where v is a reduced positive braid word for a permutation braid. This technique is known as dipping in the literature. For dipping, also known as splashing in [28, Section 3.2], we refer to [46, Section 2.4]. We refer to [10, Section 2.2] for the definition and details on the Lagrangian (-1)-closure of a braid word. In conclusion, we can use the following lemma to associate Legendrian links to both the Richardson and Le braids, which are described by braid words with some negative crossings.

Lemma 3.9. *Let $\gamma \in \mathrm{Br}_n^+$ be a positive braid word and $v \in S_n$ a permutation. Then:*

(1) *There exists a Legendrian link $\Lambda(v, \gamma) \subseteq (\mathbb{R}^3, \xi_{st})$ whose Lagrangian projection is Hamiltonian isotopic to the Lagrangian (-1)-closure of the braid word $\Delta_n^2 \beta(v)^{-1} \gamma$, for a choice of positive braid word Δ_n for the half-twist.*

(2) *$\Lambda(v, \gamma)$ is Legendrian isotopic to the Legendrian (-1)-closure of $\Delta_n \beta(w_0 v^{-1}) \gamma$.*

Proof. By [10, Prop. 2.7] the positive braid word $\Delta_n \beta(w_0 v^{-1}) \gamma$ is admissible, as defined in *loc. cit.*. In particular, there exists a Legendrian link $\Lambda'(v, \gamma) \subseteq (\mathbb{R}^3, \xi_{st})$ whose Lagrangian projection is the Lagrangian (-1)-closure of the braid word $\Delta_n \beta(w_0 v^{-1}) \gamma$. Note that *ibid.* also implies that $\Lambda'(v, \gamma)$ is Legendrian isotopic to the Legendrian (-1)-closure of $\Delta_n \beta(w_0 v^{-1}) \gamma$. By dipping according to the permutation v, introducing $\beta(v) \beta(v^{-1})$ to the Lagrangian diagram exactly between $\beta(w_0 v^{-1})$ and γ, there exists a Legendrian isotopy from $\Lambda'(v, \gamma)$ to a Legendrian link $\Lambda(v, \gamma)$ whose Lagrangian projection is the Lagrangian (-1)-closure of the braid word $\Delta_n \beta(w_0 v^{-1}) \beta(v) \beta(v^{-1}) \gamma$. This braid is a positive braid word for $\Delta_n^2 \beta(v)^{-1} \gamma$. Note that we can dip according to $\beta(v)^{-1} \beta(v)$, instead of $\beta(v) \beta(v^{-1})$. Indeed, this is because the differences in height between the strands in the front diagram provided by [10, Prop. 2.7], away from a neighborhood of the crossings, are strictly increasing as we move to the right, instead of left, as in the Ng resolution [56]. $\qquad\square$

Lemma 3.9 can be applied to Richardson and Le braids, as follows:

(a) $\gamma = w$ and $v = u$, with $u, w \in S_n$, w a k-Grassmannian permutation, and $u \leq w$ in the Bruhat order. Then Lemma 3.9 implies that $\Lambda(u, w)$ is a Legendrian representative for the Richardson braids in Definition 2.7, even if they contained negative crossings. Lemma 3.9.(2) implies that $\Lambda(u, w)$ is Legendrian isotopic to $\Lambda^+(u, w)$ as introduced in Definition 3.2.

(b) For a Le diagram ⌐, we can choose $\gamma = D_k^{(1)}(⌐)$ and $v = (D_k^{(2)}(⌐))^{-1}$. Then Lemma 3.9 constructs a Legendrian link $\Lambda((D_k^{(2)}(⌐))^{-1}, D_k^{(1)}(⌐))$ for the Le braid $D_k^{(2)}(⌐) D_k^{(1)}(⌐)$, even if it has negative crossings. By Lemma 3.9.(2), $\Lambda((D_k^{(2)}(⌐))^{-1}, D_k^{(1)}(⌐))$ is Legendrian isotopic to $\Lambda(⌐)$ as introduced in Definition 3.5.

In particular, the above gives a symplectic geometric enhancement of the smooth links presented in [29, Section 3]: they can be naturally defined as Legendrian links, from which positroid strata can be extracted by studying the Legendrian contact dg-algebra, see e.g. [8, 10, 11].

4. Braid Varieties, Richardson varieties and Brick Manifolds

In this section, we study braid varieties associated to positroid braids. In particular, we prove Theorems 1.3 and 1.4. In Subsection 4.3, we show that open Richardson varieties in type A, i.e. $G = \mathrm{GL}(n, \mathbb{C})$, are braid varieties. Subsection 4.4 shows that brick manifolds $\mathrm{brick}(\beta)$, for any choice of braid word $\beta \in \mathcal{B}$, provide *different* smooth projective compactifications of the *same* braid variety $X(\beta)$. Finally, Subsection 4.5 uses Subsection 4.4 to present a description of the homology of braid varieties in terms of brick manifolds and also establishes the relation to Khovanov-Rozansky homology.

4.1. **Preliminaries on braid varieties.** Braid varieties are defined as follows:

Definition 4.1 ([8])**.** Let β be a positive braid word $\beta \in \mathcal{B}_n^+$, $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell}$, and $\pi \in \mathrm{GL}(n, \mathbb{C})$ a permutation matrix. By definition, the *braid variety* associated to β and π is

$$X(\beta; \pi) := \{(z_1, \ldots, z_\ell) \ : \ B_\beta(z_1, \ldots, z_\ell)\pi \text{ is upper-triangular}\} \subseteq \mathbb{C}^\ell,$$

where the matrix $B_\beta(z_1, \ldots, z_\ell) \in \mathrm{GL}(n, \mathbb{C}[z_1, \ldots, z_\ell])$ is defined to be the matrix product

$$B_\beta(z_1, \ldots, z_\ell) := B_{i_1}(z_1) \cdots B_{i_\ell}(z_\ell),$$

and the braid matrices $B_i(z) \in \mathrm{GL}(n, \mathbb{C}[z])$ are defined by:

$$(B_i(z))_{jk} := \begin{cases} 1 & j = k \text{ and } j \neq i, i+1 \\ 1 & (j, k) = (i, i+1) \text{ or } (i+1, i) \\ z & j = k = i+1 \\ 0 & \text{otherwise;} \end{cases}, \quad \text{i.e. } B_i(z) := \begin{pmatrix} 1 & \cdots & & & \cdots & 0 \\ \vdots & \ddots & & & & \vdots \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ 0 & \cdots & 1 & z & \cdots & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & \cdots & & & \cdots & 1 \end{pmatrix}.$$

Here the only non-trivial (2×2)-block of $B_i(z)$ is at the ith and $(i+1)$st rows. $\qquad\square$

Braid varieties were introduced in [8, 51], where part of their geometry was studied. In particular, we proved in [8] that $X(\beta_1; \pi) \cong X(\beta_2; \pi)$ if β_1 and β_2 are related by Reidemeister III moves or braid commutation; hence the name *braid* varieties. In this article, the permutation (matrix) π will often be $\pi = w_{0,n} = [n, n-1, \ldots, 1] \in S_n$, and we sometimes abbreviate $X(\beta)$ for $X(\beta; w_{0,n})$.

4.2. **Richardson and Positroid varieties.** We introduce Richardson and positroid varieties, following [42], see also [62]. Let us consider the flag variety $\mathscr{F}\ell_n$ associated to the group $G := \mathrm{GL}(n, \mathbb{C})$. That is, $\mathscr{F}\ell_n = G/B$ where B is the Borel subgroup of upper triangular matrices. We denote by $\mathscr{F}^A$ the flag associated to a matrix $A \in G$. Namely, the i-th space in $\mathbb{C}^n$ for the flag $\mathscr{F}^A$ is spanned by the first i columns of the matrix A. Let us denote by $\mathscr{F}^{\mathrm{st}} := (0 \subseteq \langle e_1 \rangle \subseteq \langle e_1, e_2 \rangle \subseteq \cdots \subseteq \langle e_1, \ldots, e_{n-1} \rangle \subseteq \mathbb{C}^n)$ the standard flag, and by $\mathscr{F}^{\mathrm{ant}} := (0 \subseteq \langle e_n \rangle \subseteq \langle e_n, e_{n-1} \rangle \subseteq \cdots \subseteq \langle e_n, \ldots, e_2 \rangle \subseteq \mathbb{C}^n)$ the anti-standard flag. Note that we have an action of G on $G/B = \mathscr{F}\ell_n$ by left multiplication. By restricting, we also have an action of B on $\mathscr{F}\ell_n$. Similarly, if B_- is the opposite Borel subgroup of lower triangular matrices, we also have an action of B_- on $\mathscr{F}\ell_n$ by left multiplication.

Let $w \in S_n$ be a permutation. The *Schubert cell* $\overset{\circ}{X}_w$ associated to $w \in S_n$ is defined to be the B-orbit of the flag wB/B, where w is understood as a permutation matrix:

$$\overset{\circ}{X}_w := BwB/B = \{\mathscr{F} \in \mathscr{F}\ell_n \mid \dim(F_p^{\mathrm{st}} \cap F_q) = \#\{i \leq q \mid w(i) \leq p\} \text{ for all } p, q \in [1, n]\}.$$

The *opposite Schubert cell* of $w \in S_n$ is defined to be the B_--orbit of the flag wB/B:

$$\overset{\circ}{X}{}^w := B_- wB/B.$$

Similarly to $\overset{\circ}{X}_w$, the opposite Schubert cell $\overset{\circ}{X}{}^w$ admits a explicit description as follows. Let w_0 be the longest element of S_n. Then:

$$\overset{\circ}{X}{}^w = \{\mathscr{F} \in \mathscr{F}\ell_n \mid \dim(F_p \cap F_q^{\mathrm{ant}}) = \#\{i \leq q \mid w_0 w(i) \leq p\} \text{ for all } p, q = 1, \ldots, n\}.$$

Definition 4.2. Let $u, w \in S_n$ be two permutations. The *open Richardson variety* $\mathcal{R}^\circ(u, w) \subseteq \mathscr{F}\ell_n$ is the intersection:

$$\mathcal{R}^\circ(u, w) := \overset{\circ}{X}_w \cap \overset{\circ}{X}{}^u.$$

It is known that $\mathcal{R}^\circ(u, w)$ is nonempty if and only if $u \leq w$ in the Bruhat order, cf. [62], in which case it is a smooth affine variety of dimension $\ell(w) - \ell(u)$. In particular, $\mathcal{R}^\circ(w, w)$ is a single point.

Now fix $k \leq n$ and consider the Grassmannian $\mathrm{Gr}(k, n)$ of k-planes in $\mathbb{C}^n$. In fact, $\mathrm{Gr}(k, n) = G/P$, where P is the subgroup of $G = \mathrm{GL}(n, \mathbb{C})$ consisting of block-upper-triangular matrices with blocks of sizes k and $n - k$. We have the associated subgroup $W_P = S_k \times S_{n-k} \subseteq S_n$. Again, we have an action of B on $\mathrm{Gr}(n, k)$ and, given a permutation $w \in S_n$, we have an associated Schubert cell in the Grassmannian:

$$C_w := BwP/P \subseteq \mathrm{Gr}(k, n).$$

In fact, $C_w = C_{w'}$ if the cosets wW_P and $w'W_P$ coincide.

Note that we have a natural projection $\pi : \mathscr{F}\ell_n \to \mathrm{Gr}(k, n)$. In terms of flags, $\pi(\mathscr{F}) = \mathscr{F}_k$ is the k-th subspace of the flag. This map is B-equivariant and thus $\pi(\overset{\circ}{X}_w) \subseteq C_w$. The following result underscores the importance of k-Grassmannian permutations.

Lemma 4.3 (Section 5, [42]). *The map* $\pi|_{\overset{\circ}{X}_w} : \overset{\circ}{X}_w \to C_w$ *is an isomorphism of algebraic varieties if and only if w is a k-Grassmannian permutation.*

In particular, if w is a k-Grassmannian permutation and $u \leq w$ then we have:

$$\mathscr{F}\ell_n \supseteq \mathcal{R}^\circ(u, w) \cong \pi(\mathcal{R}^\circ(u, w)) \subseteq \mathrm{Gr}(k, n).$$

Definition 4.4. Let $u, w \in S_n$, with w a k-Grassmannian permutation and $u \leq w$. The *open positroid variety* $\Pi_{u,w} \subseteq \mathrm{Gr}(k, n)$ is:

$$\Pi_{u,w} := \pi(\mathcal{R}^\circ(u, w)).$$

By definition, the positroid variety $\Pi_{u,w}$ is isomorphic to the Richardson variety $\mathcal{R}^\circ(u, w)$. Since positroids can be defined inside the Grassmannian, they are specially nice amongst other Richardson varieties, see e.g. [29, 42, 43, 62].

4.3. **Open Richardson varieties as braid varieties.** The following result is the main theorem of this subsection. It states that open Richardson varieties can be described as braid varieties.

Theorem 4.5. *Let $\beta_1, \beta_2 \in \mathcal{B}_n^+$ be two reduced positive braid words, and $w_1, w_2 \in S_n$ be their Coxeter projections.*

(a) The map

$$\iota : \mathbb{C}^{\ell(\beta_1)} \times \mathbb{C}^{\ell(\beta_2)} \longrightarrow \mathscr{F}\ell_n, \qquad (z_1, z_2) \mapsto \mathscr{F}^{B_{\beta_1}^{-1}(z_1)}, \quad (z_1, z_2) \in \mathbb{C}^{\ell(\beta_1)} \times \mathbb{C}^{\ell(\beta_2)},$$

restricts to an isomorphism

$$X(\beta_1\beta_2; w_0) \longrightarrow \iota(X(\beta_1\beta_2; w_0)) \cong \overset{\circ}{X}_{w_1^{-1}} \cap \overset{\circ}{X}{}^{w_2 w_0}$$

of affine algebraic varieties.

(b) The map

$$\jmath : \mathbb{C}^{\ell(\beta_1)} \times \mathbb{C}^{\ell(\beta_2)} \longrightarrow \mathscr{F}\ell_n, \qquad (z_1, z_2) \mapsto \mathscr{F}^{w_0 B_{\beta_1}^{-1}(z_1)}, \quad (z_1, z_2) \in \mathbb{C}^{\ell(\beta_1)} \times \mathbb{C}^{\ell(\beta_2)}$$

restricts to an isomorphism

$$X(\beta_1\beta_2; w_0) \longrightarrow \jmath(X(\beta_1\beta_2; w_0)) \cong \overset{\circ}{X}{}^{w_0 w_1^{-1}} \cap \overset{\circ}{X}_{w_2^*}$$

of affine algebraic varieties.

Corollary 4.6. *Let $u, w \in S_n$ be such that $u \leq w$ in Bruhat order, and $\beta(w), \beta(u^{-1}w_0), \beta(w^*) \in \mathrm{Br}_n$ positive lifts of $w, u^{-1}w_0, w^*$. Then restrictions give the following isomorphisms of affine algebraic varieties*

$$\iota : X(\beta(w)\beta(u^{-1}w_0); w_0) \overset{\sim}{\to} \mathcal{R}^\circ(u^{-1}, w^{-1}), \qquad \jmath : X(\beta(u^{-1}w_0)\beta(w^*); w_0) \overset{\sim}{\to} \mathcal{R}^\circ(u, w).$$

Theorem 4.5, through Corollary 4.6, implies Theorem 1.3, as discussed in Subsection 4.3.3 below. Subsections 4.3.1 and 4.3.2 now prove Theorem 4.5.

Remark 4.7. We note that interpreting open Richardson varieties as braid varieties allows for a clear description of certain isomorphisms between them. In particular, the composition $\iota \circ \jmath^{-1}$ associated with $\beta_1 = \beta(u^{-1}w_0), \beta_2 = \beta(w^*)$ gives an isomorphism

$$\mathcal{R}^\circ(u, w) \overset{\sim}{\to} \mathcal{R}^\circ(w_0 w, w_0 u).$$

Also, we have an explicit isomorphism

$$\psi : X(\beta(w)\beta(u^{-1}w_0); w_0) \cong X(\beta(u^{-1}w_0)\beta(w^*); w_0)$$

given by a "cyclic rotation" of the braid word. The description of ψ and the proof that it is an isomorphism can be found in [7, Lemma 3.10]. [5] The composition $\jmath \circ \psi \circ \iota^{-1}$, with $\iota, \jmath$ as in Corollary 4.6, gives an isomorphism

$$\mathcal{R}^\circ(u^{-1}, w^{-1}) \overset{\sim}{\to} \mathcal{R}^\circ(u, w).$$

In fact, the existence of these isomorphisms implies that we could have chosen slightly different braids throughout the paper so that their varieties would be isomorphic to open Richardson and positroid varieties, at the cost of changing such isomorphisms. Namely, our positive Richardson braid is $R_n^+(u, w) = \beta(u^{-1}w_0)\beta(w^*)$, and we use the isomorphism $\jmath$ from Corollary 4.6 to realize open Richardson varieties as braid varieties. Instead, we could have used the braid $\beta(w)\beta(u^{-1}w_0)$ and its braid variety throughout the paper and applied the isomorphism $\jmath \circ \psi$ to realize open Richardson varieties as braid varieties. All the other positroid braids would have changed accordingly. $\square$

---

[5] The paper [7] uses different conventions to define braid varieties, but the same arguments apply in our setting.

4.3.1. *Technical lemmas for Theorem 4.5.* The open Schubert cell $\overset{\circ}{X}_w$ is isomorphic to an affine space of dimension $\ell(w)$, the length of w, which can be described as follows. Consider w as a permutation matrix and let E_{lm} be the (lm)-elementary matrix, so that the (s,t) entry of E_{lm} is the product $(\delta_{l,s})(\delta_{m,t})$ of Kronecker deltas. Inside the space of $(n \times n)$-matrices $M_n(\mathbb{C}) \cong \mathbb{C}^{n^2}$, consider the unique affine subspace M_w which contains $w \in M_n(\mathbb{C})$ and is spanned by all the matrices $E_{w(j),i}$ such that the pair (i,j) satisfies $(i,j) \in \mathrm{inv}(w)$.[6] It can be proven, see e.g. [62, Proposition 1.12], that the map

$$\iota : \mathrm{GL}(n,\mathbb{C}) \longrightarrow \mathscr{F}\ell_n, \quad A \mapsto \mathscr{F}^A,$$

restricts to an isomorphism between M_w and the Schubert cell $\overset{\circ}{X}_w$, i.e. $\iota(M_w) \cong \overset{\circ}{X}_w$. Let us now relate this to braid matrices. Indeed, braid matrices serve as a parametrization of the affine spaces M_w, and thus of the corresponding Schubert cells. This is the content of the following lemma, for a proof see e.g. [51, Proposition 5.1.5] and [8, Section 2].

Lemma 4.8. *Let $w \in S_n$ be a permutation and $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathcal{B}_n$ a choice of reduced positive lift for w. Then the map*

$$\mathbb{C}^{\ell(w)} \to M_w, \qquad (z_1, \ldots, z_\ell) \mapsto B_{i_1}^{-1}(z_1) \cdots B_{i_\ell}^{-1}(z_\ell)$$

is an isomorphism of affine algebraic varieties. $\square$

The opposite open Schubert cell $\overset{\circ}{X}^w$ is also an affine space, as we now show.

Lemma 4.9. *Let $w \in S_n$. The opposite Schubert cell $\overset{\circ}{X}^w$ is an affine space of dimension $\ell(ww_0) = \ell(w_0) - \ell(w)$. Moreover, if $\beta = \sigma_{j_1} \cdots \sigma_{j_s}$ is a reduced positive lift of ww_0 then the map*

$$\mathbb{C}^{\ell(ww_0)} \to \overset{\circ}{X}^w, \qquad (t_1, \ldots, t_s) \mapsto \mathscr{F}^{B_\beta(t)w_0}$$

is an isomorphism of affine algebraic varieties.

Proof. This follows from Lemma 4.8, as follows. By definition,

$$\overset{\circ}{X}^w = B_- w B/B = (w_0 B w_0) w B/B = w_0 \overset{\circ}{X}_{w_0 w}$$

so that $\overset{\circ}{X}^w$ is an affine space of dimension $\ell(w_0 w) = \ell(ww_0)$. Now, $w_0 w = (ww_0)^*$, and thus the word $\sigma_{i_1} \cdots \sigma_{i_s}$, where $\sigma_{i_k} = \sigma_{j_k}^* = \sigma_{n-j_k}$, for $1 \le k \le s$, is a reduced braid word for $\beta(w_0 w)$. Then, according to Lemma 4.8, we have a parametrization of $\overset{\circ}{X}^w$ by flags associated to matrices of the form:

$$w_0 B_{i_1}^{-1}(t_1) \cdots B_{i_s}^{-1}(t_s), \qquad (t_1, \ldots, t_s) \in \mathbb{C}^s.$$

It remains to notice that $B_i^{-1}(t) = w_0 B_{n-i}(-t) w_0$, that is proved by a direct verification. $\square$

Lemmas 4.8 and 4.9 suffice to prove Theorem 4.5. That is, we now show that if $\beta = \beta_1 \beta_2$, with β_1 and β_2 both being reduced words, then the braid variety of β is isomorphic to the intersection of the corresponding Schubert and opposite Schubert cells.

4.3.2. *Proof of Theorem 4.5.* Let us first verify that, in fact, Statements (a) and (b) are equivalent. Indeed, let $A \in \mathrm{GL}(n,\mathbb{C})$. Then,

$$\mathscr{F}^A \in \mathcal{R}^\circ(w_2 w_0, w_1^{-1}) = ((w_0 B w_0)(w_2 w_0) B/B) \cap (B w_1^{-1} B/B)$$

if and only if

$$\mathscr{F}^{w_0 A} \in (B w_2^* B/B) \cap ((w_0 B w_0)(w_0 w_1^{-1}) B/B) = \mathcal{R}^\circ(w_0 w_1^{-1}, w_2^*),$$

and the equivalence of Statements (a) and (b) follows.

Now we show (a), from which (b) will follow by the above equivalence. First, let us verify that the image $\iota(X(\beta_1 \beta_2; w_0))$ is indeed in the required intersection, i.e. that for each $z_1 \in \mathbb{C}^{\ell(\beta_1)}$, the flag $\mathscr{F}^{B_{\beta_1}^{-1}(z_1)}$ belongs to both cells $\overset{\circ}{X}_{w_1^{-1}}$ and $\overset{\circ}{X}^{w_2 w_0}$. By Lemma 4.8, and since $\text{\dh}_1 = \beta((w_1)^{-1})$, the matrix $B_{\beta_1}^{-1}(z_1)$ belongs to the affine subspace $M_{w_1^{-1}}$, and thus $\mathscr{F}^{B_{\beta_1}^{-1}(z_1)} \in \overset{\circ}{X}_{w_1^{-1}}$, as needed. For the

---

[6]Recall that an *inversion* of w is a pair (i,j) where $i < j$ and $w(i) > w(j)$, and $\mathrm{inv}(w)$ denotes the set of inversions of w; note that $\ell(w) = \#\mathrm{inv}(w)$.

inclusion $\mathscr{F}^{B_{\beta_1}^{-1}(z_1)} \in \overset{\circ}{X}{}^{w_2 w_0}$, we note that for each $z_2 \in \mathbb{C}^{l(\beta_2)}$ such that $(z_1, z_2) \in X(\beta_1 \beta_2, w_0)$, we have the identity

$$B_{\beta_1}(z_1) B_{\beta_2}(z_2) w_0 = U$$

for some upper triangular matrix U. This implies that

$$B_{\beta_1}^{-1}(z_1) = B_{\beta_2}(z_2) w_0 U^{-1},$$

and, since U^{-1} is upper triangular, we conclude that $\mathscr{F}^{B_{\beta_1}^{-1}(z_1)} = \mathscr{F}^{B_{\beta_2}(z_2) w_0 U^{-1}} = \mathscr{F}^{B_{\beta_2}(z_2) w_0}$. Then Lemma 4.9, together with the observation that β_2 is a reduced braid word for $(w_2 w_0) w_0$, shows that the flag $\mathscr{F}^{B_{\beta_2}(z_2) w_0}$ belongs to $\overset{\circ}{X}{}^{w_2 w_0}$, as needed.

Second, in order to show that ι restricts to a bijection, consider a flag $\mathscr{F} \in \overset{\circ}{X}_{w_1^{-1}} \cap \overset{\circ}{X}{}^{w_2 w_0}$. By Lemma 4.8, we can find a unique element $z_1 \in \mathbb{C}^{\ell_1}$ such that $\mathscr{F} = \mathscr{F}^{B_{\beta_1}^{-1}(z_1)}$. Finally, there exists a unique element $z_2 \in \mathbb{C}^{\ell_2}$ such that $(z_1, z_2) \in X(\beta_1 \beta_2; w_0)$. Indeed, since $\mathscr{F} \in \overset{\circ}{X}{}^{w_2 w_0}$, Lemma 4.9 implies that there exists a unique $z_2 \in \mathbb{C}^{\ell_2}$ such that $\mathscr{F} = \mathscr{F}^{B_{\beta_2}(z_2) w_0}$. So $B_{\beta_1}^{-1}(z_1) = B_{\beta_2}(z_2) w_0 U$ for an upper triangular matrix U and the result follows. $\qquad\square$

4.3.3. *Proof of Theorem 1.3.* We are now in position to prove Theorem 1.3 from the introduction. Recall that a positroid pair (u, w) yields the open positroid variety $\Pi_{u,w} \subseteq \mathrm{Gr}(k, n)$ in the Grassmannian. By [42, Theorem 5.9], there exists an algebraic isomorphism $\Pi_{u,w} \cong \mathcal{R}^\circ(u, w)$ between the positroid stratum for (u, v) and the Richardson variety $\mathcal{R}^\circ(u, w)$. By Corollary 4.6, we have the algebraic isomorphism $\mathcal{R}^\circ(u, w) \cong X(\beta(u^{-1} w_0)\beta(w^*))$ given by the restriction of $\jmath$, where, as usual, $\beta(u^{-1} w_0)$, $\beta(w^*)$ are positive braid lifts of their corresponding arguments. The composition of these two isomorphisms proves Part (i).

For Part (ii), we use Section 3. In particular, that both $X(R_n^+(u, w))$ and $X(J_k(f); w_{0,k}) \times (\mathbb{C}^*)^d$ are Legendrian invariants associated to the Legendrian links $\Lambda(u, w)$ and $\Lambda(f)$, respectively. For that, we use [10, Section 5.1], which explicitly describes the Legendrian contact dg-algebra $A_{\Lambda(\beta)}$ for a Legendrian (-1)-closure $\Lambda(\beta)$ in terms of braid matrices. It follows from *loc. cit.*, similarly to [8, Section 2.6], that there is an algebraic isomorphism $X(\beta) \cong \mathrm{Spec}\, H^0(A_{\Lambda(\beta\Delta)})$, where we choose one marked point per strand. By [12, Theorem 3.4], the quasi-isomorphism type of $A_{\Lambda(\beta)}$ is a Legendrian isotopy invariant. Therefore $X(\beta) \cong X(\beta')$ if $\Lambda(\beta)$ is Legendrian isotopic to $\Lambda(\beta')$.

Let $\overline{\Lambda(f)} \subseteq (\mathbb{R}^3, \xi_{st})$ be the Legendrian link given by the unlinked union of $\Lambda(f)$ with m unlinked max-tb Legendrian unknots, each of the unknots with one unique marked point. Then

$$\mathrm{Spec}\, H^0(A_{\overline{\Lambda(f)}}) \cong \mathrm{Spec}\, H^0(A_{\Lambda(f)}) \times (\mathbb{C}^*)^m.$$

Indeed, $H^0(A_{\Lambda(e)}) \cong \mathbb{C}[t, t^{-1}]$ for the 1-stranded braid e with one marked point $t \in \mathbb{C}^*$, which corresponds to a max-tb Legendrian unknot with one marked point, and $\mathrm{Spec}\, H^0(A_{\Lambda_1 \cup \Lambda_2}) \cong \mathrm{Spec}\, H^0(A_{\Lambda_1}) \times \mathrm{Spec}\, H^0(A_{\Lambda_2})$ is a Cartesian product if the components Λ_1, Λ_2 of the link $\Lambda_1 \cup \Lambda_2$ are unlinked from each other. By Theorem 3.6, $\Lambda(u, w)$ is Legendrian isotopic to $\overline{\Lambda(f)} \subseteq (\mathbb{R}^3, \xi_{st})$ for some $m \in \mathbb{N}$. The number m is determined by the number of destabilizations as in Lemma 3.7.(2) in Subsection 3.2, equivalently the number of destabilizations using Lemma 2.27.(2) in Subsection 2.4. Following the proof of Theorem 2.26 gives $m = n - k - \varphi$, where φ is the number of fixed points of f. Therefore, we obtain the following sequence of isomorphisms:

$$X(R_n^+(u, w)) \cong \mathrm{Spec}\, H^0(A_{\Lambda(u,w)}) \cong \mathrm{Spec}\, H^0(A_{\overline{\Lambda(f)}}) \cong$$
$$\cong \mathrm{Spec}\, H^0(A_{\Lambda(f)}) \times (\mathbb{C}^*)^{n-k-\varphi} \cong X(J_k(f)) \times (\mathbb{C}^*)^{n-k-\varphi}.$$

This proves Part (ii) and thus finishes the argument for Theorem 1.3. $\qquad\square$

4.4. Brick manifolds and compactifications of braid varieties.

This subsection discusses smooth compactifications of braid varieties. Let $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathcal{B}_n^+$ be a positive braid word. We first define the brick manifold $\mathrm{brick}(\beta)$ associated to β, following [18]. These brick varieties $\mathrm{brick}(\beta)$ will provide natural smooth compactifications of our braid varieties $X(\beta)$.

By definition, the Bott-Samelson variety $\mathrm{BS}(\beta)$ associated to β is the moduli space of collections of flags $(\mathscr{F}^0, \ldots, \mathscr{F}^\ell) \in \mathscr{F}\ell_n$ such that $\mathscr{F}^0$ is the standard flag and either $\mathscr{F}^j = \mathscr{F}^{j+1}$, or the two

contiguous flags $\mathscr{F}^j, \mathscr{F}^{j+1}$ differ precisely in the i_{j+1}-subspace. This projective variety $\mathrm{BS}(\beta)$ contains a natural subvariety, called the open Bott-Samelson variety $\mathrm{OBS}(\beta)$ in [8], defined by the additional condition that two contiguous flags must be different, i.e. $\mathscr{F}^j \neq \mathscr{F}^{j+1}$ for every $j \in [0, \ell - 1]$.

The Bott-Samelson variety $\mathrm{BS}(\beta)$ admits a natural projection map

$$m_\beta : \mathrm{BS}(\beta) \longrightarrow \mathscr{F}\ell_n, \quad m_\beta(\mathscr{F}^0, \ldots, \mathscr{F}^\ell) := \mathscr{F}^\ell,$$

onto the last, rightmost, flag.

Definition 4.10. Let $\beta \in \mathcal{B}_n^+$ be a positive braid word. By definition, the *brick variety* $\mathrm{brick}(\beta)$ associated to β is

$$\mathrm{brick}(\beta) := m_\beta^{-1}(\delta(\beta)\mathscr{F}^{\mathrm{st}}),$$

where $\delta(\beta) \in S_n$ denotes the Demazure product of β. The associated *open brick variety* is defined as

$$\mathrm{brick}^\circ(\beta) := m_\beta^{-1}(\delta(\beta)\mathscr{F}^{\mathrm{st}}) \cap \mathrm{OBS}(\beta).$$

$\square$

Here the Demazure product $\delta(\beta) \in S_n$ is the (unique) maximal permutation with respect to the Bruhat order such that β contains its positive braid lift. The brick manifold $\mathrm{brick}(\beta)$, unlike the braid variety $X(\beta)$, significantly depends on the braid word $\beta \in \mathcal{B}$, and not only on the braid $[\beta] \in \mathrm{Br}$.

Let us prove Theorem 1.4 in the introduction. Given $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathcal{B}_n$, denote its opposite braid word by $\varUpsilon := \sigma_{i_\ell} \cdots \sigma_{i_1}$. Braid varieties relate to brick varieties, up to this mirroring, as follows:

Theorem 4.11. *Let* $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathcal{B}_n$ *be a positive braid word, and consider the truncations* $\varUpsilon_j := \sigma_{i_l} \cdots \sigma_{i_{l-j+1}}$, $j \in [1, \ell]$. *The following holds:*

(i) *The algebraic map*

$$\Theta : \mathbb{C}^\ell \longrightarrow \mathscr{F}\ell_n^{\ell+1}, \quad (z_1, \ldots, z_\ell) \mapsto (\mathscr{F}^{\mathrm{st}}, \mathscr{F}^1, \ldots, \mathscr{F}^\ell),$$

where $\mathscr{F}^j$ *is the flag associated to the matrix* $B_{\varUpsilon_j}^{-1}(z_{\ell-j+1}, \ldots, z_\ell)$, *restricts to an isomorphism*

$$\Theta : X(\varUpsilon; \delta(\beta)) \xrightarrow{\cong} \mathrm{brick}^\circ(\beta),$$

of affine varieties. In particular, the braid variety $X(\varUpsilon; \delta(\beta))$ *is smooth.*

(ii) *The complement to* $X(\varUpsilon; \delta(\beta))$ *in* $\mathrm{brick}(\beta)$ *is a normal crossing divisor. Its components correspond to all possible ways to remove a letter from* β *while preserving its Demazure product.*

Proof. For Part (i), we first verify $\Theta(X(\varUpsilon; \delta(\beta))) \subseteq \mathrm{brick}^\circ(\beta)$. For that, note that

$$B_{\varUpsilon_{j+1}}^{-1}(z_{\ell-j}, \ldots, z_\ell) = B_{\varUpsilon_j}^{-1}(z_{\ell-j+1}, \ldots, z_\ell)B_{i_{j+1}}^{-1}(z_{\ell-j}),$$

and thus the two flags $\mathscr{F}^j$ and $\mathscr{F}^{j+1}$ are indeed in position i_{j+1}, as required. In order to check that

$$\mathscr{F}^{B_{\varUpsilon}^{-1}(z_1, \ldots, z_\ell)} = \delta(\beta)\mathscr{F}^{\mathrm{st}},$$

we observe that we have $B_{\varUpsilon}(z_1, \ldots, z_\ell)\delta(\beta) = U$, where U is an upper triangular matrix, and hence $B_{\varUpsilon}^{-1}(z_1, \ldots, z_\ell) = \delta(\beta)U^{-1}$, from which this conclusion follows. The fact that the map Θ restricts to an isomorphism follows from the statement of Lemma 4.12 below. The smoothness claim follows from [18, Theorem 3.3].

For Part (ii), we proceed as follows. For a subset $I \subseteq [1, \ell]$, let $\mathrm{brick}(\beta)_I^\circ \subseteq \mathrm{brick}(\beta)$ be defined by the conditions that $\mathscr{F}^{i-1} \neq \mathscr{F}^i$ if and only if $i \in I$, where $\mathscr{F}^0 = \mathscr{F}^{\mathrm{st}}$. For example, $\mathrm{brick}(\beta)_{[1,\ell]}^\circ = \mathrm{brick}(\beta)^\circ$. Now let $\mathrm{brick}(\beta)_I := \overline{\mathrm{brick}(\beta)_I^\circ}$, which is similarly defined by the condition that $\mathscr{F}^{i-1} = \mathscr{F}^i$ if $i \notin I$. Note that $\mathrm{brick}(\beta)_I \subseteq \mathrm{brick}(\beta)_J$ if $I \subseteq J$, and that $\mathrm{brick}(\beta)_I^\circ$ is nonempty if and only if $\delta(\beta_I) = \delta(\beta)$, where β_I is the subword of β indexed by I. Moreover, in this case we have natural isomorphisms

$$\mathrm{brick}(\beta_I)^\circ \xrightarrow{\sim} \mathrm{brick}(\beta)_I^\circ, \qquad \mathrm{brick}(\beta_I) \xrightarrow{\sim} \mathrm{brick}(\beta)_I.$$

Now we have the following the composition

$$\mathrm{brick}(\beta) = \mathrm{brick}(\beta)^\circ \sqcup \bigcup_{I \subsetneq [1,\ell]} \mathrm{brick}(\beta)_I = X(\varUpsilon; \delta(\beta)) \sqcup \bigcup_{\substack{I \subsetneq [1,\ell] \\ \delta(\beta_I) = \delta(\beta)}} \mathrm{brick}(\beta_I).$$

If $\delta(\beta_I) = \delta(\beta)$, then $\text{brick}(\beta_I)$ is a smooth variety of dimension $|I| - \ell(\delta(\beta))$, see [18, Theorem 3.3]. Therefore, in this case

$$\text{brick}(\beta_I) = \text{brick}(\beta)_I = \bigcap_{j \notin I} \text{brick}(\beta)_{[1,\ell]\setminus\{j\}} = \bigcap_{j \notin I} \text{brick}(\beta_{[1,\ell]\setminus\{j\}}).$$

Hence $\text{brick}(\beta_I)$ is a complete intersection, and the divisors $\text{brick}(\beta_{[1,\ell]\setminus\{j\}}) \subseteq \text{brick}(\beta)$ intersect transversely, if the intersection is non-empty. $\square$

Lemma 4.12. *Let us consider an invertible matrix $A \in \text{GL}_n$ and $i \in [1, n-1]$. Then, the map*

$$\mathbb{C} \longrightarrow \mathscr{F}\ell, \quad z \longmapsto \mathscr{F}^{AB_i^{-1}(z)},$$

yields an isomorphism from $\mathbb{C}$ to the set of all flags that are in position i with respect to $\mathscr{F}^A$.

Proof. Let $\mathscr{F} := \mathscr{F}^A$, $\mathscr{F}'$ another flag that is in position i with respect to $\mathscr{F}$ and consider the projection $\mathscr{F}'_i \longrightarrow \mathscr{F}_{i+1}/\mathscr{F}_{i-1}$. The latter space has basis $\{a_i, a_{i+1}\}$, where a_i, a_{i+1} are the ith and $(i+1)$st columns of the matrix A. Since the image is a one-dimensional subspace that cannot coincide with $\mathbb{C}a_i$, it is of the form $\mathbb{C}(a_{i+1} + za_i)$ for a unique $z \in \mathbb{C}$. The result follows. $\square$

Remark 4.13. Note that Corollary 4.6 also follows from Theorem 4.11 by results of [18], since an open Richardson variety is an open brick variety for the word considered in Corollary 4.6. Resolutions of Richardson varieties via fibers of the Bott-Samelson map first appeared in [2], reformulating constructions in [3]; the latter were also studied in [43]. $\square$

Remark 4.14. In [8, Theorem 2.34] we compared braid varieties with the *diagonal* open Bott-Samelson varieties. These are subvarieties of $\text{OBS}(\beta)$ defined by an additional condition $\mathscr{F}^0 = \mathscr{F}^l$, but this flag (which is simultaneously the first and the last one) is not fixed. Open brick varieties are defined by a different condition, as one fixes both the first and the last flags. This difference explains the additional appearance of G and B in [8, Theorem 2.34].[7] $\square$

Since the brick manifold $\text{brick}(\beta)$ depends on the braid word $\beta \in \mathcal{B}$, and the braid variety $X(\beta)$ does not, Theorem 4.11.(ii) allows us to construct many different compactifications for a given braid variety. These compactification are all smooth and projective and the compactifying divisor is a smooth normal crossing divisor.

Example 4.15. *Let us consider the equivalent braid words*

$$\beta_1 = \sigma_1\sigma_2\sigma_1\sigma_2\sigma_1, \quad \beta_2 = \sigma_1\sigma_2\sigma_2\sigma_1\sigma_2.$$

In both cases, the braid varieties are algebraic tori

$$X(\beta_1; w_0) \cong X(\beta_2; w_0) \cong (\mathbb{C}^*)^2.$$

The variety $\text{brick}(\beta_1)$ has $X(\beta_1; w_0)$ as an open stratum, 5 codimension-1 strata (all isomorphic to $\mathbb{C}^*$), and 5 codimension-2 strata, which are points. In fact, $\text{brick}(\beta_1)$ is a toric degree 5 del Pezzo surface, i.e. the toric variety associated to the pentagon, and these various strata correspond to toric orbits. In contrast, $X(\sigma_1\sigma_2^3; w_0)$ is empty and there are only 4 codimension-1 strata and 4 codimension-2 strata in $\text{brick}(\beta_2)$. In fact, $\text{brick}(\beta_2) \cong \mathbb{P}^1 \times \mathbb{P}^1$, which is a different toric variety. $\square$

Theorem 4.11 also brings forth a connection between braid varieties and the subword complexes introduced in [44]. For that, note that the open brick variety $\text{brick}^\circ(\beta)$ is the higher-dimensional stratum in the stratification of the brick variety $\text{brick}(\beta)$ given in [18, Theorem 24], and we claim that all other strata of this stratification can also be realized as braid varieties, as follows.

Let β' be a subword of β such that the Demazure product of β' coincides with that of β. Then, $X(\beta'; \delta(\beta))$ is a strata of $\text{brick}(\beta)$, given by the conditions that $\mathscr{F}^j = \mathscr{F}^{j+1}$ whenever $i_{j+1} \notin \beta'$. This stratification is dual to the *subword complex* $(\beta, \delta(\beta))$, as defined in [44]. Subword complexes are defined for arbitrary pairs (β, π), where π is an element of a finite Coxeter group and β is a word in simple generators; the latter can also be seen as a positive braid word in the corresponding braid group. In [44] it is proven that a subword complex is spherical if and only if $\delta(\beta) = \pi$. Thus, brick manifolds bijectively correspond to spherical subword complexes and they are stratified by braid varieties, with the adjacency of strata described by the dual complexes.

---

[7]Note that the Borel subgroup B is denoted by $\mathcal{B}$ in [8, Theorem 2.34].

Example 4.16. *Let us choose $n = 2$ and $\beta = \sigma_1^3$. The braid variety $X(\sigma_1^3; s_1)$ is a smooth surface in $\mathbb{C}^3$ defined by the equation*

$$X(\sigma_1^3; s_1) = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 + z_3(1 + z_1 z_2) = 0\}.$$

If $1 + z_1 z_2 = 0$, we get $z_1 = 0$, and thus come to a contradiction. Therefore, $1 + z_1 z_2 \neq 0$. Then $z_3 = -\frac{z_1}{1+z_1 z_2}$, and so $X(\sigma_1^3; s_1)$ is isomorphic to the complement

$$Y = \{(z_1, z_2) \in \mathbb{C}^2 : 1 + z_1 z_2 \neq 0\}$$

of a smooth hyperbola in $\mathbb{C}^2$. The braid variety $X(\sigma_1^2; s_1)$ is isomorphic to $\mathbb{C}^*$ and $X(\sigma_1; s_1)$ is a point. The corresponding brick manifold is $\mathrm{brick}(\sigma_1^3) \cong \mathbb{P}^1 \times \mathbb{P}^1$, and these different braid varieties stratify it as follows. Consider the homogeneous coordinates $(x, y) \in \mathbb{P}^1 \times \mathbb{P}^1$, and denote $[0:1]$ by 0 and $[1:0]$ by ∞. Note that the homogenized hyperbola $\bar{C}$ contains the points $(0, \infty)$ and $(\infty, 0)$. The stratification of the brick variety $\mathbb{P}^1 \times \mathbb{P}^1$ given by the braid varieties has the following seven strata:

- *One 2-dimensional stratum Y, which is the complement of the smooth hyperbola in $\mathbb{C}^2$.*

- *Three 1-dimensional strata, each isomorphic to $\mathbb{C}^*$. Two such strata are given by*

$$S_1 := \{(x, y) \in \mathbb{P}^1 \times \mathbb{P}^1 : x = \infty, \quad y \neq 0, \infty\}, \quad S_2 := \{(x, y) \in \mathbb{P}^1 \times \mathbb{P}^1 : x \neq 0, \infty, \quad y = \infty\},$$

  and the third one $S_3 := \{(z_1, z_2) \in \mathbb{C}^2 : 1 + z_1 z_2 = 0\}$ is the affine hyperbola itself.

- *Three 0-dimensional strata, each a point: $p_1 := (\infty, 0), p_2 := (0, \infty)$ and $p_3 := (\infty, \infty)$.*

The subvarieties $\overline{S_1} := S_1 \cup \{p_1, p_3\}$, $\overline{S_2} := S_2 \cup \{p_2, p_3\}$ and $\overline{S_3} := S_3 \cup \{p_1, p_2\}$ are all smooth projective lines $\mathbb{P}^1 \subseteq \mathrm{brick}(\sigma_1^3)$, transversely intersecting each other. The compactifying divisor of $X(\sigma_1^3; s_1) \subseteq \mathrm{brick}(\sigma_1^3)$ is the union $\overline{S_1} \cup \overline{S_2} \cup \overline{S_3}$. Its dual complex, which records the intersections of these strata, coincides with the spherical subword complex for $\beta = \sigma_1^3$. □

4.5. **Equivariant homology and Lefschetz property for braid varieties.** Theorem 4.11.(ii) also allows us to compute the equivariant homology of braid varieties, as follows.

By Theorem 4.11.(ii), we can express the weight filtration on homology of the braid variety $X(\beta; w_0)$ in terms of homologies of brick manifolds $\mathrm{brick}(\beta_I)$ as follows. Consider the big complexes

$$C_\bullet := \bigoplus_I H_*(\mathrm{brick}(\beta_I)), \quad C_\bullet^T := \bigoplus_I H_*^T(\mathrm{brick}(\beta_I))$$

where the summation is over the subsets $I \subseteq [1, \ell]$ such that $\delta(\beta_I) = \delta(\beta)$, and the differential is given by inclusions $\mathrm{brick}(\beta_I) \hookrightarrow \mathrm{brick}(\beta_J)$ for $I \subset J, |J| = |I| + 1$. The torus $T = (\mathbb{C}^*)^{n-1}$ has a natural action on braid varieties as in [8, Section 2.2], and the brick compactification is T-equivariant. One can also consider the complexes

$$C^\bullet := \bigoplus_I H^*(\mathrm{brick}(\beta_I)), \quad C_T^\bullet := \bigoplus_I H_T^*(\mathrm{brick}(\beta_I)),$$

where all inclusions of brick manifolds correspond to maps in cohomology in the opposite direction.

Proposition 4.17. *In the notation above, we have the isomorphism*

$$H_*(C_\bullet) = \mathrm{gr}^W H_*(X(\beta; w_0)), \quad H_*(C_\bullet^T) = \mathrm{gr}^W H_*^T(X(\beta; w_0)),$$

where gr^W is the associated graded for the weight filtration.

Proof. By Theorem 4.11.(ii), the homology of the complex $C_\bullet$ (resp. $C_\bullet^T$) is the E_2-page of the spectral sequence computing the homology (resp. T-equivariant homology) of $X(\beta; w_0)$. Since all brick manifolds are smooth and projective, the weight filtrations in their homology agree with the homological gradings. By [14, 15], the higher differentials preserve weights and the spectral sequence collapses at the E_2 page. Therefore we obtain the equality

$$H_*(C_\bullet) = \mathrm{gr}^W H_*(X(\beta; w_0)), \quad H_*(C_\bullet^T) = \mathrm{gr}^W H_*^T(X(\beta; w_0)).$$

□

In order to describe the complexes $C_T^\bullet$ in more detail, we need notation from the theory of Soergel bimodules, see [17] for details and references. Let

$$R := \mathbb{C}[x_1, \ldots, x_n] = H_T^*(\text{pt}).$$

We need the following R-R- bimodules. For $1 \leq i \leq n-1$, we write

$$(4.1) \qquad \mathcal{B}_i := \frac{\mathbb{C}[x_1, \ldots, x_n, x'_1, \ldots, x'_n]}{x_i + x_{i+1} = x'_i + x'_{i+1}, x_i x_{i+1} = x'_i x'_{i+1}, x_m = x'_m \quad (m \neq i, i+1)}.$$

Note that we have a natural map $b_i : \mathcal{B}_i \to R$ defined by

$$b_i(f(x_1, \ldots, x_n, x'_1, \ldots, x'_n) = f(x_1, \ldots, x_n, x_1, \ldots, x_n).$$

Given a braid word $\beta = s_{i_1} \ldots s_{i_r}$, we define the *Bott-Samelson bimodule*

$$\mathcal{B}_\beta = \mathcal{B}_{i_1} \bigotimes_R \cdots \bigotimes_R \mathcal{B}_{i_r}.$$

Alternatively, one can write $\mathcal{B}_\beta$ as a quotient of the polynomial ring $\mathbb{C}\left[x_j^{(i)}, 0 \leq \ell \leq r, 1 \leq j \leq n\right]$ by the relations of the form (4.1) between $(x_j^{(\ell)})_{j \in [1,n]}$ and $(x_j^{(\ell+1)})_{j \in [1,n]}$ for all $\ell \in [0, r-1]$.

Proposition 4.18. *In the notation above, the following hold:*

(a) *We have the isomorphism $H_T^*(\text{BS}(\beta)) \simeq \mathcal{B}_\beta$.*

(b) *If β' is a subword of β then we have a commutative diagram*

$$
\begin{array}{ccc}
H_T^*(\text{BS}(\beta)) & \xrightarrow{\simeq} & \mathcal{B}_\beta \\
\downarrow & & \downarrow \\
H_T^*(\text{BS}(\beta')) & \xrightarrow{\simeq} & \mathcal{B}_{\beta'},
\end{array}
$$

where the vertical arrows correspond to the inclusion $\text{BS}(\beta') \hookrightarrow \text{BS}(\beta)$ and to the composition of maps b_i respectively.

Proof. These facts are well-established, cf. [17], but we review the key geometric ideas for completeness. The variables $x_j^{(\ell)}$ correspond to the first Chern classes of the tautological line bundles $\mathcal{L}_j^{(\ell)} = \mathscr{F}_j^\ell / \mathscr{F}_{j-1}^\ell$ on the Bott-Samelson variety $\text{BS}(\beta)$. Since $\text{BS}(\beta)$ is a tower of $\mathbb{P}^1$-bundles, it can be proven that $x_j^{(\ell)}$ generate the equivariant cohomology. If two flags $\mathscr{F}$ and $\mathscr{F}'$ are in position s_i, and $\mathcal{L}_i = \mathscr{F}_i / \mathscr{F}_{i-1}$, $\mathcal{L}'_i = \mathscr{F}'_i / \mathscr{F}'_{i-1}$, then the rank two bundle

$$\mathscr{F}_{i+1}/\mathscr{F}_{i-1} = \mathscr{F}'_{i+1}/\mathscr{F}'_{i-1}$$

is filtered both by $\mathcal{L}_i, \mathcal{L}_{i+1}$ and by $\mathcal{L}'_i, \mathcal{L}'_{i+1}$, hence the first Chern classes $x_i = c_1(\mathcal{L}_i)$, $x'_i = c_1(\mathcal{L}'_i)$ satisfy the relations (4.1). This implies Part (a).

For Part (b), consider the inclusion of the diagonal $\text{BS}(1) \hookrightarrow \text{BS}(s_i)$ and note that the corresponding map in equivariant cohomology agrees with $b_i : \mathcal{B}_i \to R$. $\qquad \square$

Lemma 4.19. *Assume that $\delta(\beta) = w_0$, and let*

$$\overline{\mathcal{B}_\beta} = \mathcal{B}_\beta / \left(x_j^{(0)} = x_{n+1-j}^{(r)}, \quad 1 \leq j \leq n\right).$$

Then the following hold:

(a) *We have an isomorphism $H_T^*(\text{brick}(\beta)) \simeq \overline{\mathcal{B}_\beta}$.*

(b) *If β' is a subword of β with $\delta(\beta') = \delta(\beta) = w_0$ then we have a commutative diagram*

$$
\begin{array}{ccc}
H_T^*(\text{brick}(\beta)) & \xrightarrow{\simeq} & \overline{\mathcal{B}_\beta} \\
\downarrow & & \downarrow \\
H_T^*(\text{brick}(\beta')) & \xrightarrow{\simeq} & \overline{\mathcal{B}_{\beta'}}.
\end{array}
$$

Proof. The key observation is that, by [37, Corollary 6.5(b)] and [60, Corollary 4.12], the restriction map in equivariant cohomology

$$H_T^*(\mathrm{BS}(\beta)) \to H_T^*(\mathrm{brick}(\beta))$$

is surjective. In particular, $H_T^*(\mathrm{brick}(\beta))$ is generated by $x_j^{(\ell)}$ as in Proposition 4.18 subject to additional relations $x_j^{(0)} = x_{n+1-j}^{(r)}$. Part (b) is immediate from Proposition 4.18(b). $\square$

By combining the above results, we can relate the homology of braid varieties to Khovanov-Rozansky link homology [40, 41]. The following result gives an alternative proof of [63, Corollary 4].

Theorem 4.20. *Let β be a positive braid and suppose that $\delta(\beta) = w_0$. Then we have an isomorphism*

$$\mathrm{gr}^W H_T^*(X(\textrm{\reflectbox{Q}}; w_0)) \simeq \mathrm{HHH}^{a=n}(\beta\Delta) \simeq \mathrm{HHH}^{a=0}(\beta\Delta^{-1}),$$

where $\mathrm{HHH}^{a=0}$, resp. $\mathrm{HHH}^{a=n}$, is the Khovanov-Rozansky homology in lowest, resp. highest, a-degree.

Proof. We prove this by combining the above results. Similarly to Proposition 4.17 we get

$$\mathrm{gr}^W H_T^*(X(\textrm{\reflectbox{Q}}; w_0)) \simeq H^*(C_T^\bullet)$$

where

$$C_T^\bullet = \bigoplus_I H_T^*(\mathrm{brick}(\beta_I)),$$

the sum runs over subsets I such that $\delta(\beta_I) = w_0$ and the differential is given by the restriction maps in equivariant cohomology corresponding to the inclusions $\mathrm{brick}(\beta_I) \hookrightarrow \mathrm{brick}(\beta_J)$.

By Lemma 4.19 we can replace each $H_T^*(\mathrm{brick}(\beta_I))$ by $\overline{\mathcal{B}_{\beta_I}}$ and the maps between these by the maps b_i. Furthermore, one can show using the results of [47] that $\overline{\mathcal{B}_{\beta_I}} \simeq \mathrm{Hom}(T_{w_0}, \mathcal{B}_{\beta_I})$, where T_{w_0} is the Rouquier complex [17] corresponding to Δ. By the main result of [47] (see also [34, Corollary A.6]) we have $\mathrm{Hom}(T_{w_0}, \mathcal{B}_{\beta_I}) = 0$ whenever $\delta(\beta_I) \neq w_0$.

To sum up, we can replace the complex $C_T^\bullet$ by $\bigoplus_I \mathrm{Hom}(T_{w_0}, \mathcal{B}_{\beta_I})$ where the sum runs over all possible subsets I, and the restriction maps are induced by the maps b_i. This latter complex is precisely

$$\bigoplus_I \mathrm{Hom}(T_{w_0}, \mathcal{B}_{\beta_I}) = \mathrm{Hom}\left(T_{w_0}, [\mathcal{B}_{i_1} \xrightarrow{b_{i_1}} R] \otimes_R \cdots \otimes_R [\mathcal{B}_{i_r} \xrightarrow{b_{i_r}} R]\right) =$$

$$\mathrm{Hom}(T_{w_0}, T_{i_1} \bigotimes_R \cdots \bigotimes_R T_{i_r}) = \mathrm{Hom}(T_{w_0}, T_\beta) = \mathrm{Hom}(R, T_\beta T_{w_0}^{-1}) = \mathrm{HHH}^{a=0}(\beta\Delta^{-1}).$$

Finally, by the main result of [34] we conclude $\mathrm{HHH}^{a=0}(\beta\Delta^{-1}) \simeq \mathrm{HHH}^{a=n}(\beta\Delta)$. $\square$

Finally, [51] studied the *curious Lefschetz property*, as defined by [38], for cohomology rings of character varieties. In [51], a stratification of a certain vector bundle over a given character variety is shown to be equal to a stratification by vector bundles over braid varieties. In particular, [51] proves the curious Lefschetz property for each braid variety. Therefore, Corollary 4.6 implies:

Corollary 4.21. *Let $u, w \in S_n$ be such that $u \leq w$ in Bruhat order. Then the open Richardson variety $\mathcal{R}^\circ(u, w)$ for $G := SL(n, \mathbb{C})$ satisfies the curious Lefschetz property.* $\square$

This result was first conjectured by T. Lam and D. Speyer [45, Section 1.5.1], see also further discussion in a recent paper by P. Galashin and T. Lam [29].

5. CONCLUDING REMARKS ON CLUSTER STRUCTURES AND LEGENDRIAN LINKS

Let us conclude this article with a few comments and conjectures on cluster structures. The reader is referred to [25, 26] and [22, 21] for the necessary preliminaries on cluster structures, and see also [23, 27, 35, 45, 52, 53, 54, 58]. See [11, Section 2.8] and [24, Section 6] for a discussion on cluster structures on spaces, and we refer to [5, 6] and references therein for recent developments.

Let $\Lambda \subseteq (\mathbb{R}^3, \xi_{st})$ be a Legendrian link and $T \subseteq \Lambda$ a set of marked points, with at least one marked point per component. From now onward, unless otherwise specified, we assume that Λ is isotopic to the Legendrian lift Λ_β of a (-1)-closure of a positive braid β. Let us denote by $\mathfrak{M}(\Lambda, T)$ either of the two smooth affine varieties:

(1) $\mathfrak{M}(\Lambda, T) = \mathrm{Spec}(H^0(\mathcal{A}(\Lambda, T)))$, where $\mathcal{A}(\Lambda, T)$ denotes the (commutative) Legendrian contact dg-algebra associated to (Λ, T). This dg-algebra was defined in [12], see [20] for a survey.

(2) $\mathfrak{M}(\Lambda, T)$ is the moduli stack of pseudo-perfect objects of microlocal rank one in $\mathrm{Sh}_\Lambda(\mathbb{R}^2, T)$ with a trivialization of the microlocal functor on $\Lambda \setminus T$. See [11, Section 2.7] for more details.

These varieties are isomorphic, cf. [11, Section 4] and [10, Prop. 5.2], or [55]. If T is chosen so that every strand of the positive braid word β associated to $\Lambda = \Lambda_\beta$ has exactly one marked point, then $\mathfrak{M}(\Lambda_\beta, T)$ is isomorphic to the braid variety of β, cf. [8, Section 2]. In [7] we proved that the algebra of regular functions $\mathbb{C}[\mathfrak{M}(\Lambda, T)]$ is a cluster algebra satisfying many interesting properties, such as local acyclicity and the existence of a Donaldson-Thomas transformation, see also [11, 31]. This fact was conjectured in the initial version of this manuscript, prior to ibid.

Let us consider a contact isotopy $\{\phi_t\}_{t \in [0,1]}$, so that each $\phi_t \in \mathrm{Cont}^c(\mathbb{R}^3, \xi_{st})$ is a compactly supported contactomorphism and $\phi_0 = \mathrm{Id}$. Note that any Legendrian isotopy extends to a contact isotopy [33]. Then the contact isotopy $\{\phi_t\}_{t \in [0,1]}$ induces an algebraic isomorphism of affine varieties

$$\Phi_t : \mathfrak{M}(\Lambda, T) \longrightarrow \mathfrak{M}(\phi_t(\Lambda), \phi_t(T)).$$

Indeed, this is proved in [12] for the Floer-theoretic approach with the dg-algebra, and holds by [36] in the sheaf-theoretic approach. We conjecture that these are not just isomorphisms of affine varieties, but also quasi-cluster isomorphisms. See [27] for details on quasi-cluster structures and their morphisms. The precise conjecture reads as follows:

Conjecture 5.1 (Legendrian invariance of cluster structures)**.** *Let Λ_0 and Λ_1 be two Legendrian links in $(\mathbb{R}^3, \xi_{st})$ and let $\{\phi_t\}$ be a contact isotopy $\phi_t \in \mathrm{Cont}^c(\mathbb{R}^3, \xi_{st})$, $t \in [0,1]$, such that $\phi_1(\Lambda_0) = \Lambda_1$. Consider a set of marked points $T_0 \subseteq \Lambda_0$, with at least one marked point per component and write $T_1 := \phi_1(T)$. Then the induced algebraic morphism*

$$\Phi_t^* : \mathbb{C}[\mathfrak{M}(\Lambda_1, T_1)] \longrightarrow \mathbb{C}[\mathfrak{M}(\Lambda_0, T_0)]$$

is a quasi-cluster isomorphism, where both target and domain are endowed with the cluster structures constructed in [7]. That is, the quasi-cluster isomorphism type of the cluster algebra $\mathbb{C}[\mathfrak{M}(\Lambda, T)]$ is a Legendrian isotopy invariant of a Legendrian link $\Lambda \subseteq (\mathbb{R}^3, \xi_{st})$ with a set of marked points $T \subseteq \Lambda$.

Following the symplectic geometry behind the construction of the cluster structures in [7, 11], it is natural to expect a generalization of our previous work and the above conjecture to any Legendrian link $\Lambda \subseteq (\mathbb{R}^3, \xi_{st})$, even if it is not the (-1)-closure of a positive braid. A first challenge is structural: the spaces $\mathfrak{M}(\Lambda, T)$ are in general just D^--stacks and there is no definition known to us of what it means for a derived stack to admit a cluster structure. More generally, both the dg-category $\mathrm{Sh}_\Lambda(\mathbb{R}^2, T)$ of sheaves above and the stable tame isomorphism type of $\mathcal{A}(\Lambda, T)$ are Legendrian invariant. The geometry indicates that they should admit a sort of "cluster structure"; a challenge is that we do not know what it means for a (finite type) dg-category to admit a cluster structure[8] or for a dg-algebra to be a "cluster" dg-algebra. The characteristic algebra, as introduced in [56, Section 3], or the graded augmentation variety, modulo dg-homotopies, might be reasonable objects to study from this viewpoint. In either case, based on the geometry, we conjecture that such generalizations of cluster structures exist and are Legendrian invariant, up to quasi-cluster isomorphisms.

References

[1] Johan Asplund, Youngjin Bae, Orsola Capovilla-Searle, Marco Castronovo, Caitlin Leverson, and Angela Wu. Lagrangian cobordism of positroid links. *arXiv preprint arXiv:2305.16232*, 2023.

[2] Michaël Balan. Standard monomial theory for desingularized Richardson varieties in the flag variety $GL(n)/B$. *Transformation Groups*, 18(2):329–359, 2013.

[3] Michel Brion. Lectures on the geometry of flag varieties. In *Topics in cohomological studies of algebraic varieties*, pages 33–85. Springer, 2005.

[4] Ken A Brown, Kenneth R Goodearl, and Milen Yakimov. Poisson structures on affine spaces and flag varieties. i. matrix affine poisson space. *Advances in Mathematics*, 206(2):567–629, 2006.

[5] Roger Casals and Honghao Gao. Infinitely many Lagrangian fillings. *Ann. of Math. (2)*, 195(1):207–249, 2022.

[6] Roger Casals and Honghao Gao. A Lagrangian filling for every cluster seed. Preprint, 2023.

---

[8]A desirable feature would be that the "ring of regular functions" of its derived stack of pseudo-perfect objects – e.g. supposing it is isomorphic to an affine scheme – is a cluster algebra, again in the appropriate sense.

[7] Roger Casals, Eugene Gorsky, Mikhail Gorsky, Ian Le, Linhui Shen, and José Simental. Cluster structures on braid varieties. *arXiv preprint arXiv:2207.11607*, 2022.

[8] Roger Casals, Eugene Gorsky, Mikhail Gorsky, and José Simental. Algebraic weaves and braid varieties. arXiv:2012.06931, 2020.

[9] Roger Casals, Ian Le Le, Melissa Sherman-Bennett, and Daping Weng. Demazure weaves for reduced plabic graphs (with a proof that Muller-Speyer twist is Donaldson-Thomas). Preprint, 2023.

[10] Roger Casals and Lenhard Ng. Braid loops with infinite monodromy on the Legendrian contact DGA. *J. Topol.*, 15(4):1927–2016, 2022.

[11] Roger Casals and Daping Weng. Microlocal theory of legendrian links and cluster algebras. *Geometry & Topology*, pages 1–119, 2023.

[12] Yuri Chekanov. Differential algebra of Legendrian links. *Invent. Math.*, 150(3):441–483, 2002.

[13] Patrick Dehornoy. *Braids and self-distributivity*, volume 192 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2000.

[14] Pierre Deligne. Théorie de Hodge: II. *Publications Mathématiques de l'IHÉS*, 40:5–57, 1971.

[15] Pierre Deligne. Théorie de Hodge, III. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 44(1):5–77, 1974.

[16] Ben Elias. A diamond lemma for Hecke-type algebras. *Trans. Amer. Math. Soc.*, 375(3):1883–1915, 2022.

[17] Ben Elias, Shotaro Makisumi, Ulrich Thiel, and Geordie Williamson. *Introduction to Soergel bimodules*, volume 5. Springer Nature, 2020.

[18] Laura Escobar. Brick manifolds and toric varieties of brick polytopes. *The Electronic Journal of Combinatorics*, pages P2–25, 2016.

[19] John B. Etnyre. Legendrian and transversal knots. In *Handbook of knot theory*, pages 105–185. Elsevier B. V., Amsterdam, 2005.

[20] John B. Etnyre and Lenhard L. Ng. Legendrian contact homology in $\mathbb{R}^3$. In *Surveys in differential geometry 2020. Surveys in 3-manifold topology and geometry*, volume 25 of *Surv. Differ. Geom.*, pages 103–161. Int. Press, Boston, MA, [2022] ©2022.

[21] V. V. Fock and A. B. Goncharov. Cluster X-varieties, amalgamation, and Poisson-Lie groups. In *Algebraic geometry and number theory*, volume 253 of *Progr. Math.*, pages 27–68. Birkhäuser Boston, Boston, MA, 2006.

[22] Vladimir Fock and Alexander Goncharov. Moduli spaces of local systems and higher Teichmüller theory. *Publ. Math. Inst. Hautes Études Sci.*, (103):1–211, 2006.

[23] Vladimir V Fock and Alexander B Goncharov. Cluster ensembles, quantization and the dilogarithm. In *Annales scientifiques de l'École Normale Supérieure*, volume 42, pages 865–930, 2009.

[24] Sergey Fomin, Lauren Williams, and Andrei Zelevinsky. Introduction to Cluster Algebras. *arXiv preprint arXiv:2106.02160*, 2021.

[25] Sergey Fomin and Andrei Zelevinsky. Cluster algebras. I. Foundations. *J. Amer. Math. Soc.*, 15(2):497–529, 2002.

[26] Sergey Fomin and Andrei Zelevinsky. Cluster algebras. II. Finite type classification. *Invent. Math.*, 154(1):63–121, 2003.

[27] Chris Fraser. Quasi-homomorphisms of cluster algebras. *Advances in Applied Mathematics*, 81:40–77, 2016.

[28] Dmitry Fuchs. Chekanov-Eliashberg invariant of Legendrian knots: existence of augmentations. *J. Geom. Phys.*, 47(1):43–65, 2003.

[29] Pavel Galashin and Thomas Lam. Positroids, knots, and q,t-Catalan numbers. *Duke Math. J.*, to appear.

[30] Pavel Galashin, Thomas Lam, and Melissa Sherman-Bennett. Braid variety cluster structures, II: general type. *arXiv preprint arXiv:2301.07268*, 2023.

[31] Pavel Galashin, Thomas Lam, Melissa Sherman-Bennett, and David Speyer. Braid variety cluster structures, I: 3D plabic graphs. *arXiv preprint arXiv:2210.04778*, 2022.

[32] F. A. Garside. The braid group and other groups. *Quart. J. Math. Oxford Ser. (2)*, 20:235–254, 1969.

[33] Hansjörg Geiges. *An introduction to contact topology*, volume 109 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2008.

[34] Eugene Gorsky, Matthew Hogancamp, Anton Mellit, and Keita Nakagane. Serre duality for Khovanov-Rozansky homology. *Selecta Math. (N.S.)*, 25(5):Paper No. 79, 33, 2019.

[35] Mark Gross, Paul Hacking, Sean Keel, and Maxim Kontsevich. Canonical bases for cluster algebras. *Journal of the American Mathematical Society*, 31(2):497–608, 2018.

[36] Stéphane Guillermou, Masaki Kashiwara, and Pierre Schapira. Sheaf quantization of Hamiltonian isotopies and applications to nondisplaceability problems. *Duke Math. J.*, 161(2):201–245, 2012.

[37] Martin Härterich. The T-equivariant cohomology of Bott-Samelson varieties. *arXiv preprint math/0412337*, 2004.

[38] Tamás Hausel and Fernando Rodriguez-Villegas. Mixed Hodge polynomials of character varieties. *Inventiones mathematicae*, 174(3):555–624, 2008.

[39] Tamás Kálmán. Contact homology and one parameter families of Legendrian knots. *Geom. Topol.*, 9:2013–2078, 2005.

[40] Mikhail Khovanov. Triply-graded link homology and Hochschild homology of Soergel bimodules. *Internat. J. Math.*, 18(8):869–885, 2007.

[41] Mikhail Khovanov and Lev Rozansky. Matrix factorizations and link homology. II. *Geom. Topol.*, 12(3):1387–1425, 2008.

[42] Allen Knutson, Thomas Lam, and David E Speyer. Positroid varieties: juggling and geometry. *Compositio Mathematica*, 149(10):1710–1752, 2013.

[43] Allen Knutson, Thomas Lam, and David E Speyer. Projections of Richardson varieties. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2014(687):133–157, 2014.

[44] Allen Knutson and Ezra Miller. Subword complexes in Coxeter groups. *Advances in Mathematics*, 184(1):161–176, 2004.
[45] Thomas Lam and David E. Speyer. Cohomology of cluster varieties I: Locally acyclic case. *Algebra Number Theory*, 16(1):179–230, 2022.
[46] C. Leverson. Augmentations and rulings of Legendrian knots. *J. Symplectic Geom.*, 14(4):1089–1143, 2016.
[47] Nicolas Libedinsky and Geordie Williamson. Standard objects in 2-braid groups. *Proceedings of the London Mathematical Society*, 109(5):1264–1280, 2014.
[48] George Lusztig. Total positivity in reductive groups. In *Lie theory and geometry*, pages 531–568. Springer, 1994.
[49] George Lusztig. Total positivity in partial flag manifolds. *Representation Theory of the American Mathematical Society*, 2(3):70–78, 1998.
[50] Yu. I. Manin and V. V. Schechtman. Arrangements of hyperplanes, higher braid groups and higher Bruhat orders. In *Algebraic number theory*, volume 17 of *Adv. Stud. Pure Math.*, pages 289–308. Academic Press, Boston, MA, 1989.
[51] Anton Mellit. Cell decompositions of character varieties. *arXiv preprint arXiv:1905.10685*, 2019.
[52] Matthew R Mills. Maximal green sequences for quivers of finite mutation type. *Advances in Mathematics*, 319:182–210, 2017.
[53] Greg Muller. Locally acyclic cluster algebras. *Advances in Mathematics*, 233(1):207–247, 2013.
[54] Greg Muller and David Speyer. Cluster algebras of Grassmannians are locally acyclic. *Proceedings of the American Mathematical Society*, 144(8):3267–3281, 2016.
[55] Lenhard Ng, Dan Rutherford, Vivek Shende, Steven Sivek, and Eric Zaslow. Augmentations are Sheaves. *Geom. Topol.*, 24(5):2149–2286, 2020.
[56] Lenhard L. Ng. Computable Legendrian invariants. *Topology*, 42(1):55–82, 2003.
[57] Alexander Postnikov. Total positivity, Grassmannians, and networks. *http://math.mit.edu/ apost/papers/tpgrass.pdf*, 2006.
[58] Fan Qin. Bases for upper cluster algebras and tropical points. *arXiv preprint arXiv:1902.09507*, 2019.
[59] Konstanze Christina Rietsch. *Total positivity and real flag varieties*. PhD thesis, Massachusetts Institute of Technology, 1998.
[60] Vladimir Shchigolev. Bases of T-equivariant cohomology of Bott-Samelson varieties. *J. Aust. Math. Soc.*, 104(1):80–126, 2018.
[61] Vivek Shende, David Treumann, Harold Williams, and Eric Zaslow. Cluster varieties from Legendrian knots. *Duke Math. J.*, 168(15):2801–2871, 2019.
[62] David E Speyer. Richardson varieties, projected Richardson varieties and positroid varieties. *arXiv preprint arXiv:2303.04831*, 2023.
[63] Minh-Tâm Quang Trinh. From the Hecke category to the unipotent locus. *arXiv preprint arXiv:2106.07444*, 2021.

Dept. of Mathematics, University of California, Davis, One Shields Avenue, Davis, CA, USA

Email address: `casals@math.ucdavis.edu`

Dept. of Mathematics, University of California, Davis, One Shields Avenue, Davis, CA, USA

Email address: `egorskiy@math.ucdavis.edu`

Department of Mathematics, University of Vienna, Oscar-Morgenstern Platz 1, 1090 Vienna, Austria

Email address: `mikhail.gorskii@univie.ac.at`

Instituto de Matemáticas, Universidad Nacional Autónoma de México. Mexico City, Mexico.

Email address: `simental@im.unam.mx`